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Portfolio Selection Problem under Uncertainty and Risk

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PORTFOLIO SELECTION PROBLEM UNDER UNCERTAINTY AND RISK

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Abstract

The purpose of this research is the investigation of a portfolio problem in an uncertain environment. Given possible investments with random performance depending on uncertain environmental settings the objective is to establish a methodology for construction of a portfolio which is non-dominated with respect to second order stochastic dominance and whose return distribution is preferable for a least risk decision maker.
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Chapter 1

Introduction

1.1 Motivation

Important decisions are made by entrepreneurs who establish and run all kinds of businesses affecting our economy’s functions and growth. Those decisions have to be made based on very little information and sometimes almost no information at all. It is our responsibility as mathematicians to support entrepreneurs with models and solutions to make the right decision. That is why this study is so important. Throughout the last century scholars tried to model the economy in order to predict future outcomes based on information from the past. But sometimes information is not available to make such predictions. We refer to that situation as uncertain. There has not been much progress in the study of those situations, although scholars are aware of their presence. It is our purpose to make more progress in this field of study by investigating a portfolio problem under both uncertainty and risk. Since we also account for risk, we had to find an appropriate measure for risk. It was my decision to choose stochastic dominance from a paper which I worked on [9] in my Multicriteria Optimization class. I was convinced that stochastic dominance has an elegant way to deal with risk, so I wanted to learn more about it. This study is a great opportunity to challenge my mathematical skills on a problem that is very much related to real life and so is certainly a great preparation for a career as an applied mathematician.
1.2 Objectives

In this report we will examine a portfolio selection problem under uncertainty and risk. The main objective of this project is to extend knowledge of stochastic dominance to an uncertain environment. To do so we will work with simple triangular distributions which may be replaced by general distributions later on. Step by step we will examine cases with finitely many uncertainties and eventually with an interval of uncertainties. Preference rules will be established and the solutions that they provide will be investigated.

1.3 Overview

The study of economic behavior in environments of uncertainty and risk has been the center of attention of economists for a long time. It has its origin in the attempts to find an exact description of the endeavor of the individual to obtain a maximum utility, or, in the case of the investor, a maximum profit. One of the basic problems of investment management is the optimal selection of assets with the aim of maximizing future returns and constraining risks by an appropriate measure. According to Knight [4], an important American economist of the twentieth century, an investor must also face uncertainty. He argued, that risk caused by unknown outcomes with known \( \textit{ex-ante} \) probability distributions has to be distinguished from uncertainty with no known probability distributions. Though this distinction is well recognized by most economists today, there has been little progress in modeling economic problems under both risk and the so-called Knightian uncertainty.

Being easy to identify, risk is still a vague term. In the last century, there have been many attempts to define and measure risk. In particular, a widely used concept is Value at Risk (VaR) which is thoroughly analyzed by Jorion [3]. Some alternative suggestions have appeared in financial and economic literatures and their advantages and disadvantages are discussed by Levy [5] and Rockafellar [8]. For the purpose of evaluating their relevance Artzner et al. [1] proposed to identify desirable properties for measures of risk, and to call the measures satisfying these properties “coherent”. Value at Risk turned out not to satisfy those properties, but it could be proven that Conditional Value at Risk (CVaR), an alternative to VaR, is a coherent measure (see Rockafellar [8]).

VaR and CVaR are deeply connected with stochastic dominance and utility theory. As will be shown later, the comparison of two portfolios with respect to first order stochastic dominance
(FSD) is equivalent to the comparison of their Values at Risk at all confidence levels and the analogue equivalence holds for second order stochastic dominance (SSD) and Conditional Value at Risk. Concerning the expected utility theory, its first important use was that of von Neumann and Morgenstern [7] who used the assumption of expected utility maximization in their formulation of game theory. According to Levy [5], the stochastic dominance approach has been developed on their foundation of the expected utility paradigm. It was proven (see [12] and [2]) that the FSD-dominance rule is equivalent to the expected utility maximization over all non-decreasing utility functions and the SSD-dominance rule is equivalent to the expected utility maximization over all non-decreasing and concave utility functions. Those results lead to the conclusion that all rational investors prefer FSD-nondominated solutions to dominated ones and so do all rational and risk-averse investors with SSD-nondominated solutions [9].

Because of CVaR being a coherent measure of risk and since we are interested in rational and risk-averse decision-making, SSD has a higher priority than FSD. Surprisingly, SSD is equivalent to generalized Lorenz dominance (see [6] and [11]) which was developed by Lorenz in 1905, decades prior to the expected utility theory of von Neumann and Morgenstern.

1.4 Notation

This section shall help the reader as a reference to handle better the notation used in this thesis. Let $m$ be the number of assets. For asset $j = 1, \ldots, m$

\[ x_j : \text{proportion of the capital invested in asset } i \]

\[ x = (x_1, x_2, \ldots, x_m) : \text{a portfolio} \]

\[ A_j : \text{return random variable of asset } j \]

\[ A_x = x_1A_1 + \ldots + x_mA_m : \text{return random variable of portfolio } x \]

\[ F_j : \text{cumulative/first order distribution function of } A_i \]

\[ F_x = x_1F_1 + \ldots + x_mF_m : \text{cumulative/first order distribution function of } A_x \]

If $x_1 = x_2 = \ldots = x_{j-1} = x_{j+1} = x_{j+2} = \ldots = x_m = 0, x_j = 1$, then $F_j = F_{(x_1, \ldots, x_m)}$

\[ r : \text{return value of a return random variable} \]

\[ F_j(r) = Prob(A_j \leq r), F_x(r) = Prob(A_x \leq r) \]
\( F_j^{(2)} \) : second order distribution function of \( A_i \)

\[ F_x^{(2)} = x_1 F_1^{(2)} + \ldots + x_m F_m^{(2)} \] : second order distribution function of \( A_x \)

\[ F_j^{(2)}(r) = \int_{-\infty}^{r} F_j(t)dt, \quad F_x^{(2)}(r) = \int_{-\infty}^{r} F_x(t)dt \]

In consideration of contingent events some notation changes. For a contingent event \( C_i \)

\( A_j(C_i) \) : return random variable of asset \( j \)

\( A_x(C_i) = x_1 A_1(C_i) + \ldots + x_m A_m(C_i) \) : return random variable of portfolio \( x \)

\( F_j(C_i) \) : cumulative/first order distribution function of \( A_i(C_i) \)

\( F_x(C_i) = x_1 F_1(C_i) + \ldots + x_m F_m(C_i) \) : cumulative/first order distribution function of \( A_x(C_i) \)

\[ F_x(C_i)(r) = Prob(A_x(C_i) \leq r) \]

\( F_j^{(2)}(C_i) \) : second order distribution function of \( A_i(C_i) \)

\( F_x^{(2)}(C_i) = x_1 F_1^{(2)}(C_i) + \ldots + x_m F_m^{(2)}(C_i) \) : second order distribution function of \( A_x(C_i) \)

\[ F_x^{(2)}(C_i)(r) = \int_{-\infty}^{r} F_x(C_i)(t)dt \]
Chapter 2

Research Design and Methods

2.1 The portfolio model

The behavior of the economy is very complex and thus future behavior is hard to predict. Nevertheless for simplicity we consider the following model. Assume, that in our simple world only three contingent events $C_1$, $C_2$ and $C_3$ are possible

- $C_1 =$ contracting economy
- $C_2 =$ sustainable growth
- $C_3 =$ unsustainable growth

Given these possible contingencies, consider three different assets (1 = safe, 2 = conservative and 3 = aggressive). Assume, that we are interested in investing a certain amount of money in those assets by assessing $x_j$, the proportion of total capital to be invested in asset $j$, for all $j \in \{1, 2, 3\}$. We do not know the future behavior of the assets due to the uncertain contingent events, but we can say something about the behavior of each asset under each contingency occurrence. Thus given a contingent event $C_i$, we introduce the return of each asset $A_j(C_i)$, $i, j \in \{1, 2, 3\}$ as a random variable with a distribution based on some data from observed previous periods. In this example we assume, that the distribution of the return of the safe asset $A_1$ is the same for all contingencies, namely, $F_1(r) = H(r - \frac{1}{2})$ where $H$ is the Heaviside function with density the Dirac delta function $\delta(r - \frac{1}{2})$ with $r$ representing return values of that asset, i.e., the first asset’s return is always $\frac{1}{2}$. For
the conservative and aggressive assets the distribution densities are given by triangular distributions with lower limits $\chi$, modes $\psi$ and upper limits $\omega$.

\[
A_2(C_1) : \chi = \frac{1}{4}, \psi = \frac{3}{8}, \omega = 1; \quad A_2(C_2) : \chi = \frac{1}{4}, \psi = \frac{7}{8}, \omega = 1 \\
A_2(C_3) : \chi = \frac{1}{4}, \psi = \frac{5}{8}, \omega = 1 \\
A_3(C_1) : \chi = \psi = 0, \omega = \frac{1}{2}; \quad A_3(C_2) : \chi = \frac{1}{4}, \psi = \frac{1}{2}, \omega = \frac{3}{4} \\
A_3(C_3) : \chi = \frac{1}{2}, \psi = \omega = 1
\]

The goal is to create a portfolio out of these assets, which performs ”best” in some sense. In terms of optimization we face the following problem:

**optimize** $x_1 A_1(C_i) + x_2 A_2(C_i) + x_3 A_3(C_i)$

**subject to** $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \in [0, 1]$

Let decision space $X = \{(x_1, x_2, x_3)|x_1 + x_2 + x_3 = 1, \ x_j \geq 0 \text{ for all } j \in \{1, 2, 3\}\}$ be the set of all possible portfolios, where $x_i$ is the proportion of the capital invested in the asset $i$, $i = 1, 2, 3$. The question we ask is ”What portfolio are we most interested in?”. Suppose for a while that we know which contingent event is going to happen. Then the objective becomes to select a portfolio $x \in X$ with return given by the random variable $x_1 A_1 + x_2 A_2 + x_3 A_3$. First of all, every investor is certainly interested in the maximization of the profit, i.e., the investor does prefer more return to less and is hence rational. But he must also be aware of risk. We assume that our investor is risk-averse. This behavior is characterized by evaluating each additional increment of a return unit (say 1$) as less valuable than the previous one. He takes low values of return more into account than the high ones. The particular specification of this behavior is rather subjective (i.e., How much more valuable is an additional unit?). We therefore use stochastic dominance as a tool to generalize this behavior.

Let $F_1, F_2$ and $F_3$ be cumulative distribution functions of the asset’s return random variables $A_1, A_2$ and $A_3$, then the return of a portfolio $x \in X$ is represented by the return’s cumulative distribution function $F_x = x_1 F_1 + x_2 F_2 + x_3 F_3$. For comparison of two different portfolios we use
the following dominance rules:

**Definition 2.1.1.** For a single contingent event, let \( F_x \) and \( F_y \) be the cumulative distribution functions of portfolios \( x \) and \( y \), respectively.

1) \( x \) dominates \( y \) with respect to *first order stochastic dominance* (*FSD*) if and only if for all \( r \in \mathbb{R} \), \( F_x(r) \leq F_y(r) \) with at least one strict inequality.

Notation: \( x \succ_{FSD} y \).

We say portfolio \( x \) is *FSD-nondominated* if and only if there does not exist another portfolio \( y \) dominating portfolio \( x \) with respect to first order stochastic dominance.

2) \( x \) dominates \( y \) with respect to *second order stochastic dominance* (*SSD*) if and only if for all \( r \in \mathbb{R} \), \( F_x^{(2)}(r) \leq F_y^{(2)}(r) \) with at least one strict inequality, where \( F_x^{(2)}(r) = \int_{-\infty}^{r} F_x(t)dt \), for all \( r \in \mathbb{R} \).

Notation: \( x \succ_{SSD} y \).

We say portfolio \( x \) is *SSD-nondominated* if and only if there does not exist another portfolio \( y \) dominating portfolio \( x \) with respect to second order stochastic dominance.

First order stochastic dominance has an easy interpretation, since it is defined in terms of probability by cumulative distribution functions that we will also refer to as first order distribution functions. A portfolio \( x \) dominates portfolio \( y \) if for any return outcome \( r \), portfolio \( x \) gives a lower or equal probability of receiving a return outcome lower or equal than return \( r \) than portfolio \( y \) does, with at least one return outcome with strictly lower probability, i.e., for any return outcome \( r \), portfolio \( x \) gives a higher or equal probability of receiving a return outcome better than return \( r \) than portfolio \( y \) does, with at least one return outcome with strictly higher probability. A similar interpretation can be made in terms of Value at Risk as will be shown in Section 3.1. It is rather difficult to interpret second order stochastic dominance directly from its definition. Therefore, we will also show in Section 3.1 second order stochastic dominance’s interpretation in terms of Conditional
Value at Risk. Because of a deep connection between stochastic dominance and utility theory (see [12] and [2]), stochastic dominance can also be interpreted in terms of the behavior of rational and risk-averse investors. If all rational investors prefer portfolio $x$ to portfolio $y$, then portfolio $x$ is preferred to portfolio $y$ with respect to FSD. And if all rational and risk-averse investors prefer portfolio $x$ to portfolio $y$, then portfolio $x$ is preferred to portfolio $y$ with respect to SSD. With stochastic dominance we can now filter out portfolios that are most relevant for a rational risk-averse investor. Notice that each SSD-nondominated portfolio will also be FSD-nondominated. This is in particular true, since a rational non-risky investor represented by SSD is also a rational one represented by FSD. However, Definition 2.1.1 can only be applied for a fixed contingent event, i.e., when there is no uncertainty at all. Hence, we have to extend it further over all contingencies.

**Definition 2.1.2.** For each contingent event $C_i, i \in \{1, 2, 3\}$, let $F_x(C_i)$ and $F_y(C_i)$ be the cumulative distribution functions of portfolios $x$ and $y$, respectively.

1) $x$ dominates $y$ with respect to *first order stochastic dominance* (FSD) if and only if for all contingent events $C_i, i \in \{1, 2, 3\}$ and for all $r \in \mathbb{R}$, $F_x(C_i)(r) \leq F_y(C_i)(r)$ with at least one strict inequality.

Notation: $x \succ^v_{FSD} y$.

We say portfolio $x$ is *FSD-nondominated* if and only if there does not exist another portfolio $y$ dominating portfolio $x$ with respect to first order stochastic dominance.

2) $x$ dominates $y$ with respect to *second order stochastic dominance* (SSD) if and only if for all contingent events $C_i, i \in \{1, 2, 3\}$ and for all $r \in \mathbb{R}$, $F_x^{(2)}(C_i)(r) \leq F_y^{(2)}(C_i)(r)$ with at least one strict inequality, where $F_x^{(2)}(C_i)(r) = \int_{-\infty}^{r} F_x(C_i)(t)dt$, for all $r \in \mathbb{R}$.

Notation: $x \succ^v_{SSD} y$.

