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A Semi-analytical Unit Cell Synthesis Method for Design of Meta-materials with Targeted Nonlinear Deformation Response

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A SEMI-ANALYTICAL UNIT CELL SYNTHESIS METHOD FOR DESIGN OF META-MATERIALS WITH TARGETED NONLINEAR DEFORMATION RESPONSE

A Thesis
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the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mechanical Engineering

by
Shanyun Gao
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Accepted by:
Dr. Gang Li, Committee Chair
Dr. Georges Fadel
Dr. Lonny Thompson
Abstract

A Unit Cell Synthesis Method was developed recently for designing meta-materials from unit cell level to achieve prescribed nonlinear deformation response. The method starts with a process of selecting and combining a set of elemental functional geometries (EFG) and elemental structural geometries (ESG) to form the unit cell structure. A subsequent size optimization is performed to obtain an optimal design which provides the targeted nonlinear deformation behavior. While the method is proven effective in producing feasible meta-material designs, the design and optimization of the unit cells relies heavily on nonlinear finite element analysis, which makes the overall process computationally intensive and time consuming.

In this work, a semi-analytical approach is developed for predicting the large deformation response of EFGs and their combinations. In this approach, non-dimensional load and deformation parameters are proposed for the EFGs including cantilever, fixed-fixed and circular beams. The deformation parameters are then expressed as nonlinear analytical functions of the load parameters. The load parameters are generalized for each geometry to unify the analytical functions for different dimensions of the geometry. The obtained analytical functions of the EFGs' deformation behavior are then implemented in the Unit Cell Synthesis Method for meta-material unit cell design. The semi-analytical Unit Cell Synthesis Method is applied in the design of a linear elastic material based meta-material that mimics the nonlinear deformation behavior of a rubber pad under compression. The results show that the analytical deformation functions of the EFGs enable a much more efficient design process of the meta-material.
Dedication

To my family and friends.
Acknowledgments

My study in Clemson University would never be possible without the backing of my parents, for which I am forever grateful. I want to thank them for their support in every possible way, love and understanding in any condition. I would like to express my gratitude to Wenxin, for being such an incredible companion. I am thankful for her guiding my growth, and I treasure the conversations we had and the hours we spent together.

I would like to thank my adviser Dr. Li for providing me with this research opportunity, and his knowledge and support throughout this research, his recognition of my work. Thanks to Dr. Fadel, Dr. Coutris, Dr. Thompson for being my esteemed committee members, for every fruitful meeting and conversation.

I would like to thank Dr. Figliola, not only for his lectures, but for his intellectual instruction on my early graduate level study which has a great impact on my later problem solving and research.

I would like to thank my lab-mates: Qi, Jixuan, Jun, Chengjian, Ying, for their helpful suggestions and insightful discussions for our research, especially at very early stage. I also miss all the fun we had together.
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Chapter 1

Introduction

1.1 Meta-material Design for Mechanical Properties

Meta-materials are artificial materials engineered to achieve behaviors that cannot be found in nature or designed to have desired properties for certain applications [1]. Typically, a meta-material is a macroscopic composite of periodic or non-periodic micro-structures [2]. This thesis considers meta-materials with periodic micro-structures which are defined as unit cells (UC). A unit cell is the smallest repeatable structure and the basic building block of a meta-material. The structure or material layout of unit cells has a great impact on the global properties of meta-materials. The global physical properties of meta-materials emerge from the properties of the constitutive material(s) and the geometry of the unit cells as well. In recent years, extensive research work has been carried out aiming to study, develop and design meta-materials for various applications [2, 3, 4, 5, 6, 7, 8, 9, 10]. Among the many aspects of meta-materials, achieving prescribed mechanical property is the focus of this thesis. In this regard, in previous works, meta-materials have been developed to achieve negative Poisson’s ratios (Auxetic Materials) [3] and prescribed shear moduli [4], and to minimize structural compliance in two-phase [5] and three-phase materials [6]. For example, Fig. 1.1 shows an auxetic material structure which expands vertically when being stretched horizontally (negative Poisson’s ratio). More recently, another class of meta-material design problems stem from the need of mimicking the deformation behavior of nonlinear materials such as elastomers, or creating artificial materials to achieve a nonlinear deformation behavior that does not exist in constituent materials [11]. The development of such meta-materials would enable replacement of mechanical structures without the previously limiting failure modes yet exhibiting the same functionality as the original structures. In addition, these meta-materials
may greatly expand the design space in applications involving nonlinear deformation response and achieve new “pre-designed” nonlinear responses of structural components. In the recent case studied in Ref. [11], a meta-material made of titanium alloy was designed to replace the original elastomer part to avoid premature failure caused by its hysteretic nature. Several designs were successfully created to match the given nonlinear uniaxial compression curve. Figure 1.2 shows one of the meta-material unit cell geometries developed in that study and the corresponding deformation curve of the meta-material is shown in Fig. 1.3.

Figure 1.1: Auxetic meta-material.

Figure 1.2: Meta-material design for achieving a prescribed nonlinear deformation behavior.

1.2 Metamaterial Design Methods

For meta-materials with periodic micro-structures, the overall material behavior is largely determined by the UC structure and material properties of its constituent materials. Broadly speaking, there are three types of methods for the design of UCs with prescribed mechanical properties: intuitive, topology optimization and unit cell synthesis approaches. Among these approaches, the intuitive method largely relies on the designer’s experience and trial-and-error iterations. As this approach is not repeatable in general, it is not discussed here. In the following, the topology optimization and unit cell synthesis approaches are briefly reviewed.
1.2.1 Topology optimization methods

In the past two decades, topology optimization has been developed to become a powerful computational tool for meta-material design [5, 13]. In essence, topology optimization is a numerical iterative process that distributes a certain amount of material within a design domain to seek a layout satisfying the objective function which subjects to a set of constraints. The method is usually implemented along with an inverse homogenization approach, to target predetermined material properties while minimizing cost (e.g. volume). To date, major topology optimization methods include the homogenization method [13, 14], the element density SIMP method [16], the evolutionary structural optimization method [15], and the level set based method (LSM) [17].

The homogenization method stemmed from the mathematical theory of homogenization, or relation between macro and micro level properties. This theory was further developed for topology optimization by facilitating methods to determine effective properties of heterogeneous media and to enable implementation in the finite element method (FEM) [20]. In essence, the homogenization method combines homogenization theory with an FEM solver to solve an topology optimization problem. Bendsoe and Kikuchi developed the first applied homogenization method in topology optimization in Ref. [21]. Hassani and Hinton [22] developed an extensive mathematical formulation of Homogenization Method, along with its several variants. Topology design using the homogenization method is a process of adding or removing of material within a design domain to form micro-structures. These micro-structures can have voids with various shapes. One example is shown in Fig. 1.4, where the unit cell and voids are in the shape of squares. The void is defined by its height, width and angle of rotation. The micro-structure of each unit cell in the domain can be anything.
in between a pure solid and a pure void. The optimization process modifies design variables based on finite element analysis (FEA) results and update scheme chosen to improve the objective function value. Meanwhile effective properties are updated with the new micro-structures calculated from the previous iteration using homogenization theory. The iteration progress continues until the convergence criteria are satisfied.

![Design Domain](image1)

**Figure 1.4:** Unit cell with a square void [23].

One of the first adaptations of the homogenization method in topology optimization is the Solid Isotropic material with Penalization method (SIMP) [16]. The purpose of this variant is to eliminate topologies which are not manufacturable. The SIMP method accomplishes this goal by penalizing design variables if their density lies between 0 (pure void) and 1 (pure solid). The penalization is implemented by raising the element density to an exponential factor in the objective function. In the original homogenization method formulation setup, the penalization factor is 1. As this penalization factor increases, the intermediate densities are shown to be removed from the solution. One such example [24] can be seen in Fig. 1.5: the solution with penalization of 1.5 yields a defined solid topology with intermediate densities, while that with penalization of 3 yields a solid topology with little intermediate densities along the solid-void borders. Therefore, the solution with penalization factor of 3 provides a better manufacturable solution.

Evolutionary Structural Optimization (ESO) method is based on an empirical concept that a structure evolves towards an optimum by slowly removing inefficient material [25]. Since a reliable sign of potential structural failure is excessive stress or strain, a reliable sign of inefficient material use, on the other hand, is low stress or strain. Ideally the stress in every part of a structure should be around the same level of safety.
Low stressed material is assumed to be under-utilized and such material will be removed subsequently. By gradually removing material with lower stresses, the stress level in the updated design becomes more uniform. Various design constraints such as stiffness, frequency and buckling load may be imposed upon a structure. According to the types of design constraints, different rejection criteria for removing material need to be used.

