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Generalized Colorings of Graphs

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GENERALIZED COLORINGS OF GRAPHS

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

by
Honghai Xu
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Accepted by:
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Abstract

A graph coloring is an assignment of labels called “colors” to certain elements of a graph subject to certain constraints. The proper vertex coloring is the most common type of graph coloring, where each vertex of a graph is assigned one color such that no two adjacent vertices share the same color, with the objective of minimizing the number of colors used. One can obtain various generalizations of the proper vertex coloring problem, by strengthening or relaxing the constraints or changing the objective. We study several types of such generalizations in this thesis.

Series-parallel graphs are multigraphs that have no K_4 -minor. We provide bounds on their fractional and circular chromatic numbers and the defective version of these parameters. In particular we show that the fractional chromatic number of any series-parallel graph of odd girth k is exactly $2k/(k-1)$, confirming a conjecture by Wang and Yu.

We introduce a generalization of defective coloring: each vertex of a graph is assigned a fraction of each color, with the total amount of colors at each vertex summing to 1. We define the fractional defect of a vertex v to be the sum of the overlaps with each neighbor of v , and the fractional defect of the graph to be the maximum of the defects over all vertices. We provide results on the minimum fractional defect of 2-colorings of some graphs. We also propose some open questions and conjectures.

Given a (not necessarily proper) vertex coloring of a graph, a subgraph is called rainbow if all its vertices receive different colors, and monochromatic if all its vertices receive the same color. We consider several types of coloring here: a no-rainbow- F coloring of G

is a coloring of the vertices of G without rainbow subgraph isomorphic to F ; an F -WORM coloring of G is a coloring of the vertices of G without rainbow or monochromatic subgraph isomorphic to F ; an (M, R) -WORM coloring of G is a coloring of the vertices of G with neither a monochromatic subgraph isomorphic to M nor a rainbow subgraph isomorphic to R . We present some results on these concepts especially with regards to the existence of colorings, complexity, and optimization within certain graph classes. Our focus is on the case that F , M or R is a path, cycle, star, or clique.

Dedication

This thesis is dedicated to Melody Siyun Xu.

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Chapter 1

Introduction

A graph coloring is an assignment of labels called “colors” to certain elements of a graph subject to certain constraints. The proper vertex coloring is the most common type of graph coloring, where each vertex of a graph is assigned one color such that adjacent vertices receive different colors, with the objective of minimizing the number of colors used.

One can obtain various generalizations of the proper vertex coloring problem, by strengthening or relaxing the constraints or changing the objective. For example, in a *distance d -coloring* (see, for example, [55, 32, 48]), no two vertices within distance d of each other share the same color; in a *defective coloring* (see, for example, [20, 21]), a vertex can receive the same color as some of its neighbors do; in a *fractional coloring* (see, for example, [57, 67]), each vertex receives a set of colors instead of one color.

We study several types of such generalizations in this thesis. For comprehensive surveys of graph coloring problems, we refer readers to [49, 69, 17].

1.1 Definitions and Notations

Our definitions and notations are fairly standard. For additional background and examples, see [78].

A graph G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, such that each edge is an unordered pair of distinct vertices; thus, each edge is associated with two vertices called its *endpoints*. For brevity we write uv instead of (u, v) for an edge with endpoints u and v . If uv is an edge, then vertices u and v are *adjacent* and are *neighbors*, and they are each *incident* to uv . Edges are *incident* if they have a common endpoint.

More generally, a *multigraph* G consists of a set $V(G)$ of vertices and a multiset $E(G)$ of edges, such that each edge is an unordered pair of (not necessarily distinct) vertices. *Multiple edges* are edges having the same pair of endpoints. A *loop* is an edge whose endpoints are equal. When discussing multigraphs, we may emphasize the absence of multiple edges and loops by calling a graph a *simple graph*.

The number of vertices of a graph G is its *order*. We say a graph is *trivial* if its order is 0 or 1. The number of edges of a graph G is its *size*. We say a graph is *empty* if its size is 0. The *degree* $d(v)$ of a vertex v is the number of edges incident to v . The *minimum degree* $\delta(G)$ of a graph G is $\min\{d(v)|v \in V(G)\}$. The *maximum degree* $\Delta(G)$ of a graph G is $\max\{d(v)|v \in V(G)\}$. If every vertex of a graph G has degree k , then G is *k-regular*. In particular, a 3-regular graph is also called a *cubic graph*. A *clique* in a graph is a set of pairwise adjacent vertices. The *clique number* $\omega(G)$ of a graph G is the maximum size of a clique in G . An *independent set* in a graph is a set of pairwise nonadjacent vertices. The *independence number* $\alpha(G)$ of a graph G is the maximum size of an independent set in G .

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H , then we say that G is *isomorphic to* H , written $G \cong H$.

The *complement* \overline{G} of a graph G is the graph with vertex set $V(G)$ such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A graph H is called a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G and $H \neq G$, then H is a *proper subgraph* of G . If H is a subgraph of G and $V(H) = V(G)$, then H is a *spanning subgraph* of G . If H is a subgraph of G , and H contains all the edges $uv \in E(G)$ with $u, v \in V(H)$,

then H is an *induced subgraph* G ; we say that $V(H)$ (or $E(H)$) *induces* H . A graph G is *H -free* if G has no induced subgraph isomorphic to H . The *open neighborhood* of a vertex v in a graph G , written $N_G(v)$ or simply $N(v)$, is the subgraph of G induced by all neighbors of v . The *closed neighborhood* of a vertex v in a graph G , written $N_G[v]$ or simply $N[v]$, is the subgraph of G induced by v and all neighbors of v .

In a graph G , the *subdivision* of an edge uv is the operation that replaces uv with a path u, w, v through a new vertex w ; while the *contraction* of an edge uv , written G/uv , is the operation that replaces u and v with a new vertex such that the new vertex is incident to the edges, other than uv , that were incident to u or v . We write $G - e$ for the subgraph of G obtained by deleting an edge e , and $G - M$ for the subgraph of G obtained by deleting a set of edges M . We write $G - v$ for the subgraph of G obtained by deleting a vertex v and all its incident edges, and $G - S$ for the subgraph of G obtained by deleting a set of vertices S and all their incident edges. A graph H is a *minor* of G if H can be formed from G by deleting vertices or edges or by contracting edges. A graph H is a *subdivision* of G if H can be formed from G by successive edge subdivisions.

A *complete graph* is a graph whose vertices are all pairwise adjacent. The complete graph with n vertices is denoted K_n ; in particular, K_3 is also called a *triangle*. A *path* is a graph of the form $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$, where $n \geq 1$ and the v_i are all distinct. The path with n vertices is denoted P_n . A *cycle* is a graph of the form $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$, where $n \geq 3$ and the v_i are all distinct. The cycle with n vertices is denoted C_n . The number of edges of a path (or cycle) is called its *length*. A *cyclic graph* is a graph that contains a cycle. A *forest* (*acyclic graph*) is a graph that does not contain any cycle. A *chord* of a cycle C is an edge not in C whose endpoints lie in C . A *chordal graph* is a graph in which all cycles with four or more vertices have a chord. Equivalently, every induced cycle in the graph has at most three vertices.

A graph G is *connected* if there is a path between every pair of distinct vertices of G .

The *components* of a graph are its maximal connected subgraphs. A *tree* is a connected forest. A connected graph G is said to be *k-connected* if it has more than k vertices and remains connected whenever fewer than k vertices (and their incident edges) are removed.

Let $r \geq 2$ be an integer. A graph G is *r-partite* if $V(G)$ admits a partition into r independent sets. These independent sets are called *partite sets* of G . Usually we say *bipartite* instead of “2-partite”, and *tripartite* instead of “3-partite”. An *r-partite* graph G in which every two vertices from different partite sets are adjacent is called a *complete r-partite graph*. Equivalently, every component of \bar{G} is a complete graph. We write K_{n_1, \dots, n_r} for the complete *r-partite* graph with partite sets of size n_1, \dots, n_r . A complete bipartite graph is also called a *biclique*. The complete bipartite graph $K_{1,r}$ is also called a *star*. The *Turán graph* $T_{n,r}$ is the complete *r-partite* graph with n vertices whose partite sets differ in size by at most 1.

The *cartesian product* of G and H , written $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. The *m-by-n rooks graph* is $K_m \square K_n$. The *m-by-n grid graph* is $P_m \square P_n$. The *prism* of order $2n$ is $K_2 \square C_n$.

The *disjoint union* of graphs G_1, G_2, \dots, G_k , written $G_1 \cup G_2 \cup \dots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^k V(G_i)$ and edge set $\bigcup_{i=1}^k E(G_i)$. The *join* of graphs G and H , written $G \vee H$, is the graph obtained from the disjoint union $G \cup H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

The *Petersen graph* is a graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

A *k-tree* is a graph obtained by starting with K_k and repeatedly adding a vertex and adding all possible edges between the new vertex and a *k-clique*. A *partial k-tree* is a spanning subgraph of a *k-tree*.

A graph is *planar* if it can be drawn on the plane without crossing edges. Such a drawing is called a *planar embedding* of the graph. A *plane graph* is a particular planar

embedding of a planar graph. The *dual graph* G^* of a plane graph G is a plane multigraph such that there is a bijection f from the set of faces of G to the set of vertices of G^* . The edges of G^* correspond to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then e^* is an edge of G^* with endpoints $f(X)$ and $f(Y)$. The *weak dual* of a plane graph G is the graph obtained from the dual graph G^* by deleting the vertex that corresponds to the unbounded face of G . A graph is *outerplanar* if it admits a planar embedding in which every vertex lies on the boundary of the outer face. An outerplanar graph is *maximal outerplanar* if it does not allow addition of edges while preserving outerplanarity.

The *distance* from u to v , written $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path from u to v in G . It is defined to be infinity if G does not contain such a path. The *eccentricity* of a vertex u , written $e(u)$, is $\max_{v \in V(G)} d(u, v)$. The *diameter* of a graph G , written $\text{diam}(G)$, is $\max_{u, v \in V(G)} d(u, v)$. The *girth* of a graph G , written $g(G)$, is the length of a shortest cycle contained in G . It is defined to be infinity if G does not contain any cycles. The *odd girth* of a graph is the length of a shortest odd cycle contained in the graph. It is defined to be infinity if the graph does not contain any odd cycles.

A *vertex cover* in a graph is a set of vertices that contains at least one endpoint of every edge. The *vertex cover number* $\beta(G)$ of a graph G is the minimum size of a vertex cover in G . A *matching* in a graph is a set of edges without common vertices. The endpoints of the edges of a matching M are *saturated* by M . A *perfect matching* is a matching that saturates all vertices of the graph. The *matching number* $m(G)$ of a graph G is the maximum size of a matching in G . A set of vertices S is *dominating* if every vertex not in S has a neighbor in S . The *domination number* $\gamma(G)$ of a graph G is the minimum size of a dominating set in G .

A *graph coloring* is an assignment of labels to certain elements of a graph subject to certain constraints. The labels are called *colors*. In particular, a *vertex coloring* is an assignment of colors to vertices of a graph. In this thesis, we consider only vertex colorings.

Given a vertex coloring of a graph, we say that the vertices having the same color form a *color class*. A *k-coloring* of a graph G is a vertex coloring of G using k colors. A *proper k-coloring* of a graph G is a k -coloring of G such that each vertex of G receives exactly one color and adjacent vertices receive different colors. Note that a proper k -coloring is equivalent to a partition of the vertex set into k independent sets. A graph is *k-colorable* if it has a proper k -coloring. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k -colorable.

Given a (not necessarily proper) vertex coloring of a graph G where each vertex of G receives one color, we say a subgraph of G is *rainbow* (or *heterochromatic*) if all its vertices receive distinct colors, and *monochromatic* if all its vertices receive the same color.

In this thesis, we mostly deal with graphs. But sometimes we need to consider a generalization of graphs: a *hypergraph* \mathcal{H} consists of a set $V(\mathcal{H})$ of *vertices* and a set $E(\mathcal{H})$ of *hyperedges*, such that each hyperedge is a nonempty set of vertices. A hypergraph \mathcal{H} is *r-uniform* if every hyperedge of \mathcal{H} contains r vertices. (So a simple graph is just a 2-uniform hypergraph.) The *degree* $d(v)$ of a vertex v is the number of hyperedges that contain v . A hypergraph \mathcal{H} is *k-regular* if every vertex of \mathcal{H} has degree k . More generally, if we allow $E(\mathcal{H})$ to be a multiset, then \mathcal{H} will be a *multihypergraph* instead of hypergraph.

1.2 Thesis Organization

The rest of this thesis is organized as follows:

In Chapter 2, we study several types of generalized vertex colorings of series-parallel graphs. The main result is that the fractional chromatic number of a series-parallel graph of odd girth k is exactly $2 + 2/(k - 1)$, confirming a conjecture by Wang and Yu [77]. We also provide additional results on defective fractional coloring and defective circular coloring of series-parallel graphs addressing conjectures and results in the literature. In particular, we answer a question of Klostermeyer by showing that for every d there is a series-parallel graph whose d -defective fractional and circular chromatic numbers are both 3.

In Chapter 3, we introduce a generalization of defective coloring: each vertex of a graph is assigned a fraction of each color, with the total amount of colors at each vertex summing to 1. We define the fractional defect of a vertex v to be the sum of the overlaps with each neighbor of v , and the fractional defect of the graph to be the maximum of the defects over all vertices. We provide results on the minimum fractional defect of 2-colorings of some graphs. For example, we show that the minimum fractional defect of 2-colorings of the complete tripartite graph $K_{a,b,c}$ with $a \leq b \leq c$ is $bc/(b+c-a)$.

The next few chapters are devoted to the problem of coloring the vertices of a graph while forbidding rainbow or monochromatic subgraphs.

In Chapter 4, we define a no-rainbow- F coloring of G as a coloring of the vertices of G without rainbow subgraph isomorphic to F , and the F -upper chromatic number of G as the maximum number of colors in such a coloring. We present some results on this parameter for certain graph classes. The focus is on the case that F is a star or triangle. For example, we show that the K_3 -upper chromatic number of any maximal outerplanar graph on n vertices is $\lfloor n/2 \rfloor + 1$.

In Chapter 5, we define an F -WORM coloring of G as a coloring of the vertices of G without rainbow or monochromatic subgraph isomorphic to F . We present some results on this concept especially as regards to the existence, complexity, and optimization within certain graph classes. The focus is on the case that $F = P_3$.

In Chapter 6, we consider some other cases of WORM coloring, in particular the cases that F is a cycle and that F is a complete graph.

In Chapter 7, we consider a generalization of WORM coloring. Specifically, for graphs M and R , we define an (M, R) -WORM coloring of G to be a coloring of the vertices of G with neither a monochromatic subgraph isomorphic to M nor a rainbow subgraph isomorphic to R . The focus is on the case that $M = K_2$.

In Chapter 8, we briefly summarize the main results of the thesis and propose some future directions of research.

Chapter 2

Fractional, Circular, and Defective Coloring of Series-Parallel Graphs

2.1 Introduction

This chapter is based on joint work with Wayne Goddard [42]. As all proofs in the original paper are provided, we do not give specific references to that paper.

A *two-terminal series-parallel graph* $(G; l, r)$ is a multigraph with two distinguished vertices l and r called the *terminals*, formed recursively as follows:

- $(K_2; l_0, r_0)$ is a two-terminal series-parallel graph.
- *Series join*: let $(G_1; l_1, r_1)$ and $(G_2; l_2, r_2)$ be two-terminal series-parallel graphs. We define $G_1 \bullet G_2$ to be the graph obtained from the union of G_1 and G_2 by identifying r_1 and l_2 into a single vertex, and choosing (l_1, r_2) as the new terminal pair. Then $G_1 \bullet G_2$ is a two-terminal series-parallel graph.
- *Parallel join*: let $(G_1; l_1, r_1)$ and $(G_2; l_2, r_2)$ be two-terminal series-parallel graphs. We define $G_1 // G_2$ to be the graph obtained from the union of G_1 and G_2 by identifying l_1

and l_2 into a single vertex l , identifying r_1 and r_2 into a single vertex r , and choosing (l, r) as the new terminal pair. Then $G_1 // G_2$ is a two-terminal series-parallel graph.

- There are no other two-terminal series-parallel graphs.

For convenience, we shall use the following notations: for a two-terminal series-parallel graph G , we let $G^{<n>}$ denote the series join of n copies of G , and let $G_{<n>}$ denote the parallel join of n copies of G . For example, the 5-cycle C_5 with non-adjacent terminals is denoted by $(K_2 \bullet K_2) // (K_2 \bullet K_2 \bullet K_2)$, or alternatively $(K_2)^{<2>} // (K_2)^{<3>}$.

A *series-parallel graph* is a multigraph without a K_4 -minor (as used for example in [50]). It is well known that every block of a series-parallel graph is a two-terminal series-parallel graph for some choice of distinguished vertices.

We will consider the following colorings. A (k, q) -*fractional coloring* [67] is an assignment of q colors to each vertex, where the colors are drawn from a palette of k colors, such that adjacent vertices receive disjoint q -sets. A (k, q) -*circular coloring* [72] (originally called star coloring) is an assignment of one color to each vertex, where the colors are drawn from \mathbb{Z}_k , such that adjacent vertices receive colors that are at least $q \pmod{k}$ apart. It is well known that if there is a (k, q) -circular coloring, then there is a (k, q) -fractional coloring. For a survey of circular colorings, see Zhu [80]. The *fractional chromatic number* $\chi_f(G)$ and the *circular chromatic number* $\chi_c(G)$ of a graph G are defined as the infimum of k/q taken over all fractional colorings and all circular colorings of G respectively. It is well known that the infimum is achieved; that is, one can replace infimum by minimum. Also, by definition, we have $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$ for every graph G .

A d -*defective coloring* [20] (also called a d -improper coloring) is an assignment of one color to each vertex such that every vertex has at most d neighbors of the same color. Equivalently, a graph has a d -defective coloring if one can remove the edges of a subgraph of maximum degree d such that the result is a proper coloring. Similarly, a d -*defective (k, q) -fractional coloring* [29] is an assignment of q colors to each vertex, where the colors are

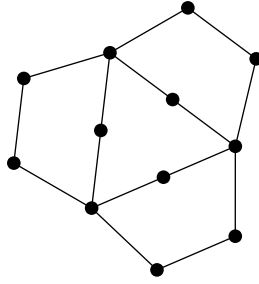


Figure 2.1: A series-parallel graph with circular chromatic number $8/3$

drawn from a palette of k colors, such that every vertex v is adjacent to at most d vertices u where the q -set of v overlaps the q -set of u . A d -defective (k, q) -circular coloring [53] is an assignment of one color to each vertex, where the colors are drawn from \mathbb{Z}_k , such that every vertex v is adjacent to at most d vertices u where the difference between the color of v and the color of u is less than $q \pmod{k}$. Since the number of subgraphs of a graph is finite, it follows similarly that one can define the d -defective fractional chromatic number and the d -defective circular chromatic number as the minimum of k/q taken over all d -defective (k, q) -fractional colorings and all d -defective (k, q) -circular colorings of a graph respectively. (Defective circular colorings have been generalized by Mihók et al. [60] by considering alternative requirements on the graph induced by the improper edges.)

It is well known that series-parallel graphs are 3-colorable. Thus, if such a graph has a triangle, then its fractional and circular chromatic number are 3. Hell and Zhu [47] showed that a triangle-free series-parallel graph has circular chromatic number at most $8/3$, and that this is best possible because of the graph of Figure 2.1. Pan and Zhu provided bounds for series-parallel graphs of higher girth in [62], and proved that their bounds are best possible in [63].

Outerplanar graphs may be characterized as graphs without a K_4 -minor or a $K_{2,3}$ -minor (see, for example, [15]). Thus, outerplanar graphs form a subclass of the series-parallel graphs. The results simplify for such graphs. Klostermeyer and Zhang [54] (and later Kemnitz and Wellmann [51]) observed that every outerplanar graph of odd girth k has

circular chromatic number (and fractional chromatic number) exactly $2k/(k-1)$. This was extended by Wang et al. [76] who showed that the same result holds for circular list colorings of outerplanar graphs. The defective choosability of outerplanar and series-parallel graphs was studied by Woodall [79].

We proceed as follows: in Section 2.2 we show that the fractional chromatic number of any series-parallel graph of odd girth k is exactly $2k/(k-1)$. In Section 2.3 we first note that a series-parallel graph of girth 5 is 2-colorable with defect 1, and then provide constructions that show that in many cases the upper bound for the defective version of the parameter is the same as the upper bound for the ordinary fractional or circular chromatic numbers. In Section 2.4 we propose some related questions.

2.2 Proper Fractional Colorings

Wang and Yu [77] (assuming a typo in the paper) conjectured that the fractional chromatic number of any series-parallel graph of odd girth at least k is at most $2k/(k-1)$. The main goal of this section is to show that the fractional chromatic number of any series-parallel graph of odd girth k is exactly $2k/(k-1)$, which proves their conjecture.

2.2.1 Combining Intervals

We need the following definitions and notations. Fix k to be an odd integer; say $k = 2\ell + 1$. Fix a palette of k colors, and let \mathcal{L} be the set of all subsets of ℓ colors. Given integers i and j such that $0 \leq i, j \leq \ell$, define $i \oplus j$ as the set of all $|S_1 \cap S_2|$ such that $S_1, S_2, S_3 \in \mathcal{L}$ with $|S_1 \cap S_3| = i$ and $|S_2 \cap S_3| = j$. Given integers a and b , let $[a, b]$ denote the set of consecutive integers $\{a, a+1, \dots, b\}$, and call it an *integer interval*.

The proof of our main result is based on the following lemma.

Lemma 1 (a) $i \oplus j$ is the nonempty integer interval $[\max(\ell - i - j - 1, i + j - \ell), \min(\ell - i + j, \ell + i - j)]$.

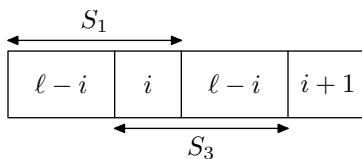


Figure 2.2: S_1 and S_3 as subsets of ℓ colors

(b) Given any integer intervals I_1 and I_2 , the set $I_1 \oplus I_2 = \{i \oplus j : i \in I_1 \text{ and } j \in I_2\}$ is an integer interval.

Proof. (a) It is easy to verify that $\max(\ell - i - j - 1, i + j - \ell) \leq \min(\ell - i + j, \ell + i - j)$, and so the above interval is nonempty. Suppose that $S_1, S_3 \in \mathcal{L}$ with $|S_1 \cap S_3| = i$. Note that there are $k + i - 2\ell = i + 1$ elements outside $S_1 \cup S_3$. See Figure 2.2.

To maximize $|S_1 \cap S_2|$, we take j elements from S_3 using S_1 as much as possible, and then $\ell - j$ elements outside S_3 using S_1 as much as possible. The overlap $|S_1 \cap S_2|$ is $\min(i, j) + \min(\ell - j, \ell - i)$, which simplifies to $\min(\ell - i + j, \ell + i - j)$. To minimize $|S_1 \cap S_2|$, we take j elements from S_3 avoiding S_1 as much as possible, and then $\ell - j$ elements outside S_3 avoiding S_1 as much as possible. The overlap $|S_1 \cap S_2|$ is $\max(j - (\ell - i), 0) + \max(\ell - j - (i + 1), 0)$ which simplifies to $\max(\ell - i - j - 1, i + j - \ell)$, since $(j - (\ell - i)) + (\ell - j - (i + 1)) = -1$ and therefore at least one of $j - (\ell - i)$ and $\ell - j - (i + 1)$ is nonnegative.

