A LAMINATED RING ON ELASTIC FOUNDATION MODEL WITH APPLICATION TO ANALYSIS OF TIRE-ROAD CONTACT

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A LAMINATED RING ON ELASTIC FOUNDATION MODEL WITH APPLICATION TO ANALYSIS OF TIRE-ROAD CONTACT

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Automotive Engineering

by
Chunjian Wang
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Accepted by:
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ABSTRACT

A deformable Ring on Elastic Foundation (REF) model has been widely used for modeling rotating structures such as gears, bearings and tires. This dissertation extends the existing work with three main contributions focusing on applications to tires: 1) The elastic foundation is endowed with unilateral stiffness, which leads to a nonlinear deformation dependent stiffness typical of some non-pneumatic tires; 2) The model of the deformable ring is extended from simple Timoshenko rings to a laminated orthotropic ring model, with bending, shear stiffness and extensibility in the internal layer, and transverse stiffness in the external layer; 3) Facilitated by the laminated ring model, a feedback compensation algorithm is proposed for analysis of frictionless contact of the tire with arbitrary uneven surfaces in both static and rolling contact.

To being with, a general orthotropic and extensible thick ring is considered and modeled using Timoshenko beam theory. The nonlinear foundation is modeled as two-parameter elastic one with a linear torsional stiffness but a unilateral radial stiffness that vanishes in either compressed or tensioned zones. Then, the governing equations for the static deformation problem are derived via the principle of virtual work. The fully linear foundation case can be solved analytically using a Fourier expansion method. It is found that compared with the unilateral foundation case, the linear foundation applies excessive forces in regions where the unilateral foundation would have collapsed or tensioned. An iterative compensation method has been presented to solve for the static deformation of the ring on unilateral foundation that avoids directly solving the coupled and complex...
nonlinear differential equations. Using the proposed approach, the deformation response to an arbitrary in-plane force could be solved in a uniform way.

Then, the approach developed for the solution for the static problem has been extended to solve the nonlinear dynamic vibration problem by combining it with an implicit Newmark scheme for time-integration. The iterative compensation method is embedded into every time step of the Newmark scheme. Compared with Finite Element Analysis (FEA), the advantages of the propose method are: 1) time consuming modeling and meshing work is avoided, especially at the stage where only parametric design studies are of interest; 2) proposed method has much smaller size of matrices; 3) time consuming matrix inversions imbedded in the iterations of nonlinear FEA are avoided. Instead, the iteration involved in the proposed approach is only algebraic.

For analysis of tire-road contact, the deformable REF model is extended to a laminated REF model, where on top of the Timoshenko ring accounting for bending, shear and extensibility, an additional external layer is added to take the transverse compliance into account and simulate the tread layer of tires. Then, the analytical solution for the static deformation response of this laminated REF model due to an arbitrary external force is detailed. Facilitated by the laminated ring model, a feedback compensation approach that penalizes geometry errors in the contact region is proposed to solve the frictionless tire-road contact problem. This feedback compensation algorithm offers a uniform and easy-to-implement approach for arbitrary road profiles. The foundation pre-deformation is also considered for modeling the inflation pressure in pneumatic tires or pre-tension/compression of the foundation in non-pneumatic tires.
Finally, the rotational effects are included in the Equation of Motions (EOMs) of the laminated REF model and the tire-road contact analysis is extended into quasi-static and dynamic rolling contact cases. The solution procedure utilizes the combination of the feedback compensation algorithm for contact and iterative compensation method for spatial deformation with unilateral foundation along with the implicit Newmark scheme for time integration. In quasi-static rolling contact with a flat surface, this unilateral foundation model leads to a rolling ring supported by a partially distributed and space-fixed foundation. For this situation, a method for modal analysis and prediction of critical velocities and standing waves is also given in this dissertation.

In most cases analyzed, comparisons with alternative finite element analysis (FEA) software solutions showed that the proposed modeling approach gives very similar results with much reduced computational time and avoids the need for extensive FEA oriented pre-processing. As such, the modeling approach and solution methods proposed in this dissertation are ideally suited for rapidly covering the design space for tire designs with the objective to meet ride comfort requirements over various road surfaces.
DEDICATION

To my beloved wife, Yan, my precious and lovely daughter, Evy, and all my family members. Without their companion and support, none of my PhD work, including this dissertation, is possible.
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First, I would like to express my sincere appreciation to my advisor Dr. Beshah Ayalew whose excellent mentorship guides me and helps me throughout my PhD program. His enthusiasm on the investigation of unknown areas and attitude to pursue high quality work impressed me and affected me; his critical feedback always helps; and his patience, trust, encouragement and continuous support enable me to persist on my research and finally gain something.

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CHAPTER 1 : INTRODUCTION

1.1 Introduction of Ring on Elastic Foundation (REF) Model

Ring on Elastic Foundation (REF) is a very common structure in rotating components such as gears [1], bearings [2] and tires [3] [4] [5], etc. These broad applications have motivated intensive studies on models for REF structures. One of the methods to study this problem assumes a rigid ring, which ignores detailed deformation and dynamics inside the ring but focuses on the overall performance such as dynamic transmissibility to a higher-level system. An example of this rigid REF model is the well-known SWIFT model [6], [7] for tire dynamic modeling in vehicle dynamic simulations. In this model, a radial spring is attached to the rigid ring to approximate normal compliance of the ring deformation. Discrete multi-tread elements are used for the contact modeling.

In contrast with the rigid ring model, flexible ring model [5] consists of a deformable circular beam. This allows investigations into ring deformation [8] and high frequency dynamics inside the ring [9] [10]. This flexible ring model can be a tensioned string [11], which only accounts for the tensile strain but no bending stiffness, or Euler-Bernoulli beam [5] [12] [13] with bending stiffness, or Timoshenko beam [14] [15] [16] which also considers shear. Extensible or inextensible assumptions for the beam models are also discussed in [9] [10].

The other important part of the REF model is the elastic foundation. The most popularly used assumption is the linear and uniformly distributed stiffness all over the
elastic foundation. The stiffness can be only in the radial direction [11] [12], or include both radial and torsional stiffness [17] [13], or even more stiffness components [1]. Compared with these uniformly distributed stiffness models, the foundation stiffness can be non-uniform in some cases [18] [2] [19]. A more complex situation involves the loading or deformation dependent foundation stiffness such as that described in [20].

The analysis of contact with these REF models are investigated and achieved via different schemes. One approach considers the contact as a set of boundary conditions of the EOMs, as in [21]. This method is improved by inclusion of transverse compliance of the beam in [22]. Alternatively, extra discrete spring elements are added to the outer edge of the ring for the contact analysis [23] [24]. These contact analyses can give the solutions for contact pressure and force transmissibility on flat or uneven surfaces.

1.2 Research Motivation and Contributions

One of the motivations comes from the development of non-pneumatic tire [25], where a shear beam is used to constitute the tire belt and collapsible spokes are used for the foundation. Prediction of the tire static and dynamic road contact performance requires a thick ring on collapsible foundation model along with a capability of solving contact problems. Finite Element (FE) models are applicable for the analysis but are time consuming in FEA oriented pre-processing in the tire design stage, which covers large design space for parameter studies. Moreover, if achievable, analytical-based methods as pursued in this dissertation give more insights on the physics of such REF structures and the contact problems involving them.
Some work has been published on the related topics but the existing results and methods still have limitations in solving the problems raised by non-pneumatic tire in the following aspects: 1) the static and dynamic deformation of a deformable ring on unilateral foundation have not been solved completely for a general thick ring subject to arbitrary external force; 2) the static and dynamic contact responses with arbitrary and uneven surfaces have not completely been investigated for this structure; 3) the possibility of standing waves and the prediction of the critical velocity has not been studied for the deformable ring on a unilateral foundation.

This dissertation extends the existing work and solves the problems mentioned above with the following main contributions:

1. An iterative compensation method is presented to solve the deformation of a general thick ring on unilateral foundation;

2. This iterative compensation method is embedded into an implicit Newmark method to solve the dynamic response for a thick ring on unilateral foundation.

3. A laminated ring model and a feedback compensation scheme are presented as a uniform approach for analysis of contact with arbitrary uneven surfaces.

4. Rotational effects are explicitly included to give a comprehensive REF model that can be used for dynamic contact analysis, with linear or nonlinear foundations.

5. A method of prediction of the critical velocity and standing waves is given for the REF on a unilateral foundation model.
1.3 Dissertation Organization

The dissertation is organized according to the journal papers that have already been completed.


Chapter 2 presents the contribution[16], which details the iterative compensation method for arbitrary external force induced static deformation of a Timoshenko ring on unilateral foundation. Chapter 3 presents the contribution [26], which extends the problem in Chapter 2 to the dynamic case by embedding this iterative compensation method into an implicit Newmark scheme. Chapter 4 presents the contribution [27], which details the laminated ring model and a feedback p-controller algorithm for contact
analysis with arbitrary uneven surface. Only static contact is analyzed in this chapter and a linear foundation model is used. Chapter 5 presents the contribution [28], in which, rotation effects are explicitly included in the EOMs, and the combination of p-controller algorithm, implicit Newmark scheme and iterative compensation method is applied to extend the analysis into quasi-static and dynamic rolling contact. The critical velocity for this unilateral foundation model in rolling contact with flat surface is also analyzed. Finally, Chapter 6 concludes the dissertation along with discussions of some extensions that may be pursued in the future.
2.1 Abstract

A thick ring on a unilateral elastic foundation can be used to model important applications such as non-pneumatic tires or bushing bearings. This paper presents a reduced-order compensation scheme for computing the static deformation response of a thick ring supported by a unilateral elastic foundation to an arbitrary applied force. The ring considered is an orthotropic and extensible ring that can be treated as a Timoshenko beam. The elastic foundation is a two-parameter foundation with a linear torsional stiffness but a unilateral radial stiffness whose value vanishes when compressed or tensioned. The paper first derives the deformation response for the linear foundation case for which Fourier expansion techniques can be applied to obtain an analytical solution. Then, the nonlinear unilateral foundation problem is solved via an iterative compensation scheme that identifies regions with vanishing radial stiffness and applies a compensation force to the linear foundation model to counteract the excessive foundation forces that would not be there with the unilateral foundation. This scheme avoids the need for solving the complex set of nonlinear differential equations and gives a computationally efficient tool for rapidly analyzing and designing such systems. Representative results are compared with Finite Element Analysis (FEA) results to illustrate the validity of the proposed approach.

2.2 Introduction
The flexible Ring on Elastic Foundation (REF) model [9] [17] is a classical one that has been studied for decades. This is because of its broad and important applications such as automotive tires [29] [4] [5], railway wheels and gears [1], and others [12].

Different criteria can be used to classify the focus of existing works that analyze REFs. The simplest categories are the treatment of the static deformation problem [12] [30] vs. the dynamic problem as free vibration [1] or forced vibration [17] [13]. Considering the ring mechanics, the ring has been treated as a tensioned string that has direct tensile strain but no bending stiffness [31]; as an Euler-Bernoulli beam or thin ring whose plane section remains plane and always normal to the neutral axis of the ring after deformation [12] [5] [13]; or as a Timoshenko beam or thick ring [14] which takes the shear deformation into account by assuming that the normal of a plane section is subject to rotation. Further distinctions exist between extensional and inextensional rings. [10] and [9] studied the vibration problem for both a rotating thin ring and a thick ring, and pointed out that the inextensional assumptions in thick ring theories are improper because extensional coupling effects are as important as shearing effects especially for a rotating ring.

As an important component of the REF model, the treatment of the elastic foundation can be used as another criterion to classify the existing research. Numerous works, including all of the ones cited above, assume a linear and uniformly distributed stiffness for the whole elastic foundation, independent of location and deformation state. The distributed stiffness can be modeled with one parameter [12] [31] where the foundation has a stiffness in only the radial direction; or with two parameters [17] [13]
involving both radial and torsional stiffness values; or even more parameters [1], where in addition to the radial and torsional stiffness, a stiffness associated with the distortion of the foundation due to in-plane rotation of a cross-section of the ring is included.

Although much has been gained from linear and uniform elastic foundation assumptions, not all REF problems have a perfect linear elastic foundation with uniformly distributed stiffness. Examples for non-uniform distribution include planetary gearing where tooth meshes for the ring and planets are not equally spaced [2] and tires with non-uniformity [18]. For these type of problems, [2] studied the free vibration of rings on a general elastic foundation, whose stiffness distribution can be variable circumferentially in the radial, tangential, or inclined orientation, and gave the closed-form expression for natural frequencies and vibration modes. [18] studied the natural frequencies and mode shapes of rings supported by a number of radial spring elements with a constant radial stiffness using modal expansion and receptance method. However, in both works, the distributions of the stiffnesses for the elastic foundations still did not change with the deformation of the ring resting on them.

A more complicated problem was invoked by considering deformation-dependent elastic foundations, such as those with unilateral stiffness whose values vanish when compressed or tensioned. The difficulty in solving this group of problems is in the fact that the compressed or tensioned region is not known in advance. It depends on the loading and consequently on the deformation. An example for the application is the non-pneumatic tire presented in [25], whose structure is a deformable shear ring supported by collapsible spokes which offer stiffness only in tension [32]. Another example is a
bushing bearing, whose external sleeve can be modeled as a ring on a tensionless foundation and the internal sleeve as a ring on a collapsible foundation. Not too many works exist that deal with such rings on unilateral foundations. [20] worked on the forced response of a thin and inextensible ring on a tensionless two-parameter foundation under a time varying in-plane load. To solve the unilateral problem, an auxiliary function is used in the coefficients of the equations to track and reflect the status of the foundation. This auxiliary function makes the differential equations of the system nonlinear and difficult to solve. Furthermore, the tangential displacement of the ring is obtained from the inextensible assumption, which cannot be adopted in a more general case, such as that with extensible Timoshenko ring. [33] studied the static deformation and the contact pressure of a non-pneumatic tire resting on collapsible spokes, when it contacts against a rigid plane ground. The governing differential equations were derived only for the thick ring modeled via Timoshenko beam theory by treating the supporting force by collapsible spokes as radial distributed force which vanishes in collapsed spoke regions. The ring was divided into three regions according to the post-deformation status of the spokes (tensioned or buckled spokes) and contact status with the ground (contact region or free region). Closed-form expressions for the deformation and contact stress are given in terms of angular bounds of these three regions, which then need to be solved numerically. However, the method has limitations in two aspects: 1) For more complex loading cases, such as that with multiple forces applied at multiple locations, the number of regions into which the ring must be divided grows with the number of the load regions. Multiple
unknown angular bounds would then need to be solved numerically. 2) It is difficult to extend this method to practical dynamic cases.

The present paper studies the deformation of a thick ring on an elastic two-parameter foundation where the radial stiffness is unilateral. The deformation response to an arbitrary in-plane force is considered. The ring is modeled as an orthotropic and extensible circular Timoshenko beam. As the first step, the linear foundation problem is solved analytically using Fourier expansion techniques for both the radial and tangential directions. It is then shown that the linear foundation case includes an excessive foundation force compared with that of the unilateral foundation. An iterative compensation scheme is then set up to both find the region of vanishing radial stiffness with the unilateral foundation and that of the required compensation force to counter-act the excessive force predicted via the linear foundation. The method is an intuitive and efficient alternative to numerically solving the coupled and complex system of nonlinear differential equations for a flexible ring on a unilateral foundation. In addition, compared with discretization-based numerical methods such as nonlinear FEA, the proposed scheme avoids the time-consuming modeling and meshing work, which makes it attractive specially for rapid parameter studies at the design stage. Compared with the method in [33], the method proposed in this paper is capable of handling arbitrary force distributions and directions, without increasing complexity. Furthermore, since the proposed scheme is Fourier expansion-based, it can be easily extended to the dynamic cases (both forced response and dynamic contact) as we illustrate in our other work [26].
The rest of this chapter is organized as follows. Section 2.3 restates the problem and gives the governing equations. Section 2.4 gives the analytical solutions for the linear foundation problem and extends them to the unilateral case. Then, in Section 2.5, discussions of some example results are given and compared with FEA results. Conclusions and future work are given in Section 2.6.

2.3 Statement of Problem and Governing Equations

Figure 1 shows a schematic of the model for a thick ring on a two-parameter elastic foundation. The ring with thickness $h$ is assumed to have a radius $R$ at its centroid. The width perpendicular to the plane of the ring is $b$. The uniformly distributed radial and tangential stiffnesses are assumed to be $K_r$ and $K_\theta$, respectively. They have units of stiffness per radian. For a linear elastic foundation, the distributed radial stiffness $K_r$ is constant. However, for the unilateral elastic foundation, the radial stiffness vanishes when the elastic foundation is tensioned or compressed. A polar coordinate system with origin located at the ring center is adopted. The center of the ring is fixed and friction is neglected.
Figure 1: Thick ring on a two-parameter elastic foundation

The radial and tangential displacements at the centroid are assumed to be $u_r(R, \theta)$ and $u_\theta(R, \theta)$, respectively. Following [34], the cross-section of the ring is assumed to have a rotation $\phi(R, \theta)$ at the centroid at circumferential position $\theta$ and keeps its plane after deformation. Then, the radial and tangential displacements at an arbitrary point on the ring with radius $r$ and circumferential position $\theta$, $u_r(r, \theta)$ and $u_\theta(r, \theta)$, can be represented by:

$$
\begin{align*}
    u_r(r, \theta) &= u_r(R, \theta) \\
    u_\theta(r, \theta) &= u_\theta(R, \theta) + (r - R)\phi(R, \theta)
\end{align*}
$$

The strain-displacement relationships in polar coordinates are [35]:
\[ \varepsilon_{rr}(r, \theta) = \frac{\partial}{\partial r} u_{r}(r, \theta) \]
\[ \varepsilon_{\theta\theta}(r, \theta) = \frac{1}{r} \frac{\partial}{\partial \theta} u_{\theta}(r, \theta) + \frac{1}{r} u_{r}(r, \theta) \]
\[ \gamma_{r\theta}(r, \theta) = \frac{1}{r} \frac{\partial}{\partial \theta} u_{r}(r, \theta) + \frac{\partial}{\partial r} u_{\theta}(r, \theta) - \frac{1}{r} u_{\theta}(r, \theta) \]

where \( \varepsilon_{rr}(r, \theta) \), \( \varepsilon_{\theta\theta}(r, \theta) \), \( \gamma_{r\theta}(r, \theta) \) are the radial, tangential and shear strains, respectively.

The ring is assumed to be orthotropic and homogeneous. We consider that the ring material axes coincide with the polar coordinate system adopted. The stress-strain relationships are [35]:

\[ \sigma_{rr}(r, \theta) = \frac{E_{r} \left( \nu_{\theta} \varepsilon_{00}(r, \theta) + \varepsilon_{rr}(r, \theta) \right)}{-\nu_{\theta} \nu_{\theta} + 1} \]
\[ \sigma_{\theta\theta}(r, \theta) = \frac{E_{\theta} \left( \nu_{r} \varepsilon_{rr}(r, \theta) + \varepsilon_{\theta\theta}(r, \theta) \right)}{-\nu_{r} \nu_{\theta} + 1} \]
\[ \tau_{r\theta}(r, \theta) = G \gamma_{r\theta}(r, \theta) \]

where \( \sigma_{rr}(r, \theta) \), \( \sigma_{\theta\theta}(r, \theta) \) and \( \tau_{r\theta}(r, \theta) \) are radial, tangential and shear stresses, respectively. \( E_{r} \), \( E_{\theta} \) and \( G \) are the elastic moduli in the radial and tangential directions and the shear modulus, respectively. \( \nu \) is the Poisson’s ratio.

The strain energy stored in the ring is given by:

\[ U_{1} = \frac{b}{2} \int_{-\pi}^{\pi} \int_{R-\frac{h}{2}}^{R+\frac{h}{2}} \left( \sigma_{rr} \varepsilon_{rr} + \sigma_{\theta\theta} \varepsilon_{\theta\theta} + \tau_{r\theta} \gamma_{r\theta} \right) r \, dr \, d\theta \]  \hspace{1cm} (2.4)

The strain energy stored in the elastic foundation is given by:

\[ U_{2} = \frac{1}{2} \int_{-\pi}^{\pi} \left( K_{r} \left( u_{r} \left( R - \frac{h}{2}, \theta \right) \right)^{2} + K_{\theta} \left( u_{\theta} \left( R - \frac{h}{2}, \theta \right) \right)^{2} \right) d\theta \]  \hspace{1cm} (2.5)
Note here that the radial and tangential displacements for internal edge of the ring $u_r \left( R - \frac{h}{2}, \theta \right)$ and $u_\theta \left( R - \frac{h}{2}, \theta \right)$ couple the ring and the elastic foundation.

The work by the applied forces is obtained from:

$$W = b \int_{-\pi}^{\pi} \left( q_r \left( R + \frac{h}{2}, \theta \right) + q_\theta \left( R + \frac{h}{2}, \theta \right) \right) \left( R + \frac{h}{2} \right) \, d\theta$$

(2.6)

where $q_r = q_r \left( R + \frac{h}{2}, \theta \right)$ and $q_\theta = q_\theta \left( R + \frac{h}{2}, \theta \right)$ are distributed forces applied to the external edge of the ring (at the radial location $R + \frac{h}{2}$) in the radial and tangential directions, respectively. The units of $q_r$ and $q_\theta$ are in $\text{Force} \, \text{Area}$.

Invoking the principle of virtual work [36]:

$$\delta (U_1 + U_2) = \delta W$$

(2.7)

After substitution of (2.1) to (2.6) into (2.7) and some manipulations according to Euler–Lagrange equation, the governing equations for the present thick ring on an elastic foundation problem can be derived. Here, for brevity, $\nu_{r\theta}$ and $\nu_{\theta r}$ are set to zero for the governing equations given as (2.8). However, the proposed approach will hold for the general case where these are not zero.
\[-\frac{GA}{b} \frac{\partial}{\partial \theta} \phi - \frac{GA}{b} \frac{\partial^2}{\partial \theta^2} u_r + \left( \frac{EA_\theta}{Rb} + \frac{GA}{Rb} \right) \frac{\partial}{\partial \theta} u_\theta \]

\[+ \left( \frac{EA_\theta}{Rb} + \frac{K_\theta}{b} \right) u_r = \left( R + \frac{h}{2} \right) q_r \]

\[\left( \frac{GA}{b} - \frac{1}{2} \frac{K_\theta h}{b} \right) \phi + \frac{EA_\theta}{Rb} \frac{\partial^2}{\partial \theta^2} u_r + \left( \frac{EA_\theta}{Rb} - \frac{GA}{Rb} \right) \frac{\partial}{\partial \theta} u_\theta \]

\[+ \left( \frac{GA}{Rb} + \frac{K_\theta}{b} \right) u_\theta = \left( R + \frac{h}{2} \right) q_\theta \]

\[\frac{GA R}{b} + \frac{1}{4} \frac{K_\theta h^2}{b} \phi + \frac{EA_\theta h^2}{Rb} \frac{\partial^2}{\partial \theta^2} \phi + \frac{GA}{b} \frac{\partial}{\partial \theta} u_r \]

\[+ \left( \frac{GA}{b} - \frac{1}{2} \frac{K_\theta h}{b} \right) u_\theta = \frac{1}{2} \left( R + \frac{h}{2} \right) h q_\theta \]

where the following notations are used:

\[u_r = u_r (R, \theta) \]
\[u_\theta = u_\theta (R, \theta) \]
\[\phi = \phi (R, \theta) \]
\[q_r = q_r \left( R + \frac{h}{2}, \theta \right) \]
\[q_\theta = q_\theta \left( R + \frac{h}{2}, \theta \right) \]

and

\[EA_\theta = E_\theta \times A \]
\[GA = G \times A \]
\[EI_\theta = E_\theta \times I \]

where \( A = b \times h \) is the cross-sectional area of the ring and \( I = \frac{1}{12} bh^3 \) is the area moment of inertia of the cross-section. The following approximations are used in the manipulations to obtain the governing equations above, considering that \( R \gg h \).
\[
\int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} \frac{1}{r} \, dr \leq \frac{1}{R} \int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} \, dr = \frac{h}{R} \\
\int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} \frac{r - R}{r} \, dr \leq \frac{1}{R} \int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} (r - R) \, dr = 0 \\
\int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} \frac{(r - R)^2}{r} \, dr \leq \frac{1}{R} \int_{R^{-\frac{1}{2}}}^{R^{+\frac{1}{2}}} (r - R)^2 \, dr = \frac{1}{12} \frac{h^3}{R}
\]

2.4 Method of Solution

In this section, the governing equations (2.8) are first solved analytically considering a linear foundation with constant circumferential distribution of the foundation stiffnesses. The approach we use is to first get Fourier series expansion of the applied arbitrary force, and then the governing equations are solved harmonic-wise both in the radial and tangential directions. The solution for the total displacements will then be the superposition of the harmonic contributions in both directions. Then the solutions are extended to the unilateral foundation case via the iterative compensation scheme proposed in this work.

2.4.1 Solution for Linear Elastic Foundation

An arbitrary circumferentially distributed (with respect to \( \theta \)) or concentrated force \( F(\theta) \) can be decomposed into its radial and tangential components as:

\[
\bar{F}(\theta) = F_r(\theta) \bar{r} + F_\theta(\theta) \bar{\theta}
\]

where \( F_r(\theta) \) and \( F_\theta(\theta) \) are force components in the radial and tangential directions, respectively. Using Fourier expansion on \([-\pi, \pi]\), these components can be approximated by:
where \( N \) is the cut-off harmonic number, \( Q_{n,r,c} \), \( Q_{n,r,s} \), \( Q_{n,\theta,c} \) and \( Q_{n,\theta,s} \) are corresponding coefficients of the \( n^{th} \) harmonic force for radial, tangential and cosine, sine components. The subscript \( r \) or \( \theta \) indicates the coefficient is for radial or tangential direction, respectively, while \( c \) or \( s \) represents cosine or sine component, respectively. \( \theta_0 \) is used to define the rotation of the local cylindrical coordinate system with respect to the global Cartesian coordinate system. In this paper, \( \theta_0 \) is always set to 0, which indicates \( \theta = 0 \) corresponds to the bottom point of the ring.

Considering only the \( n^{th} \) harmonic force, the force per unit area applied on the external edge of the ring (where the radial location is \( R + \frac{h}{2} \)) can be written for the radial and tangential directions independently, where each direction consists of cosine and sine components:

\[
q_{nr}
\left(R + \frac{h}{2}, \theta\right) = q_{n,r,c}
\left(R + \frac{h}{2}, \theta\right) + q_{n,r,s}
\left(R + \frac{h}{2}, \theta\right)
\]

\[
q_{n\theta}
\left(R + \frac{h}{2}, \theta\right) = q_{n,\theta,c}
\left(R + \frac{h}{2}, \theta\right) + q_{n,\theta,s}
\left(R + \frac{h}{2}, \theta\right)
\]  

The cosine or sine component in each direction can be expressed in terms of the width of the ring, the radial coordinate and the corresponding force component:
The four components of the n\textsuperscript{th} harmonic force in (2.15) can be solved separately and then superposed to get the deformation for the n\textsuperscript{th} harmonic. In order to solve for the response to the first force component in (2.15), which is the cosine part in the radial direction, q_r and q_θ in (2.8) can be replaced with q_{nr} and q_{nθ}:

\[
q_{nr} = q_{n,r,c} \left( R + \frac{h}{2}, \theta \right) = \frac{1}{b(R + \frac{h}{2})} Q_{n,r,c} \cos(n(\theta - \theta_0))
\]

(2.16)

\[
q_{nθ} = 0
\]

Noting that the right hand side of the first equation in (2.8) has a cosine function, combined with the order of differentiation to the variables, it is easy to obtain the assumed ansatz:

\[
\begin{align*}
\nu_{r,n,r,c}(R, \theta) &= Cu_{r,c}(n) \cos(n(\theta - \theta_0)) \\
\nu_{θ,n,r,c}(R, \theta) &= Cu_{θ,c}(n) \sin(n(\theta - \theta_0)) \\
\phi_{n,r,c}(R, \theta) &= C\phi_{r,c}(n) \sin(n(\theta - \theta_0))
\end{align*}
\]

(2.17)

where \( u_{r,n,r,c}(R, \theta), u_{θ,n,r,c}(R, \theta) \) are displacement solutions in radial and tangential direction for the centroid and \( \phi_{n,r,c}(R, \theta) \) is the cross-section rotation. The subscripts \( n,r,c \) indicate that they are the solution components due to the cosine part of n\textsuperscript{th}
harmonic force in the radial direction. $Cu_{r,c}$, $Cu_{r,c}$ and $C\phi_{r,c}$ are modal coefficients to be solved for. They are indexed by the harmonic (circumferential mode) number $n$. Substituting (2.16)(2.17) into (2.8) and noting that the coefficients of $\cos \left(n(\theta - \theta_0)\right)$ or $\sin \left(n(\theta - \theta_0)\right)$ in all three equations in (2.8) should be equal, one obtains 3 equations with 3 unknown coefficients $Cu_{r,c}$, $Cu_{r,c}$ and $C\phi_{r,c}$. These can be solved explicitly and the results are given in the Appendix A. The responses to the other three components of the $n^{th}$ harmonic force in (2.15) can be solved for in a similar way.