We say portfolio $x$ is *SSD-nondominated* if and only if there does not exist another portfolio $y$ dominating portfolio $x$ with respect to second order stochastic dominance.
In consideration of Definition 2.1.2, our next goal is to apply a preference rule identifying a nondominated portfolio that is most preferred. We are in particular interested in a rule providing a unique preferred solution. Possible preference rules are shown in Section 2.3 with some computational results.

2.2 Asset’s return distributions

In this section we investigate the distribution functions of the three assets introduced in Section 2.1 and compare their behavior for contracting, sustainable and unsustainable economy by applying the domination rules defined in Section 2.1.

First, consider the contracting economy case. Figure 2.1 depicts densities of three assets considered for this contingent event. The graphs of the density functions of the conservative and aggressive asset’s return random variables have a shape of triangles representing triangular distributions. We use these density functions to compute the probabilities that the return random variables attain certain return values, e.g., for \( f(r) \) being the density function of \( A_3(C_1) \), the probability that the aggressive asset yields a return between 0.25 and 0.5 units (units can be dollars, euros or any other currency) during the contracting economy is \( \text{Prob}(0.25 \leq A_3(C_1) \leq 0.5) = \int_{0.25}^{0.5} f(r) \, dr = 0.25 \). The vertical line at 1/2 is a schematic representation of the Dirac delta function, i.e., the probability that asset 3 yields a return of 0.5 units is 1. First order distribution functions are determined by the probabilities \( F_j(C_1)(r) = \text{Prob}(A_j(C_1) \leq r), \ j = 1, 2, 3 \) using corresponding density functions and second order distribution functions are computed by \( F^{(2)}_j(C_1)(r) = \int_{-\infty}^{r} F_j(C_1)(t) \, dt \). Therefore, distribution functions of pure strategies, i.e., investments in only one asset, for the contracting economy are

\[
F_{(1,0,0)}(r) = \begin{cases} 
0 & r < 1/2 \\
1 & 1/2 \leq r 
\end{cases}
\]

\[
F^{(2)}_{(1,0,0)}(r) = \begin{cases} 
0 & r < 1/2 \\
r - 1/2 & 1/2 \leq r 
\end{cases}
\]

Figure 2.1: Densities for a contracting economy.
As can be seen in Figure 2.2, both first and second order distribution function values for the safe or conservative asset are lower than those for the aggressive asset. Thus, according to Definition 2.1.1, during the contracting economy safe pure strategy, i.e., the investment of every penny in the safe asset, and conservative pure strategy dominate the pure strategy investing everything in the aggressive asset with respect to both first and second order stochastic dominance. However, safe
and conservative pure strategies do not dominate each other, since for some return values the safe strategy has lower distribution function values whereas for other return values the conservative strategy does.

Next, consider the case with sustainable economy growth. Figure 2.3 depicts densities of three assets considered for this contingent event. Distribution functions of pure strategies for a sustainable economy are

\[ F_{(1,0,0)}(r) = \begin{cases} 
0 & r < 1/2 \\
1/2 & 1/2 \leq r 
\end{cases} \]

\[ F_{(1,0,0)}^{(2)}(r) = \begin{cases} 
0 & r < 1/2 \\
r - 1/2 & 1/2 \leq r 
\end{cases} \]

Figure 2.3: Densities for a sustainable economy.
Figure 2.4: Distribution functions for a sustainable economy: (a) $F(r)$-graphs (b) $F^{(2)}(r)$-graphs.

We conclude from the distribution functions in Figure 2.4 that for a sustainable economy, in terms of first order stochastic dominance only the conservative pure strategy dominates the aggressive one and no other domination relations are observable. In terms of second order stochastic dominance, however, additionally the safe pure strategy dominates the aggressive one. This example shows that second order stochastic dominance filters out more portfolios than first order stochastic dominance.

Finally, consider the case with an unsustainable economy growth. Figure 3 depicts densities of three assets considered for this contingent event. The distribution functions of pure strategies for the unsustainable economy are

$$F_{(1,0,0)}(r) = \begin{cases} 
0 & r < 1/2 \\
1 & 1/2 \leq r 
\end{cases}$$

and

$$F^{(2)}_{(1,0,0)}(r) = \begin{cases} 
0 & r < 1/2 \\
r - 1/2 & 1/2 \leq r 
\end{cases}$$

Figure 2.5: Densities for an unsustainable economy
The distribution functions in Figure 2.6 indicate that for an unsustainable economy, in terms of both first and second order stochastic dominance the aggressive pure strategy dominates the other two pure strategies. Notice that the value of any convex combination of two numbers is not lower than the value of the lowest number. For that reason, any strategy investing money in the safe or conservative assets is dominated by the aggressive pure strategy for unsustainable economy. Between the safe and conservative pure strategies there are no domination relations.

Figure 2.6: Distribution functions for an unsustainable economy: (a) \( F(r) \)-graphs (b) \( F^{(2)}(r) \)-graphs.
Since integration is linear, we can also compute first order distribution functions \( F_x(r) \) and second order distribution functions \( F^{(2)}_x(r) \) for every portfolio \( x \in X \) by

\[
F_{(x_1,x_2,x_3)}(r) = x_1 \cdot F_{(1,0,0)}(r) + x_2 \cdot F_{(0,1,0)}(r) + x_3 \cdot F_{(0,0,1)}(r)
\]

and

\[
F^{(2)}_{(x_1,x_2,x_3)}(r) = x_1 \cdot F^{(2)}_{(1,0,0)}(r) + x_2 \cdot F^{(2)}_{(0,1,0)}(r) + x_3 \cdot F^{(2)}_{(0,0,1)}(r)
\]

### 2.3 An example with finitely many feasible portfolios

The previous section provides enough tools to compare some portfolios. Consider the following portfolios:

\[
p_1 = (1, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = (0, 0, 1), \quad p_4 = (1/2, 1/2, 0), \quad p_5 = (1/2, 0, 1/2),
\]
\[
p_6 = (0, 1/2, 1/2), \quad p_7 = (1/3, 1/3, 1/3), \quad p_8 = (2/3, 0, 1/3), \quad p_9 = (2/3, 1/3, 0), \quad p_{10} = (0, 2/3, 1/3),
\]
\[
p_{11} = (1/3, 2/3, 0), \quad p_{12} = (0, 1/3, 2/3), \quad p_{13} = (1/3, 0, 2/3).
\]

Since the three investigated assets are only taking values in the interval \([0, 1]\), it suffices to make comparisons inside this interval.

We decided to apply the following preference rules. Depending on the choice of domination rule between FSD and SSD, use either first order distribution functions \( F_{p_i}(r) \), \( i \in \{1, \ldots, 13\} \) or second order distribution functions \( F^{(2)}_{p_i}(r) \), \( i \in \{1, \ldots, 13\} \), respectively. The main idea of this preference rule is to compare distribution functions of considered portfolios to a utopian portfolio whose distribution function is given by the lower envelope of all possible distribution functions. For this purpose we compute the difference functions \( F_{p_i} - I \) or \( F^{(2)}_{p_i} - I^{(2)} \), \( i \in \{1, \ldots, 13\} \), respectively, where \( I \) is the lower envelope of all first order distribution functions and \( I^{(2)} \) is the lower envelope of all second order distribution functions. Then we would like to compute either the \( L_1 \)- or \( L_2 \)-norm of the difference function over all possible return values, i.e., the interval \([0, 1]\). The real complication here is that it is non-trivial to automate the computation of the lower envelope. But since first order distribution functions are right-continuous, monotone increasing and bounded and second order distribution functions are continuous, monotone increasing and bounded, we tried to avoid this complication by approximating the norms. First, discretize the interval \([0, 1]\) with some step-size (in this example step-size is \(1/60000\)) and evaluate the distribution functions for each portfolio \( p_i \), \( i \in \{1, \ldots, 13\} \) at the resulting grid points. Further, for each contingency compute the lower envelope given by the minimal distribution function values at each grid point. Since in this example all pure strategies are considered as well, we only need to use their distribution functions to compute the
lower envelope (for a proof of this see next section). Then compute the approximation of the norms by the following formulas.

\[
\|F_x - I\|_1 \approx \sum_{i=0}^{59999} \frac{1}{60000} \left( \sum_{j=1}^{3} x_j F_j\left(\frac{i}{60000}\right) - I\left(\frac{i}{60000}\right) \right)
\]

\[
\|F_x - I\|_2 \approx \sqrt{\sum_{i=0}^{59999} \left( \sum_{j=1}^{3} x_j F_j\left(\frac{i}{60000}\right) - I\left(\frac{i}{60000}\right) \right)^2}
\]

\[
\|F_x^{(2)} - I^{(2)}\|_1 \approx \sum_{i=0}^{59999} \frac{1}{60000} \left( \sum_{j=1}^{3} x_j F_j^{(2)}\left(\frac{i}{60000}\right) - I^{(2)}\left(\frac{i}{60000}\right) \right)
\]

\[
\|F_x^{(2)} - I^{(2)}\|_2 \approx \sqrt{\sum_{i=0}^{59999} \left( \sum_{j=1}^{3} x_j F_j^{(2)}\left(\frac{i}{60000}\right) - I^{(2)}\left(\frac{i}{60000}\right) \right)^2}
\]

Finally, take the \(l_2\)-norm of the resulting contingency vector, with entries equal to the norm approximations for each contingent event, and choose a portfolio, for which the resulting value is minimal.

We refer to the functions computing those values as \(L_1\)-FSD, \(L_1\)-SSD, \(L_2\)-FSD or \(L_2\)-SSD objective function with discretization step-size \(\frac{1}{60000}\).

The proposed preference rules have the property that if the objective function value of portfolio \(x \in X\) is lower then the objective function value of portfolio \(y \in X\), then \(y\) can not dominate \(x\). Without loss of generality consider the \(L_2\)-FSD objective function. Lower objective function value means that portfolio \(x\) must have a lower \(L_2\)-norm of the first order distribution difference function for at least one contingent event than portfolio \(y\) does, since otherwise the \(l_2\)-norm of the contingency vector and thus the objective function value of \(x\) would be equal or greater.

Analogously, lower norm-value implies that there must exist at least one \(i \in \{0, ..., 59999\}\) such that \(F_x\left(\frac{i}{60000}\right) = \sum_{j=1}^{3} x_j F_j\left(\frac{i}{60000}\right) < \sum_{j=1}^{3} y_j F_j\left(\frac{i}{60000}\right) = F_y\left(\frac{i}{60000}\right)\) contradicting the possibility of portfolio \(y\) dominating portfolio \(x\) by Definition 2.1.2 (if you compute the norm exactly, then there would be at least one \(r \in [0, 1]\) such that this is true because of the right-continuity). This also means that a dominating portfolio in the sense of Definition 2.1.2 can not have a higher objective function value than a dominated one. As a conclusion, amongst all portfolios minimizing the objective function value there must exist at least one nondominated.

After applying introduced preference rules the computational results in Table 2.1 yield portfolio \(p_2\) as preferred one regardless of what objective function we choose. Hence, the pure
strategy investing all the money in the conservative asset must be non-dominated.

The following questions occur. First, we would like to know whether our preference rules always yield the same solution and, second, whether this procedure always leads to a pure strategy result. To answer these questions we introduce a fourth contingency \( C_4 \) with densities depicted in Figure 2.7. The densities of the safe and conservative asset are given by two vertical lines representing Dirac delta functions.

The distribution functions of pure strategies for contingency \( C_4 \) are

\[
F_{(1,0,0)}(r) = \begin{cases} 
0 & r < \frac{2}{3} \\
1 & \frac{2}{3} \leq r
\end{cases}
\]

\[
F_{(1,0,0)}^{(2)}(r) = \begin{cases} 
0 & r < \frac{2}{3} \\
r - \frac{2}{3} & \frac{2}{3} \leq r
\end{cases}
\]

Figure 2.7: Densities for contingency \( C_4 \).
Table 2.2: Approximated objective function values for different preference rules (four contingencies)

<table>
<thead>
<tr>
<th></th>
<th>$L_1$-FSD</th>
<th>$L_1$-SSD</th>
<th>$L_2$-FSD</th>
<th>$L_2$-SSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>0.4439</td>
<td>0.1307</td>
<td>0.7373</td>
<td>0.2097</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.4739</td>
<td>0.1617</td>
<td>0.6791</td>
<td>0.2398</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.4759</td>
<td>0.2527</td>
<td>0.6258</td>
<td>0.2990</td>
</tr>
<tr>
<td>$p_4$</td>
<td>0.4242</td>
<td>0.1291</td>
<td>0.6133</td>
<td>0.1998</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.3863</td>
<td>0.1559</td>
<td>0.5477</td>
<td>0.2040</td>
</tr>
<tr>
<td>$p_6$</td>
<td>0.3539</td>
<td>0.1524</td>
<td>0.4908</td>
<td>0.1940</td>
</tr>
<tr>
<td>$p_7$</td>
<td>0.3605</td>
<td>0.1290</td>
<td>0.5024</td>
<td>0.1782</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.3890</td>
<td>0.1354</td>
<td>0.5864</td>
<td>0.1919</td>
</tr>
<tr>
<td>$p_9$</td>
<td>0.4229</td>
<td>0.2527</td>
<td>0.6258</td>
<td>0.2990</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>0.3539</td>
<td>0.1524</td>
<td>0.4908</td>
<td>0.1940</td>
</tr>
<tr>
<td>$p_{11}$</td>
<td>0.3605</td>
<td>0.1290</td>
<td>0.5024</td>
<td>0.1782</td>
</tr>
<tr>
<td>$p_{12}$</td>
<td>0.3890</td>
<td>0.1354</td>
<td>0.5864</td>
<td>0.1919</td>
</tr>
<tr>
<td>$p_{13}$</td>
<td>0.4229</td>
<td>0.2527</td>
<td>0.6258</td>
<td>0.2990</td>
</tr>
</tbody>
</table>

This time the application of the four preference rules results in different mixed preferred portfolios highlighted in Table 2.2. We conclude that portfolios $p_6$, $p_7$, and $p_9$ are nondominated. In order to find a preferred portfolio it would be sufficient to compute objective function values for nondominated portfolios only, but it is difficult to indicate all of them. However, we can derive nondominated portfolios in the sense of Definition 2.1.2 from nondominated portfolios for fixed contingent events. For both contracting and sustainable economies, FSD- and SSD-nondominated portfolios in the sense of Definition 2.1.1 are the same (see Figure 2.8(a)-(d)). For an unsustainable economy, the pure aggressive strategy $p_1$ dominates the other two pure strategies (see Figure 2.6) and hence every portfolio different from $p_1$. Contingency $C_4$’s FSD- and SSD-nondominated portfolios
Figure 2.8: Nondominated portfolios’ distributions: (a) $F(r)$-graphs under contracting economy (b) $F^{(2)}(r)$-graphs under contracting economy (c) $F(r)$-graphs under sustainable economy (d) $F^{(2)}(r)$-graphs under sustainable economy (e) $F(r)$-graphs under unsustainable economy (f) $F^{(2)}(r)$-graphs under unsustainable economy.
are \( p_1, p_3, p_5, p_8 \) and \( p_{13} \) (see Figure 2.8(e)-(f)). Consequently, by Definition 2.1.2 portfolios \( p_1, p_2, p_3, p_4, p_5, p_8, p_9, p_{11} \) and \( p_{13} \) are also nondominated. However, unifying nondominated portfolios over all fixed contingent events does not give us all nondominated portfolios in the sense of Definition 2.1.2, since portfolios \( p_6 \) and \( p_7 \) would also have to be nondominated.