To complete the proof, note that we can get any value between the two extremes, by choosing differently.

(b) This follows from noting that the upper and lower limits of $i \oplus j$ change by at most 1 when we change either i or j by 1. \square

2.2.2 Coloring Two-terminal Series-parallel Graphs

For a two-terminal series-parallel graph G , let $o(G)$ denote the length of the shortest odd path between the two terminals of G if such a path exists, and let $o(G) = \infty$ otherwise.

Similarly, let $e(G)$ denote the length of the shortest even path between the two terminals of G if such a path exists, and let $e(G) = \infty$ otherwise. Let $I_\ell(G) = [\ell - e(G)/2, (o(G) - 1)/2] \cap [0, \ell]$. For any series-parallel graph G with odd girth at least k , clearly $o(G) + e(G) \geq k$, and hence $\ell - e(G)/2 \leq (o(G) - 1)/2$. Therefore, $I_\ell(G)$ is nonempty for such a graph.

Theorem 2 *Let G be a two-terminal series-parallel graph with odd girth at least k , where $k = 2\ell + 1$. Then there is a (k, ℓ) -fractional coloring of G . Furthermore, the color sets for the two terminals of G can be specified as any pair (S_1, S_2) such that $|S_1 \cap S_2| \in I_\ell(G)$.*

Proof. We prove the theorem by induction. The base case is $G = K_2$. Here $o(G) = 1$ and $e(G) = \infty$, so $I_\ell(G) = \{0\}$. Choosing disjoint color sets S_1 and S_2 for the two terminals yields the requisite coloring.

Suppose G is obtained from graphs G_1 and G_2 by the parallel join. Let color sets S_1 and S_2 with $|S_1 \cap S_2| \in I_\ell(G)$ be specified for the two terminals of G . Note that $\ell - e(G_j)/2 \leq \max(\ell - e(G)/2, 0) \leq |S_1 \cap S_2| \leq \min((o(G) - 1)/2, \ell) \leq (o(G_j) - 1)/2$ for $j = 1, 2$; therefore $|S_1 \cap S_2| \in I_\ell(G_1) \cap I_\ell(G_2)$. By the inductive hypothesis, graphs G_1 and G_2 have the desired coloring with S_1 and S_2 at their terminals. This yields the requisite coloring of G .

Suppose G is obtained from graphs G_1 and G_2 by the series join. Let color sets S_1 and S_2 with $|S_1 \cap S_2| \in I_\ell(G)$ be specified for the two terminals of G . To complete the proof by induction, we need to show that we can find a color set S_3 with $|S_1 \cap S_3| \in I_\ell(G_1)$ and $|S_2 \cap S_3| \in I_\ell(G_2)$, since then we can color G_1 with S_1 and S_3 at its terminals and G_2 with S_3 and S_2 at its terminals to obtain the requisite coloring. This means that $|S_1 \cap S_2| \in |S_1 \cap S_3| \oplus |S_2 \cap S_3|$. That is, it suffices to show that $I_\ell(G) \subseteq I_\ell(G_1) \oplus I_\ell(G_2)$. By Lemma 1, it suffices to show that the extrema of $I_\ell(G)$ are contained in $I_\ell(G_1) \oplus I_\ell(G_2)$.

Consider the upper limit of $I_\ell(G)$. Assume first that $(o(G) - 1)/2 \geq \ell$, so that the upper limit of $I_\ell(G)$ is ℓ . Since $o(G) = \min(o(G_1) + e(G_2), e(G_1) + o(G_2))$, we have $(o(G_1) - 1)/2 \geq \ell - e(G_2)/2$ and $(o(G_2) - 1)/2 \geq \ell - e(G_1)/2$. Therefore, $x = \max(0, \ell -$

$\min(e(G_1), e(G_2))/2) \in I_\ell(G_1) \cap I_\ell(G_2)$. By Lemma 1, we have $\ell \in x \oplus x$; that is, $\ell \in I_\ell(G_1) \oplus I_\ell(G_2)$.

Assume second that $(o(G)-1)/2 < \ell$, so that the upper limit of $I_\ell(G)$ is $(o(G)-1)/2$. Without loss of generality we may assume $o(G) = o(G_1) + e(G_2)$. Then we have $(o(G_1) - 1)/2 < \ell - e(G_2)/2$. Take $i = (o(G_1) - 1)/2$ and $j = \ell - e(G_2)/2$; then by Lemma 1, it follows that $(o(G) - 1)/2 = \ell + i - j \in i \oplus j \subseteq I_\ell(G_1) \oplus I_\ell(G_2)$.

Consider the lower limit of $I_\ell(G)$. Assume first that $\ell - e(G)/2 < 0$. Then, define $\ell - I_\ell(G_1)$ to be the set $\{j : j = \ell - i, i \in I_\ell(G_1)\}$. Note that both $\ell - I_\ell(G_1)$ and $I_\ell(G_2)$ are nonempty, and $\ell - I_\ell(G_1) = [\ell + (1 - o(G_1))/2, e(G_1)/2] \cap [0, \ell]$. It is easy to verify that $\ell - I_\ell(G_1)$ and $I_\ell(G_2)$ intersect; say containing integer j . Hence, $0 = (\ell - j) + j - \ell \in (\ell - j) \oplus j \subseteq I_\ell(G_1) \oplus I_\ell(G_2)$.

So suppose $\ell - e(G)/2 \geq 0$. Without loss of generality we may assume $e(G) = o(G_1) + o(G_2)$. Then take $i = (o(G_1) - 1)/2$ and $j = (o(G_2) - 1)/2$, and by Lemma 1, we have $\ell - e(G)/2 = \ell - i - j - 1 \in i \oplus j \subseteq I_\ell(G_1) \oplus I_\ell(G_2)$. \square

Our main result follows from Theorem 2.

Theorem 3 *If G is a series-parallel graph of odd girth k then $\chi_f(G) = 2k/(k - 1)$.*

Proof. The expression $2k/(k - 1)$ is a lower bound, since that is the fractional chromatic number of the k -cycle (see, for example, [67]). The upper bound follows from Theorem 2 and the fact that the fractional chromatic number of a graph is the maximum of the fractional chromatic numbers of its blocks. \square

In particular, Theorem 3 shows that the fractional chromatic number of a series-parallel graph is polynomial-time computable.

We point out here that Feder and Subi [31] independently obtained the same result by using a different method.

2.3 Defective Colorings

2.3.1 Girth 5

We show below (the probably known fact) that for all d there is a triangle-free series-parallel graph whose d -defective chromatic number is 3. Indeed, we note in Theorem 7 that the same is true for fractional chromatic number. However, for girth 5 the situation changes.

We will need the following observation from [6]:

Observation 4 [6] *A cyclic series-parallel graph G with girth g contains a path with $\lfloor (g-1)/2 \rfloor$ vertices each with degree 2 in G .*

Theorem 5 *A series-parallel graph G of girth 5 is 1-defective 2-colorable.*

Proof. The graph G either contains a vertex of degree 1 (in which case induction is immediate), or is cyclic and therefore by the above observation, contains two adjacent vertices of degree 2, say x and y . Apply the induction hypothesis to $G - \{x, y\}$. Then color x the opposite color to its neighbor in $G - \{x, y\}$ and similarly with y . \square

We note that Borodin et al. [6] considered the case where the defect condition is different for each color. A $[d_1, \dots, d_k]$ -coloring is a k -coloring of the vertices such that for each i , the vertices of color i induce a graph of maximum degree at most d_i . They showed that a series-parallel graph of girth 7 has a $[1, 0]$ -coloring, and this is best possible; indeed that for all k there is a series-parallel graph of girth 6 that does not have a $[k, 0]$ -coloring.

2.3.2 Defective Fractional and Circular Colorings

We start with a construction.

Lemma 6 *For all d and $k \geq 3$, there is a series-parallel graph of odd girth k such that removing the edges of a subgraph of maximum degree d cannot destroy every k -cycle.*

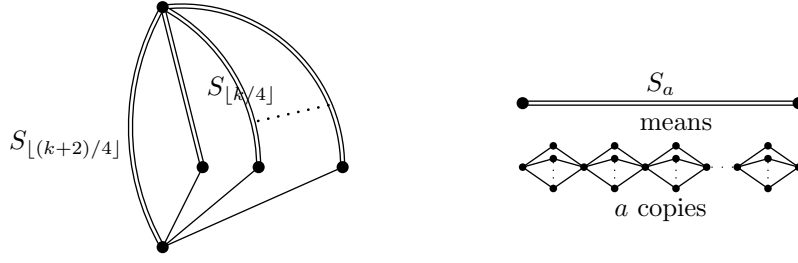


Figure 2.3: A series-parallel graph G_k ($k \geq 5$) with given odd girth and large d -defective fractional chromatic number

Proof. Assume first that $k = 3$. Construct graph G_3 as follows. Start with a complete bipartite graph $K_{2,d+1}$ and then for every edge, join its ends by $2d + 1$ disjoint paths of length 2. In our notation, $G_3 = [((K_2 \bullet K_2)_{<2d+1>} // K_2)_{<2>}]_{<d+1>}$. Removal of the edges of a subgraph of maximum degree d from G_3 must leave at least one triangle.

Assume second that $k \geq 5$. Let graph $S_a = ((K_2 \bullet K_2)_{<2d+1>})_{<a>}$. (Thus S_a has diameter $2a$.) Then let graph $G_k = (S_{[k/4]} \bullet K_2)_{<d+1>} // S_{[(k+2)/4]}$. See Figure 2.3. Clearly G_k has odd girth k , and removal of the edges of a subgraph of maximum degree d from G_k must leave at least one k -cycle. \square

From this it follows:

Theorem 7 *For all d and $k \geq 3$, the maximum d -defective fractional chromatic number of a series-parallel graph of odd girth k is $2k/(k - 1)$.*

Proof. The upper bound follows from Theorem 3. The lower bound follows from the above construction. \square

In particular, this theorem shows that for every d there is a series-parallel graph whose d -defective fractional and circular chromatic numbers are both 3. This gives a negative answer to Klostermeyer's question [53] whether every series-parallel graph has a 2-defective $(5, 2)$ -circular coloring.

The above construction carries over partially to d -defective circular chromatic number. In particular:

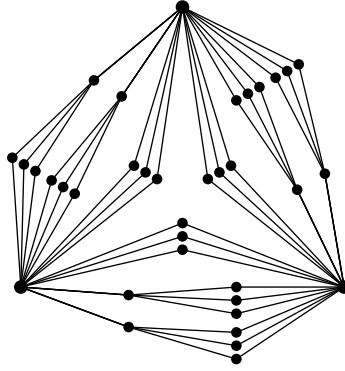


Figure 2.4: The graph H_1 whose 1-defective circular chromatic number is $8/3$

Theorem 8 *For all d , the maximum d -defective circular chromatic number of a triangle-free series-parallel graph is $8/3$.*

Proof. The upper bound follows from the result of Hell and Zhu (see Theorem 1.1 in [47]).

For the lower bound, let graph $F_d = [(K_2 \bullet K_2)_{\langle 2d+1 \rangle} \bullet K_2]_{\langle d+1 \rangle} // (K_2 \bullet K_2)_{\langle 2d+1 \rangle}$, graph $G_d = [K_2 \bullet (K_2 \bullet K_2)_{\langle 2d+1 \rangle}]_{\langle d+1 \rangle} // (K_2 \bullet K_2)_{\langle 2d+1 \rangle}$, and graph $H_d = (F_d \bullet F_d) // G_d$. The graph H_1 is shown in Figure 2.4. Clearly H_d is triangle-free. It can readily be shown that if we remove the edges of a subgraph of maximum degree d from H_d , the remaining graph still contains a copy of the graph of Figure 2.1. Hence H_d has d -defective circular chromatic number $8/3$. \square

The above construction also provides a counterexample to Klostermeyer's claimed result [53] that every triangle-free series-parallel graph has a 2-defective $(5, 2)$ -circular coloring.

By starting with the graphs constructed by Pan and Zhu [63], one can similarly show that the maximum d -defective circular chromatic number of a series-parallel graph of **odd girth** k is at least the maximum circular chromatic number of a series-parallel graph of **girth** k . But there does not seem any reason to believe that the values are equal, since the question of the maximum circular chromatic number of a series-parallel graph of **odd girth** k is unresolved.

2.4 Related Questions

Note that simple series-parallel graphs are also the partial 2-trees (see, for example, [25]). So it is natural to consider partial k -trees in general. For example, Chlebíková [18] showed that: for $k \geq 3$, every triangle-free partial k -tree has chromatic number at most k . So one question is whether this is best possible? Also what happens for fractional/circular coloring and/or higher girth/odd girth?

Chapter 3

Colorings with Fractional Defect

3.1 Introduction

This chapter is based on joint work with Wayne Goddard [39]. As all proofs in the original paper are provided, we do not give specific references to that paper.

In a proper vertex coloring of a graph, every vertex is assigned one color and that color is different from each of its neighbors. We consider here a two-fold generalization of this: a vertex can receive multiple colors and can overlap slightly with each neighbor. Specifically, each vertex is assigned a fraction of each color, with the total amount of colors at each vertex summing to 1. The (*fractional*) *defect* of a vertex v is defined to be the sum of the overlaps over all colors and all neighbors of v ; the (*fractional*) *defect* of the graph is the maximum of the defects over all vertices. We say that a vertex is *monochromatic* if it has only one color, and an edge is *monochromatic* if both of its endpoints are monochromatic and they have the same color. Note that if every vertex is monochromatic, then our fractional defect coincides with the usual definition of defect (see for example [20]).

The idea of assigning vertices multiple colors has been used most notably in fractional colorings (e.g. [64, 57]), but also for example in t -tone colorings [23]. Like in t -tone colorings (and unlike in fractional colorings), we consider here the situation where one pays

for each color used, regardless of how much the color is used. Note that for proper colorings, allowing one to color a vertex with multiple colors does not yield anything new. For, one can just choose for each vertex v one color present at v and recolor it entirely that color, and therefore the minimum number of colors needed is just the chromatic number. Similarly, with the usual definition of the defect of a vertex as the number of neighbors that share a color, there is no advantage to using more than one color at a vertex. But we consider colorings where a vertex overlaps only slightly with each neighbor.

Consider, for example, the Hajós graph. Figure 3.1 gives a 2-coloring of this graph with defect $4/3$ (and this is best possible in that any 2-coloring has at least this much defect). For another example, consider the complete graph on 3 vertices. Any 2-coloring of K_3 has defect at least 1, but there are multiple optimal colorings: color one vertex red, one vertex blue, and the third vertex any combination of red and blue.

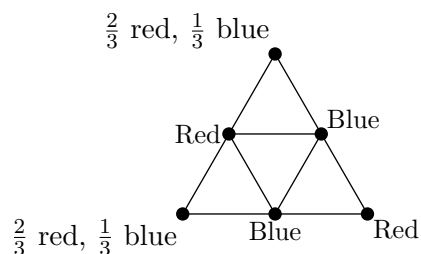


Figure 3.1: An optimal 2-coloring of the Hajós graph

Our objective is to minimize the defect of the graph. Specifically, for a given number of colors, what is the minimum defect that can be obtained? If the number of colors is the chromatic number, then of course there need be no defect. But if the number of colors is smaller, then there is a defect.

In the rest of the chapter we proceed as follows: in Section 3.2 we introduce notation and provide elementary results about monochromatic vertices. Thereafter, in Section 3.3, we consider calculating the parameter in 2-colorings for several graph families, including fans, wheels, complete multipartite graphs, rooks graphs, and regular graphs. We give exact values in some cases and bounds in others. We also pose several conjectures. Finally in

Section 3.4 we observe that the decision problem is NP-hard.

3.2 Preliminaries

Consider coloring a graph G by using k colors. For color j , let $f_j(v)$ be the usage of color j on vertex v . For each edge vw in G , we call $\sum_{j=1}^k \min(f_j(v), f_j(w))$ the *overlap* between v and w (or alternately, the *edge defect* of vw).

The *defect* of vertex v is given by

$$df(v) = \sum_{w \in N(v)} \sum_{j=1}^k \min(f_j(v), f_j(w)). \quad (3.1)$$

In general, the problem is to minimize

$$\max_v df(v)$$

over all colorings such that $f_j(v)$ is nonnegative and $\sum_{j=1}^k f_j(v) = 1$ for all vertices v . We denote this minimum by $D(G, k)$, and call it the *minimum defect*. We call a k -coloring *optimal* if it achieves the minimum defect $D(G, k)$.

Note that the existence of the minimum defect is guaranteed, since the objective function above is continuous and the feasible region is a closed bounded set. Further, the calculation is at least finite, since, for example, we can prescribe which of $f_j(v)$ and $f_j(w)$ are smaller in every min in Equation 3.1 for each vertex v , and thus $D(G, k)$ is the minimum over exponentially many linear programs.

A related parameter, called the *total defect*, is $\sum_{v \in V} df(v)$. Note that the total defect is equal to twice the sum of all edge defects. We define $TD(G, k)$ to be the minimum of $\sum_{v \in V} df(v)$ over all colorings, and call it the *minimum total defect*. We prove that there always exists a coloring that achieves the minimum total defect in which every vertex is monochromatic:

Lemma 9 *For graph G and number of colors k , there is a k -coloring that achieves $TD(G, k)$ in which every vertex is monochromatic.*

Proof. Consider any vertex v that is not monochromatic: say $f_1(v), f_2(v) > 0$ with $f_1(v) + f_2(v) = A$. Then consider adjusting the coloring such that $f_1(v) = x$ and $f_2(v) = A - x$. As a function of x , the defect of v with any neighbor w is a (piecewise-linear) concave-down function. Thus, the total defect of the graph, as a function of x , is a concave-down function, and so its minimum is attained at an endpoint. This means that one can either replace color 1 by color 2 or replace color 2 by color 1 at v without increasing the total defect. Repeated application of this replacement yields a coloring with every vertex monochromatic. \square

From the above lemma, it follows that:

Corollary 10 $TD(K_n, k) = \lfloor n/k \rfloor (2n - k - \lfloor n/k \rfloor k)$.

Proof. By Lemma 9, there is a coloring f that achieves $TD(K_n, k)$ in which every vertex is monochromatic. Note that the total defect in such a coloring is equal to twice the number of monochromatic edges. If there exist two color classes whose sizes differ by at least 2, say there are a_1 vertices having color 1, a_2 vertices having color 2, and $a_1 \geq a_2 + 2$, then we recolor one vertex that has color 1 with color 2. Let M denote the increase in the number of monochromatic edges. We have

$$\binom{a_1 - 1}{2} - \binom{a_1}{2} + \binom{a_2 + 1}{2} - \binom{a_2}{2} = a_2 - a_1 + 1 < 0.$$

This contradicts that f achieves minimum total defect. So we may assume that the sizes of the color classes in f differ by at most 1. Thus there are $n - \lfloor n/k \rfloor k$ color classes having size $\lfloor n/k \rfloor + 1$ and $k - n + \lfloor n/k \rfloor k$ color classes having size $\lfloor n/k \rfloor$. For simplicity, let q denote $\lfloor n/k \rfloor$. Then the number of monochromatic edges is

$$\binom{q}{2} (k - n + qk) + \binom{q+1}{2} (n - qk) = q \left(n - \frac{(q+1)k}{2} \right).$$

So $TD(K_n, k) = q(2n - k - qk)$, and the result follows. \square

There are several fundamental results about monochromatic vertices for minimum defect $D(G, k)$. One is that we may assume that there is a monochromatic vertex of each color.

Lemma 11 *Let k be an integer and G be a graph with at least k vertices. Then there is an optimal k -coloring of G that has at least one monochromatic vertex for each color.*

Proof. Consider any optimal k -coloring, and consider each color $j = 1, 2, \dots, k$ in turn. Each time, define vertex v_j as any vertex other than v_1, \dots, v_{j-1} with the largest usage of color j ; then recolor v_j (if needed) such that $f_j(v_j) = 1$ and $f_i(v_j) = 0$ for all $i \neq j$. Such a recoloring does not increase the defect at any vertex. So we will reach an optimal coloring with the desired property. \square

We next show that the minimum defect is either 0 or at least 1.

Lemma 12 *For any graph G and positive integer k , if $D(G, k) > 0$ then $D(G, k) \geq 1$.*

Proof. If every vertex is monochromatic, then the defect is an integer and so the result is immediate. So consider any vertex v that is not monochromatic. If for any color j we have $f_j(v) \geq f_j(w)$ for all neighbors w of v , then we can recolor v to be monochromatically color j without increasing the defect of any vertex. So we may assume that for every color j at v , vertex v has a neighbor w_j with $f_j(w_j) \geq f_j(v)$. It follows that

$$df(v) = \sum_{w \in N(v)} \sum_{j=1}^k \min(f_j(v), f_j(w)) \geq \sum_{j=1}^k \min(f_j(v), f_j(w_j)) = \sum_{j=1}^k f_j(v) = 1.$$

The result follows. \square

It follows from Lemma 12 that the minimum defect in a 2-coloring of a nonbipartite graph is at least 1. One example of equality is the odd cycle C_{2n+1} : let $v_1, v_2, \dots, v_{2n+1}$

denote the vertices of C_{2n+1} . Color v_i red if i is an odd integer and blue otherwise. That coloring has defect exactly 1.

Proposition 13 *The complete graph K_n has $D(K_n, k) = \lceil n/k \rceil - 1$.*

Proof. This defect is achieved by (inter alia) coloring each vertex with a single color and using each color as equitably as possible. (This is trivially the best coloring for total defect.)

To see that $\lceil n/k \rceil - 1$ is best possible, we proceed by induction on n , noting that the result is trivial if $n \leq k$. So assume $n > k$. By Lemma 11, there is an optimal k -coloring of K_n that has at least one monochromatic vertex v_j for each color $1 \leq j \leq k$. Let $A = \{v_1, \dots, v_k\}$. Then the defect of any other vertex w in G equals 1 plus the defect of w in $G - A$. By the induction hypothesis, there exists a vertex in $G - A$ that has defect at least $\lceil (n - k)/k \rceil - 1$ in $G - A$. This proves the lower bound. \square

3.3 Two-Colorings of Some Graph Families

We now consider 2-colorings. Unless otherwise specified, we assume the colors are red and blue, and denote the red usage at vertex v by $r(v)$ (so that the blue usage is $1 - r(v)$).

3.3.1 Fans

The *fan*, denoted by F_n , is the graph obtained from a path of order n by adding a new vertex u and adding an edge between u and every vertex of the path.

Lemma 14 *In any 2-coloring of F_3 it holds that $df(v) + df(w) \geq 2$ where v and w are the dominating vertices.*

Proof. Suppose the dominating vertices are v and w and the other two vertices are a and b . Let e_{xy} denote the overlap $\min(r(x), r(y)) + \min(1 - r(x), 1 - r(y))$ between vertices x and y .