The solutions for the sine component in the radial direction are given by:

$$u_{r,n,r,s}(R, \theta) = Cu_{r,c}(n) \sin\left(n(\theta - \theta_0)\right)$$
$$u_{\theta,n,r,s}(R, \theta) = Cu_{r,c}(n) \cos\left(n(\theta - \theta_0)\right)$$
$$\phi_{n,r,c}(R, \theta) = C\phi_{r,c}(n) \cos\left(n(\theta - \theta_0)\right)$$

And the solutions for the cosine and sine components in the tangential direction are given by:

$$u_{r,n,\theta,c}(R, \theta) = C\theta_{r,c}(n) \sin\left(n(\theta - \theta_0)\right)$$
$$u_{\theta,n,\theta,c}(R, \theta) = C\theta_{r,c}(n) \cos\left(n(\theta - \theta_0)\right)$$
$$\phi_{n,\theta,c}(R, \theta) = C\phi_{r,c}(n) \cos\left(n(\theta - \theta_0)\right)$$

The corresponding modal coefficients are given in the Appendix A.

The final displacement solutions for the governing equations in (2.8) are then written as follows:
2.4.2 Solution for Unilateral Elastic Foundation

The radial displacement solution $u_r(R, \theta)$ obtained from (2.20) considers support by the foundation whether the displacement is positive and negative. With a collapsible foundation, there is no support in regions where $u_r(R, \theta)$ is negative. In this region, an excessive force is included by the linear foundation assumption. The amount of this excessive force $F_e(\theta)$ is proportional to the extent of negative deformation:

$$F_e(\theta) = \begin{cases} u_r(R, \theta) Kr & \text{where } u_r(R, \theta) < 0 \\ 0, & \text{where } u_r(R, \theta) \geq 0 \end{cases} \quad (2.21)$$

For tensionless foundation, a similar excessive force exists in regions where $u_r(R, \theta)$ is positive. But the remainder of the analysis will be the same as the case of the collapsible foundation.

In order to counteract the excessive force and obtain the deformation for the unilateral foundation case, a compensation force $F_{cp}(\theta)$ which has the same magnitude as the excessive force can be applied to the linear elastic foundation:

$$F_{cp}(\theta) = F_e(\theta) \quad (2.22)$$

$F_{cp}$ can be expanded into Fourier series on $[-\pi, \pi]$ using the same harmonic numbers as $F(\theta)$:
Because the unilateral property of the foundation only exists in the radial direction, this compensation force will only have radial components.

The compensation force is then applied to the external edge of the foundation, at internal ring radius \( R - \frac{h}{2} \). The distributed compensation force per unit area in both the radial and tangential directions is given by:

\[
q_{cp,n,r,c} \left( R - \frac{h}{2}, \theta \right) = \frac{1}{b(R - \frac{h}{2})} H_{n,r,c} \cos\left(n(\theta - \theta_0)\right)
\]

\[
q_{cp,n,r,s} \left( R - \frac{h}{2}, \theta \right) = \frac{1}{b(R - \frac{h}{2})} H_{n,r,s} \sin\left(n(\theta - \theta_0)\right)
\]

\[
q_{cp,n,\theta,c} \left( R - \frac{h}{2}, \theta \right) = 0
\]

\[
q_{cp,n,\theta,s} \left( R - \frac{h}{2}, \theta \right) = 0
\]

Substitution of (2.24) into (2.8) for \( q_{nr} \) and \( q_{n\theta} \) leads to the governing equations for the compensation displacements driven by the compensation force. Note that the external ring radius \( R + \frac{h}{2} \) in the right hand sides of (2.8) needs to be replaced by the internal ring radius \( R - \frac{h}{2} \). The governing equations can then be solved using the same approach as above as they are still linear differential equations. In so doing, the nonlinear unilateral foundation problem is approximated by applying a compensation force to the linear foundation model instead of directly solving the system of coupled nonlinear differential equations for the problem.
However, instead of solving the linear differential equations again with the compensation force, the following observations can be used to get the compensation displacements to further reduce the computational load. Since the radius on the right hand sides of (2.8) (updated) will be canceled out by those in the denominators of (2.24), the displacement solutions are only affected by the magnitude of the forces regardless of how they are applied on the external edge of the foundations or the outside of the ring. So, based on (2.17) and (2.18), the cosine and sine components of the compensation displacements caused by the compensation force can be obtained directly by:

\[ u_{r,cp,n,r,c}(R, \theta) = \frac{H_{n,r,c}}{Q_{n,r,c}} C_{r,c}(n) \cos \left( n(\theta - \theta_0) \right) \]
\[ u_{\theta,cp,n,r,c}(R, \theta) = \frac{H_{n,r,c}}{Q_{n,r,c}} C_{\theta,c}(n) \sin \left( n(\theta - \theta_0) \right) \]
\[ \phi_{cp,n,r,c}(R, \theta) = \frac{H_{n,r,c}}{Q_{n,r,c}} C_{\phi,c}(n) \sin \left( n(\theta - \theta_0) \right) \]
\[ u_{r,cp,n,r,s}(R, \theta) = \frac{H_{n,r,s}}{Q_{n,r,s}} C_{r,s}(n) \sin \left( n(\theta - \theta_0) \right) \]
\[ u_{\theta,cp,n,r,s}(R, \theta) = \frac{H_{n,r,s}}{Q_{n,r,s}} C_{\theta,s}(n) \cos \left( n(\theta - \theta_0) \right) \]
\[ \phi_{cp,n,r,s}(R, \theta) = \frac{H_{n,r,s}}{Q_{n,r,s}} C_{\phi,s}(n) \cos \left( n(\theta - \theta_0) \right) \]

By using these simple algebraic equations, even the linear differential equations need to be solved only once.

Correspondingly, the compensation displacements are obtained from:
\[ u_{r, cp}(R, \theta) = \sum_{n=-N}^{N} \left[ u_{r, cp,n,r,c}(R, \theta) + u_{r, cp,n,r,s}(R, \theta) \right] \]

\[ u_{\theta, cp}(R, \theta) = \sum_{n=-N}^{N} \left[ u_{\theta, cp,n,r,c}(R, \theta) + u_{\theta, cp,n,r,s}(R, \theta) \right] \]  \hspace{1cm} (2.26)

\[ \phi_{cp}(R, \theta) = \sum_{n=-N}^{N} \left[ \phi_{cp,n,r,c}(R, \theta) + \phi_{cp,n,r,s}(R, \theta) \right] \]

The new total displacements are the summation of the original solutions (2.20) and the compensation displacements (2.26):

\[ \hat{u}_r(R, \theta) = u_r(R, \theta) + u_{r, cp}(R, \theta) \]

\[ \hat{u}_{\theta, cp}(R, \theta) = u_{\theta}(R, \theta) + u_{\theta, cp}(R, \theta) \]  \hspace{1cm} (2.27)

\[ \hat{\phi}_{cp}(R, \theta) = \phi(R, \theta) + \phi_{cp}(R, \theta) \]

Due to the change of radial displacement from \( u_r(R, \theta) \) to \( \hat{u}_r(R, \theta) \), the excessive force in (2.21) needs to be updated as well. Thus, the procedure from (2.21) to (2.27) needs to be repeated, thereby setting up an iterative scheme. This scheme is considered to converge when values of the compensated displacements in (2.27) do not change anymore with further iteration steps. The converged displacements in (2.27) can then be taken as the solution for the unilateral foundation case.

2.5 Results and Discussions

In this section, two examples are included to demonstrate the utility of the proposed method. In the first example, a radial point force with both positive and negative magnitudes is considered and the results obtained using the iterative compensation approach introduced above are compared with those obtained via FEA. In the second one, a more general distributed force, with both radial, tangential and sine, cosine components are applied and the solutions are shown.
2.5.1 Response to a Concentrated Force

As a basic comparison, we consider a radial concentrated force $F_C$ applied at the bottom of the ring. A Gaussian function representation allows one to model different degrees of "concentration" of this force:

$$f_c(\theta) = Q \frac{1}{\sqrt{\pi} \sigma^2} e^{-\frac{(\theta-\theta_0)^2}{\sigma^2}}$$  \hspace{1cm} (2.28)

where $Q = \pm 1000N$ is the magnitude of the force, $\sigma$ is a parameter that determines how concentrated the applied force is. The smaller $\sigma$ is, the more concentrated the force is. Figure 2 shows the effect of $\sigma$ to the distribution density of a unit force. It can be seen that with $\sigma \rightarrow 0$, $f_c(\theta_0) \rightarrow \infty$, and the force becomes an ideal concentrated force with 0 distribution width. The integration of the distribution density function around the ring always equals to the concentrated force:

$$\int_{-\pi}^{\pi} f_c(\theta) d\theta = F_c$$ \hspace{1cm} (2.29)

$f_c(\theta)$ can be expanded into Fourier series in $[-\pi, \pi]$:

$$f_c(\theta) = \sum_{n=-N}^{N} Q_n \cos(n(\theta - \theta_0)) = \frac{1}{2\pi} Q \sum_{n=-N}^{N} e^{-\frac{4n^2\sigma^2}{\pi}} \cos(n(\theta - \theta_0))$$ \hspace{1cm} (2.30)

that is:

$$Q_n = \frac{1}{2\pi} Q e^{-\frac{4n^2\sigma^2}{\pi}}$$ \hspace{1cm} (2.31)

The displacement responses due to this concentrated force are then solved by using the iterative scheme described above. A collapsible unilateral foundation is considered and compared with a linear foundation. Parameter values used for the results given below are
listed in Table 1. In this case, negligible tangential stiffness is considered for the comparison with FEA results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definitions</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>Centroid Radius</td>
<td>m</td>
<td>0.2</td>
</tr>
<tr>
<td>b</td>
<td>Model Width</td>
<td>m</td>
<td>0.06</td>
</tr>
<tr>
<td>h</td>
<td>Ring Thickness</td>
<td>m</td>
<td>0.02</td>
</tr>
<tr>
<td>$K_r$</td>
<td>Radial Stiffness Per Radian of Foundation</td>
<td>$N \cdot m \cdot \text{radian}$</td>
<td>$1e5$</td>
</tr>
<tr>
<td>$K_θ$</td>
<td>Torsional Stiffness Per Radian of Foundation</td>
<td>$N \cdot m \cdot \text{radian}$</td>
<td>1</td>
</tr>
<tr>
<td>$E_θ$</td>
<td>Extensional Modulus of the Ring</td>
<td>Pa</td>
<td>$1e10$</td>
</tr>
<tr>
<td>G</td>
<td>Shear Modulus of the Ring</td>
<td>Pa</td>
<td>$4e6$</td>
</tr>
<tr>
<td>$k_s$</td>
<td>Stiffness of Discretized Spring Element</td>
<td>$N \cdot m$</td>
<td>$5.236e3$</td>
</tr>
<tr>
<td>$F_c$</td>
<td>Magnitude of Concentrated Force</td>
<td>N</td>
<td>$\pm1000$</td>
</tr>
<tr>
<td>$σ$</td>
<td>Distribution Factor of Gaussian Function</td>
<td>NA</td>
<td>0.02</td>
</tr>
</tbody>
</table>

A 2-D FE model of the ring on a unilateral foundation is built using the commercial finite element software Abaqus/Standard. The thick ring is meshed using the second-order quadrilateral plane stress element with reduced integration, CPS8R, with 6 layers of elements in the radial direction and 120 divisions in the circumferential direction. In the examples, only radial stiffness is taken into account for the foundation, which is modeled using 120 evenly distributed and discretized spring elements SpringA. The property of these spring elements can be set to linear or nonlinear to represent the linear or unilateral elastic foundation. The stiffness of every linear spring element (or the effective non-vanished stiffness for the nonlinear spring element representing unilateral foundation) is calculated by:
\[ k_s = \frac{2\pi K_r}{N_s} \]  

where \( N_s = 120 \) is the number of the spring elements.

Figure 3 shows the deformation of the ring centroid for both collapsible and linear foundation with \( F_c = -1000 \text{N} \) (pointing to the ring center). The deformations are amplified by 10 times for clarity. It can be seen that the solutions obtained by the proposed iterative approach match the results by the FEA very well, both for the linear and unilateral foundation cases.

For the unilateral foundation case, it is expected that the magnitude of displacements will be different under a positive and negative force with same magnitude. Figure 4 shows the deformation responses under \( F_c = \pm 1000 \text{N} \) for the ring on collapsible foundation. In this figure, the displacements by positive force are shown with flipped sign so that they can be easily compared with those by negative force. It can be seen that for the collapsible foundation case, a negative force will lead to a larger magnitude of \( u_r \) at the position where the force is applied because negative force will directly compress the local area of the foundation to make it collapse while the positive force will stretch it so the effective stiffness there is higher than that for a negative force.

### 2.5.2 Response to a More Complex Distributed Force

A more complex distributed force is also considered as shown in Figure 5. This arbitrarily assigned distributed force consists of both tangential and radial components, and the corresponding distribution density is plotted separately. Due to the asymmetry of the distribution, the Fourier expansion of this general distributed force will include both
cosine and sine components. The same parameters as in Table 1 are used for the REF model except for the higher, more practical value of \( K_\theta = 0.5 \times 10^5 \text{ N/m per radian} \). The deformation results solved by this general force are plotted in Figure 6. Again the deformation is amplified by 10 times for clarity. In addition to the deformation of both linear and unilateral foundation cases, the identified collapsed regions can be seen in this figure, corresponding roughly to the circumferential regions of the applied distributed force (more on the left than on the right).

Finally, we remark that with a cut-off harmonic number \( N = 50 \), even for the complex applied force cases the iterative computations are completed within seconds on a modern PC. This makes the proposed methods suitable for parametric design studies where the different beam and foundation material properties, geometric dimensions, etc. can be quickly optimized.

### 2.6 Conclusion and Discussions

In this paper, the analysis of the well-studied REF model has been extended to the unilateral foundation case. This paper focused on the static deformation problem of a Timoshenko ring resting on a two-parameter foundation with unilateral radial stiffness and subject to arbitrary in-plane force. For the case of a linear foundation, the governing equations have been derived and solved analytically for both the radial and tangential displacements as well as for the section rotation. It is then observed that for a unilateral foundation, the ring on the linear foundation can be treated as supported by a distributed excessive force to the linear foundation model. Then, the nonlinear unilateral foundation problem can be solved via the compensation of this excessive force, thereby
setting up a simple iterative scheme. Comparison with FEA results for a concentrated radial force illustrated the validity and accuracy of the proposed approach. Then a more complex distributed force with both radial and tangential components is considered and the deformation as well as the determined collapsed region are shown.

This approach achieves the solution to the nonlinear unilateral foundation problem in a physically-motivated, fast and elegant way. The computation involves simple iterations based on linear analytical solutions. Direct numerical solution of the governing nonlinear differential equations is avoided. Compared with FEA, the proposed method achieves the nonlinear solution without time-consuming modeling and meshing work. Compared with the existing semi-analytical method in [15], the proposed method is capable of solving for the deformation response to an arbitrary complex distributed force in a unified way. Furthermore, in our continuing work, we shall show how the method can be easily extended to solve the static contact problem with arbitrary surface profiles. In a companion paper [26], the authors present the application of this iterative approach in the analysis of the dynamic forced response problem of a ring on a unilateral elastic foundation. Therein lies the major benefit of the iterative approach. It avoids the need for solving the complex, coupled and nonlinear differential equations for the unilateral elastic foundation. It can therefore be used to accelerate parametric design sensitivity studies and optimizations to select the proper ring and foundation materials and geometric properties for the envisaged application of the REF.
Figure 2: Effect of $\sigma$ to the distribution density of a unit force

Figure 3: Comparison of deformed centroid: collapsible foundation vs. linear foundation
Figure 4: Comparison of Deformed Centroid: Response by Positive and Negative Force

Figure 5: Distribution Density of a General Distributed Force
Figure 6: Deformation of Ring Centroid Due to the Distributed Force Given in Figure 5
CHAPTER 3: FORCED IN-PLANE VIBRATION OF THICK RING ON UNILATERAL ELASTIC FOUNDATION

3.1 Abstract

Most existing studies of the deformable ring on elastic foundation rely on the assumption of a linear foundation which is insufficient in some cases. This paper analyzes the in-plane dynamics of a thick ring on a nonlinear elastic foundation, where the stiffness of the elastic foundation changes with the deformation of the ring. Specifically, we consider a two-parameter unilateral elastic foundation, where the stiffness of the foundation is treated as linear in the torsional direction but unilateral (i.e. collapsible or tensionless) in the radial direction. The thick ring is modeled as an orthotropic and extensible Timoshenko beam. An arbitrary time-varying in-plane forcing function is considered. We use an implicit Newmark scheme to solve the partial differential equations of the established equations of motion in the time domain. An iterative compensation scheme is embedded into every time step to solve the unilateral foundation problem in the spatial domain. The proposed combination leads to a simple and efficient iterative scheme for solving the coupled and complex nonlinear dynamics of the thick ring on a unilateral elastic foundation. The dynamic axle force transmission is also analyzed. Pulse and chirp signal response results are included to illustrate the utility of the proposed scheme for analyzing the response of the ring on unilateral foundation in the time and frequency domains.

3.2 Introduction

The in-plane vibration of a deformable ring on elastic foundation (REF) model [9] [17] has been intensively studied due to its broad and important applications such as in
tires [3] [4] [5], wheels and gears [1] [2]. Most of the existing studies involving REFs assume a distributed elastic foundation, whose stiffness is uniformly constant around the ring. This assumption leads to an ideal linear REF model if the ring center is fixed, which can be solved analytically [13] [17]. The ring resting on this linear elastic foundation has been treated using different beam models and theories. The simplest model proposed for the ring is a tensioned string that has direct tensile strain but no bending stiffness [11]. More practically, a thin ring is modeled using Euler-Bernoulli beam theory, where both the tensile and the bending stiffnesses are taken into account by assuming that plane sections remain plane and always normal to the neural axis of the ring after deformation [5] [13]. Thick rings are often modeled using Timoshenko beam theory, which takes the shear deformation into account by assuming the normal of a plane section is subject to rotation on top of the tensile and bending effects [14]. The effect of the extensibility of the ring has also been addressed by combining both thin and thick ring models with linear and uniform elastic foundations [10] [9].

Allaei et al. [18], Wu and Parker [2] extended the studies to the linear but non-uniform elastic foundations. Allaei et al. [18] studied the natural frequencies and mode shapes of rings supported by a number of radial spring elements attached at arbitrary locations. Wu and Parker [2] studied the free vibration of rings on a general elastic foundation, whose stiffness distribution can be different circumferentially, and gave the closed-form expression for natural frequencies and vibration modes. In this case, however, the distribution of the stiffness is fixed and known. The application of this non-
uniform elastic foundation includes tires with non-uniformity and planetary gears where tooth meshes for the ring and the planets are not equally spaced.

The present paper deals with the in-plane vibration of a deformable ring resting on a more complex foundation, which is uniform circumferentially but with unilateral stiffness, i.e., a form of nonlinear elastic foundation. The stiffness offered by this kind of unilateral foundation can vanish locally where it is subjected to tension or compression. This deformation-dependent stiffness makes dynamics of the ring on unilateral foundation nonlinear and thus more difficult to solve than linear REF models. An example for the application is the non-pneumatic tire presented by [25], whose structure is a deformable shear ring supported by collapsible spokes which offer stiffness only in tension [32]. Another example is the bushing bearing, whose external sleeve can be modeled as a ring on a tensionless foundation and the internal sleeve as a ring on a collapsible foundation. However, the existing body of literature on the analysis of a ring on unilateral foundation is rather limited. Celep [20] studied the forced response of a thin and inextensible circular ring on tensionless two-parameter foundation under a time varying in-plane load. Direct numerical integration was used to solve the nonlinear differential equations. The tangential displacement of the ring was obtained from the inextensible ring assumption. This approach cannot be adopted to the more general extensible Timoshenko ring. Gasmi et al. [33] studied a Timoshenko ring resting on collapsible foundation, but their work was only limited to the static problem and cannot be straightforwardly extended to the dynamic case.
In our companion paper [16], which focuses on the static deformation of a thick ring on unilateral foundation, we proposed an iterative compensation scheme to analyze the unilateral foundation problem. The approach was built from the analytical solution of the linear foundation case by first computing the excessive force that would not be there with a unilateral foundation. Then, a compensation force is applied to the linear foundation model to counter-act this excessive force thereby setting up a simple algebraic iteration scheme to solve for the final deformation for the unilateral foundation case. In this paper, the iterative compensation approach is extended to solve the dynamic problem where the forced vibration of a thick ring on a unilateral elastic foundation is investigated. The ring is modeled as an orthotropic circular Timoshenko beam and the foundation is assumed to be a two-parameter elastic foundation, which has both radial and tangential stiffness but the one in the radial direction is unilateral. Linear viscous damping is incorporated into the Equations of Motions (EOMs) using Rayleigh’s dissipation functions via the extended Hamilton’s principle [37]. We adopted the implicit discrete integration scheme via the Newmark method to solve the time response for the unilateral foundation problem. At every time step, the iterative compensation approach mentioned above is implemented to obtain the spatial response. This iterative compensation scheme solves the unilateral foundation problem via simple algebraic iterations, and completely avoids time-consuming matrix inverse implementations needed in conventional implicit integration methods incorporated in most finite element analysis (FEA) packages. The proposed method is therefore suited for rapid design space analysis.
of mechanical structures such as tires and bushings that can be modeled using the simple
to parameterize ring on a unilateral foundation.

The rest of the paper is organized as follows. Section 3.3 states the problem and
develops the Equation of Motions (EOMs). Sections 3.4 gives the solution for both the
linear and unilateral foundation cases. In Section 3.5, pulse and chirp signal responses are
shown as examples to illustrate the effectiveness of the proposed method along with
comparisons with FEA results. Section 3.6 concludes the paper and discusses the future
work.

3.3 Statement of Problem and Equation of Motions

Figure 7 shows the schematic of the REF model. The ring with thickness \( h \) is
assumed to have a radius \( R \) at its centroid. The width perpendicular to the plane of the
ring is \( b \). The uniformly distributed radial and tangential stiffnesses are assumed to be \( K_r \)
and \( K_\theta \), respectively. These have units of stiffness per radian. For a linear elastic
foundation, the distributed radial stiffness \( K_r \) is constant. However, for the unilateral
elastic foundation, the radial stiffness vanishes where the elastic foundation is tensioned
or compressed. A local polar coordinate system with the origin located at the ring center
is adopted. The center of the ring is fixed.

At time \( t \), the radial and tangential displacements at the centroid of the ring are
assumed to be \( u_r(R,\theta,t) \) and \( u_\theta(R,\theta,t) \), respectively. Following [38], the cross-section
of the ring is assumed to have rotation \( \phi(R,\theta,t) \) at circumferential position \( \theta \) of the
centroid and keeps its plane after deformation. Then, the radial and tangential
displacements at an arbitrary point on the ring with radius \( r \) and circumferential position \( \theta \), \( u_r(r, \theta, t) \) and \( u_\theta(r, \theta, t) \), can be represented by:

\[
\begin{align*}
    u_r(r, \theta, t) &= u_r(R, \theta, t) \\
    u_\theta(r, \theta, t) &= u_\theta(R, \theta, t) + (r - R) \phi(R, \theta, t)
\end{align*}
\]  

(3.1)

**Figure 7:** Timoshenko ring on two-parameter elastic foundation

The strain-displacement relationships in polar coordinates are [39]:

\[
\begin{align*}
    \epsilon_{rr}(r, \theta, t) &= \frac{\partial}{\partial r} u_r(r, \theta, t) \\
    \epsilon_{\theta\theta}(r, \theta, t) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta(r, \theta, t) + \frac{1}{r} u_r(r, \theta, t) \\
    \gamma_{r\theta}(r, \theta, t) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_r(r, \theta, t) + \frac{\partial}{\partial r} u_\theta(r, \theta, t) - \frac{1}{r} u_\theta(r, \theta, t)
\end{align*}
\]  

(3.2)

where \( \epsilon_{rr}(r, \theta, t) \), \( \epsilon_{\theta\theta}(r, \theta, t) \), \( \gamma_{r\theta}(r, \theta, t) \) are the radial, tangential and shear strains, respectively.
The ring is assumed to be orthotropic and homogeneous. The stress-strain relationships are [39]:

\[
\begin{align*}
\sigma_{rr}(r, \theta, t) &= \frac{E_r \left( \nu_{\theta r} \epsilon_{\theta r}(r, \theta, t) + \epsilon_{rr}(r, \theta, t) \right)}{-\nu_{\theta r} \nu_{r r} + 1}, \\
\sigma_{\theta \theta}(r, \theta, t) &= \frac{E_\theta \left( \nu_{r \theta} \epsilon_{r \theta}(r, \theta, t) + \epsilon_{\theta \theta}(r, \theta, t) \right)}{-\nu_{r \theta} \nu_{r r} + 1}, \\
\tau_{r \theta}(r, \theta, t) &= G \gamma_{r \theta}(r, \theta, t)
\end{align*}
\]

(3.3)

where \( \sigma_{rr}(r, \theta, t) \), \( \sigma_{\theta \theta}(r, \theta, t) \) and \( \tau_{r \theta}(r, \theta, t) \) are radial, tangential and shear stresses, respectively. \( E_r \), \( E_\theta \) and \( G \) are elastic modulus in the radial and tangential directions and the shear modulus, respectively. \( \nu \) represents the Poisson’s ratio, and the subscript \( ij \) indicates the effect is from direction \( i \) to direction \( j \). In the following manipulations, \( \nu_{r \theta} \) and \( \nu_{\theta r} \) are set to zero merely for clarity of presentation. However, their actual values can be substituted in practical use.

The strain energy change in the ring from time \( t_0 \) to \( t_1 \) is obtained from:

\[
U_1 = \frac{b}{2} \int_{\theta_0}^{\theta_1} \int_{-\pi}^{\pi} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \left( \sigma_{rr} \epsilon_{rr} + \sigma_{\theta \theta} \epsilon_{\theta \theta} + \tau_{r \theta} \gamma_{r \theta} \right) r \, dr \, d\theta \, dt
\]

(3.4)

The strain energy change in the elastic foundation from time \( t_0 \) to \( t_1 \) is obtained from:

\[
U_2 = \frac{1}{2} \int_{\theta_0}^{\theta_1} \int_{-\pi}^{\pi} \left( K_r \left( u_r \left( R - \frac{h}{2}, \theta, t \right) \right)^2 + K_\theta \left( u_\theta \left( R - \frac{h}{2}, \theta, t \right) \right)^2 \right) d\theta \, dt
\]

(3.5)

Note here that the radial and tangential displacements for the internal edge of the ring \( u_r \left( R - \frac{h}{2}, \theta, t \right) \) and \( u_\theta \left( R - \frac{h}{2}, \theta, t \right) \) couple the ring and the elastic foundation.

The kinetic energy change of the ring within time interval \([t_0, t_1]\) is:
\[
T = \frac{\rho b}{2} \int_0^h \int_{-\pi}^\pi \int_{R-h/2}^{R+h/2} \left[ \left( \frac{\partial}{\partial t} u_r(r, \theta, t) \right)^2 + \left( \frac{\partial}{\partial t} u_\theta(r, \theta, t) \right)^2 \right] r \, dr \, d\theta \, dt \tag{3.6}
\]

where \( \rho \) is the mass density of the ring.