### 2.4 An example of optimization over all portfolios

In the previous section we examined 13 different portfolios. In this section we consider all possible portfolios \( x \in X \) investing money in three assets under four uncertain contingent events \( C_1, C_2, C_3 \) from Section 2.1 and \( C_4 \) from Section 2.3. We have also decided to drop the \( L_1 \)-FSD and \( L_1 \)-SSD preference rules, since the \( L_1 \)-norm is just the integration of functions such that a function with high values for one part of the domain and low values for the other one can be evaluated the same as a function with average values everywhere. The \( L_2 \)-norm, however, will prefer a function with average values. Hence, \( L_2 \)-norm preference rules will look for portfolios with distribution functions not deviating much from the utopian distribution function, i.e., the lower envelope.

The real valued function \( f(z) = z^2 \) defined on \([0, \infty)\) is strictly monotone increasing and so, according to the methodology introduced in Section 2.3, for a fixed contingent event the preferred portfolio with respect to FSD can be computed by minimizing the following function.

\[
g(x) = (\|F_x - I\|_2)^2 = \int_a^b (F_x(r) - I(r))^2 \, dr = \int_0^1 2F_{(1,0,0)}(r)F_{(0,1,0)}(r) dr \cdot x_1 x_2 + \int_0^1 2F_{(1,0,1)}(r) dr \cdot x_1 x_3 + \int_0^1 2F_{(0,1,0)}(r)F_{(0,0,1)}(r) dr \cdot x_2 x_3 - \int_0^1 2F_{(1,0,0)}(r)I(r) dr \cdot x_1 - \int_0^1 2F_{(0,1,0)}(r)I(r) dr \cdot x_2 - \int_0^1 2F_{(0,0,1)}(r)I(r) dr \cdot x_3 + \int_0^1 I(r)^2 \, dr
\]

For each contingent event the integrals in \( g \) can be easily computed, since all distribution functions are known (see Sections 2.2 and 2.3), and their values will be all positive, since distribution functions take positive values only. In order to get the lower envelope of all portfolio’s distribution functions
Computational results for the $L_2$-FSD preference rule

<table>
<thead>
<tr>
<th>$f_0^1(F_{(1,0,0)}(r))^2dr$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$f_0^1(F_{(0,1,0)}(r))^2dr$</td>
<td>263</td>
<td>143</td>
<td>23</td>
<td>7</td>
</tr>
<tr>
<td>$f_0^1(F_{(0,0,1)}(r))^2dr$</td>
<td>23</td>
<td>53</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$f_0^1(2F_{(1,0,0)}(r)F_{(0,1,0)}(r)dr$</td>
<td>$\frac{37}{45}$</td>
<td>$\frac{101}{150}$</td>
<td>$\frac{77}{105}$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$f_0^1(2F_{(0,1,0)}(r)F_{(0,0,1)}(r)dr$</td>
<td>1</td>
<td>$\frac{11}{12}$</td>
<td>$\frac{1}{3}$</td>
<td>26</td>
</tr>
<tr>
<td>$f_0^1(2F_{(1,0,0)}(r)I(r)dr$</td>
<td>26323</td>
<td>$\frac{11}{20}$</td>
<td>5249</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$f_0^1(2F_{(0,1,0)}(r)I(r)dr$</td>
<td>37</td>
<td>101</td>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td>$f_0^1(2F_{(0,0,1)}(r)I(r)dr$</td>
<td>263</td>
<td>3559</td>
<td>5249</td>
<td>26</td>
</tr>
<tr>
<td>$f_0^1(I(r)^2dr$</td>
<td>263</td>
<td>3559</td>
<td>1</td>
<td>121</td>
</tr>
</tbody>
</table>

Table 2.3: Computational results for the $L_2$-FSD preference rule

represented by the ideal distribution $I(r)$ we rewrite it as

$$I(r) = \min_{x \in X}\{x_1F_{(1,0,0)}(r) + x_2F_{(0,1,0)}(r) + x_3F_{(0,0,1)}(r)\} = \min\{F_{(1,0,0)}(r), F_{(0,1,0)}(r), F_{(0,0,1)}(r)\}$$

The equality above is essentially true, because for any triple of positive real numbers $a \leq b \leq c$ the value of $v(x) = x_1a + x_2b + x_3c$ will be minimized over $X$ for $x = (1, 0, 0)$. Assuming that this is not true, there must exist an $x^* = (x_1^*, x_2^*, x_3^*) \in X$ such that $v(x) > v(x^*)$. But then

$$v(x^*) = x_1^*a + x_2^*b + x_3^*c \geq a(x_1^* + x_2^* + x_3^*) = a = v(x)$$

which is a contradiction to $v(x) > v(x^*)$. Consequently, the lower envelope can be computed by taking into consideration only the distribution functions of pure strategies. A general proof for an arbitrary number of assets works the same.

Table 2.3 depicts the values of all integrals that have to be computed for each contingency in order to define the functions $g$ (each contingent event has a different function $g$). We use MATLAB to carry out those computations. For each fixed contingency, function $g$ is the square of the $L_2$-norm of the difference function $F_2 - I$ and, since the last step of the $L_2$-FSD preference rule requires to take the $l_2$-norm of the resulting contingency vector (see Section 2.3), we only need to sum up functions $g$ over all contingencies and take the square root to obtain the exact $L_2$-FSD objective function. We use MATLAB again to carry out the optimization step. Starting with $x = (0, 0, 0)$, the minimization
Table 2.4: Optimal solutions of approximated $L_2$-FSD objective functions

<table>
<thead>
<tr>
<th>step-size $1/n$</th>
<th>optimal portfolio $x^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>(0.2499, 0.3220, 0.4280)</td>
</tr>
<tr>
<td>1/100</td>
<td>(0.1495, 0.3621, 0.4884)</td>
</tr>
<tr>
<td>1/1000</td>
<td>(0.1382, 0.3686, 0.4932)</td>
</tr>
<tr>
<td>1/10000</td>
<td>(0.1371, 0.3692, 0.4937)</td>
</tr>
<tr>
<td>1/100000</td>
<td>(0.1370, 0.3693, 0.4938)</td>
</tr>
</tbody>
</table>

of the exact objective function with respect to $(x_1, x_2, x_3) \in X$ using fmincon with default settings returns the following optimal solution.

$$x^* = (0.1370, 0.3693, 0.4938)$$

In comparison to the approximated $L_2$-FSD objective function in Section 2.3 the computational effort to compute the exact objective function seems to be higher. However, if we can automate the process of computing the integrals in Table 2.3, the optimization process will become extremely quick, because then we only need to optimize a quadratic function over a convex space $X$. Also, the solution will be exact. The approximated objective function does not need the preceding computational results. But for a high order discretization the evaluation of the approximate objective function requires more time, since depending on the choice of step-size many operations have to be made for each objective function evaluation. Additionally, the solution from the optimization of the approximated objective function is not exact, but it does converge to the exact optimal solution as can be seen in Table 2.4. This general pattern will be established later.

The same calculations can be executed for SSD. We simply have to exchange $F_x(r)$ by $F_x^{(2)}(r)$ and $I(r)$ by $I^{(2)}(r)$ in the definition of $g$ and compute appropriate integrals for the exact objective function. Table 2.5 displays the values of all those integrals that we need for the coefficients of function $g$.

Starting with $x = (0, 0, 0)$, minimization of the exact $L_2$-SSD objective function with respect to $(x_1, x_2, x_3) \in X$ using fmincon with default settings returns the following optimal solution.

$$x^* = (0.3838, 0.3424, 0.2738)$$
Using the approximated $L_2$-SSD objective function the optimal solution provided by \textit{fmincon} starting at $x = (0, 0, 0)$ again converges to $x^*$. Table 2.6 shows that for smaller discretization approximated optimal solutions get closer to the exact one.

<table>
<thead>
<tr>
<th>step-size 1/n</th>
<th>optimal portfolio $x^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>(0.4322, 0.3319, 0.2358)</td>
</tr>
<tr>
<td>1/100</td>
<td>(0.3876, 0.3415, 0.2709)</td>
</tr>
<tr>
<td>1/1000</td>
<td>(0.3842, 0.3423, 0.2735)</td>
</tr>
<tr>
<td>1/10000</td>
<td>(0.3839, 0.3424, 0.2738)</td>
</tr>
<tr>
<td>1/100000</td>
<td>(0.3838, 0.3424, 0.2738)</td>
</tr>
</tbody>
</table>

### 2.5 Optimization over an interval of uncertainty values

Notice that it is possible to consider other ideal distribution functions than the lower envelope. The main idea of subtracting the lower envelope is to find a portfolio with distribution function values that are "overall" as small as possible. If we choose an alternative ideal distribution function, then the preferred portfolio might change. As an example consider two cumulative distribution functions, displayed in Figure 2.9, both starting at 0 with value 0 and ending at 1 with value 1. Function 1 represents an asset, where all return outcomes from 0 to 1 are equally probable.
Function 2, however, represents an asset with only two possible return outcomes, namely 0 and 1, each with probability $\frac{1}{2}$. Consider the squared $L_2$-norm of their convex combination.

$$\int_0^1 \left( xr + (1-x) \frac{1}{2} \right)^2 \, dr = \left( \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \right) x^2 + \frac{1}{4} = \frac{1}{6} x^2 + \frac{1}{4}$$

This convex function is minimal for $x = 0$ and so Function 2 would be the distribution function of the preferred portfolio if the ideal distribution was 0. By subtracting the lower envelope, however, the preferred portfolio turns out to be $(\frac{1}{2}, \frac{1}{2})$. Even though a change of the ideal distribution can lead to different results, since it is easier to work with 0 as the ideal distribution function, we will adopt this viewpoint in all that follows. Besides, the distribution function 0 represents a return random variable that yields the highest possible return with probability 1 and that is truly desirable.

In this section we consider three assets with densities depending on the uncertain parameter $u \in [0, 1]$. The first asset’s return $A_1(u)$ shall be equal to the safe asset’s return from Section 2.1 for any value of $u$. The second asset’s return $A_2(u)$ and third asset’s return $A_3(u)$ shall be given by triangular distributions with lower limits $\chi(u)$, modes $\psi(u)$ and upper limits $\omega(u)$.

$$A_2(u): \quad \chi(u) = \frac{1}{4}, \psi(u) = u^3 - 3u^2 + \frac{9}{4}u + \frac{3}{8}, \omega(u) = 1$$

$$A_3(u): \quad \chi(u) = \frac{u}{2}, \psi(u) = u, \omega(u) = \frac{u + 1}{2}$$
On closer examination notice that those assets are equivalent to the ones considered before. The first contingent event \( C_1 \) is the same as the event with uncertainty parameter \( u = 0 \), \( C_2 \) is identical to the event with \( u = 1/2 \), and \( C_3 \) is identical to the event with \( u = 1 \). From now on \( u \in [0, 1] \) represents contingent events.

For each fixed uncertainty value \( u \in [0, 1] \) the return densities of the assets are different. Recall from Section 2.4 the function \( g(x) \), the square of the \( L_2 \)-norm of the difference function \( F_x - I \), which was different depending on what contingency event we have. In order to compute the \( L_2 \)-FSD objective function we could simply sum up functions \( g(x) \) over all contingencies and take the square root to complete the last step of the \( L_2 \)-FSD preference rule in Section 2.3, which was the application of the \( l_2 \)-norm to the contingency vector with entries equal to the square root of \( g(x) \). Since we do not deal with finitely many contingencies any more, in order to account for all uncertainty values we take the \( L_2 \)-norm of the square root of functions \( g(x) \) over all uncertainty values \( u \).

Each function \( g \) has the form

\[
g(x) = \int_0^1 (F_x(r))^2 dr = c_{x_1} x_1^2 + c_{x_2} x_2^2 + c_{x_3} x_3^2 + c_{x_1 x_2} x_1 x_2 + c_{x_1 x_3} x_1 x_3 + c_{x_2 x_3} x_2 x_3
\]

For each value of \( u \) the coefficients \( c_{x_1}, c_{x_2}, c_{x_3}, c_{x_1 x_2}, c_{x_1 x_3} \) and \( c_{x_2 x_3} \) are different, i.e., they are functions of \( u \) and, since the nodes of the triangular distributions are continuous functions of \( u \), those coefficients must also be continuous functions of \( u \). The real valued function \( f(z) = z^2 \) defined on \([0, \infty)\) is strictly monotone increasing and so the minimization of the \( L_2 \)-FSD objective function is equivalent to the minimization of its square. For that reason the \( L_2 \)-FSD objective function has the form

\[
\int_0^1 g(x) \, du = \int_0^1 c_{x_1} \, du \cdot x_1^2 + \int_0^1 c_{x_2} \, du \cdot x_2^2 + \int_0^1 c_{x_3} \, du \cdot x_3^2 + \int_0^1 c_{x_1 x_2} \, du \cdot x_1 x_2 \\
+ \int_0^1 c_{x_1 x_3} \, du \cdot x_1 x_3 + \int_0^1 c_{x_2 x_3} \, du \cdot x_2 x_3
\]

The integrals above can be computed by generalizing the density function of a triangular distribution in terms of \( u \) and using Maple or MATLAB to do the integration. Finally, we optimize the objective function.

The procedure for the \( L_2 \)-SSD preference rule is the same. Exchange \( F_x(r) \) by \( F_x^{(2)}(r) \) in
Table 2.7: $L_2$-FSD and $L_2$-SSD objective function computational results

<table>
<thead>
<tr>
<th></th>
<th>FSD</th>
<th>SSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^1 c_{x_1}^1 , du$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{27}$</td>
</tr>
<tr>
<td>$\int_0^1 c_{x_2}^2 , du$</td>
<td>$6127 , 25200$</td>
<td>$137281789 , 9684176800$</td>
</tr>
<tr>
<td>$\int_0^1 c_{x_3}^3 , du$</td>
<td>$79 , 180$</td>
<td>$18337 , 302300$</td>
</tr>
<tr>
<td>$\int_0^1 c_{x_1, x_2} , du$</td>
<td>$0.6360952030 , 0.04739518635$</td>
<td></td>
</tr>
<tr>
<td>$\int_0^1 c_{x_1, x_3} , du$</td>
<td>$\frac{23}{18} - \frac{2}{3} \ln(2) , \frac{1153}{14400} + \frac{1}{120} \ln(2)$</td>
<td></td>
</tr>
<tr>
<td>$\int_0^1 c_{x_2, x_3} , du$</td>
<td>$0.5740267828 , 0.05086988902$</td>
<td></td>
</tr>
</tbody>
</table>

the definition of $g$, compute corresponding integrals for the $L_2$-SSD objective function and optimize.