Then $df(v) + df(w) = e_{va} + e_{vb} + e_{wa} + e_{wb} + 2e_{vw}$; further, it follows from Corollary 10) that a triangle has total defect at least 2, and so we have $e_{va} + e_{wa} + e_{vw} \geq 1$ and $e_{vb} + e_{wb} + e_{vw} \geq 1$. The result follows. \square

Note that F_1 is just K_2 and F_2 is just K_3 , and so it holds that $D(F_1, 2) = 0$ and $D(F_2, 2) = 1$. For the general cases of F_n , we have the following:

Proposition 15 *The minimum defect in a 2-coloring of F_n ($n \geq 3$) is*

$$D(F_n, 2) = \frac{2\lfloor n/3 \rfloor}{\lfloor n/3 \rfloor + 1}.$$

Proof. Let $v_1v_2 \dots v_n$ denote the path of order n , and let u denote the dominating vertex.

We prove the upper bound by the following construction. Set $x = 2/(\lfloor n/3 \rfloor + 1)$. Let $r(u) = 1$. Let $r(v_i) = x$ if i is a multiple of 3, and 0 otherwise. It can readily be checked that every vertex v_i has defect at most $2 - x$, and that vertex u has defect $\lfloor n/3 \rfloor x$. The result follows since both these values equal the claimed upper bound.

To prove the lower bound, it suffices to show that $D(F_n, 2) \geq 2n/(n + 3)$ if n is a multiple of 3. We partition the path P_n into $n/3$ copies of P_3 ; thus each P_3 along with vertex u forms a copy of F_3 . It follows from Lemma 14 that $df(u) + \sum_{i=1}^{n/3} df(v_{3i-1}) \geq 2n/3$, whence the result. \square

Note that the defect $D(F_n, 2)$ tends to 2 as n increases. The fan is outerplanar. Several researchers [7, 59] showed that one can ordinarily 2-color an outerplanar graph with defect at most 2. However, we conjecture that this bound can be improved slightly in the following sense:

Conjecture 1 *$D(G, 2) < 2$ for any outerplanar graph G .*

3.3.2 Wheels

The *wheel*, denoted by W_n , is the graph formed from a cycle of order n by adding a new vertex and joining it to every vertex of the cycle. The vertex of degree n is called the *hub* of the wheel.

Proposition 16 *For $n \geq 3$, the minimum defect in a 2-coloring of W_n is*

$$D(W_n, 2) = \frac{2\lceil n/3 \rceil}{\lceil n/3 \rceil + 1}.$$

Proof. Let $x = 2/(\lceil n/3 \rceil + 1)$ and let D be a minimum independent dominating set of the cycle. For a vertex v on the cycle, let $r(v) = x$ if $v \in D$, and $r(v) = 0$ otherwise. Let $r(h) = 1$ for the hub h . It can readily be checked that every vertex on the cycle has defect at most $2 - x$, and that the hub has defect $x|D|$. The upper bound follows, since $2 - x = x|D| = 2\lceil n/3 \rceil/(\lceil n/3 \rceil + 1)$.

Next we prove the lower bound. When $n = 3k$, the lower bound follows directly from Proposition 15. Indeed, $D(W_{3k}, 2) \geq D(F_{3k}, 2) = 2k/(k + 1)$. So we need to establish the lower bound for $n = 3k + 1$ and $n = 3k + 2$.

Consider an optimal coloring of W_n with hub h and cycle $v_1, v_2, \dots, v_n, v_1$. By Lemma 11, we may assume there exist vertices u and u' such that $r(u) = 0$ and $r(u') = 1$. There are two possible cases.

(a) If $h \notin \{u, u'\}$, then we can form $k - 1$ edge-disjoint copies of P_3 without using vertex h, u , or u' . Let S denote the set of centers of these copies. By Lemma 14, it follows that the total defect of $S \cup \{h\}$ within these copies is at least $2(k - 1)$. Further, vertices u and u' together contribute defect 1 to the hub h . It follows that, in the graph as a whole, $df(h) + \sum_{s \in S} df(s) \geq 2k - 1$, and so G has defect at least $(2k - 1)/k$. If $k \geq 2$, then $(2k - 1)/k \geq (2k + 2)/(k + 2)$, and we are done.

So consider the case when $k = 1$. Assume first that $n = 5$. Suppose u and u' are consecutive on the cycle; say $u = v_1$ and $u' = v_2$. Then $G - \{u, u'\}$ is a copy of F_3 . Since u

and u' together contribute defect 1 to h , it follows from Lemma 14 that $df(h) + df(v_4) \geq 3$, and so G has defect at least $3/2$. So assume without loss of generality that $u = v_1$ and $u' = v_3$. If h has two neighbors that are at least as red as h , and two other neighbors that are at most as red as h , then h has defect at least 2. So, without loss of generality, we may assume that $r(v_2), r(v_4), r(v_5) \leq r(h)$. Then, the defect that h receives from $\{v_2, v_5\}$ is $2 - 2r(h) + r(v_2) + r(v_5)$, and the defect that u receives from $\{v_2, v_5\}$ is $2 - r(v_2) - r(v_5)$. That is, the sum of the defects that h and u receive from $\{v_2, v_5\}$ is at least 2. Since h also receives defect 1 from u and u' , it follows that $df(u) + df(h) \geq 3$, and the result follows. The argument for $n = 4$ is similar and omitted.

(b) If $h \in \{u, u'\}$, then, without loss of generality, we may assume that $u = v_1$ and $u' = h$. Note that v_n receives defect 1 from $\{v_1, h\}$. Let index j be such that v_j is the redder vertex of v_{n-1} and v_n . Then v_{n-1} and v_n have at least $1 - r(v_j)$ of blue overlap and so $df(v_n) \geq 2 - r(v_j)$.

Further, one can form k edge-disjoint copies of P_3 without using vertex h or v_j . Let S denote the set of centers of these copies. By Lemma 14 and noting that the hub h receives defect $r(v_j)$ from vertex v_j , it follows that $df(h) + \sum_{s \in S} df(s) \geq 2k + r(v_j)$.

Thus $df(h) + df(v_n) + \sum_{s \in S} df(s) \geq 2k + 2$, and the result follows. \square

3.3.3 Complete multipartite graphs and compositions

We consider here complete multipartite graphs. These can be thought of as taking a complete graph and replacing each vertex by an independent set with the same adjacency. In general, we define $G[aK_1]$ to be the *composition* of G with the empty graph on a vertices; that is, the graph obtained by replacing every vertex v of G with a vertex set I_v of size a such that a vertex of I_v is adjacent to a vertex of I_w if and only if v and w are adjacent in G .

There are two simple bounds:

Proposition 17 *For any graph G ,*

(a) $TD(G[aK_1], k) \geq a^2 TD(G, k)$.

(b) $D(G[aK_1], k) \leq aD(G, k)$.

Proof. (a) Let n denote the order of G . Consider a k -coloring of $G[aK_1]$ that achieves $TD(G[aK_1], k)$. Note that $G[aK_1]$ contains a^n copies of G (by choosing one vertex from each set I_v of size a). The sum of the total defects of those a^n graphs is at least $a^n TD(G, k)$. Since each edge of $G[aK_1]$ is contained in exactly a^{n-2} of those graphs, the result follows by averaging.

(b) Take an optimal coloring of G and replicate it. □

We let $K_a^{(m)}$ denote the complete m -partite graph with a vertices in each partite set; that is $K_a^{(m)} = K_m[aK_1]$. It follows that:

Proposition 18 *If m is a multiple of k , then the complete multipartite graph $K_a^{(m)}$ can be k -colored with defect $(m/k - 1)a$, and this is best possible.*

But if m is not a multiple of k , the result is not clear. We have the following conjecture:

Conjecture 2 *The minimum defect in a k -coloring of $K_a^{(m)}$ is $(\lceil m/k \rceil - 1)a$.*

In fact, we do not have an example that precludes it being the case that it always holds that $D(G[aK_1], k) = aD(G, k)$.

We shall prove Conjecture 2 for 2 colors. We need the following definitions. Define a vertex x as *large* if $r(x) > 1/2$ and *small* if $r(x) < 1/2$. Also we let $N(x)$ denote the set of neighbors of x , $U(x)$ denote the set of vertices y in $N(x)$ with $r(y) \geq r(x)$, and $L(x)$ denote the set of vertices y in $N(x)$ with $r(y) < r(x)$.

We also need the following observations and lemmas. Some of them are very easy to verify and so the proofs are omitted:

Observation 19 *If $r(x) = 1/2$, then $df(x) \geq |N(x)|/2$.*

Observation 20 *If two vertices are both large (or both small), then the overlap between them is greater than $1/2$.*

Observation 21 $df(x) \geq \min(|U(x)|, |L(x)|)$.

Lemma 22 $df(x) \geq |N(x)|/2$ if either

(a) x is large and $|U(x)| \geq |L(x)|$,

or (b) x is small and $|U(x)| \leq |L(x)|$.

Proof. It suffices to prove it for the case that x is large. We pair each vertex in $L(x)$ with a vertex in $U(x)$. Then each pair contributes at least 1 to $df(x)$. By Observation 20, each of the remaining vertices in $U(x)$ contributes more than $1/2$ to $df(x)$. Hence $df(x) \geq |N(x)|/2$.

□

Lemma 23 *If x is large and y is small, then $\max(df(x), df(y)) \geq |N(x) \cap N(y)|/2$.*

Proof. If $|U(x)| \geq |L(x)|$, then we have $df(x) \geq |N(x)|/2 \geq |N(x) \cap N(y)|/2$ by Lemma 22. So we may assume $|U(x)| < |L(x)|$. Similarly we may assume $|U(y)| > |L(y)|$. Note that we can increase $r(x)$ to 1 and decrease $r(y)$ to 0 without increasing the defect of either vertex. It follows that $df(x) + df(y)$ is at least their common degree, whence the result. □

Lemma 24 *If all neighbors of x are large (small), then $r(x)$ can be changed to 0 (1) without increasing the defect of any vertex.*

Proof. It suffices to prove it for the case that all neighbors of x are large. Let v be any neighbor of x . The overlap between them is $1 - |r(v) - r(x)|$. If $r(x)$ is changed to 0, then the overlap becomes $1 - r(v)$. Since $r(v) > 1/2$, we have $1 - |r(v) - r(x)| \geq 1 - r(v)$ and the conclusion follows. □

Proposition 25 *The minimum defect in a 2-coloring of $K_a^{(m)}$ is $(\lceil m/2 \rceil - 1)a$.*

Proof. Such defect is attained by coloring all vertices in $\lfloor m/2 \rfloor$ of the partite sets with red, and the remaining vertices blue. So we need to prove that this is best possible.

If m is even, Proposition 17 shows that $TD(K_a^{(m)}, 2) \geq m(m/2 - 1)a^2$, and thus some vertex has defect at least $(m/2 - 1)a$. So assume m is odd.

If there is a vertex v in the graph with $r(v) = 1/2$, then $df(v) \geq (m - 1)a/2 = (\lceil m/2 \rceil - 1)a$ by Observation 19. Also, if there is a partite set that contains both a large vertex and a small vertex, then the result follows from Lemma 23.

Hence, we may assume every partite set contains either only large vertices or only small vertices. Without loss of generality, assume at least $(m + 1)/2$ partite sets contain only large vertices. Let x be the large vertex with *minimum* $r(x)$. Note that $|U(x)| \geq (m - 1)a/2 \geq |L(x)|$, and therefore $df(x) \geq (m - 1)a/2 = (\lceil m/2 \rceil - 1)a$ by Lemma 22. \square

Proposition 26 *The minimum defect in a 2-coloring of the complete tripartite graph $K_{a,b,c}$ with $a \leq b \leq c$ is $bc/(b + c - a)$.*

Proof. Let A , B , and C denote the partite sets of order a , b , and c , respectively. The upper bound is attained by coloring all vertices v in A with $r(v) = 0$, all vertices in C with $r(v) = 1$, and all vertices in B with $r(v) = x$, where x is chosen to give the vertices in A and B the same defect, namely $x = (b - a)/(b + c - a)$.

Now we prove the lower bound. Let x_1, x_2, \dots, x_a be the vertices in A with $r(x_1) \leq r(x_2) \leq \dots \leq r(x_a)$, y_1, y_2, \dots, y_b be the vertices in B with $r(y_1) \leq r(y_2) \leq \dots \leq r(y_b)$, and z_1, z_2, \dots, z_c be the vertices in C with $r(z_1) \leq r(z_2) \leq \dots \leq r(z_c)$.

Case 1: $a \leq b \leq c \leq a + b$.

Then we have $(b + c)/2 \geq (a + c)/2 \geq (a + b)/2 \geq bc/(b + c - a)$. If there is a vertex v in the graph with $r(v) = 1/2$, then the conclusion follows from Observation 19.

Also, if there is a partite set that contains both a large vertex and a small vertex, then the conclusion follows from Lemma 23. Hence we may assume every partite set contains either only large vertices or only small vertices, and by symmetry we only need to consider the following four cases:

Case 1.1: all vertices in the graph are large.

By Observation 20, we have $df(x_i) \geq (b+c)/2$ for every $1 \leq i \leq a$. So the conclusion follows.

Case 1.2: all vertices in A are small and all the other vertices are large.

Let u be the large vertex with *minimum* $r(u)$. By Lemma 22, $df(u) \geq (a+b)/2$ and the conclusion follows.

Case 1.3: all vertices in B are small and all the other vertices are large.

By Lemma 24, we may assume $r(y_j) = 0$ for every $1 \leq j \leq b$. If $r(x_1) \leq r(z_1)$, then by Lemma 22, $df(x_1) \geq (b+c)/2$. So assume $r(x_1) > r(z_1)$. We have

$$\begin{aligned}
df(x_a) &= b(1 - r(x_a)) + \sum_{k=1}^c (1 - |r(x_a) - r(z_k)|) \\
&\geq b(1 - r(x_a)) + \sum_{k=1}^c (r(x_a) + r(z_k) - 1) \\
&= (b - c)(1 - r(x_a)) + \sum_{k=1}^c r(z_k) \\
&\geq (b - c)(1 - r(x_a)) + c r(z_1),
\end{aligned}$$

and

$$\begin{aligned}
df(z_1) &= \sum_{i=1}^a (1 - (r(x_i) - r(z_1))) + b(1 - r(z_1)) \\
&= \sum_{i=1}^a (1 - r(x_i)) + (a - b)r(z_1) + b \\
&\geq a(1 - r(x_a)) + (a - b)r(z_1) + b.
\end{aligned}$$

Hence, $(b-a)df(x_a) + cdf(z_1) \geq [(b-a)(b-c) + ac](1 - r(x_a)) + bc \geq bc$. It follows that $\max(df(x_a), df(z_1)) \geq bc/(b+c-a)$.

Case 1.4: all vertices in C are small and all the other vertices are large.

By Lemma 24, we may assume $r(z_k) = 0$ for every $1 \leq k \leq c$. If $r(x_1) \leq r(y_1)$, then we have

$$\begin{aligned} df(x_1) &= c(1 - r(x_1)) + \sum_{j=1}^b (1 - (r(y_j) - r(x_1))) \\ &= c + (b-c)r(x_1) + \sum_{j=1}^b (1 - r(y_j)) \\ &\geq c + (b-c)r(x_1), \end{aligned}$$

and

$$\begin{aligned} df(y_1) &= \sum_{i=1}^a (1 - |r(x_i) - r(y_1)|) + c(1 - r(y_1)) \\ &\geq \sum_{i=1}^a (r(x_i) + r(y_1) - 1) + c(1 - r(y_1)) \\ &= (c-a)(1 - r(y_1)) + \sum_{i=1}^a r(x_i) \\ &\geq \sum_{i=1}^a r(x_i) \\ &\geq a r(x_1). \end{aligned}$$

Hence, $bdf(x_1) + (c-a)df(y_1) \geq bc + [b(b-c) + (c-a)a]r(x_1) = bc + (b+a-c)(b-a)r(x_1) \geq bc$. It follows that $\max(df(x_1), df(y_1)) \geq bc/(b+c-a)$.

Similarly, if $r(x_1) > r(y_1)$, then it can be verified that

$$df(x_1) \geq (c-b)(1 - x_1) + \sum_{j=1}^b r(y_j) \geq b r(y_1),$$

and

$$df(y_1) = \sum_{i=1}^a (1 - r(x_i)) + c + (a - c)r(y_1) \geq c + (a - c)r(y_1).$$

Hence, $(c - a)df(x_1) + bdf(y_1) \geq bc$. It follows that $\max(df(x_1), df(y_1)) \geq bc/(b + c - a)$.

Case 2: $a \leq b < a + b < c$.

Then we have $(b + c)/2 \geq (a + c)/2 > c/2 > bc/(b + c - a)$. By Observation 19 and Lemma 23, we only consider the case that the vertices of $A \cup B$ are either all large or all small. Without loss of generality, assume they are all large. Then by Lemma 24, we may assume $r(z_k) = 0$ for every $1 \leq k \leq c$.

If $r(x_1) \leq r(y_1)$, then $df(x_1) \geq b$ by Observation 21. So assume $r(x_1) > r(y_1)$. But then by the same argument as that in Case 1.4, we have $\max(df(x_1), df(y_1)) \geq bc/(b + c - a)$.

□

For another composition, consider $C_m[aK_1]$ where m is odd. We now prove that $D(C_m[2K_1], 2) = 2$. There are at least two different optimal colorings. The first such coloring is obtained by taking an optimal coloring for C_m and replicating it. The second such coloring is obtained by, for each copy of $2K_1$, coloring one vertex red and one vertex blue.

Proposition 27 *For m odd, $D(C_m[2K_1], 2) = 2$.*

Proof. Consider a 2-coloring of $C_m[2K_1]$. We need to show that the defect is at least 2. As in the proof of Proposition 25, we may assume that every copy of $2K_1$ contains either two large vertices or two small vertices. Since m is odd, it follows that there must be two adjacent copies of the same type. Without loss of generality, assume u_1 and u_2 are adjacent to v_1 and v_2 with all four vertices being large. If any $x \in \{u_1, u_2, v_1, v_2\}$ has $|U(x)| \geq 2$, then the lower bound follows from Lemma 22(a). Therefore we may assume that $|U(x)| \leq 1$

for every $x \in \{u_1, u_2, v_1, v_2\}$. This means that each u_i is redder than some v_j and vice versa, a contradiction. \square

3.3.4 Rooks graphs and Cartesian products

Recall that the Cartesian product $G \square H$ is the graph whose vertex set is $V(G) \times V(H)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

We will need the obvious lower bound for the total defect of Cartesian products.

Proposition 28 *Let G and H be graphs of order m and n respectively. Then*

$$TD(G \square H, k) \geq m TD(H, k) + n TD(G, k).$$

Proof. The defect of a vertex in the product is the sum of the defects in its copies of G and H . \square

Recall that rooks graphs are the Cartesian product of complete graphs. We denote the vertices of $K_m \square K_n$ by (i, j) with $1 \leq i \leq m, 1 \leq j \leq n$.

Lemma 29 *The rooks graph $K_m \square K_n$ can be 2-colored with defect $\lceil m/2 \rceil + \lceil n/2 \rceil - 2$.*

Proof. Color vertex (i, j) red if i and j have the same parity and blue otherwise. \square

Corollary 30 *Let m and n be even integers. Then $D(K_m \square K_n, 2) = m/2 + n/2 - 2$.*

Proof. The upper bound follows from Lemma 29. The lower bound follows from Proposition 28, since $TD(K_s, 2) = s(s/2 - 1)$ for s even (Corollary 10), and thus $TD(K_m \square K_n, 2) \geq mn(n/2 - 1) + nm(m/2 - 1)$. \square

We show below that the upper bound in Lemma 29 is not always optimal. In fact we conjecture that it is never optimal when m and n are both odd, except for the case that $m = n = 3$.

Lemma 31 $D(K_3 \square K_3, 2) = 2$.

Proof. The upper bound is from Lemma 29.

We have two proofs of the lower bound, one by computer and one by hand. Both proofs entail converting the question to a set of linear programs.

Observe that given a coloring, one can generate an acyclic orientation by orienting each edge from smaller to larger proportion of red (with ties broken by vertex number). Further, if N_1 is the set of neighbors of vertex v with more red and N_2 is the set of neighbors of v with less red, then Equation 3.1 simplifies to

$$df(v) = |N_1|r(v) + |N_2|b(v) + \sum_{w \in N_2} r(w) + \sum_{w \in N_1} b(w),$$

where $b(x) = 1 - r(x)$.

We continue by enumerating the acyclic orientations. For each such orientation, we add the constraints that $r(u) \leq r(v)$ for all arcs uv . That is, minimizing the defect for a given orientation is a linear program.

Further, if any vertex has in- and out-degree 2 for the orientation, the defect is definitely at least 2 (by Observation 21). With several pages of calculation or by using a computer, one can show that $K_3 \square K_3$ has eight acyclic orientations (up to symmetry) that need to be considered, and then solve the eight associated linear programs. We omit the details. □

In contrast, we found a coloring of $K_3 \square K_5$ that beats the bound of Lemma 29:

Lemma 32 $D(K_3 \square K_5, 2) \leq 38/13$.

Proof. A 2-coloring of $K_3 \square K_5$ is shown in the matrix below. The element (i, j) of the matrix is the red-usage on vertex (i, j) .

$$\begin{bmatrix} 0 & 8/13 & 0 \\ 0 & 0 & 8/13 \\ 1 & 11/13 & 0 \\ 1 & 0 & 11/13 \\ 6/13 & 1 & 1 \end{bmatrix}$$

It can be verified that the defect of the coloring is $38/13$. \square

The above coloring can be extended to show that Lemma 29 is not optimal for $m = 3$ and n odd, $n \geq 5$, and indeed that $D(K_3 \square K_n, 2) \leq n/2 + 11/26$ in this case. However, this is still not best possible. For example, one can get defect $42/11$ for $K_3 \square K_7$ and defect $14/3$ for $K_3 \square K_9$ by the colorings illustrated in the matrices below:

$$\begin{bmatrix} 1 & 0 & 1 \\ 4/11 & 1 & 1 \\ 0 & 8/11 & 0 \\ 1 & 1 & 4/11 \\ 1/11 & 0 & 1 \\ 0 & 8/11 & 0 \\ 1 & 0 & 1/11 \end{bmatrix} \quad \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

We used simulated annealing computer search (that is, a randomized search for a coloring) to find upper bounds. Though we have no exact values, it seems to us that the search results suggest the following:

Conjecture 3 (a) *If $m + n$ is odd, then $D(K_m \square K_n, 2) = (m + n - 3)/2$.*

(b) *If mn is odd and greater than 9, then $D(K_m \square K_n, 2) < \lceil m/2 \rceil + \lceil n/2 \rceil - 2$.*

Note that Conjecture 3 (a) is trivially true for the case that $m = 2$ (or $n = 2$), since $D(K_2 \square K_n, 2) \geq D(K_n, 2) = (n - 1)/2$.

Proposition 28 yields the following lower bounds:

Corollary 33

(a) *If both m and n are odd, $D(K_m \square K_n, 2) \geq (m + n)/2 - 2 + 1/(2m) + 1/(2n)$.*

(b) *If m is even and n is odd, $D(K_m \square K_n, 2) \geq (m + n)/2 - 2 + 1/(2n)$.*

For more colors we have the trivial observation that $D(K_n \square G, k) = \lceil n/k \rceil - 1$ for any k -partite graph G , as a corollary of Proposition 13.

3.3.5 Regular graphs

Lovász [58] showed that we can ordinarily 2-color a cubic graph with defect at most 1. Therefore $D(G, 2) = 1$ for all nonbipartite cubic graphs G .