The virtual work of the applied forces is obtained by:

\[
W = b \int_0^h \int_{-\pi}^\pi \left( q_r \left( R + \frac{h}{2}, \theta, t \right) + q_\theta \left( R + \frac{h}{2}, \theta, t \right) \right) \left( R + \frac{h}{2} \right) d\theta \, dt \tag{3.7}
\]

where \( q_r = q_r \left( R + \frac{h}{2}, \theta, t \right) \) and \( q_\theta = q_\theta \left( R + \frac{h}{2}, \theta, t \right) \) are time variant distributed forces applied to the external edge of the ring (at the radial location \( R + \frac{h}{2} \)) in radial and tangential directions, respectively. The units of \( q_r \) and \( q_\theta \) are in \( \frac{\text{Force}}{\text{Area}} \).

Invoking the Hamilton’s principle [39]:

\[
\delta \left( U_1 + U_2 - T \right) = \delta W \tag{3.8}
\]

the EOM for the conservative system can be obtained via (3.8).

We then introduce viscous damping using the concept of the Rayleigh’s dissipation function [40]:

\[
\zeta_{r}\! \! \! \! \! \! \! \! \! = \frac{1}{2} C_{r} \left( \frac{\partial}{\partial t} u_r(r, \theta, t) \right)^2 \\
\zeta_{\theta}\! \! \! \! \! \! \! \! \! = \frac{1}{2} C_{\theta} \left( \frac{\partial}{\partial t} u_\theta(r, \theta, t) \right)^2 \\
\zeta_{r}\! \! \! \! \! \! \! \! \! = \frac{1}{2} C_{r} \left( \frac{\partial}{\partial t} u_r \left( R - \frac{h}{2}, \theta, t \right) \right)^2 \\
\zeta_{\theta}\! \! \! \! \! \! \! \! \! = \frac{1}{2} C_{\theta} \left( \frac{\partial}{\partial t} u_\theta \left( R - \frac{h}{2}, \theta, t \right) \right)^2 
\tag{3.9}
\]
where \( \zeta_{Rr}, \zeta_{R\theta}, \zeta_{Er} \) and \( \zeta_{E\theta} \) are Rayleigh’s dissipation functions for the radial, tangential direction of the ring and radial, tangential direction of the elastic foundation, respectively; and \( C_{Rr}, C_{R\theta}, C_{Er} \) and \( C_{E\theta} \) represent the viscous damping densities in the radial, tangential directions for the ring, and in radial, tangential directions for the elastic foundation, respectively. The unit of \( C_{Rr} \) and \( C_{R\theta} \) is \( \text{N} \text{m}^3 / \text{m} / \text{s} \), while the unit of \( C_{Er} \) and \( C_{E\theta} \) is \( \text{N} \text{radian m} / \text{s} \). According to the extended framework of Hamilton’s principle [37], the EOM for the non-conservative system, i.e. system with dampings, can be obtained by:

\[
\delta (U_1 + U_2 - T) + (R_R + R_E) = \delta W \tag{3.10}
\]

where \( R_R \) and \( R_E \) are defined as:

\[
R_R = \int_0^l \int_{\pi/2}^{3\pi/2} \left[ \left( \frac{\partial \zeta_{Er}}{\partial (\frac{\partial u_r}{\partial r})} \frac{\partial u_r}{\partial r} \right) + \left( \frac{\partial \zeta_{E\theta}}{\partial (\frac{\partial u_\theta}{\partial \theta})} \frac{\partial u_\theta}{\partial \theta} \right) \right] r dr d\theta dt \tag{3.11}
\]

\[
R_E = \int_0^l \int_{\pi/2}^{3\pi/2} \left[ \left( \frac{\partial \zeta_{Er}}{\partial (\frac{\partial u_r}{\partial r})} \frac{\partial u_r}{\partial r} \right) + \left( \frac{\partial \zeta_{E\theta}}{\partial (\frac{\partial u_\theta}{\partial \theta})} \frac{\partial u_\theta}{\partial \theta} \right) \right] \left( R - \frac{h}{2} \right) dr d\theta dt
\]

After substitution of (3.1) to (3.7) and (3.11) into (3.10) and some manipulations according to the Euler–Lagrange equation, the final EOMs are found to be as follows:
\[-\frac{GA}{b} \frac{\partial}{\partial \theta} \phi - \frac{GA}{Rb} \frac{\partial^2}{\partial \theta^2} u_r + \left( \frac{EA_0}{Rb} + \frac{GA}{Rb} \right) \frac{\partial}{\partial \theta} u_\theta + \left( \frac{EA_0}{Rb} + \frac{K_r}{b} \right) u_r,\]

\[+ \rho Rh \frac{\partial^2}{\partial t^2} u_r + \left( Rh C_{r_0} + \frac{C_{E_h}}{b} \right) \frac{\partial}{\partial t} u_r - q_r \left( R + \frac{h}{2} \right) \]

\[\left( -\frac{GA}{b} - \frac{1}{2} K_\theta \frac{h}{b} \right) \phi - \frac{EA_0}{Rb} \frac{\partial^2}{\partial \theta^2} u_\theta + \left( -\frac{EA_0}{Rb} - \frac{GA}{Rb} \right) \frac{\partial}{\partial \theta} u_r + \left( \frac{GA}{Rb} + \frac{K_\theta}{b} \right) u_\theta,\]

\[+ \rho Rh \frac{\partial^2}{\partial t^2} u_\theta + \left( Rh C_{r_0} + \frac{C_{E_h}}{b} \right) \frac{\partial}{\partial t} u_\theta + \frac{1}{12} \rho h^3 \frac{\partial^2}{\partial t^2} \phi + \left( \frac{1}{12} h^3 C_{r_0} - \frac{1}{2} \frac{C_{E_h}}{b} \right) \frac{\partial}{\partial t} \phi + q_\theta \left( R + \frac{h}{2} \right) \]

(3.12)

\[\left( \frac{GAR}{b} + \frac{1}{4} K_\theta \frac{h^2}{b} \right) \phi - \frac{EI_\theta}{Rb} \frac{\partial^2}{\partial \theta^2} \phi + \frac{GA}{b} \frac{\partial}{\partial \theta} u_r + \left( -\frac{GA}{b} - \frac{1}{2} K_\theta \frac{h}{b} \right) u_\theta + \frac{1}{12} \rho h^3 \frac{\partial^2}{\partial t^2} u_\theta \]

\[+ \left( \frac{1}{12} h^3 C_{r_0} + \frac{1}{2} \frac{C_{E_h}}{b} \right) \frac{\partial}{\partial t} u_\theta + \frac{1}{12} \rho Rh^3 \frac{\partial^2}{\partial t^2} \phi + \left( \frac{1}{12} Rh^3 C_{r_0} + \frac{1}{4} \frac{C_{E_h}}{b} \right) \frac{\partial}{\partial t} \phi - \frac{1}{2} q_\theta \left( R + \frac{h}{2} \right) \]

where:

\[u_r = u_r \left( R, \theta, t \right)\]
\[u_\theta = u_\theta \left( R, \theta, t \right)\]
\[\phi = \phi \left( R, \theta, t \right)\]
\[q_r = q_r \left( R + \frac{h}{2}, \theta, t \right)\]
\[q_\theta = q_\theta \left( R + \frac{h}{2}, \theta, t \right)\]

and

\[EA_0 = E_\theta A\]
\[GA = G A\]
\[EI_\theta = E_\theta I\]

(3.14)

with \( A = bh \) is the cross-sectional area of the ring and \( I = \frac{1}{12} bh^3 \) is the area moment of inertia of the cross-section. The following approximations are used in the manipulations to obtain the governing equations, considering the case that \( R \gg h \):
3.4 Solving the EOMs for the Ring on Elastic Foundation Problem

While our objective is to solve the EOMs for the dynamics of the ring on a unilateral elastic foundation, we step through the solution for the linear elastic foundation case as it is a building block for the iterative approach we construct later.

3.4.1 Solutions for Linear Elastic Foundation

In this section, the EOMs (3.12) are solved for the case of linear foundation assumptions with constant stiffness around the ring. Our approach is to get the solutions for the static problem first, then extend them to the dynamic problem.

We start by writing an arbitrary time-variant force as a product of a time-varying coefficient \( q(t) \) and a spatially-distributed static force vector \( \vec{F}(\theta) \):

\[
\vec{F}(\theta,t) = q(t) \vec{F}(\theta)
\]

The static force vector \( \vec{F}(\theta) \) can be decomposed into radial and tangential components and then expanded into a Fourier series on \([-\pi, \pi]\):

\[
\vec{F}(\theta) = F_r(\theta)\vec{r} + F_\theta(\theta)\vec{\theta}
\]

\[
F_r(\theta) = \sum_{n=-N}^{N} Q_{r,n} \cos(n(\theta - \theta_0)) + \sum_{n=-N}^{N} Q_{r,n} \sin(n(\theta - \theta_0))
\]

\[
F_\theta(\theta) = \sum_{n=-N}^{N} Q_{\theta,n} \cos(n(\theta - \theta_0)) + \sum_{n=-N}^{N} Q_{\theta,n} \sin(n(\theta - \theta_0))
\]
where \( N \) is the cut-off harmonic number, \( Q_{n,r,c}, Q_{n,r,s}, Q_{n,\theta,c} \) and \( Q_{n,\theta,s} \) are corresponding coefficients of the \( n^{th} \) harmonic force. The subscript \( r \) or \( \theta \) indicates the coefficient is for radial or tangential direction, respectively; while \( c \) or \( s \) represents cosine or sine components, respectively. \( \theta_0 \) is used to define the rotation of the local polar coordinate system with respect to the global Cartesian coordinate system, and \( \theta = \theta_0 \) corresponds to the bottom point of the ring.

Then, the distributed force per unit area applied to the external edge of the ring can be written in terms of corresponding components as:

\[
q_r \left( R + \frac{h}{2}, \theta, t \right) = \sum_{n=1}^{N} \left[ q_{n,r,c} \left( R + \frac{h}{2}, \theta, t \right) + q_{n,r,s} \left( R + \frac{h}{2}, \theta, t \right) \right]
\]

\[
q_\theta \left( R + \frac{h}{2}, \theta, t \right) = \sum_{n=1}^{N} \left[ q_{n,\theta,c} \left( R + \frac{h}{2}, \theta, t \right) + q_{n,\theta,s} \left( R + \frac{h}{2}, \theta, t \right) \right]
\]

The expressions for the components of the distributed force are given by:

\[
q_{n,r,c} \left( R + \frac{h}{2}, \theta, t \right) = \frac{1}{b(R + \frac{h}{2})} q(t) Q_{n,r,c} \cos \left( n(\theta - \theta_0) \right)
\]

\[
q_{n,r,s} \left( R + \frac{h}{2}, \theta, t \right) = \frac{1}{b(R + \frac{h}{2})} q(t) Q_{n,r,s} \sin \left( n(\theta - \theta_0) \right)
\]

\[
q_{n,\theta,c} \left( R + \frac{h}{2}, \theta, t \right) = \frac{1}{b(R + \frac{h}{2})} q(t) Q_{n,\theta,c} \cos \left( n(\theta - \theta_0) \right)
\]

\[
q_{n,\theta,s} \left( R + \frac{h}{2}, \theta, t \right) = \frac{1}{b(R + \frac{h}{2})} q(t) Q_{n,\theta,s} \sin \left( n(\theta - \theta_0) \right)
\]

The governing differential equations for the static case are obtained from (3.12) by setting the time derivatives in (3.12) to 0, and letting \( q(t) = 1 \):
In our companion paper [16] the solutions for (3.20) were derived to be as follows:

\[
\begin{align*}
&u_r(R, \theta) = \sum_{n=N}^{\infty} \frac{u_{r,n}}{R} \frac{u_{r,n,R}}{R} (R, \theta) + \frac{u_{r,n,R}}{R} (R, \theta) + \frac{u_{r,n,R}}{R} (R, \theta) + \frac{u_{r,n,R}}{R} (R, \theta) + \frac{u_{r,n,R}}{R} (R, \theta) \\
&u_{\theta}(R, \theta) = \sum_{n=N}^{\infty} \frac{u_{\theta,n}}{R} \frac{u_{\theta,n,R}}{R} (R, \theta) + \frac{u_{\theta,n,R}}{R} (R, \theta) + \frac{u_{\theta,n,R}}{R} (R, \theta) + \frac{u_{\theta,n,R}}{R} (R, \theta) + \frac{u_{\theta,n,R}}{R} (R, \theta) \\
&\phi(R, \theta) = \sum_{n=N}^{\infty} \frac{\phi_{n}}{R} \frac{\phi_{n,R}}{R} (R, \theta) + \frac{\phi_{n,R}}{R} (R, \theta) + \frac{\phi_{n,R}}{R} (R, \theta) + \frac{\phi_{n,R}}{R} (R, \theta) + \frac{\phi_{n,R}}{R} (R, \theta)
\end{align*}
\]

where static solutions \(u_r(R, \theta), u_{\theta}(R, \theta)\) and \(\phi(R, \theta)\) are superposed in terms of corresponding components for the \(n^{th}\) harmonic, which is represented by \(u_{r,n}(R, \theta), u_{\theta,n}(R, \theta)\) and \(\phi_n(R, \theta)\), respectively. Then, every \(n^{th}\) harmonic solution is superposed in terms of the radial/tangential directions and the cosine/sine components. The detailed expressions for these components are given in Appendix B.

Using (3.21), the dynamic solutions for the \(n^{th}\) harmonic component is assumed in the following form:

\[
\begin{align*}
u_{r,n,R}(R, \theta, t) &= a_{n,R} (i) u_{r,n,R}(R, \theta) + a_{n,R} (i) u_{r,n,R}(R, \theta) + a_{n,R} (i) u_{r,n,R}(R, \theta) + a_{n,R} (i) u_{r,n,R}(R, \theta) + a_{n,R} (i) u_{r,n,R}(R, \theta) \\
u_{\theta,n,R}(R, \theta, t) &= b_{n,R} (i) u_{\theta,n,R}(R, \theta) + b_{n,R} (i) u_{\theta,n,R}(R, \theta) + b_{n,R} (i) u_{\theta,n,R}(R, \theta) + b_{n,R} (i) u_{\theta,n,R}(R, \theta) + b_{n,R} (i) u_{\theta,n,R}(R, \theta) \\
\phi_n(R, \theta, t) &= c_{n,R} (i) \phi_n(R, \theta) + c_{n,R} (i) \phi_n(R, \theta) + c_{n,R} (i) \phi_n(R, \theta) + c_{n,R} (i) \phi_n(R, \theta) + c_{n,R} (i) \phi_n(R, \theta)
\end{align*}
\]

where \(a, b\) and \(c\) are time coefficients for the dynamic solution of \(u_r, u_{\theta}\) and \(\phi\), respectively. The subscripts indicate the harmonic number, direction and cosine/sine part.

To outline the solution procedure for the time coefficients, only the cosine component in
the radial direction of $n^{th}$ harmonic is shown here for simplicity. The coefficients for the other components can be solved for in a similar way. The dynamic solution for the cosine part in the radial direction of the $n^{th}$ harmonic component is:

$$u_{r,n,r,c}(R, \theta, t) = a_{r,n,r,c}(t)u_{r,n,r,c}(R, \theta)$$
$$u_{\theta,n,r,c}(R, \theta, t) = b_{n,r,c}(t)u_{\theta,n,r,c}(R, \theta)$$
$$\phi_{n,r,c}(R, \theta, t) = c_{n,r,c}(t)\phi_{n,r,c}(R, \theta)$$  \hspace{1cm} (3.23)

Substitution of (3.23) and the cosine component in the radial direction for $n^{th}$ harmonic of (3.18) into (3.12) generates the following equations for the time coefficients $a_{n,r,c}(t)$, $b_{n,r,c}(t)$ and $c_{n,r,c}(t)$:

$$\rho Rb \frac{d^2}{dt^2} r \phi_{n,r,c}(t) u_{n,r,c}(R, \theta) + \left( \frac{GA}{b} \frac{\partial^2}{\partial \theta^2} u_{n,r,c}(R, \theta) \right) a_{n,r,c}(t) u_{n,r,c}(R, \theta) + \left( \frac{1}{12} h^2 C_{m} - \frac{1}{2} C_{r} h \right) \frac{d^2}{dt^2} c_{n,r,c}(t) \phi_{n,r,c}(R, \theta) + \left( \frac{1}{12} h^2 C_{m} - \frac{1}{2} C_{r} h \right) \frac{d^2}{dt^2} b_{n,r,c}(t) \phi_{n,r,c}(R, \theta) = 0$$

Note that the static solution components in (3.23), i.e., the factors $u_{r,n,r,c}(R, \theta)$, $u_{\theta,n,r,c}(R, \theta)$ and $\phi_{n,r,c}(R, \theta)$, are independent of time (constants) as given in our previous paper [16]. Then, (3.24) can be written more compactly as:
A straightforward comparison of (3.24) and (3.25) leads to the definitions of the constant coefficients in (3.25). These are known constants from the static solutions.

The dynamic linear REF model (3.25) can be written in the mass-spring-damper form:

\[ M_{n,r,c} \dot{A}_{n,r,c}(t) + C_{n,r,c} \dot{V}_{n,r,c}(t) + K_{n,r,c} X_{n,r,c}(t) = \Gamma_{n,r,c}(t) \quad (3.26)\]

where the matrices and vectors are given by:

\[
A_{n,r,c}(t) = \begin{bmatrix}
\frac{\partial^2}{\partial t^2} a_{n,r,c}(t) \\
\frac{\partial^2}{\partial t^2} b_{n,r,c}(t) \\
\frac{\partial^2}{\partial t^2} c_{n,r,c}(t)
\end{bmatrix}
V_{n,r,c}(t) = \begin{bmatrix}
\frac{\partial}{\partial t} a_{n,r,c}(t) \\
\frac{\partial}{\partial t} b_{n,r,c}(t) \\
\frac{\partial}{\partial t} c_{n,r,c}(t)
\end{bmatrix}
X_{n,r,c}(t) = \begin{bmatrix}
a_{n,r,c}(t) \\
b_{n,r,c}(t) \\
c_{n,r,c}(t)
\end{bmatrix}
\]

\[
M_{n,r,c} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
C_{n,r,c} = \begin{bmatrix}
c_{nr,c,11} & 0 & 0 \\
0 & c_{nr,c,22} & c_{nr,c,23} \\
0 & c_{nr,c,32} & c_{nr,c,33}
\end{bmatrix}
K_{n,r,c} = \begin{bmatrix}
k_{nr,c,11} & k_{nr,c,12} & k_{nr,c,13} \\
k_{nr,c,21} & k_{nr,c,22} & k_{nr,c,23} \\
k_{nr,c,31} & k_{nr,c,32} & k_{nr,c,33}
\end{bmatrix}
\]

\[
\Gamma_{n,r,c}(t) = q(t) \Gamma_{n,r,c}^T = q(t) \begin{bmatrix}
Q_{n,r,c} \\
\rho RhbCur_{r,c}(n)
\end{bmatrix}
\]
time coefficients \( a_{n,r,c}(t) \), \( b_{n,r,c}(t) \) and \( c_{n,r,c}(t) \), the classical Newmark method [41] is adopted to implicitly solve the time-integration of the mass-spring-damper system (3.26):

\[
V_{n,r,c}(t + \Delta t) = V_{n,r,c}(t) + \left((1 - \alpha) A_{n,r,c}(t + \Delta t) + \alpha A_{n,r,c}(t)\right) \Delta t
\]

\[
X_{n,r,c}(t + \Delta t) = X_{n,r,c}(t) + \Delta t V_{n,r,c}(t) + \left(\beta A_{n,r,c}(t + \Delta t) + \left(\frac{1}{2} - \beta\right) A_{n,r,c}(t)\right) \Delta t^2
\]  

(3.28)

where \( \Delta t \) is the time step. \( \alpha \) and \( \beta \) are weight coefficients, and the average acceleration method with \( \alpha = \frac{1}{2} \), \( \beta = \frac{1}{4} \) is adopted. Combining (3.26) and (3.28), the displacement increment for every time step can be obtained by:

\[
\Delta X_{n,r,c}(t + \Delta t) = \frac{1}{2} \Delta t \left( C_{n,r,c} \alpha \Delta t^2 + 2 C_{n,r,c} \beta \Delta t^2 - C_{n,r,c} \Delta t^2 - M_{n,r,c} \Delta t \right) A_{n,r,c}(t)
\]

\[
+ \frac{1}{2} \Delta t \left( 2 C_{n,r,c} \alpha \Delta t - 2 C_{n,r,c} \Delta t - 2 M_{n,r,c} \Delta t \right) V_{n,r,c}(t)
\]

\[
- \frac{\Delta t^2 \beta}{-K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \Delta \Gamma_{n,r,c}(t + \Delta t)
\]  

(3.29)

Then the obtained displacement for the current time step:

\[
X_{n,r,c}(t + \Delta t) = X_{n,r,c}(t) + \Delta X_{n,r,c}(t + \Delta t)
\]  

(3.30)

\( A_{n,r,c}(t + \Delta t) \) and \( V_{n,r,c}(t + \Delta t) \) can be solved by substitution of (3.30) into (3.28). The incremental forcing vector \( \Delta \Gamma_{n,r,c}(t + \Delta t) \) is given as

\[
\Delta \Gamma_{n,r,c}(t + \Delta t) = \Gamma_{n,r,c}(t + \Delta t) - \Gamma_{n,r,c}(t) = \left[q(t + \Delta t) - q(t)\right] \Gamma_{n,r,c}^r
\]

\[
= \Delta q \left(t + \Delta t\right)
\]

\[
= \begin{bmatrix}
\frac{Q_{n,r,c}}{\rho R h b C \left.r_{r,c}(n)\right.}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0
0
\end{bmatrix}
\]

(3.31)
Noted that the state of the previous time step $A_{n,r,c}(t)$, $V_{n,r,c}(t)$ and the incremental forcing vector of the current time step $\Delta \Gamma_{n,r,c}(t + \Delta t)$ are used to get the current displacement $X_{n,r,c}(t + \Delta t)$ in (3.29) and (3.30). This is an implicit time iteration scheme. With any known initial conditions $X_{n,r,c}(0)$, $V_{n,r,c}(0)$, $A_{n,r,c}(0)$ and any time-varying forcing coefficient $q(t)$, EOMs (3.12) can be solved with this scheme.

Substitution of the $a_{n,r,c}(t)$, $b_{n,r,c}(t)$, $c_{n,r,c}(t)$ solved from (3.30) into (3.23), leads to the time-variant displacements of the ring’s centroid. Consequently, the time-variant displacements of any point on the ring can be obtained via (3.1). The accelerations and velocities are obtained analogously using the time derivatives of $a_{n,r,c}(t)$, $b_{n,r,c}(t)$ and $c_{n,r,c}(t)$.

The time coefficients for the other three components of the displacements in (3.22) can be solved for in the exact same way. Finally, the total time-variant displacements for the centroid are obtained as:

$$u_r(R, \theta, t) = \sum_{n=-N}^{N} u_{r,n}(R, \theta, t)$$

$$u_\theta(R, \theta, t) = \sum_{n=-N}^{N} u_{\theta,n}(R, \theta, t); \quad (3.32)$$

$$\phi(R, \theta, t) = \sum_{n=-N}^{N} \phi_n(R, \theta, t);$$

### 3.4.2 Solution for Ring on Unilateral Elastic Foundation

In order to get the time domain solution for the unilateral foundation case, it is assumed that the status for previous time step at time $t$, $\bar{X}(t)$, $\bar{V}(t)$ and $\bar{A}(t)$ are already known. If the increments from time $t$ to $t + \Delta t$ for the unilateral foundation can be solved
for, the solution for any time point can be obtained via a time iterative scheme. Considering the transition from time $t$ to $t + \Delta t$, based on the linear foundation model, the displacement increment given by (3.29) can be written in terms of the state of the previous time step, and the mass, damping and linear stiffness matrices:

$$\Delta \hat{X}_{n,r,c}(t + \Delta t) = \frac{1}{2} \Delta t \left( C_{n,r,c} \alpha \Delta t^2 + 2 C_{n,r,c} \beta \Delta t^2 - C_{n,r,c} \Delta t^2 - M_{n,r,c} \Delta t \right) \hat{X}_{n,r,c}(t)$$

$$+ \frac{1}{2} \Delta t \left( 2 C_{n,r,c} \alpha \Delta t - 2 C_{n,r,c} \Delta t - 2 M_{n,r,c} \right) \hat{v}_{n,r,c}(t)$$

$$- \frac{\Delta t^2 \beta}{-K_{n,r,c} \Delta t^2 + C_{n,r,c} \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \Delta \Gamma_{n,r,c}(t + \Delta t)$$

(3.33)

Then, the new displacement of the current time step is:

$$\hat{X}_{n,r,c}(t + \Delta t) = \hat{X}_{n,r,c}(t) + \Delta \hat{X}_{n,r,c}(t + \Delta t)$$

(3.34)

Correspondingly, $\hat{u}_r(R, \theta, t + \Delta t)$ and $\Delta \hat{u}_r(R, \theta, t + \Delta t)$ are obtained, although the linear stiffness matrix $K_{n,r,c}$ is utilized. Compared with collapsible foundation, where foundation force vanishes in the region where $\hat{u}_r(R, \theta, t + \Delta t) < 0$, the transition from time $t$ to $t + \Delta t$ solved in (3.33) and (3.34) includes effects by an excessive force in that region. The magnitude of this excessive force at time $t + \Delta t$ is proportional to the displacement increment $\Delta \hat{u}_r(R, \theta, t + \Delta t)$, but the region is determined by the total displacement $\hat{u}_r(R, \theta, t + \Delta t)$. That is,

$$F^{t+\Delta t}_e(\theta) = \begin{cases} 
\Delta \hat{u}_r(R, \theta, t + \Delta t) K_r, & \{\theta \mid \hat{u}_r(R, \theta, t + \Delta t) < 0\} \\
0, & \{\theta \mid \hat{u}_r(R, \theta, t + \Delta t) \geq 0\}
\end{cases}$$

(3.35)

For tensionless foundation case, this excessive force exists in the opposite region where $\hat{u}_r(R, \theta, t + \Delta t) > 0$, but all the remaining analysis will be the same.
Similar to the static case we outlined in [16], a compensation force can be applied to the linear foundation model to counter-act the excessive force at the current time step:

\[ F_{cp}^{t+\Delta t}(\theta) = F_e^{t+\Delta t}(\theta) \]  

(3.36)

Applying Fourier expansion to (3.36) within \([-\pi, \pi]\), and using the same mode numbers as \(\bar{F}(\theta)\):

\[ F_{cp}^{t+\Delta t}(\theta) = \sum_{n=-N}^{N} F_{cp,n}^{t+\Delta t}(\theta) = \sum_{n=-N}^{N} \left[ H_{n,r,c}^{t+\Delta t} \cos(n(\theta - \theta_0)) + H_{n,r,s}^{t+\Delta t} \sin(n(\theta - \theta_0)) \right] \]  

(3.37)

where \(F_{cp,n}^{t+\Delta t}\) is the compensation force of \(n^{th}\) harmonic for the current time step; \(H_{n,r,c}^{t+\Delta t}\) and \(H_{n,r,s}^{t+\Delta t}\) are the Fourier coefficients. Because the unilateral property of the foundation only exists in the radial direction, this compensation force only has radial components. Again, we neglect the sine part in the following derivation, merely for simplicity and clarity.