Table 2.7 displays the coefficients of the objective function for both the $L_2$-FSD and the $L_2$-SSD preference rule.

The optimization of the $L_2$-FSD objective function using default settings of $fmincon$ starting with 0 provides the solution

$$x^* = (0, 1, 0).$$

The optimization of the $L_2$-SSD-objective function using default settings of $fmincon$ starting with 0 provides the solution

$$x^* = (0, 1, 0).$$

Interestingly, both preference rules produce the same result, namely that all the money should be invested in the conservative asset. It is not always the case that a pure strategy is preferred. The example in Section 2.4 shows that portfolios with mixed investments can be better.

In the next chapter, we will establish the theoretical background for the $L_2$-FSD and the $L_2$-SSD preference rules. In Chapter 4, we will show a way to evaluate the preferred solution provided by these preference rules and, finally, Chapter 5 will conclude the overall methodology discussed in this thesis.
Chapter 3

Theoretical Background

3.1 Connection between FSD or SSD and VaR or CVaR

In this section we will show that the comparison of two portfolios with respect to first order stochastic dominance (FSD) and second order stochastic dominance (SSD) is equivalent to the comparison of their Values at Risk and Conditional Value at Risk, respectively, at all confidence levels. First, we will establish this result for discrete random variables with equal probability outcomes and then show the general case using some published theorems.

Let \( R \) be a random variable representing the gain/return of a portfolio. Our goal is to maximize the return and thus to minimize the negative return \(-R\). According to Rockafellar [8], for a confidence level \( \alpha \in (0, 1) \) the Value at Risk of a portfolio is the smallest return value \( r \) such that the probability that the negative return \(-R\) does not exceed \( r \) is not smaller than \( \alpha \).

\[
\text{VaR}_\alpha(-R) = \inf\{r \in \mathbb{R} | \mathbb{P}(-R \leq r) \geq \alpha\} \\
= \inf\{r \in \mathbb{R} : \mathbb{P}(-R > r) \leq 1 - \alpha\} \\
= \inf\{r \in \mathbb{R} : \mathbb{P}(R < -r) \leq 1 - \alpha\} \\
= \inf\{-r \in \mathbb{R} : \mathbb{P}(R < r) \leq 1 - \alpha\} \\
= -\sup\{r \in \mathbb{R} : \mathbb{P}(R < r) \leq 1 - \alpha\} \\
= -\inf\{r \in \mathbb{R} : \mathbb{P}(R < r) > 1 - \alpha\}
\]
Therefore, for the return random variable \( R \) we use the definition of Value at Risk at confidence level \( \alpha \)

\[
VaR_\alpha(R) = \inf\{ r \in \mathbb{R} : P(R < r) > \alpha \}
\]

and obtain the connection \( VaR_\alpha(-R) = -VaR_{1-\alpha}(R) \). Notice that for continuous distributions the definitions for \( R \) and \(-R\) are the same, but this is not true in general, e.g., they are not the same for discrete distributions. According to Rockafellar [8], Conditional Value at Risk is always connected with negative returns. In order to stay consistent with him we will always consider Value at Risk and Conditional Value at Risk for the negative return random variable \(-R\) throughout this section.

Let \( R \) be a discrete random variable defined by \( T \) return values, say without loss of generality \( r_1 < r_2 < \ldots < r_T \) (the concluding result is also true for \( \leq \) instead of \(<\)), with equal outcome probabilities i.e., \( P(R = r_i) = \frac{1}{T} \) for all \( i = 1, \ldots, T \). For \( 0 < \epsilon \leq \frac{1}{T} \) and integer \( 0 < k \leq T \)

\[
P(R < r_{(T-k+1)} + \epsilon) = \frac{T - k + 1}{T} = 1 - \frac{k - 1}{T} \quad \text{and} \quad P(R < r_{(T-k+1)}) = \frac{T - k}{T} = 1 - \frac{k}{T}
\]

Hence for \( \frac{k-1}{T} < \alpha \leq \frac{k}{T} \) the negative Value at Risk is equal to the \( k \)-th best return value \( r_{(T-k+1)} \).

Theorem 3.1.1 will show that the comparison of all \( k \)-th best return values, \( 1 \leq k \leq T \) of portfolios \( x \) and \( y \) by rank is equivalent to the application of the FSD-domination rule in Definition 2.1.1.

Consequently, comparison of two portfolios \( x \) and \( y \) with respect to FSD is equivalent to the comparison of their Values at Risk at all confidence levels, which means, that if portfolio \( x \) has no larger Values at Risk at all confidence levels \( \alpha \in (0,1) \) than portfolio \( y \) with at least one \( \alpha \) for which strict inequality holds, then \( x \) dominates \( y \) with respect to FSD. Also, if there exists an \( \alpha \in (0,1) \) such that the Value at Risk of \( x \) is greater than the Value at Risk of \( y \), then \( x \) can not dominate \( y \) with respect to FSD.

Conditional Value at Risk of a portfolio with return \( R \) at confidence level \( \alpha \in (0,1) \) is defined as

\[
CVaR_\alpha(-R) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} VaR_t(-R) dt
\]

If \( R \) is discrete again, then for \( \alpha = \frac{k}{T} \), where \( 0 \leq k < T \) is an integer, the Conditional Value at Risk is equal to the negative mean of the \((T-k)\) worst return values.

\[
CVaR_{\frac{k}{T}}(-R) = \frac{1}{1 - \frac{k}{T}} \int_{\frac{k}{T}}^{1} VaR_t(-R) dt = \frac{1}{1 - \frac{k}{T}} \frac{1}{T} \sum_{i=k+1}^{T} -r_{(T-i+1)} = \frac{1}{T-k} \sum_{i=1}^{T-k} r_i
\]
Next, suppose that portfolio $x$ is represented by the return variable $R_x$ with outcomes $r_1 \leq r_2 \leq \ldots \leq r_T$, whereas portfolio $y$ is represented by return variable $R_y$ with outcomes $\bar{r}_1 \leq \bar{r}_2 \leq \ldots \leq \bar{r}_T$. If portfolio $x$ has no larger Conditional Values at Risk, i.e., $CVaR_\alpha(-R_x)$, at all confidence levels $\alpha \in (0, 1)$ than portfolio $y$, i.e., $CVaR_\alpha(-R_y)$, with at least one strict inequality, then

$$
\sum_{i=1}^{T-k} r_i \geq \sum_{i=1}^{T-k} \bar{r}_i, \text{ for all } 0 \leq k < T \text{ with at least one strict inequality.} \tag{1}
$$

On the other hand if the above inequalities hold, then fixing $k$ for $0 < \epsilon \leq \frac{1}{T}$ we obtain

$$
\sum_{i=1}^{T-k-1} r_i + (1-\epsilon)r_{(T-k)} \geq \sum_{i=1}^{T-k-1} \bar{r}_i + (1-\epsilon)\bar{r}_{(T-k)}
$$

Since both the LHS and the RHS of the above equation is linear in $\epsilon$ and $\geq$ holds for $\epsilon \in \{0, 1\}$, the strict inequality $>$ does not hold for $\epsilon \in (0, 1)$. Thus, for $0 < \epsilon \leq \frac{1}{T}$

$$
CVaR_{k+\epsilon}(-R_x) = -\frac{1}{T-k-\epsilon} \left( \sum_{i=1}^{T-k-1} r_i + (1-\epsilon)r_{(T-k)} \right) \leq CVaR_{k+\epsilon}(-R_y)
$$

Theorem 3.1.1 will reveal that the comparison of return values shown in (1) is equivalent to the application of the SSD-domination rule from Section 2.1. Hence, a portfolio $x$ is preferred to portfolio $y$ with respect to the minimization of Conditional Value at Risk at all confidence levels if and only if it is preferred with respect to SSD.

**Theorem 3.1.1.** Let $R_x$ and $R_y$ be random variables defined on $\{\Omega, \mathbb{F}, P\}$, with $\Omega = \{\omega_1, \ldots, \omega_T\}$, $\mathbb{F}$ a power set of $\Omega$ and $P(\omega_i) = \frac{1}{T}$ for all $i \in \{1, \ldots, T\}$.

Let $\Pi_1, \Pi_2$ be two permutations of $\{1, \ldots, T\}$ and $(r_1, \ldots, r_T) = (R_x(\omega_{\Pi_1(1)}), \ldots, R_x(\omega_{\Pi_1(T)}))$, $(\bar{r}_1, \ldots, \bar{r}_T) = (R_y(\omega_{\Pi_2(1)}), \ldots, R_y(\omega_{\Pi_2(T)}))$ such that $r_1 \leq \ldots \leq r_T$ and $\bar{r}_1 \leq \ldots \leq \bar{r}_T$ (ordered outcomes in ascending order), then

$x \succ_{FSD} y$ if and only if $r_i \geq \bar{r}_i$ for all $i \in \{1, \ldots, T\}$, with at least one inequality.

$x \succ_{SSD} y$ if and only if $\sum_{j=1}^{i} r_j \geq \sum_{j=1}^{i} \bar{r}_j$ for all $i \in \{1, \ldots, T\}$, with at least one strict inequality.
Proof. Proof of the result for FSD.

Suppose \( F_x(r) \leq F_y(r) \) for all \( r \in \mathbb{R} \) with at least one strict inequality. We can not have \( r_i = \tau_i \) for all \( i \in \{1,...,T\} \), since then the cumulative distribution functions would be the same. So suppose, that for some \( i \in \{1,...,T\} \), \( r_i < \tau_i \). Then we can choose \( r_i < r < \tau_i \), so that \( P(R_x \leq r) \geq \frac{i}{T} \) and \( P(R_y \leq r) < \frac{i}{T} \), since all the outcomes \( r_j \)'s and \( \tau_j \)'s are ordered. But then we get \( F_y(r) = P(R_y \leq r) < \frac{i}{T} \leq P(R_x \leq r) = F_x(r) \), which contradicts our assumption. This implies that \( r_i \geq \tau_i \) for all \( i \in \{1,...,T\} \) with at least one strict inequality.

Suppose \( r_i \geq \tau_i \) for all \( i \in \{1,...,T\} \) with at least one strict inequality. Choose any \( r \in \mathbb{R} \). Let \( \tau_i \leq r < \tau_{i+1} \), then \( P(R_y \leq r) = \frac{i}{T} \). From the assumption we get that \( r < \tau_{i+1} \leq r_{i+1} \) and so \( P(R_x \leq r) \leq \frac{i}{T} \). By putting everything together we obtain, that \( F_x(r) \leq F_y(r) \). For \( r < \tau_1 \) the argument is the same and for \( r \geq \tau_T \) \( F_y(r) = 1 \geq F_x(r) \), since \( P(R_x \leq t) \leq 1 \) for all \( t \in \mathbb{R} \).

Now we only need to prove that strict inequality holds for at least one \( r \in \mathbb{R} \). By assumption, there exists an \( i \in \{1,...,T\} \) such that \( r_i > \tau_i \). Choose \( r_i > r > \tau_i \), then \( P(R_x \leq r) < \frac{i}{T} \leq P(R_y \leq r) \), which completes the proof.

Proof of the result for SSD.

Suppose \( F_x^{(2)}(r) \leq F_y^{(2)}(r) \) for all \( r \in \mathbb{R} \). Without loss of generality let \( r_n \leq r < r_{n+1} \) and \( \tau_m \leq r < \tau_{m+1} \).

\[
\int_{-\infty}^{r} F_x(t) dt \leq \int_{-\infty}^{r} F_y(t) dt \\
\sum_{i=1}^{n-1} \frac{i}{T} (\tau_{i+1} - r_i) + \frac{n}{T} (r - r_n) \leq \sum_{j=1}^{m-1} \frac{j}{T} (\tau_{j+1} - \tau_j) + \frac{m}{T} (r - \tau_m) \\
\frac{1}{T} \sum_{i=1}^{n} r_i + \frac{n}{T} r \leq -\frac{1}{T} \sum_{j=1}^{m} \tau_j + \frac{m}{T} r \\
\sum_{i=1}^{n} r_i + m r \geq \sum_{j=1}^{m} \tau_j + n r
\]
Now suppose, that \( \sum_{j=1}^{m} r_j > \sum_{j=1}^{m} r_j \), then

\[
\sum_{i=1}^{n} r_i > \sum_{j=1}^{m} r_j + (n - m)r
\]

For \( n = m \), \( \sum_{i=1}^{n} r_i > \sum_{j=1}^{m} r_j \) is evidently not true.

For \( n > m \), \( \sum_{i=m+1}^{n} r_i > (n - m)r \) can not be true, since \( r_i \leq r \) for \( i \leq n \).

For \( n < m \), \( \sum_{i=n+1}^{m} -r_i > (m - n)(-r) \) can not be true, since \( r_i > r \) and so \( -r_i < -r \) for \( i \geq n + 1 \).

This implies that \( \sum_{j=1}^{m} r_j \leq \sum_{j=1}^{m} r_j \). And since we can choose \( r \) such that \( m \) is any number between \( 1 \) and \( T \), this result holds for any \( 1 \leq m \leq T \).

To show, that there exists an \( 1 \leq m \leq T \) such that \( \sum_{j=1}^{m} r_j < \sum_{j=1}^{m} r_j \), choose an \( r \in \mathbb{R} \) such that \( F^{(2)}_x(r) < F^{(2)}_y(r) \) and do the same proof again. Since we have \( < \) instead of \( \leq \), we can suppose that \( \sum_{j=1}^{m} r_j \geq \sum_{j=1}^{m} r_j \) and will get a contradiction by the same arguments as before.

Suppose \( \sum_{j=1}^{i} r_j \geq \sum_{j=1}^{i} r_j \) for all \( i \in \{1, ..., T\} \).

\[
F^{(2)}_x(r) = -\frac{1}{T} \sum_{i=1}^{n} r_i \geq \frac{n}{T} r, \text{ where } r_n \leq r \leq r_{n+1}
\]

\[
\leq -\frac{1}{T} \sum_{i=1}^{n} r_i + \frac{n}{T} r
\]

Now let \( \tau_m \leq r < \tau_{m+1} \).

For \( n = m \), \( -\frac{1}{T} \sum_{i=1}^{n} r_i + \frac{n}{T} r = F^{(2)}_y(r) \).

For \( n < m \), \( -\frac{1}{T} \sum_{i=1}^{n} r_i + \frac{n}{T} r = F^{(2)}_y(r) + \sum_{i=n+1}^{m} -\frac{r}{T} r - \frac{m-n}{T} r \leq F^{(2)}_y(r) \), since \( r_i - r \leq 0 \) for \( i \leq m \).
For \( n > m \), \( \sum_{i=1}^{n} \bar{r}_i + \frac{n}{T} r = F_y^{(2)}(r) + \sum_{i=m+1}^{n} \frac{1}{T} \bar{r}_i + \frac{n-m}{T} r \leq F_y^{(2)}(r) \), since \( r - \bar{r}_i < 0 \) for \( i > m \).

