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Synchronization of Pulse-Coupled Oscillators to a Global Pacemaker

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Abstract

Pulse-coupled oscillators (PCOs) are limit cycle oscillators coupled by exchanging pulses at discrete time instants. Their importance in biology and engineering has motivated numerous studies aiming to understand the basic synchronization properties of a network of PCOs. In this work, we study synchronization of PCOs subject to a global pacemaker (or global cue) and local interactions between slave oscillators. We characterize solutions and give synchronization conditions using the phase response curve (PRC) as the design element. It is shown that global synchronization is feasible when using an advance-delay PRC if the influence of the global cue is strong enough. Numerical examples are provided to illustrate the analytical findings.

Keywords:

Pulse-coupled oscillators, Synchronization, Phase models, Phase response curve

1. Introduction

Pulse-coupled oscillators (PCOs) are limit cycle oscillators that are coupled together to form a network by exchanging pulses at discrete time instants. A pulse has two effects on the network state: 1) it resets the phase at the originating oscillator, and 2) it induces a jump on the phase of the receiving oscillators. The magnitude of the impulsive jump induced is, in general, phase dependent and is given in the form of a phase response curve (PRC) Q [1]. Moreover, it is customary to include a coupling strength l to scale the effect of the PRC. In this setting, the value of l can be interpreted as the extra energy needed to synchronize the system, as is indeed the case when PCOs are realized using passive circuits, or as an extra gain present at the receiver side.

The dynamics of a network of PCOs, and thus its synchronization properties, are fully determined by

the interaction topology (communication network) \mathcal{R} , the number of oscillators in the network N , the initial phases x_0 , and the feedback strategy given by Q and l , i.e., the PRC and the coupling strength. Despite the simple formulation and behavior of an isolated firing oscillator, a network of PCOs is able to exhibit intricate collective dynamics. For this reason, PCOs have emerged as a powerful modeling and design tool in complex networked biological and engineering systems. Examples of biological systems that have been modeled using PCOs include cardiac pacemakers [2], crickets that chirp in unison [3], and rhythmic flashing of fireflies [4]. While one of the most important applications of PCOs in engineering is time synchronization in sensor networks [5, 6, 7].

In a network of PCOs, the role of each agent, i.e., leader or slave, also determines the resulting dynamics. In fact, in the achievement of synchronization the interplay between a global cue and local interactions between agents is an important feature [8]. For example, in the mammalian olfactory bulb, ensembles of neurons synchronize to discriminate odors by utilizing intercellular interplays among individual neurons while at the same time receiving a global

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driving odorant stimulus via the odorant receptors [9]. In engineering, the coordination of a network of unmanned ground vehicles is achieved by means of the interplay between individual vehicles and external coordination from the central resources [10].

In this work, we study the synchronization properties of a network of PCOs when there is a leader node, or global cue, that can reach every other node and does not react to any incoming pulse. In particular, this work refines and extends the results in [8] and [11]. In [8], the weak coupling assumption [12] is used to transform the impulsive dynamics of a PCO network into an ordinary differential equation via averaging. Synchronization is proven to emerge for arbitrary initial conditions when an advance-delay PRC is used; however, the PRC is restricted to be a continuous function, which introduces a zero crossing point that precludes global synchronization. An important finding in [8] is that for a network of PCOs, global synchronization to a leader is feasible only when the leader can reach every other node. However, when the initial conditions are restricted to half of the circle, the leader reaching a single node is a sufficient condition for synchronization. In [11], hybrid dynamical systems theory is used to allow the PRC to be a discontinuous mapping. However, the weak coupling assumption is also used, which limits the applicability in an artificial network of PCOs. Moreover, no guideline is given regarding the strength of the global coupling. In this paper, we remove the weak coupling assumption and prove that global synchronization is feasible when using a set-valued advance-delay PRC. Moreover, we provide an explicit bound for the global coupling that ensures global synchronization. We exploit the hybrid nature of pulse-coupled networks [13] to pose the synchronization problem as a set stabilization problem, which we solve using tools from hybrid systems theory.