Level Set Method (LSM) in topology optimization employs a level set model embedded in a scalar function of a higher dimension to represent a structural boundary. The method’s principle is based on implicitly expressing structural boundary as the zero level set of a higher dimensional level set function (LSF) [27]. For instance, Fig. 1.6 shows the representation of a two-dimensional boundary with a three-dimensional level set surface, where $\phi$ is used to denote different parts of the reference domain. Design boundary is denoted by $\phi = 0$, solid region is when $\phi > 0$ and void region is when $\phi < 0$. Then the motion of the design boundary is mathematically described as a Hamilton-Jacobi partial differential equation (H-J PDE), in which the normal velocity field to enable the evolution of the design boundary is often achieved using the shape derivative method [28]. As a result, the LSM can provide unique advantages in optimizing topology of a structure, in particular, smooth boundary and distinct interface, as well as shape fidelity and topological flexibility [29].

While topology optimization methods are effective in designing structures with target mechanical properties in the linear regime, upon further investigation, it is deemed difficult, if not infeasible, to use these methods to design meta-materials with prescribed nonlinear mechanical properties. To date, consideration of nonlinear mechanics in the topology optimization of material architectures remains a challenge, primarily due to (1) the lack of unit cell upscaling methods [30]; (2) robustness issues such as dependence of the optimization results on the initial guess [30, 31]; and (3) numerical instability induced by low density elements in the nonlinear computational analysis [32]. The linear elasticity formulation assumes small deformation and hence allows simplification in terms of using Cauchy stress tensor and infinitesimal strain.
In terms of geometric nonlinearity, however, it is necessary to use Second Piola Kirchhoff stress tensor and the Green-Lagrange strain tensor. Some researchers [33, 34] have raised concerns about geometric nonlinearity in topology optimization, but such nonlinearity has not yet been employed to a nonlinear inverse homogenization problem. For such issue in implementing the inverse homogenization, multiple prescribed stiffness tensors that are dependent on loading conditions must be considered. Components of these stiffness tensors represent the tangents to stress-strain relations of the nonlinear target behavior. Because of this added complexity, implementation of nonlinear inverse homogenization remains unexplored. Other than homogenization approach, research work in [32] has been reported to have used a numerical technique design truss-based continuum material structure with prescribed nonlinear properties using topology optimization. Although the procedure accomplished matching prescribed finite strain nonlinear deformation, the continuum structures are designed only to exhibit a given Poisson’s ratio with finite axial strain. Therefore, meta-material design for prescribed nonlinear deformation behavior with stress constraints has not been done by using the topology optimization techniques.

Other limitations of topology optimization are corresponding to the consideration of unit cell aspect ratios and periodicity of boundary conditions in various directions. In traditional topology design problems, design domain is determined at the problem formulation stage. However, since a given meta-material design is unknown before solving, the aspect ratio of unit cells should be considered as a free variable. For issues relating periodic boundary conditions, even when a unit cell boundary is adjacent to more than one unit cells, implementation of the offset boundary conditions must be launched. But it is rarely done in previous
work [4].

### 1.2.2 Unit-Cell Synthesis Method

Most recently, a Unit Cell Synthesis Method is developed by Satterfield and Kulkarni [11, 12] to help designers design meta-materials from unit cell level with focus on matching predetermined nonlinear deformation response. The method is expected to serve as an easy alternative to topology optimization. As the method relies on a fundamental understanding of nonlinear mechanics, it provides a simple and yet effective approach to solve topology design problems with nonlinear deformation behavior objectives. The key principle of this method is to achieve a given nonlinear deformation behavior of the meta-material through a combination of different geometric nonlinearities associated with different elemental geometries that undergo bending deformation. The nonlinear deformation responses of elemental structural components are obtained from nonlinear finite element analysis. Then, by comparing the target nonlinear deformation behavior with those of elemental geometric entities, the elemental components are selected to construct unit cells. Finally the optimized meta-material design is achieved through a size optimization procedure.

Figure 1.7 illustrates the workflow of the design process. The scope of this method is limited to 2-D geometries that are extruded in the third dimension. The method consists of an iterative process of selecting and placing elemental components in a unit cell and then perform tessellation of unit cells into a meta-material. Series and parallel connections are used to evaluate the softening or stiffening deformation behavior of combined geometric entities and to form the conceptual design of a representative volume element (RVE).

The basic building block of meta-material is unit cell, and unit cell is formed by combining elemental geometric entities mentioned above. These elemental structures are defined as Elemental Functional Geometries (EFG) and their deformation behavior is used to satisfy a desired response. In the Unit Cell Synthesis Method, the first step is to prepare a repository of EFGs. As shown in Fig. 1.8, examples of EFGs included cantilever, fixed-fixed and oval beams. Their general large deformation behaviors are depicted in the figure.

The EFGs can be combined in different ways. In their work, the combinations are categorized by their configuration orders. If a unit cell is only composed of one EFG, it is defined as a $0^{th}$ order configuration. When two EFGs are connected, it is referred to as $1^{st}$ order configuration. The $1^{st}$ order connection can be categorized into parallel and series connections, as shown in Fig. 1.9. Furthermore, when two $1^{st}$ order configurations, or a $1^{st}$ order and a $0^{th}$ order configuration are combined, the structure becomes a $2^{nd}$ order
After the initial unit cell structure is determined, and meta-material tessellation is completed, a subsequent size optimization is performed to obtain an optimal design which satisfies the targeted mechanical property. The optimization procedure converges the deformation response of the meta-material towards that of the target response. Finite element analysis is utilized to calculate large deformation responses of the meta-material structure in each iteration.

Figure 1.7: Unit Cell Synthesis Method Workflow.
While the unit cell synthesis method’s capability of producing feasible UC designs matching target nonlinear deformation responses is demonstrated in Ref. [11], there are several shortcomings that may limit
the application of the method: (1) the basic principle of the approach is to construct UC topology by combining elemental functional geometric elements. However, without quantitative or analytical solution of the deformation behavior of such simple geometries, the initial designs are only educated guesses; and (2) the entire design process heavily relies on nonlinear FEA which is time consuming. In view of these issues, this thesis aims to simplify the calculations in the method by providing analytical nonlinear force-displacement relations of the EFGs undergoing large deformations.

### 1.3 Research Objectives

The primary research objectives of this thesis are to:

- Develop a systematic approach to obtain analytical force-displacement functions of the EFGs subjected to large deformations.
- Implement the force-displacement functions in the unit cell synthesis method to design meta-materials with prescribed nonlinear deformation response.

### 1.4 Thesis Outline

This thesis is organized into four chapters. The first chapter provides an introduction to the background, motivation and research objectives.

Chapter 2 describes the method to obtain large deformation functions of the EFGs. Non-dimensional parameters containing geometric and material properties of the EFGs are proposed and optimized. In addition, analytical solutions of series and parallel combinations of the EFGs are obtained for the purpose of unit cell design.

Chapter 3 presents the implementation of the obtained analytical force-displacement functions in the unit cell synthesis method to design meta-material with prescribed nonlinear deformation behavior. The modified unit cell method is referred to as the Semi-Analytical Unit Cell Synthesis Method. In the case study carried out in this work, two feasible UC designs are produced by using the method.

Chapter 4 presents the conclusions and future work.
Chapter 2

Analytical Functions of Large Deformation Behavior of Elemental Functional Geometries

2.1 Introduction

For the sake of brevity, a simplified workflow of the unit cell synthesis method is depicted in Fig. 2.1. The first step is to prepare a repository of elemental structures as shown in Fig. 1.8, defined as elemental functional geometries (EFG), whose deformation responses can be predetermined. The basic idea of the design method is to construct unit cell geometry by combining the EFGs. For example, in Fig. 2.2, the unit cell topology on the right is formed by connecting a cantilever beam and a curved (circular) beam. The unit cell’s structure implies that its deformation behavior depends on that of the cantilever beam and the curved beam as well as the way they are connected.

An EFG is defined as a geometry (1) that is simple in shape and (2) whose deformation behavior can be predetermined and can be used to meet a desired response. The deformation responses of the EFGs are tunable by modifying their geometric parameters such as length and aspect ratio, or by changing their material properties such as Young’s moduli. Examples of EFGs include cantilever beams, fixed-fixed beams, and oval structures (curved beams). In the Unit Cell Synthesis Method, the deformation behavior of the EFGs in case of large deformation is predetermined by using finite element analysis. It is known that nonlinear FEA
is typically time consuming. Furthermore, in the size optimization step of the Unit Cell Synthesis Method, tens of thousands of nonlinear FEA simulations are required, which leads to a very high computational cost. Therefore, it is highly desirable to develop analytical functions of the EFGs’ large deformation behavior, and replace the nonlinear FEA runs with simple calculations using the analytical functions, which are orders of magnitude faster.