Since there exists an \( 1 \leq n \leq T \) such that \( \sum_{j=1}^{n} r_j < \sum_{j=1}^{n} r_j \), by applying the same procedure with \( r_n \leq r \leq r_{n+1} \), we will get the required strict inequality.

For a general connection between stochastic dominance and VaR and CVaR consider the quantile function \( Q(P) \) (Haim Levy [5]). For \( R \) being the return of a portfolio taking values in \( [a, b] \), the \( P^{th} \) quantile is defined as

\[
Q(P) = \begin{cases} 
\inf\{l \in \mathbb{R} : P(R \leq l) > P\} & P = 0 \\
\inf\{l \in \mathbb{R} : P(R \leq l) \geq P\} & 0 < P \leq 1 
\end{cases}
\]

The following is proven by Haim Levy [5], for two portfolios \( x \) and \( y \).

**Theorem 3.1.2.** Let \( F_x \) and \( F_y \) be the cumulative distributions of the return on two portfolios \( x \) and \( y \) with quantiles \( Q_x(P) \) and \( Q_y(P) \), respectively. Then \( x \succ_{FSD} y \), if and only if

\[
Q_x(P) \geq Q_y(P) \text{ for all } 0 \leq P \leq 1
\]

with strict inequality for at least one value \( P_0 \) for which a strict inequality holds.

**Theorem 3.1.3.** Let \( F_x \) and \( F_y \) be the two distributions under consideration with quantiles \( Q_x(P) \) and \( Q_y(P) \), respectively. Then \( x \succ_{SSD} y \), if and only if

\[
\int_0^P [Q_x(t) - Q_y(t)] dt \geq 0 \text{ for all } 0 \leq P \leq 1
\]

with strict inequality for at least one \( P_0 \).

Notice that if the distribution function of a portfolio is continuous, then for a confidence level \( \alpha \in (0, 1) \)

\[
VaR_\alpha = -Q(1 - \alpha)
\]
CVaR $\alpha = 1 \int_{1-\alpha}^{1} -Q(1-t)dt = \frac{1}{1-\alpha} \int_{0}^{1-\alpha} Q(s)ds = -\frac{1}{1-\alpha} \int_{0}^{1} Q(s)ds$

By Theorems 3.1.2 and 3.1.3, the comparison of two portfolios $x$ and $y$ with continuous distribution functions of the return with respect to FSD and SSD is equivalent to the comparison of their Values at Risk and Conditional Values at Risk at all confidence levels, respectively.

### 3.2 Uniqueness of preferred solution

We have seen in Chapter 2 how to find a preferred solution. It is, however, important to know whether the methodology from Chapter 2 also provides a unique preferred solution. In this section we will investigate the $L_2$-FSD and $L_2$-SSD objective functions and finally show under which circumstances their minimization yields a unique preferred solution.

Without loss of generality, we will concentrate on the $L_2$-FSD preference rule and will not distinguish between $L_2$-FSD and $L_2$-SSD objective functions but simply talk about the objective function unless a distinction is necessary, since the $L_2$-FSD preference rule only differs from the $L_2$-SSD preference rule by using second order instead of first order distribution functions.

Recall from Section 2.4 that in order to identify the objective function we had to compute function $g(x)$, the square of the $L_2$-norm of the difference function $F_x - I$, for each possible uncertain event first. For a single contingency, the objective function will be

$$g(x) = \int_{a}^{b} (F_x(r) - I(r))^2 dr$$

with $[a, b], a, b \in \mathbb{R}$ being the interval of all possible return values. We will see in Section 3.3 that $g(x)$ can be uniformly approximated by

$$g_n(x) = \sum_{i=1}^{n} \frac{b-a}{n} \left( \sum_{j=1}^{m} x_j F_j(s^n_i) - I(s^n_i) \right)^2$$

where $(s^n_i)$ is a discretization of the interval $[a, b]$ with stepsize $\frac{1}{n}(b-a)$ i.e., $s^n_1 = a < s^n_2 = a + \frac{1}{n}(b-a) < s^n_3 = a + \frac{2}{n}(b-a) < ... < s^n_{n+1} = b$. This means that for any $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$, such that for all $n \geq N$

$$|g_n(x) - g(x)| < \epsilon \text{ for all } x \in X$$
**Theorem 3.2.1.** $g_n(x)$ is convex and the set of minimal points of $g_n(x)$ in $X$ is convex.

For the proof of this theorem, we will use the Cauchy-Binet formula stating the following.

**Proposition 3.2.2 (Cauchy-Binet formula).** Suppose $A$ is a $m \times n$ matrix and $B$ is an $n \times m$ matrix. If $S$ is a subset of $\{1, ..., n\}$ with $m$ elements, we write $A_S$ for the $m \times m$ matrix whose columns are those columns of $A$ that have indices from $S$. Similarly, we write $B_S$ for the $m \times m$ matrix whose rows are those rows of $B$ that have indices from $S$. The Cauchy-Binet formula then states

$$\det(AB) = \sum_S \det(A_S) \det(B_S)$$

where the sum extends over all possible subsets $S$ of $\{1, ..., n\}$ with $m$ elements (there are $\binom{n}{m}$ of them).

**Proof of Theorem 3.2.1.** Consider the Hessian of the function $g_n(x)$.

$$\nabla^2 g_n = \frac{2(b-a)}{n} \begin{bmatrix} \sum_{i=1}^n F_1(s_i^n)^2 & \sum_{i=1}^n F_1(s_i^n)F_2(s_i^n) & \cdots & \sum_{i=1}^n F_1(s_i^n)F_m(s_i^n) \\ \sum_{i=1}^n F_2(s_i^n)F_1(s_i^n) & \sum_{i=1}^n F_2(s_i^n)^2 & \cdots & \sum_{i=1}^n F_2(s_i^n)F_m(s_i^n) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n F_m(s_i^n)F_1(s_i^n) & \sum_{i=1}^n F_m(s_i^n)^2 & \cdots & \sum_{i=1}^n F_m(s_i^n)F_m(s_i^n) \end{bmatrix}$$

It has the form $\nabla^2 g_n = \frac{2(b-a)}{n} A \cdot A^T$, where

$$A = \begin{bmatrix} F_1(s_1^n) & F_1(s_2^n) & \cdots & F_1(s_n^n) \\ F_2(s_1^n) & F_2(s_2^n) & \cdots & F_2(s_n^n) \\ \vdots & \vdots & \ddots & \vdots \\ F_m(s_1^n) & F_m(s_2^n) & \cdots & F_m(s_n^n) \end{bmatrix}$$

By the Cauchy-Binet formula in Proposition 3.2.2, for $m \leq n$,

$$\det(\nabla^2 g_n) = \left(\frac{2(b-a)}{n}\right)^m \sum_S \det(A_S) \det(A_S^T) = \left(\frac{2(b-a)}{n}\right)^m \sum_S \det(A_S)^2 \geq 0$$

If $m > n$, then we extend matrix $A$ by adding $m-n$ zero columns to obtain a square matrix
$A \in \mathbb{R}^{m \times m}.$

\[
A = \begin{bmatrix}
F_1(s^n_1) & F_1(s^n_2) & \cdots & F_1(s^n_n) & 0 & \cdots & 0 \\
F_2(s^n_1) & F_2(s^n_2) & \cdots & F_2(s^n_n) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
F_m(s^n_1) & F_m(s^n_2) & \cdots & F_m(s^n_n) & 0 & \cdots & 0
\end{bmatrix}
\]

Hence, we can apply the Cauchy-Binet formula.

\[
\det(\nabla^2_x g_n) = \det(\frac{2(b-a)}{n} A \cdot A^T) = \det(\frac{2(b-a)}{n} A \cdot A^T) = \left(\frac{2(b-a)}{n}\right)^m \det(A) \det(A^T) = 0
\]

By the same argument, the determinant of each upper-left square submatrix of $\nabla^2_x g_n$ is also nonnegative and so, $\nabla^2_x g_n$ is positive semi-definite. For this reason, according to Ruszczynski [10], each function $g_n(x)$ is convex and since $X$ is an affine-linear convex set, the set of minimal solutions of $g_n(x)$ is convex.

\[\square\]

**Corollary 3.2.3.** If real-valued functions $F_j, j = 1, \ldots, m$ are continuous everywhere but in finitely many points and bounded on the interval $[a, b]$, then $g(x)$ is convex and the set of minimal points of $g(x)$ in $X$ is convex.

**Proof.** For a real-valued function $f$ that is continuous everywhere but in finitely many points and bounded on the interval $[a, b]$, the Riemann integral exists and

\[
\int_a^b f(r) \, dr = \lim_{n \to \infty} \sum_{i=1}^n f(s^n_i)(s^n_{i+1} - s^n_i) = \lim_{n \to \infty} \frac{(b-a)}{n} \sum_{i=1}^n f(s^n_i)
\]

For that reason, $\nabla^2_x g_n$ converges to $\nabla^2_x g$ as $n$ goes to infinity. The determinant as a function is continuous and so, $\nabla^2_x g$ is positive semi-definite, too. Thus, according to Ruszczynski [10] again, $g(x)$ is convex and the set of minimal points of $g(x)$ in $X$ is convex.

\[\square\]

Corollary 3.2.3 applies to first and second order distribution functions of triangular distributions, since those are always continuous, and also to distribution functions of discrete distributions with finitely many outcomes. Hence, all functions $g(x)$ from Chapter 2 are convex and the set of
minimal points of \( g(x) \) in \( X \) is convex. Notice that second order distribution functions are always continuous regardless what random variables are considered.

On the way to the uniqueness of the preferred solution produced by the proposed \( L_2 \)-FSD and \( L_2 \)-SSD preference rules we first consider the case with a single contingent event, i.e., a case without uncertainty.

**Theorem 3.2.4.** For two return random variables \( A_1 \) and \( A_2 \) with return outcomes taking values in the interval \([a,b]\) and different distribution functions \( F_1(r) \) and \( F_2(r) \), respectively, with finitely many discontinuous points, the objective function \( g(x) \) has a unique minimum in \( X \).

**Proof.** For the case with two variables the decision space is \( X = \{(x_1, (1-x_1))| x_1 \in [0,1]\} \) and so, the function to be minimized is

\[
g(x) = \int_a^b (F_2(r) - I(r))^2 dr = \int_a^b (x_1(F_1(r) - I(r)) + (1 - x_1)(F_2(r) - I(r))^2 dr
\]

\[= \int_a^b (F_1(r) - I(r))^2 dr \cdot x_1^2 + 2 \int_a^b (F_1(r) - I(r))(F_2(r) - I(r)) dr \cdot x_1 + \int_a^b (F_2(r) - I(r))^2 dr \cdot (1-x_1)^2
\]

Assume that \( h(x_1) = g(x) \) does not have a unique minimum in \([0,1]\), i.e., there are at least two values of \( x_1 \) with function values equal to the minimal function value \( \min x \in \mathbb{R} \), then by convexity of \( g(x) \) from Corollary 3.2.3 there must be a subinterval \( I \) of \([0,1]\) for which the function values of \( h(x_1) \) are all the same, namely \( c \). Therefore for all \( x_1 \in I \)

\[
h(x_1) = \int_a^b (F_1(r) - I(r))^2 dr \cdot x_1^2 + 2 \int_a^b (F_1(r) - I(r))(F_2(r) - I(r)) dr \cdot x_1
\]

\[= \int_a^b (F_1(r) - I(r))^2 dr \cdot x_1^2 + 2 \int_a^b (F_1(r) - I(r))(F_2(r) - I(r)) dr \cdot x_1 + \int_a^b (F_2(r) - I(r))^2 dr \cdot (1-x_1)^2
\]

\[= 0 \cdot x_1^2 + 0 \cdot x_1 + \min = \min
\]
Consequently, we can derive three equations.

\[ \int_{a}^{b} (F_1(r) - I(r))^2 dr - 2 \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr + \int_{a}^{b} (F_2(r) - I(r))^2 dr = 0 \] (2)

\[ 2 \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr - 2 \int_{a}^{b} (F_2(r) - I(r))^2 dr = 0 \] (3)

\[ \int_{a}^{b} (F_2(r) - I(r))^2 dr = \text{min} \] (4)

Using Equations (3) and (4), we obtain

\[ \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr = \text{min} \] (5)

And finally, using Equations (4) and (5) in the Equation (2), we obtain

\[ \int_{a}^{b} (F_1(r) - I(r))^2 dr = \text{min} = \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr = \int_{a}^{b} (F_2(r) - I(r))^2 dr \]

The inner product of two functions \( f, g \) in the \( L_p \)-space (Lebesgue space) associated with the Borel measure on the interval \([a, b]\) is defined as

\[ \langle f, g \rangle = \int_{a}^{b} f(r)g(r) \, dr \]

which induces the norm as

\[ \|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{a}^{b} f^2(r) \, dr} \]

We know that Lebesgue spaces are Hilbert spaces where the Cauchy-Schwarz-inequality holds.

\[ \|F_1 - I\|^2 = \int_{a}^{b} (F_1(r) - I(r))^2 dr = \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr = \langle F_1 - I, F_2 - I \rangle \leq \|F_1 - I\| \|F_2 - I\| \]

implies the inequality \( \|F_1 - I\| \leq \|F_2 - I\| \) and

\[ \|F_2 - I\|^2 = \int_{a}^{b} (F_2(r) - I(r))^2 dr = \int_{a}^{b} (F_1(r) - I(r))(F_2(r) - I(r))dr = \langle F_1 - I, F_2 - I \rangle \leq \|F_1 - I\| \|F_2 - I\| \]

implies the inequality \( \|F_2 - I\| \leq \|F_1 - I\| \) meaning that equality holds. Consequently, if there is
more than one minimum, then the distribution functions $F_1$ and $F_2$ have to fulfill

$$\langle F_1 - I, F_2 - I \rangle = \|F_1 - I\| \|F_2 - I\|$$

By the Cauchy-Schwarz inequality, the above equality holds if and only if $F_1 - I = \lambda (F_2 - I)$ almost surely for some constant $\lambda \in \mathbb{R}$. Since a distribution function is continuous from the right hand side, we can omit the almost surely part. From $\int_a^b (F_1(r) - I(r))^2 dr = \int_a^b (F_2(r) - I(r))^2 dr$ we conclude that $\lambda = 1$ and hence, $F_1 - I$ and $F_2 - I$ are equal, which contradicts the condition that the distribution functions are different and finishes the proof of the theorem.

Notice that this theorem works for both first and second order distribution functions. Next, we extend Theorem 3.2.4 to finitely many return random variables (assets).