1.1. Basic Notation and Definitions

In this work, \mathbb{R} denotes the real numbers, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers, $\mathbb{R}_{<0(>0)}$ denotes the negative (positive) real numbers, $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers, \mathbb{R}^n denotes the Euclidean space of dimension n , and $\mathbb{R}^{n \times n}$ denotes the set of $n \times n$ square matrices with real coefficients. For a countable set χ , we denote its car-

dinality as $|\chi|$; for two sets Λ_1 and Λ_2 , we denote their difference as $\Lambda_1 \setminus \Lambda_2$. A set-valued mapping $\Phi : A \rightrightarrows B$ associates to the element $\alpha \in A$ the set $\Phi(\alpha) \subseteq B$; the graph of a set-valued mapping is the set: $\text{graph}(\Phi) := \{(\alpha, \beta) \in A \times B : \beta \in \Phi(\alpha)\}$. A set-valued mapping Φ is outer semi-continuous if and only if its graph is closed [14, Theorem 5.7(a)].

1.2. Hybrid Systems Preliminaries

We follow the framework given in [15]. A hybrid system \mathcal{H} consists of continuous-time dynamics (flows), discrete-time dynamics (jumps), and sets on which these dynamics apply:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases} \quad (1)$$

where x is the state, the flow map F and the jump map G are set-valued mappings, $\mathcal{C} \subseteq \mathbb{R}^n$ is the flow set, and $\mathcal{D} \subseteq \mathbb{R}^n$ is the jump set, $(F, \mathcal{C}, G, \mathcal{D})$ is the data of \mathcal{H} . A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a hybrid time domain if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times j$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times j$, $[t_j, t_{j+1}) \times j$, or $[t_j, \infty) \times j$. A solution to \mathcal{H} is a function $\phi : \text{dom } \phi \rightarrow \mathbb{R}^n$ where $\text{dom } \phi$ is a hybrid time domain and for each fixed j , $t \mapsto \phi(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \text{dom } \phi\}$. ϕ is called a hybrid arc, and is such that: for each $j \in \mathbb{N}$ for which I_j has nonempty interior $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I_j$, $\phi(t, j) \in \mathcal{C}$ for all $t \in [\min I_j, \sup I_j]$; for each $(t, j) \in \text{dom } \phi$ for which $(t, j+1) \in \text{dom } \phi$, $\phi(t, j+1) \in G(\phi(t, j))$, $\phi(t, j) \in \mathcal{D}$. A solution ϕ is nontrivial if its domain contains at least one point different from $(0, 0)$, is maximal if it cannot be extended, and is complete if its domain is unbounded.

1.3. Graph Theory

Throughout this paper we use several concepts from algebraic graph theory [16]. Consider a network with $N \in \mathbb{Z}_{\geq 0}$ agents. The communication between agents is modeled by a weighted directed graph $\mathcal{R} = \{\mathcal{V}, \mathcal{E}_{\mathcal{R}}, \mathcal{A}_{\mathcal{R}}\}$, where $\mathcal{V} = \{1, \dots, N\}$ is the node set of the graph. $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of the graph, whose elements are such that $(i, k) \in \mathcal{E}_{\mathcal{R}}$ if and only if node k receives the pulse emitted

by node i ; we assume that the self edge $(i, i) \notin \mathcal{E}_{\mathcal{R}}$. $\mathcal{A}_{\mathcal{R}} = [a_{ik}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix of \mathcal{R} with $a_{ik} \in \{0, l\}$, where $a_{ik} = l \in (0, 1]$ if and only if $(i, k) \in \mathcal{E}_{\mathcal{R}}$. For a node i , $\mathcal{N}^{i-} = \{k \in \mathcal{V} : (i, k) \in \mathcal{E}_{\mathcal{G}}\}$ denotes the in-neighbors of node i , i.e., the set of nodes whose pulses are received by i , and $\mathcal{N}^{i+} = \{k \in \mathcal{V} : (k, i) \in \mathcal{E}_{\mathcal{G}}\}$ denotes the out-neighbors of node i , i.e., the set of nodes that receive pulses emitted by i .