This compensation force needs to be applied to the linear foundation model at the current time step to obtain the displacements for the unilateral foundation case. According to (3.31) and using the analogy between \(Q_{n,r,c}\Delta q(t + \Delta t)\) and \(H_{n,r,c}^{t+\Delta t}\), the forcing vector due to the \(n^{th}\) harmonic compensation force for the current time step is given by:

\[
\Gamma_{cp,n,r,c}^{t+\Delta t} = \begin{bmatrix}
H_{n,r,c}^{t+\Delta t} \\
\rho R h b C u r_{r,c}(n) \\
0 \\
0
\end{bmatrix}
\]  

(3.38)
Now, the summated forcing vectors applied to the linear foundation model at the current time step are:

$$\Delta \Gamma_{n,r,c}(t+\Delta t) + \Gamma^{\Delta t}_{ep,n,r,c} = \left[ \Delta q(t+\Delta t) + \frac{H^{i+\Delta t}_{n,r,c}}{Q_{n,r,c}} \right] \Gamma^t_{n,r,c}$$  \hspace{1cm} (3.39)

Then, the compensated displacement increment is obtained by replacing the external forcing vector $\Delta \Gamma_{n,r,c}(t+\Delta t)$ in (3.33) with the summation of the external and compensation forcing vectors:

$$\Delta \hat{X}_{n,r,c}(t+\Delta t) = \frac{1}{2} \left[ \frac{\Delta t (C_{n,r,c} \alpha \Delta t^2 + 2 C_{n,r,c} \beta \Delta t^3 - C_{n,r,c} \Delta t^2 - M_{n,r,c} \Delta t)}{-K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \right] \hat{\Lambda}_{n,r,c}(t) \hspace{1cm} (3.40)$$

$$+ \frac{1}{2} \left[ \frac{\Delta t [2 C_{n,r,c} \alpha \Delta t - 2 C_{n,r,c} \Delta t - 2 M_{n,r,c}]}{-K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \right] \hat{V}_{n,r,c}(t)$$

$$- \frac{\Delta t^2}{-K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \left[ \Delta q(t+\Delta t) + \frac{H^{i+\Delta t}_{n,r,c}}{Q_{n,r,c}} \right] \Gamma^t_{n,r,c}$$

The new increment in (3.40) and the corresponding new displacement via (3.34) lead to new references to obtain the excessive force in (3.35). This leads to an iterative scheme within each time step which is deemed to converge when the re-computation of (3.34) to (3.40) does not change $\Delta \hat{X}_{n,r,c}(t+\Delta t)$ any more. Once converged, $\hat{X}_{n,r,c}(t+\Delta t)$ is plugged into (?) to get new $\hat{\Lambda}_{n,r,c}(t+\Delta \text{right})$ and $\hat{V}_{n,r,c}(t+\Delta t)$ for the iteration to the next time step.

The three remaining components in (3.22) for the unilateral foundation can be solved for by following the same procedure. The final solution for the unilateral case is obtained via the reuse of (3.22) and (3.32).

It is worth mentioning that the third component in the right hand side of (3.40) can be obtained algebraically from the third component in the right hand side of (3.33),
via the proportional relationships of the forcing vectors. Furthermore, due to the fact that the constant linear stiffness matrix is used, all the matrices \((K_{n,r,c}, C_{n,r,c}, M_{n,r,c})\) used in (3.40) are constant and time-invariant. This means that they can be pre-computed at the beginning of the whole iteration. Compared with conventional nonlinear implicit time integration method, where the time consuming matrix inversion needs to be implemented in every nonlinear iteration of each time step, tremendous numerical efficiency is obtained by this method.

3.4.3 Axle Force Transmission

The foundation is essentially modeled as continuous springs with viscous damping in radial and tangential directions via \(K_r, K_\theta\) and viscous damping \(C_{Er}, C_{E\theta}\). The force transmitted to the axle/ring center by the foundation is therefore readily related to the velocity and deformation of the foundation. The radial and tangential displacements of the foundation, which are same as the radial and tangential displacement of the internal rim of the ring, are given via (3.1):

\[
\begin{align*}
    u_{r,f}(\theta, t) &= u_r\left(R - \frac{h}{2}, \theta, t\right) = u_r(R, \theta, t) \\
    u_{\theta,f}(\theta, t) &= u_\theta\left(R - \frac{h}{2}, \theta, t\right) = u_\theta(R, \theta, t) - \frac{h}{2} \phi(R, \theta, t)
\end{align*}
\]

(3.41)

Consequently, the velocities of the foundation in both the radial and tangential direction are given by:

\[
\begin{align*}
    \frac{\partial}{\partial t} u_{r,f}(\theta, t) &= \frac{\partial}{\partial t} u_r(R, \theta, t) \\
    \frac{\partial}{\partial t} u_{\theta,f}(\theta, t) &= \frac{\partial}{\partial t} u_\theta(R, \theta, t) - \frac{h}{2} \frac{\partial}{\partial t} \phi(R, \theta, t)
\end{align*}
\]

(3.42)
Considering that the excitation force only includes the radial cosine components, and according to the above analysis, the centroid displacements and velocities are given by:

\[
\begin{align*}
\mathbf{u}_r(R, \theta, t) &= \sum_{n=-N}^{N} a_{n,r,c}(t)u_{r,n,r,c}(R, \theta) \\
\mathbf{u}_\theta(R, \theta, t) &= \sum_{n=-N}^{N} b_{n,r,c}(t)u_{\theta,n,r,c}(R, \theta) \\
\mathbf{\phi}(R, \theta, t) &= \sum_{n=-N}^{N} c_{n,r,c}(t)\phi_{n,r,c}(R, \theta)
\end{align*}
\] (3.43)

The static displacement solutions appearing above are derived in our previous paper [16]:

\[
\begin{align*}
u_{r,n,r,c}(R, \theta) &= Cur_{r,c}(n)\cos(n(\theta - \theta_0)) \\
u_{\theta,n,r,c}(R, \theta) &= Cu_{\theta,r,c}(n)\sin(n(\theta - \theta_0)) \\
\phi_{n,r,c}(R, \theta) &= C\phi_{r,c}(n)\sin(n(\theta - \theta_0))
\end{align*}
\] (3.44)

The modal coefficients are listed in Appendix A.

We consider the force transmitted in the opposite direction of \( \theta_0 \). The force induced by radial stiffness and damping, and that by the tangential direction, are obtained, respectively, by:
\[ F_{A,r} = \int_{-\pi+\theta_0}^{\pi+\theta_0} \left[ u_{r,f}(\theta,t)K_r + \frac{\partial}{\partial t}u_{\theta,f}(\theta,t)C_{Er} \right] \cos(\theta - \theta_0) \, d\theta \]
\[ F_{A,f} = \int_{-\pi+\theta_0}^{\pi+\theta_0} \left[ u_{\theta,f}(\theta,t)K_r + \frac{\partial}{\partial t}u_{\theta,f}(\theta,t)C_{Er} \right] \sin(\theta - \theta_0) \, d\theta \]  

Substitution of (3.41)(3.44) into (3.45) leads to:

\[ F_{A,r} = \left[ \sum_{n=-N, n \neq \pm 1}^{N} \frac{2\sin(\pi n)Cu_{r,c}(n)}{n^2 - 1} - 2\pi Cu_{r,c}(1) \right] \left[ a_{n,r,c}(t)K_r + \frac{d}{dt}a_{n,r,c}(t)C_{Er} \right] \]
\[ F_{A,f} = \left[ \sum_{n=-N, n \neq \pm 1}^{N} \frac{2\sin(\pi n)Cu_{\theta,r,c}(n)}{n^2 - 1} + 2\pi Cu_{\theta,r,c}(1) \right] \left[ b_{n,r,c}(t)K_{\theta} + \frac{d}{dt}b_{n,r,c}(t)C_{E\theta} \right] \]

Noted that when \( n \neq \pm 1, \sin(n\pi) = 0 \), which indicates that only the modes with \( n = \pm 1 \) contribute to the axle force, (3.46) is simplified to:

\[ F_{A,r} = -2\pi Cu_{r,c}(1) \left[ a_{n,r,c}(t)K_r + \frac{d}{dt}a_{n,r,c}(t)C_{Er} \right] \]  
\[ F_{A,f} = 2\pi Cu_{\theta,r,c}(1) \left[ b_{n,r,c}(t)K_{\theta} + \frac{d}{dt}b_{n,r,c}(t)C_{E\theta} \right] - h\pi C_{\phi_{r,c}}(1) \left[ c_{n,r,c}(t)K_{\theta} + \frac{d}{dt}c_{n,r,c}(t)C_{E\theta} \right] \]  

For the unilateral foundation, \( K_r \) is effective only when \( u_{r,f}(\theta,t) > 0 \) (for collapsible foundation) or \( u_{r,f}(\theta,t) < 0 \) (for tensionless foundation). So, (3.47) becomes:

\[ \hat{F}_{A,r} = -\theta Cu_{r,c}(1) \left[ \dot{a}_{n,r,c}(t)K_r + \frac{d}{dt}\dot{a}_{n,r,c}(t)C_{Er} \right] \left\{ \theta \mid \theta \in u_{r,f}(\theta,t) > 0 \text{ or } u_{\theta,f}(\theta,t) < 0 \right\} \]
\[ \hat{F}_{A,f} = 2\pi Cu_{\theta,r,c}(1) \left[ \dot{b}_{n,r,c}(t)K_{\theta} + \frac{d}{dt}\dot{b}_{n,r,c}(t)C_{E\theta} \right] - h\pi C_{\phi_{r,c}}(1) \left[ \dot{c}_{n,r,c}(t)K_{\theta} + \frac{d}{dt}\dot{c}_{n,r,c}(t)C_{E\theta} \right] \]  

For both the linear and unilateral foundation cases, the total axle force is the summation of the two components:

\[ F_A = F_{A,r} + F_{A,f} \]  

(3.49)
3.5 Illustrative Results

In this section, two examples are given to show the effectiveness of the approach presented above. First, the results for the pulse force response are shown and compared with Finite Element Analyses (FEA) results; then chirp signal response is shown and used to analyze the frequency response and axle force transmissibility.

3.5.1 Pulse Response and Comparison with FEA Results

A pulse signal in the form of a radial concentrated force is applied to the bottom of the ring and the response for both the linear foundation and unilateral foundation cases are computed and compared. The applied time-varying radial concentrated force can be represented by:

\[
f_c(\theta, t) = q(t)\frac{1}{2\pi} Q \sum_{n=-N}^{N} e^{-\frac{n^2}{\sigma^2}} \cos\left(n(\theta - \theta_0)\right)
\]

(3.50)

where time-varying function \( q(t) \) controls the signal shape in the time domain. \( \sigma \) is a parameter that determines how the force is concentrated. \( Q \) determines the magnitude of the force, in this case \( Q = -1000 \)N is used, where the minus sign indicates the force is pointing to the ring's center. A collapsible unilateral foundation is considered. The rest of the parameters used are listed in Table 2.

Table 2: Value of Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definitions</th>
<th>Units</th>
<th>Values</th>
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<tbody>
<tr>
<td>( R )</td>
<td>Centroid Radius</td>
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<tr>
<td>( b )</td>
<td>Model Width</td>
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<td>( h )</td>
<td>Ring Thickness</td>
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<td>Description</td>
<td>Unit</td>
<td>Value</td>
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<td>------------</td>
<td>--------------------------------------------------</td>
<td>---------------</td>
<td>-----------</td>
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<tr>
<td>$K_\theta$</td>
<td>Torsional Stiffness Per Radian of Foundation</td>
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<tr>
<td>$G$</td>
<td>Shear Modulus of the Ring</td>
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<td>$\sigma$</td>
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<tr>
<td>$C_{Er}$</td>
<td>Viscous Damping Density in Radial Direction of Foundation</td>
<td>N/radian (m/s)</td>
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</tr>
<tr>
<td>$C_{E\theta}$</td>
<td>Viscous Damping Density in Tangential Direction of Foundation</td>
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<td>0</td>
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<tr>
<td>$C_{Rr}$</td>
<td>Viscous Damping Density in Radial Direction of Ring</td>
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<tr>
<td>$C_{R\theta}$</td>
<td>Viscous Damping Density in Tangential Direction of Ring</td>
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<tr>
<td>$\rho$</td>
<td>Mass Density of Ring</td>
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</tbody>
</table>

An equivalent 2D Finite Element model is established using Abaqus software to validate the presented solution method. The thick ring is meshed using the Timoshenko beam element B21, with 360 divisions circumferentially. The radial stiffness of the foundation is modeled via 360 evenly distributed and independent linear or nonlinear Spring2 elements, and the torsional stiffness is generated by another 360 linear Spring2 elements. The stiffness of every linear spring element, or the effective non-vanished stiffness for the nonlinear spring element representing the unilateral foundation, is obtained by:

$$
k_r = \frac{2\pi K_r}{N_r}
$$

$$
k_\theta = \frac{2\pi K_\theta}{N_\theta}
$$

(3.51)
where \( N_r = N_\theta = 360 \) are number of spring elements in the radial and circumferential directions, respectively. The dynamic response is obtained using Abaqus/Standard solver.

The time responses of the loaded point (bottom point) displacement, as well as the axle force are shown in Figure 8 and compared with FEA results. It is expected that small deviations will exist between the results by the presented method and those by FEA because the modeling approaches differ substantially. It can be seen that the errors are bounded for linear foundation cases, but these errors accumulate for the unilateral (nonlinear) foundation cases as is typical for time integrations of nonlinear systems [42]. However, for the present analysis, the proposed scheme gives very close results to those of the FEA, especially in computing the transmitted axle force.

The deformation of the ring’s centroid at select time points are plotted in Figure 9, with the magnitudes of displacements amplified by 5 times. At around \( t = 0.07 \)s, the compression of the loaded point in the unilateral foundation case approaches its maximum magnitude, and bounces back at around \( t = 0.09 \)s. For every time point, the identified collapsible regions of the foundation are also shown in magenta circles on top of the deformed centroid of collapsible foundation case.

3.5.2 Chirp Signal Response

The frequency response and axle force transmissibility of the unilateral foundation problem can be studied via the chirp signal response. In this example, a chirp signal with frequency range from 10Hz to 100Hz is used. The peak to peak magnitude of the chirp signal is set to 40N. The chirp signal is offset by a static force with magnitude
±1000N. These opposite offsets lead to chirp forces in different directions. The time response to these opposite chirp forces by both the unilateral and linear foundations are analyzed in the frequency domain and the results are shown in Figure 10. The left column shows the results for collapsible foundation and the right column is for linear foundation. The frequency content for the applied chirp force is shown in the bottom row. It can be seen that the dominant components are between 20Hz to 90Hz and float around a mean value. And the static offset (positive or negative 1000N) of the chirp force do not affect the magnitude of the frequency components at all. The top and the middle rows show the frequency analyses results for the displacement of the loaded point and for the axle force, respectively. For the unilateral foundation, it can be seen that the resonance frequencies of the loaded point oscillation and axle force are different for the chirp force offset by a positive or negative magnitude. However, for the linear foundation case, they are the same for either positive or negative offset. The different resonance frequencies for the unilateral foundation case indicate the different stiffness when the ring on unilateral foundation is compressed or stretched.
Figure 8: Pulse Response and Comparison with FEA Results

Figure 9: Deformation of Ring Centroid at Different Time Points
Figure 10: Comparison of Frequency Response to Chirp Signal with Different Offset

3.6 Conclusions

This paper considered the in-plane dynamics of a ring on a unilateral elastic foundation model. It dealt with a two-parameter elastic foundation that is linear torsionally, but unilateral in the radial direction where the radial stiffness of the foundation vanishes when compressed or tensioned. The difficulty of this nonlinear problem lies in the fact that the stiffness of the foundation changes with the deformation state of the ring and vice versa. A general orthotropic and extensible thick ring is considered and modeled via Timoshenko beam theory. An approach that combines a new
iterative spatial compensation scheme with Newmark implicit time integration method is presented to solve the in-plane vibration problem.

Compared with discretization-based numerical methods such as nonlinear FEA, the advantage of the proposed iterative approach are three-fold: 1) Time-consuming modeling and meshing work is avoided. This is specially attractive in the parametric studies at the design stage for the application (e.g. non-pnuematic tires and bushing bearings); 2) Only matrices of dimensions of 3x3 are involved in computations at every time step, corresponding to the 3 variables of the Timoshenko beam; while the dimensions of the matrices in FEA depend on the number of the nodes, which is generally a lot higher than 3 to sufficiently describe a thick ring; 3) The iterations at every time step are based on the linear foundation model, where system matrices, specially the stiffness matrix, is constant and time invariant. So the matrix need only be inverted only once at the beginning of the whole computation. However, in the nonlinear implicit FEA methods, the nonlinear stiffness matrix is time variant and the time consuming matrix inversion needs to be implemented at every nonlinear iteration of each time step.

In our future work, this approach will be extended to solve even more complex dynamics problems involving a ring on unilateral foundation. This includes the dynamic contact problem where the computation efficiency of the approach can be exploited to solve for the response of the system involving contact with arbitrary geometry surfaces.
CHAPTER 4: A LAMINATED RING ON ELASTIC FOUNDATION MODEL AND A FEEDBACK CONTACT ALGORITHM FOR TIRE-ROAD CONTACT ANALYSIS

4.1 Abstract

This paper proposes: 1) a 2D tire model that extends the deformable Ring on Elastic Foundation (REF) model by treating the ring as a laminated beam, and, 2) a feedback compensation approach to solve the tire-road contact problem as facilitated by the laminated REF model. The internal layer of the laminated ring is formulated using Timoshenko beam theory that can also be easily regressed to an Euler beam. The external layer of the laminated ring is modeled as a circular beam that primarily takes into account the strain energy contributed by the tire tread in the transverse or radial direction. The elastic foundations are assumed to have a pre-deformation in radial direction, which can model tire inflation pressure in pneumatic tires or foundation pre-compression/pre-tension for non-pneumatic tires. The analytical solution for the static deformation response of this laminated REF model due to an arbitrary external force is detailed first. Then a feedback p-controller algorithm that penalizes geometry errors in the contact region is outlined as a unified approach that can be used to solve frictionless contact problems between a tire and arbitrary road profiles. The performance of the proposed model and algorithm are compared against those obtained from a detailed Finite-Element Analysis (FEA). Both flat surface and cleat contact responses are shown to illustrate the utility of this laminated REF model and the contact algorithm.

4.2 Introduction

The study of tire-road contact is a crucial aspect of vehicle design for ensuring acceptable ride comfort, vehicle handling, tire wear and noise levels[43]. Tire-road
contact pressures are also important inputs for pavement studies [44]. For these reasons, tire-road contact performance has been heavily studied. Many investigations involve field measurements [45], [46], [47], [48], [49], [50]. However, at the early tire design stage, model-based analysis is generally the preferred approach to conduct cost-effective design iterations. To this end, finite element models [43], [51], are broadly adopted for their relative accuracy, but the pre-processing activities including the geometric modeling, mesh generation, material property specifications are very time intensive. A good complement to the finite element method are the so-called the Ring on Elastic Foundation (REF) models, which are also given attention due to for their analytical expedience while retaining acceptable accuracy [5], [9], [52], [53]. In REF models, the tire is usually approximated as an Euler beam (thin ring) [5] or a Timoshenko beam (thick ring) [54], and the sidewall is treated as the elastic foundation modeled via torsional and radial stiffnesses with viscoelastic damping.

In order to model the contact problems while using REF models, different schemes have been presented in previous works. One approach is to model the contact as a set of boundary conditions in the Equation of Motions (EOMs) of the REF model, as in [21]. The contact problem is then solved by dividing the ring, an inextensible Euler beam, into free regions and contact regions and combining the boundary conditions in the contact region with the EOMs of the ring. The limitation of this method lies in the fact that more contact regions need to be formulated if the contact is with more complex surface profiles beyond single flat surfaces, and the equations for the boundary conditions need to be updated for each contact region. Another idea presented in [23] is to use
additional elastic spring elements on the outer surface of the ring to model the compliance of the tire tread. The compatibility condition requires that the displacements of the tire belt and tread must conform to the geometry of the road surface considering their relative motion with respect to it. This method of additional spring elements is adopted and extended in [24] to study the contact with an uneven road for both static and dynamic responses, based on the analytical solution for the response of an Euler ring on elastic foundation [29]. The transient tire models presented in [55] and [56] also use the same idea to solve the contact problem. The disadvantage of this method is that the added spring elements require separate treatments via additional equations for the force-deformation relationships in addition to that of the beam. This can be improved by incorporating the transverse stiffness of the tread rubber into the equations of the beam, as shown in [22], where a higher-order beam theory (higher than a Timoshenko beam) with transverse compliance has been presented. The transverse compliance is used to take the transverse deformation into account for the contact problem and is shown to be especially important when the transverse compliance is significant as is the case in most tires with tread. However, therein, this transverse deformation is modeled as symmetric with the beam’s centroid, which is not exactly the case in the tire belt and tread. Besides, the contact problem solved in [22] only includes the ring in contact with two opposing flat surfaces. The contact is solved by imposing boundary conditions in different regions similar to [21], and therefore, the same limitations remain when analyzing contact with more complex non-flat surfaces.
In this paper, we build on the idea in [22] of incorporating the transverse compliance but correct the unrealistic symmetric deformation by proposing a laminated beam model. This new laminated beam model includes a Timoshenko (which can regress to an Euler beam) as the internal layer and an external beam layer which only accounts for the transverse deformation. This model avoids the need for adding spring elements to a beam structure as proposed by [23], [24], [55] and [56]. The key benefit of the laminated beam model of a tire is in the simplified treatment for the tire-road contact deformation, as we shall explore in this paper. We will first outline the solution for the static deformation of the laminated REF model subject to an arbitrary external force and possible pre-deformation of the elastic foundation. Using these preliminaries, we propose a feedback compensation algorithm, which acts like a proportional-controller or p-controller, for the solution of the frictionless tire-road contact response. In this algorithm, the contact pressure is computed iteratively as proportional to the geometry error between deformation of the ring and road profile but with a unilateral constraint. The algorithm offers simplicity, ease of implementation and a uniform solution approach for arbitrary road profiles. Illustrating results and validating comparisons are outlined against detailed finite-element software solutions. Both flat surface and cleat contact responses are shown to illustrate the effectiveness of this laminated REF model and p-controller algorithm.

In this paper, only the static tire-road contact problem will considered to keep the material focused on the main contributions. We also limit discussions to linear foundations. An upcoming companion paper by the authors will provide analysis of the
dynamic contact case, including rotational effects and nonlinear foundations, using the laminated REF model and the contact algorithm proposed here.

The rest of the paper is organized as follows. Section 4.3 describes the schematic of the laminated REF model and develops the governing equations. Section 4.4 gives the analytical solution for the static deformation due to arbitrary external forces. In Section 4.5, the p-controller algorithm is presented to solve the tire-road contact problem. Results and discussions are given in Section 4.6. Section 4.7 concludes the paper and briefly discusses the continuing applications of the modeling framework and contact algorithm outlined in this contribution.

4.3 Schematic of the Laminated REF Model and Governing Equations

4.3.1 Schematic of the Laminated REF Model

Figure 11 shows the schematic of the Laminated REF model, which consists of two layers of the ring. The internal ring with thickness h and initial centroid radius R is used to model the tire belt. The external ring with thickness h₂ models the tread band/rubber. The width perpendicular to the plane of the ring is b. A global Cartesian coordinate frame and a cylindrical coordinate frame with the origin located at the ring center are adopted. The circumferential coordinate of the cylindrical frame takes values in \([-\pi, \pi]\), and the circumferential location of the bottom point of the ring is assumed to be 0. The center of the ring is fixed. The uniformly distributed radial and tangential stiffnesses representing the elastic foundation are assumed to be \(K_r\) and \(K_\theta\), respectively. These have units of stiffness per radian. The elastic foundation is assumed to have an
initial radius $R_0$, which can be greater or smaller than its assembled radius in the REF model. This leads to a pre-compression or pre-tension on the foundation. This can, for example, represent the pre-inflation in pneumatic tires since it similarly generates a distributed pressure on the internal edge of the ring. The pre-compression or pre-tension can also model pre-deformation in the spokes of non-pneumatic tires [25].

Figure 11: Schematic of Laminated REF Model

We start by applying Timoshenko beam theory to model the internal ring. The radial and tangential displacements at circumferential position $\theta$ of the internal ring centroid are assumed to be $u_r(R, \theta)$ and $u_\theta(R, \theta)$, respectively. The cross-section of the internal ring at the same location is assumed to have a rotation $\phi(R, \theta)$ and keeps its plane after deformation. Then, the radial and tangential displacements at an arbitrary point on the internal ring with radius $r$ (which belongs to $R - \frac{h}{2} \leq r \leq R + \frac{h}{2}$) and circumferential position $\theta$, can be represented by:
If we set $\phi(R, \theta) \equiv 0$, the internal ring model regresses to the Euler-Bernoulli beam (or Euler beam for short).

A transverse compliance is considered in the external ring to take into account the radial stiffness of tread band rubber. Similar to [22], this transverse deformation inside the external ring is assumed to change linearly with depth. However, two additional continuity conditions are needed for the laminated beam formulation. The radial displacement at the edge between the internal and the external rings should be consistent, i.e. the continuity of the radial displacement is assumed to be guaranteed inside the whole laminated ring. And the cross-section rotation inside the external ring is assumed to be the same as that of the internal ring, i.e. the continuity of the cross-section rotation is also satisfied. Then, the radial and tangential displacements at an arbitrary point in the external ring where $R + \frac{h}{2} \leq r \leq R + \frac{h}{2} + h_2$ are written as:

$$
\begin{align*}
    u_{r,1}(r, \theta) &= u_r(R, \theta) \\
    u_{\theta,1}(r, \theta) &= u_\theta(R, \theta) + (r - R)\phi(R, \theta)
\end{align*}
$$

(4.1)

where $u(r, \theta)$ is a non-dimensional variable which indicates the linear gradient of the transverse displacement inside the external ring. Its worth mentioning that the transverse compliance associated with this transverse displacement is important when addressing the tire-road contact problem, as also pointed out in [22].

**4.3.2 Governing Equations of the Laminated REF Model**

$$
\begin{align*}
    u_{r,2}(r, \theta) &= u_r(R, \theta) + \left[ r - \left( R + \frac{h}{2} \right) \right] \psi(R, \theta) \\
    u_{\theta,2}(r, \theta) &= u_\theta(R, \theta) + (r - R)\phi(R, \theta)
\end{align*}
$$

(4.2)
The strain-displacement relationships can be written for both the laminated (internal and external) ring in polar coordinate form as [35]:

\[
\begin{align*}
\epsilon_{rr,i}(r, \theta) &= \frac{\partial}{\partial r} u_{r,i}(r, \theta) \\
\epsilon_{\theta\theta,i}(r, \theta) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_{\theta,i}(r, \theta) + \frac{1}{r} u_{r,i}(r, \theta) \\
\gamma_{r\theta,i}(r, \theta) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_{r,i}(r, \theta) + \frac{\partial}{\partial r} u_{\theta,i}(r, \theta) - \frac{1}{r} u_{r,i}(r, \theta)
\end{align*}
\]

where \(\epsilon_{rr}, \epsilon_{\theta\theta}\) and \(\gamma_{r\theta}\) are the radial, tangential and shear strains, respectively. Subscript \(i = 1, 2\) represents the internal and the external ring, respectively.

Both the internal and the external ring can be treated generally as orthotropic and homogeneous. The ring material axes are assumed to coincide with the polar coordinate system adopted. Then, the stress-strain relationships are given by [35]:

\[
\begin{align*}
\sigma_{rr,i}(r, \theta) &= \frac{E_{r,i}(v_{r\theta,i} \epsilon_{r\theta,i}(r, \theta) + \epsilon_{rr,i}(r, \theta))}{-v_{r\theta,i} + 1} \\
\sigma_{\theta\theta,i}(r, \theta) &= \frac{E_{\theta,i}(v_{r\theta,i} \epsilon_{r\theta,i}(r, \theta) + \epsilon_{\theta\theta,i}(r, \theta))}{-v_{r\theta,i} + 1} \\
\tau_{r\theta,i}(r, \theta) &= G_{\theta,i} \gamma_{r\theta,i}(r, \theta)
\end{align*}
\]

where \(\sigma_{rr}, \sigma_{\theta\theta}\) and \(\tau_{r\theta}\) are the radial, tangential and shear stresses, respectively. \(E_r, E_\theta\) and \(G\) are the elastic moduli in the radial and tangential directions and the shear modulus, respectively. \(\nu\) represents the Poisson's ratio.

In the following presentation, for the internal ring which models the tire belt, the Poisson’s ratios \(\nu_{r\theta,1}\) and \(\nu_{\theta r,1}\) will be set to zero for brevity of the resulting governing equations included in this paper. However, the analysis approach to be outlined in this paper will stay the same in the case that these Poisson’s ratios are non-zero. For the
external ring, which represents the tread band rubber, the material is assumed to be isotropic with $\nu_{r\theta} = \nu_{\theta r} = \nu_2 = 0.5$, $E_{r,2} = E_{\theta,2} = E_2$ and $G_2 = \frac{E_2}{2(1+\nu_2)}$, where $E_2$, $G_2$ and $\nu_2$ are Young’s modulus, shear modulus and Poisson’s ratio of the tread rubber material.