In this work, we propose an approach to obtain analytical functions of nonlinear deformation response of compliant EFGs. Inspired by the work of Bisshopp et al [18, 19], this approach defines non-dimensional load and deformation parameters to describe EFG’s deformation response. It can be shown
that, EFGs with different geometric dimensions and material properties give different non-dimensional load-deformation curves. However, through a process of modifying and optimizing the load parameter, we show that the nonlinear deformation curves of an EFG with different dimensions and material properties can be reduced to a single response curve. Next, this converged deformation curve is expressed in terms of the non-dimensional parameters through a polynomial fitting. Finally, the analytical solution of deformation is obtained. In the following section, the cantilever beam EFG is taken as an example to illustrate the approach.

2.2 Large Deflection of Cantilever Beams

As shown in Fig. 2.3, the classical problem of the deflection of a cantilever beam under the action of a vertical concentrated load at the free end is taken as the example problem. The beam is assumed to have a uniform rectangle cross section and made of a linear elastic material with Young’s modulus $E$.

![Cantilever beam](image)

Figure 2.3: Cantilever beam subjected a vertical concentrated load at the free end.

For the cantilever beam, the non-dimensional load parameter $\beta_o$ and non-dimensional deflection parameter $\eta$ are defined in Bisshopp’s work as shown in Eqs. (2.1, 2.2) as

$$\beta_o = \frac{FL^2}{2EI}$$ (2.1)

$$\eta = \frac{y}{L}$$ (2.2)

where $I = \frac{h^3}{12}$ denotes the moment of inertia of the beam cross section about the neutral axis. For two dimensional problems, the beam thickness is assumed to be 1. It was shown that, by using the parameters $\beta_o$ and $\eta$, it is possible to obtain a more general understanding of the beam deflection. For example, different combinations of $E$, $I$, $F$ and $L$ may give the same value of $\beta_o$ and consequently, yield the same tip deflection. Note that, the coefficient 2 in the denominator was introduced in the process of deriving deflection’s expression.
Having defined the two non-dimensional parameters, Bisshopp et al approximated cantilever beam’s large deflection behavior by using elliptic integrals, which can only be evaluated numerically. Unfortunately, Bisshopp et al’s solution is not applicable to the unit cell design. First, in Bisshopp et al’s work, the beam is assumed to be thin and long, i.e. aspect ratio of the beam ($\alpha = \frac{L}{h}$) is large. However, in meta-material unit cells, aspect ratio of a beam may not be very large, and Bisshopp et al’s assumption becomes invalid. Second, the evaluation of the elliptical integrals are quite involved: in order to obtain the deformation, the beam’s angle of rotation must be calculated first by solving an integral equation, and then the beam’s deflection can be calculated next by numerically evaluating the elliptic integrals of the first and second kinds. Finally, Bisshopp et al’s solution is only for cantilever beams. Similar solutions do not exist for other EFGs.

Therefore, we perform nonlinear FEA of the cantilever beam using ANSYS Mechanical APDL to study the effectiveness of the above non-dimensional parameters, especially when aspect ratio is not very large. As shown in Fig. 2.4, cantilever beams with 5 different aspect ratios are modeled and simulated. It is observed that the 5 curves start to diverge when the load becomes larger. The deformation parameter is larger for beams with smaller aspect ratios. It can be concluded from the results that the aspect ratio has a negative effect on the non-dimensional deformation parameter, and the original load parameter $\beta_o$ does not work well for small aspect ratios because of the effect of aspect ratio.

![Figure 2.4: Cantilever beam deformation curves with different aspect ratios.](image)

Based on the above observation and analysis, in Eq. (2.3), $\beta_o$ is rewritten and expressed by force,
Young’s Modulus, beam height and aspect ratio, and finally the load parameter is decomposed into two parts.

\[ \beta_o = \frac{FL^2}{2EI} = \frac{FL^2}{2E \cdot \frac{h^2}{12}} = \frac{6F}{Eh} \cdot \left( \frac{L}{h} \right)^2 = \frac{6F}{Eh} \cdot \alpha^2 \] \hspace{1cm} (2.3)

Since it is observed that the aspect ratio has a negative effect on deformation, reducing the impact of aspect ratio is desirable. Therefore, a modified non-dimensional load parameter with a reduced exponent of the aspect ratio is proposed as:

\[ \beta = \frac{6F}{Eh} \cdot \alpha^{2-n} \] \hspace{1cm} (2.4)

The parameter \( n \) can be determined by an optimization process. The objective, which is expressed in Eq. (2.5) is to minimize the summation of deformation differences between the five curves.

\[ \min : f = \sum_{i=1}^{5} \sum_{j=1}^{10} (\eta_{ij} - \overline{\eta})^2 \quad i : i\text{-th aspect ratio}, \ j : j\text{-th load parameter} \] \hspace{1cm} (2.5)

Figure 2.5 is a plot of the summation of deformation differences with respect to \( n \)’s value, and the minimum value appears around 0.1. After the optimization and rounding, \( n \) is determined to be 0.1. Thus the
new non-dimensional load parameter $\beta$ is set to be:

$$\beta = \frac{FL^2}{2EI \cdot \alpha^{0.1}}$$  \hfill (2.6)

The new set of non-dimensional parameters are tested by conducting another series of FEA simulations. By plotting load-deformation curves based on the modified load parameter $\beta$, we observe in Fig. 2.6 that the five curves nearly converge to one, and thus verified the validity of the obtained non-dimensional load parameter. The relation of the load parameter and deformation parameter can be expressed in a polynomial form by carrying out a polynomial fitting in Matlab. Figure 2.7 indicates that a $3^{rd}$ order polynomial works well in describing the large deformation behavior of cantilever beam. The polynomial function is given in Eq. (2.7). The coefficients of the polynomial function is listed in Table 2.1. It should be noted that Eq. (2.7) is valid within a certain range of the aspect ratio. In this case, the function is accurate when aspect ratio is between 5 and 30. When the aspect ratio gets larger, however, the original non-dimensional load parameter $\beta_o$ is better in generalizing beam’s large deformation behavior. And the polynomial expression needs to be obtained again accordingly.

$$\eta = \sum_{i=0}^{3} a_i \cdot \beta^i \quad 5 \leq \alpha \leq 30$$  \hfill (2.7)

<table>
<thead>
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</tr>
</tbody>
</table>

The polynomial representation of the nonlinear load-deflection curve is simple, accurate, and requires little computational cost. More importantly, the approach described above for the cantilever beam is applicable for other EFGs to obtain polynomial load-deflection functions. The results for the fixed-fixed beam and curved beam (circular beam) EFGs are presented in the following section.
2.3 Other Elemental Functional Geometries

In this section, analytical load-deflection functions are presented for two different EFGs: fixed-fixed beam and curved (circular) beam. Note that, for designing unit cells with one directional deformation, only the vertical deflection of the beams is considered.
2.3.1 Fixed-fixed beam

Figure 2.8 shows a beam fixed at both ends, with a vertical concentrated force applied in the middle. The beam has a uniform rectangle cross section, with $h$ denoting the beam height, $L$ denoting the beam length and $y$ denoting the vertical deformation.

![Fixed-Fixed Beam](image)

Figure 2.8: Fixed-Fixed Beam.

For fixed-fixed beam, the original load parameter $\beta_o$, the non-dimensional deformation parameter $\eta$ and aspect ratio $\alpha$ are given as:

$$\beta_o = \frac{FL^2}{EI}, \quad \eta = \frac{y}{L}, \quad \alpha = \frac{L}{h}$$  \hspace{1cm} (2.8)

When the original load parameter $\beta_o$ is used, the load-deflection curves for beam aspect ratio between 30 and 50 are shown in Fig. 2.9. It is shown that, the load-deflection curves are sensitive to the beam aspect ratio. In this regard, by following the optimization procedure described in the previous section, a modified load parameter $\beta$ is obtained in Eq. (2.9).

$$\beta = \frac{FL^2}{EI \cdot \alpha^{1.5}}$$  \hspace{1cm} (2.9)

With the modified load parameter $\beta$, the load-deflection curves converge to a narrow band, as shown in Fig. 2.10. The narrow band is then fitted to a 6th order polynomial function which is given by:

$$\eta = \sum_{i=0}^{6} a_i \cdot \beta^i \quad 30 \leq \alpha \leq 50$$  \hspace{1cm} (2.10)

The coefficients of the polynomial are listed in Table 2.2. Figure 2.11 shows the comparison between the obtained polynomial load-deformation curve and the FEA simulation results.
Figure 2.9: Fixed-fixed beam deformation curves with different beam aspect ratios.

Figure 2.10: Fixed-fixed beam deformation curves with modified load parameter.