2. Model and Problem Formulation

Mirollo and Strogatz [17] presented the classical formulation of a network of PCOs. The network is formed by N oscillators, where each oscillator $i \in \{1, 2, \dots, N\}$ follows

$$\dot{z}_i = f(x_i), \quad (2)$$

where $f : [0, 1] \rightarrow [0, 1]$ is smooth, monotonically increasing, and concave down, i.e., $f'(x_i) > 0$, $f''(x_i) < 0$, and $x_i \in [0, 1]$ is a phase-like variable such that

$$\frac{\partial x_i}{\partial t} = \frac{1}{T} = \omega \quad (3)$$

and $x_i = 1$ ($x_i = 0$) when the oscillator is at the end (start) of the cycle, i.e., when $z_i = 1$ ($z_i = 0$). Therefore, $f(0) = 0$ and $f(1) = 1$ holds. The oscillators are assumed to interact by a simple form of pulse coupling: when an oscillator fires it increases the state of all the other oscillators by an amount ϵ , or forces them to fire, whichever is less. That is,

$$\begin{aligned} z_i(t) = 1 &\Rightarrow z_i(t^+) = 0 \\ &\Rightarrow z_j(t^+) = \min(1, z_j(t) + \epsilon), \quad \forall j \neq i. \end{aligned} \quad (4)$$

In the following, we reformulate the PCO model in the hybrid systems framework, which allows us to consider an arbitrary feedback mapping (in contrast to the constant ϵ) and include explicitly the structure of an underlying communication graph. The particular network structure considered is the one where an omnipresent leader is part of the network, which we denote as the global cue or master node. In this setup, the network consists of a global cue and N slave oscillators aiming to synchronize their phases to the phase of the global cue. We assume that the slave os-

cillators interact on a given graph $\mathcal{R} = \{\mathcal{V}, \mathcal{E}_{\mathcal{R}}, \mathcal{A}_{\mathcal{R}}\}$, not necessarily connected. The phase of each slave oscillator evolves continuously following its natural frequency, and jumps impulsively upon receiving a pulse. The global cue is not affected by pulses, thus, its phase evolution is determined only by its natural frequency. Pulses are generated following an integrate-and-fire process, i.e., when its phase reaches the limit (2π in this case), the oscillator fires, i.e., emits a pulse, and resets its phase to 0. When an oscillator receives a pulse, it updates its phase according to the coupling strength and the PRC, which is defined in the framework of hybrid systems as follows:

Definition 1 (Phase Response Curve). A phase response curve (PRC), or phase resetting curve [1, 18], describes the change in the phase of an oscillator due to a pulse stimulus, as a function of the phase at which the pulse is received. A phase response curve $Q : [0, 2\pi] \Rightarrow \mathbb{R}_{\geq 0}$ is called an advance-only PRC. Similarly, a phase response curve $Q : [0, 2\pi] \Rightarrow \mathbb{R}_{\leq 0}$ is called a delay-only PRC. A phase response curve $Q : [0, 2\pi] \Rightarrow \mathbb{R}$ is called an advance-delay PRC if there exists $\bar{q}_1 \in Q(q_1)$ and $\bar{q}_2 \in Q(q_2)$, with q_1 and q_2 in $[0, 2\pi]$, satisfying $\bar{q}_1 > 0$ and $\bar{q}_2 < 0$.

Remark 1. In the mathematical neuroscience literature, advance-only PRCs are referred to as Type I PRCs. Similarly, advance-delays PRCs are referred to as Type II PRCs. To deal with a delay-only PRC the system is modeled as an inhibitory system, i.e., a system where the coupling strength is negative, coupled using a Type I PRC [12, 19].

2.1. Data

The network of N slave oscillators and 1 global cue is modeled as a hybrid system with state x given by:

$$x := [x_g, x_1, \dots, x_N]^T \quad (5)$$

where $x_i \in [0, 2\pi]$ denotes the phase of the i th slave oscillator and x_g denotes the phase of the global cue. The dynamics are given by:

$$\left. \begin{aligned} \dot{x}_g &= \omega_g \\ \dot{x}_i &= \omega \end{aligned} \right\} =: F(x), \quad x \in \mathcal{C} \quad (6)$$

$$\left. \begin{aligned} x_g^+ &= 0 \\ x_i^+ &\in \text{sat}_0^{2\pi}(x_i + g_i Q_g(x_i)) \end{aligned} \right\} =: G_g(x), \quad x \in \mathcal{D}_g \quad (7)$$