**Theorem 3.2.5.** For finitely many return random variables $A_i, i = 1, ..., m$ with return outcomes taking values in the interval $[a, b]$ and with almost, i.e., up to finitely many discontinuous points, continuous distribution functions $F_i(r), i = 1, ..., m$ for which the set $S = \{1, ..., m\}$ can not be subdivided into two disjoint subsets $S_1, S_2 \subset S$ such that for some $v \in \mathbb{R}^m_+$ with $\sum_{i \in S_1} v_i = \sum_{j \in S_2} v_j = 1$,

$$\sum_{i \in S_1} v_i F_i(r) = \sum_{j \in S_2} v_j F_j(r)$$

we refer to this property as “convex dependence”),

the objective function $g(x)$ has a unique minimum in $X$.

**Proof.** Assume that the objective function $g(x) = \int_a^b (F_x(r) - I(r))^2 dr$ has at least two minima given by portfolios $x$ and $y$, then we can consider two return random variables with distribution functions $F_x(r)$ and $F_y(r)$, respectively. By Corollary 3.2.3, any convex combination of $x$ and $y$ minimizes $g(x)$. According to Theorem 3.2.4, we conclude that the distribution functions $F_x(r)$ and $F_y(r)$ are the same.

$$F_x(r) - F_y(r) = (x_1 - y_1) F_1(r) + (x_2 - y_2) F_2(r) + \ldots + (x_m - y_m) F_m(r) = 0$$

where $(x_1 - y_1) + (x_2 - y_2) + \ldots + (x_m - y_m) = \sum_{i=1}^m x_i - \sum_{i=1}^m y_i = 1 - 1 = 0$

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We subdivide the set $S$ in $S_1 = \{ i \in S | x_i - y_i \geq 0 \}$ and $S_2 = \{ i \in S | x_i - y_i < 0 \}$ so that

$$\sum_{i \in S_1} (x_i - y_i) F_i(r) = -\sum_{j \in S_2} (x_j - y_j) F_j(r) \quad (5)$$

Since $\sum_{i \in S_1} (x_i - y_i) = -\sum_{j \in S_2} (x_i - y_i) > 0$, because by assumption, $x$ and $y$ are different portfolios,

$$\frac{\sum_{i \in S_1} (x_i - y_i)}{\sum_{i \in S_1} (x_i - y_i)} = -\frac{\sum_{j \in S_2} (x_j - y_j)}{\sum_{i \in S_1} (x_i - y_i)} = 1$$

and so, by dividing both sides of the Equation (5) by $\sum_{i \in S_1} (x_i - y_i)$ and defining $v_i = \frac{(x_i - y_i)}{\sum_{i \in S_1} (x_i - y_i)}$, $i \in S_1$ and $v_j = -\frac{(x_j - y_j)}{\sum_{i \in S_1} (x_i - y_i)}$, $j \in S_2$, we obtain the proposed “convex dependence”.

Generally speaking, Theorem 3.2.5 says that the minimal solution will be unique if distribution functions of pure strategies are “convex independent”. This condition is usually easy to check by considering the lower limits of triangular distributions or, in general, the lowest return values that are possible. It is easy to see that if all lowest return values are different, then there is no “convex dependence”, since the lowest return values of convex combinations of two disjoint groups of return random variables have to be different. For example, we know from Section 2.2 that for a contracting economy the lowest return values are 0, 0.25 and 0.5 and hence, for a contracting economy, distribution functions are “convex independent”.

Further, we account for finitely many contingencies. According to Section 2.4, for the case with finitely many contingent events, the objective function is the square root of the sum of functions $g(x)$ over all contingencies. Since we are only interested in the uniqueness of the minimal solution and the square-root function is strictly monotone increasing, we can drop the square root. Thus, the question becomes when the sum of functions of the form $g(x)$ has a unique minimum. Notice that since functions of the form $g(x)$ are all convex, the sum of them is convex, too. Besides, according to Ruszczynski [10], the set of minimal solutions of the objective function is convex.

**Theorem 3.2.6.** Let $A_i, i = 1, ..., m$ be return random variables with return outcomes taking values in the interval $[a, b]$ and with distribution functions $F_i(r), i = 1, ..., m$ with finitely many discontinuous points. If for at least one contingent event the distribution functions are “convex independent”, then the objective function for finitely many contingencies has a unique minimum over $X$. 

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Proof. First, consider the single contingency case with two return random variables \( m = 2 \). We know from the proof of Theorem 3.2.4 that each function of the form

\[
g(x) = \alpha x_1^2 + \beta x_1 + \gamma, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{R}, x_1 \in [0, 1]
\]

can only have more than one minimum, if the distribution functions of both asset’s return random variables \( F_1 \) and \( F_2 \) are the same. But by assuming “convex independence”, these are different and the minimum is unique. From the proof of Theorem 3.2.4 we also know that

\[
\alpha = \int_a^b (F_1(r) - I(r))^2 dr - 2 \int_a^b (F_1(r) - I(r))(F_2(r) - I(r)) dr + \int_a^b (F_2(r) - I(r))^2 dr
\]

By the Cauchy-Schwarz inequality \( \alpha \) must be nonnegative, since otherwise

\[
\|F_1 - I\|^2 + \|F_2 - I\|^2 < 2(F_1 - I, F_2 - I) \leq 2\|F_1 - I\\|\|F_2 - I\|
\]

which implies the contradicting result

\[
(\|F_1 - I\| - \|F_2 - I\|)^2 < 0
\]

And if \( \alpha \) is equal to zero, then again by the Cauchy-Schwarz inequality

\[
\|F_1 - I\|^2 + \|F_2 - I\|^2 = 2(F_1 - I, F_2 - I) \leq 2\|F_1 - I\\|\|F_2 - I\|
\]

If the last inequality in the above equation was strict, then we would get the same contradiction as before, which means that equality holds. Hence, by the Cauchy-Schwarz inequality, \( F_1 - I = \lambda(F_2 - I) \) almost surely for some constant \( \lambda \in \mathbb{R} \) and since a distribution function is continuous from the right, we can omit the almost surely part. Because of equality we derive

\[
(\|F_1 - I\| - \|F_2 - I\|)^2 = 0
\]

which is equivalent to saying \( \int_a^b (F_1(r) - I(r))^2 dr = \int_a^b (F_2(r) - I(r))^2 dr \) meaning that \( \lambda = 1 \). As a conclusion, \( \alpha \) being equal to zero implies that \( F_1 \) and \( F_2 \) are the same distribution functions and so, by assuming “convex independence”, \( \alpha \) is strictly positive.

The objective function for finitely many contingencies also has the form \( \alpha^* x_1^2 + \beta^* x_1 + \)
Since the objective function is sum of functions of the form \( g(x) \), the coefficient \( \alpha^* \) is simply the sum of coefficients \( \alpha \) from the \( g(x) \)-functions corresponding to all contingent events considered. If “convex independence” holds for at least one contingent event, then at least one function \( g(x) \) will have coefficient \( \alpha \) being strictly positive. Consequently, the coefficient \( \alpha^* \) of the objective function will be strictly positive and as second derivative of the objective function it will make the objective function strictly convex. Three situations can be distinguished. If the global minimum (i.e., for \( x_1 \in \mathbb{R} \)) of \( \alpha^* x_1^2 + \beta^* x_1 + \gamma^* \) is attained in the interval \([0, 1]\), then by strict convexity there will be a unique minimum in \([0, 1]\). If the global minimum is attained to the left of the interval \([0, 1]\), then the function \( \alpha^* x_1^2 + \beta^* x_1 + \gamma^* \) will be strictly increasing on the interval \([0, 1]\) and the unique minimum will be attained for \( x_1 = 0 \). Similarly, if the global minimum is attained to the right of the interval \([0, 1]\), then the function \( \alpha^* x_1^2 + \beta^* x_1 + \gamma^* \) will be strictly decreasing on the interval \([0, 1]\) and the unique minimum will be attained for \( x_1 = 1 \). This completes the proof for two return random variables.

In the last step, we extend the number of return random variables to finitely many. Suppose that the objective function has more than one minimum, say \( x \) and \( y \). As in proof of Theorem 3.2.5, we change the number of return random variables to two having the same distribution functions as \( x \) and \( y \), respectively. By definition of “convex dependence”, two distribution functions are “convex dependent” if and only if they are equal. Hence, from the two return random variables case we conclude that the distribution functions of \( x \) and \( y \) are “convex dependent” and hence equal for each contingent event. Analogue to the proof of Theorem 3.2.5, we conclude that for each contingent event the distribution functions of \( m \) return random variables are “convex dependent”. Thus, if for at least one contingent event there is no “convex dependence”, then the objective function will have a unique minimum.

We discussed prior to Theorem 3.2.6 that for a contracting economy the distribution functions are “convex independent”. Hence, by Theorem 3.2.6 the objective function for finitely many contingencies including the contracting economy event, e.g. the one considered in Section 2.4, has a unique minimum.

Finally we consider the objective function introduced in Section 2.5.
Theorem 3.2.7. Depending on the uncertain parameter \( u \in [0, 1] \), let \( A_i(u), i = 1, ..., m \) be return random variables with return outcomes taking values in the interval \([a, b]\) and distribution functions \( F_i(u, r), i = 1, ..., m \) with finitely many discontinuous points. Let \( \int_a^b F_i(u, r) \, dr \) be integrable over \( u \in [0, 1] \) for all \( i = 1, ..., m \). If for any subinterval of the interval \([0, 1]\), \( I = [i_1, i_2] \subset [0, 1] \), \( i_1 < i_2 \), the distribution functions are “convex independent” for all \( u \in I \), then the objective function for an interval of uncertainties has a unique minimum over \( X \).

Proof. Considering the two return random variables case \((m = 2)\), for a fixed uncertain parameter \( u \in [0, 1] \)

\[ g(u, x) = \alpha(u)x_1^2 + \beta(u)x_1 + \gamma(u), \text{ where } \alpha(u), \beta(u), \gamma(u) \in \mathbb{R}, x_1 \in [0, 1] \]

We integrate over \( u \) in order to obtain the objective function. Assuming “convex independence” for all uncertainty parameters \( u \in I \), we have \( \int_{i_1}^{i_2} \alpha(u) \, du > 0 \), since otherwise, for some uncertainty values \( u \in I \), the coefficients \( \alpha(u) \) would be equal to zero and thus, contradict the assumption of “convex independence” (see proof of Theorem 3.2.6). We also know from the proof of Theorem 3.2.6 that \( \alpha(u) \geq 0 \) for all \( u \in [0, 1] \). Hence, for the objective function

\[ \int_0^1 g(u, x) \, du = \alpha^*x_1^2 + \beta^*x_1 + \gamma^*, \alpha^*, \beta^*, \gamma^* \in \mathbb{R} \]

the coefficient \( \alpha^* \) is strictly positive, which means that the objective function has a unique minimum in \( X \) (see proof of Theorem 3.2.6).

Similar to the proof of Theorem 3.2.5, we extend the number of return random variables to finitely many. Suppose that the objective function has more than one minimum, say \( x \) and \( y \). We change the number of return random variables to two having the same distribution functions as \( x \) and \( y \). By definition of “convex dependence”, two distribution functions are “convex dependent” if and only if they are equal. So, from the two return random variables case we conclude that for the distribution functions of \( x \) and \( y \) there exists no interval of the form \( I = [i_1, i_2] \subset [0, 1], i_1 < i_2 \) such that for all uncertainty values in \( I \), the distribution functions of \( x \) and \( y \) are “convex independent” and hence, not equal. Analogous to the proof of Theorem 3.2.5, there exists no such interval \( I \) such that for all uncertainty values in \( I \), the distribution functions of \( m \) return random variables are not “convex dependent”, i.e., each such interval contains at least one uncertainty value for which the distribution functions of \( m \) return random variables are “convex dependent”. Thus, we can conclude
that if for any subinterval of the interval \([0, 1], I = [i_1, i_2] \subset [0, 1], i_1 < i_2\) the distribution functions are “convex independent” for all \(u \in I\), then the objective function for an interval of uncertainties has a unique minimum over \(X\).

Consider the example from Section 2.5. The uncertainty value \(u = 0\) represents the contracting economy discussed throughout this section. The three distribution functions for this event are “convex independent”, since the lowest return values of those functions are different, namely 0, 0.25 and 0.5. In the example of Section 2.5 the lowest return values \(u/2, 0.25\) and 0.5 are continuous in \(u\) and so, there certainly exists an interval, e.g. \(I = [0, 0.1]\), for which the distribution functions are “convex independent”, which guarantees a unique minimum by Theorem 3.2.7. In any case with continuous lowest return values, it suffices to find one uncertainty value \(u \in [0, 1]\) for which “convex independence” holds in order to assure a unique preferred solution/minimum.

### 3.3 Approximation theory

In this section we will investigate the approximation approach used in Section 2.3 for both \(L_2\)-FSD and \(L_2\)-SSD preference rule. Our major interest lies in answering the question how accurate the results from the application of the approximated preference rule are and to what extent we can control the error. As in the previous section, we don’t distinguish between \(L_2\)-FSD and \(L_2\)-SSD objective functions, but simply talk about the objective function unless a distinction is necessary.