Table 2.2: fixed-fixed beam polynomial coefficients

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.824596</td>
<td>-6.420578</td>
<td>31.696244</td>
<td>-85.785177</td>
<td>117.731101</td>
<td>-64.009050</td>
</tr>
</tbody>
</table>
2.3.2 Circular beam (pulling up)

Figure 2.12 shows a circular beam being pulled up by a vertical concentrated force. One end of the beam is fixed and the other is constrained horizontally. The circular beam has a uniform rectangle cross section, with $h$ denoting the beam height, $R$ denoting the outer radius and $y$ denoting the vertical deflection of the loaded end.

For the circular beam pulling up case, the original load parameter $\beta_o$, the non-dimensional deforma-
tion parameter $\eta$ and aspect ratio $\alpha$ are given as:

$$\beta_o = \frac{FR^2}{EI} \quad \eta = \frac{y}{R} \quad \alpha = \frac{R}{h} \quad (2.11)$$

When the original load parameter $\beta_o$ is used, the load-deflection curves for beam aspect ratio between 20 and 40 are shown in Fig. 2.13. It is shown that, the load-deflection curves are sensitive to the beam aspect ratio. In this regard, by following the optimization procedure described in Section 2.2, a modified load parameter $\beta$ is obtained as

$$\beta = \frac{FR^2}{EI \cdot \alpha^{0.03}} \quad (2.12)$$

With the modified load parameter $\beta$, the load-deflection curves converge to a narrow band, as shown in Fig. 2.10. The narrow band is then fitted to a $3^{rd}$ order polynomial function which is given by:

$$\eta = \sum_{i=0}^{3} a_i \cdot \beta^i \quad 20 \leq \alpha \leq 40 \quad (2.13)$$

where the coefficients of the polynomial, $a_i$, are listed in Table 2.3. Figure 2.15 shows the comparison between the obtained polynomial load-deformation curve and the FEA simulation results.

![Figure 2.13: Circular beam (pulling up) load-deformation curves with different beam aspect ratios.](image)
Figure 2.14: Circular beam (pulling up) load-deformation curves with modified load parameter.

Table 2.3: Circular beam (pulling up) polynomial coefficients

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.034044</td>
<td>-0.008561</td>
<td>0.002196</td>
</tr>
</tbody>
</table>

Figure 2.15: Circular beam (pulling up) load-deformation polynomial function compared with FEA result.
2.3.3 Circular beam (pushing down)

Figure 2.16 shows a circular beam being pushed down by a vertical concentrated force, with identical boundary conditions as shown in Fig. 2.12.

For the circular beam pushing down case, the original load parameter $\beta_o$, the non-dimensional deformation parameter $\eta$ and aspect ratio $\alpha$ are given as:

$$\beta_o = \frac{FR^2}{EI} \quad \eta = \frac{y}{R} \quad \alpha = \frac{R}{h}$$

(2.14)

When the original load parameter $\beta_o$ is used, the load-deflection curves for beam aspect ratio between 10 and 40 are shown in Fig. 2.17. It is shown that, the load-deflection curves are sensitive to the beam aspect ratio. In this regard, by following the optimization procedure described in Section 2.2, a modified load parameter $\beta$ is obtained as:

$$\beta = \frac{FR^2 \cdot \alpha^{0.03}}{EI}$$

(2.15)

With the modified load parameter $\beta$, the load-deflection curves converge to a narrow band, as shown in Fig. 2.18. The narrow band is then fitted to a $3^{rd}$ order polynomial function which is given by:

$$\eta = \sum_{i=0}^{3} a_i \cdot \beta^i \quad 10 \leq \alpha \leq 40$$

(2.16)

where the coefficients of the polynomial, $a_i$, are listed in Table 2.4. Figure 2.19 shows the comparison between the obtained polynomial load-deformation curve and the FEA simulation results.
Figure 2.17: Circular beam (pushing down) load-deformation curves with different beam aspect ratios.

Figure 2.18: Circular beam (pushing down) load-deformation curves with modified load parameter.

Table 2.4: Circular beam (pushing down) polynomial coefficients

<table>
<thead>
<tr>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.024060</td>
<td>-0.002342</td>
<td>(1.10 \times 10^{-4})</td>
</tr>
</tbody>
</table>
Figure 2.19: Circular beam (pushing down) load-deformation polynomial function compared with FEA result.

### 2.3.4 EFG Repository Summary

For the three EFGs with the four loading conditions, the modified non-dimensional load parameter, deformation parameter, the polynomial load-deformation functions with their aspect ratio ranges are summarized in Table 2.5. The polynomial coefficients are listed Table 2.6.

<table>
<thead>
<tr>
<th>EFG case</th>
<th>load ((\beta))</th>
<th>deformation ((\eta))</th>
<th>polynomial</th>
<th>aspect ratio ((\alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canti</td>
<td>(\frac{FL^2}{2EI\cdot\alpha})</td>
<td>(\frac{y}{L})</td>
<td>(\eta = \sum_{i=0}^{3} a_{ci} \cdot \beta^i)</td>
<td>([5,30])</td>
</tr>
<tr>
<td>FF</td>
<td>(\frac{FL^2}{EI\cdot\alpha})</td>
<td>(\frac{y}{L})</td>
<td>(\eta = \sum_{i=0}^{3} a_{ffi} \cdot \beta^i)</td>
<td>([30,50])</td>
</tr>
<tr>
<td>CirUp</td>
<td>(\frac{FR^2}{EI\cdot\alpha})</td>
<td>(\frac{y}{H})</td>
<td>(\eta = \sum_{i=0}^{3} a_{cui} \cdot \beta^i)</td>
<td>([20,40])</td>
</tr>
<tr>
<td>CirDown</td>
<td>(\frac{FR^2}{EI\cdot\alpha} e^{-\alpha 3})</td>
<td>(\frac{y}{H})</td>
<td>(\eta = \sum_{i=0}^{3} a_{cdi} \cdot \beta^i)</td>
<td>([10,40])</td>
</tr>
</tbody>
</table>

While the Unit Cell Synthesis approach for meta-material design is in its early stage of development, so far the EFG repository only contains 3 different EFGs and 4 loading cases. The repository is currently being expanded to allow a wide spectrum of nonlinear deformation behaviors.
As shown in the figures in the previous sections, the EFGs exhibit either a stiffening or a softening behavior when they undergo large deformation. The cantilever beam and fixed-fixed beam EFGs become stiffer when the deflection increases, but with different stiffening rates [39]. The circular beam, depending on the direction of the load, can have either a stiffening (pulled up) or a softening behavior (pushed down). These nonlinear deformation characteristics of the EFGs are limited in varieties for designing metamaterial unit cells to match target nonlinear deformation behavior. One solution to this limitation is to include more EFGs as mentioned above. On the other hand, with these 4 basic stiffening and softening deformation behaviors of the EFGs, we can also achieve more nonlinear deformation characteristics by combining these EFGs to form more complex structures in the unit cells.

Table 2.6: polynomial coefficients summary

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
<td>$a_{ci}$</td>
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<td>0.075602</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$a_{ff}$</td>
<td>0.824596</td>
<td>-6.420578</td>
<td>31.696244</td>
<td>-85.785177</td>
<td>117.731101</td>
<td>-64.009050</td>
<td></td>
</tr>
<tr>
<td>$a_{cu}$</td>
<td>0.034044</td>
<td>-0.008561</td>
<td>0.002196</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{cd}$</td>
<td>0.024060</td>
<td>-0.002342</td>
<td>$1.10 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.4 Combination of EFGs

As discussed in Chapter 1, the EFGs can be combined in different ways. By using the analogy of series and parallel connections of springs, the EFGs with nonlinear stiffness can be combined in series or parallel to create a different nonlinear stiffness or compliance. Two spring systems with different connections are shown below.

For the two spring systems, their effective stiffness $k_{eff}$ and effective compliance $C_{eff}$ are expressed in the form of:

$$k_{eff,p} = k_1 + k_2$$
$$k_{eff,s} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

$$C_{eff,p} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$
$$C_{eff,s} = C_1 + C_2$$

where $k_{eff,p}$ and $k_{eff,s}$ denote effective stiffness of two springs in parallel and in series, respectively, while $C_{eff,p}$ and $C_{eff,s}$ are the corresponding compliances. Similarly, effective stiffness or compliance of connected EFGs can be derived and represented in terms of individual EFG’s stiffness or compliance. In the pre-
previous sections, we have obtained the analytical functions of the EFGs’ load-deformation relations. For each individual EFG, the relation between the non-dimensional load parameter and non-dimensional deformation parameter can be written as $\eta = f(\beta)$. Since for a given EFG, its dimension, size and material properties are known, its load-deformation relation can be written as: $y = L \cdot f(c \cdot F)$, where $y$ is deformation, $L$ is beam length and $c$ is the constant multiplying factor of the load in the non-dimensional load parameter. Therefore, effective compliance can be obtained by differentiating $y$ with respect to $F$ as:

$$C(F) = \frac{dy(F)}{dF} = L \cdot \frac{df(c \cdot F)}{dF}$$  \hspace{1cm} (2.19)

Note that $f$ is a polynomial whose variable is $F$, therefore $C$ is also a function of $F$, which in most cases is not a constant.