$$\left. \begin{aligned} x_g^+ &= x_g \\ x_i^+ &= 0 \\ x_k^+ &\in \text{sat}_0^{2\pi}(x_k + a_{ik}Q_l(x_k)) \end{aligned} \right\} =: G_i(x), \quad x \in \mathcal{D}_i \quad (8)$$

where $\mathcal{C} := \{x \in [0, 2\pi]^{N+1}\}$, $\mathcal{D}_g := \{x \in \mathcal{C} : x_g = 2\pi\}$, $\mathcal{D}_i := \{x \in \mathcal{C} : x_i = 2\pi\}$, $\omega \in \mathbb{R}_{>0}$ denotes the natural frequency of the slave oscillators, $\omega_g \in \mathbb{R}_{>0}$ is the natural frequency of the global cue, $a_{ij} \in \{0, l\}$ is the corresponding entry from $\mathcal{A}_{\mathcal{R}}$, $g_i \in [0, 1]$ is the global coupling strength, $Q_g : [0, 2\pi] \rightrightarrows \mathbb{R}$, and $Q_l : [0, 2\pi] \rightrightarrows \mathbb{R}$ are the global and local PRCs, and $\text{sat}_0^{2\pi}$ is the linear function with slope one that saturates at 2π from above and 0 from below. Then, we define the jump set as:

$$\mathcal{D} := \mathcal{D}_g \cup \bigcup_{i \in \mathcal{V}} \mathcal{D}_i \quad (9)$$

and the corresponding jump map as:

$$G(x) := \bigcup_{i \in \mathcal{V} \cup \{g\} : x \in \mathcal{D}_i} G_i(x). \quad (10)$$

In the following we will refer to the model (6)-(10) as the hybrid system \mathcal{H}_g . To continue the analysis we utilize the following assumption.

Assumption 1. The PRCs Q_g and Q_l are such that: $Q_q(0) = Q_q(2\pi) = \{0\}$, $q \in \{g, l\}$. Moreover, Q_g and Q_l are outer semi-continuous set-valued mappings, bounded on $[0, 2\pi]$, and such that $Q_g(x_i)$ and $Q_l(x_i)$ are non empty for all $x_i \in [0, 2\pi]$.

Assumption 1 guarantees that the hybrid system \mathcal{H}_g as defined above is well-posed [15]. Moreover, the condition $Q_q(0) = Q_q(2\pi) = \{0\}$, $q \in \{g, l\}$ restricts any undesired avalanche-type behavior.

Remark 2. An important concept used in the analysis of PCOs is absorption [17], which leads to synchronization in finite time. It should be noted that in our model (6)-(10) absorption is modeled by the saturation function.

Remark 3. Note that the use of the saturation at 2π from above is consistent with the use of the min function in the Mirollo and Strogatz model [17]. In the same line, the use of the saturation at 0 from below is a natural extension for advance-delay PRCs.

Remark 4. It should be noted that the Mirollo and Strogatz model [17] uses a constant ϵ to advance the phase. Although ϵ is required to be small, a large network is susceptible to firing avalanches since multiple incoming pulses accumulate [17]. Our model handles this undesired behavior by requiring $Q_q(0) = Q_q(2\pi) = \{0\}$, $q \in \{g, l\}$.

3. Synchronization

In this section we analyze the synchronization properties of the hybrid system \mathcal{H}_g . To conduct the analysis, we use the following:

Assumption 2. The global cue and the slave oscillators have identical natural frequencies, i.e., $\omega_g = \omega$.

Assumption 3. The global coupling is identical and strictly positive, i.e., $g_i = g > 0$, $\forall i \in \mathcal{V}$.

Remark 5. Note that in [8] it is stated that the condition $g_i = g > 0$, $\forall i \in \mathcal{V}$ is necessary to ensure synchronization when the oscillators are distributed in the whole interval $[0, 2\pi]$. Moreover, the condition $\omega_g = \omega$ is required to ensure perfect synchronization in phase. If $\omega_g \neq \omega$, then a weaker notion of synchronization is needed.