Since the tread rubber layer consists of separated tread blocks, the circumferential strain energy and the shear energy for this layer are considered negligible compared with that of the tire belt. So only the radial strain energy is considered for the external ring. The total strain energy in the laminated ring is obtained by:

$$U_1 = \frac{b}{2} \int_{-\pi}^{\pi} \int_{\frac{R-h}{2}}^{\frac{R-h}{2}} (\sigma_{rr,1} \varepsilon_{rr,1} + \sigma_{\theta \theta,1} \varepsilon_{\theta \theta,1} + \tau_{r \theta,1} \gamma_{r \theta,1}) r dr d\theta + \frac{b}{2} \int_{-\pi}^{\pi} \int_{\frac{R-h}{2}}^{\frac{R-h}{2}} (\sigma_{rr,2} \varepsilon_{rr,2}) r dr d\theta$$

(4.5)

The strain energy in the elastic foundation is given by:

$$U_2 = \frac{1}{2} \int_{-\pi}^{\pi} \left( K_r \left( R - \frac{h}{2} + u_{r,1} \left( R - \frac{h}{2}, \theta \right) - R_0 \right)^2 + K_\theta \left( u_{\theta,1} \left( R - \frac{h}{2}, \theta \right) \left( R - \frac{h}{2}, \theta \right) \right)^2 \right) d\theta$$

(4.6)

where, here, we consider only radial pre-deformations of the foundation as the more practical scenario, but tangential pre-deformations can also be readily added in a similar manner. Note here that the radial and tangential displacements for the internal edge of the internal ring $u_{r,1} \left( R - \frac{h}{2}, \theta \right)$ and $u_{\theta,1} \left( R - \frac{h}{2}, \theta \right)$ couple the ring and the elastic foundation.

The work done by the applied external forces is given by:

$$W = b \int_{-\pi}^{\pi} \left( q_r u_{r,2} \left( R + \frac{h}{2} + h_2, \theta \right) + q_\theta u_{\theta,2} \left( R + \frac{h}{2} + h_2, \theta \right) \right) \left( R + \frac{h}{2} + h_2 \right) d\theta$$

(4.7)
where \( q_r = q_r \left( R + \frac{h}{2} + h_2, \theta \right) \) and \( q_\theta = q_\theta \left( R + \frac{h}{2} + h_2, \theta \right) \) are, respectively, the radial and tangential components of general, and distributed forces applied to the external surface of the external ring (at radial location is \( R + \frac{h}{2} + h_2 \)). The units of \( q_r \) and \( q_\theta \) are in \( \text{Force per Area} \).

We then apply the principle of virtual work\([57]\):

\[
\delta (U_1 + U_2) = \delta W
\]  

(4.8)

After substitution of (4.1) to (4.7) into (4.8) and some manipulations according to the Euler-Lagrange equation, the final governing equations are found to be as follows:

\[
\begin{align*}
C112 \cdot & \frac{\partial^2}{\partial \theta^2} u_r + C110 \cdot u_r + C121 \cdot \frac{\partial}{\partial \theta} u_\theta + C141 \cdot \frac{\partial^2}{\partial \theta^2} \phi + C10 = \left( R + \frac{h}{2} + h_2 \right) q_r \\
C211 \cdot & \frac{\partial}{\partial \theta} u_r + C222 \cdot \frac{\partial^2}{\partial \theta^2} u_\theta + C220 \cdot u_\theta + C240 \cdot \phi + C20 = \left( R + \frac{h}{2} + h_2 \right) q_\theta \\
C310 \cdot u_r + C321 \cdot \frac{\partial}{\partial \theta} u_\theta + C330 \cdot \psi + C341 \cdot \frac{\partial^2}{\partial \theta^2} \phi + C30 = \left( R + \frac{h}{2} + h_2 \right) h_2 q_r \\
C411 \cdot & \frac{\partial}{\partial \theta} u_r + C420 \cdot u_\theta + C442 \cdot \frac{\partial^2}{\partial \theta^2} \phi + C440 \cdot \phi + C40 = \left( R + \frac{h}{2} + h_2 \right) \left( \frac{h}{2} + h_2 \right) q_\theta
\end{align*}
\]  

(4.9)

where the following shorthand is adopted for the displacement variables introduced in (4.1) and (4.2):

\[
\begin{align*}
u_r &= u_r (R, \theta) \\
u_\theta &= u_\theta (R, \theta) \\
\psi &= \psi (R, \theta) \\
\phi &= \phi (R, \theta)
\end{align*}
\]  

(4.10)

And, similarly for the applied force components introduced in (4.7):
\[
q_r = q_r \left( R + \frac{h}{2} + h_z, \theta \right)
\]
\[
q_\theta = q_\theta \left( R + \frac{h}{2} + h_z, \theta \right)
\]

(4.11)

The coefficients and constants in (4.9) whose names start with \( C \) and followed by numbers chiefly consist of geometric and material parameters. Their detailed expressions are listed in Appendix C, where four new stiffness parameters \( E_{A_\theta}, E_{I_\theta}, G_A \) and \( E_{A_r,2} \) are introduced whose definitions are:

\[
E_{A_\theta} = E_{\theta,1} A_1
\]
\[
E_{I_\theta} = E_{\theta,1} I_1
\]
\[
G_A = G_1 \cdot A_1
\]
\[
E_{A_r,2} = E_{r,2} A_2 = E_2 A_2
\]

(4.12)

with \( A_1 = bh \) and \( A_2 = bh_2 \) the cross-sectional areas of the internal and external ring, respectively. And \( I_1 = \frac{1}{12} bh^3 \) is the area moment of inertia of the internal ring cross-section.

The following approximations are used in the manipulations to obtain the governing equations given by (4.9) above:

\[
\int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{1}{r} dr \approx \frac{1}{R} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{dr}{r} = \frac{h}{R}
\]

\[
\int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{(r - R)}{r} dr \approx \frac{1}{R} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{(r - R) dr}{r} = 0
\]

\[
\int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{(r - R)^2}{r} dr \approx \frac{1}{R} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{(r - R)^2 dr}{r} = \frac{1}{12} \frac{h^3}{R}
\]

(4.13)

\[
\int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{1}{r} dr \approx \frac{1}{R + \frac{h}{2} + h_z} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{dr}{r + \frac{h}{2} + h_z} = \frac{h_z}{R + \frac{h}{2} + h_z}
\]

\[
\int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{1}{r} dr \approx \frac{1}{R + \frac{h}{2} + h_z} \int_{R - \frac{h}{2}}^{R + \frac{h}{2}} \frac{dr}{r + \frac{h}{2} + h_z} = \frac{h_z}{R + \frac{h}{2} + h_z}
\]
Considering that \( R \gg h \) and \( R \gg h_2 \), the above approximations are reasonable.

The development of governing equations here has been based on an extensible Timoshenko beam for the internal layer of the laminated ring. This beam includes cross-section shear. However, in some published works, an inextensible Euler ring (without cross-section shear) has been used for modeling the tire belt of pneumatic tires ([24], [55]). The inextensible ring case can also be treated within our modeling approach by simply assigning a relatively high value for the circumferential stiffness \( EA_\theta \) of the ring. And a relatively high value for \( GA \) models a ring with minimal shear, which reduces it to the Euler ring model. The effects of these extended cases will be illustrated towards the end of Section 4.6.

4.4 Analytical Solutions for Static Deformation of Laminated REF Model

The static deformation response of a single layer of a Timoshenko REF has been studied in our recent paper [16]. However, in the present paper, the governing equations (4.9) derived above are specifically for a laminated REF and include the following important additions: 1) an additional variable \( \psi \) corresponding to the transverse compliance of the external layer of the ring, and 2) new terms representing the radial pre-deformation of the elastic foundation appear in the left-hand sides of the governing equations. This section details the solution procedure for the laminated REF model; the result will be embedded in the contact algorithm to be outlined in the next section.

The static arbitrary distributed force vector \( \mathbf{F}(\theta) \) can be decomposed into radial and tangential components and then expanded into a Fourier series on \([-\pi, \pi]\):
\[ \vec{F}(\theta) = F_r(\theta) \hat{r} + F_\theta(\theta) \hat{\theta} \]

\[ F_r(\theta) = \sum_{n=-N}^{N} Q_{n,r,c} \cos(n\theta) + \sum_{n=-N}^{N} Q_{n,r,s} \sin(n\theta) \]  
\[ F_\theta(\theta) = \sum_{n=-N}^{N} Q_{n,\theta,c} \cos(n\theta) + \sum_{n=-N}^{N} Q_{n,\theta,s} \sin(n\theta) \]  

where, \( n \) is a circumferential mode (harmonic) number, \( N \) is the cut-off harmonic number and \( Q_{n,r,c}, Q_{n,r,s}, Q_{n,\theta,c} \) and \( Q_{n,\theta,s} \) are corresponding coefficients of the \( n \)th harmonic force component. The subscript \( r \) or \( \theta \) indicates whether the coefficient is for radial or tangential direction, respectively; while \( c \) or \( s \) represents cosine or sine components, respectively. Then, \( q_r \) and \( q_\theta \) introduced in (4.7) can be written in terms of corresponding components as:

\[ q_r = \sum_{n=-N}^{N} [q_{n,r,c} + q_{n,r,s}] \]  
\[ q_\theta = \sum_{n=-N}^{N} [q_{n,\theta,c} + q_{n,\theta,s}] \]  

where:

\[ q_{n,r,c} = q_{n,r,c} \left( R + \frac{h}{2} + h_2, \theta \right) = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} Q_{n,r,c} \cos(n\theta) \]  
\[ q_{n,r,s} = q_{n,r,s} \left( R + \frac{h}{2} + h_2, \theta \right) = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} Q_{n,r,s} \sin(n\theta) \]  
\[ q_{n,\theta,c} = q_{n,\theta,c} \left( R + \frac{h}{2} + h_2, \theta \right) = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} Q_{n,\theta,c} \cos(n\theta) \]  
\[ q_{n,\theta,s} = q_{n,\theta,s} \left( R + \frac{h}{2} + h_2, \theta \right) = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} Q_{n,\theta,s} \sin(n\theta) \]
Correspondingly, the solutions for (4.9) can be considered to consist of radial, tangential and cosine, sine components, as well as components due to the pre-deformation of the foundation:

\[
\begin{align*}
    u_r &= u_{r,N,r,e} + u_{r,N,r,s} + u_{r,N,\theta, c} + u_{r,N,\theta, s} + u_{r,0} \\
    u_\theta &= u_{\theta,N,r,e} + u_{\theta,N,r,s} + u_{\theta,N,\theta, c} + u_{\theta,N,\theta, s} + u_{\theta,0} \\
    \psi &= \psi_{N,r,e} + \psi_{N,r,s} + \psi_{N,\theta, c} + \psi_{N,\theta, s} + \psi_0 \\
    \phi &= \phi_{N,r,e} + \phi_{N,r,s} + \phi_{N,\theta, c} + \phi_{N,\theta, s} + \phi_0
\end{align*}
\]  
\text{(4.17)}

Substituting (4.15), (4.16) and (4.17) into (4.9) and decoupling the equations into the corresponding components appearing in (4.17), the original governing equations (4.9) can be transformed into 5 groups of governing equations. The group of governing equations for the components due to foundation pre-deformation (last terms in(4.17)) will be in the form:

\[
\begin{align*}
    C112 \cdot \frac{\partial}{\partial \theta^2} u_{r,0} + C110 \cdot u_{r,0} + C121 \cdot \frac{\partial}{\partial \theta} u_{\theta,0} + C141 \cdot \frac{\partial}{\partial \theta} \phi_0 + C10 &= 0 \\
    C211 \cdot \frac{\partial}{\partial \theta} u_{r,0} + C222 \cdot \frac{\partial}{\partial \theta^2} u_{\theta,0} + C220 \cdot u_{\theta,0} + C240 \cdot \phi_0 + C20 &= 0 \\
    C310 \cdot u_{r,0} + C321 \cdot \frac{\partial}{\partial \theta} u_{\theta,0} + C330 \cdot \psi_0 + C341 \cdot \frac{\partial}{\partial \theta} \phi_0 + C30 &= 0 \\
    C411 \cdot \frac{\partial}{\partial \theta} u_{r,0} + C420 \cdot u_{\theta,0} + C442 \cdot \frac{\partial^2}{\partial \theta^2} \phi_0 + C440 \cdot \phi_0 + C40 &= 0
\end{align*}
\]  
\text{(4.18)}

Since the displacements solely due to foundation pre-deformation are the same circumferentially, derivatives of these displacements with respect to \( \theta \) all vanish. (4.18) is actually:
The unknown displacements due to foundation pre-deformation, $u_{r,0}$, $u_{θ,0}$, $ψ_0$ and $ϕ_0$, can be easily solved from (4.19). When modeling pneumatic tires with the REF model, an equivalent initial inflation pressure applied at the internal edge of the ring can be calculated based on $u_{r,0}$ as follows:

$$p_0 = \frac{K_i \left[ R_0 - \left( R - \frac{h}{2} + u_{r,0} \right) \right]}{\left( R - \frac{h}{2} + u_{r,0} \right)b}$$

(4.20)

The remaining 4 groups of governing equations for the other components will take a similar form. For example, those for the radial and cosine components (first terms in (4.17)) are given by:

$$C110\cdot u_{r,0} + C10 = 0$$
$$C220\cdot u_{θ,0} + C240\cdot φ_0 + C20 = 0$$
$$C310\cdot u_{r,0} + C330\cdot ψ_0 + C30 = 0$$
$$C420\cdot u_{θ,0} + C440\cdot φ_0 + C40 = 0$$

(4.19)

Please note that the constant terms $C10, C20, C30$ and $C40$ no longer appear in (4.21). Similar to our approach in [16], the solutions for equations (4.21) can be sought by superposition of every harmonic component:
After substitution of (4.16) and (4.22) into (4.21), solution coefficients for each component \( \text{Cur}_{\theta,r,c}(n) \), \( \text{Cu} \theta_{\theta,r,c}(n) \), \( \text{C} \psi_{\theta,r,c}(n) \) and \( \text{C} \phi_{\theta,r,c}(n) \) can be solved for and their detailed expressions are given in Appendix C.

The other components of the solution (2nd through 4th terms in (4.17)) (e.g., tangential direction and sine parts) can be solved for similarly. Finally, the total displacement response of the laminated REF system (4.9) due to the arbitrary force (4.14) can be assembled using (4.17).

According to (4.1), the foundation radial and tangential displacements can be obtained by:

\[
\begin{align*}
    u_{r,F} & = u_{r,1} \left( R - \frac{h}{2}, \theta \right) = u_r \\
    u_{\theta,F} & = u_{\theta,1} \left( R - \frac{h}{2}, \theta \right) = u_\theta - \frac{h}{2} \phi
\end{align*}
\]  

(4.23)

Given the displacement response of the laminated REF and foundation, the reaction forces at the axle (center of the ring) in the vertical and horizontal directions are often of interest and are given by:
In this section, we outline an iterative feedback compensation scheme to solve the frictionless tire-road contact problem using the laminated REF model. We will use the analytical solution for the static force-deformation relationship of the laminated REF model obtained in Section 4.4 above.

Given any external force \( F_i(\theta) \), the radial and tangential displacements of the point on the external edge of the laminated ring with cylindrical coordinate \( (R + \frac{h}{2} + h_2, \theta) \) can be obtained from the solution obtained above and the definition given earlier in equation (4.2):

\[
\begin{align*}
    u_{r,2} \left( R + \frac{h}{2} + h_2, \theta \right) &= u_r(R, \theta) + h_2 \psi(R, \theta) \\
    u_{\theta,2} \left( R + \frac{h}{2} + h_2, \theta \right) &= u_\theta(R, \theta) + \left( \frac{h}{2} + h_2 \right) \phi(R, \theta)
\end{align*}
\] (4.25)

After deformation, the new radial and circumferential locations in the same cylindrical coordinate system are obtained by:

\[
\begin{align*}
    r_d \left( R + \frac{h}{2} + h_2, \theta \right) &= \sqrt{\left( R + \frac{h}{2} + h_2 + u_{r,2} \right)^2 + \left( u_{\theta,2} \right)^2} \\
    \theta_d \left( R + \frac{h}{2} + h_2, \theta \right) &= \theta + \arctan \left( \frac{u_{\theta,2}}{R + \frac{h}{2} + h_2 + u_{r,2}} \right)
\end{align*}
\] (4.26)
where, we use the short hand \( u_{r,2} = u_{r,2} \left( R + \frac{h}{2} + h_2, \theta \right) \) and \( u_{\theta,2} = u_{\theta,2} \left( R + \frac{h}{2} + h_2, \theta \right) \).

A mapping from new circumferential coordinate to the new radial coordinate for the deformed external edge of the laminated ring can be defined by:

\[
    r_{M_2}(\alpha) : \theta_d \rightarrow r_d, \quad \alpha \in \left[ -\pi, \pi \right]
\]  

(4.27)

where, \( \alpha \) is a free-variable representing the argument of the mapping function. In this case, it simply refers to \( \theta_d \).

A rigid road surface profile with Cartesian coordinate \((X,Y)\) can be translated into the cylindrical coordinate by:

\[
    r_s = \sqrt{X^2 + Y^2} \\
    \theta_s = \arctan \left( -\frac{X}{Y} \right)
\]

(4.28)

We assume that the dimension of the features of the rigid surface is relatively small compared with the centroid radius of the internal ring \( R \) and that the contact regions are near the bottom side of the laminated ring, as shown in Figure 12. This restricts the reach of the rigid surface, \( |\theta_s| < \frac{\pi}{2} \). A mapping from the circumferential coordinate to the radial coordinate for the rigid surface can be defined as:

\[
    r_{M_5}(\alpha) : \begin{cases} 
        \theta_s \rightarrow r_s & \text{min}(\theta_s) \leq \alpha \leq \max(\theta_s) \\
        C_t & -\pi \leq \alpha \leq \text{min}(\theta_s) \cup \text{max}(\theta_s) \leq \alpha \leq \pi 
    \end{cases}
\]

(4.29)

where \( C_t \) is a large constant \( C_t \gg R \).
Then, the geometry error between the ring and the contacting rigid surface can be written as:

\[ GE(\alpha) = r_{Md}(\alpha) - r_{MS}(\alpha), \quad (\alpha \in [-\pi, \pi]) \]  

(4.30)

In order to get a new contact force that can compensate for the geometry error, the magnitude of adjusting force is calculated based on the geometry error and a simple gain:

\[ F_{GE}(\alpha) = GE(\alpha) \cdot K_{GE} \]  

(4.31)

The idea is that larger adjusting force is needed to remove larger geometry error. In the region where \( GE(\alpha) > 0 \), penetration happens which should not be allowed and should be removed by applying an additional contact force as given by (4.31). This region is identified as the Contact Region for the current iteration step. In the region where \( GE(\alpha) \leq 0 \) and \( \min(\theta_S) \leq \alpha \leq \max(\theta_S) \), the ring and rigid surface are off contact, so that the existing force in this region should be reduced until it vanishes. This reduction of the contact force is also achieved by (4.31) via the sign of \( GE(\alpha) \) in this region. In the region where \( -\pi \leq \alpha \leq \min(\theta_S) \cup \max(\theta_S) \leq \alpha \leq \pi \), the large constant \( C_t \) guarantees that \( GE(\alpha) \leq 0 \). Because of the limitation that the minimum contact force is zero (unilateral property of the contact), no contact force will appear in this region.

Figure 12: Contact between Laminated REF Model and Rigid Surface
In (4.27) and consequently (4.31), the \( \alpha \) coordinate on the ring represents the circumferential coordinate after deformation, i.e., \( \theta_d \). Another mapping can be defined to translate \( \theta_d \) back to the un-deformed coordinate \( \theta \):

\[
\theta_o(\alpha): \theta_d \rightarrow \theta
\]

(4.32)

Then, the contact force adjustment in the deformed coordinate \( F_{GE}(\theta_d) \) obtained from (4.31), can be added to the force at the current iteration step \( F_i(\theta) \) after translating \( \theta_d \) back to \( \theta \) to get the new contact force:

\[
F_{i+1}(\theta) = \begin{cases} 
F_i(\theta) + F_{GE}(\theta_o(\theta_d)) & \theta \in \{F_i(\theta) + F_{GE}(\theta_o(\theta_d)) > 0\} \\
0 & \text{otherwise}
\end{cases}
\]

(4.33)

The subscript \( i \) is the index for the current iteration step.

A new iteration starts from the new contact force (4.33) and repeats the computations given by (4.25) to (4.33). The algorithm stops when the maximum absolute value of geometry error in the Contact Region is smaller than a pre-defined threshold, in which case the iteration is considered converged. Since the algorithm to get the contact force from the geometry error acts similar to a feedback proportional controller, it is named here as a p-controller algorithm. The whole schematic to solve static tire-road contact problem via the p-controller algorithm is shown in Figure 13.
After convergence, the distributed contact force and the deformation of the laminated REF model in the last iteration step (4.33) can be regarded as the solved contact force and deformation, respectively, the axle reaction forces can be easily obtained via (4.24).

4.6 Results and Discussions

In this section, simulations of static tire contact against a flat surface and a cleat are given to demonstrate the utility of the proposed method. The contact responses, such as the axle reaction forces, ring deformation, as well as the contact pressure will be generated. Parameters used for the model are given in Table 3. We validate our proposed model and solution algorithm by comparing responses obtained via a model constructed in Finite-Element Analysis (FEA) software replicating the same set of basic parameters listed in this table.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definitions</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>Initial Centroid Radius of Internal Ring</td>
<td>m</td>
<td>0.3</td>
</tr>
<tr>
<td>b</td>
<td>Model Width</td>
<td>m</td>
<td>0.2</td>
</tr>
<tr>
<td>h</td>
<td>Thickness of Internal Ring</td>
<td>m</td>
<td>0.012</td>
</tr>
<tr>
<td>h₂</td>
<td>Thickness of External Ring</td>
<td>m</td>
<td>0.01</td>
</tr>
<tr>
<td>Kᵣ</td>
<td>Radial Stiffness Per Radian of Elastic Foundation</td>
<td>N/m · radian</td>
<td>5e5</td>
</tr>
<tr>
<td>K₀</td>
<td>Torsional Stiffness Per Radian of Elastic Foundation</td>
<td>N/m · radian</td>
<td>5e4</td>
</tr>
<tr>
<td>Eᴬ₀</td>
<td>Axial Stiffness of Internal Ring Cross-section</td>
<td>N</td>
<td>1.2e6</td>
</tr>
<tr>
<td>Eᴵ₀</td>
<td>Flexural Rigidity of Internal Ring</td>
<td>N · m²</td>
<td>14.4</td>
</tr>
<tr>
<td>Gᴬ</td>
<td>Shear Stiffness of Internal Ring Cross-section</td>
<td>N</td>
<td>1.2e4</td>
</tr>
<tr>
<td>Eᴬᵣ,₂</td>
<td>Transverse Stiffness of External Ring Cross-section</td>
<td>N</td>
<td>1e4</td>
</tr>
<tr>
<td>R₀</td>
<td>Initial Radius of the Elastic Foundation</td>
<td>m</td>
<td>0.324</td>
</tr>
</tbody>
</table>

For the given parameters, a pre-compression can be identified in radial direction of the foundation, since \( R₀ > R - \frac{h}{2} \). Using equations (4.19) and (4.20), the radial displacement due to the pre-deformation of the foundation as well as the corresponding equivalent pressure can be obtained to be:

\[
\begin{align*}
    u_{r,₀} &= 3.33\text{mm} \\
    p₀ &= 2.24\text{bar}
\end{align*}
\] (4.34)
The comparison 2D Finite Element (FE) model is developed in the commercial FE software Abaqus. Both the internal and external rings are represented using 8-node quadrilateral plane stress element CPS8R, with 6 layers of elements in the radial direction for internal ring and 4 layers for external ring. There are 360 divisions in the circumferential direction for both the internal and external ring. The material properties for the internal layer of the ring are set to Lamina type in the cylindrical coordinate system, so that the orthotropic material properties can be assigned with their respective parameters in Abaqus software:

\[
E_2 = \frac{EA_{11}}{bh} = 500 \text{MPa}
\]

\[
\nu_{12} = 0
\]

\[
G_{12} = \frac{GA}{bh} = 5 \text{MPa}
\]

where \(E_2\) is the young’s modular in circumferential direction, \(\nu_{12}\) is the Poisson’s ratio between the radial and circumferential direction, \(G_{12}\) is the shear modular between the two orthogonal directions. Subscript 1 indicates the internal ring. The other unlisted material property options in the Lamina type are set to very high values to obtain very high rigidity. An example is the Young’s modulus in radial direction \(E_1\) which is assumed to be infinity in the Timoshenko ring. For the external ring, although the isotropic material property is assumed, the strain energy in the circumferential direction as well as the shear energy are neglected in the derivation of the governing equations, which corresponds to vanished \(E_2\) and \(G_{12}\). So Lamina type material model is also used for this layer of the ring. However, zero Young’s modulus or shear modulus in material properties is not allowed in Abaqus. So, very small values were used to approximate the
desired behavior. However, there is a trade-off, since, if too small values were used for E2 and G12, the elements will be severely distorted in the contact region due to too small circumferential and shear stiffnesses, which deviates from the real physics. With this in mind, the following values were selected.

\[ E_{12} = \frac{E_{A_2}}{bh_2} = 5 \text{MPa} \]
\[ E_{22} = 0.5 \text{MPa} \] \hspace{1cm} (4.36)
\[ G_{12} = 0.5 \text{MPa} \]

where subscript 2 indicates the external ring, E1 is the Young’s modulus in the radial direction. The other unlisted material property options are set to very high values in Abaqus, which are the same as that of the internal ring.

The elastic foundation in the FEA (Abaqus) is modeled using evenly and circumferentially distributed 360 independent Spring2 elements for the radial stiffness and another 360 independent Spring2 elements for the tangential stiffness. The stiffness of every radial and tangential Spring2 element is calculated, respectively, by:

\[ k_{s,r} = \frac{2\pi K_r}{N_{s,r}} \]
\[ k_{s,t} = \frac{2\pi K_\theta}{N_{s,t}} \] \hspace{1cm} (4.37)

where \( N_{s,r} = 360 \) and \( N_{s,t} = 360 \) are the number of radial and tangential spring elements, respectively. In Abaqus, the initial force due to pre-deformation of the radial spring elements can be applied as initial forces of the spring elements.
Table 4 gives the first set of comparisons: the prediction of the vertical stiffness, i.e., the vertical force response under different deflections with fixed axle in contact with a flat surface. Note that, the deflection is calculated with respect to the initial external radius of the laminated ring before the pre-deformation of the foundation is applied. So even at zero deflection, there is a non-zero vertical force response, which is caused by the expansion of the ring due to the release of foundation pre-deformation. It can be seen that the model predicted vertical forces are quite consistent with those obtained by the FEA model. The distributions of the contact pressure under different deflections are given and compared with FEA results in Figure 14. The distributions as well as the magnitudes of the contact pressure obtained by both methods are close, with only minor deviations. These deviations include the effect of the trade-offs in selecting small values in $E_2$ and $G_{12}$ for the tread layer in Abaqus as mentioned above. Furthermore, the internal ring in the our proposed laminated REF model is assumed as a pure Timoshenko beam, whose cross-section keeps plane after deformation, while in the FE model the cross-section of the ring does not necessarily remain plane. In other words, the FE model is not and should not exactly equivalent to the laminated REF model, although it is configured to be as close to as possible for validating our proposed model.

Table 4- Vertical Force at Axle under Different Deflections

<table>
<thead>
<tr>
<th>Deflection (mm)</th>
<th>Axle Force $F_Z$ by Model (N)</th>
<th>Axle Force $F_Z$ by FEA (N)</th>
<th>Percentage Error w.r.t. FEA</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>707.0</td>
<td>706.735</td>
<td>0.04%</td>
</tr>
<tr>
<td>10</td>
<td>3542.9</td>
<td>3539.2</td>
<td>0.10%</td>
</tr>
</tbody>
</table>
Next, a cleat contact case is shown to demonstrate the application of the proposed method in dealing with contact with uneven or complex surface profiles. A cleat with 40mm width and 10mm height is placed under the wheel center. Deflections are applied and the deformed shapes of the laminated ring are obtained from both of laminated REF model and the FE model. The deformed shape at 20 mm deflection as well as the cleat surface profile is shown in Figure 15. It can be seen that the dashed-blue and solid red lines, which show the deformed centroid of the internal ring predicted by our model and by FEA, respectively, are almost on top of each other. Figure 16 gives more information by plotting radial and tangential displacements of the centroid under different deflections.

Figure 14: Contact Pressure Distribution with Different Deflections on Flat Surface
Both figures show the contact deformation response predicted by the developed laminated REF model is quite close to the FEA results.