### 2.4.1 EFGs connected in parallel

To calculate the nonlinear deformation of a structure consisting of two EFGs connected in parallel, the effective compliance or stiffness must be evaluated first. The expressions for effective compliance can be written as

$$C_{eff,p} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$  \hspace{1cm} (2.20)

where $C_1$ and $C_2$ are the compliance of each EFG. They are functions of the load acting on them:

$$C_1 = C_1(F_1) \quad C_2 = C_2(F_2)$$  \hspace{1cm} (2.21)
The deformation of the combined structure is then given by

\[ y_{eff}(F) = C_{eff,p} \cdot F \]  \hspace{1cm} \text{(2.22)}

The displacement of the combined structure can be obtained by using Eqs. (2.20-2.22). In this section, a geometric model of a fixed-fixed beam and a circular beam connected in parallel is taken as an example. The model is shown in Fig. 2.22.

![Fixed-fixed beam and circular beam connected in parallel.](Image)

The goal is to find the effective vertical displacement \( y_{eff} \). It is clear from the figure that, given the load, both beams will deform the same in the vertical direction. The analytical deformation functions of fixed-fixed beam and circular beam (pushed down) are assumed to be \( y_f = g_f(F_f) \) and \( y_{cd} = g_{cd}(F_{cd}) \), respectively. Not that, the subscripts \( f \) and \( cd \) represent fixed-fixed beam and circular beam (pushed down) respectively. Then a system of equations can be written as

\[ F = F_f + F_{cd} \]  \hspace{1cm} \text{(2.23)}

\[ y_{eff} = g_f(F_f) = g_{cd}(F_{cd}) \]  \hspace{1cm} \text{(2.24)}

By solving the simultaneous equations, we are able to obtain the distribution of total force \( F \) on each beam, i.e. \( F_f \) and \( F_{cd} \), respectively. Then, the deformation of the combined structure can be calculated by substituting either \( F_f \) or \( F_{cd} \) into Eq. (2.24).

By using ANSYS (Fig. 2.23), FEA simulation of the model is performed to validate the analytical solution. By comparing the results generated from the analytical solution and those from ANSYS, the effec-
tiveness of the analytical functions is verified. Four different combinations of the two beams’ aspect ratios are modeled to test the accuracy of the analytical solution. It is shown in Figs. 2.24, 2.25 that the curves produced by FEA and analytical functions are almost identical. The results demonstrate that the combination of the analytical deformation functions and the analogy of the series and parallel connections of the EFGs enable accurate prediction of the deformation of the combined structures.

Figure 2.23: Fixed-fixed beam and circular beam connected in parallel: FEA validation.

Figure 2.24: Comparison of FEA and analytical solutions: (left) fixed-fixed beam and circular beam aspect ratios are 40 and 30, respectively; (right) fixed-fixed beam and circular beam aspect ratios are 30 and 30, respectively.
2.4.2 EFGs connected in series

Similar to the parallel connection, when the EFGs are connected in series, the structure’s effective compliance must be solved first. The expressions for effective compliance can be written as

\[
C_{\text{eff},s} = C_1 + C_2 \quad (2.25)
\]

where \( C_1 \) and \( C_2 \) are the compliance of each EFG. They are functions of the load acting on them:

\[
C_1 = C_1(F_1) \quad C_2 = C_2(F_2) \quad (2.26)
\]

The deformation of the combined structure is then given by

\[
y_{\text{eff}}(F) = C_{\text{eff},p} \cdot F \quad (2.27)
\]

The displacement of the combined structure can be obtained by using Eqs. (2.25-2.27). Figure 2.26 shows an example of a series connection: a cantilever beam can be regarded as two shorter cantilevers, denoted as beam 1 and beam 2 in the figure, connected in series. However, the loading condition on beam 1 is no longer the case as displayed in Fig. 2.3, which means vertical deformation of beam 1 is not only due to force \( F \).

The actual loading situation of beam 1 and beam 2 is illustrated in Fig. 2.27. The corresponding displacement breakdown is depicted in Fig. 2.28. The magnitude of the moment acting on the right end of beam 1 is \( F \cdot L_2 \). The vertical displacement of the right end of beam 2 is decomposed into three components.
in Eq. 2.28, where \( y_1 \) is displacement of beam 1 caused by both force \( F \) and moment \( M \), \( L_2 \cdot \sin \theta \) indicates the displacement due to beam 2’s rigid body rotation and \( y_2 \) is beam 2 deflection due to \( F \).

\[
y_{total} = y_1 + L_2 \cdot \sin \theta + y_2 \cdot \cos \theta
\]

(2.28)

Since the moment plays a role in the total displacement, only knowing the load-deformation relation is no longer sufficient. In order to calculate the total displacement, a new function of displacement in terms of both force and moment is necessary. In addition, analytical function of the angle of rotation \( \theta \) is also required. Using the idea of the non-dimensional load parameter optimization described in Section 2.2, along with the non-dimensional load (force) parameter \( \beta_F \), a non-dimensional moment parameter is defined here as \( \beta_M \):

\[
\beta_F = \frac{FL^2}{2EI \cdot \alpha^{0.1}} \quad \beta_M = \frac{M \cdot L}{2EI}
\]

(2.29)
where the 2 in the denominator has no physical meaning and only serves to make the expression consistent with the non-dimensional force parameter. The force and moment parameters are used to generate load-deformation curves with different aspect ratios. Whether $\beta_M$ requires aspect ratio based modification or not is determined by observing the plot of deformation data points with varying aspect ratios. In Figs. 2.29 and 2.30, data points of non-dimensional deformation parameter and angle of rotation with aspect ratios varying from 10 to 30 are plotted. It is observed from the plots that the moment parameter is independent of aspect ratio. Therefore, $\beta_M$ needs no modification. A two-variable surface fitting process is then conducted to obtain analytical expressions of the vertical displacement and the angle of rotation. The multivariable polynomials are obtained as shown Eqs. (2.30, 2.31). The coefficients are listed in Table 2.7.

$$\eta(\beta_F, \beta_M) = a_{00} + a_{10} \cdot \beta_F + a_{01} \cdot \beta_M + a_{20} \cdot \beta_F^2 + a_{11} \cdot \beta_F \beta_M + a_{02} \cdot \beta_M^2$$

$$+ a_{30} \cdot \beta_F^3 + a_{21} \cdot \beta_F^2 \beta_M + a_{12} \cdot \beta_F \beta_M^2 + a_{03} \cdot \beta_M^3$$ \hspace{1cm} (2.30)

$$\theta(\beta_F, \beta_M) = b_{00} + b_{10} \cdot \beta_F + b_{01} \cdot \beta_M + b_{20} \cdot \beta_F^2 + b_{11} \cdot \beta_F \beta_M + b_{02} \cdot \beta_M^2$$

$$+ b_{30} \cdot \beta_F^3 + b_{21} \cdot \beta_F^2 \beta_M + b_{12} \cdot \beta_F \beta_M^2 + b_{03} \cdot \beta_M^3$$ \hspace{1cm} (2.31)

<table>
<thead>
<tr>
<th>$a_{ij}$</th>
<th>0</th>
<th>0.971</th>
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<th>-1.19</th>
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<th>0.01745</th>
<th>0.2468</th>
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<tbody>
<tr>
<td>$b_{ij}$</td>
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<td>2.17</td>
<td>-0.4394</td>
<td>-1.72</td>
<td>-1.599</td>
<td>-0.02995</td>
<td>0.1931</td>
<td>0.6523</td>
<td>0.4717</td>
</tr>
</tbody>
</table>

Table 2.7: Polynomial coefficients

ANSYS Mechanical APDL is once again used to provide nonlinear FEA results of displacement and angle of rotation for beams with different aspect ratios. The FEA results are also plotted in Figs. 2.29, 2.30 for comparison. It is observed that FEA solutions match very well with the fitted surface. The numerical results confirm the effectiveness of polynomial functions obtained.

With the comprehensive cantilever beam model, we are able to find the solution of the deformation for two cantilever beams connected in series. Equation (2.28) can be rewritten as Eq. (2.32):

$$y_{total} = y_1(F, FL_2) + L_2 \cdot \sin \theta(F, FL_2) + y_2(F, 0) \cdot \cos \theta(F, FL_2)$$ \hspace{1cm} (2.32)
Figure 2.29: Displacement surface fitting.