To analyze synchronization, define the difference between the global cue and the i th slave oscillator as $\xi_i := x_g - x_i$ and the vector of differences as $\xi = [\xi_1, \dots, \xi_N]$. We consider synchronization achieved whenever $|\xi_i| = 0$ or $|\xi_i| = 2\pi \forall i$. Hence the synchronization set can be written as:

$$\mathcal{A} := \{x \in \mathcal{C} : \xi_i = 0 \text{ or } \xi_i = \pm 2\pi \forall i \in \mathcal{V}\}. \quad (11)$$

The synchronization condition is as follows.

Theorem 1. Consider the network of PCOs given by \mathcal{H}_g . If:

1. Assumptions 1, 2, and 3 hold,
2. Q_g and Q_l are such that if $x_i \in (\pi, 2\pi)$, then $Q_q(x_i) \subset \mathbb{R}_{>0}$, and if $x_i \in (0, \pi)$, then $Q_q(x_i) \subset \mathbb{R}_{<0}$, $\forall q \in \{g, l\}$,
3. the influence of the global cue is strong enough compared with the local coupling,

and moreover, the PRCs satisfy $0 \notin Q_q(\pi)$, $q \in \{g, l\}$, then the network synchronizes from every initial condition $x(0, 0) \in \mathcal{C}$.

PROOF. We will prove synchronization in two steps. First we will show that there exists a forward invariant neighborhood of \mathcal{A} , denoted as \mathcal{B} , such that if the state belongs to \mathcal{B} , then the network synchronizes irrespective of the strength of the couplings. Secondly, we will show that the network eventually reaches \mathcal{B} in finite time from every initial condition, if the global coupling is strong enough.

In the following, we make use of the concept of containing arc. Given an arc α , i.e., a connected subset of $[0, 2\pi]$ with 0 and 2π mapped to each other, with associated length $d(\alpha)$, α is a containing arc if and only if $x_i \in \alpha$, $\forall i \in \mathcal{V} \cup \{g\}$. Given x , the set of all arcs containing x , henceforth called the set of containing arcs, is denoted $\Upsilon(x)$.

As a first step, we will show that if the smallest containing arc α has length $d(\alpha) < \pi$ the network synchronizes to the global cue for all gQ_g and lQ_l . To this end, consider the Lyapunov candidate

$$W(x) := \inf_{\alpha \in \Upsilon(x)} d(\alpha), \quad (12)$$

and for every $\mu \in \mathbb{R}_{\geq 0}$ define the set $L_v(\mu) := \{x \in \mathcal{C} : W(x) = \mu\}$. We focus on the initial conditions contained in $\mathcal{B} := \{x \in \mathcal{C} : W(x) < \pi\}$. It is clear that during flows $W(x)$ remains unaltered since there is no interaction and the natural frequencies are identical. Suppose that $x \in (\mathcal{D} \cap \mathcal{B}) \setminus \mathcal{A}$ and that an oscillator $i \in \mathcal{V} \cup \{g\}$ will fire. First, note that α maps the nodes into a portion of the unit circle, hence the jump of i from 0 to 2π does not affect its length. From condition 2, every oscillator m such that $m \in \mathcal{N}^{i+}$ is attracted to i after the firing and then, since $d(\alpha) < \pi$, the length of the smallest containing arc cannot increase, i.e., $W^+ - W \leq 0$. In particular, if g jumps, since every $i \in \mathcal{V}$ belongs to \mathcal{N}^{g+} , we have that after a jump of the global cue $W^+ - W < 0$. Therefore, $W^+ - W \leq 0$ for all $x \in (\mathcal{D} \cap \mathcal{B}) \setminus \mathcal{A}$. Moreover, since every solution is complete and the global cue jumps periodically, for every $\mu > 0$ no complete solution to \mathcal{H}_g remains in $\mathcal{B} \cap L_v(\mu)$. Since the hybrid system \mathcal{H}_g is well-posed, we can rely on the invariance principle to establish synchronization.

In particular, directly applying Theorem 23 in [15] gives asymptotic stability of the set \mathcal{A} with basin of attraction containing \mathcal{B} .