For a 4-regular graph, Lovász’s result shows that one can ordinarily 2-color it with defect at most 2. We conjecture that this can be improved. Proposition 27 shows that the composition $G = C_m[2K_1]$ where m is odd has $D(G, 2) = 2$. Using simulated annealing, the computer can find a 2-coloring with defect smaller than 2 for all 4-regular graphs on up to 14 vertices, except for the compositions of odd cycles, and the two graphs K_5 and $K_3 \square K_3$, which we saw earlier have minimum defect 2. We give a conjecture for the general behavior:

Conjecture 4 *Apart from $G = C_m[2K_1]$ where m is odd, it holds that $D(G) < 2$ for all but finitely many connected 4-regular graphs.*

3.4 Complexity

Unsurprisingly, it is NP-hard to determine if there is a coloring with defect at most some specified d .

One way to see this is that fractional defect 2-coloring is NP-hard even for $d = 1$. One can extend Lemma 11 to show that in graphs of minimum degree at least 3, a 2-coloring with defect 1 can only be a coloring with monochromatic vertices. Thus the fractional defect 2-coloring problem is equivalent to the ordinary defective 2-coloring problem in such graphs. The latter problem was shown to be NP-hard by Cowen [22]. (Actually, we need ordinary 1-defect coloring to be NP-hard in graphs with minimum degree at least 3. But one can transform a graph to having minimum degree at least 3 without changing the coloring property by adding, for each vertex v , a copy of K_4 and joining v to one vertex of the K_4 .)

Chapter 4

Vertex Colorings without Rainbow Subgraphs

4.1 Introduction

This chapter is based on joint work with Wayne Goddard [41]. As all proofs in the original paper are provided, we do not give specific references to that paper.

Given a (not necessarily proper) vertex coloring of a graph G , recall that a subgraph is rainbow if all its vertices receive distinct colors and monochromatic if all its vertices receive the same color. For a graph F , we refer to a (not necessarily proper) vertex coloring of G without rainbow subgraphs isomorphic to F as a *no-rainbow- F coloring* of G (*valid coloring* for short); we define the *F -upper chromatic number* of G as the maximum number of colors that can be used in a valid coloring. We denote this maximum by $NR_F(G)$. A valid coloring is *optimal* if it uses exactly $NR_F(G)$ colors.

There are many papers on the edge-coloring version, where the parameter is called the *anti-Ramsey number*. Note that this parameter is also exactly 1 less than the *rainbow number*, which is the minimum number of colors such that every edge-coloring of G with at least that many colors produces a rainbow F . For the edge-coloring case, the most

studied situation is when G is complete and F is a cycle, clique, tree, or matching. For example, a *Gallai-coloring* is an edge-coloring of the complete graph without a rainbow triangle [44, 43]. For a survey of anti-Ramsey theory, see [33].

In contrast, not much has been written about the vertex-coloring case. There are two papers on avoiding rainbow *induced* subgraphs: [3] and [52]. More recently, the special case where F is P_3 was considered by Bujtás et al. [10] (under the name *3-consecutive upper chromatic number*), and then the case where F is $K_{1,k}$ was considered by Bujtás et al. [9] (under the name *star-[k] upper chromatic number*). Besides these, a related question that has been studied is coloring embedded graphs with no rainbow *faces*, see for example [24, 65].

Graph colorings without rainbow (monochromatic) subgraphs fall within the theory of mixed hypergraphs introduced by Voloshin (see, for example, [74]; see [75] for an overview of the theory, and see [71] for a survey of results and open problems).

In general, a *mixed hypergraph* \mathcal{H} is a triple $(X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set*, and \mathcal{C} and \mathcal{D} are families of subsets of X , called the \mathcal{C} -edges and \mathcal{D} -edges, respectively. A *proper coloring* of \mathcal{H} is an assignment of one color to each vertex in X such that each \mathcal{C} -edge has at least two vertices with a *Common* color, and each \mathcal{D} -edge has at least two vertices with *Distinct* colors. The case that \mathcal{C} is an empty set is just the proper coloring of hypergraphs; while the case that \mathcal{D} is an empty set leads to the notion called *C-coloring* (see [12]).

Note that the theory of mixed hypergraphs provides a general framework for graph colorings without rainbow (monochromatic) subgraphs: say we color a graph G with vertex set V forbidding monochromatic subgraph M and rainbow subgraph R , then consider the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where $X = V$, \mathcal{C} consists of the vertex subsets of cardinality $|V(R)|$ which induces a subgraph containing R in G , and \mathcal{D} consists of the vertex subsets of cardinality $|V(M)|$ which induces a subgraph containing M in G . One can see that there is a bijection between the colorings of G and the proper colorings of \mathcal{H} .

In this chapter, we investigate the F -upper chromatic number for certain graph classes. We proceed as follows: in Section 4.2 we present some basic observations. Then in

Section 4.3 we consider the case that F is a path on three vertices, in Section 4.4 the case that F is a triangle, and in Section 4.5 the case that F is the star $K_{1,r}$.

4.2 Preliminaries

Bujtás et al. [9] observed the following when F is a star, but the results hold in general:

- For fixed F , the parameter is monotonic: if H is a spanning subgraph of G , then $NR_F(G) \leq NR_F(H)$.
- If F is connected and G is disconnected, then $NR_F(G)$ is the sum of the NR_F 's of the components of G .
- The chromatic spectrum has no gaps: G has a coloring without a rainbow F using k colors for $1 \leq k \leq NR_F(G)$. Simply take an optimal coloring and successively merge color classes.
- $NR_F(G) = |V(G)|$ if and only if G is F -free.
- $NR_F(G) \geq |F| - 1$, provided G has that many vertices.

For a natural lower bound, one can define an F -*bi-cover* of a graph as a set of vertices that contains at least two vertices from every copy of F . It follows that one can obtain a no-rainbow- F coloring by giving all vertices in an F -bi-cover the same color and giving all other vertices unique colors. For example, if G is a connected graph of order at least 3, then a P_3 -bi-cover is the complement of a *packing*. (A packing is a set of vertices at pairwise distance at least 3; the *packing number* $\rho(G)$ is the maximum size of a packing.) The lower bound $NR_{P_3}(G) \geq \rho(G) + 1$ follows. A vertex cover is an F -bi-cover for any connected non-star graph F . In this case we have $NR_F(G) \geq \alpha(G) + 1$ (where $\alpha(G)$ denotes the independence number) provided G is not empty.

We say that a set S *bi-covers* a subgraph H if at least two vertices of H are in S . For positive integer s , define $b_F(s)$ to be the maximum number of copies of F that can be bi-covered by using a set of size s . Note that $b_F(1) = 0$.

Proposition 34 *Suppose that graph G of order n contains f copies of F and that $b_F(s) \leq a(s - 1)$ for all s . Then $NR_F(G) \leq n - f/a$.*

Proof. Consider a no-rainbow- F coloring. Say one uses k colors, being used s_1, \dots, s_k times respectively. Then $k = n - \sum_{i=1}^k (s_i - 1)$. Since every copy of F has to be bi-covered by some color class, $\sum_{i=1}^k b_F(s_i) \geq f$. It follows that $k \leq n - f/a$. \square

4.3 Forbidden P_3

The parameter $NR_{P_3}(G)$ can also be thought of as the maximum number of colors in a coloring such that each vertex sees at most one color other than its own.

4.3.1 Fundamentals

There are two natural lower bounds.

Proposition 35

(a) *For a graph G , $NR_{P_3}(G) \geq \text{diam}(G)/2 + 1$.*

(b) *For any nonempty graph G , $NR_{P_3}(G) \geq \rho(G) + 1$.*

Proof. (a) Let x be a vertex of eccentricity $\text{diam}(G)$, and color each vertex v by $\lceil d(x, v)/2 \rceil$, where $d(x, v)$ denotes the distance from x to v .

(b) See the previous section. (Give every vertex in a maximum packing a unique color, and give all other vertices the same color.) \square

Bujtás et al. [10] showed a partial converse to the first lower bound:

Proposition 36 [10] *If G has diameter 2, then $NR_{P_3}(G) = 2$.*

For example, rooks graphs have diameter 2 and so they have P_3 -upper chromatic number 2.

Bujtás et al. [10] showed that the P_3 -upper chromatic number of a connected graph is at most one more than its vertex cover number:

Theorem 37 [10] *For every connected graph G , it holds that $NR_{P_3}(G) \leq \beta(G) + 1$.*

Bujtás et al. [10] also showed that for every tree T , $NR_{P_3}(T)$ is one more than the matching number of T :

Theorem 38 [10] *For every tree T , it holds that $NR_{P_3}(T) = m(T) + 1$.*

From the above theorem, we obtain the following upper bound for $NR_{P_3}(G)$:

Corollary 39 *For a connected graph G of order n , $NR_{P_3}(G) \leq \lfloor n/2 \rfloor + 1$.*

Proof. Let T be a spanning tree of G . By theorem 38, $NR_{P_3}(T) = m(T) + 1 \leq \lfloor n/2 \rfloor + 1$. Since $NR_{P_3}(G) \leq NR_{P_3}(T)$, the result follows. \square

It is natural to ask what graphs achieve equality in Corollary 39.

The *corona* $cor(G)$ of a graph G is the graph obtained from G by adding, for each vertex v in G , a new vertex v' and the edge vv' . The new vertices are called the *leaves* of the corona. Note that coronas achieve equality in Corollary 39:

Observation 40 *If G is connected, then $NR_{P_3}(cor(G)) = |G| + 1$.*

Proof. The lower bound follows from Proposition 35(b). That is, give all the leaves unique colors and give all the original vertices of G the same color. The upper bound is from Corollary 39. \square

Note that in an optimal no-rainbow- P_3 coloring of a graph, the subgraphs that are induced by the color classes do not have to be connected. For example, Figure 4.1 shows a graph G with $NR_{P_3}(G) = 4$, which is uniquely attained by giving each white vertex a unique color and all black vertices the same color.

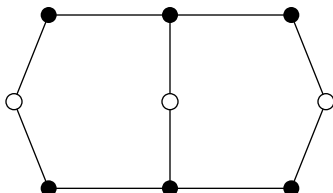


Figure 4.1: A graph whose optimal no-rainbow- P_3 coloring has a disconnected color class

4.3.2 Complexity

We next show that calculating $NR_{P_3}(G)$ is NP-hard. We will need the following construction: for graph G , define graph $M(G)$ by adding, for every vertex v in G , a new vertex v' adjacent to v , and adding edges to make $C = \{v' : v \in V(G)\}$ a clique.

Observation 41 *For any nontrivial graph G it holds that $NR_{P_3}(M(G)) = \rho(G) + 1$.*

Proof. Note that $\rho(M(G)) = \rho(G)$. So the lower bound follows from Proposition 35(b). To prove the upper bound, consider a coloring of $M(G)$ with no rainbow P_3 . Note that the clique C contains at most two colors. There are two cases.

First, consider that C contains two colors. Note that for every vertex v in $V(G)$, there is a vertex w' such that v' and w' receive different colors. It follows that v receives one of the two colors in C . That is, the coloring uses two colors.

Second, consider that C contains only one color, say red. Let v and w be vertices of $V(G)$ such that neither is red and they have different colors. Then they cannot be adjacent, since that would make vwv' rainbow, nor can they have a common neighbor x , since x would see three colors. It follows that if we take one vertex of each non-red color, we obtain a packing. That is, the number of non-red colors is at most $\rho(G)$, as required. \square

As a consequence it follows that computing $NR_{P_3}(G)$ is NP-hard, since computing the packing number is NP-hard [14].

In contrast, Bujtás et al. [10] showed that determining whether a graph G has $NR_{P_3}(G) = 3$ or $NR_{P_3}(G) = 4$ is solvable in polynomial time.

4.3.3 Graph Families and Operations

4.3.3.1 Clones

In general, if v and w have the same neighbors (themselves excluded), then $NR_F(G) \geq NR_F(G - v)$, since one can take any coloring of $G - v$ and give v the same color as w . But we have equality for $F = P_3$:

Observation 42 *Assume vertices v and w are such that $N(v) - \{w\} = N(w) - \{v\} \neq \emptyset$. Then $NR_{P_3}(G) = NR_{P_3}(G - v)$.*

Proof. Consider a valid coloring of G . Let x be any common neighbor of v and w . If v and w have different colors, then x must have the same color as one of them. If x has the same color as v , then the coloring restricted to $G - v$ is a valid coloring with every color of G . If x has the same color as w , then the coloring restricted to $G - w$ is a valid coloring with every color of G . Note that $G - w = G - v$ and so the conclusion follows. \square

4.3.3.2 Maximal Outerplanar Graphs

We now consider avoiding rainbow P_3 in maximal outerplanar graphs. The minimum value of $NR_{P_3}(G)$ for an outerplanar graph of order n is obtained by the fan (having value 2). The maximum value for a maximal outerplanar graph of order n is given by the following:

Theorem 43 *The maximum value of $NR_{P_3}(G)$ for a maximal outerplanar graph G of order $n \geq 3$ is $\lfloor n/3 \rfloor + 1$.*

Proof. We prove the lower bound by the following construction: start with a cycle $v_1v_2 \dots v_nv_1$. For $1 \leq i \leq \lfloor n/3 \rfloor$, assign v_{3i} a distinct color. Then use one additional color for all the remaining vertices, and add edges between them until we have a maximal outerplanar graph G . Clearly, exactly $\lfloor n/3 \rfloor + 1$ colors are used and there is no rainbow P_3 .

We prove the upper bound by induction on n . It suffices to show that $NR_{P_3}(G) \leq n/3 + 1$. It is easy to verify the result for $n = 3$. For larger n , the outer cycle of G has a chord.

The first case is that there is a chord, say uv , with different colors on its ends. Let V_1 and V_2 be the vertex sets of the components of $G - \{u, v\}$. Let G_i be the subgraph of G induced by the vertices $V_i \cup \{u, v\}$. Note that G_i is a maximal outerplanar graph. By the induction hypothesis, G_i has at most $|G_i|/3 + 1$ colors. But G_1 and G_2 share two colors. So the total number of colors in G is at most $(|V_1| + 2)/3 + 1 + (|V_2| + 2)/3 + 1 - 2 = (n + 2)/3 < n/3 + 1$.

The second case is that every chord is monochromatic. Since the chords induce a connected subgraph of G , it follows that all the vertices with degree at least 3 in G have the same color, say red. Let X be the set consisting of one vertex of each remaining color.

Since the vertices with degree 2 are independent, it follows that X is independent. Further, vertices x_1 and x_2 of X cannot have a common neighbor, since that vertex would be red and we would have a rainbow P_3 . It follows that $|X| \leq \rho(C_n) \leq \lfloor n/3 \rfloor$, and so the colors in G is at most $\lfloor n/3 \rfloor + 1$. \square

Note that there are maximal outerplanar graphs where $NR_{P_3}(G) > \rho(G) + 1$.

4.3.3.3 Cubic Graphs

We consider now avoiding rainbow P_3 in cubic graphs. It is unclear what the minimum and maximum values of $NR_{P_3}(G)$ (for cubic graphs G of fixed order) are. Some data generated by a computer search is shown in Table 4.1:

order	4	6	8	10	12	14	16	18
min	2	2	2	2	3	3	4	4
max	2	2	3	4	4	5	6	7

Table 4.1: Extremal values of $NR_{P_3}(G)$ for cubic graphs G of fixed order

Computer search shows that, for $n \leq 18$, the minimum value of the parameter is one more than the minimum value of the packing number. However, it is unclear what the asymptotics of the packing number are. Favaron [30] showed that $\rho(G) \geq n/8$ for a cubic graph G of order n other than the Petersen graph, but it is unclear if this bound is sharp in general. Furthermore, it is unclear under what circumstances a graph has parameter equal to the packing number lower bound.

We consider next the maximum value of the parameter for cubic graphs of order n .

Theorem 44 *For any connected cubic graph G on $n \geq 6$ vertices, $NR_{P_3}(G) \leq 2n/5$.*

Proof. We extend a counting idea suggested in [9]. For a color c , define $CN(c)$ as the number of closed neighborhoods that c is in. (Equivalently, the number of vertices dominated by a vertex of color c .) Let $A = \sum_c CN(c)$; that is, A is the number of pairs (c, v) where c is a color that occurs in $N[v]$. The requirement of no rainbow P_3 means that each closed neighborhood has at most 2 colors in it, and so $A \leq 2n$. To prove the theorem, it suffices to show that the average value of $CN(c)$ is at least 5.

Since G is cubic, it is immediate that $CN(c) \geq 4$ for all colors c . Call a color c *sparse* if $CN(c) = 4$. Say vertex v has color c . Then all other vertices with color c , if any, must be neighbors u of v such that $N[u] = N[v]$. Since the graph is not K_4 , it follows that there are at most two vertices with color c . The remaining neighbors of v (which are also the neighbors of the other vertex of color c , if any) must be the same color, say c' .

We claim that $CN(c') \geq 6$. By connectivity there is a vertex w that is not in $N[v]$ but has a neighbor x in $N[v]$. Since $N[w]$ does not contain color c but vxw is a P_3 , it follows that x and w are both color c' . Since $n \geq 6$, there must be a vertex that is not

in $N[v] \cup \{w\}$ and is adjacent to a vertex of color c' in $N[v] \cup \{w\}$. So $\text{CN}(c') \geq 6$. In particular, $\text{CN}(c) + \text{CN}(c') \geq 10$.

Now, suppose that the same color surrounds multiple sparse colors. Say, we have c_1, \dots, c_b such that $c'_1 = \dots c'_b = d$. Then we claim that $\text{CN}(d) \geq 4b$. This follows by noting that the $N[v_i]$ are disjoint if v_i has color c_i , and all of $N[v_i]$ is dominated by a vertex of color d . It follows that $\text{CN}(d) + \sum_i \text{CN}(c_i) \geq 8b \geq 5(b+1)$, since $b \geq 2$.

So, by partitioning the sparse colors into sets based on the surrounding color, it follows that the average value of $\text{CN}(c)$ is at least 5, whence the result. \square

The computer data verifies that the maximum value is $\lfloor 2n/5 \rfloor$ for $6 \leq n \leq 18$. However, the bound in Theorem 44 might not be sharp in general. Let H be the graph of order 5 obtained from K_4 by subdividing one edge. Let I_0 be built from two copies of H by adding an edge joining the vertices of degree 2. Computer search confirms that for $n = 10$ this is the unique graph that achieves the maximum value. In general, let graph I_j be the cubic graph built from two copies of H by adding j copies of $K_4 - e$ between the copies of H . For $n = 14$, the graph I_1 is a graph that achieves the maximum value. For $n = 18$, the graph I_2 is a graph that achieves the maximum value. But there is one other graph that achieves the maximum value: take three copies of H and one copy of K_3 and add edges to make a connected cubic graph. See Figure 4.2. It can be checked that $NR_{P_3}(I_j)$ is $3n/8 + O(1)$. It is unclear if this is best possible.

4.4 Forbidden Triangles

We consider now colorings that forbid a rainbow copy of the other connected graph on three vertices, a triangle. That is, we consider colorings where every triangle has a monochromatic edge.

We saw earlier that $NR_{K_3}(G) \geq \alpha(G) + 1$, provided G is nonempty. In particular, we note that if every edge of the graph is in a triangle, then a subset S is a K_3 -bi-cover if

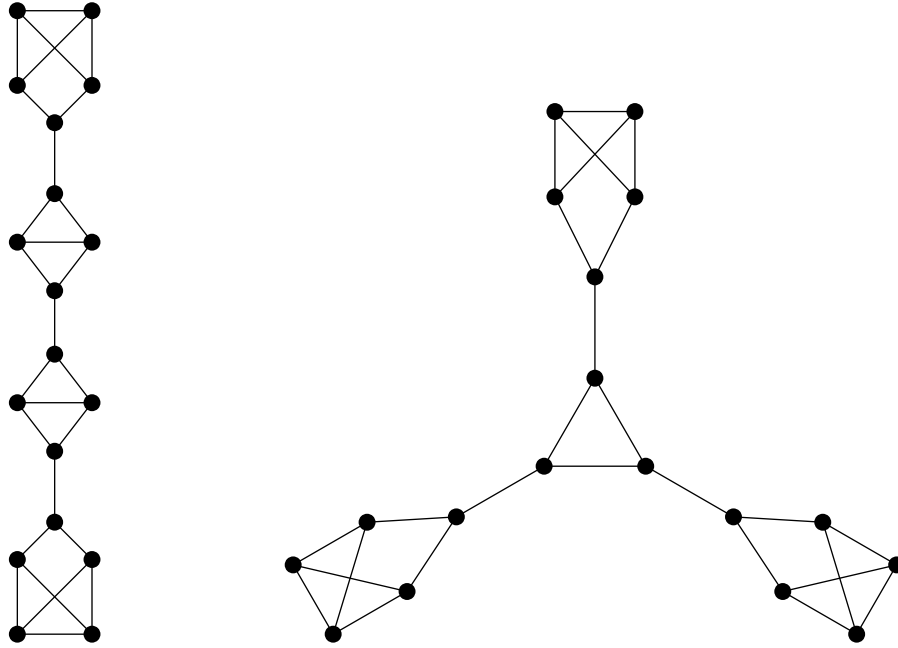


Figure 4.2: The two cubic graphs of order 18 with maximum NR_{P_3}

and only if S is a vertex cover. Note that when F is a complete graph, every color class in an optimal no-rainbow- F coloring of a graph must induce a connected subgraph. (Assume to the contrary that color red induces a disconnected subgraph; then change the vertices in one red component to a new color pink. There cannot be a red vertex and a pink vertex together in a clique, since the pink vertex and red vertex were not adjacent.)

One can again investigate the minimum and maximum values of the parameter for graphs of fixed order in particular classes. For example, the extremal values of $NR_{K_3}(G)$ for cubic graphs G of order n are straightforward. The maximum is n , achieved by a triangle-free graph. The minimum is $2n/3$, achieved by a cubic graph with $n/3$ disjoint triangles.

4.4.1 Maximal Outerplanar Graphs

We show that the value of $NR_{K_3}(G)$ for a maximal outerplanar graph G of fixed order does not depend on the structure of G :

Theorem 45 *Let G be a maximal outerplanar graph of order n . Then it holds that*

$$NR_{K_3}(G) = \lfloor n/2 \rfloor + 1.$$

Proof. We prove the upper bound $n/2 + 1$ by induction on n . It is easy to verify the result for $n = 3$. For larger n , the outer cycle of G has a chord.

The first case is that there is a chord, say uv , with different colors on its ends. Say the removal of $\{u, v\}$ from G yields components with vertex sets V_1 and V_2 . Let G_i be the subgraph of G induced by the vertices $V_i \cup \{u, v\}$. Note that each G_i is a maximal outerplanar graph. But G_1 and G_2 share two colors. So, by the induction hypothesis, the total number of colors in G is at most $(|V_1| + 2)/2 + 1 + (|V_2| + 2)/2 + 1 - 2 = n/2 + 1$.

The second case is that every chord is monochromatic. Since the chords induce a connected subgraph of G , it follows that all the vertices with degree at least 3 in G have the same color, say red. Since the vertices with degree 2 are independent, it follows that the number of colors in G is at most $n/2 + 1$.

We prove the lower bound by induction. It is easy to verify that the result is true for $n \leq 4$; so assume $n \geq 5$. Note that the weak dual of G is a tree T of order $n - 2$ and maximum degree at most 3. Let b be a penultimate vertex on a longest path in T . There are two cases.