Figure 2.30: Angle of rotation surface fitting.
where \( y_1 \) and \( y_2 \) are calculated as:

\[
y_1 (F, L_2) = L_1 \cdot \eta \left( \frac{F L_1^2}{2EI \cdot \alpha_1}, \frac{F L_2 L_1}{2EI} \right) \quad y_2 (F, 0) = L_2 \cdot \eta \left( \frac{F L_2^2}{2EI \cdot \alpha_2}, 0 \right)
\] (2.33)

Figure 2.4.2 shows two curves representing the analytical results and FEA results obtained from ANSYS. The results show that, while the analytical approach is reasonably accurate compared to the nonlinear FEA, visible discrepancy exists when deformation is large. The cause of the discrepancy is likely to be the effect of the horizontal displacement of the beam tips, which is not included in the analytical model.

![Figure 2.31: FEA validation: two cantilever beams in series.](image)

### 2.5 Summary

In this chapter, a systematic approach is developed to obtain analytical functions of the EFGs’ load-deformation relations. Analytical load-deformation functions are obtained for three EFGs with four loading conditions. The analytical solution’s accuracy is validated by using the results from nonlinear FEA.

To expand the design space of meta-material unit cells, the EFGs are combined in series or in parallel to produce new deformation behavior. Depending on how they are connected, the combined geometry’s deformation behavior can be calculated from individual EFG’s deformation response. Two geometry models, which are respectively EFGs connected in parallel and in series, are used as examples to demonstrate the
process of obtaining analytical load-deformation functions of the combined structure. The solutions are verified by FEA results.
Chapter 3

Semi-Analytical Unit Cell Synthesis Method

3.1 Introduction

As discussed in Chapter 1, although the recently developed Unit Cell Synthesis Method has been successful in producing meta-material designs that can deform following a target nonlinear uniaxial compression curve, there remains several drawbacks, such as lacking quantitative understanding of nonlinear deformation behavior of EFGs, initial UC topology relying on educated guess and a computationally intensive and time consuming size optimization process.

In view of such an engineering gap, in this work, the analytical load-deformation functions developed for the EFGs and their combinations are employed in the Unit Cell Synthesis Method to provide efficient quantitative prediction of the nonlinear deformation behavior of EFGs and to replace the FEA simulations in the size optimization step of the unit cell design. The method, which carries out the unit cell synthesis approach using the analytical load-deformation functions, is referred to as the Semi-Analytical Unit Cell Synthesis Method. As in the original unit cell synthesis method, the basic principle is to achieve an overall nonlinear deformation response of bulk meta-material by using a combination of geometric nonlinearities associated with different EFGs. The nonlinear deformation characteristics of these elemental geometries are determined from the nonlinear mechanics of these EFGs. The unit cell is constructed by selecting EFGs by comparing their nonlinear deformation behavior with the target deformation response. Then size optimiza-
tion is carried out to obtain the optimal meta-material UC design. The Semi-Analytical Unit Cell Synthesis Method utilizes the approximated analytical solutions of EFGs’ nonlinear deformation behavior instead of nonlinear FEA simulations. For this reason, it is able to produce unit cell designs without running any FEA simulation, which leads to greatly accelerated design circles.

3.2 Method

Figure 3.1 illustrates the workflow of the Semi-Analytical Unit Cell Synthesis Method for UC design. The scope of this method is specified to 2-D geometries that are extruded in the third dimension, i.e. 3-D lattices are not considered. The method includes a process of selecting and assembling EFGs to form unit cell, then followed by a size optimization to get the final optimal design. The entire design process is divided into four steps.

![Design workflow diagram](image)

Figure 3.1: Design workflow.

3.2.1 Step 1: EFG repository preparation

Step 1 is EFG repository preparation, which has been explained in length in Chapter 2. EFGs are the building blocks of UCs. For this thesis, the EFG repository consists of three EFGs with four loading conditions: cantilever beam (Canti), fixed-fixed beam (FF), circular beam pulling up (CirUp), and circular
beam pushing down (CirDown).

3.2.2 Step 2: EFG selection and combination

Step 2 is to select EFGs from the repository and to construct the UC by combining the chosen EFGs. EFGs can be connected in two configurations: series connection and parallel connection. In general, assembling EFGs in parallel can increase the stiffness and placing them in series makes the overall stiffness of structure smaller. EFG selection is based on the targeted deformation curve: for a stiffening curve, EFGs with stiffening effect are likely to meet the desired behavior, and vice versa. Among the load-deformation curves of the EFGs, circular beam pushing down shows a softening response while all the other three become stiffer with increased load.

3.2.3 Step 3: ESG design to form UC

Elemental Structural Geometry (ESG) is introduced here as a structural entity to help construct UC along with EFGs. ESGs are structural components which serve as support/rigid connection of the EFGs and adjacent UCs. They have high stiffness and do not interfere with the deformation of EFGs. The former feature works to isolate tunable properties of EFGs, while the latter contributes to completing the UC and enable tessellation into meta-material.

As an example, a UC design is shown in Fig. 3.2, where EFGs and ESGs are exhibited in different colors. The UC is constructed in a manner that the Canti and CirUp EFGs are first connected in series, and then two of these structures are placed in parallel. The ESGs are necessary to support connectivity function while accomplishing tessellation and in transmitting the loads from upper layers.

![Figure 3.2: An example of UC construction.](image)
3.2.4 Step 4: UC size optimization

As the material property is predetermined and basic UC layout is completed, a size optimization is required to achieve the objective of matching the targeted nonlinear response. Taking the UC design in Fig. 3.2 as an example, all design variables of the UC are depicted in Fig. 3.3. Each independent variable has a varying range determined by manufacturing constraints and geometry feasibility. In Fig. 3.3, $H$ and $W$ are unit cell dimension parameters which are determined first.

![Figure 3.3: UC size parameters.](image)

In this step, if the size optimization process yields a UC design which satisfies both optimizing objective and design constraints, then the UC structure is deemed feasible. If no design variable combination can meet both the objective and design constraints, the UC topology is not a feasible design. A new UC geometry needs to be formed from the possible combinations of the EFGs and the size optimization process is repeated for the new iteration.

3.3 Case Study

A case study from [11] is introduced here to further explain the design process as well as to test the performance of the semi-analytical unit cell synthesis method. The design purpose is to seek a new material which can replace a carbon black filled Styrene-Butadiene Rubber (SBR) tank track pad. The motivation comes from the material failure of SBR caused by its hysteretic loss due to the elastomer’s viscoelastic nature, and the high cost associated with the tank track rubber repairs and replacements [36]. Besides avoiding hysteretic loss, mimicking SBR’s mechanical property is the principal goal. The meta-material is required to exhibit high strains at low stress levels. Since no alternative traditional material exists to fulfill the requirement of large compliance and low hysteretic loss, developing a meta-material composed of linear elastic...
material which is inherently non-hysteretic is proposed. Since the predominant mode of deformation of the track pad is compression, the design objective of the meta-material is to achieve a defined nonlinear behavior under compression similar to the SBR in use. The material nonlinearity of the SBR is to be reproduced by utilizing the geometric nonlinearity of the EFGs in the unit cells. The design objective of meta-material is to achieve a pre-determined nonlinear deformation behavior under uniaxial compression. Figure 3.4 shows the stress strain relationship of the SBR in uniaxial compression obtained via experimental testing and subsequent curve fitting using a 2-parameter Ogden model. The specific range of the compressive behavior is indicated in Fig. 3.4. The target property values that the designed meta-material will be evaluated against are shown in Table 3.1. And titanium alloy is chosen to be the meta-material constitutive material.

Figure 3.4: Targeted stress-strain curve under uniaxial compression.

<table>
<thead>
<tr>
<th>Applied Pressure MPa</th>
<th>Meta-Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>-0.3817</td>
<td>0.05</td>
</tr>
<tr>
<td>-0.8384</td>
<td>0.10</td>
</tr>
<tr>
<td>-2.0632</td>
<td>0.20</td>
</tr>
<tr>
<td>-3.9327</td>
<td>0.30</td>
</tr>
</tbody>
</table>
3.4 “Canti” UC design

Since Chapter 2 has covered the preparation of EFG repository process, the first step in the method is not discussed again. All the UC designs in this thesis choose EFGs from the EFG repository shown in Fig. 1.8 and Table 2.5.

3.4.1 EFG selection and combination

Since the targeted curve shows a stiffening effect with increasing load, EFGs with similar deformation behavior are sought after. First we can select cantilever beam (Canti) as the single constructing EFG. Therefore a UC design called ”Canti” is proposed here to meet the objective curve.

3.4.2 ESG design to form UC

ESGs are introduced in this step as a structural entity to help construct UC along with EFGs. ESGs are structural components which serve as support/rigid connection of the EFGs and adjacent UCs. They have high stiffness and do not interfere with the deformation of EFGs. The former feature works to isolate tunable properties of EFGs, while the latter contributes to completing the UC and enable tessellation into metamaterial.