Now consider $x(0, 0) \in \mathcal{C} \setminus \mathcal{B}$ and define the following family of functions:

$$V_i(\xi) = \min(|\xi_i|, 2\pi - |\xi_i|). \quad (13)$$

Note that the V_i are continuous functions and they are unchanged during flows, due to Assumption 2. Define $U(\xi) := \sum_{i \in \mathcal{V}} V_i(\xi)$ as the total distance to the global cue. Note that U is continuous and positive definite with respect to \mathcal{A} . A sufficient condition for $W(x) < \pi$ is $U(\xi) < \pi$. Hence, if there exists a time instant (T, J) such that $U(\xi(T, J)) < \pi$, then the network synchronizes.

Suppose $x \in \mathcal{D}$. We analyze the change in $V_i(\xi)$ when x_g jumps. We have that $x_g = 2\pi$ and $x_i \in [0, 2\pi]$, then:

$$\begin{aligned} V_i &= \min\{2\pi - x_i, x_i\} \\ V_i^+ &\in \min\{\text{sat}_0^{2\pi}(x_i + gQ_g(x_i)), \\ &\quad 2\pi - \text{sat}_0^{2\pi}(x_i + gQ_g(x_i))\}. \end{aligned} \quad (14)$$

Since the phase is advanced if $x_i \in (\pi, 2\pi)$, and the phase is delayed if $x_i \in (0, \pi)$, then, $V_i > V_i^+$ holds for all $V_i > 0$ independent of the value of x_i before x_g jumps. Hence, $U^+ - U < 0$ after a jump of the global cue. Therefore, if gQ_g is strong enough such that $U < \pi$ after one jump of the global cue, the network enters \mathcal{B} and synchronizes, irrespective of the strength of the local coupling lQ_l .

When gQ_g is not as strong to force $U < \pi$ on one jump, the network still enters the set \mathcal{B} if gQ_g is strong enough compared with lQ_l . To prove this claim, we can rely on the practical robustness property of well-posed hybrid systems [15, Theorem 17]. In fact, we can consider the local coupling lQ_l as a perturbation to a nominal hybrid system for which $lQ_l(x_i) = \{0\}$ for all $x_i \in [0, 2\pi]$. From the previous analysis, we know that when $lQ_l(x_i) = \{0\}$ the network synchronizes since the global cue jumps persistently and $U^+ - U < 0$ after a jump of the global cue whenever $x \notin \mathcal{A}$. Then, when lQ_l is weak the network converges to a neighborhood of the set \mathcal{A} , whose size can be made arbitrarily small by restricting lQ_l . In particular, given a global coupling gQ_g ,

we can restrict the local coupling lQ_l so that the network converges to the set \mathcal{B} . Therefore, the network synchronizes for every $x(0, 0) \in \mathcal{C}$. \square

Remark 6. Note that Theorem 1 does not impose any connectivity requirement on the communication graph \mathcal{R} or uses any assumption on the monotonicity of the PRCs Q_g and Q_l . This suggests that Theorem 1 is valid for a class of oscillators larger than monotone oscillators, for which the PRC is monotone [20].

Theorem 1 emphasizes the role of the global coupling, which should be strong enough compared with the local coupling. A practical bound can be stated as follows.

Corollary 1. Consider the network of PCOs given by \mathcal{H}_g and suppose conditions 1 and 2 in Theorem 1 hold. If the global coupling gQ_g is such that:

1. for all $x \in [\frac{\pi}{2}, \pi)$, if $q \in Q_g(x)$ then $gq < \frac{\pi}{2} - x$
2. for all $x \in (\pi, \frac{3\pi}{2}]$, if $\bar{q} \in Q_g(x)$ then $g\bar{q} > \frac{3\pi}{2} - x$
3. if $q \in Q_g(\pi)$ then $gq \in [-\pi, -\frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$

then the network synchronizes for all $x(0, 0) \in \mathcal{C}$ irrespective of the strength of the local coupling lQ_l .

PROOF. Conditions 1, 2, and 3 ensure that after the first jump of the global cue the oscillators will be contained in an arc α of length $d(\alpha) < \pi$. Hence the network synchronizes for all $x(0, 0) \in \mathcal{C}$. \square

The previous Corollary gives a practical bound since the global coupling does not need to be stronger than the coupling characterized in Corollary 1.