The first case is that b has degree 2, with leaf neighbor a . Say b lies in triangle xyz of G and a in triangle xyu , with vertex y of degree 3. Then let $G' = G - \{u, y\}$. Note that G' is maximal outerplanar. Let ϕ be a valid coloring of G' . Then ϕ can be extended to a valid coloring of G by giving u a new color and giving y the same color as x . The lower bound follows by induction.

The second case is that b has degree 3, with leaf neighbors a and a' . (See Figure 4.3.) Say vertex b lies in triangle xyz of G , vertex a in triangle xyu and vertex a' in triangle yzv . Then let $G' = G - \{u, v\}$. Note that G' is maximal outerplanar. Let ϕ be a valid coloring

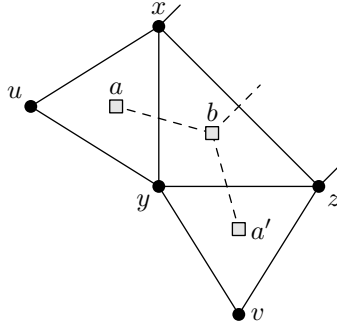


Figure 4.3: Part of a maximal outerplanar graph and its weak dual

of G' . We need to show how to introduce one new color. If vertex y has the same color as either x or z , then this is immediate. So assume y has a different color to both x and z . Since triangle xyz is not rainbow, this means that x and z have the same color. Then one can proceed by recoloring y to be the same color as x and z , and then giving both u and v unique colors. It follows that $NR_{K_3}(G) \geq \lfloor n/2 \rfloor + 1$, as required. \square

Note that the above result does not extend to 2-trees. For example, consider the graph G obtained from the complete bipartite graph $K_{2,n-2}$ by adding an edge between the two vertices in the partite set of size 2. The graph G is 2-tree, and it is easy to verify that $NR_{K_3}(G) = n - 1$.

4.4.2 Rooks Graphs

Define R_m as the rooks graph given by the cartesian product $K_m \square K_m$. The following is probably known:

Proposition 46 *Consider a coloring of the rooks graph R_m such that every row and column contains at most two colors. Then the number of colors used is at most $\max(4, m + 1)$.*

Proof. Suppose first that there is both a row and a column that are monochromatic, say red. If red does not appear elsewhere, then the rest of the graph is monochromatic, while if red does appear elsewhere, the bound follows by induction. So we may assume that every row say contains exactly two colors; say row i has A_i and B_i for $1 \leq i \leq m$.

Suppose two rows—say i and j —have disjoint colors. Then every column contains one color of $\{A_i, B_i\}$ and one of $\{A_j, B_j\}$ and thus the total number of colors used is at most 4. So we may assume that every pair of rows share a color. If we construct an auxiliary graph H with the colors as vertices and join two vertices if they are together in some row, then this means that every pair of edges in H share an endpoint. Thus, H is either a star or a triangle. The former means there is one color that occurs in every row, which means at most $m + 1$ colors total; and the latter means at most three colors total. \square

Theorem 47

$$NR_{K_3}(R_m) = \begin{cases} 4, & \text{if } m = 2, \\ m + 1, & \text{if } m \geq 3. \end{cases}$$

Proof. For $m = 2$, the rooks graph has no triangle, whence the result. In general, $m + 1$ is a lower bound by the independence number bound. The upper bound follows from Proposition 46. \square

4.4.3 Complexity

It is straightforward to show that calculating $NR_{K_3}(G)$ is NP-hard. For example, one can reduce from the problem of calculating the independence number (which is known to be NP-hard; see, for example, [35]) as follows:

Observation 48 *Consider the graph G' obtained by adding one new vertex adjacent to all vertices of G . Then $NR_{K_3}(G') = \alpha(G) + 1$.*

Proof. Note that $\alpha(G') = \alpha(G)$. Thus, $NR_{K_3}(G') \geq \alpha(G') + 1 = \alpha(G) + 1$. For the upper bound, consider an arbitrary no-rainbow- K_3 coloring of G' . Suppose the dominating vertex has color red. Then for every other color, choose one vertex; let S be the resultant set. Then since there is no rainbow K_3 , S must be independent, and so the total number of numbers is $|S| + 1 \leq \alpha(G) + 1$. \square

4.5 Forbidden Stars

We consider here the star $K_{1,r}$. The parameter $NR_{K_{1,r}}(G)$ is equal to the maximum number of colors in a coloring such that each vertex sees at most $r - 1$ colors other than its own. This parameter was studied by Bujtás et al. [9]. They showed that:

Theorem 49 [9]

- (a) For graph G of order n and minimum degree δ , it holds that $NR_{K_{1,r}}(G) \leq nr/(\delta + 1)$.
- (b) For graph G of order n and vertex cover number β , it holds that $NR_{K_{1,r}}(G) \leq 1 + (r-1)\beta$.
- (c) For graph G of domination number γ , it holds that $NR_{K_{1,r}}(G) \leq r\gamma$.

As an example of a specific result, it was shown in [38] that $NR_{K_{1,r}}(G) = 2(r - 1)$ for the complete bipartite graph $G = K_{m,m}$ when $m \geq r \geq 2$.

Bujtás et al. [9] ask: when is $NR_{K_{1,r}}(G) = r$? They showed that G having diameter at most 2 is necessary (e.g. for stars) but not sufficient.

4.5.1 Trees

We show first that Theorem 38 generalizes to all stars. Indeed, it is true for any forbidden rainbow subgraph:

Theorem 50 For a tree T and any connected graph F , $NR_F(T)$ equals 1 more than the maximum number of edges in an F -free subgraph of T .

Proof. Let H be an arbitrary F -free subgraph of T . Let T' be the spanning subgraph of T with the edges of H removed. By giving each component of T' a different color, we get a valid coloring of T : every rainbow connected subgraph of T is a subgraph of H . Since the number of colors used equals 1 more than the number of edges in H and H is arbitrary, $NR_F(T)$ is at least 1 more than the maximum number of edges in an F -free subgraph of T .

Conversely, take an optimal coloring of T . Let B be the set of edges whose ends have different colors. Consider an edge $e = uv$ in B ; say u is red and v is blue. If red appears in the component of $T - e$ containing v , recolor all red vertices in that component with color blue. (Note that this does not decrease the total number of colors.) If this recoloring increases the number of colors in some copy of F , then that copy must now contain both a red vertex and a newly-blue vertex; but by connectivity that means it must also contain v , and thus is not rainbow. That is, we may assume that all color classes are connected. Then the set B induces an F -free subgraph of T . And the number of colors in T equals the number of components of $T - B$, which is $|B| + 1$. It follows that $NR_F(T)$ is at most 1 more than the maximum number of edges in an F -free subgraph of T . \square

4.5.2 Specific Results for Forbidden $K_{1,3}$

4.5.2.1 Rooks Graphs

For a rooks graph, there is a connection between forbidding stars and cliques.

Proposition 51 *For any rooks graph R_m , it holds that*

$$NR_{K_{1,r}}(R_m) \leq \max\{r, NR_{K_r}(R_m)\}.$$

Proof. The requirement for no rainbow $K_{1,r}$ is that every vertex sees at most r colors including its own. Suppose some row of R_m contains r colors. Then every vertex in that row sees the same r colors. That is, the graph is r -colored. So we may assume that each row and column contains at most $r - 1$ colors. That is, there is no rainbow K_r . \square

It follows that:

Theorem 52

$$NR_{K_{1,3}}(R_m) = \begin{cases} 4, & \text{if } m = 2, \\ m + 1, & \text{if } m \geq 3. \end{cases}$$

Proof. For $m = 2$, the rooks graph has no $K_{1,3}$ nor K_3 , whence the result. In general, $m + 1$ is a lower bound by giving each vertex on one diagonal a unique color and coloring all other vertices the same. The upper bound follows from Theorem 47 and Proposition 51. \square

4.5.2.2 Cubic Graphs

We consider here the question of forbidding rainbow $K_{1,3}$ in cubic graphs. Let G be a cubic graph of order n . By Theorem 49(a), it holds that $NR_{K_{1,3}}(G) \leq 3n/4$. Equality can be obtained by taking disjoint copies of $K_4 - e$ and adding edges to make a connected cubic graph. Figure 4.4 shows the case that $n = 20$:

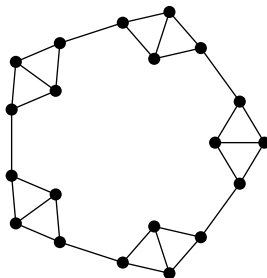


Figure 4.4: The cubic graph of order 20 with maximum $NR_{K_{1,3}}$

For $n < 10$, it can readily be shown that the minimum value of $NR_{K_{1,3}}(G)$ for cubic graphs G of order n is $n/2 + 1$. But we conjecture:

Conjecture 5 *For $n \geq 10$, the minimum value of $NR_{K_{1,3}}(G)$ over all cubic graphs G of order n is $n/2$.*

Computer search confirms this conjecture for $n \leq 14$. Note that $NR_{K_{1,3}}(G) \geq n/2$ for every cubic graph G of order n that has a perfect matching, since we can give one color to each pair of matched vertices. Recall that a prism is the cartesian product of a cycle with K_2 . If Conjecture 5 is true, then the prism of order n is a graph that achieves the minimum value when n is not a multiple of 4 (figure 4.5 shows the case that $n = 14$), as we shall prove below.

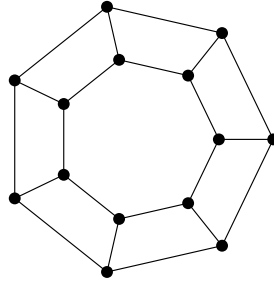


Figure 4.5: A cubic graph of order 14 with minimum $NR_{K_{1,3}}$

We will need the following observations about bi-covers:

Observation 53 *For the ladder $P_m \square K_2$, it holds that $b_{K_{1,3}}(s) \leq 2(s - 1)$.*

Proof. The result is by induction. Think of the P_m as the rows and the K_2 as columns. Consider a set S of vertices. If S is contained within one of the columns, then the result is immediate. So assume S includes vertices from at least two columns.

Let S' be the vertices of S in the leftmost column that S occupies. Then by going through the cases, one can check that: The number of copies of $K_{1,3}$ that are bi-covered by S but not by $S \setminus S'$, is at most $2|S'|$. The bound follows by induction. \square

Proposition 54 *For the prism $C_m \square K_2$, it holds that $b_{K_{1,3}}(s) \leq 2(s-1)$ provided $s < 2m/3$. Further if m is odd, then $b_{K_{1,3}}(s) \leq 2s - 1$ for all s .*

Proof. Consider a set S of s vertices. If there are two consecutive K_2 -fibers without a vertex of S , then the result follows from Observation 53. Further, if there are three consecutive K_2 -fibers with only one vertex of S between them, say v , then we can remove vertex v , apply the above observation, and noting that v can contribute to the bi-cover of at most 2 copies, again obtain the result. So we may assume that every three consecutive K_2 -fibers contain at least two vertices of S ; in particular, $s \geq 2m/3$.

Now, note that since the graph is cubic, every vertex of S lies in exactly 4 copies of $K_{1,3}$. So it is immediate that $b_{K_{1,3}}(s) \leq 4s/2 = 2s$. So suppose that S bi-covers exactly $2s$

copies of $K_{1,3}$. Then by the calculation, it must be that S covers each of the $2s$ copies exactly twice, and covers no other copy at all.

In particular, consider a vertex v in S . Since v dominates itself, one of its neighbors must be in S , say w . There are two cases. If vw is a K_2 -fiber, then since neither v nor w is triple dominated, it follows that neither adjacent K_2 -fiber contains a vertex of S . But since both these fibers are dominated, it follows that in the next K_2 -fibers, both vertices are in S . By repeated application of this, it follows that every alternate fiber contains two vertices of S . This is only possible if m is even.

The second case is that vw lies within a C_m -fiber. Then by similar reasoning, no other vertex in $N(v) \cup N(w)$ is in S . But they dominate the other vertex of their K_2 -fibers. So in the two adjacent K_2 -fibers, the vertex not in $N(v) \cup N(w)$ is in S . Since that vertex is doubly dominated by S , it follows that its neighbor outside $N(v) \cup N(w)$ is in S . By repeated application of this, it follows that S consists of one vertex from each K_2 -fiber and that S induces a matching. This is only possible if m is a multiple of four. \square

Theorem 55 For $m \geq 3$,

$$NR_{K_{1,3}}(C_m \square K_2) = \begin{cases} m + 1, & \text{if } m \text{ is even or } m = 3, \\ m, & \text{if } m \text{ is odd and } m \geq 5. \end{cases}$$

Proof. For $C_3 \square K_2$, color two vertices in one triangle red and two vertices in another triangle green, and then give the other two vertices unique colors. For $C_m \square K_2$ when m is even, color every alternate K_2 -fiber red and then give the remaining m vertices unique colors. For $C_m \square K_2$ when m is odd, give each K_2 -fiber a different color.

It remains to prove the upper bound. The upper bound for $m = 3$ is straightforward; so assume $m > 3$.

We note first that if it were true that $b_{K_{1,3}}(s) \leq 2(s - 1)$, then an upper bound of m would follow from Proposition 34. Indeed, by the proof of that proposition, that bound

follows provided every color class bi-covers at most $2(c - 1)$ copies of $K_{1,3}$, where c is the number of times that color is used.

So assume some color, say red used c times, bi-covers more than $2(c - 1)$ copies of $K_{1,3}$. By Proposition 54, red is used at least $2m/3$ times. If there was another such color, then the total number of colors would be at most $2m - 2(2m/3) + 2 = 2m/3 + 2$, which is less than $m + 1$ (since $m > 3$), and the result follows. So we may assume that red is the only such color.

Assume first that m is even. Then by the argument in Proposition 54, it holds that red bi-covers at most $2c$ copies of $K_{1,3}$. By repeating the proof of Proposition 34, it follows that at most $m + 1$ colors are used, and so the result follows. If m is odd, then by Proposition 54, it holds that red bi-covers at most $2c - 1$ copies of $K_{1,3}$. By repeating the proof of Proposition 34, it follows that at most $m + 1/2$ colors are used, and so by integrality the result follows. \square

4.5.2.3 Maximal Outerplanar Graphs

The minimum value of $NR_{K_{1,3}}(G)$ for an outerplanar graph of order n is obtained by the fan (having value 3). For the maximum, we need to restrict to maximal outerplanar graphs. It is unclear what the maximum value is. Some data generated from a computer search is shown in Table 4.2:

order	3	4	5	6	7	8	9	10	11	12	13	14
max	3	3	3	4	5	5	6	6	7	7	8	9

Table 4.2: Maximum values of $NR_{K_{1,3}}(G)$ for maximal outerplanar graphs of fixed order

Figure 4.6 shows the unique graphs achieving the maximum for $n = 7$ and $n = 9$. The black vertices are all colored the same and each white vertex gets a unique color.

However, while this data suggests that the maximum is $n/2 + O(1)$, that is not correct. It is possible to construct maximal outerplanar graphs where $NR_{K_{1,3}} = 4n/7 + 1$.

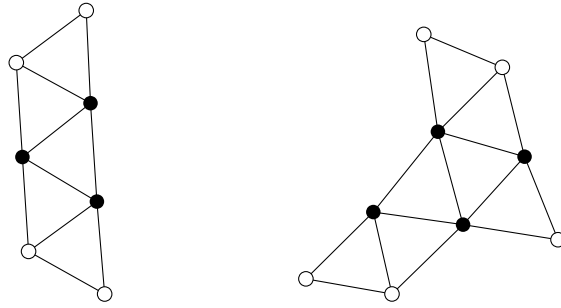


Figure 4.6: Maximal outerplanar graphs with maximum $NR_{K_{1,3}}$

Let $s \geq 2$. Start with a cycle \mathcal{C} of $3s$ vertices and partition the vertex set into copies of P_3 . For each copy abc , introduce vertices d, e, f , and g , and add edges $ad, ed, ae, be, bf, fg, cf$, and cg . Finally, add edges incident with the cycle \mathcal{C} to make it a triangulation. Let M_s denote the resultant graph. For example, M_3 is shown in Figure 4.7. The graph M_s can be colored by giving all the vertices on the cycle \mathcal{C} the same color, and all other vertices unique colors. This shows that $NR_{K_{1,3}}(M_s) \geq 4|M_s|/7 + 1$. We omit the details here but it can be verified that $NR_{K_{1,3}}(M_s) = 4|M_s|/7 + 1$.

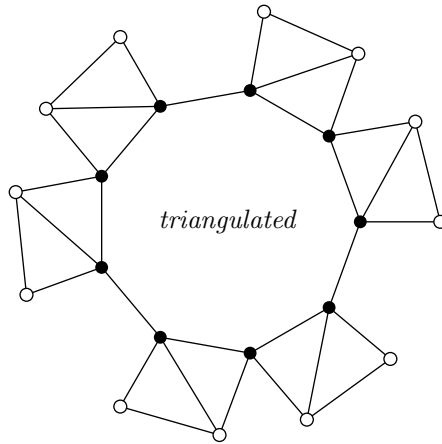


Figure 4.7: The maximal outerplanar graph M_3 with $NR_{K_{1,3}} = 13$

4.6 Conclusion

We have considered colorings without rainbow stars or cliques. Besides the specific open problems and conjectures presented here, a future direction of research would be colorings without other rainbow subgraphs, say trees, cycles, or bicliques. One avenue that looks interesting is coloring grids and other products while forbidding rainbow subgraphs.

Chapter 5

WORM Colorings Forbidding Paths

5.1 Introduction

This chapter is based on joint work with Wayne Goddard and Kirsti Wash [38]. As all proofs in the original paper are provided, we do not give specific references to that paper.

Let graphs F and G be given. Consider a coloring of the vertices of G . It is easy to avoid monochromatic subgraphs of G isomorphic to F : color every vertex in G a different color. It is also easy to avoid rainbow subgraphs of G isomorphic to F : color every vertex the same color. But things are more challenging if one tries to avoid both simultaneously. For example, if $G = K_5$, then any coloring of G yields either a monochromatic or a rainbow P_3 . On the other hand, if we color $G = K_4$ giving two vertices red and two vertices blue, we avoid both a rainbow and a monochromatic P_3 .

So we define an *F-WORM coloring of G* as a coloring of the vertices of G Without a Rainbow or Monochromatic subgraph isomorphic to F . We assume the graph F has at least 3 vertices, since any subgraph on 1 or 2 vertices is automatically rainbow or monochromatic. Also, if G is k -colorable with k less than the order of F (and F is nonempty), then

a proper k -coloring of G is an F -WORM coloring.

As is mentioned in Chapter 4, this coloring is a special case of colorings of mixed hypergraphs introduced by Voloshin; see for example [74, 75, 71]. Also note that the idea of forbidding a monochromatic subgraph or a rainbow subgraph has been extensively studied for edge-colorings; see for example [1, 2]. Similarly, there is much work on vertex colorings with various local constraints, especially avoiding monochromatic subgraphs; see for example [19, 58, 69].

In this chapter we establish some basic properties of WORM colorings. We also consider, for a graph that has such a coloring, what the range of colors is. To this end, we define $W^+(G, F)$ as the maximum number of colors and $W^-(G, F)$ as the minimum number of colors in an F -WORM coloring of graph G . We focus on the fundamental results and the case that F is the path P_3 . Some further results where a cycle or clique is forbidden are given in Chapter 6.

We proceed as follows: in Section 5.2 we show that if a graph has a P_3 -WORM coloring then it has one using two colors, from which it follows that the decision problem is NP-hard. In Sections 5.3 and 5.4 we consider the existence and range of P_3 -WORM colorings for several graph families including bipartite graphs, Cartesian products, cubic graphs, outerplanar graphs, and trees. Finally, we consider some related complexity results in Section 5.5 and an extremal question in Section 5.6.

5.2 Basics

In this section we consider the maximum and minimum number of colors in a P_3 -WORM coloring. In particular, we show that if a graph has a P_3 -WORM coloring then it has such a coloring with at most two colors. Note that we consider only connected graphs G , since if G is disconnected then the existence and range of colors for G is determined by the existence and range of colors for the components. For example, $W^+(G, P_3)$ is the sum of $W^+(G_i, P_3)$ over all components G_i of G .

In Chapter 4, we considered no-rainbow colorings. Note that for every graph F , a F -WORM coloring is also a no-rainbow- F coloring. So $W^+(G, F) \leq NR_F(G)$, provided that G has an F -WORM coloring. Therefore, from Theorem 37, it follows that:

Corollary 56 *If a graph G of order n has a P_3 -WORM coloring, then*

$$W^+(G, P_3) \leq \lfloor n/2 \rfloor + 1.$$

Also, from Theorem 38, it follows that:

Corollary 57 *If a connected graph G has a P_3 -WORM coloring, then*

$$W^+(G, P_3) \leq \beta(G) + 1.$$

We next consider $W^-(G, P_3)$:

Theorem 58 *A graph G has a P_3 -WORM coloring if and only if G has a P_3 -WORM coloring using only two colors.*

Proof. Consider a P_3 -WORM coloring of the graph G . Say an edge is monochromatic if its two ends have the same color. By the lack of monochromatic P_3 's, the monochromatic edges form a matching. Let H be the spanning subgraph of G with the monochromatic edges removed. Consider any edge uv in H ; say u is color i and v is color j . Then by the lack of rainbow P_3 's, every neighbor of u is color j and every neighbor of v is color i . It follows that H is bipartite. If we 2-color G by the bipartition of H , the monochromatic edges still form a matching, and so this is a P_3 -WORM coloring. \square

Recall that a 1-defective 2-coloring of a graph G is a 2-coloring such that each vertex has at most one neighbor of its color. It follows that:

A 1-defective 2-coloring is equivalent to a P_3 -WORM 2-coloring.

Cowen [22] proved that determining whether a graph has a 1-defective 2-coloring is NP-complete. It follows that determining whether a graph has a P_3 -WORM coloring is NP-complete.

It is not true, however, that if a graph has a P_3 -WORM coloring using k colors, then it has one using j colors for every $2 < j < k$. Indeed, we now construct a graph H_k that has a P_3 -WORM coloring using k colors and one using 2 colors, but for no other number of colors.

For $k \geq 3$ we construct graph H_k as follows: let $s = \max(3, k-2)$. For every ordered pair of distinct i and j , with $i, j \in \{1, \dots, k\}$, create disjoint sets B_i^j of s vertices. For each i , define $C_i = \bigcup_{j \neq i} B_i^j$. Then add all s^2 possible edges between sets B_i^j and B_j^i for all $i \neq j$. For each triple of distinct integers i, j, j' , add exactly one edge between B_i^j and $B_i^{j'}$ such that for each i the subgraph induced by C_i has maximum degree 1. One possibility for the graph H_4 is shown in Figure 5.1.

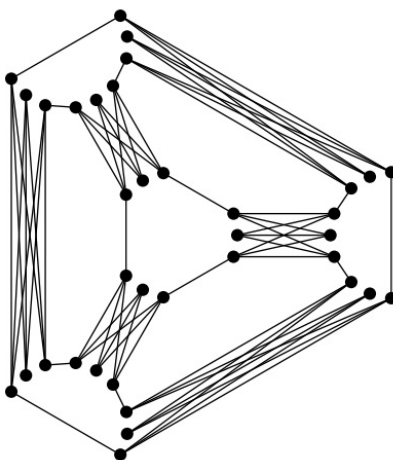


Figure 5.1: The graph H_4 whose P_3 -WORM colorings use either 2 or 4 colors

Observation 59 For $k \geq 3$, every P_3 -WORM coloring of H_k uses either 2 or k colors.