The Canti UC design is shown in Fig. 3.5, with EFGs and ESGs exhibited in different colors. The UC is assembled in a manner that two identical cantilever beams are connected in parallel. Therefore, the UC structure can be regarded as the equivalent spring system shown in Fig. 3.6. The ESGs are necessary to support connectivity function as well as to accomplish tessellation and pass the loads from upper layers. Figure 3.7 presents the conceptual UC tessellation. Since each UC in the meta-material structure is required to undergo similar deformation when acting under a compressive load, UC tessellation is carried out in the following manner: the UC in the upper layer is offset by half of the UC width such that the ESG of the top UC provides the necessary boundary conditions to its EFGs and at the same time transmits force down to the EFGs of the UC underneath. Thus, the completed tessellation of the UCs for the meta-material structure is as shown in Fig. 3.7.

3.4.3 UC size optimization

As the material of the EFGs (Table 3.2) has been selected and basic UC layout is fixed, a size optimization is required to achieve the objective of matching the target nonlinear response. For the Canti
UC design, all design variables of the UC are depicted in Fig. 3.8. Among these variables, $t_2$, $t_3$, $g$ are independent, and $t_1$ is twice the summation of $t_3$ and $g$. Each independent variable has a varying range determined by geometry feasibility and manufacturing constraints. The varying range of all design variables are listed in Table 3.5. $H$ and $W$ are unit cell dimension parameters which are known beforehand, given in Table 3.3.

From Fig. 3.8, the length of the cantilever beam $L$ and the aspect ratio $\alpha$ can be represented by Eqs. (3.1, 3.2). As described in Chapter 2, the analytical functions of the beams’ deformation are expressed by
Table 3.2: Material properties of EFGs in Canti UC design

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Titanium Alloy</td>
<td>Ti₃Al – 8V – 6Cr – 4Mo – 4Zr – 0.05Pd</td>
</tr>
<tr>
<td>Young’s Modulus (E)</td>
<td>102 GPa</td>
</tr>
<tr>
<td>Poisson Ratio (ν)</td>
<td>0.32</td>
</tr>
<tr>
<td>Yield Strength (δₚ)</td>
<td>1103 MPa</td>
</tr>
</tbody>
</table>

Table 3.3: Constant design parameters for Canti UC design

<table>
<thead>
<tr>
<th>Constant Design Parameters</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>0.0032</td>
</tr>
<tr>
<td>W</td>
<td>0.0205</td>
</tr>
<tr>
<td>Hₜotal</td>
<td>0.0227</td>
</tr>
</tbody>
</table>

using these parameters.

\[ L = W - t₁ - g \]  \hspace{1cm} (3.1)

\[ α = \frac{L}{t₂} \]  \hspace{1cm} (3.2)

For the size optimization, the effective mechanical properties of the UC needs to be defined. For the purpose of matching a stress-strain curve, a meta-strain is defined as the percentage of the bulk meta-material’s deformation as shown in Fig. 3.9: \( ϵ = δ/H \), where \( δ \) is the UC’s vertical displacement and \( H \) is the original height of meta-material. Note that \( H \) here is not the same as the UC height \( H \). The objective function can then be mathematically written as,

\[ \text{min strain} - \text{error} : f = \sum_{i=1}^{N} (\epsilonᵢ^{t} - \epsilonᵢ)^2 \]  \hspace{1cm} (3.3)

where \( \epsilonᵢ^{t} \) and \( \epsilonᵢ \) are target strain and meta-strain of meta-material at \( i^{th} \) load step for a total of \( N \) steps. The size optimization is launched in a commercial optimizer modeFRONTIER 4.4.2. The optimization workflow is shown in Fig. 3.10.

The optimization process starts with creating input variables with ranges and constraints. Then initial design of experiments (DoE) are generated using the Uniform Latin Hypercube (ULH) DoE algorithm, which ensures that for each variable the values are distributed randomly and uniformly [37]. These designs are used
Figure 3.9: Meta-strain of the meta-material.

Figure 3.10: Size optimization workflow.
to calculate output variables, \( strain \) – \( error \), and aspect ratios of cantilever beam. Constraints are applied to aspect ratios because of the valid range for each EFG’s analytical load-deformation function. NSGA-II, short for Non-Dominated Sorting Genetic Algorithm II is selected as the optimization algorithm to minimize \( strain \) – \( error \). Generally, the optimizing process is initiated with generating DoE in modeFRONTIER, then these design points are imported to a Matlab program to calculate deformation which enter modeFRONTIER as output variables to see if the design is feasible. The optimization algorithm decides which trajectory the design point is going, and improves the design result as generation number grows. Genetic Algorithm (GA) is an evolutionary algorithm that is based on a biological systems’ improved fitness through evolution. A large population size and a large number of generations enhance the possibility of achieving a global optimum solution, but processing time can be substantially increased accordingly [38]. However, since evaluating analytical functions is much faster than the nonlinear FEA simulations used before, computational cost is not much a concern here. Table 3.4 lists the parameters used in the optimization algorithm.

Table 3.4: Optimization algorithm parameters for Canti UC design

<table>
<thead>
<tr>
<th>Optimization Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Initial DoE</td>
<td>24</td>
</tr>
<tr>
<td>Number of Generations</td>
<td>100</td>
</tr>
<tr>
<td>Cross-over Probability</td>
<td>0.9</td>
</tr>
<tr>
<td>Mutation Probability</td>
<td>1.0</td>
</tr>
<tr>
<td>Total Design Points</td>
<td>2400</td>
</tr>
</tbody>
</table>

Table 3.5: Design variable ranges for Canti UC design

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Lower Bound ((m))</th>
<th>Upper Bound ((m))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>(1 \times 10^{-4})</td>
<td>(6 \times 10^{-4})</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>(1 \times 10^{-3})</td>
<td>(1.8 \times 10^{-4})</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>(1 \times 10^{-3})</td>
<td>(3 \times 10^{-3})</td>
</tr>
</tbody>
</table>

The evolution histories of variables \( t_2, t_3, g \) and \( strain \) – \( error \) are shown in Figs 3.11 to 3.14, respectively. As the generation number grows, these variables converge to the optimal design. The converging process takes less than 5 minutes on a desktop computer with an Intel(R) Core(TM) i7-4790 CPU @ 3.6GHz processor.

Table 3.6 shows the optimization results for a feasible design. All parameters except \( strain \) – \( error \) are in meters. Figure 3.15 shows the nonlinear deformation response of the optimum design as compared to the target curve. In Fig. 3.16 we add a curve which is the FEA result to validate the UC design. From the
plots, it can be concluded that the size optimization procedure is able to produce a Canti UC design whose compressive deformation matches the target deformation response. The comparison with FEA result has confirmed the result’s accuracy.

<table>
<thead>
<tr>
<th>strain-error</th>
<th>$H_{total}$</th>
<th>$H$</th>
<th>$W$</th>
<th>$g$</th>
<th>$t_2$</th>
<th>$t_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.8634 \times 10^{-4}$</td>
<td>0.0227</td>
<td>0.0032</td>
<td>0.0205</td>
<td>$4.9032 \times 10^{-4}$</td>
<td>$1.0000 \times 10^{-3}$</td>
<td>$2.3774 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

It should be noted that a constraint for the objective function needs to be incorporated in the optimization process. If the optimizer is able to generate sufficient design points that satisfy the objective constraint, the unit cell design is deemed feasible. Since the target curve has a total of 4 datum points, along with the objective function expression given in Eq. (3.5), the constraint for the objective can be written in Eq. (3.4). While it is expected that a good meta-material unit cell design can attain strain error much lower than the given constraint value, the prescribed value is expected to give a good indication of the feasibility of the
Figure 3.15: Target curve compared with deformation curve of the optimal Canti UC design.

Figure 3.16: Canti UC design: FEA validation.

The summary of design points generated by the size optimization process is shown in Fig. 3.17. It is

\[
strain - error \leq 4 \times 10^{-4}
\]  

(3.4)

The summary of design points generated by the size optimization process is shown in Fig. 3.17. It is
observed that nearly 33% of the design points (2,400) satisfy all constraints. Therefore, the Canti UC design is regarded as feasible.

Figure 3.17: Canti UC design summary.

3.5 ”CantiCirup” UC Design

An alternative meta-material UC design is proposed in this section. Although the Canti UC design shows good performance in matching the target deformation curve, it is still slightly stiffer than the target response. Based on theoretical analysis of combined EFGs connected in series, adding a new EFG in series can increase the compliance of the combined structure. Therefore, in order to obtain a better performance, incorporating a new EFG to form a series connection in the Canti UC design is attempted in this section.