4. Numerical Experiments

To illustrate the applicability and conservativeness of the analytical results previously derived, we use the hybrid systems simulator [21] to simulate the networks shown in Figure 1 when the PRCs Q_g and Q_l are identical and given by

$$Q(x) = \begin{cases} 2\pi - x, & \text{if } x > \pi \\ \{\pi, -\pi\}, & \text{if } x = \pi \\ -x, & \text{if } x < \pi \end{cases} \quad (15)$$

which corresponds to the set-valued regularization of the discontinuous function $Q(x) = 2\pi - x$, $x \in [\pi, 2\pi]$; $Q(x) = -x$, $x \in [0, \pi)$. Note that (15) is an outer semi-continuous set-valued mapping and bounded on \mathcal{D} ; hence Assumption 1 holds.

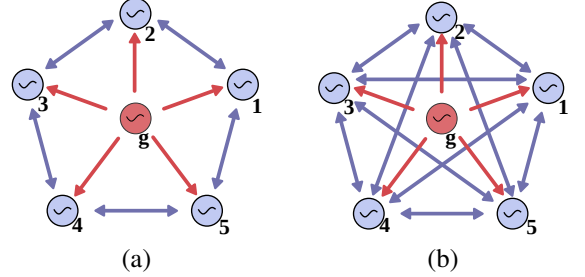
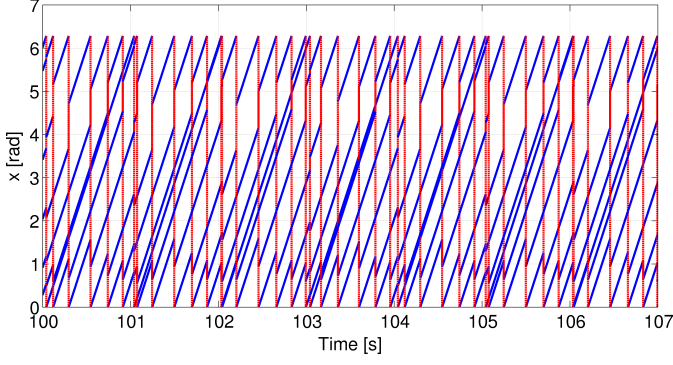


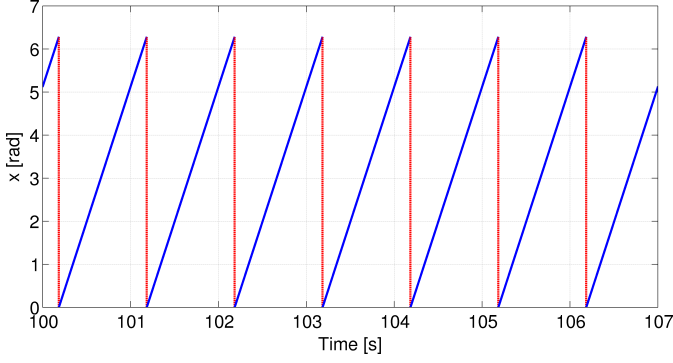
Figure 1: Networks used in the numerical simulations. (a): Bidirectional 5-nodes cycle network plus an omnipresent global cue (red node in the middle); (b): All-to-all 5-nodes network plus an omnipresent global cue (red node in the middle).

To study synchronization, natural frequencies were set as $w_i = w = 2\pi$. For the first example consider the network in Figure 1(a), which consists of a 5-slave-nodes bidirectional cycle plus an omnipresent global cue (in red). We simulate the network from a random initial condition and coupling given by $g = 0.1$ and $l = 0.4$. Figure 2(a) shows the simulation results. It can be seen that the network does not synchronize. In fact, the local coupling $l = 0.4$ is not strong enough to synchronize the cycle network globally in the absence of a global cue [22]. Moreover, the global coupling $g = 0.1$ is not strong enough to synchronize the slave oscillators to the global cue. Hence, the global coupling can be regarded as a periodic perturbation to the slave system that precludes convergence to a phase-locked state. Figure 2(b) shows the simulation results, from the same initial condition, when the coupling is given by $g = 0.51$ and $l = 0.4$. In this case the slave network synchronizes to the global cue. Although the local coupling is not strong enough to synchronize the cycle network locally, the global coupling is sufficiently attractive to preclude the slave system to converge to a phase-locked state and forces the slave system to follow the global cue. Note that when $g = 0.51$, conditions in Corollary 1 hold.