Proof. Consider a P_3 -WORM coloring of H_k . Note that the subgraph induced by $B_i^j \cup B_j^i$ is $K_{s,s}$. It is easy to show that for $s \geq 3$ the only P_3 -WORM coloring of $K_{s,s}$ is the

bipartition. It follows that for each i and j , all s vertices in B_i^j receive the same color; further, the color of B_i^j is different from the color of B_j^i . Because there is a P_3 that goes from B_j^i to B_i^j to $B_i^{j'}$, it must be that $B_i^{j'}$ receives either the color of B_i^j or the color of B_j^i . That is, there are precisely two colors on all of $C_i \cup C_j$.

So suppose some C_i receives two colors. Then every other C_j is colored with a subset of these two colors. Otherwise, assume every C_i is monochromatic. It follows that every C_i is a different color, and thus we use k colors. \square

5.3 Some Calculations

Now we consider P_3 -WORM colorings for some specific families of graphs.

5.3.1 Bipartite graphs

As observed earlier, the bipartite coloring of a bipartite graph is automatically a P_3 -WORM coloring. So we focus on the maximum number of colors a WORM-coloring may use. We observed above that every P_3 -WORM coloring of $K_{n,n}$ uses two colors. Indeed, we now observe the following slightly more general result:

Proposition 60 *For $n \geq m \geq 2$, $W^+(K_{n,n}, K_{1,m}) = 2m - 2$.*

Proof. One can achieve $2m - 2$ colors by using disjoint sets of $m - 1$ different colors on each partite set. So we need to prove the upper bound.

Let A and B denote the partite sets of $K_{n,n}$ and consider a $K_{1,m}$ -WORM coloring. Suppose that one partite set, say A , receives at least m different colors. Picking m vertices in A with different colors, it follows that every vertex v in B must be one of these m colors. Furthermore, v cannot see m distinct colors different from it. That is, the coloring uses exactly m colors. On the other hand, if every partite set has at most $m - 1$ colors, the total

number of colors is at most $2(m-1)$. It follows that $W^+(K_{n,n}, K_{1,m}) \leq \max(m, 2m-2) = 2m-2$. \square

5.3.2 Cartesian products

We next consider a P_3 -WORM coloring of Cartesian products $G \square H$.

Theorem 61 *If G and H are nontrivial connected graphs and $G \square H$ has a P_3 -WORM coloring, then it uses only two colors.*

Proof. It suffices to prove the result when G and H are trees. We proceed by induction. Clearly when $G = H = K_2$, we have $W^+(C_4, P_3) = 2$.

So assume that at least one of the factors, say G , has order at least 3. Let u be a leaf of G , with neighbor u' , and let $G' = G - \{u\}$. By the inductive hypothesis, every P_3 -WORM coloring of $G' \square H$ uses only two colors. Consider any vertex v of H . Since G is not K_2 , vertex u' has at least one neighbor in G' , and so vertex (u', v) is the center of a P_3 in $G' \square H$. This means that the vertex (u', v) has a neighbor x of a different color in $G' \square H$, and thus (u, v) must get either the color of (u', v) or x . \square

5.3.3 Cubic graphs

Let G be a connected cubic graph. We know from [58] that G has a 2-coloring where every vertex has at most one neighbor of the same color. This coloring is a P_3 -WORM. So the natural question is: What is the minimum and maximum value of $W^+(G, P_3)$ for connected cubic graphs G of the order n ?

There are many cubic graphs G that have $W^+(G, P_3) = 2$. One general family is the ladder $C_m \square K_2$. (See Theorem 61.)

A computer search of small cases suggests the following:

Conjecture 6 For every cubic graph G of order n , $W^+(G, P_3) \leq n/4 + 1$.

This upper bound is achieved by several graphs including the following graph: for $s \geq 2$, create B_s by taking s copies of $K_4 - e$ and adding edges to make the graph cubic and connected. For example, B_5 is illustrated in Figure 5.2.

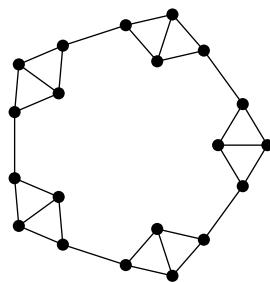


Figure 5.2: A cubic graph of order 20 with $W^+(G, P_3) = 6$

Proposition 62 For $s \geq 2$, $W^+(B_s, P_3) = s + 1$.

Proof. Consider a P_3 -WORM coloring of B_s . It is easy to show that each copy of $K_4 - e$ has exactly two colors, and one of those colors is also present at the end of each edge leading out of the copy. Thus $s + 1$ is an upper bound. An optimal coloring is obtained by coloring each central pair from a $K_4 - e$ with a new color, and coloring all other vertices the same color. \square

Another interesting case is where the forbidden graph is the star on three edges. Bujtás et al. [9] showed that for every r -regular graph G of order n , $NR_{K_{1,r}}(G) \leq rn/(r+1)$. Hence, $W^+(G, K_{1,3}) \leq 3n/4$ for cubic graphs G of order n . This upper bound is achieved uniquely by the above graph B_s .

5.3.4 Maximal Outerplanar graphs

In this section, we characterize the maximal outerplanar graphs (*MOPs*) that have a P_3 -WORM coloring. But first we note that if a maximal outerplanar graph G has a P_3 -WORM coloring, then $W^+(G, P_3) = W^-(G, P_3) = 2$.

Observation 63 *If a MOP has a P_3 -WORM coloring, then that coloring uses two colors.*

Proof. Consider some triangle $T_0 = \{x, y, z\}$. It must have exactly two colors; say red and blue. If this is the whole graph we are done. Otherwise, there is another triangle T_1 that overlaps T_0 in two vertices. Say T_1 has vertices $\{x, y, w\}$. Then if x and y have different colors, w must be one of their colors. Further, if x and y are both red say, since zxw is a P_3 it must be that w is the same color as z . That is, all vertices of T_1 are red or blue. Repeating the argument we see that all vertices in the graph are red or blue. \square

We now consider the necessary conditions for a MOP to have a P_3 -WORM coloring. Note that by Theorem 58, this is equivalent to determining which MOPs have a 1-defective 2-coloring. Let F_6 denote the fan given by the join $K_1 \vee P_6$. This graph and the Hajós graph (also known as a 3-sun) are shown in Figure 5.3.

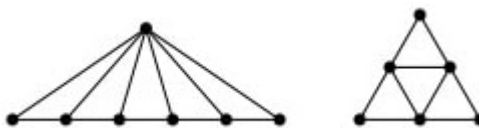


Figure 5.3: Two MOPs: the fan F_6 and the Hajós graph

Proposition 64 *Neither the fan F_6 nor the Hajós graph has a P_3 -WORM coloring.*

Proof. Consider a 2-coloring of the fan F_6 . Let v be the central vertex; say v is colored red. Then at most one other vertex can be colored red. It follows that there must be 3 consecutive non-red vertices on the path. Thus, the coloring is not WORM.

Consider a 2-coloring of the Hajós graph. It is immediate that two of the central vertices must be one color, say red, and the other central vertex the other color, say blue. Now let u and v be the two vertices of degree 2 that have a blue neighbor. Then, coloring either of them red creates a red P_3 , but coloring both of them blue creates a blue P_3 . \square

So it is necessary that the MOP has maximum degree at most 5 and contains no copy of the Hajós graph. For example, one such MOP is drawn in Figure 5.4. (The vertices of degree 5 are in white.)

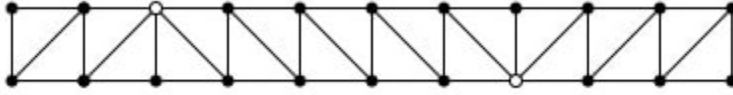


Figure 5.4: A MOP that has no F_6 or Hajós subgraph

The *interior graph* of a MOP G , denoted by C_G , is the subgraph of G induced by the chords. (This is well-defined as a MOP has a unique Hamiltonian cycle.)

Proposition 65 *A MOP G contains a copy of Hajós graph if and only if the interior graph C_G has a cycle.*

Proof. If G contains a copy of Hajós graph, then C_G contains a triangle. Conversely, assume C_G contains a cycle; then it must contain a triangle. Since every chord of G is contained in two adjacent triangles, it follows that G contains a copy of the Hajós graph. \square

A *caterpillar* is a tree in which every vertex is within distance one of a path; that path is called a *central path*. Hedetniemi et al. [46] showed that if the interior graph of a MOP is acyclic, then it is a caterpillar. Let V_5 denote the set of vertices of degree 5 in a MOP G . Equivalently, V_5 is the vertices of degree 3 in C_G . Define a *stem* as a path in C_G whose ends are in V_5 and whose interior vertices are not.

Theorem 66 *A MOP G has a 1-defective 2-coloring (equivalently a P_3 -WORM coloring) if and only if*

- (a) G has maximum degree at most 5,
- (b) the interior graph C_G is a caterpillar, and
- (c) every stem of C_G has odd length.

Proof. We first prove necessity. Let G be a MOP with a P_3 -WORM coloring. By Propositions 64 and 65, G has maximum degree at most 5 and C_G is a caterpillar. Let P denote a central path of the caterpillar C_G . We are done unless C_G has a stem; so assume $P_{u,v}$ is a stem with ends u and v .

Let the path through $N(u)$ be $u_1u_2u_3u_4u_5$. Note that u_1 and u_5 are neighbors of u on the outer cycle. Further, by the lack of Hajós subgraph, the edge u_2u_3 is not in C_G ; that is, u_2 and u_3 are consecutive on the outer cycle. Similarly, so are u_3 and u_4 . It follows that u_3 has degree exactly 3 in G , and so u_3 is a leaf in C_G .

Now consider the P_3 -WORM coloring of G . By Observation 63, this coloring uses two colors, say 1 and 2. It is easy to see that u must have the same color as u_3 , while u_1, u_2, u_4, u_5 have the other color. In particular, u has no neighbor on P of the same color. Similarly, v has no neighbor on P of the same color. Further, since the subgraph of G induced by the vertices of $P_{u,v}$ is its square, all other vertices do have neighbors on P of the same color. Indeed, assume u has color 1; then the coloring pattern of $P_{u,v}$ must be either $1, 2, 2, 1, 1, \dots, 2, 2, 1$ or $1, 2, 2, 1, 1, \dots, 1, 1, 2$. Hence, the stem $P_{u,v}$ must have odd length.

Now we prove sufficiency. Assume G has maximum degree at most 5, and the interior graph C_G is a caterpillar with every stem of C_G having odd length. We color P with two colors such that every vertex of V_5 has no neighbor on P of the same color and all other vertices (except possibly the end-vertices of P) do have neighbors on P of the same color. Then we give each vertex of $C_G - P$ the same color as its neighbor in C_G .

It remains to color the (at most two) vertices of degree 2 in G . Let x be a vertex of degree 2 in G , and let y and z be its neighbors. Let t be the other vertex with which y and z forms a triangle. Say t is adjacent to z on the outer cycle. By the construction of the coloring so far, it follows that either y has the same color as t , in which case we can give x the same color as z , or y has the same color as z , in which case we can give x the other color. Thus we can extend the coloring to the whole graph, as required. \square

5.4 Trees

The following observation will facilitate bounds for $W^+(T, P_3)$ when T is a tree.

Observation 67 *Consider a tree T . A P_3 -WORM coloring of some of the vertices, such that the colored vertices induce a connected subgraph, can be extended to a P_3 -WORM coloring of the whole tree.*

Proof. Assume we have a P_3 -WORM coloring of $U \subseteq V(T)$ such that U induces a connected subgraph of T . Consider any uncolored vertex v that is adjacent to some colored vertex w_v (since T is a tree, w_v is unique). If w_v sees a color c different from its own color, then assign v the color c . If w_v has no colored neighbor or its only colored neighbor has the same color as it, then give v any other color. In both cases we do not create a monochromatic or rainbow P_3 . Repeat until all vertices colored. \square

We consider next a tree algorithm. There are general results (see for example [68]) that show that there is a linear-time algorithm to compute the parameter $W^+(T, P_3)$ for a tree T , and indeed for bounded treewidth. Nevertheless, we give the details of an algorithm below, and then use it to calculate the value of $W^+(T, P_3)$ for a spider (sometimes called an octopus). We do the standard postorder traversal algorithm. That is, we root the tree at some vertex r and then calculate a vector at each vertex representing the values of several parameters on the subtree rooted at that vertex.

For vertex v , define T_v to be the subtree rooted at v and $k(v)$ to be the number of children of v . Define $p(v)$ to be the maximum number of colors in a P_3 -WORM coloring of T_v with the constraint that v has a child of the same color (“partnered”); and define $s(v)$ to be the maximum number of colors in a P_3 -WORM coloring of T_v with the constraint that v has no child of the same color (“solitary”). By Observation 67, such a coloring exists (that is, $p(v)$ and $s(v)$ are defined) except for the case of $p(v)$ when $k(v) = 0$.

Define $\ell_p(v) = 2$ if $k(v) \geq 2$ and 1 otherwise; define $\ell_s(v) = 2$ if $k(v) \geq 1$, and 1 otherwise. Note that $\ell_p(v)$ and $\ell_s(v)$ denote the number of colors in $N[v]$ in a partnered and solitary coloring of T_v respectively. Let $P(v) = p(v) - \ell_p(v)$ and $S(v) = s(v) - \ell_s(v)$.

Theorem 68 *If vertex v has children c_1, \dots, c_k , $k \geq 1$, then*

$$p(v) = \begin{cases} \max_{1 \leq i \leq k} \left\{ 1 + s(c_i) + \sum_{j \neq i} \max(P(c_j), S(c_j)) \right\}, & \text{if } k \geq 2, \\ s(c_1), & \text{otherwise;} \end{cases}$$

$$s(v) = 2 + \sum_{i=1}^k \max(P(c_i), S(c_i)).$$

Proof. Consider a P_3 -WORM coloring of T_v . Say v has children c_1, \dots, c_k . To maximize the colors, the color-set used in T_{c_i} should be as disjoint as possible from the color-set used in T_{c_j} . But note that there has to be some overlap.

Specifically, if v is solitary, then all its children have the same color. In the tree T_{c_i} , any child of c_i has the same color as either c_i or v . So the maximum number of colors that appear only in $T_{c_i} - \{c_i\}$ is $\max(P(c_i), S(c_i))$. Further, if v is partnered, say with c_i , then there are $s(c_i)$ colors in the subtree T_{c_i} . There is 1 color for all other children c_j of v . As above, the maximum number of colors that appear only in $T_{c_j} - \{c_j\}$ is $\max(P(c_j), S(c_j))$.
□

Since these maxima can be computed in time proportional to $k(v)$, and $W^+(T, P_3) = \max(p(r), s(r))$, we obtain a linear-time algorithm to calculate $W^+(T, P_3)$ for a tree T .

As an application, we determine the value of $W^+(T, P_3)$ for an octopus:

Theorem 69 *Let X be a star with $k \geq 2$ leaves, and let T be the subdivision of X where the i^{th} edge of X is subdivided $a_i \geq 0$ times for $1 \leq i \leq k$. Then*

$$W^+(T, P_3) = 2 + \sum_{i=1}^k \left\lceil \frac{a_i - 1}{2} \right\rceil + x,$$

where x is 1 if at least one a_i is odd and 0 otherwise.

Proof. This follows from Theorem 68 by considering the children c_1, \dots, c_k of the original center. It is easy to check that $P(c_i) = \lceil (a_i - 1)/2 \rceil$ and $S(c_i) = \lceil (a_i - 2)/2 \rceil$. □

5.5 WORM is Easy Sometimes

We observed earlier that determining whether a graph has a P_3 -WORM coloring is NP-hard. There are at least a few cases of forbidden graphs where the problem has a polynomial-time algorithm. The first case is trivial:

Observation 70 *For $F = mK_1$, graph G has an F -WORM-coloring if and only if G has at most $(m - 1)^2$ vertices.*

Here is another forbidden graph with a characterization:

Observation 71 *Let F be the one-edge graph on three vertices. A graph G has an F -WORM coloring if and only if G is (a) bipartite, (b) a subgraph of the join $K_2 \vee mK_1$ for some $m \geq 1$, or (c) K_4 .*

Proof. Clearly a graph with order at most 4 has an F -WORM coloring using only 2 colors. We already know that a proper 2-coloring of a bipartite graph is an F -WORM coloring. If G is a subgraph of $K_2 \vee mK_1$, color the vertices of the K_2 with one color and the other vertices a second color. This gives an F -WORM coloring.

Conversely, let G be a graph with an F -WORM coloring and suppose G is not bipartite. Then consider any vertex u with at least two neighbors. Since there is no monochromatic F , it must be that u has a different color to at least one of its neighbors, say v . Since there is no rainbow F , it follows that every other vertex has the same color as either u or v .

But since G is not bipartite, this means there exists an edge xy where x and y have the same color, say color 1. By the lack of monochromatic F it follows that every other vertex has the other color, say 2. If the vertices of color 2 form an independent set, then we have a subgraph of the join $K_2 \vee mK_1$ for some $m > 0$. Otherwise, by the same reasoning there are exactly two vertices of color 2 and we have a subgraph of K_4 . \square

5.6 Extremal Questions

The classical Turán problem asks for the maximum number of edges of a graph with n vertices that does not contain some given subgraph H . This maximum is called the *Turán number* of H . Here we consider an analogue of the classical Turán problem: what is the maximum number of edges of a graph with n vertices if the graph admits a F -WORM coloring? We will let $wex(n, F)$ denote this maximum. We have the following result when $F = P_3$.

Theorem 72 For $n \geq 1$,

$$wex(n, P_3) = \begin{cases} \frac{n(n+2)}{4} & \text{if } n \text{ is a multiple of } 4 \\ \frac{n^2+2n-4}{4} & \text{if } n \equiv 2 \pmod{4} \\ \frac{(n-1)(n+3)}{4} & \text{otherwise.} \end{cases}$$

Proof. Consider a graph G that has a P_3 -WORM coloring. By Theorem 58, there is such a coloring using only two colors, say red and blue. Such a coloring is a P_3 -WORM coloring if and only if there is no monochromatic P_3 . It follows that the maximum number of edges in G is obtained by taking some complete bipartite graph and adding a maximum matching within each partite set. Without loss of generality, let a and $n - a$ be the sizes of the two partite sets. Then the total number of edges in the resultant graph is $\lfloor a/2 \rfloor + \lfloor (n - a)/2 \rfloor + a(n - a)$.

If n is a multiple of 4, then the number of edges is maximized when $a = n/2$. If n is odd, then the number of edges is maximized when $a = (n - 1)/2$ or $a = (n + 1)/2$. If n is even but not a multiple of 4, then the number of edges is maximized when $a = n/2 - 1$, or $a = n/2$, or $a = n/2 + 1$. The results follows. \square

Chapter 6

WORM colorings Forbidding Cycles or Cliques

6.1 Introduction

This chapter is mainly based on joint work with Wayne Goddard and Kirsti Wash [37]. As all proofs in the original paper are provided, we do not give specific references to that paper.

In Chapter 5 we considered WORM colorings forbidding a path. Here we considered WORM colorings forbidding a cycle or a clique. Specifically, in Section 6.2 we calculate or bound $W^+(G, K_3)$ and $W^-(G, K_3)$ for some families of graphs G . We also show that the question of determining if a graph has a K_3 -WORM coloring is NP-complete. In Section 6.3 we consider similar questions where C_4 and other cycles are forbidden, while in Section 6.4 a clique or biclique is forbidden. In Section 6.5 we consider the question of coloring a graph to minimize the total number of monochromatic and rainbow subgraphs.

6.2 Forbidding a Triangle

We consider here the case that the forbidden graph F is the triangle. As noted before, the largest complete graph that has a K_3 -WORM coloring is K_4 . Of course, if graph G does not have a triangle, then one can color every vertex the same color or every vertex a different color and have a WORM coloring. Thus the interesting graphs are those with clique numbers 3 and 4. We focus our attention on two classes of graphs: (maximal) outerplanar graphs and cubic graphs. We also show that, as expected, determining whether a graph has a K_3 -WORM coloring is NP-complete.

6.2.1 Outerplanar graphs

The *vertex arboricity* of graph G is the minimum number of subsets into which the set of vertices of G can be partitioned so that each subset induces a forest. Every outerplanar graph has vertex arboricity at most 2 (see [16]); that is, one can 2-color the vertices without any monochromatic cycle. It follows that outerplanar graph has a K_3 -WORM coloring with 2 colors. Thus the main question is about W^+ in outerplanar graphs.

Note that the value of $W^+(G, K_3)$ is not the same as the maximum number of colors without a rainbow K_3 . For example, consider the Hajós graph H . Then $W^+(G, H) = 3$ (see below), but one can use four colors to avoid a rainbow K_3 by coloring all the vertices of the central triangle the same color. We focus on maximal outerplanar graphs (MOPs).

Theorem 73 *For a MOP G , it holds that $W^+(G, K_3) = m(D(G)) + 2$ where $D(G)$ is the weak dual of G and $m(H)$ is the matching number of graph H .*

Proof. Consider a MOP G and a k -coloring of its vertices. Let F be the set of monochromatic edges. Note that the coloring is a WORM-coloring if and only if every triangle of G has precisely one edge in F . Let H be the spanning subgraph with edge set F .

Since G is chordal, if H contains a cycle then it contains a triangle. So H is a forest. Furthermore, since k is at most the number of components of H , we have $k \leq n - h$, where n is the number of vertices in G and h is the number of edges in H .

Now, let M be the edges of H that are chords in G . Each edge in M corresponds to an edge in the weak dual $D(G)$. Since H contains exactly one edge from each triangle, this means that the edges of M correspond to a matching M' in $D(G)$. The edges of H not in M correspond to unsaturated vertices in $D(G)$. Since $D(G)$ has $n - 2$ vertices, we have $h = (n - 2) - |M'|$. And thus $k \leq n - ((n - 2) - |M'|) = |M'| + 2 \leq m(D(G)) + 2$. This proves the upper bound.

Next we prove the lower bound. Note that $D(G)$ is a tree. It is known that every tree with at least one edge has a maximum matching that saturates all its non-leaf vertices (see, for example, [56]). So consider a maximum matching M' of $D(G)$ that saturates all its non-leaf vertices. That M' corresponds to a set M of chords of G . Each vertex of $D(G)$ not incident with M' corresponds to a triangle in G that has two sides on the outer face. So, for each such triangle, we can choose one of the edges on the outer face, and add to M to form a set of edges F in G that contain exactly one edge from each triangle.

Take the spanning subgraph with edge-set F and give every component a different color. Then we have a WORM-coloring with, by the same arithmetic as above, $|M'| + 2$ colors. □

It follows that the maximum value of $W^+(G, K_3)$, taken over all MOPs G of order n , is $\lfloor n/2 \rfloor + 1$. This is attained by multiple graphs, including the fan (obtained from a path by adding one vertex adjacent to every vertex on the path) and the MOPs of maximum diameter (obtained from an n -cycle by first adding a noncrossing matching of $\lfloor n/2 \rfloor - 1$ chords and then adding more chords to make the result into a MOP). The MOPs with the minimum value of $W^+(G, K_3)$ for their order are those whose duals are the trees of maximum degree 3 with minimum matching number.