Let \(f_i, i \in \{1, \ldots, m\}\), be continuous real valued functions defined on the interval \([a, b]\). On a compact set \(X \subset \mathbb{R}^m\), consider the function

\[
g(x) = \int_a^b \left( \sum_{j=1}^m x_j f_j(r) \right)^2 \, dr, \quad x \in X
\]

Function \(g(x)\) can be approximated by the function

\[
g_n(x) = \sum_{i=1}^n \frac{b-a}{n} \left( \sum_{j=1}^m x_j f_j(s^n_i) \right)^2, \quad x \in X
\]

where \((s^n_i)\) is a discretization of the interval \([a, b]\) with stepsize \(\frac{1}{n}(b-a)\), i.e., \(s^n_1 = a < s^n_2 = a + \frac{1}{n}(b-a) < s^n_3 = a + \frac{2}{n}(b-a) < \ldots < s^n_{n+1} = b\).
Theorem 3.3.1. Function $g_n$ converges uniformly to the function $g$, i.e., for any $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$, such that for all $n \geq N$

$$|g_n(x) - g(x)| < \epsilon \text{ for all } x \in X$$

Proof. Consider any sequence $(x^k, r^k) \in X \times [a, b], k \in \mathbb{N}$ converging to $(x, r) \in X \times [a, b]$ for $k$ increasing. We observe that

$$|x_i^k f_i(r^k) - x_i f_i(r)| \leq |x_i^k f_i(r^k) - x_i f_i(r^k)| + |x_i f_i(r^k) - x_i f_i(r)|$$

$$\leq |x_i^k - x_i||f_i(r^k)| + x_i |f_i(r^k) - f_i(r)|$$

Since, by assumption, each function $f_i, i = 1, ..., m$, is continuous on $[a, b]$, it is also bounded, say by a constant $c_i > 0, i = 1, ..., m$, and $|f_i(r^k) - f_i(r)|$ converges to 0 as $r^k$ converges to $r$.

$$|x_i^k f_i(r^k) - x_i f_i(r)| \leq c|x_i^k - x_i| + x_i |f_i(r^k) - f_i(r)| \to 0$$

Hence, each function $x_i f_i, i = 1, ..., m$, is continuous on $X \times [a, b]$, which makes $f$ defined by

$$f(x, r) = \left(\sum_{j=1}^{m} x_j f_j(r)\right)^2$$

also continuous on $X \times [a, b]$. Notice that because $X$ is compact, $X \times [a, b]$ is a compact set, too. Consequently, $f$ is uniformly continuous on $X \times [a, b]$, which means by definition that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|z - \overline{z}\|< \delta \text{ implies } |f(z) - f(\overline{z})| < \frac{\epsilon}{b - a} \text{ for all } z, \overline{z} \in X \times [a, b]$$

This means that for $n > \frac{b - a}{\delta}$

$$|g_n(x) - g(x)| = \sum_{i=1}^{n} \frac{b - a}{n} f(x, s_i^n) - \int_{a}^{b} f(x, r)dr = \sum_{i=1}^{n} \int_{s_{i-1}^{n}}^{s_i^{n}} f(x, s_i^n) - f(x, r)dr$$

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\[
\leq \sum_{i=1}^{n} \int_{s_{r-1}^{n}}^{s_{r}^{n}} \left| f(x, s_{r}^{n}) - f(x, r) \right| dr < \sum_{i=1}^{n} \frac{b-a}{n} \frac{\epsilon}{b-a} = \epsilon \text{ for all } x \in X
\]

We conclude that \( g_n \) converges uniformly to \( g \).

\[\square\]

Suppose that for a single contingency, we have \( m - 1 \) assets with return random variables given by triangular distribution functions \( F_i(r), i = 1, \ldots, m - 1 \) with return outcomes taking values in the interval \([a, b]\). Notice that first order distribution functions of those assets are continuous. Second order distribution functions, however, are always continuous independent of what random variables we consider. We set \( f_i(r) = F_i(r), i = 1, \ldots, m - 1 \). Since these functions are continuous, the lower envelope function \( f_m(r) = \min_{i=1,\ldots,m-1} f_i(r) \) representing the ideal distribution is continuous, too. The set \( X = \{(x_1, \ldots, x_{m-1}, -1) \in \mathbb{R}_{+}^{m-1} \times \{-1\} | \sum_{i=1}^{m-1} x_i = 1\} \) is convex, closed and bounded and hence, compact. By setting up the function \( g(x) \) this way we get

\[
g(x) = \int_{a}^{b} \left( \sum_{j=1}^{m-1} x_j F_j(r) - \min_{i=1,\ldots,m-1} f_i(r) \right)^2 dr, \ x \in X
\]

which is exactly the square of the \( L_2 \)-FSD or \( L_2 \)-SSD objective function, respectively, from Section 2.3 for a single contingency. Theorem 3.3.1 tells us that the approximation

\[
\sqrt{\sum_{i=1}^{n} \frac{b-a}{n} (\sum_{j=1}^{m-1} x_j F_j(s_{r}^{n}) - \min_{i=1,\ldots,m-1} f_i(s_{r}^{n}))}^2, \ x \in X
\]

converges uniformly to the exact objective function for a single contingent event.

For finitely many contingencies, the objective function is the square root of the sum of functions of the same form as \( g(x) \). By definition of uniform convergence, if functions \( g_n(x) \) converge uniformly to \( g(x) \) and functions \( h_n(x) \) converge uniformly to \( h(x) \) then so do functions \( g_n(x) + h_n(x) \) to \( g(x) + h(x) \) as \( n \) increases. For any \( \epsilon > 0 \), there exist integers \( N_1, N_2 \in \mathbb{N} \) such that for all \( n \geq \max(N_1, N_2) \)

\[
|g_n(x) - g(x)| < \frac{\epsilon}{2} \text{ and } |h_n(x) - h(x)| < \frac{\epsilon}{2} \text{ for all } x \in X
\]
This implies that for all \( n \geq \max(N_1, N_2) \)

\[ |g_n(x) + h_n(x) - (g(x) + h(x))| \leq |g_n(x) - g(x)| + |h_n(x) - h(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

for all \( x \in X \)

Neither taking square nor square root changes the minimum of a positive function, since both operations are monotone increasing. Therefore, for the investigation of minima, we consider the square objective function \( G(x) \), which is simply the sum of functions of the same form as \( g(x) \), instead of the objective function from Section 2.3. This function is uniformly approximated by sums of functions of the form \( g_n(x) \) denoted by \( G_n(x) \).

**Theorem 3.3.2.** Let \( x^n \in X \) be minima of functions \( G_n(x) \) defined on \( X \) and \( x^* \in X \) be a unique minimum of a continuous function \( G(x) \) defined on \( X \), where \( X \subset \mathbb{R}^m \) is a compact set. If \( G_n(x) \) is uniformly converging to \( G(x) \), then \( x^n \) converges to \( x^* \) as \( n \) goes to infinity.

**Proof.** We know from uniform continuity, that \( G_n(x^*) \) converges to \( G(x^*) \) as \( n \) goes to infinity. Because of minimality of \( x^n \) for \( G_n(x) \), \( G_n(x^n) \leq G_n(x^*) \). Thus, we conclude that for each \( \epsilon > 0 \) there is some integer \( N \in \mathbb{N} \) such that

\[ G_n(x^n) \leq G(x^*) + \epsilon \]

for all \( n \geq N \)

\( G(x) \) is continuous on \( X \) and has a unique minimum \( x^* \). Hence, for any \( \delta > 0 \) there is a \( \tau > 0 \) such that

\[ S_\tau = \{ x \in X | G(x) < \tau + G(x^*) \} \subset (U_\delta(x^*) \cap X) \]

where \( U_\delta(x^*) \) is an open ball in \( \mathbb{R}^m \) around \( x^* \) with radius \( \delta \).

Suppose the last statement is not true, then for some \( \delta > 0 \) there exists a sequence \( (y^n)_{n \in \mathbb{N}} \subset X \setminus U_\delta(x^*) \) with \( G(y^n) \) converging to \( G(x^*) \) as \( n \) goes to infinity. \( X \) is compact and so, there exists \( K \subset \mathbb{N} \) such that the subsequence \( (y^n)_{n \in K} \) converges to a point \( y \in X \setminus U_\delta(x^*) \), because \( X \setminus U_\delta(x^*) \) is compact, too. Since \( G(x) \) is continuous, \( G(y) \) equals \( G(x^*) \) even though \( y \neq x^* \), which contradicts the uniqueness of the minimum of \( G(x) \).

Hence, for any \( \delta > 0 \) there is some \( \tau > 0 \) such that \( S_\tau \subset U_\delta(x^*) \). By uniform convergence, we can choose an integer \( N \in \mathbb{N} \) such that \( |G_n(x) - G(x)| < \frac{\epsilon}{2} \) for all \( x \in X, n \geq N \). Consequently,
for any \( x \notin U_\delta(x^*) \)

\[
G(x) \geq G(x^*) + \tau \text{ implies } G_n(x) > G(x^*) + \frac{\tau}{2} \text{ for all } n \geq N
\]

and

\[
G_n(x^*) < G(x^*) + \frac{\tau}{2} \text{ for all } n \geq N
\]

This implies that for any \( \delta > 0 \) there is always an integer \( N \in \mathbb{N} \) such that for any \( x \notin U_\delta(x^*) \)

\[
G_n(x^*) < G(x) \text{ for all } n \geq N
\]

stating that the minima \( x^n \) of \( G_n(x) \) lie inside the ball \( U_\delta(x^*) \) for all \( n \geq N \). For decreasing \( \delta > 0 \), this ball is shrinking and so, \( x^n \) is converging to \( x^* \).

\[ \square \]

Both \( L_2 \)-FSD and \( L_2 \)-SSD preference rule provide all conditions needed for Theorem 3.3.2 but uniqueness of the minimum. Prior to the Theorem 3.3.2 we discussed functions \( g(x) \) and \( g_n(x) \) defined on a compact set \( X \). Those functions are continuous in \( x \) by definition. For that reason, functions \( G(x) \) and \( G_n(x) \) as sums of functions of the form \( g(x) \) and \( g_n(x) \), respectively, are continuous as well. Functions \( G_n(x) \) are also uniformly converging to \( G(x) \). We make use of Section 3.2 to ensure the uniqueness of minimum of \( G(x) \). We have seen in Section 3.2 that the example from Section 2.3 fulfills the uniqueness condition. Hence, by Theorem 3.3.2 the minima of objective function approximations from Section 2.3 are converging to the minimum of the objective function. This behavior is observable on the computational results in Section 2.4.

In order to get more control of the approximation approach we consider positive real valued Lipschitz-continuous functions \( f_i \), \( i \in \{1, \ldots, m\} \), defined on the interval \([a, b]\). Since each function \( f_i \) is Lipschitz-continuous with constant \( c_i \), \( i \in \{1, \ldots, m\} \), it must also be bounded from above by \( u_i \in \mathbb{R}_+ \), \( i = 1, \ldots, n \). For \( (x,r), (\overline{x}, \overline{r}) \in X \times [a,b] \), where \( X = \{(x_1, \ldots, x_m, -1) \in \mathbb{R}_+^{m-1} \times \{-1\} | \sum_{i=1}^{m-1} x_i = 1 \} \),

\[
|x_i f_i(r) - \overline{x}_i f_i(\overline{r})| \leq |x_i f_i(r) - \overline{x}_i f_i(r) + \overline{x}_i f_i(r) - \overline{x}_i f_i(\overline{r})| \leq |f_i(r)||x_i - \overline{x}_i| + |\overline{x}_i||f_i(r) - f_i(\overline{r})|
\]

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\[ \leq (u_i + c_i)\| (x_1, x_2, \ldots, x_n, r) - (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}, r) \| \]

This makes the function \( h(x, r) = \sum_{i=1}^{m} x_i f_i(r) : X \times [a, b] \to \mathbb{R} \) Lipschitz-continuous with constant \( c = \sum_{i=1}^{m} (u_i + c_i) \) and bounded from above by \( u = \max_{i \in \{1, \ldots, m\}} \{ u_i \} \) and from below by \(-u\). For \( z, \overline{z} \in X \times [a, b] \)

\[ |f(z) - f(\overline{z})| = |h(z) + h(\overline{z})||h(z) - h(\overline{z})| \leq 2uc\|z - \overline{z}\| \]

We conclude that \( f = h^2 \) is also Lipschitz-continuous with constant \( C = 2uc \). Similar to the proof of Theorem 3.3.1 choose any \( \epsilon > 0 \), then for \( n > \frac{2(b-a)^2C}{\epsilon} \)

\[ |g_n(x) - g(x)| \leq \sum_{i=1}^{n} \int_{s_{i-1}^n}^{s_i^n} |f(x, s_i^n) - f(x, r)|dr \leq \sum_{i=1}^{n} \int_{s_{i-1}^n}^{s_i^n} 2uc\|(x, s_i^n) - (x, r)\|dr \]

\[ \leq \sum_{i=1}^{n} \frac{b-a}{n} 2uc \frac{b-a}{n} \leq \sum_{i=1}^{n} C \left( \frac{b-a}{n} \right)^2 < \frac{\epsilon}{2} \] for all \( x \in X \)

Consequently, for \( n > \frac{2(b-a)^2C}{\epsilon^2} \)

\[ g(x^n) < g_n(x^n) + \frac{\epsilon}{2} \leq g_n(x^n) + \frac{\epsilon}{2} < g(x^*) + \epsilon \text{ implies } 0 \leq g(x^n) - g(x^*) < \epsilon \]

Notice that for a single contingency, function \( g(x) \) represents the square of the objective function and yields the same minima as the objective function itself. If \( g(x) \) was the objective function then the above formula would tell us how large \( n \) should be chosen in order to get an approximated minimum, where the objective function value deviates at most by \( \epsilon \) from the exact minimal objective function value. For the usual objective function \( \sqrt{g(x)} \), the above formula has to be adjusted. Given two positive numbers \( p, q \in \mathbb{R}_+ \)

\[ 0 \leq p - q < \epsilon \text{ implies } 0 \leq \sqrt{p} - \sqrt{q} < \sqrt{\epsilon} \]

because the slope of the square root function is decreasing for larger values. Thus, we derive from the above formula that the objective function value of the approximated minimum deviates at most by \( \epsilon \) from the exact minimal objective function value for \( n > \frac{2(b-a)^2C}{\epsilon^2} \).
For finitely many contingencies, the objective function is the square root of the sum of functions of the form \( g(x) \). Let \( l \in \mathbb{N} \) be the number of contingencies and for a fixed contingent event \( j \in \{1, \ldots, l\} \) let function \( f \) be Lipschitz-continuous with constant \( C_j \). For a maximal deviation of \( \epsilon \) from the minimal objective function value, it can be shown similar to the previous formula that it suffices to have

\[
n > \max_{j \in \{1, \ldots, l\}} \frac{2l(b-a)^2C_j}{\epsilon^2}
\]

Recall the example from Section 2.3 where we had \( l = 3 \) contingencies and \( m-1 = 3 \) assets to pick from. For the \( L_2 \)-SSD objective function, we have to consider second order distribution functions which are always Lipschitz-continuous with constant 1, since first order distribution functions are bounded by 1. Because of being antiderivative of first order distribution functions, second order distribution functions are also bounded by \( b - a = 1 - 0 = 1 \). The ideal distribution function being the lower envelope shares all properties of the second order distribution functions, i.e., it is Lipschitz-continuous with constant 1 and bounded by 1. Hence, for each contingent event \( j \in \{1, 2, 3\} \) function \( f \) is Lipschitz-continuous with constant \( C_j = 2 \cdot 1 \cdot \sum_{i=1}^{4} (1 + 1) = 16 \). Finally, in order to assure maximal deviation of \( \epsilon = 0.01 \) it suffices to choose

\[
n > \max_{j \in \{1,2,3\}} \frac{2 \cdot 3 \cdot (1 - 0)^2 \cdot 16}{0.01^2} = 960000
\]

The formula for the \( L_2 \)-SSD preference rule can always be applied provided that the ideal distribution function is Lipschitz-continuous on the interval \([a, b]\) which is the case for the lower envelope. If distribution functions \( f_i(r), i \in \{1, ..., m\} \), are not continuous for finitely many points, which might happen for first order distribution functions or ideals, then the formula can be further adjusted by choosing \( n \) big enough such that finitely many terms \( \int_{s^n_{i-1}}^{s^n_i} |f(x, s^n_i) - f(x, r)|dr \) in the Equation (6) covering areas with discontinuous points are small enough.
Chapter 4

Interpretation

4.1 Investigation of the future performance of the optimal solution

In Section 2.5 it turned out that the pure strategy investing all the money in the conservative asset is the most preferred one. The next question that occurs is ”What does the performance of this strategy look like in the future?” One possible way of comparing the value of a portfolio in the long term is to consider its discounted value a.k.a. the present value. For simplification, we assume that for all the time in the future the uncertainty parameter $u$ (you can think of it as the prime interest rate) is constant and that the optimal portfolio’s return $R$, which is aleatory uncertain, stays the same all the time, too. For the discount parameter $d$, the discounted value $DV$ of a portfolio with constant return $R$ is computed using the formula

\[ DV = \int_{0}^{\infty} e^{-dt} R \, dt = | - \frac{1}{d} e^{-dt} R |_{0}^{\infty} = \frac{1}{d} R \]

Notice that the discounted value is a random variable, since the return $R$ is a random variable. The density of the second asset’s return is dependent on the values of $u$. Thus, $DV$’s density also depends on $u$. Indeed, since the return is assumed to be constant all the time through, the discounted value is equal to $\frac{1}{d}$ times the return. Supposing that the discounting parameter is $d = 0.06$, we compute the mean and variance of $DV$, both depending on the values of $u$. Figure 4.1 displays their graphs with the horizontal axis representing the values of $u$.  