As a second example, consider the the network in Figure 1(b), which consists of a 5-slave-nodes all-to-all network plus an omnipresent global cue (in red).



(a)



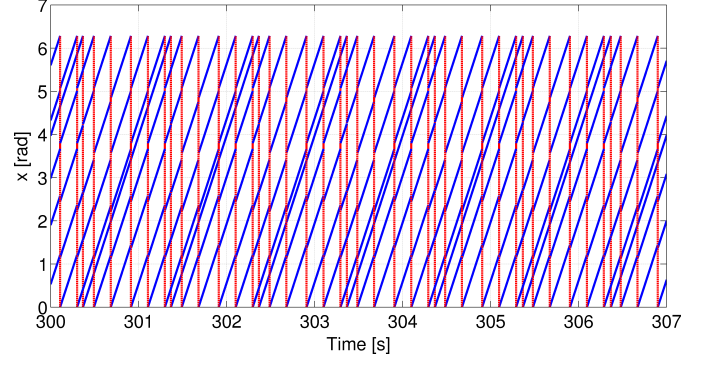
(b)

Figure 2: Simulation results for the network in Figure 1(a) with initial condition $x_i(0, 0) = \frac{2\pi}{5}i$ and $x_g(0, 0) = \frac{2\pi}{6}$. (a): Results when the coupling is given by $g = 0.1$ and $l = 0.4$, since the global coupling is not strong enough the network do not synchronize; (b): Results when the coupling given by $g = 0.51$ and $l = 0.4$, the global cue is attractive enough and the network synchronizes.

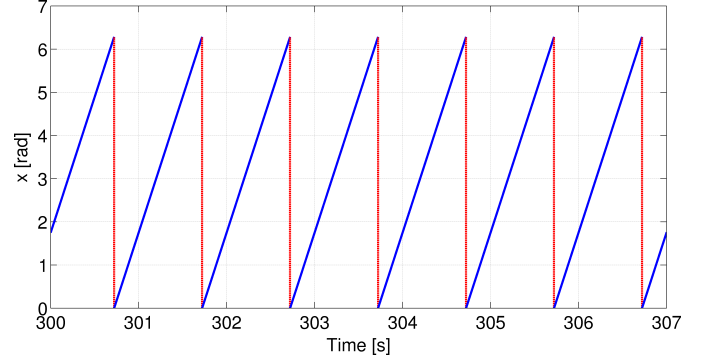
We simulate the network from a random initial condition and coupling given by $g = 0.05$ and $l = 0.05$. The simulation results are shown in Figure 3(a). It can be seen that the network does not synchronize. As with the previous example, in this case the global coupling is too weak and it can be regarded as a perturbation precluding convergence to a phase-locked state. Figure 3(b) shows the simulation results when the coupling is given by $g = 0.2$ and $l = 0.05$. In this case, the global coupling is strong enough to force the slave system to follow the global cue.

5. Concluding Remarks

Synchronization properties of networks of pulse-coupled oscillators (PCOs) subject to a global cue and local interactions between slave oscillators is stud-



(a)



(b)

Figure 3: Simulation results for the network in Figure 1(b) with initial condition $x_i(0, 0) = \frac{2\pi}{5}i$ and $x_g(0, 0) = \frac{2\pi}{6}$. (a): Results when the coupling is given by $g = 0.05$ and $l = 0.05$, since the global coupling is not strong enough the network do not synchronize; (b): Results when the coupling is given by $g = 0.2$ and $l = 0.05$, the global cue is attractive enough and the network synchronizes.

ied. It is shown that global synchronization is feasible in a network of identical oscillators if every slave oscillator is connected to the global cue and the influence of the global coupling is strong enough compared with the local coupling. The phase response curve is assumed to be of the advance-delay type, yet no requirements on its monotonicity are stated.

Future work includes finding a precise relationship between local and global coupling based on the exact interaction topology, as well as extending the results to non-identical oscillators.

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