Figure 15: Cleat Contact Deformation of Laminated REF Model for Applied Deflection of 20 mm

Figure 16: Details of the Centroid Radial and Tangential Displacement Responses under Different Deflections in Cleat Contact Case

The developed laminated REF model can be used to predict the tire axle force response when it slowly rolls over a cleat. Some such results as shown in Figure 17 for a
rolling speed of 10kph. The vertical and longitudinal axle force $F_Z$ and $F_X$ are calculated when the laminated REF model is located at different relative positions with respect to the cleat (under different deflections). Then, the force variations $\Delta F_Z$ and $\Delta F_X$ (to isolate the cleat response) are obtained by subtracting the corresponding force response in flat surface contact under the respective applied deflections.

Figure 17: Axle Force Variation When Slowly Rolling Over Cleat under Different Deflections
Finally, as pointed out earlier, while the proposed model and contact algorithm are developed using the extensible Timoshenko beam assumption for the internal ring, the framework can be easily extended to treat other beam models. For example, we can model an inextensible beam by setting high value for $EA_\theta$ and an Euler beam by using high value for $GA$. Table 5 shows the parameters for 4 different beam models for the internal ring considering the extensibility and shear. The axle force responses for these cases are compared in Figure 18. It can be seen that inclusion of the beam shear (Timoshenko cases) in the model results in a relative lower force variation when going over the cleat. The popularly used inextensible assumption ([24], [55]) also affects the axle force variations, particularly for the Timoshenko cases. It should be clear that

Figure 18: Axle Force Variation When Slowly Rolling Over Cleat for Different Beam Models (20mm Deflection)
designers can do similar parametric study that covers all these cases using the framework of our proposed model and contact algorithm.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Extensible Timoshenko Ring</th>
<th>Inextensible Timoshenko Ring</th>
<th>Extensible Euler Ring</th>
<th>Inextensible Euler Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_A$</td>
<td>$1.2e6$N</td>
<td>$1.2e11$N</td>
<td>$1.2e6$N</td>
<td>$1.2e11$N</td>
</tr>
<tr>
<td>$G_A$</td>
<td>$1.2e4$N</td>
<td>$1.2e4$N</td>
<td>$1.2e9$N</td>
<td>$1.2e9$N</td>
</tr>
</tbody>
</table>

### 4.7 Summary and Conclusions

In this paper, we have presented a laminated REF model and a feedback compensation contact algorithm for the simulation of tire-road contact response with arbitrary road surface profiles. The governing equations of the laminated REF model are obtained based on general extensible Timoshenko ring models, which can be easily simplified to inextensible Euler ring by assigning relatively high values for $E_A$ and $G_A$. The transverse stiffness of the tire tread is taken account into via external beam layer of the laminate so that the deformations can be incorporated into the governing equations and solved simultaneously with the variables of the internal beam layer. Continuity conditions have been specified that couple the displacements in the two layers. This avoids the need for formulation of additional spring elements for modeling the tread as widely adopted in the existing literatures that deal with tire-road contact problems. The laminated REF model facilitates the use of the proposed feedback contact algorithm. The advantage of this new contact algorithm includes simple logic, easy implementation and uniform approach for arbitrary road profiles. Static contact results with both flat surface
and cleat are given and validated via FEA results demonstrate the utility and accuracy of this method. It is also shown that the effects of the extensibility and shear of the ring (choice of design variables affecting each) on the response of the system can be easily studied using the proposed laminated REF model and contact algorithm.

The framework proposed here can be extended further to include more dynamic scenarios. To this end, the authors have outlined an implicit Newmark scheme to treat the forced response for an Timoshenko ring (on a unilateral foundation) [26]. Upcoming publications by the authors will build on these preliminaries and re-visit the equations of motion for the dynamic contact case including rotational effects.
CHAPTER 5 : ROLLING CONTACT INDUCED DYNAMICS OF A DEFORMABLE LAMINATED RING ON UNILATERAL ELASTIC FOUNDATION

5.1 Abstract

This paper analyzes the structural dynamics of a rotating laminated ring supported by unilateral elastic foundation. Such a foundation exhibits linear stiffness in the tangential direction but unilateral stiffness (i.e. collapsible or tensionless) in the radial direction. Frictionless rolling contact is considered as the excitation. The laminated ring is comprised of an internal layer of orthotropic extensible Timoshenko beam and an external layer of beam accounting for transverse deformation, i.e., the bending, shear, extensibility and transverse compliance are all included in the laminated ring model. Rotational effects are also included explicitly in the derivation of the Equation of Motions (EOMs). For the forced dynamic response for the linear foundation case analytical solutions are given. For the unilateral elastic foundation case, an iterative compensation method is used to solve for the spatial deformation and an implicit Newmark scheme is adopted to solve for the time domain dynamics. The frictionless rolling contact problem is then solved via a feedback compensation algorithm that acts on geometric errors for the deformable ring rolling an arbitrary uneven surface. Finally, the critical velocity of the rolling laminated ring is analyzed by conducting modal analysis of the structure on sub-cases of quasi-static rolling contact on flat surfaces. This general Ring on Elastic Foundation (REF) model can be used for analysis of the rolling contact response of pneumatic and non-pneumatic tires, as well as other rolling contact problems involving REF structures such as bearings and gears.

5.2 Introduction
Ring on Elastic Foundation (REF) is a very common structure in rotating components such as tires, bearings, gears, etc. These broad applications have motivated intensive studies on the dynamics of REF structures. One class of these studies assumes a rigid ring, which ignores the detailed dynamics of the ring and focuses on the dynamic transmissibility of forces and vibration to other systems connected to the REF. An example of this rigid REF model is the SWIFT model [6], [7] well-known for tire dynamics modeling in vehicle dynamics simulations. In this model, a radial spring is attached to the rigid ring to approximate normal compliance due to the deformation of the ring. Discrete multi-tread elements are used for contact modeling. In contrast with the rigid ring models, the flexible ring model [5] consists of a deformable circular beam and allows investigations into the high frequency dynamics of the ring, via modal analysis [9] and prediction of standing waves [10].

The simplest flexible ring model is a tensioned string [11], which only accounts for the tensile strain without bending stiffness. Combined with a nest of radially arranged linear springs and dampers, this simple REF model was used to predict tire’s vertical load-deflection characteristics and free rolling resistance. To take the ring bending stiffness into account, a circular Euler-Bernoulli beam was adopted in [5], [12], [13]. This Euler-Bernoulli REF model has been more popular for its simplicity and acceptable accuracy in predicting responses for relatively thin rings. For thick rings, shear deformation is incorporated by the using Timoshenko beam theory in [14], [15], [16]. Therein, the cross-section plane is assumed to have a rotation in contrast to the Euler-Bernoulli beam model where the cross section plane is always normal to the neural axis.
In addition to the shear deformation, the extensibility of the ring in the circumferential direction is another assumption that might affect the response of the thick ring. Both thin and thick rings with extensible or inextensible assumptions were analyzed in [9], [10], where it is concluded that for rotating ring dynamics, the extensional coupling effects are as important as the shearing effects so that assumptions of inextensibility are improper for thick ring cases.

The flexible REF model is also used to study the contact problems involving the ring structure using different schemes. One approach considers the contact as a set of boundary conditions of the Equation of Motions (EOMs), as in [21]. The ring is divided into a non-contact region and a contact region. Displacement boundary conditions are applied within the contact region. However, there are two limitations here: 1) More contact regions need to be formulated if the contact is with more complex surface profile than that of a flat surface. In the dynamic contact situation with complex uneven surfaces, the number and location of the contact regions vary from time to time and cannot be known apriori. Formulating boundary conditions for every contact region at every time step becomes an iterative, if not impractical, undertaking. 2) The transverse deformation of the ring is missing in the contact analysis. The effect of this can be significant as discussed in [22]. Another approach adds discrete spring elements to the outer edge of the ring to model the transverse compliance of the beam [23], [24]. The compatibility condition in the contact requires separate treatments via additional equations for the force-deformation relationships of the added spring elements, which complicates the contact analysis. The idea presented by [22] uses a higher-order beam theory (higher than
a Timoshenko beam) for facilitating a solution to the static contact problem, where transverse compliance is incorporated into the beam equations themselves and thereby simplify the contact analysis. However, in this ring model, the transverse compliance of every point of the beam is assumed to be proportional to the un-deformed distance to the beam centroid. This assumption leads to a symmetric shrink or stretch of the beam’s transverse thickness with respect to the beam centroid, even when the loading of the beam is not symmetric at all. So this is not a realistic assumption. Furthermore, the contact is formulated as displacement boundary conditions in [22] as in [21] so that the limitation mentioned above still exist. In our previous work [27], we have presented a laminated beam model which consists of an internal layer of extensible Timoshenko beam and an external layer of transvers deformable beam. This preserves the benefit of higher-order beam model for the simplicity but corrects the unrealistic symmetric transverse deformation assumption. Based on this laminated ring model, we proposed a feedback compensation algorithm in [27] which gives a uniform method to solve frictionless contact problems of the flexible ring with arbitrary uneven surface. The laminated ring model and the contact algorithm will be revisited in this paper in the context of the rotating ring and dynamic rolling contact.

Besides the ring itself, the other important part of the REF model is the elastic foundation, which has also been treated using different assumptions. One of the most popular assumptions is the linear and uniformly distributed stiffness for the whole elastic foundation. The simplest assumption in this case only considers the radial stiffness [11], [12]. However, a two-parameter elastic foundation model, which includes both radial and
tangential stiffnesses, has been more broadly adopted [17], [13]. A more complex three-parameter foundation model was used in [1], where in addition to the radial and tangential stiffnesses, a stiffness associated with the distortion of the foundation due to in-plane rotation of a cross-section was included. The case of the linear and uniform assumption for the foundation stiffness, regardless of whether one, two or three parameters are used to describe it, leads to a problem that can be solved analytically. This will also be shown in this paper considering a laminated ring on a two-parameter foundation.

Deviations from uniformity of the stiffness of the elastic foundation have also been investigated in the existing literature. Natural frequencies and mode shapes of rings supported by arbitrarily located radial spring elements were studied in [18]. This problem is extended by [2] to a two-parameter, circumferentially varying foundation. Further extensions to a rotating ring with space-fixed and non-uniformly distributed discrete stiffnesses were outlined in [19]. However, these studies addressing non-uniformity of elastic foundation stiffness are still limited to pre-known stiffness distributions and both time and space fixed foundations. A more complex type of foundation was presented by considering a unilateral foundation, whose stiffness vanishes in the tensioned region (tensionless) or compressed region (collapsible). The difficulty in solving this group of problems lies in the fact that the tensioned or compressed region is not known in advance. The extent of the region depends on the loading and nonlinearly on the foundation’s stiffness distribution. One of the applications of this unilateral foundation is the non-
pneumatic tire presented in [25], where the foundation consists of collapsible spokes which offer stiffness only in tension.

There is rather limited published work analyzing flexible rings on such unilateral foundations. In [20], the forced response of a thin ring on a tensionless foundation was investigated via direct numerical integration. The assumption of an inextensible ring, which relates the tangential displacement of the ring with the radial displacement, was used to facilitate the solution of the resulting EOMs. However, this assumption oversimplifies of the problem. A more general case which considers an extensible Timoshenko ring supported by a collapsible foundation was studied by [33], by treating the supporting force of the collapsible spokes as radial distributed forces. Similar to the idea mentioned above from [21], the ring was divided in different regions and the contact is treated as boundary conditions in the contact region. However, in contrast to the two different regions (contact and non-contact region) used in [21], additional region had to be included for regions where neither contact force nor foundation supporting force exist. This addition region is obviously where the foundation supporting force vanishes due to the collapsible property. This gives the solution for the unilateral foundation problem but keeps the limitations cited above for [21] which prevent it from efficiently and uniformly handling dynamic contact with arbitrary uneven surfaces. In our previous work [26], we solved the forced vibration problem of flexible ring on such a unilateral foundation using iterative compensation method and an implicit Newmark scheme. The approach outlined is able to deal with arbitrary time varying forces in a uniform way. However, the ring model in [26] was limited to a non-rotating Timoshenko ring and the dynamic contact
problem was not addressed. The present paper removes both restrictions by treating a rotating laminated ring on a unilateral foundation and in rolling contact with uneven surfaces.

To recap, the laminated ring model we presented in [27] covers a good general case that includes the bending, shear, extensibility of the ring as well as the transverse compliance of the ring structure. The feedback compensation contact algorithm proposed there also works seamlessly with the laminated ring model to give a uniform way to solve the contact problem of the ring with rigid surface of arbitrary profile. However, the elastic foundation analyzed in [27] was limited to a linear foundation and the problem solved there is limited to static contact cases. In this paper, we address the rolling dynamic contact response of the laminated ring on a unilateral foundation, by combining the iterative compensation method proposed in [16] for the unilateral foundation problem and the implicit Newmark scheme we adopted in [26] for solving the dynamic problem. Rotation effects will be explicitly included in this paper in the derivation of the EOMs to obtain a very general model that can be used for the analysis of rolling contact induced dynamics of tires and other similar REF structures, with linear or nonlinear foundations.

The rest of the paper is organized as follows. Section 5.3 describes the schematic of the rotating laminated REF model and develops the Equation of Motions (EOMs). Section Error! Reference source not found. gives the analytical solutions for both the static deformation and dynamic response due to arbitrary external forces for the case of the linear foundation. Section Error! Reference source not found. extends the solution to the unilateral foundation model. In Section Error! Reference source not found., a
feedback p-controller algorithm is utilized to solve the dynamic contact problem with arbitrary surface profiles. Section 5.7 outlines a method for modal analysis and prediction of critical velocities for the linearized cases of the laminated REF on unilateral foundation. Results for the rolling contact response and modal/critical velocity analysis are given in Section 5.8. Finally, Section 5.9 concludes the paper.

5.3 Schematic of the Rotating Laminated REF Model and Equation of Motions

Figure 19 shows the schematic of rotating Laminated REF model. The laminated ring model consists of an internal layer of Timoshenko ring with thickness $h$ and centroid radius $R$, as well as an external layer of transverse deformable ring with thickness $h_2$. The width of the model in the direction perpendicular to the plane of the ring is $b$. The radial and tangential stiffnesses of the elastic foundation are $K_r$ and $K_\theta$, respectively. These have units of stiffness per radian. The ring is rotating at angular velocity $\Omega$ with respect to a fixed center. A Cartesian coordinate frame and a non-rotating cylindrical coordinate frame are adopted, with origins both located at the center. The circumferential coordinate in the cylindrical frame $\theta$ takes values in $[-\pi, \pi]$, and the circumferential location of the bottom point of the ring is assumed to be $\theta = 0$. The ring may be in contact with a translating surface of arbitrary profile at the bottom (to be detailed later).
Starting from our laminated ring formulation in [27], with the goal of extending it to the dynamic and rotating case, the radial and tangential displacements of an arbitrary point in the laminated ring at time $t$ are:

$$
\begin{align*}
&u_r(r, \theta, t) = \begin{cases} 
u_r(R, \theta, t) & R - \frac{h}{2} < r < R + \frac{h}{2} \\ u_r(R, \theta, t) + \left[ r - \left( R + \frac{h}{2} \right) \right] \phi(R, \theta, t) & R + \frac{h}{2} < r < R + \frac{h}{2} + h_z \end{cases} \\
u_\theta(r, \theta, t) &= u_\theta(R, \theta, t) + (r - R) \phi(R, \theta, t)
\end{align*}
$$

where $r, \theta$ are radial and circumferential coordinates of the arbitrary point, respectively. $u_r(R, \theta, t), u_\theta(R, \theta, t), \phi(R, \theta, t)$ and $\psi(R, \theta, t)$ are respectively, radial, tangential displacements, cross-section rotation and linear gradient of the transverse displacement at the internal ring centroid (at radius $R$ and circumferential coordinate $\theta$).
The strain-displacement relationships are given in polar coordinate form as [35]:

\[
\begin{align*}
\epsilon_r(r, \theta, t) &= \frac{\partial}{\partial r} u_r(r, \theta, t) \\
\epsilon_{\theta\theta}(r, \theta, t) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta(r, \theta, t) + \frac{1}{r} u_r(r, \theta, t) \\
\gamma_{r\theta}(r, \theta, t) &= \frac{1}{r} \frac{\partial}{\partial \theta} u_r(r, \theta, t) + \frac{\partial}{\partial r} u_\theta(r, \theta, t) - \frac{1}{r} u_\theta(r, \theta, t)
\end{align*}
\]  

(5.2)

Since \( u_r(r, \theta, t) \) is a piecewise continuous function, strain expressions obtained from (5.2) will be different for the internal and the external ring. We shall use notations \( \epsilon_{rr,i}, \epsilon_{\theta\theta,i} \) and \( \gamma_{r\theta,i} \) for the radial, tangential and shear strains in both layers, and \( \sigma_{rr,i}, \sigma_{\theta\theta,i} \) and \( \tau_{r\theta,i} \) for the corresponding stresses, where subscript \( i = 1, 2 \) represents the internal and the external ring, respectively.

Considering the general case that the ring is orthotropic and homogeneous, and material axes coincide with the cylindrical coordinate system adopted, the stress-strain relationships are given by:

\[
\begin{align*}
\sigma_{rr,i}(r, \theta, t) &= E_{r,i} \epsilon_{rr,i}(r, \theta, t) \\
\sigma_{\theta\theta,i}(r, \theta, t) &= E_{\theta,i} \epsilon_{\theta\theta,i}(r, \theta, t) \\
\tau_{r\theta,i}(r, \theta, t) &= G_i \gamma_{r\theta,i}(r, \theta, t)
\end{align*}
\]  

(5.3)

\( E_{r,i}, E_{\theta,i} \) and \( G_i \) are corresponding elastic moduli in the radial and tangential directions and the shear modulus, respectively.

We consider the tire application where circumferential strain energy and the shear energy for external tread rubber layer are considered negligible compared with that of the tire belt. Then the total strain energy change in the laminated ring from time \( t_0 \) to \( t_1 \) is obtained by:
\[ U_1 = \frac{b}{2} \int_0^\pi \int_{-\pi}^\pi \int_{-\pi}^{\pi/2} \left( \sigma_{rr}, \epsilon_{rr}, \sigma_{\theta\theta}, \epsilon_{\theta\theta}, \tau_{r\theta}, \gamma_{r\theta} \right) r dr d\theta dt \]
\[ + \frac{b}{2} \int_0^\pi \int_{-\pi}^\pi \int_{-\pi}^{\pi/2} \left( \sigma_{rr}, \epsilon_{rr} \right) r dr d\theta dt \]

The strain energy change in the elastic foundation (assumed elastic at this stage) from time \( t_0 \) to \( t_1 \) is obtained by:

\[ U_2 = \frac{1}{2} \int_{t_0}^{t_1} \int_0^\pi \left[ K_r \left( u_r \left( R - \frac{h}{2}, \theta, t \right) \right)^2 + K_\theta \left( u_\theta \left( R - \frac{h}{2}, \theta, t \right) \right)^2 \right] d\theta dt \]

Then kinetic energy of the rotating ring is obtained by:

\[ T = \rho_1 \frac{b^2}{2} \int_0^\pi \int_{-\pi}^\pi \int_{-\pi}^{\pi/2} (v_x^2 + v_y^2) r dr d\theta dt + \rho_2 \frac{b^2}{2} \int_0^\pi \int_{-\pi}^\pi \int_{-\pi}^{\pi/2} (v_x^2 + v_y^2) r dr d\theta dt \]

where \( \rho_1, \rho_2 \) are the mass densities of the internal and external rings, \( v_x \) and \( v_y \) are velocities in global Cartesian coordinate frame of any mass point of the ring and are obtained from:

\[ v_x = \frac{d}{dt} x(r, \theta, t) \]
\[ v_y = \frac{d}{dt} y(r, \theta, t) \]
\[ x(r, \theta, t) = \left( R + u_r(r, \theta, t) \right) \cos \theta(t) - u_\theta(r, \theta, t) \sin \theta(t) \]
\[ y(r, \theta, t) = \left( R + u_r(r, \theta, t) \right) \sin \theta(t) + u_\theta(r, \theta, t) \cos \theta(t) \]

The work done by external forces is given by:

\[ W = b \int_0^\pi \int_{-\pi}^\pi \left[ q_r u_r \left( R + \frac{h}{2}, \theta, t \right) + q_\theta u_\theta \left( R + \frac{h}{2}, \theta, t \right) \right] \left( R + \frac{h}{2}, \theta, t \right) d\theta dr \]
where \( q_r = q_r \left( R + \frac{h}{2} + h_2, \theta, t \right) \) and \( q_\theta = q_\theta \left( R + \frac{h}{2} + h_2, \theta, t \right) \) are respectively, the radial and tangential components of general time-varying and distributed forces applied to the external surface of the external ring (at radial coordinate \( R + \frac{h}{2} + h_2 \)). The units of \( q_r \) and \( q_\theta \) are in \( \frac{\text{Force}}{\text{Area}} \).

Viscous damping is introduced using Rayleigh’s dissipation functions [40]:

\[
\zeta_{Rr,2} = \frac{1}{2} C_{Rr,2} \left( \frac{\partial}{\partial t} \epsilon_{r,2} \left( r, \theta, t \right) \right)^2 \\
\zeta_{Er} = \frac{1}{2} C_{Er} \left( \frac{\partial}{\partial t} u_r \left( R - \frac{h}{2}, \theta, t \right) \right)^2 \\
\zeta_{E\theta} = \frac{1}{2} C_{E\theta} \left( \frac{\partial}{\partial t} u_\theta \left( R - \frac{h}{2}, \theta, t \right) \right)^2
\]

(5.9)

where \( \zeta_{Rr,2}, \zeta_{Er} \) and \( \zeta_{E\theta} \) are Rayleigh's dissipation functions for the radial direction of the external ring, and the radial and tangential direction of the elastic foundation, respectively; \( C_{Rr,2}, C_{Er} \) and \( C_{E\theta} \) represent the viscous damping densities in the radial direction in the external ring, and in radial and tangential directions in the elastic foundation, respectively. The unit of \( C_{Rr,2} \) is \( \frac{N}{m \ (m/s)} \), while the unit of \( C_{Er} \) and \( C_{E\theta} \) is in \( \frac{N}{\text{radian \ (m/s)}} \). Please note that the damping effects inside the internal layer of the laminated ring are lumped into those of the elastic foundation.

The EOMs of the rotating laminated REF model can be obtained by applying the extended Hamilton's principle [37]:

\[
\delta \left( U_1 + U_2 - T \right) + \left( R_R + R_E \right) = \delta W
\]

(5.10)

where \( R_R \) and \( R_E \) are defined in terms of Rayleigh’s dissipation functions as:
where the following shorthand is adopted for the displacement variables introduced in (5.1):

\[
\begin{align*}
&u_r = u_r (R, \theta, t) \\
&u_\phi = u_\phi (R, \theta, t) \\
&\psi = \psi (R, \theta, t) \\
&\phi = \phi (R, \theta, t)
\end{align*}
\]  

The coefficients whose names start with C or CT and followed by numbers are constant coefficients, which chiefly consist of geometric and material parameters, as well as ring rotation speed \( \Omega \), which is the time derivative of circumferential position \( \theta(t) \):
\[ \Omega = \frac{d}{dt} \theta(t) \]  
(5.14)

The detailed expressions of all coefficients appearing in (5.12) are listed in the Appendix D, where four new stiffness parameters \( E_{A\theta}, E_{I\theta}, G_A \) and \( E_{A_{r,2}} \) are defined as:

\[
\begin{align*}
E_{A\theta} &= E_{\theta,1}A_1 = E_{\theta,1}bh \\
E_{I\theta} &= E_{\theta,1}I_1 = \frac{1}{12} E_{\theta,1}bh^3 \\
G_A &= G_A A_1 = G_A bh \\
E_{A_{r,2}} &= E_{r,2}A_2 = E_{r,2}bh_2 
\end{align*}
\]
(5.15)

The EOMs (5.12) include 4 deformation variables \( u_r, u_\theta, \psi \) and \( \psi \), in contrast with just the first three appearing in our previous works [16] and [26]. The additional variable \( \psi \) is introduced for transverse compliance of the external layer of the ring. Compared with [27], where the same laminated REF model is used to analyze the static problem, time derivative terms and rotational effects are included in the EOMs (5.12). In summary, (5.12) models the rotating dynamics for a laminated REF and serves as a generalization of the special cases treated previously.

### 5.4 Forced Response: Analytical Solutions for Laminated Ring on Linear Elastic Foundation

We start by considering the governing differential equations for the quasi-static case (by setting the time derivatives in (5.12) to zero) which have the same structure as those in [27], with differences in the rotational contributions:
\[ C_{112} \frac{\partial^2}{\partial \theta^2} u_r + C_{110} \cdot u_r + C_{121} \frac{\partial}{\partial \theta} u_\theta + C_{141} \frac{\partial}{\partial \theta} \phi + C_{110} = \left( R + \frac{h}{2} + h_2 \right) q_r \]

\[ C_{211} \frac{\partial}{\partial \theta} u_r + C_{222} \frac{\partial^2}{\partial \theta^2} u_\theta + C_{220} \cdot u_\theta + C_{240} \cdot \phi + C_{20} = \left( R + \frac{h}{2} + h_2 \right) q_\theta \]

\[ C_{310} \cdot u_r + C_{321} \frac{\partial}{\partial \theta} u_\theta + C_{330} \cdot \psi + C_{341} \frac{\partial}{\partial \theta} \phi + C_{30} = \left( R + \frac{h}{2} + h_2 \right) h_2 q_r \]

\[ C_{411} \frac{\partial}{\partial \theta} u_r + C_{420} \cdot u_\theta + C_{442} \frac{\partial^2}{\partial \theta^2} \phi + C_{440} \cdot \phi + C_{40} = \left( R + \frac{h}{2} + h_2 \right) \left( \frac{h}{2} + h_2 \right) q_\theta \]

(5.16)

As in [27], \( q_r \) and \( q_\theta \) in the right hand sides of governing equations (5.16) can be expanded in corresponding radial/tangential and cosine/sine components:

\[ q_r = \sum_{n=-N}^{N} \left[ q_{n,r,c} + q_{n,r,s} \right] = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} \sum_{n=-N}^{N} \left[ Q_{n,r,c} \cos(n\theta) + Q_{n,r,s} \sin(n\theta) \right] \]

\[ q_\theta = \sum_{n=-N}^{N} \left[ q_{n,\theta,c} + q_{n,\theta,s} \right] = \frac{1}{b \left( R + \frac{h}{2} + h_2 \right)} \sum_{n=-N}^{N} \left[ Q_{n,\theta,c} \cos(n\theta) + Q_{n,\theta,s} \sin(n\theta) \right] \]

(5.17)

where, \( n \) is a circumferential mode (harmonic) number, \( N \) is the cut-off harmonic number. The subscript \( r \) or \( \theta \) indicate whether the term is for radial or tangential direction, respectively; while \( c \) or \( s \) represents cosine or sine components, respectively. \( Q_{n,r,c}, Q_{n,r,s}, Q_{n,\theta,c} \) and \( Q_{n,\theta,s} \) are corresponding Fourier coefficients of the \( n^{th} \) harmonic force component of an arbitrary distributed force vector \( \vec{F}(\theta) \) which can be given by:

\[ \vec{F}(\theta) = \left( \sum_{n=-N}^{N} Q_{n,r,c} \cos(n\theta) + \sum_{n=-N}^{N} Q_{n,r,s} \sin(n\theta) \right) \hat{r} + \left( \sum_{n=-N}^{N} Q_{n,\theta,c} \cos(n\theta) + \sum_{n=-N}^{N} Q_{n,\theta,s} \sin(n\theta) \right) \hat{\theta} \]

(5.18)

The solution of (5.16) to this applied force can be sought in the form of corresponding components:
where, besides the first four terms corresponding to the radial/tangential and cosine/sine components in the solution of each displacement variable, there is an initial component with subscripts 0, which is introduced by constant terms C10, C20, C30 and C40 in (5.16). These initial displacement components are generated by centrifugal forces since these constant terms are functions of $\Omega$ (See Appendix D). For each of the radial/tangential and cosine/sine components, the expressions were given in terms of superposition of every $n^{th}$ harmonic component. Take radial and cosine terms, for example:

\[
\begin{align*}
    u_r(R,\theta) &= u_{r,N,r,c}(R,\theta) + u_{r,N,\theta,c}(R,\theta) + u_{r,0,r,c}(R,\theta) + u_{r,0}\nonumber \\
    u_\theta(R,\theta) &= u_{\theta,N,r,c}(R,\theta) + u_{\theta,N,\theta,c}(R,\theta) + u_{\theta,0,r,c}(R,\theta) + u_{\theta,0}\nonumber \\
    \psi(R,\theta) &= \psi_{N,r,c}(R,\theta) + \psi_{N,\theta,c}(R,\theta) + \psi_{N,0,r,c}(R,\theta) + \psi_0\nonumber \\
    \phi(R,\theta) &= \phi_{N,r,c}(R,\theta) + \phi_{N,\theta,c}(R,\theta) + \phi_{N,0,r,c}(R,\theta) + \phi_0
\end{align*}
\] (5.19)

where, Cur$_{r,c}$(n), Cu$_{\theta,c}$(n), C$\psi_{r,c}$(n) and C$\phi_{r,c}$(n) are corresponding modal coefficients which are functions of harmonic number $n$. They can be easily solved for by substitution of (5.17), (5.19) and (5.20) into (5.16) and decoupling the obtained equations into corresponding radial/tangential, cosine/sine and constant components at every harmonic level.
Next, we consider the **dynamic case** that the arbitrary external force (5.18) becomes time variant:

\[
F(\theta,t) = \left[ \sum_{n=-N}^{N} Q_{n,r}(t)\cos(n\theta) + \sum_{n=-N}^{N} Q_{n,s}(t)\sin(n\theta) \right] + \left[ \sum_{n=-N}^{N} Q_{n,\theta,r}(t)\cos(n\theta) + \sum_{n=-N}^{N} Q_{n,\theta,s}(t)\sin(n\theta) \right] \partial \theta
\]  

(5.21)

Note that in (5.21) the time variant force vector is decomposed into Fourier series with respect to the fixed (non-rotating) cylindrical coordinate system. However, due to the rotation of the ring, the displacement variables for every harmonic component rotates at speed \( \Omega \) with respect to the fixed cylindrical coordinate system. Again using radial and cosine terms for example:

\[
u_{r,N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} u_{r,n,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cu_{r,c}(n)\cos(n(\theta + \Omega t))
\]

(5.22)

\[
u_{\theta,N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} u_{\theta,n,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cu_{r,c}(n)\sin(n(\theta + \Omega t))
\]

\[
\psi_{N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} \psi_{n,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cy_{r,c}(n)\cos(n(\theta + \Omega t))
\]

\[
\phi_{N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} \phi_{n,r,c}(R,\theta,t) = \sum_{n=-N}^{N} C\phi_{r,c}(n)\sin(n(\theta + \Omega t))
\]

Applying trigonometric expands in (5.22) leads to the following corresponding terms in the dynamic solution of the rotating ring:

\[
u_{r,N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cu_{r,c}(n)\left[ \cos(\Omega nt)\cos(n\theta) - \sin(\Omega nt)\sin(n\theta) \right] = \sum_{n=-N}^{N} Cu_{r,c}(n)\left[ a_{n,r,c}(t)\cos(n\theta) + a_{n,s,r,c}(t)\sin(n\theta) \right]
\]

\[
u_{\theta,N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cu_{r,c}(n)\left[ \sin(\Omega nt)\cos(n\theta) + \cos(\Omega nt)\sin(n\theta) \right] = \sum_{n=-N}^{N} Cu_{r,c}(n)\left[ b_{n,\theta,r,c}(t)\cos(n\theta) + b_{n,\theta,s,r,c}(t)\sin(n\theta) \right]
\]

(5.23)

\[
\psi_{N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} Cy_{r,c}(n)\left[ c_{n,r,c}(t)\cos(n\theta) + c_{n,s,r,c}(t)\sin(n\theta) \right]
\]

\[
\phi_{N,r,c}(R,\theta,t) = \sum_{n=-N}^{N} C\phi_{r,c}(n)\left[ d_{n,r,c}(t)\cos(n\theta) + d_{n,s,r,c}(t)\sin(n\theta) \right]
\]

where time variant coefficients \( a, b, c \) and \( d \) are those to be solved in the dynamic case.