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We see in Figure 4.1 that the mean discounted value of our preferred portfolio is at least 9 units. The variance in Figure 4.1, which represents the deviation from the mean discounted value, stays in the area between 6.5 and 7.5 indicating that deviation does not change much for all uncertainties. But we get a much better interpretation by considering the Value at Risk of the discounted value at the confidence level $\alpha = 0.05$. Figure 4.2 shows that there is at most a 5% chance that the discounted value drops under the 5.3 units mark. With this knowledge, an economist can decide whether to invest his money into the proposed preferred portfolio or not.

![Figure 4.1: Performance of the discounted value: (a) mean (b) variance.](image_url)

![Figure 4.2: VaR of the discounted value at confidence level $\alpha = 0.05$.](image_url)
Chapter 5

Conclusions and Discussion

The study shows an approach that accounts for both risk and uncertainty. We have established preference rules that can be applied to bounded return random variables with distribution functions having up to finitely many discontinuous points and yield a unique preferred solution under certain circumstances (see "convex independence" in Section 3.2). We have applied those preference rules to triangular distributions, which are typically used when sample data is sparse, and the computational part worked out very well. However, we had to know what distribution functions look like. In case of triangular distributions, the first order distribution functions are quadratic polynomials and the second order distribution functions are cubic polynomials, which made it easy to carry out the integration. It is important to know how to integrate the distribution functions. If we don’t know how to integrate them, but know how to evaluate the distribution functions, then we can still use the approximation approach for the case with finitely many uncertainties. The major disadvantage of the approximation approach is that it takes much longer to evaluate the objective function, especially if we choose the error to be very small. For that reason, discrete random variables with more than just a few outcomes are problematic. Hence, even though in theory the proposed approach works very well for discrete distributions, from computational point of view a different approach might be considered.

An applicant of the proposed approach has to be aware of what interval he integrates over. We suggest to integrate over the smallest interval \([a, b]\) containing every possible outcome of all return random variables. According to Section 3.1 this interval suffices to assure the proven equivalence between stochastic dominance and Value at Risk or Conditional Value at Risk, respectively. If you
extend this interval to the left, then the preferred solution will not change, because for values smaller
than \( a \), all distribution functions have function values equal to 0. If you extend this interval to the
right, then the preferred solution will not change for the \( L_2 \)-FSD preference rule, because for values
larger than \( b \), all first order distribution functions have function values equal to 1. For the \( L_2 \)-SSD
preference rule, however, the preferred solution might change, because at point \( b \) function values
of second order distribution functions can be different and increase linearly with slope 1 for values
larger than \( b \).

A decision maker might want to participate in the decision process, e.g., if he feels that
some contingency events are more likely. That kind of participation can be imposed by introducing
positive weights, e.g., the functions of the form \( g(x) \) in Section 3.2 could be multiplied with some
strictly positive scalars, which does not affect the theoretical results.

5.1 Recommendations for further research

We have seen how well the proposed approach performs for triangular distributions. Based
on the established theory a computer scientist or engineer might take into consideration automating
the search for a preferred solution. The issues involved in this process have to be investigated.

Our approach is set up for bounded return random variables. Theoretically, it also works
for unbounded return random variables, since, except for the approximation approach, we didn’t
require boundedness. Nevertheless, unbounded return random variables might cause other issues,
which have to be investigated as well.

Overall, our approach is only one of the many possible. We mentioned that it has some
computational issues with discrete distributions. Alternative preference rules have to be introduced
to see how good or bad it really is.
Appendices
Appendix A  MATLAB-Programs

The purpose of this Appendix is to help the reader to understand and repeat the computations that have been made in this thesis. All MATLAB-functions that have been used, in particular, in Chapters 2 and 4, can be downloaded at the website

http://www.math.clemson.edu/grad_students/dimitrn.htm

We will further explain how to use those programs and what they stand for.

First, consider the functions \texttt{fsd\_cont1.m}, \texttt{fsd\_cont2.m}, \texttt{fsd\_cont3.m}, \texttt{fsd\_cont4.m}, \texttt{fsd\_ideal\_for\_cont.m} and \texttt{fsd\_min.m}. These functions have been used to compute the approximated \(L_1\)-FSD and \(L_2\)-FSD objective function values. As depicted in Program 1, MATLAB-function \texttt{fsd\_cont4.m} takes vectors of the form \(x = [x(1), x(2), x(3)]\) representing portfolios and returns a vector, with entries equal to the first order distribution function values of portfolio \(x\) for contingency \(C_4\) at grid points given by the vector \(v\). Functions \texttt{fsd\_cont1.m}, \texttt{fsd\_cont2.m} and \texttt{fsd\_cont3.m} have the same functionality. The only difference is that they compute first order distribution function values for contingencies \(C_1\), \(C_2\) and \(C_3\). Program 2 displays the MATLAB-function \texttt{fsd\_ideal\_for\_cont.m} which computes for contingency \(C_{\text{cont}}\) the vector, with entries equal to the values of the lower envelope of all first order distribution functions, \(I\), at grid points given by the vector \(v\) in MATLAB-functions \texttt{fsd\_cont1.m}, ..., \texttt{fsd\_cont4.m}. Using all these functions, we compute the approximated \(L_1\)-FSD and \(L_2\)-FSD objective function values of a portfolio \(x\) by the MATLAB-function \texttt{fsd\_min.m} depicted in Program 3. Depending on what discretization we choose, the division operation has to be adjusted accordingly. In order to compute the objective function for less contingencies, we simply uncomment the lines in \texttt{fsd\_min.m} representing the contingencies that we want to drop. The MATLAB-function \texttt{fsd\_min.m} is called this way because this is the function to be minimized with \texttt{fmincon} in the approximated approach described in Section 2.4.

The MATLAB-functions \texttt{ssd\_cont1.m}, \texttt{ssd\_cont2.m}, \texttt{ssd\_cont3.m}, \texttt{ssd\_cont4.m}, \texttt{ssd\_ideal\_for\_cont.m} and \texttt{ssd\_min.m} are used the same way as the ones for first order stochastic dominance. The only difference is that, in order to apply the approximated \(L_1\)-SSD and \(L_2\)-SSD preference rules, they evaluate second order distribution function values instead of first order distribution function values.
Program 1 \textit{f\textunderscore fsd\textunderscore cont4.m} function in MATLAB.

\begin{verbatim}
function Fx=f\_fsd\_cont4(x)

v=(0:59999)/60000;
Fx=x(1)*(2/3<v);
Fx=Fx+x(2)*(5/12<v);
Fx=Fx+x(3)*(1/2<v).*(4*v.^2-4*v+1);
end
\end{verbatim}

Program 2 \textit{fsd\_ideal\_for\_cont.m} function in MATLAB.

\begin{verbatim}
function m = fsd\_ideal\_for\_cont( cont )
switch cont
    case {1}
        m = min(f\_fsd\_cont1([1;0;0]), min(f\_fsd\_cont1([0;1;0]), f\_fsd\_cont1([0;0;1])));
    case {2}
        m = min(f\_fsd\_cont2([1;0;0]), min(f\_fsd\_cont2([0;1;0]), f\_fsd\_cont2([0;0;1])));
    case {3}
        m = min(f\_fsd\_cont3([1;0;0]), min(f\_fsd\_cont3([0;1;0]), f\_fsd\_cont3([0;0;1])));
    case {4}
        m = min(f\_fsd\_cont4([1;0;0]), min(f\_fsd\_cont4([0;1;0]), f\_fsd\_cont4([0;0;1])));
    otherwise
        m = 0;
end
end
\end{verbatim}

Program 3 \textit{fsd\_min.m} function in MATLAB.

\begin{verbatim}
function f = fsd\_min(x)

% L1-FSD
f = (sum((f\_fsd\_cont1(x)-fsd\_ideal\_for\_cont(1))))^2;
f = f + (sum((f\_fsd\_cont2(x)-fsd\_ideal\_for\_cont(2))))^2;
f = f + (sum((f\_fsd\_cont3(x)-fsd\_ideal\_for\_cont(3))))^2;
f = f + (sum((f\_fsd\_cont4(x)-fsd\_ideal\_for\_cont(4))))^2;
end
f = sqrt(f/3600000000);

% L2-FSD
%f = (sum((f\_fsd\_cont1(x)-fsd\_ideal\_for\_cont(1)).^2));
%f = f + (sum((f\_fsd\_cont2(x)-fsd\_ideal\_for\_cont(2)).^2));
%f = f + (sum((f\_fsd\_cont3(x)-fsd\_ideal\_for\_cont(3)).^2));
%f = f + (sum((f\_fsd\_cont4(x)-fsd\_ideal\_for\_cont(4)).^2));
%f = sqrt(f/60000);
end
\end{verbatim}
The MATLAB-functions `exact_fsd_min.m` and `exact_ssd_min.m` compute the exact $L_2$-FSD objective function values or $L_2$-SSD objective function values, respectively, for portfolio vectors of the form $x = [x(1), x(2), x(3)]$. They are used in Section 2.4 for minimization with `fmincon`. In order to compute the coefficients of the objective functions displayed in Tables 2.3 and 2.5, we used predefined MATLAB-functions `conv`, `polyval` and `polyint`. We show in Program 4 how to compute the coefficient $\int_{0}^{1} 2F_{(1,0,0)}^{2}(r)F_{(0,1,0)}(r)dr$ for contingency $C_1$. For the computation of the lower envelope, we had to know where distribution functions intersect. Hence, we had to use the predefined MATLAB-function `roots`. The predefined MATLAB-function `rats` was used to find exact fraction representations of the coefficients. For the $L_2$-SSD objective function, it was difficult to find them for every coefficient. Therefore, we stored the coefficients without this conversion inside the vectors $c_{1ssd}$, $c_{2ssd}$, $c_{3ssd}$ and $c_{4ssd}$. It is not important to make the conversion, though. We simply wanted to know what the exact coefficients look like, since we integrate polynomials over intervals with fractional endpoints.

For the computational results in Table 2.7 we used Maple to do the integration. We will not provide a Maple-file. However, we will explain how to do those computations on the example of coefficient $\int_{0}^{1} c_{x_2}^2 du$. Consider the general triangular first order distribution function depending on the lower limits $\chi$, modes $\psi$ and upper limits $\omega$.

$$F_{\chi,\psi,\omega}(r) = \begin{cases} 
0 & \text{for } x \leq \chi \\
\frac{(r-\chi)^2}{(\omega-\chi)(\psi-\chi)} & \text{for } \chi \leq x \leq \psi \\
1 - \frac{(\omega-r)^2}{(\omega-\chi)(\omega-\psi)} & \text{for } \psi \leq x \leq \omega \\
1 & \text{for } \omega \leq x 
\end{cases}$$

First, substitute the nodes $\chi$, $\psi$ and $\omega$ by functions of uncertain parameter $u \in [0,1]$. Since we compute the coefficient of $x_2^2$ and $x_2$ is the proportion of capital invested in the conservative asset, we substitute $\chi$ by $\frac{1}{4}$, $\psi$ by $u^3 - 3u^2 + \frac{3}{4}u + \frac{3}{8}$ and $\omega$ by 1 (see Section 2.5) to obtain the first order distribution function $F_2(r, u)$ depending on uncertain parameter $u \in [0,1]$. We know from Sections 2.4 and 2.5 that $c_{x_2}^2 = \int_{0}^{1} (F_2(r, u))^2 dr$. Hence, we square $F_2(r, u)$ and carry out the integration in Maple with respect to $r$ over the interval $[a, b] = [0,1]$. Finally, we integrate the resulting function with respect to $u$ over the interval $[0,1]$. The MATLAB-functions `fsd_maple_fmin` and `ssd_maple_fmin` compute the objective function values for the $L_2$-FSD and $L_2$-SSD preference rules.
**Program 4** Computation of $\int_0^1 2F_{(1,0,0)}(r)F_{(0,1,0)}(r)dr$ for contingency $C_1$ in MATLAB.

\[ p1a = [0 0 0] \]
\[ p1b = [0 0 1] \]
\[ p2a = [0 0 0] \]
\[ p2b = [32 -16 2]/3 \]
\[ p2c = [-32 64 -17]/15 \]

\[
i = (\text{polyval}(\text{polyint}(\text{conv}(p1a,p2a)),1/4) - \text{polyval}(\text{polyint}(\text{conv}(p1a,p2b)),0))
\]
\[
i = i + (\text{polyval}(\text{polyint}(\text{conv}(p1a,p2b)),3/8) - \text{polyval}(\text{polyint}(\text{conv}(p1a,p2b)),1/4))
\]
\[
i = i + (\text{polyval}(\text{polyint}(\text{conv}(p1a,p2c)),1/2) - \text{polyval}(\text{polyint}(\text{conv}(p1a,p2c)),3/8))
\]
\[
i = i + (\text{polyval}(\text{polyint}(\text{conv}(p1b,p2c)),1) - \text{polyval}(\text{polyint}(\text{conv}(p1b,p2c)),1/2))
\]
\[
\text{answer} = \text{rats}(2*i1,30)
\]

for an interval of uncertainties, respectively. These functions have been used for minimization in Section 2.5.

In Chapter 4 we used MATLAB-functions `mean.m`, `variance.m` and `VaR.m` to display the mean, variance and Value at Risk at confidence level $\alpha = 0.05$ of the discounted value. The discounted value is a multiple of the return’s random variable of the conservative asset. Its mean and variance have been computed in Maple from the first two moments of the return's random variable by using the general density function of the triangular distribution

\[
f(r|\chi,\psi,\omega) = \begin{cases} 
0 & \text{for } x \leq \chi \\
\frac{2(r-\chi)}{2(\omega-\chi)} & \text{for } \chi \leq x \leq \psi \\
\frac{2(\omega-r)}{(\omega-\chi)(\omega-\psi)} & \text{for } \psi \leq x \leq \omega \\
0 & \text{for } \omega \leq x 
\end{cases}
\]

The Value at Risk was computed by taking the inverse of $F_2(r, u)$ with respect to $r$ and evaluating the resulting function at $\alpha = 0.05$. 

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Bibliography