3.5.1 EFG selection and UC design

In the new design, a circular beam EFG is selected to add to the original Canti UC design. The UC topology is depicted in Fig. 3.18. The UC is constructed in a manner that a cantilever and a circular beam are first connected in series, and then two of the combined structures are connected in parallel. The ESGs are designed as in the Canti UC design to provide connectivity while carrying out tessellation and in transmitting the loads from one UC layer to another. Figure 3.19 presents the conceptual UC tessellation process. As a result, the equivalent loading situation and equivalent spring system are illustrated in Fig. 3.20 and Fig. 3.21, respectively. The new UC design is referred to as “CantiCirup” UC design in this section.
3.5.2 UC Size optimization

The constituent material for the CantiCirup design is the same as that used in the Canti UC design. The material properties are listed in Table 3.2. All design variables of the UC are shown in Fig. 3.18. Among these variables, $t_2$, $t_3$, $t_4$, $R$, $g$ are independent, and $t_1$ is twice the summation of $t_3$ and $g$. Integrating the circular beam EFG adds 2 more design variables: the beam width $t_4$ and the outer radius $R$. Each independent variable has a range determined by manufacturing constraints and geometry feasibility. Table 3.7 lists the design variables’ upper and lower bounds. The objective function is written below as

$$\text{min strain} - \text{error} : f = \sum_{i=1}^{N} (\epsilon_i^f - \epsilon_i)^2$$  (3.5)
Table 3.7: Design variable ranges for CantiCirp UC

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Lower Bound (m)</th>
<th>Upper Bound (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>( 1 \times 10^{-4} )</td>
<td>( 1.8 \times 10^{-3} )</td>
</tr>
<tr>
<td>( R )</td>
<td>( 2 \times 10^{-4} )</td>
<td>( 2.5 \times 10^{-3} )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( 6 \times 10^{-5} )</td>
<td>( 2 \times 10^{-3} )</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>( 6 \times 10^{-5} )</td>
<td>( 3 \times 10^{-3} )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( 6 \times 10^{-5} )</td>
<td>( 6 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

Figure 3.22: modeFRONTIER work-flow.

From Fig. 3.18, the length \( L \) and aspect ratio \( \alpha \) of cantilever beam can be written as Eq. (3.6, 3.7), and aspect ratio of the circular beam \( \alpha_{cu} \) can be represented as Eq. (3.8). Then these parameters can be used to obtain analytical solution of the deformation of the UC by implementing the analytical approach described in Chapter 2.
\[ L = W - R - t_1 - g \]  

(3.6)

\[ \alpha_c = \frac{L}{t_2} \]  

(3.7)

\[ \alpha_{cu} = \frac{R}{t_4} \]  

(3.8)

Then initial design of experiments (DoE) are again generated using Uniform Latin Hypercube (ULH) DoE algorithm. These designs are used to calculate output variables: strain - error and the aspect ratios of the EFGs. Constraints are applied to keep aspect ratios stay in the valid range for each EFG’s analytical solution. The optimization parameters are shown in Table 3.8. The evolution histories of design variables are shown in Fig. 3.23 to Fig. 3.28. The converging process takes approximately 40 minutes on a desktop computer with an Intel(R) Core(TM) i7-4790 CPU @ 3.6GHz processor.

Table 3.8: Optimization algorithm parameters for CantiCirup design

<table>
<thead>
<tr>
<th>Optimization Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Initial DoE</td>
<td>50</td>
</tr>
<tr>
<td>Number of Generations</td>
<td>100</td>
</tr>
<tr>
<td>Cross-over Probability</td>
<td>0.9</td>
</tr>
<tr>
<td>Mutation Probability</td>
<td>1.0</td>
</tr>
<tr>
<td>Total Design Points</td>
<td>5000</td>
</tr>
</tbody>
</table>

Figure 3.23: \( g \) convergence history.  
Figure 3.24: \( R \) convergence history.

Three design constraints are imposed in the UC optimization. The first constraint ensures that there
Figure 3.25: $t_2$ convergence history.  
Figure 3.26: $t_3$ convergence history.  
Figure 3.27: $t_4$ convergence history.  
Figure 3.28: Canti Aspect Ratio.

There is no contact in the meta-material structure when it undergoes 20% vertical deformation (i.e. meta-strain=0.2), as shown in Eq. (3.9).

$$R + t_4 \leq 0.00256$$ \hspace{1cm} (3.9)

The second design constraint guarantees that the cantilever beam thickness is less than the summation of the minor radius and the thickness of the circular beam. This ensures that the topology of the UC geometry is correct when it is constructed. The formulation is given in Eq. (3.10).

$$t_2 - R - t_4 \leq 1 \times 10^{-5}$$ \hspace{1cm} (3.10)

The third design constraint, as shown in Eq. (3.11), also ensures correctness of the UC geometry by constraining the radius of the circular beam such that it does not intersect with the ESGs on the sides.

$$R + t_1 + g \leq W \equiv R + 2g + 3t_3 - 0.0205 \leq 0$$ \hspace{1cm} (3.11)
Table 3.9 shows the optimization results of a feasible design. All parameters except \(\text{strain} - \text{error} \) are in millimeters. Figure 3.29 shows the nonlinear deformation response of the optimum design as compared to the target curve. In Fig. 3.30, we add a curve which is the FEA result to validate the UC design.

The plots show that the optimized CantiCirup UC design is able to match the target nonlinear deformation response. The comparison with FEA result has confirmed the result’s accuracy. The value of \(\text{strain} - \text{error} \) is further decreased, which indicates a better solution is obtained. With the same constraint for the objective function expressed in Eq. (3.4), the size optimization process produces a design summary as shown in Fig. 3.31. Among the 5000 design points, 46% of them satisfy all the design constraints. Therefore, this UC design is deemed as a feasible design.

### Table 3.9: Optimum design parameters

<table>
<thead>
<tr>
<th>strain-error</th>
<th>( H_{\text{total}} )</th>
<th>( H )</th>
<th>( W )</th>
<th>( g )</th>
<th>( R )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2.4979 \times 10^{-4} )</td>
<td>22.700</td>
<td>3.200</td>
<td>2.050</td>
<td>1.800</td>
<td>2.0453</td>
<td>1.1441</td>
<td>2.5543</td>
<td>5.0946</td>
</tr>
</tbody>
</table>

Figure 3.29: Target curve compared with deformation curve of the optimal CantiCirup UC design.

It can be concluded from the CantiCirup UC results that incorporating more EFGs in the UC increases the number of design variables, enlarges the design space and increase the tuning ability of the meta-material deformation behavior. With 3 additional design variables, the CantiCirup design offers bet-
Figure 3.30: CantiCirUp UC design: FEA validation.

Figure 3.31: CantiCirUp UC design summary.

ter solutions than the Canti UC design. However, combining multiple EFGs leads to a more complicated optimization problem due to additional design constraints. For CantiCirup design, the three constraints are necessary for constructing a topologically correct UC geometry. These design constraints largely depend on the selected EFGs and their combinations. Therefore, a good understanding of the UC configuration is required for formulating the optimization problem.
Chapter 4

Conclusions and Future Work

4.1 Conclusions

In this work, we have developed a systematic approach to obtain analytical load-deformation functions of the EFGs subjected to large deformations. In this approach, for each EFG and load condition, a non-dimensional deformation parameter is expressed as a polynomial function of a non-dimensional load parameter. The non-dimensional load parameter is optimized so that the nonlinear deformation behavior of EFGs with different aspect ratios can be described by a single polynomial function. In this thesis, nonlinear deformation polynomial functions are obtained for three EFGs with four loading conditions. Furthermore, nonlinear deformation polynomial functions are obtained for multiple EFGs connected in series or in parallel. Two example structures are studied for their deformation behavior. By comparing the analytical solutions with the nonlinear FEA results, the accuracy of the analytical load-deformation functions is verified.

We have implemented the analytical load-deformation functions in the Unit Cell Synthesis Method to design meta-materials with prescribed nonlinear deformation response. The new meta-material design method is named Semi-Analytical Unit Cell Synthesis Method. The method enables the development of unit cell structures by means of combining EFGs with tunable nonlinear deformation characteristics. The design process is initiated by selecting EFGs which have deformation response similar to the target response. The EFGs are connected in a certain way for stiffness tuning. The combined structure are connected with ESGs to form unit cell topology. Subsequent size optimization is performed on the unit cell design, converging the unit cell response to the target curve. In the thesis, a case study is presented to demonstrate the design and optimization process, and the effectiveness of the semi-analytical unit cell synthesis method.
4.2 Future Work

While it has been demonstrated that the semi-analytical unit cell synthesis approach is effective in designing metamaterials with prescribed nonlinear deformation behavior, several improvements are expected to be made. In this regard, for our future work, we would like to

1. include additional EFGs and expand the EFG repository by using the procedure discussed in Chapter 2.
2. study different loading conditions and deformation in multiple directions for the EFGs.
3. include stress distribution and manufacturability in the optimization objective, and implement a multi-objective optimization process to obtain better design.
4. automate the EFGs selection process by using the EFGs’ load-deformation functions.
References