The additional subscript \( i = 1,2 \) indicates the corresponding time coefficients are for
cosine or sine part, respectively. After substitution of (5.23) and radial and cosine component of (5.21) into (5.12) and decoupling the consequent equations (by coefficients of \(\cos(n\theta)\) or \(\sin(n\theta)\)), one obtains equations that can be written into the form of standard second-order dynamic system:

\[
M_{n,r,c}A_{n,r,c}(t) + C_{n,r,c}V_{n,r,c}(t) + K_{n,r,c}X_{n,r,c}(t) = \Gamma_{n,r,c}(t)
\]  

(5.24)

where the unknown vectors are given by:

\[
A_{n,r,c}(t) = \frac{\partial}{\partial t} \left[ a_{1,n,r,c}(t) \ a_{2,n,r,c}(t) \ b_{1,n,r,c}(t) \ b_{2,n,r,c}(t) \ c_{1,n,r,c}(t) \ c_{2,n,r,c}(t) \ d_{1,n,r,c}(t) \ d_{2,n,r,c}(t) \right]^T
\]

\[
V_{n,r,c}(t) = \frac{\partial}{\partial t} \left[ a_{1,n,r,c}(t) \ a_{2,n,r,c}(t) \ b_{1,n,r,c}(t) \ b_{2,n,r,c}(t) \ c_{1,n,r,c}(t) \ c_{2,n,r,c}(t) \ d_{1,n,r,c}(t) \ d_{2,n,r,c}(t) \right]^T
\]

\[
X_{n,r,c}(t) = \left[ a_{1,n,r,c}(t) \ a_{2,n,r,c}(t) \ b_{1,n,r,c}(t) \ b_{2,n,r,c}(t) \ c_{1,n,r,c}(t) \ c_{2,n,r,c}(t) \ d_{1,n,r,c}(t) \ d_{2,n,r,c}(t) \right]^T
\]

(5.25)

Then, a standard State-Space form can be obtained via further manipulation of (5.24):

\[
\dot{Y}_{n,r,c}(t) = AM_{n,r,c}Y_{n,r,c}(t) + UM_{n,r,c}(t)
\]  

(5.26)

where the state vector is:

\[
Y_{n,r,c}(t) = \left[ a_{1,n,r,c}(t) \ \frac{\partial}{\partial t} a_{1,n,r,c}(t) \ a_{2,n,r,c}(t) \ \frac{\partial}{\partial t} a_{2,n,r,c}(t) \ b_{1,n,r,c}(t) \ \frac{\partial}{\partial t} b_{1,n,r,c}(t) \ b_{2,n,r,c}(t) \ \frac{\partial}{\partial t} b_{2,n,r,c}(t) \ c_{1,n,r,c}(t) \ \frac{\partial}{\partial t} c_{1,n,r,c}(t) \ c_{2,n,r,c}(t) \ \frac{\partial}{\partial t} c_{2,n,r,c}(t) \ d_{1,n,r,c}(t) \ \frac{\partial}{\partial t} d_{1,n,r,c}(t) \ d_{2,n,r,c}(t) \ \frac{\partial}{\partial t} d_{2,n,r,c}(t) \right]^T
\]

(5.27)

The analytical solution for (5.26) exist and is given by:

\[
Y_{n,r,c}(t) = e^{AM_{n,r,c}t}Y_{n,r,c}(0) + \int_0^t UM_{n,r,c}(\tau)e^{AM_{n,r,c}(t-\tau)}d\tau
\]  

(5.28)

where \(Y_{n,r,c}(0)\) is the initial value of the state vector. The components of the solved for state vector \(Y_{n,r,c}(t)\) can then be used to construct the dynamic solutions (5.23) for the radial and cosine term.
Repeating the steps from (5.22) to (5.28) for the other terms (radial and sine, tangential and cosine, etc) and combining the results, the full dynamic forced response can be assembled:

\[ u_r = u_{r,N,r,r}(R, \theta, t) + u_{r,N,r,s}(R, \theta, t) + u_{r,N,\theta,r}(R, \theta, t) + u_{r,0} \]

\[ u_\theta = u_{\theta,N,r,r}(R, \theta, t) + u_{\theta,N,r,s}(R, \theta, t) + u_{\theta,N,\theta,r}(R, \theta, t) + u_{\theta,0} \]

\[ \psi = \psi_{N,r,r}(R, \theta, t) + \psi_{N,r,s}(R, \theta, t) + \psi_{N,\theta,r}(R, \theta, t) + \psi_0 \]

\[ \phi = \phi_{N,r,r}(R, \theta, t) + \phi_{N,r,s}(R, \theta, t) + \phi_{N,\theta,r}(R, \theta, t) + \phi_0 \]  

Note that the initial components \( u_{r,0}, u_{\theta,0}, \psi_0 \) and \( \phi_0 \) are only determined by constant terms (due to centrifugal) and are not subject to time variant forces, so they are still time invariant even in the dynamic case.

Often the axle reaction forces are of interest and can be obtained by considering the motion of foundation. According to (5.1), the foundation radial and tangential displacements can be obtained by:

\[ u_{r,F} = u_r \left( R - \frac{h}{2}, \theta, t \right) = u_r \]

\[ u_{\theta,F} = u_\theta \left( R - \frac{h}{2}, \theta, t \right) = u_\theta - \frac{h}{2} \phi \]  

Then, the axle reaction forces in the vertical and horizontal directions are given by:

\[ F_z = \int_{-\pi}^{\pi} \left( -u_{r,F} \cdot K_r + \frac{\partial}{\partial t} u_{r,F} \cdot C_{Er} \right) \cos(\theta) + \left( u_{\theta,F} \cdot K_\theta + \frac{\partial}{\partial t} u_{\theta,F} \cdot C_{E\theta} \right) \sin(\theta) \, d\theta \]

\[ F_x = \int_{-\pi}^{\pi} \left( u_{r,F} \cdot K_r + \frac{\partial}{\partial t} u_{r,F} \cdot C_{Er} \right) \sin(\theta) + \left( u_{\theta,F} \cdot K_\theta + \frac{\partial}{\partial t} u_{\theta,F} \cdot C_{E\theta} \right) \cos(\theta) \, d\theta \]  

5.5 Forced Response: Solution for Laminated Ring on Unilateral Elastic Foundation
The dynamic solution for a non-rotating Timoshenko ring on unilateral foundation was given in our previous work [26] using an implicit Newmark scheme [41] for time integration and an interactive compensation method to solve the unilateral foundation problem within every time step. These methods will be extended in this section to solve the dynamic case for the rotating laminated ring on unilateral foundation problem.

Considering the second-order dynamic system (5.24), the classical Newmark method assumes that the velocity \( V_{n,r,c}(t + \Delta t) \) and position vectors \( X_{n,r,c}(t + \Delta t) \) of the next time step are determined by weighted combinations of the accelerations of both the next and current time steps \( (A_{n,r,c}(t + \Delta t) \) and \( A_{n,r,c}(t) \)):

\[
V_{n,r,c}(t + \Delta t) = V_{n,r,c}(t) + [(1 - \alpha)A_{n,r,c}(t + \Delta t) + \alpha A_{n,r,c}(t)]\Delta t \\
X_{n,r,c}(t + \Delta t) = X_{n,r,c}(t) + \Delta V_{n,r,c}(t) + \left[ \beta A_{n,r,c}(t + \Delta t) + \left( \frac{1}{2} - \beta \right) A_{n,r,c}(t) \right]\Delta t^2
\]

where \( \Delta t \) is the time step. \( \alpha \) and \( \beta \) are weighing coefficients. In the present work, the average acceleration method with \( \alpha = \frac{1}{2}, \beta = \frac{1}{4} \) is adopted.

Assuming the states of current time step for the unilateral foundation system are already known as \( \tilde{X}(t), \tilde{V}(t) \) and \( \tilde{A}(t) \), combining (5.24) and (5.32), the displacement increments for the next time step can be obtained by (considering the radial and cosine terms as examples):
\[
\Delta \hat{X}_{n,rc}(t + \Delta t) = \hat{X}_{n,rc}(t + \Delta t) - \hat{X}_{n,rc}(t)
\]

\[
= \frac{1}{2} \Delta t \left( C_{n,rc} \alpha \Delta t^2 + 2 C_{n,rc} \beta \Delta t^2 - C_{n,rc} \alpha \Delta t - C_{n,rc} \Delta t - M_{n,rc} \Delta t \right) \hat{A}_{n,rc}(t)
\]

\[
+ \frac{1}{2} \Delta t \left( 2 C_{n,rc} \alpha \Delta t - 2 C_{n,rc} \Delta t - 2 M_{n,rc} \right)
\]

\[
- K_{n,rc} \beta \Delta t^2 + C_{n,rc} \alpha \Delta t - C_{n,rc} \Delta t - M_{n,rc}
\]

\[
\left[ \Gamma_{n,rc}(t + \Delta t) - \Gamma_{n,rc}(t) \right]
\]

(5.33)

Then, the radial and cosine displacement component \( \hat{u}_{r,n,rc}(R, \theta, t + \Delta t) \) and its increment \( \Delta \hat{u}_{r,n,rc}(R, \theta, t + \Delta t) \) are obtained. Consequently, the total displacement and increment can be obtained and are assumed to be \( \hat{u}_r(R, \theta, t + \Delta t) \) and \( \Delta \hat{u}_r(R, \theta, t + \Delta t) \). However, the matrices \( M, K, C \) in (5.33) are still for a linear system with a linear elastic foundation. For a radially collapsible foundation, the foundation forces vanish in the area \( \hat{u}_r(R, \theta, t + \Delta t) < 0 \). In this case, the transition from time \( t \) to \( t + \Delta t \) solved via (5.33) includes effects by excessive forces in that region. The magnitude of this excessive force at time \( t + \Delta t \) is proportional to the displacement increment \( \Delta \hat{u}_r(R, \theta, t + \Delta t) \), but the region is determined by the total displacement \( \hat{u}_r(R, \theta, t + \Delta t) \). The excessive force is given by:

\[
F_e(\theta, t + \Delta t) = \begin{cases} 
\Delta \hat{u}_r(R, \theta, t + \Delta t) K_r & \{ \theta | \theta \in \hat{u}_r(R, \theta, t + \Delta t) < 0 \} \\
0 & \{ \theta | \theta \in \hat{u}_r(R, \theta, t + \Delta t) \geq 0 \}
\end{cases}
\]

(5.34)

For tensionless foundation case, this excessive force exists in the opposite region where \( \hat{u}_r(R, \theta, t + \Delta t) > 0 \), but all remaining analyses will remain the same. Then, the compensation to the excessive force is applied to the linear foundation system (5.24) expanded in the Fourier series form:
\[ F_{cp}(\theta, t + \Delta t) = F_e(\theta, t + \Delta t) = \sum_{n=0}^{N} F_{e,n}(\theta, t + \Delta t) = \sum_{n=0}^{N} \left[ H_{n,r,e}(t + \Delta t) \cos(n\theta) + H_{n,r,e}(t + \Delta t) \sin(n\theta) \right] \]

(5.35)

Since the unilateral property of the foundation is only assumed in the radial direction, this compensation force only has radial components. Noting the common terms in (5.35) and (5.21), the compensation force vectors can be obtained by:

\[
\Gamma_{cp,n,r,e}(t + \Delta t) = \frac{H_{n,r,e}(t + \Delta t)}{Q_{n,r,e}(t + \Delta t)} \Gamma_{n,r,e}(t + \Delta t)
\]

\[
\Gamma_{cp,n,r,e}(t + \Delta t) = \frac{H_{n,r,e}(t + \Delta t)}{Q_{n,r,e}(t + \Delta t)} \Gamma_{n,r,e}(t + \Delta t)
\]

(5.36)

The subscript \( cp \) stands for compensation force. By analogy, the compensated displacement increment is obtained by updating the external forcing vector with the compensation force vector:

\[
\Delta \hat{X}_{n,r,c}(t + \Delta t) = \hat{X}_{n,r,c}(t + \Delta t) - \hat{X}_{n,r,c}(t)
\]

\[
= \frac{1}{2} \Delta t \left( \frac{C_{n,r,c} \Delta t^2 + 2C_{n,r,c} \beta \Delta t^2 - C_{n,r,c} \Delta t^2 - M_{n,r,c} \Delta t}{-K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \hat{\dot{X}}_{n,r,c}(t) \right)
\]

\[
+ \frac{1}{2} \Delta t \left( \frac{2C_{n,r,c} \alpha \Delta t - 2C_{n,r,c} \Delta t - 2M_{n,r,c}}{K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c}} \hat{\dot{V}}_{n,r,c}(t) \right)
\]

\[
\Delta t^2 \beta \left( -K_{n,r,c} \beta \Delta t^2 + C_{n,r,c} \alpha \Delta t - C_{n,r,c} \Delta t - M_{n,r,c} \right)
\]

(5.37)

The new increment and displacement obtained via (5.37) leads to new references that can used to obtain the excessive force (5.34). Repeated iterations from (5.34) to (5.37) lead to the iterative compensation within every time step to find the spatial solution response for the unilateral foundation problem. Once converged, \( \hat{X}_{n,r,c}(t + \Delta t) \)
can be plugged into the Newmark scheme (5.32) to solve for the new $\hat{V}_{n,r,c}(t + \Delta t)$ and $\hat{A}_{n,r,c}(t + \Delta t)$ to start the iteration to the next time step.

The axle force response in the unilateral foundation can be obtained by the same integration as in (5.31) but within the effective regions where the foundation force does not vanish. The axle forces for the collapsible foundation case, for instance, are obtained by:

$$ F_z = - \int_{\theta_{t_0,z} \leq \theta} \left[ u_{zF} \cdot K_z + \frac{\partial}{\partial t} u_{zF} \cdot C_{zE} \right] \cos(\theta) d\theta + \int_{-\pi}^{\pi} \left[ u_{\theta_0} \cdot K_{\theta} + \frac{\partial}{\partial t} u_{\theta_0} \cdot C_{E\theta} \right] \sin(\theta) d\theta $$

$$ F_x = \int_{\theta_{t_0,x} \leq \theta} \left[ u_{xF} \cdot K_x + \frac{\partial}{\partial t} u_{xF} \cdot C_{xF} \right] \sin(\theta) d\theta + \int_{-\pi}^{\pi} \left[ u_{\theta_0} \cdot K_{\theta} + \frac{\partial}{\partial t} u_{\theta_0} \cdot C_{E\theta} \right] \cos(\theta) d\theta $$

(5.38)

5.6 Contact Response of the Rolling Laminated Ring: Feedback Compensation Algorithm

We proposed a feedback compensation scheme to solve the tire-road contact problem using the laminated REF model in our previous work [27]. The problem solved there is limited to static contact only. In this paper, with the dynamic solution for rotating laminated ring on unilateral foundation addressed in Section 5.5, the feedback compensation algorithm is extended to solve the rolling contact dynamics of the laminated ring with an arbitrary rigid surface profile.

The general idea for the feedback compensation algorithm, which we called a P-Controller Algorithm, is to iteratively compensate the distributed contact force using the geometry error between the rigid surface profile and the deformed shape of the laminated ring, until the maximum geometry error is reduced below an acceptable tolerance value.
At the beginning of every time step, the surface profile is updated according to the relative motion between the ring and the surface. The ring deformation is also updated based on a no-contact assumption. Then, geometry error is calculated so that the adjusted contact force is obtained based on this error. At every iteration step, the deformed shape of the laminated ring has to be re-calculated according to this adjusted contact force. The dynamic Force-Deformation relationship established in Sections 5.4 (for the linear foundation case) and 5.5 (for the unilateral foundation case) serves as a modular block at every iteration step for the contact solution. Once the convergence of the contact force as well as the ring shape is achieved at the current time step, the ring deformation is saved and used for next time step. The schematic of the overall dynamic contact algorithm is shown in Figure 20.
Figure 20: Schematic for Solving Rolling Contact of the Laminated Ring on a Linear or Unilateral Foundation via P-Controller Algorithm

The geometry error is obtained based on the deformed ring shape and the projected surface profile with respect to the non-rotating cylindrical coordinate. The surface profile projection from a Cartesian coordinate \((X, Y)\) to the cylindrical coordinate is:

\[
\begin{align*}
    r_s &= \sqrt{X^2 + Y^2} \\
    \theta_s &= \arctan \left( \frac{-X}{Y} \right)
\end{align*}
\]  

(5.39)

where \(r_s, \theta_s\) are the projected radial and circumferential coordinates of the surface profile, respectively. In general, the surface profile lies on the one side of the ring, such as the tire-road contact case, so that \(\max |\theta_s|\) exists and we assume \(\max |\theta_s| < \frac{\pi}{2}\). Then, we define the radial coordinate of the surface profile as a mapping of the circumferential coordinate:

\[
\begin{align*}
    r_{ms}(\alpha) : \{ \theta_s \rightarrow r_s, \min(\theta_s) \leq \alpha \leq \max(\theta_s) \} \\
    Ct \quad -\pi \leq \alpha \leq \min(\theta_e) \cup \max(\theta_e) \leq \alpha \leq \pi
\end{align*}
\]  

(5.40)

where \(Ct\) is a large constant \(Ct \gg R\).

The deformed ring shape can be obtained via (5.1) using the radial coordinate \((R + \frac{h}{2} + h_2)\) of the external edge of the laminated ring:

\[
\begin{align*}
    u_{r,e} &= u_r \left( R + \frac{h}{2} + h_2, \theta, t \right) = u_r (R, \theta, t) + h_2 \phi_e (R, \theta, t) \\
    u_{\theta,e} &= u_\theta \left( R + \frac{h}{2} + h_2, \theta, t \right) = u_\theta (R, \theta, t) + \left( \frac{h}{2} + h_2 \right) \phi (R, \theta, t)
\end{align*}
\]  

(5.41)
Then, the radial and circumferential coordinate of the deformed ring shape in the non-rotating cylindrical coordinate system:

\[ r_d = \sqrt{(R + \frac{h}{2} + h_z + u_{r,e})^2 + (u_{\theta,e})^2} \]

\[ \theta_d = \theta + \arctan \left( \frac{u_{\theta,e}}{R + \frac{h}{2} + h_z + u_{r,e}} \right) \] (5.42)

Similar to (5.40), \( r_d \) is defined as a mapping function of \( \theta_d \):

\[ r_{Md} (\alpha) : \theta_d \rightarrow r_d, \quad \alpha \in [-\pi, \pi] \] (5.43)

Now, with (5.40) and (5.43), the distributed geometry error with respect to the cylindrical coordinate system can be obtained by:

\[ GE(\alpha) = r_{Md} (\alpha) - r_{MS} (\alpha), \quad (\alpha \in [-\pi, \pi]) \] (5.44)

Consequently the distributed adjusting force is given by:

\[ F_{GE}(\alpha) = GE(\alpha) \cdot K_{GE} \] (5.45)

In the region where \( GE(\alpha) > 0 \), penetration happens which should not be allowed and should be removed by applying additional contact force as given by (5.45). This region is identified as the Contact Region for the current iteration step. In the region, where \( GE(\alpha) \leq 0 \) and \( \min(\theta_S) \leq \alpha \leq \max(\theta_S) \), the ring and rigid surface are off contact, so that the existing force in this region should be reduced until zero. This reduction of the contact force is also achieved by (5.45) via the sign of \( GE(\alpha) \) in this region. In the region where \( -\pi \leq \alpha \leq \min(\theta_S) \cup \max(\theta_S) \leq \alpha \leq \pi \), the large constant \( C_t \) guarantees that
GE(α) ≤ 0. Because of the limitation that the minimum contact force is zero (unilateral property of the contact), no contact force will appear in this region.

The distributed adjusting force computed by (5.45) is with respect to the post-deformed circumferential coordinate, while the forcing function (5.21) used in the Force-Deformation relationships of the previous sections (Sections 3 and 4) are with respect to the un-deformed circumferential coordinate. Therefore another mapping needs to be defined to translate θ_d back to the un-deformed coordinate θ:

\[ \theta_o(\alpha) : \theta_d \rightarrow \theta \quad (5.46) \]

Then, the adjusting force (5.45) can be added to the force at the current iteration step \( F_i(\theta, t) \) after translating \( \theta_d \) back to \( \theta \) to get the new contact force:

\[
F_{i+1}(\theta, t) = \begin{cases} 
F_i(\theta, t) + F_{GE}(\theta_o(\theta_d)) & \theta \in \{F_i(\theta, t) + F_{GE}(\theta_o(\theta_d)) > 0\} \\
0 & \text{otherwise}
\end{cases}
\]

(5.47)

The subscript \( i \) is the index for the current iteration step. Then the obtained new contact force (5.47) is sent to the Force-Deformation module to update the deformation of the ring.

5.7 Modal Analysis and Critical Velocity

In rotating ring structures, the critical velocity where standing waves could be generated is always an important consideration for safety concerns [58], [59], [60], [61], [62], [63], [64]. Intensive investigations have been conducted in these cited works for linear REF models whose foundation stiffness is constant circumferentially. In the unilateral foundation case, however, the ring is only supported by part of the foundation
since foundation stiffness may vanish in certain regions due to the contact induced deformation of the ring.

To conduct a relevant modal analysis, consider the case that the laminated ring is in quasi-static rolling contact with a flat surface. The non-collapsed region of the foundation supports the rotating ring. This region itself is non-rotating with respect to the fixed coordinate system. The critical velocity analysis for this setting can then be achieved in two steps. In the first step, the collapsed region of the foundation is identified for the quasi-static rolling contact with a flat surface. This is facilitated by the contact algorithm developed above and unilateral foundation force-deformation computations outlined in the previous sections. In the following, we assume the collapsed foundation region is identified to be in \([\theta_{k1}, \theta_{k2}]\) with respect to the fixed cylindrical coordinate system.

In the second step, modal analysis is implemented to get the angular frequencies and correspondingly critical velocities. The modal analysis of this rotating flexible ring coupled to space-fixed stiffness has been studied in [19] using the Galerkin’s method, but the partially distributed and space-fixed stiffness were assumed known in advance. Our first step replaces this assumption. In our second step, direct modal expansion method will be used for consistency with most of the solutions approach discussed in the previous sections. The modal expansion for the collapsed region of the foundation stiffness is substituted into the EOMs to solve for the damped angular frequencies for the rotating laminated ring. Finally, the critical velocity for a certain mode is obtained as the minimal
flexural wave propagation velocity [64] in the circumferential direction of the ring using the minimal damped angular frequency for that mode.

The supporting effective radial stiffness of the partially collapsed foundation can be written in the form:

\[ K_r(\theta) = \left(1 - C_{Kr}(\theta)\right)K_r \]

\[ C_{Kr}(\theta) = \begin{cases} 0 & -\pi < \theta < \theta_{k1} \cup \theta_{k2} < \theta < \pi \\ 1 & \theta_{k1} < \theta < \theta_{k2} \end{cases} \]

(5.48)

where \( C_{Kr}(\theta) \) can be written in terms of Fourier series by:

\[ C_{Kr}(\theta) = \sum_{n=-N}^{N} \left[ C_{Kr,n,r,c}(\theta) + C_{Kr,n,r,s}(\theta) \right] = \sum_{n=-N}^{N} \left[ a_{Kn} \cos(n\theta) + b_{Kn} \sin(n\theta) \right] \]

(5.49)

where Fourier coefficients \( a_{Kn}, b_{Kn} \) can be easily obtained with known \( \theta_{k1}, \theta_{k2} \). In the frictionless flat surface contact case we consider, \( \theta_{k1} = -\theta_{k2} \) so that \( b_{Kn} = 0 \) and consequently \( C_{Kr,n,r,s}(\theta) = 0 \). Substitution of (5.48)(5.49) into (5.24) and dropping forcing vector leads to the EOMs for the free vibration of laminated ring supported by the partially collapsed foundation:

\[ M_{n,r,c}A_{n,r,c}(t) + C_{n,r,c}V_{n,r,c}(t) + K_{C,n,r,c}X_{n,r,c}(t) = 0 \]

(5.50)

where matrix \( K_{C,n,r,c} \) can be obtained by replacing the foundation radial stiffness \( K_r \) in the odd lines of the \( K_{n,r,c} \) matrix with \( [1 - a_{Kn}]K_r \). In (5.50), the effect of the partially collapsed foundation is incorporated mode-wise via Fourier series of the missing radial stiffness.