6.2.2 Cubic graphs

We now consider 3-regular graphs. Lovász [58] showed that a cubic graph G has a 2-coloring where every vertex has at most one neighbor of the same color. Thus, $W^-(G, K_3)$ is 2 if G has a triangle, and 1 otherwise.

So, as in outerplanar graphs, the focus is on $W^+(G, K_3)$. Although we do not have a general formula, we can determine the extremal values. The maximum value for a given order is trivial: $W^+(G, K_3)$ is the order of G if the graph is triangle-free. The following result determines the minimum value for a given order:

Theorem 74 *For every connected cubic graph G of order $n \geq 6$, $W^+(G, K_3) \geq 2n/3$.*

Proof. Let H be the spanning subgraph of G whose edges are those edges of G that lie in a triangle. Then H is a union of disjoint copies of K_3 and $K_4 - e$ (and isolated vertices). It follows that there is a matching M of cardinality at most $n/3$ whose removal destroys all triangles in G . For each edge e in M , create one color c_e and color both ends of e with color c_e . Give every other vertex in G a distinct color. This gives a K_3 -WORM coloring that uses at least $2n/3$ colors. \square

There is equality in the above bound if and only if the graph G has a 2-factor (a spanning 2-regular graph) consisting of triangles.

6.2.3 Cartesian products

Since a K_3 in a Cartesian product has to lie completely within one of the fibers, it is not surprising that the product $G \square H$ has a K_3 -WORM coloring if and only if both graphs G and H have K_3 -WORM colorings. Indeed, a coloring of the product $G \square H$ can be produced in the standard way of combining colors. Thus:

Observation 75 *Assume graphs G and H have K_3 -WORM colorings. Then*

$$(a) W^-(G \square H, K_3) = \max\{W^-(G, K_3), W^-(H, K_3)\}.$$

$$(b) W^+(G \square H, K_3) \geq W^+(G, K_3) \times W^+(H, K_3).$$

Proof. For the upper bound on $W^-(G \square H)$, take the minimum colorings of graphs G and H and think of them as integers in the range 1 to $W^-(G)$ and 1 to $W^-(H)$. Then color vertex (g, h) of $G \square H$ by the sum of the colors of g and h , arithmetic modulo $\max\{W^-(G), W^-(H)\}$.

For the lower bound on $W^+(G \square H)$, take the maximum colorings of graphs G and H and color vertex (g, h) with the ordered pair of colors. \square

6.2.4 Complexity

We show next that determining whether a graph has a K_3 -WORM coloring is hard.

We will need the following gadget: let G_7 be the graph given by the join of K_2 and C_5 . The graph G_7 has a K_3 -WORM coloring: color the two dominating vertices red and the other five vertices blue. In fact, this coloring is unique:

Observation 76 *In any K_3 -WORM coloring of G_7 , the two dominating vertices receive the same color.*

Proof. Suppose the two dominating vertices received different colors, say red and blue. Then by the lack of rainbow triangles, every vertex on the 5-cycle must be red or blue. But then there must be two consecutive vertices with the same color, which yields a monochromatic triangle, a contradiction. \square

Theorem 77 *Determining whether a graph G has a K_3 -WORM coloring is NP-complete.*

Proof. We reduce from NAE-3SAT (not all equal 3SAT) (see [35]). Given a boolean formula in conjunctive normal form with three literals per clause, the NAE-3SAT problem is whether there is a truth assignment with at least one true literal and one false literal for each clause.

Given a boolean formula ϕ , we build a graph G_ϕ as follows: start with two master vertices M_1 and M_2 joined by an edge. For each variable x , create two vertices labeled x and \bar{x} joined by an edge and join both to both M_1 and M_2 . Then, pick one variable, say x_1 , and add a C_5 all vertices of which are joined to x_1 and to M_1 . Now, for each clause c , create a triangle T_c of three vertices. For each vertex of each clause triangle, join it to its corresponding literal and add a 5-cycle adjacent to both those vertices. An example is illustrated in Figure 6.1.

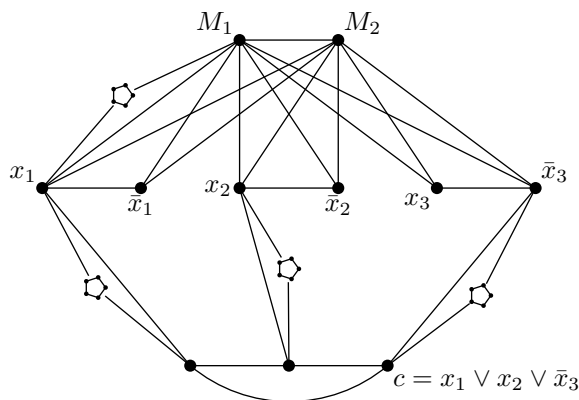


Figure 6.1: Reduction of K_3 -WORM coloring from NAE-3SAT

Now, we claim that G_ϕ has a K_3 -WORM coloring if and only if ϕ has an NAE assignment. So assume that G_ϕ has such a coloring. The four vertices $x_1, \bar{x}_1, M_1,$ and M_2 form a K_4 . So two of these vertices are one color, say red, and two are another color, say blue. By Observation 76, x_1 and M_1 have the same color. So it must be that M_1 and M_2 have different colors. Further, for every variable x we have that $\{x, \bar{x}, M_1, M_2\}$ forms a K_4 . That is, for every pair x and \bar{x} , exactly one is red and one is blue.

By Observation 76, it follows that the vertices in the clause triangle are colored by the same colors as the constituent literals. To avoid monochromatic triangles, it must be that not all the literals are equal. That is, we have an NAE assignment.

Conversely, given an NAE assignment for ϕ , we can color G_ϕ with colors red and blue as follows. Color literal vertices red if they are true and blue if they are false. Color M_1 the same color as x_1 and M_2 the other color. Color the C_5 's the opposite color of their

neighbors.

This shows that we have reduced NAE-3SAT to the K_3 -WORM coloring problem, as required. \square

6.2.5 $W^-(G, K_3)$ can be arbitrarily large

In Chapter 5 we showed that if a graph has a P_3 -WORM coloring, then it has such a coloring using at most 2 colors. We originally conjectured that a similar result holds for K_3 -WORM coloring. In [37], we proposed the following conjecture:

Conjecture 7 *If a graph has a K_3 -WORM coloring, then it has a K_3 -WORM coloring using (at most) 2 colors.*

It turns out that this conjecture is false, as is shown by Bujtás and Tuza [11]. Indeed, they showed the following:

Theorem 78 [11] *for every $k \geq 3$ there exists a graph F_k such that $W^-(F_k, K_3) = k$.*

They also showed that the gap in the chromatic spectrum can be arbitrarily large:

Theorem 79 [11] *For every $k \geq 4$, there exists a graph H_k such that $W^-(H_k, K_3) = 2$ and $W^+(H_k, K_3) \geq k$, and H_k does not have a K_3 -WORM coloring with j colors for all $3 \leq j \leq k - 1$.*

Interested readers can refer to [11] for the proofs of the above two theorems and for other interesting results and questions.

6.3 Forbidding a 4-Cycle or All Cycles

In this section we consider WORM colorings where a cycle is forbidden. Specifically, we consider C_4 as a forbidden subgraph, as well as extending the notion of WORM colorings to forbid all cycles.

For a set \mathcal{F} of graphs, we define an \mathcal{F} -WORM coloring as one with no rainbow nor monochromatic copy of any graph in \mathcal{F} . We define $W^+(G, \mathcal{F})$ as the maximum number of colors and $W^-(G, \mathcal{F})$ as the minimum number of colors in an \mathcal{F} -WORM coloring of graph G . Let \mathcal{C} denote the set of all cycles.

A set of vertices S is said to be a *decycling set* of a graph G if $G - S$ is acyclic. The *decycling number* of G , written $\nabla(G)$, is the smallest size of a decycling set of G . (See [5] for a discussion of the decycling number.)

Proposition 80 *For a graph G of order n , $W^+(G, \mathcal{C}) \leq n - \nabla(G)$.*

Proof. Suppose we have a \mathcal{C} -WORM coloring of G with k colors. If we keep one vertex of each color and delete the other vertices from G , then what remains must be an acyclic graph. It follows that $n - k \geq \nabla(G)$, that is, $k \leq n - \nabla(G)$. \square

6.3.1 Outerplanar and Cubic Graphs

As is mentioned earlier, every outerplanar graph G has vertex arboricity at most 2. That partition shows that G has a \mathcal{C} -WORM coloring. The next result shows that $W^+(G, \mathcal{C})$ for such a graph is at least one more than the diameter of G .

Theorem 81 *For a connected outerplanar graph G , we have $W^+(G, \mathcal{C}) \geq \text{diam}(G) + 1$.*

Proof. Let G be a connected outerplanar graph and v_0 be a vertex with maximum eccentricity. For $i \geq 0$, let V_i be the set of vertices that are at distance i from v_0 . Define a coloring c by giving color i to vertices in V_i . Clearly this coloring uses $\text{diam}(G) + 1$ colors.

It is known that in an outerplanar graph the set of vertices at distance i from a vertex v_0 form a linear forest. (A cycle in V_i would yield a subdivision of K_4 in G ; see, for example, [59].) So this coloring c has no monochromatic cycle.

Further, a rainbow subgraph has at most one vertex from each V_i . Since there is no edge between V_i and V_j for $|i - j| \geq 2$, it follows that all rainbow subgraphs are paths. In particular there is no rainbow cycle. \square

By Theorem 73, we have equality in the above theorem for MOPs of maximum diameter (that is, MOPs of diameter $\lfloor n/2 \rfloor$). However, the bound does not appear sharp if one forbids only a specific cycle, such as C_4 . For example, computer search shows that $W^+(G, C_4) = 7$ for every MOP G of order 9.

Similarly, since cubic graphs have vertex arboricity 2 (or by Lovász' result [58]), it follows that $W^-(G, \mathcal{C}) \leq 2$ for a cubic graph G . Here is one calculation for $W^+(G, \mathcal{C})$:

Observation 82 *If G is the Petersen graph, then $W^+(G, \mathcal{C}) = 7$.*

Proof. Since the order of the Petersen graph is 10, and the decycling number of the Petersen graph is 3 (see, for example, [4]), it follows from Proposition 80 that $W^+(G, \mathcal{C}) \leq 7$.

For the lower bound, take a maximum independent set I (which has size 4) and color all of its vertices red while giving all other vertices distinct colors. Since this is a proper coloring, there is no monochromatic cycle. Also there is no rainbow cycle, since every cycle in G intersects I in at least two vertices. \square

We consider next the question for cubic graphs forbidding only the 4-cycle. The maximum value of $W^+(G, C_4)$ for cubic graphs G on n vertices is n . This value is achieved by C_4 -free cubic graphs. For the minimum value of $W^+(G, C_4)$ for cubic graphs G of order n , we have the following conjecture:

Conjecture 8 *For every cubic graph G of order n , $W^+(G, C_4) \geq 2n/3$.*

This value is achieved by the following graph: take r disjoint copies of $K_{3,3} - e$ and add edges so that the resultant graph B_r is cubic and connected. For example, B_2 is drawn in Figure 6.2. The graph B_r has $6r$ vertices and one can calculate that $W^+(B_r, C_4) = 4r$. A

computer search shows that B_2 is indeed the unique (connected) cubic graph G of order 12 with minimum $W^+(G, C_4)$.

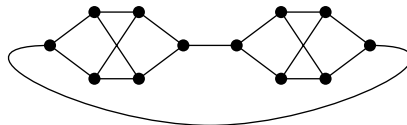


Figure 6.2: The graph having minimum $W^+(G, C_4)$ over cubic graphs of order 12

6.3.2 Cartesian Products

For the Cartesian product $G \square H$ to have a C_4 -WORM coloring, it is of course necessary that both G and H have a C_4 -WORM coloring. However, we have not been able to show that this condition is sufficient, nor could we find an example to the contrary.

We first consider grids. Let $G_{m,n}$ denote the grid formed by the cartesian product of P_m and P_n . We need the observation below. Recall that a set S bi-covers a subgraph H if at least two vertices of H are in S . For positive integer s , $b_F(s)$ is the maximum number of copies of F that can be bi-covered by using a set of size s .

Observation 83 *For any grid and $s > 0$, $b_{C_4}(s) \leq 2(s - 1)$.*

Proof. We prove this bound by induction. Let S be a set of s vertices. The bound is immediate when S is contained in only one row. Now suppose S intersects at least two rows. Let S_1 be a maximal set of consecutive vertices of S in the topmost row of S . By the induction hypothesis, the number of C_4 's that contain at least two vertices in $S \setminus S_1$ is at most $2(|S| - |S_1|) - 2$. Further, the number of C_4 's that contain at least one vertex in S_1 and at least two vertices in S is at most $2|S_1|$: there are $|S_1| - 1$ possible copies above S_1 and at most $|S_1| + 1$ copies below. Hence, the number of C_4 's that S bi-covers is at most $2(|S| - |S_1|) - 2 + 2|S_1| = 2|S| - 2$. \square

For grid graphs we have the following theorem. It was stated as a conjecture in [37], and the lower bound was also proved there. The upper bound was proved in [40].

Theorem 84 (see [37, 40]) For G the $m \times n$ grid,

$$W^+(G, C_4) = \left\lfloor \frac{(m+1)(n+1)}{2} \right\rfloor - 1.$$

Proof. We first prove the lower bound by constructing a suitable coloring. Think of the grid as having m rows numbered 1 up to m , and n columns numbered 1 up to n . There are two cases.

Consider first the case that at least one of m or n is odd, say m . Then in each odd-numbered row, give every vertex a unique color. For even-numbered rows, give every vertex the same color. This coloring has no monochromatic cycle, no rainbow cycle, and uses $n(m+1)/2 + (m-1)/2 = (m+1)(n+1)/2 - 1$ colors.

Consider second the case that both m and n are even. Then color all but the last two rows as before: in each odd-numbered row give every vertex a unique color, and in even-numbered rows give every vertex the same color. For the last two rows, do the same coloring but based on columns: that is, in each odd-numbered column give both vertices a unique color, and in even-numbered columns give both vertices the same color. The coloring for $m = n = 6$ with 23 colors is illustrated in Figure 6.3, where \bullet means that the vertex receives a distinct color.

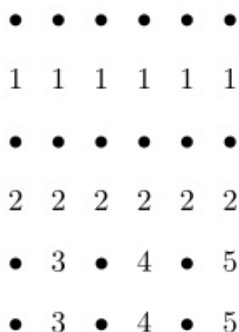


Figure 6.3: An optimal C_4 -WORM coloring of the 6×6 grid

This coloring has no monochromatic cycle, no rainbow cycle, and uses $n(m-2)/2 +$

$$(m-2)/2 + 3(n/2) = (m+1)(n+1)/2 - 3/2 \text{ colors.}$$

The upper bound follows from Proposition 34 and Observation 83: in the $m \times n$ grid, there are $(m-1)(n-1)$ copies of C_4 , and so $W^+(G; C_4) \leq mn - (m-1)(n-1)/2 = (m+1)(n+1)/2 - 1$. \square

We next consider rooks graphs.

Observation 85 *The largest rooks graph that has a WORM C_4 -coloring is $K_9 \square K_9$.*

Proof. Note that any coloring of K_{10} must contain either a monochromatic C_4 or a rainbow C_4 . On the other hand, a C_4 -WORM coloring of $K_9 \square K_9$ is shown in the matrix below.

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 3 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 \end{bmatrix}$$

(Note that the above is equivalent to a partition of the edge-set of $K_{9,9}$ into three C_4 -free cubic graphs. As such it was inspired by the construction used to prove the bipartite Ramsey number $b(2, 2, 2) \geq 11$; see [28, 36].) \square

6.4 Forbidding a Clique or Biclique

We consider first the case that the forbidden subgraph is a complete graph. It is essentially a special case of the result on complete (l, m) -uniform mixed hypergraphs proved by Tuza and Voloshin in [70]:

Theorem 86 (a) K_n has a K_m -WORM coloring if and only if $n \leq (m - 1)^2$.

(b) In this range, $W^+(K_n, K_m) = m - 1$ and $W^-(K_n, K_m) = \lceil n/(m - 1) \rceil$.

Proof. (a) Assume K_n has a K_m -WORM coloring. Let r be the number of colors; since there is no rainbow K_m , $r \leq m - 1$. Since there is no monochromatic K_m , each color class contains at most $m - 1$ vertices. Hence $n \leq (m - 1)^2$. On the other hand, if $n \leq (m - 1)^2$, then we use $m - 1$ colors as equitably as possible. This coloring has no monochromatic or rainbow K_m .

(b) Let r be the number of colors in a K_m -WORM coloring of K_n . If $r \geq m$, then there is a rainbow K_m . So $W^+ \leq m - 1$. Further, each color is used at most $m - 1$ times. Therefore, $r \geq n/(m - 1)$.

(We note that we can have r as any value between $\lceil n/(m - 1) \rceil$ and $m - 1$: just use each of the r colors as equitably as possible.) \square

Part (a) of the above theorem can be extended to say that: if graph G has chromatic number at most $(m - 1)^2$, then it has a K_m -WORM coloring. One just needs to color G with $m - 1$ colors, giving all the vertices in $m - 1$ color classes the same color.

From this one can determine the maximum number of edges $wex(n, K_m)$ in a graph of n vertices that has a K_m -WORM coloring.

Theorem 87 The value $wex(n, K_m)$ equals the maximum number of edges in a $K_{(m-1)^2+1}$ -free graph.

Proof. Let G be a graph on n vertices with a K_m -WORM coloring. By Theorem 86, G does not contain $K_{(m-1)^2+1}$ as a subgraph. Thus $wex(n, K_m)$ is at most the Turán number $ex(n, K_{(m-1)^2+1})$. Further, the Turán graphs are complete $(m - 1)^2$ -partite graphs and thus, by the above discussion, have a K_m -WORM coloring. \square

We consider next the case that the forbidden subgraph is a complete bipartite graph. Trivially, the bipartite coloring of a bipartite graph is automatically an F -WORM col-

oring. So we focus on the maximum number of colors. In Chapter 5 we proved that $W^+(K_{n,n}, K_{1,m}) = 2m - 2$ for $n \geq m \geq 2$. One can extend that as follows:

Theorem 88 For $n \geq m \geq 2$, $W^+(K_{n,n}, K_{m,m}) = n + m - 1$.

Proof. For the lower bound, color red $n - m + 2$ vertices in one of the partite set, and give every other vertex a distinct color. This uses $2n + 1 - (n - m + 2) = n + m - 1$ colors, and is a $K_{m,m}$ -WORM coloring (since one partite set contains only $m - 1$ colors).

On the other hand, consider a $K_{m,m}$ -WORM coloring of $K_{n,n}$ that uses at least $m + n$ colors. Then each partite set has at least m colors that do not appear in the other partite set. This immediately gives a rainbow $K_{m,m}$. That is, $W^+(K_{n,n}, K_{m,m}) \leq n + m - 1$.

□

6.5 Minimal Colorings

For graphs that do not have WORM colorings, one can ask how close to WORM can one get. We call a subgraph *bad* if it is rainbow or monochromatic, and define $B(G, F)$ as the minimum number of bad subgraphs isomorphic to F in a coloring of G . In particular, $B(G, F) = 0$ means there is an F -WORM coloring of G .

Theorem 86 says that $B(K_n, K_m) \geq 1$ if $n \geq (m - 1)^2 + 1$. One can ask what is the exact value of $B(K_n, K_m)$ in this case. We show that an optimal coloring uses $m - 1$ colors as equitably as possible.

Theorem 89 Let $n = (m - 1)k + j$ with $0 \leq j \leq m - 2$. Then $B(K_n, K_m) = j \binom{k+1}{m} + (m - 1 - j) \binom{k}{m}$.

Proof. Let an optimal coloring of K_n be given. Assume that color i is used a_i times with $a_1 \leq a_2 \leq \dots \leq a_r$.

Suppose that $r \geq m$. Assume we recolor one vertex that has color 1 with color 2; let M denote the increase in the the number of monochromatic K_m 's and R the decrease in the number of rainbow K_m 's. Then

$$\begin{aligned} M &= \binom{a_1 - 1}{m} - \binom{a_1}{m} + \binom{a_2 + 1}{m} - \binom{a_2}{m} \\ &\leq \binom{a_2 + 1}{m} - \binom{a_2}{m} = \binom{a_2}{m-1} \leq a_2^{m-1}, \end{aligned}$$

and

$$R = a_2 \sum_{3 \leq i_1 \leq \dots \leq i_{m-2} \leq r} a_{i_1} \dots a_{i_{m-2}} \geq a_2 a_3 \dots a_m \geq a_2^{m-1}.$$

That is, $M \leq R$. Hence the total number of bad K_m 's will not increase.

By repeating the argument, it follows that we may recolor all the vertices with color 1 and so reduce the total number of colors used. That is, we may assume that $r \leq m - 1$. In this case, rainbow K_m 's are impossible and thus we have only to minimize the number of monochromatic K_m 's. By convexity of the function $f(x) = \binom{x}{m}$, optimality is achieved when $m - 1$ colors are used as equitably as possible. \square

Chapter 7

Vertex Colorings without Rainbow or Monochromatic Subgraphs

7.1 Introduction

This chapter is based on joint work with Wayne Goddard [40]. As all proofs in the original paper are provided, we do not give specific references to that paper.

In Chapter 5 and Chapter 6, we considered WORM colorings: these forbid both a rainbow and a monochromatic copy of a specific subgraph. But it is more flexible to allow different restrictions. For graphs M and R , we define an (M, R) -WORM coloring of G to be a coloring of the vertices of G with neither a monochromatic subgraph isomorphic to M nor a rainbow subgraph isomorphic to R . As is mentioned in Chapter 4, this coloring is a special case of colorings of mixed hypergraphs introduced by Voloshin; see for example [74, 75, 71]. Note that when $M = R$, this is just the WORM colorings we studied in Chapter 5 and Chapter 6.

One special case of such colorings has a distinguished history. Erdős et al. [26] defined the *local chromatic number* of a graph as the maximum order of a rainbow star that must appear in all proper colorings. In our notation, this is the minimum r such that the

graph has an $(K_2, K_{1,r+1})$ -WORM coloring. For a survey on this parameter, see [61].

A related question studied in the edge case is the *rainbow Ramsey number* (or *constrained Ramsey number*); this is defined as the minimum N such that any coloring of the edges of K_N produces either a monochromatic M or a rainbow R . See [27].

One case of (M, R) -WORM colorings is trivial: if we forbid a rainbow K_2 , then every component of the graph must be monochromatic. Similarly, if we forbid a rainbow kK_1 , then this is equivalent to using less than k colors. So we will assume that the subgraph R has at least three vertices and at least one edge. On the other hand, taking $M = K_2$ is equivalent to insisting that the coloring is proper. Also, taking $M = kK_1$ is equivalent to using each color less than k times.

Having two competing restrictions leads naturally to considering both the minimum and maximum number of colors in such a coloring. So we define the *upper chromatic number* $W^+(G; M, R)$ as the maximum number of colors, and the *lower chromatic number* $W^-(G; M, R)$ as the minimum number of colors, in an (M, R) -WORM coloring of G (if the graph has such a coloring). For bounds, it will be useful to also let $m^-(G; M)$ be the minimum number of colors without a monochromatic M , and $r^+(G; R)$ be the maximum number of colors without a rainbow R . Note that

$$m^-(G; M) \leq W^-(G; M, R) \leq W^+(G; M, R) \leq r^+(G; R),$$

provided G has an (M, R) -WORM coloring.