Equation (5.50) includes 8 second-order dynamic equations with 8 unknown time-dependent variables (coefficients). It can be deduced that at each rotational speed \( \Omega \), there
are 16 eigen values with 8 sets of distinct frequency values. The critical velocity is related to the minimal damped frequency value at each rotational speed. Assuming this minimal damped angular frequency for mode number \( n \) is \( \omega_{n,\text{min}} \), then the critical velocity for this mode is given by:

\[
v_{cr,n} = \frac{\omega_{n,\text{min}} R}{n}
\]

The minimal damped angular velocity is generally due to the ring flexural deformation \( u_{r,N,r,c}(R, \theta, t) \). Therefore, critical velocity represents a circumferential wave propagation speed for a specific mode of this deformation. Comparing the critical velocity for any mode with the rolling velocity (or rotational speed), one can determine whether a mode will possibly be excited. This will be illustrated in the results section below.

### 5.8 Results and Discussions

In this section, the results for rolling contact as well as modal/critical velocity analysis are given as examples of the application of developed model, solution methods and contact algorithm. We consider a tire application where the parameters for the laminated ring and the foundation are given in Table 6. For contact analysis, we consider two cases: first a rigid flat surface, and then a rigid cleat of 20mm in length and 10mm in height are placed under the ring/tire and select downward deflections are applied to the axle center. The contact responses of this laminated REF model are solved for both collapsible foundation and linear foundation using the same parameters (other than the unilateral property for the collapsible foundation). Then the results of the two cases are
compared. For the modal/critical velocity analysis, the laminated ring is assumed in the quasi-static flat surface contact so that rotating ring is supported by a partially distributed space-fixed foundation. Throughout this section, the rolling velocity of the ring in the longitudinal direction \( v_{\text{rolling}} \) is assumed to be related with rotation speed \( \Omega \) via the external radius of the ring:

\[
v_{\text{rolling}} = \Omega \left( R + \frac{h}{2} + h_2 \right)
\]

Table 6- Parameters.

<table>
<thead>
<tr>
<th>Parameters ( R )</th>
<th>Definitions</th>
<th>Units ( m )</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>Model Width</td>
<td>( m )</td>
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</tr>
<tr>
<td>( h )</td>
<td>Thickness of Internal Ring</td>
<td>( m )</td>
<td>0.012</td>
</tr>
<tr>
<td>( h_2 )</td>
<td>Thickness of External Ring</td>
<td>( m )</td>
<td>0.01</td>
</tr>
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<td>( K_r )</td>
<td>Radial Stiffness Per Radian of Elastic Foundation</td>
<td>( \frac{N}{m \cdot \text{radian}} )</td>
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</tr>
<tr>
<td>( K_\theta )</td>
<td>Torsional Stiffness Per Radian of Elastic Foundation</td>
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</tr>
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<td>( E_A )</td>
<td>Axial Stiffness of Internal Ring Cross-section</td>
<td>( N )</td>
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<tr>
<td>( E_I )</td>
<td>Flexural Rigidity of Internal Ring</td>
<td>( N \cdot m^2 )</td>
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</tr>
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<td>( G_A )</td>
<td>Shear Stiffness of Internal Ring Cross-section</td>
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<tr>
<td>( E_{A_r} )</td>
<td>Transverse Stiffness of External Ring Cross-section</td>
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<td>Mass Density of Internal Ring</td>
<td>( \frac{\text{kg}}{m^3} )</td>
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</tr>
<tr>
<td>( \rho_2 )</td>
<td>Mass Density of External Ring</td>
<td>( \frac{\text{kg}}{m^3} )</td>
<td>1.2e3</td>
</tr>
<tr>
<td>( C_{Er} )</td>
<td>Viscous Damping Density in Radial Direction of Internal Ring</td>
<td>( \frac{N}{\text{radian (m/s)}} )</td>
<td>100</td>
</tr>
<tr>
<td>$C_{E\theta}$</td>
<td>Viscous Damping Density in Tangential Direction of Internal Ring</td>
<td>$\frac{N \text{ radian (m/s)}}{}$</td>
<td>100</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------------------------------------------------------------</td>
<td>---------------------------------</td>
<td>-----</td>
</tr>
<tr>
<td>$C_{Rr,2}$</td>
<td>Viscous Strain Damping Density in Radial Direction of External Ring</td>
<td>$\frac{N \text{ m (m/s)}}{}$</td>
<td>5e2</td>
</tr>
</tbody>
</table>

### 5.8.1 Results for Rolling Contact Problem

First, we consider quasi-static rolling contact against a rigid flat surface. As a special situation in this case, if the rotation speed $\Omega = 0$, the static contact result is obtained. The corresponding results are shown in the left plot in Figure 21, where the vertical force responses given different deflections (downward deflections applied to the axle center). This gives the static vertical stiffness of the REF structure. It is as expected that the collapsible foundation model has much lower vertical stiffness than that of the linear foundation model with same parameters since radial supporting force in the collapsed region vanishes. Both models show approximately linear vertical stiffness in the applied range of axle deflections. The right plot in Figure 21 shows the force increment with respect to the static flat surface contact (corresponding to $\Omega = 0$) case when the rings are rotating at different speeds. It is noticed that the magnitude of vertical force increments due to ring rotations are slightly higher for the collapsible foundation than for the linear foundation for all of the applied axle deflections.

Results for dynamic rolling contact against a rigid cleat surface are shown in Figure 22, where the laminated REF models roll over the cleat at different velocities and the force variations (increments due to cleat) in the vertical and longitudinal directions are plotted. An initial 20mm axle deflection is applied for all the results here. As may be expected, the maximum $\Delta F_z$ and $\Delta F_x$ increase with the rolling velocity. However, these
increasing rates with respect to rolling velocities are significantly different for the linear and collapsible foundations. It can be seen that the maximum $\Delta F_z$ with the collapsible foundation is much lower than that with the linear foundation at 20kph, then becomes close at 60kph, and becomes much higher than with the linear foundation at 100kph. Although the maximum $\Delta F_z$ also increases with respect to the rolling velocity with the linear foundation, it increases much faster with the collapsible foundation. Maximum $\Delta F_x$ has the same trend as that of maximum $\Delta F_z$. This trend is shown in a more clear way in Figure 23 where only the maximum $\Delta F_z$ and $\Delta F_x$ at different rolling velocities are plotted and compared for the two foundations. The 0kph rolling velocity here corresponds to the situation that the laminated ring quasi-statically rolls over the cleat. Note that the much faster increase of force variations with the collapsible foundation only exits in dynamic cleat contact case at higher rolling speeds.

Figure 21: Static Vertical Stiffness and Effect of the Rotation for Linear and Collapsible Foundation Models
Figure 22: Force Variations at Different Rolling Velocities (at 20mm Axle Deflection)
Figure 23: Maximum Force Variations at Different Rolling Velocity for Cleat Contact (at 20mm Axle Deflection)

The maximum force variations are less sensitive to the applied axle deflections, as shown in Figure 24, where the left column shows the $\Delta F_z$ and $\Delta F_x$ at different deflections with the linear foundation while the right column shows those with the collapsible foundation. The rolling velocity of the laminate REF models is set at 60kph. The trends of the maximum $\Delta F_z$ and $\Delta F_x$ with respect to applied axle deflections are plotted in Figure 25. It can be seen that the magnitude changes of the maximum $\Delta F_z$ and $\Delta F_x$ are limited when the deflection changes from 10mm to 25mm, compared to the magnitudes change when rolling velocity increases from 20kph to 100kph as plotted in Figure 23.
Figure 24: Force Variations at Different Deflections (Cleat Contact at Rolling Velocity 60kph)
5.8.2 Results for Modal and Critical Velocity Analysis

For a first step, we solved the quasi-static flat surface contact problem of the REF with collapsible foundation via algorithms developed in Section 5.5 and Section 5.6 to identify the collapsed region of the foundation. This is done for initial applied axle deflection 20mm and rolling velocities from 0 to 200kph. The corresponding collapsed region \([\theta_{k1}, \theta_{k2}]\) is identified as plotted in Figure 26. It can be seen that with higher rolling velocity, smaller collapsed region is obtained. This is due to the centrifugal effect of ring rotation. For the linear foundation, this first step is neglected since there is no collapsed region.
Then in the second step, the obtained collapsed region is substituted into (5.49) and (5.50), and following the procedures described in Section 5.7, the damped angular frequencies of any mode can be solved for, for each rolling velocity considered (from 0kph until 200kph). Figure 27 shows all 16 damped angular frequencies for mode $n = 6$ at different rolling velocities. The frequency splitting (bifurcation) due to ring rotation can be observed in Figure 27. This effect also discussed in detail in [17]. For every mode number at every rolling velocity, there is a minimal damped angular frequency, which corresponds to a nominal critical velocity via (5.51). These can be shown in 3D plots for selected mode range and given rolling velocity range, as in Figure 28.

For every rolling velocity the ring is running at, there is a nominal critical velocity for every mode number, which can be higher or lower than the actual rolling velocity. The relative relationship between the nominal critical velocity and actual rolling velocity determines whether this certain mode will be excited or not. By doing this comparison at every actual rolling velocity, a group of possibly excited modes can be identified. This is shown in Figure 29, where the results for both the collapsible foundation and linear
foundation are plotted. In this figure, if mode number is 0, it indicates no mode will be excited. It can be seen that if the rolling velocity is no higher than 90kph, no modes will be excited for either the collapsible foundation or linear foundation model. Starting from 100kph, modes 5 to 8 are excited in both models. Then more modes will be excited with even higher rolling velocity. At rolling velocity 160kph or higher, all studied modes are excited in collapsible foundation model; for linear foundation model, this velocity becomes 170kph. For an overall performance, only minor difference appears between these two foundation models, at least for the design parameters selected for these examples.
Figure 27: Damped Angular Frequencies at Different Rolling Velocities

(Mode Number $n=6$)
Figure 28: Minimal Damped Angular Frequencies and Nominal Critical Velocities at the Selected Rolling Velocity Range and Specified Mode Range
Figure 29: Possibly Excited Mode Ranges at Different Rolling Velocities
(Initial Deflection=20mm)

5.9 Summary and Conclusions

This paper outlined generalized models, solution methods and a contact algorithm for analyzing the structural dynamics of a rotating laminated ring on an elastic foundation (laminated REF), where the foundation may be unilateral (collapsible or tensionless) in some apriori unknown regions. The laminated REF is excited by a rolling contact with an arbitrary rigid surface profile. The laminated ring consists of two layers consisting of an extensible Timoshenko beam as the internal ring and a transversely compliant beam as the external ring. Together the two layers model bending, shear, extensibility as well as
the transverse compliance in the laminated ring structure. The general EOMs of the laminated REF model are derived. For the linear foundation case, analytical solutions are given for the static and dynamic deformation response to arbitrary applied force. The unilateral foundation case is addressed by the combination of an iterative compensation scheme and an implicit Newmark scheme. Then contact of the REF with arbitrary rigid surface profile is solved via a feedback compensation algorithm.

Modal analysis is also offered for the case where the laminated REF is rolling quasi-statically on the flat rigid surface with an applied axle deflection. The modal analysis is then used for analysis of critical velocity, which is generally a concern for high speed rotating components. Illustrative critical velocity analyses conducted using parameters from a tire application showed that the excited mode ranges for the laminated REF on both linear and nonlinear foundations have similar trends, at least for the model parameters tested.

Illustrative comparisons have also been included between the vertical force responses of the ring with the unilateral foundation and with the linear foundation when rolling over a cleat and a flat surface. It is found that the variation of the vertical force for the ring with the unilateral foundation is much more sensitive to the rolling velocity than the ring with linear foundation when in rolling contact with a cleat. The force variations for both foundation types are closer when in rolling contact with a flat surface, at least for the model parameters tested.

In closing, we state that even though our motivations are drawn from non-pneumatic tires involving collapsible foundations, the Ring on Elastic Foundation (REF)
models and solution methods outlined in this paper are quite general and can be used for the analysis of quasi-static and dynamic rolling contact responses of rotating REF structures such as pneumatic tires, bearings, and gears.
CHAPTER 6 : CONCLUSIONS AND FUTURE WORK

This dissertation extends the existing study on REF model into a more general laminated ring on unilateral foundation case and gives a contact solution method when against any uneven surface. Modal and critical velocity of this rolling ring when partially supported by space-fixed foundation is also analyzed.

In order to solve the unilateral foundation problem an iterative compensation method is proposed based on the analytical solution of linear foundation model. Then it is extended to solve forced vibration problem of a thick ring on unilateral foundation, by incorporation with an implicit Newmark scheme. For the contact analysis, a laminated ring model as well as a feedback P-Controller algorithm is presented, which gives a simple and easy-to-implement method for contact problem with any uneven surface. Finally, this laminated ring and P-Controller algorithm are combined with iterative compensation method and implicit Newmark scheme to solve the dynamic rolling contact response of the laminated ring on unilateral foundation model. Critical velocity of model when rolling on the flat surface is also analyzed via modal analysis for this rotating ring partially supported by space-fixed foundation. The developed model and solution methods can be used for static and dynamic contact analysis on non-pneumatic tires as well as the other general rotating components.

In summary, this dissertation gives the model and methodology for static, dynamic and contact analysis for a very general REF model. As the future work, the unique characteristics of these unilateral foundation model needs to be studied via the
developed model and methods, effect of model parameters to the performances is also to be investigated.
APPENDIX A: STATIC SOLUTION FOR TIMOSHENKO REF

\[
\begin{align*}
    u_{r,n,c}(R, \theta) &= Cur_{r,c}(n) \cos\left(n(\theta - \theta_0)\right) \\
    u_{\theta,n,c}(R, \theta) &= Cu\theta_{r,c}(n) \sin\left(n(\theta - \theta_0)\right) \\
    \phi_{n,c}(R, \theta) &= C\varphi_{r,c}(n) \sin\left(n(\theta - \theta_0)\right) \\
    u_{r,n,s}(R, \theta) &= Cur_{r,s}(n) \sin\left(n(\theta - \theta_0)\right) \\
    u_{\theta,n,s}(R, \theta) &= Cu\theta_{r,s}(n) \cos\left(n(\theta - \theta_0)\right) \\
    \phi_{n,s}(R, \theta) &= C\varphi_{r,s}(n) \cos\left(n(\theta - \theta_0)\right)
\end{align*}
\]

\[u_{r,n,c}(R, \theta) = \frac{(Zur4n^4 + Zur2n^2 + Zur0)}{ZD6} \] 
\[u_{\theta,n,c}(R, \theta) = \frac{(Zur3n^3 + Zur1n)}{ZD6} \] 
\[\phi_{n,c}(R, \theta) = \frac{(Zr\varphi3n^3 + Zr\varphi1n)}{ZD6} \] 
\[u_{r,n,s}(R, \theta) = \frac{(Zur4n^4 + Zur2n^2 + Zur0)}{ZD6} \] 
\[u_{\theta,n,s}(R, \theta) = \frac{(Zur3n^3 + Zur1n)}{ZD6} \] 
\[\phi_{n,s}(R, \theta) = \frac{(Zr\varphi3n^3 + Zr\varphi1n)}{ZD6} \]

(7.1)

\[Cur_{r,c}(n) = \frac{(Zur4n^4 + Zur2n^2 + Zur0)}{ZD6} \] 
\[Cu\theta_{r,c}(n) = \frac{(Zur3n^3 + Zur1n)}{ZD6} \] 
\[C\varphi_{r,c}(n) = \frac{(Zr\varphi3n^3 + Zr\varphi1n)}{ZD6} \] 
\[Cur_{r,s}(n) = \frac{(Zur4n^4 + Zur2n^2 + Zur0)}{ZD6} \] 
\[Cu\theta_{r,s}(n) = \frac{(Zur3n^3 + Zur1n)}{ZD6} \] 
\[C\varphi_{r,s}(n) = \frac{(Zr\varphi3n^3 + Zr\varphi1n)}{ZD6} \]

(7.2)
\[
\text{Cur}_{\theta,c}(n) = \frac{\left( Z\theta u r 3 n^3 + Z\theta u r 1 n \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,c}
\]
\[
\text{Cu}_{\theta,c}(n) = \frac{\left( Z\theta u t 4 n^4 + Z\theta u t 2 n^2 + Z\theta u t 0 \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,c}
\]
\[
\text{C}_{\theta,c}(n) = \frac{\left( Z\theta \phi 4 n^4 + Z\theta \phi 2 n^2 + Z\theta \phi 0 \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,c}
\]
\[
\text{Cur}_{\theta,s}(n) = \frac{\left( Z\theta u r 3 n^3 + Z\theta u r 1 n \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,s}
\]
\[
\text{Cu}_{\theta,s}(n) = \frac{\left( Z\theta u t 4 n^4 + Z\theta u t 2 n^2 + Z\theta u t 0 \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,s}
\]
\[
\text{C}_{\theta,s}(n) = \frac{\left( Z\theta u t 4 n^4 + Z\theta u t 2 n^2 + Z\theta u t 0 \right)}{ZD6} n^6 + ZD4 n^4 + ZD2 n^2 + ZDQ_{\theta n,s}
\]
\[ \begin{align*}
ZD6 &= 4GA_0EA_0EI_0 \\
ZD4 &= GARh^2EA_0K_0 + 4GAReE_0EI_0 + 4K_rReEA_0EI_0 - 8GAEA_0EI_0 \\
ZD2 &= K_rR^2h^2EA_0K_0 + 4GAK_rR^2E_0EI_0 + 4GAR^2hEA_0K_0 \\
&\quad - 2GARh^2EA_0K_0 + 4K_rR^2E_0EI_0K_0 + 4GAR^2K_rReEI_0 + 4ReEA_0EI_0K_0 \\
&\quad + 4GA_EA_0EI_0 \\
ZD0 &= (4K_rR^4 - 4K_rR^3h + 4K_rR^2h^2 + 4R^3EA_0 - 4R^2hEA_0 + Rh^2EA_0)K_0GA \\
Zrur4 &= 4ReEA_0EI_0 \\
Zrur2 &= R\left(Rh^2EA_0K_0 + 4GAR^2EA_0 + 4ReEI_0K_0 + 4GAEI_0\right) \\
Zrur0 &= R^2GA_0K_0\left(4R^2 - 4Rh + h^2\right) \\
Zrut3 &= (-4GA - 4EA_0)ReEI_0 \\
Zrut1 &= (2GARR^2K_0 - GArh^2E_0K_0 - Rh^2EA_0K_0 - 4GAR^2EA_0)R \\
Zr\phi 3 &= 4GAR^2EA_0 \\
Zr\phi 1 &= 2R^2\left(2GARK_0 - GAhK_0 - hEA_0K_0 - 2GAEA_0\right) \\
Z\theta ur3 &= -2R\left(GARhEA_0 - 2GAEI_0 - 2EA_0EI_0\right) \\
Z\theta ur1 &= 4GAR^3hK_0 + 2GAR^2h^2K_0 + 2R^2h^2EA_0K_0 + 4GAR^3EA_0 \\
&\quad + 2GAR^2hEA_0 \\
Z\theta ut4 &= 4GAEI_0R \\
Z\theta ut2 &= 2\left(GARh^2K_0 - GAReEA_0 + 2K_rReEI_0 + 2EA_0EI_0\right)R \\
Z\theta ut0 &= 2K_rR^3h^2K_0 + 4GAK_rR^4 + 2GAK_rR^3h + 2R^2h^2EA_0K_0 \\
&\quad + 4GAR^3EA_0 + 2GAR^2hEA_0 \\
Z\theta \phi 4 &= 2GARhEA_0 \\
Z\theta \phi 2 &= 2\left(2GARhK_0 + K_rRhEA_0 - 2GAREA_0 - 2GAhEA_0\right)R \\
Z\theta \phi 0 &= 4K_rR^3hK_0 + 4GAK_rR^3 + 2GAK_rR^2h + 4R^2hEA_0K_0 \\
&\quad + 4GAR^2EA_0 + 2GARhEA_0
\end{align*}\]
APPENDIX B: ELEMENTS FOR DAMPING AND STIFFNESS MATRICES

\[ c_{arc,11} = \frac{C_{R} Rb h + C_{Er}}{R \rho \overline{h} b} \]

\[ c_{arc,22} = \frac{12 R^2 h C_{R0} - bh^3 C_{R0} + 12 R C_{E0} + 6 h C_{E0}}{\rho [12 R^2 - h^2]} b \]

\[ c_{arc,23} = -3 \frac{C \phi_{r,c}(n) C_{E0} (2 R + h)}{\rho C u \theta_{r,c}(n) [12 R^2 - h^2]} b \]

\[ c_{arc,32} = -12 \frac{C u \theta_{r,c}(n) C_{E0} (h + 6 R)}{\rho h^2 C \phi_{r,c}(n) [12 R^2 - h^2]} b \]

\[ c_{arc,33} = \frac{12 R^2 h C_{R0} - bh^3 C_{R0} + 36 RC_{E0} + 6 h C_{E0}}{\rho [12 R^2 - h^2]} b \]

\[ k_{arc,11} = \frac{G A n^2 + K_{r} R + E A_{0}}{\rho R^2 h b} \]

\[ k_{arc,12} = \frac{n (G A + E A_{0}) C u \theta_{r,c}(n)}{\rho R^2 h C u \theta_{r,c}(n) b} \]

\[ k_{arc,13} = -\frac{G A C \phi_{r,c}(n) n}{\rho R h C u \theta_{r,c}(n) b} \]

\[ k_{arc,21} = 12 \frac{n C u \theta_{r,c}(n) (2 G A - E A_{0})}{\rho h C u \theta_{r,c}(n) [12 R^2 - h^2]} b \]

\[ k_{arc,22} = 6 \frac{2 n^2 E A_{0} + 2 R K_{0} + h K_{0} + 4 G A}{\rho h} [12 R^2 - h^2] b \]

\[ k_{arc,23} = -3 \frac{C \phi_{r,c}(n) (2 R^2 h K_{0} + R h^2 K_{0} + 8 G A R^2 + 4 n^2 E I_{0})}{\rho h C u \theta_{r,c}(n) [12 R^2 - h^2]} R b \]

\[ k_{arc,31} = -12 \frac{n C u \theta_{r,c}(n) (12 G A R^2 + G A h^2 - h^2 E A_{0})}{\rho h^3 C \phi_{r,c}(n) [12 R^2 - h^2]} R b \]

\[ k_{arc,32} = -12 \frac{C u \theta_{r,c}(n) (h^2 n^2 E A_{0} + 6 R^2 h K_{0} + R h^2 K_{0} + 12 G A R^2 + G A h^2)}{\rho h^3 C \phi_{r,c}(n) [12 R^2 - h^2]} R b \]

\[ k_{arc,33} = 6 \frac{6 R h^2 K_{0} + h^3 K_{0} + 24 G A R^2 + 2 G A h^2 + 24 n^2 E I_{0}}{\rho h^3 [12 R^2 - h^2]} b \]

(8.1)
APPENDIX C: EOM COEFFICIENTS FOR LAMINATED REF

\[ C_{112} = -\frac{GA}{Rb} \]

\[ CT_{112} = Rh \rho_1 + \left( Rh_2 + \frac{hh_2}{2} + \frac{h_2^2}{2} \right) \rho_2 \]

\[ CT_{111} = \frac{C_{Er}}{b} \]

\[ C_{110} = \frac{K_R + EA_o}{Rb} \]  

\[ C_{120} = \frac{EA_o + GA}{Rb} \]  

\[ CT_{132} = \left( \frac{Rh_2^2}{2} + \frac{hh_2^2}{4} + \frac{h_2^3}{3} \right) \rho_2 \]

\[ C_{141} = GA \]

\[ C_{211} = -\frac{EA_o + GA}{Rb} \]

\[ C_{222} = -\frac{EA_o}{Rb} \]

\[ CT_{222} = Rh \rho_1 + \left( Rh_2 + \frac{hh_2}{2} + \frac{h_2^2}{2} \right) \rho_2 \]

\[ CT_{221} = \frac{C_{Eo}}{b} \]

\[ C_{220} = \frac{RK_o + GA}{Rb} \]

\[ CT_{242} = \frac{h^3 \rho_1}{12} + \left( \frac{Rhh_2}{2} + \frac{Rh_2^2}{2} + \frac{hh_2^2}{4} + \frac{h_2^3}{2} + \frac{h_2^3}{3} \right) \rho_2 \]

\[ CT_{241} = \frac{C_{Eo} h}{2b} \]

\[ C_{240} = hK_o + 2GA \]  

(9.1)  

(9.2)
\[\begin{align*}
CT^{312} &= \frac{h_2^2 \rho_3}{12} (6R + 3h + 4h_2) \\
C^{310} &= -\frac{v_2 E A r_2}{b \left(v_2^2 - 1\right)} \\
C^{321} &= -\frac{v_2 E A r_2}{b \left(v_2^2 - 1\right)} \\
CT^{332} &= \frac{h_2^3 \rho_3}{12} (4R + 2h + 3h_2) \\
CT^{331} &= \left(\frac{h_2^2}{2} + \left(R + \frac{h}{2}\right)h_2\right) c_{r_2} \\
C^{330} &= -\frac{(v_2 h_2 + 2R + h + h_2) E A r_2}{2b \left(v_2^2 - 1\right)} \\
C^{341} &= -\frac{(h + h_2) v_2 E A r_2}{2b \left(v_2^2 - 1\right)} \\
\end{align*}\]  
(9.3)

\[\begin{align*}
C^{411} &= \frac{G A}{b} \\
CT^{422} &= \frac{h^3 \rho_3}{12} + \left(\frac{R h h_2}{2} + \frac{R h_2^2}{2} + \frac{h^2 h_2}{4} + \frac{h h_2^2}{2} + \frac{h_2^3}{3}\right) \rho_2 \\
CT^{421} &= -\frac{C_{e_2} h}{2b} \\
C^{420} &= -\frac{h K_\theta + 2G A}{2b} \\
C^{442} &= -\frac{E I_\theta}{R b} \\
CT^{442} &= \frac{\rho_1 R h^3}{12} + \left(\frac{R h_2^3}{3} + \frac{h^3 h_2}{8} + \frac{3 h^2 h_2^2}{8} + \frac{h h_2^3}{2} + \frac{R h^2 h_2}{4} + \frac{R h h_2^2}{2} + \frac{h_2^4}{4}\right) \rho_2 \\
CT^{441} &= \frac{C_{e_2} h^2}{4b} \\
C^{440} &= \frac{h^2 K_\theta + 4G A \cdot R}{4b} \\
\end{align*}\]  
(9.4)
\[ ZDn6 = -C112 \cdot C222 \cdot C442 \cdot b \]
\[ ZDn4 = (C110 \cdot C222 \cdot C442 + C112 \cdot C220 \cdot C442 + C112 \cdot C222 \cdot C440 + C112 \cdot C220 \cdot C440 - C121 \cdot C211 \cdot C442)b \]
\[ ZDn2 = (\frac{-C110 \cdot C222 \cdot C442 - C110 \cdot C222 \cdot C440 - C112 \cdot C220 \cdot C440 + C112 \cdot C240 \cdot C420}{+C121 \cdot C211 \cdot C440 - C121 \cdot C240 \cdot C411})b \]
\[ ZDn0 = C110 \cdot C220 \cdot C440 \cdot b - C110 \cdot C240 \cdot C420 \cdot b \]
\[ ZD = ZDn6 \cdot n^6 + ZDn4 \cdot n^4 + ZDn2 \cdot n^2 + ZDn0 \]
\[ Nur = \left( C222 \cdot C442 \cdot n^4 - C220 \cdot C442 \cdot n^2 - C222 \cdot C440 \cdot n^2 + C220 \cdot C440 \cdot C240 \cdot C420 \right)Q_{arc} \]
\[ Nu\theta = Q_{arc} \cdot n \left( -C211 \cdot C442 \cdot n^2 + C211 \cdot C440 \cdot C240 \cdot C411 \right) \]
\[ N\phi = -nQ_{arc} \left( C222 \cdot C441 \cdot n^2 + C211 \cdot C420 \cdot C220 \cdot C411 \right) \]
\[ Cur_{r,\phi}(n) = \frac{Nur}{ZD} \]
\[ Cu\theta_{r,\phi}(n) = \frac{Nu\theta}{ZD} \]
\[ C\phi_{r,\phi}(n) = \frac{N\phi}{ZD} \]
\[ C\psi_{r,\phi}(n) = -\frac{C321 \cdot Cu\theta_{r,\phi}(n) \cdot b \cdot n + C341 \cdot C\phi_{r,\phi}(n) \cdot b \cdot n + C310 \cdot Cur_{r,\phi}(n) \cdot b - Q_{arc} \cdot h_2}{C330 \cdot b} \] (9.5)
REFERENCES


5. Gong, S., *A Study of In-plane Dynamics of Tires*. 1993: Delft University of Technology, Faculty of Mechanical Engineering and Marine Technology.