We proceed as follows: in Section 7.2 we give basic observations. In Section 7.3 we provide one general upper bound when R is a path. In Section 7.4 we consider proper colorings without rainbow P_3 , P_4 , or C_4 . Finally, in Section 7.5 we provide a few results for other cases.

7.2 Preliminaries

We start with some simple observations. If G is bipartite then the bipartition is immediately an (M, R) -WORM coloring. Indeed, if G is k -colorable with $k < |R|$, then a proper k -coloring of G is an (M, R) -WORM coloring. Also:

Observation 90 *Fix graphs M and R and let G be a graph.*

(a) *If G has an (M, R) -WORM coloring, then so does $G - e$ where e is any edge and $G - v$ where v is any vertex. Further, $W^+(G - e; M, R) \geq W^+(G; M, R)$ and $W^+(G - v; M, R) \geq W^+(G; M, R) - 1$, with similar results for the lower chromatic number.*

(b) *If M and R are connected but G is disconnected, then $W^+(G; M, R)$ is the sum of the parameter for the components, and $W^-(G; M, R)$ is the maximum of the parameter for the components.*

(c) *It holds that $W^+(G; M, R) = |V(G)|$ if and only if G is R -free.*

(d) *It holds that $W^+(G; M, R) \geq |R| - 1$ if G is $|R| - 1$ colorable (and has at least that many vertices).*

7.2.1 General M

It should be noted that maximizing the number of colors while avoiding a rainbow subgraph can produce a large monochromatic subgraph. For example:

Observation 91 *For all connected graphs M , there exists a graph G such that*

$$W^+(G; M, P_3) < r^+(G; P_3).$$

Proof. In Chapter 4 we considered the corona $cor(G)$ of a graph G ; this is the graph obtained from G by adding, for each vertex v in G , a new vertex v' and the edge vv' . It was shown that $r^+(G; P_3) = |G| + 1$. In fact, we note here that if G is connected, then one can

readily show by induction that the optimal coloring is unique and gives every vertex of G the same color. In particular, it follows that the no-rainbow- P_3 coloring of $\text{cor}(M)$ with the maximum number of colors contains a monochromatic copy of M . \square

7.3 A Result on Rainbow Paths

We showed in Chapter 5 that a nontrivial graph G has a (P_3, P_3) -WORM coloring if and only if it has one using at most two colors. We prove an analogue for general paths. This result is a slight generalization of Theorem 10 in [70].

Theorem 92 *Fix some graph M ; if graph G has an (M, P_r) -WORM coloring, then G has one using at most $r - 1$ colors.*

Proof. Consider an (M, P_r) -WORM coloring f of G . Let G_M be the spanning subgraph of G whose edges are monochromatic and G_R the spanning subgraph whose edges are rainbow. It follows that G_M does not contain M , and that G_R does not contain P_r . By the Gallai-Hasse-Roy-Vitaver theorem [34, 45, 66, 73], since G_R does not contain P_r , it has chromatic number at most $r - 1$.

Now, let g be a proper coloring of G_R using at most $r - 1$ colors and consider g as a coloring of G . Note that the monochromatic edges under g are a subset of the monochromatic edges under f . Therefore, g is a (M, P_r) -WORM coloring of G using at most $r - 1$ colors. \square

It follows that:

Corollary 93 *For any graph M and $r > 0$, graph G has an (M, P_r) -WORM coloring if and only if $m^-(G, M) \leq r - 1$. If so, $W^-(G; M, P_r) = m^-(G, M)$.*

On the other hand, Theorem 92 does not extend to stars. For example, Erdős et al. [26] constructed a shift graph that has arbitrarily large chromatic number but can be properly colored without a rainbow $K_{1,3}$. That is:

Theorem 94 For $r \geq 3$ and $k \geq 1$, there is a graph G with $W^-(G; K_2, K_{1,r}) \geq k$.

Nor does Theorem 92 generalize to K_3 ; see [11].

7.4 Proper Colorings

Recall that $W^+(G; K_2, R)$ is the maximum number and $W^-(G; K_2, R)$ is the minimum number of colors in a proper coloring without a rainbow R .

7.4.1 Two simple cases

Two cases for $M = K_2$ are immediate:

Observation 95 A graph G has a (K_2, P_3) -WORM coloring if and only if it is bipartite. If so, $W^+(G; K_2, P_3) = W^-(G; K_2, P_3) = 2$, provided G is connected and nonempty.

Proof. If we have a (K_2, P_3) -WORM coloring, then for each vertex v all its neighbors must have the same color, which is different to v 's color. It follows that every path must alternate colors. □

In a proper coloring of a graph, all cliques are rainbow. Thus it follows:

Observation 96 A graph G has a (K_2, K_m) -WORM coloring if and only if it is K_m -free. If so, $W^+(G; K_2, K_m) = |G|$ while $W^-(G; K_2, K_m) = \chi(G)$.

7.4.2 No rainbow $K_{1,3}$

Consider first that G is bipartite. Then in maximizing the colors, it is easy to see that one may assume the colors in the partite sets are disjoint. (If red is used in both partite sets, then change it to pink in one of the sets.) In particular, unless G is a star, one can use

at least two colors in each partite set. (This result generalizes to R any star.) For example, it follows that $W^+(K_{m,m}; K_2, K_{1,3}) = 4$ for $m \geq 2$.

Indeed, it is natural to consider the *open neighborhood hypergraph* $ON(G)$ of the graph G . This is the multihypergraph with vertex set $V(G)$ and a hyperedge for every open neighborhood in G . In general, since we have a proper coloring, the requirement of no rainbow $K_{1,r}$ is equivalent to every hyperedge in $ON(G)$ receiving at most $r - 1$ colors. In the case that G is bipartite, the two problems are equivalent:

Observation 97 *For any graph G , the parameter $W^+(G; K_2, K_{1,r})$ is at most the maximum number of colors in a coloring of $ON(G)$ with every hyperedge receiving at most $r - 1$ colors. Furthermore, there is equality if G is bipartite.*

Proof. When G is bipartite, the $ON(G)$ can be partitioned into two disjoint multihypergraphs and so will have disjoint colors in the multihypergraphs. It follows that the coloring back in G will be proper. □

Observation 98 *If G is a 2-tree of order at least 3, then $W^+(G; K_2, K_{1,3}) = 3$.*

Proof. It is well known that any 2-tree is 3-colorable. Furthermore, it follows readily by induction that a $(K_2, K_{1,3})$ -WORM coloring can use only three colors: when we add a vertex v and join it to adjacent vertices x and y , they already have a common neighbor z , and so v must get the same color as z . □

Osang showed that determining whether a graph has a $(K_2, K_{1,3})$ -WORM coloring is hard:

Theorem 99 [61] *Determining whether a graph has a $(K_2, K_{1,3})$ -WORM coloring is NP-complete.*

7.4.2.1 Cubic graphs

We next consider cubic graphs. It is easy to see that K_4 does have a $(K_2, K_{1,3})$ -WORM coloring. On the other hand, by Brooks' Theorem [8], every cubic graph other than K_4 has a proper coloring using at most 3 colors. Note that such a proper coloring is also a $(K_2, K_{1,3})$ -WORM coloring. Further, they have a coloring using two colors if and only if they are bipartite. So the only interesting question is the behavior of $W^+(G; K_2, K_{1,3})$.

Observation 100 *If G is cubic of order n , then $W^+(G; K_2, K_{1,3}) \leq 2n/3$.*

Proof. Since G is cubic, the open neighborhood hypergraph $ON(G)$ is 3-regular and 3-uniform. Further we need a coloring of $ON(G)$ where every hyperedge has at least one pair of vertices the same color. Consider some color used more than once, say red. If there are r red vertices, then at most $3r/2$ hyperedges can have at least two red vertices. (Each vertex is contained in at most three hyperedges.)

It follows that if the i^{th} non-unique color is used r_i times, then we need $\sum_i r_i \geq 2n/3$. Let B be the number of vertices that can be discarded and still have one vertex of each color. Then $B = \sum_i (r_i - 1)$ and by above $B \geq n/3$. It follows that the total number of colors is at most $2n/3$. \square

Equality in Observation 100 is obtained by taking disjoint copies of $K_{3,3} - e$ and adding edges to make the graph connected. See Figure 7.1.

Consider next the minimum value of $W^+(G; K_2, K_{1,3})$ for cubic graphs of order n . We noted above that bipartite graphs in general have a value of at least 4. Computer search shows that this parameter is at least 3 for $n \leq 18$. Indeed, it finds only three graphs where the parameter is 3: one of order 6 (the prism), one of order 10, and one of order 14 (the generalized Petersen graph). These three graphs are shown in Figure 7.2.

The general case remains open.

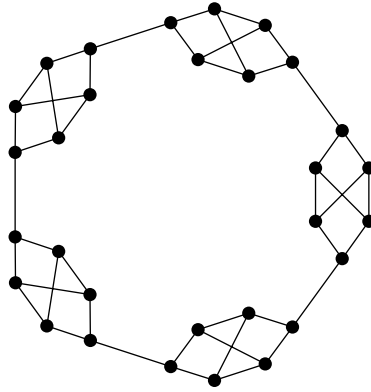


Figure 7.1: A cubic graph G with $W^+(G; K_2, K_{1,3})$ two-thirds its order

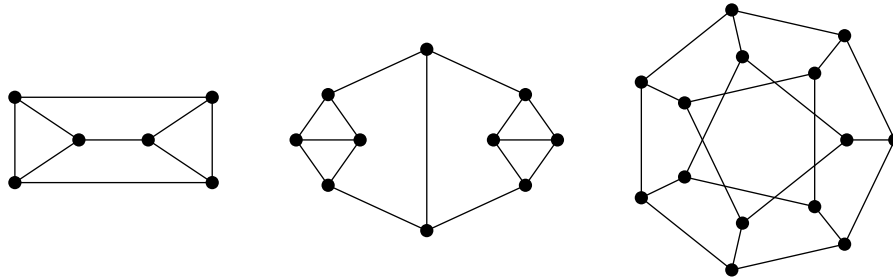


Figure 7.2: The known cubic graphs with $W^+(G; K_2, K_{1,3}) = 3$

7.4.3 Forbidding rainbow P_4

We consider proper colorings without rainbow P_4 's, then Theorem 92 applies. That is, a graph G has a (K_2, P_4) -WORM coloring if and only if G has chromatic number at most 3. Since it is NP-complete to determine if a graph has a proper 3-coloring [35], it follows that it is also NP-complete to determine if a graph has a (K_2, P_4) -WORM coloring. Further, if such a coloring exists, then $W^-(G; K_2, P_4)$ is just the chromatic number of G . So we consider only the upper chromatic number here.

Observation 101 *If graph G is bipartite of order n , then $W^+(G; K_2, P_4) \geq n/2 + 1$.*

Proof. In the smaller partite set, give all vertices the same color, and in the other partite set, give all vertices unique colors. Note that every copy of P_4 contains two vertices from both partite sets. □

Observation 102 *If connected graph G of order n has a perfect matching, then it holds that $W^+(G; K_2, P_4) \leq n/2 + 1$.*

Proof. Number the edges of the perfect matching $e_1, \dots, e_{n/2}$ such that for all $i > 1$, at least one endpoint of e_i is connected to some e_j for $j < i$ through another edge. Then e_i , e_j , and the connecting edge form a P_4 . It follows that e_j and e_i share a color. Thus the total number of colors used is at most $2 + (n/2 - 1) = n/2 + 1$. \square

For example, equality is obtained in both observations for any connected bipartite graph with a perfect matching, such as the balanced complete bipartite graph or the path/cycle of even order. Equality is also obtained in Observation 101 for the tree of diameter three where the two central vertices have the same degree. Also, there are nonbipartite graphs that achieve equality in Observation 102; for example, the graph shown in Figure 7.3.

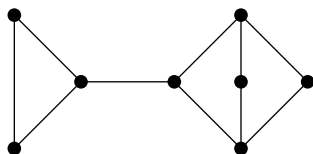


Figure 7.3: A nonbipartite graph G with a perfect matching and maximum $W^+(G; K_2, P_4)$

We determine next the parameter for the odd cycle:

Observation 103 *If n is odd, then $W^+(C_n; K_2, P_4)$ is 3 for $n \leq 5$, and $(n-1)/2$ for $n \geq 7$.*

Proof. The result for $n = 3$ is trivial and for $n = 5$ is easily checked. So assume $n \geq 7$. For the lower bound, color a maximum independent set red, give a new color to every vertex with two red neighbors, and color each vertex with one red neighbor the same color as on the other side of its red neighbor. For example, the coloring for C_{13} is shown in Figure 7.4 (where the red vertices are shaded).

We now prove the upper bound. Two same-colored vertices distance 2 apart bi-cover two copies of P_4 , while two same-colored vertices distance 3 apart bi-cover one copy. It follows that if a color is used k times, it can bi-cover at most $2(k-1)$ copies of P_4 , except if

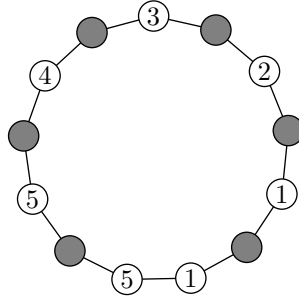


Figure 7.4: Coloring showing $W^+(C_{13}; K_2, P_4)$

the vertices of that color form a maximum independent set, when it bi-covers $2k - 1$ copies. Since there are n copies of P_4 in total, by Proposition 34 it follows that the total number of colors is at most $n/2$, unless some color is a maximum independent set. So say red is a maximum independent set. Let b and e be the two red vertices at distance 3; say the portion of the cycle containing them is $abcdef$. By considering the $a-d$ copy of P_4 , it follows that a must have the same color as c or d . Similarly, f must have the same color as c or d . Thus the total number of colors is at most $1 + (n - (n - 1)/2) - 2 = (n - 1)/2$. \square

In contrast to Observation 101, we get the following:

Theorem 104 *If connected graph G has every vertex in a triangle, then $W^+(G; K_2, P_4) = 3$ if such a coloring exists.*

Proof. Note that every triangle is properly colored. We show that every triangle receives the same three colors. Consider two triangles T_1 and T_2 . If T_1 and T_2 share two vertices, then the third vertex in each share a color. Consider the case that T_1 and T_2 share one vertex. Then by considering the four P_4 's using all vertices but one, it readily follows that the triangles must have the same colors.

Now, assume that T_1 and T_2 are disjoint but joined by an edge e . Suppose they do not have the same three colors. Then there is vertex u_1 in T_1 and u_2 in T_2 that do not share a color with the other triangle. If u_1 and u_2 are the ends of e , then any P_4 starting with e is rainbow. If u_1 and u_2 are not the ends of e , then there is a P_4 whose ends are u_1 and u_2

and that P_4 must be rainbow. Either way, we obtain a contradiction.

Since the graph is connected, it follows that every triangle is colored with the same three colors. Since this includes all the vertices, the result follows. \square

From the above theorem, it follows that $W^+(G; K_2, P_4) = 3$ for every maximal outerplanar graph G .

7.4.3.1 Cubic Graphs

There are many cubic graphs with $W^+(G; K_2, P_4) = 3$. These include, for example, the cubic graphs that do not contain $K_{1,3}$ as an induced subgraph (equivalently the ones where every vertex is in a triangle). See Theorem 104.

For the largest value of the parameter, computer evidence suggests:

Conjecture 9 *If G is a connected cubic graph of order n , then $W^+(G; K_2, P_4) \leq n/2 + 1$, with equality exactly when G is bipartite.*

Certainly, by Observations 101 and 102 (and the fact that regular bipartite graphs have perfect matchings), that value is achieved by all bipartite cubic graphs.

7.4.4 Forbidding rainbow C_4

We conclude this section by considering proper colorings without rainbow 4-cycles.

Observation 105 *For every maximal outerplanar graph G , it holds that*

$$W^-(G; K_2, C_4) = W^+(G; K_2, C_4) = 3.$$

Proof. Consider two triangles sharing an edge. Then, to avoid a rainbow C_4 , the two vertices not on the edge must have the same color. It follows that all triangles have the same three colors. \square

We next consider cubic graphs. It is easy to see that K_4 does have a (K_2, C_4) -WORM coloring. On the other hand, by Brooks' Theorem [8], every cubic graph other than K_4 has a proper coloring using at most 3 colors. Note that such a proper coloring is also a (K_2, C_4) -WORM coloring. Thus, $W^-(G; K_2, C_4) = 2$ if G is a bipartite cubic graph, and $W^-(G; K_2, C_4) = 3$ if G is a nonbipartite cubic graph other than K_4 . Further, the upper bound for $W^+(G; K_2, C_4)$ is trivial: one can have a cubic graph without a 4-cycle.

Computer evidence suggests that:

Conjecture 10 *If G is a connected cubic graph of order $n \geq 6$, then $W^+(G; K_2, C_4) \geq n/2$.*

This lower bound is achievable. For n even, a *Mobius ladder* is defined by taking the cycle on n vertices and joining every pair of opposite vertices. Note that a Mobius ladder is bipartite when its order is not a multiple of 4; while a prism is bipartite when its order is a multiple of 4.

Observation 106 *If G is a nonbipartite Mobius ladder or nonbipartite prism of order n , then it holds that $W^+(G; K_2, C_4) = n/2$.*

Proof. We first exhibit the coloring. Let $m = n/2$. Say the vertices of the prism are u_1, \dots, u_m and v_1, \dots, v_m , where u_i has neighbors u_{i-1} , u_{i+1} , and v_i (arithmetic modulo m) and similarly for v_i . Then for $1 \leq i \leq m$, give vertices u_i and v_{i+1} color i .

Say the vertices of the Mobius ladder are w_1, \dots, w_n where w_i has neighbors w_{i-1} , w_{i+1} , and w_{i+m} (arithmetic modulo n). Then for $2 \leq i \leq m$, give vertices w_i and w_{i+m-1} color i , give vertex w_1 color 1 and give vertex w_n color 2. For example, the coloring for the case $n = 12$ is shown in Figure 7.5.

Now, for the upper bound, consider a color that is used r times. A color bi-covers a copy of C_4 if it contains vertices from consecutive rungs (where a rung is an edge in two C_4 's). Since the graph is not bipartite, the color cannot be present in every rung. It

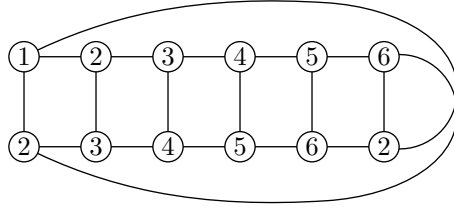


Figure 7.5: Coloring of Mobius ladder

follows that it can bi-cover at most $r - 1$ copies of C_4 . Now, there are m copies of C_4 (note that the prism of C_4 is bipartite so excluded). It follows from Proposition 34 that the number of colors is at most $n - n/2 = n/2$. \square

7.5 Other Results

7.5.1 Paths and paths

The natural strategy to color a long path without a rainbow P_r yields the following result. It can also be obtained as a special case of the result on interval mixed hypergraphs given in [13]:

Observation 107 *For any $m \geq 3$, it holds that $W^+(P_n; P_m, P_r) = r^+(P_n; P_r) = \lfloor (r - 2)n/(r - 1) \rfloor + 1$.*

Proof. Give the first $r - 1$ vertices different colors, then the next vertex the same color as the previous vertex, then the next $r - 2$ vertices different colors, and so on. This coloring has a monochromatic P_2 but not a monochromatic P_3 , and is easily seen to be best possible (as every copy of P_r must contain two vertices of the same color). \square

7.5.2 Bicliques and bicliques

Next we revisit the case that G , M , and R are bicliques. For $n \geq b$ it was proved that $W^+(K_{n,n}; K_{1,b}, K_{1,b}) = 2b - 2$ in [38] and that $W^+(K_{n,n}; K_{b,b}, K_{b,b}) = n + b - 1$ in [37]. The case for stars is special, but it is straight-forward to generalize the latter:

Theorem 108 *Let $m \leq n$ and $2 \leq a \leq b$ with $m \geq a$ and $n \geq b$. Then*

$$r^+(K_{m,n}; K_{a,b}) = \max(a + n - 1, b - 1 + \min(m, b - 1)).$$

Proof. Consider a coloring $K_{m,n}$ without a rainbow $K_{a,b}$ and assume there are at least $a + b$ colors. If we can choose a colors from one partite set and b colors from the other that are disjoint, then we would obtain a rainbow $K_{a,b}$. So either there is a partite set that has at most $a - 1$ colors that do not appear in the other partite set, or each partite set has at most $b - 1$ colors. In the first case, the maximum number of colors possible is $a + n - 1$. In the second case, the maximum number of colors possible is $b - 1 + \min(m, b - 1)$. The theorem follows. \square

Note that in the above proof, the optimal number of colors can be achieved by making the sets of colors in the two partite sets disjoint. Thus, one obtains a similar value for $W^+(K_{m,n}; M, K_{a,b})$ where M is any nontrivial biclique.

7.6 Other Directions

We conclude with some thoughts on future directions. Apart from the specific open problems raised here, a direction that looks interesting is the case where M and R are both stars. Also of interest is the case that the host graph is a product graph.

Chapter 8

Conclusion and Future Directions of Research

In this thesis, we studied several types of generalized vertex colorings of graphs.

We studied fractional and circular chromatic numbers and the defective version of these parameters for series-parallel graphs. In particular, we showed that the fractional chromatic number of any series-parallel graph of odd girth k is exactly $2k/(k-1)$, confirming a conjecture by Wang and Yu. We also showed that for every d there is a series-parallel graph whose d -defective fractional and circular chromatic numbers are both 3, answering a question of Klostermeyer. Note that simple series-parallel graphs are also the partial 2-trees. So it is natural to consider partial k -trees in general. Chlebíková [18] showed that for $k \geq 3$, every triangle-free partial k -tree has chromatic number at most k . It would be interesting to investigate if this is best possible and what happens for fractional/circular coloring and/or higher girth/odd girth.

We introduced the concept of fractional defect. We established some basic properties of this concept, and investigated the minimum fractional defect over a few families of graphs, giving exact values in some cases and bounds in others. The problem is NP-hard in general, and it seems to be nontrivial even for some graphs that have simple structures. It is

quite possible that more advanced tools or techniques are required to make progress on the conjectures and open problems that we proposed. One of the problems that seems interesting to us is to determine the minimum fractional defect for the rooks graph $K_m \square K_n$ where one of m and n is odd.

We considered several types of vertex colorings of a graph forbidding rainbow or monochromatic subgraphs. We presented some results especially with regards to the existence of colorings, complexity, and optimization within certain graph classes. Our focus was on the case that the forbidden subgraph is a path, cycle, star, or clique. One possible direction of future research would be to attack the specific conjectures and open problems raised in Chapters 4, 5, 6, 7 of the thesis, for example, those conjectures that concern extremal values of number of colors in no-rainbow-colorings of cubic graphs of fixed order. Another possible direction of future research would be to consider other types of forbidden subgraphs. Finally, the gap of chromatic spectrum seems to be an interesting direction. In particular, Bujtás and Tuza [11] proposed the following conjecture: for every integer $k \geq 4$, there exists a K_3 -WORM-colorable K_4 -free graph G such that $W^-(G, K_3) = k$. This conjecture, if true, would extend Theorem 79.

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