Aperture Coupling and Penetration in Various Configurations

Jason Keen
Clemson University, jkeen@clemson.edu

Follow this and additional works at: https://tigerprints.clemson.edu/all_dissertations
Part of the Electrical and Computer Engineering Commons

Recommended Citation
https://tigerprints.clemson.edu/all_dissertations/300

This Dissertation is brought to you for free and open access by the Dissertations at TigerPrints. It has been accepted for inclusion in All Dissertations by an authorized administrator of TigerPrints. For more information, please contact kokeefe@clemson.edu.
APERTURE COUPLING AND PENETRATION IN VARIOUS CONFIGURATIONS

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Electrical Engineering

by
Jason M. Keen
December 2008

Accepted By:
Darren M. Dawson, Committee Chair
J. Bruce Rafert
Robert L. Taylor
Peter A. Barnes
ABSTRACT

The problem of a slot in a perfectly conducting surface is addressed for a variety of configurations using both integral equation and transmission line techniques. The use of Bethe hole theory to model short slots is discussed and utilized where appropriate. Primary problems of discussion are a slot array, wires coupling through slots in a ground plane, and wires coupling through slots in a bent ground plane. Additionally, the use of the extended BLT formulation for incorporation of slot effects in a transmission line problem is addressed. In this manuscript, the work of Bethe is extended to include cross-aperture coupling for the first time. Further, techniques for coupling Bethe hole theory with the method of moments are presented. Other major contributions include method of moments formulations for apertures in wedge geometries.
DEDICATION

I dedicate this work to my daughter Elizabeth and my son Connor. Every day is made brighter through their presence in my life.
This work would not have been possible without the full support of many people. Drs. Chalmers Butler and Fred Tesche provided insights, questions, and arguments that taught me more than any class. Dr. Frankie Felder along with the rest of the graduate school provided me the opportunity to move this work forward. My parents Blaise and Cheryl Keen gave me a push when I needed it most. Finally, I wish to acknowledge my wife Amanda. Without her and her loving support, I would be lost in the world.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE PAGE</td>
<td>i</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>DEDICATION</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>viii</td>
</tr>
<tr>
<td>PREFACE</td>
<td>1</td>
</tr>
<tr>
<td>ALTERNATE METHODS OF CALCULATING THE FIELDS FROM ARRAYS OF NARROW SLOTS</td>
<td>2</td>
</tr>
<tr>
<td>Introduction</td>
<td>2</td>
</tr>
<tr>
<td>Integral Equation Formulation</td>
<td>5</td>
</tr>
<tr>
<td>Reduction to Polarizabilities</td>
<td>6</td>
</tr>
<tr>
<td>Slot Arrays</td>
<td>10</td>
</tr>
<tr>
<td>Results</td>
<td>19</td>
</tr>
<tr>
<td>Conclusion</td>
<td>25</td>
</tr>
<tr>
<td>References Cited</td>
<td>25</td>
</tr>
<tr>
<td>SOLUTION TECHNIQUES FOR WIRES COUPLED THROUGH SLOTS IN CONDUCTING GROUND PLANES</td>
<td>35</td>
</tr>
<tr>
<td>Introduction</td>
<td>35</td>
</tr>
<tr>
<td>Structure Description</td>
<td>36</td>
</tr>
<tr>
<td>Integral Equation Formulations</td>
<td>38</td>
</tr>
<tr>
<td>Simple Transmission Line Formulation</td>
<td>39</td>
</tr>
<tr>
<td>Modified Transmission Line Formulation</td>
<td>41</td>
</tr>
<tr>
<td>Results</td>
<td>43</td>
</tr>
<tr>
<td>Conclusion</td>
<td>47</td>
</tr>
<tr>
<td>References Cited</td>
<td>48</td>
</tr>
<tr>
<td>ON THE INTEGRAL EQUATION SOLUTION OF A WIRE PARALLEL TO A BEND IN A CONDUCTING GROUND PLANE</td>
<td>60</td>
</tr>
<tr>
<td>Introduction</td>
<td>60</td>
</tr>
<tr>
<td>AN INTEGRAL EQUATION SOLUTION FOR TWO WIRES COUPLED THROUGH A SLOT IN A BENT GROUND PLANE</td>
<td>80</td>
</tr>
<tr>
<td>Introduction</td>
<td>80</td>
</tr>
<tr>
<td>Potential-Based Green’s Function</td>
<td>82</td>
</tr>
<tr>
<td>Integral Equation</td>
<td>87</td>
</tr>
<tr>
<td>Results</td>
<td>100</td>
</tr>
<tr>
<td>Conclusion</td>
<td>101</td>
</tr>
<tr>
<td>References Cited</td>
<td>102</td>
</tr>
</tbody>
</table>

| ON THE USE OF THE BLT EQUATION FOR THE COUPLING OF SLOTS TO TRANSMISSION LINES | 111 |
| Introduction | 111 |
| BLT Development | 112 |
| Application Specialization | 117 |
| Results | 122 |
| Conclusion | 124 |
| References Cited | 124 |

<p>| CONCLUSIONS | 132 |</p>
<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Case 1: Percentage Difference in Effective Dipole Moments between Solution Techniques</td>
<td>31</td>
</tr>
<tr>
<td>2</td>
<td>Case 2: Percentage Difference in Effective Dipole Moments between Solution Techniques</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>Case 3: Percentage Difference in Effective Dipole Moments between Solution Techniques</td>
<td>32</td>
</tr>
<tr>
<td>4</td>
<td>Case 4: Percentage Difference in Effective Dipole Moments between Solution Techniques</td>
<td>33</td>
</tr>
<tr>
<td>5</td>
<td>Modified Case 4: Percentage Difference in Effective Dipole Moments between Solution Techniques</td>
<td>34</td>
</tr>
<tr>
<td>6</td>
<td>Physical Parameters for Measured Data</td>
<td>58</td>
</tr>
<tr>
<td>7</td>
<td>Physical Parameters for Parametric Study Data</td>
<td>58</td>
</tr>
<tr>
<td>8</td>
<td>Physical Parameters for Small Aperture Solution Data</td>
<td>59</td>
</tr>
<tr>
<td>9</td>
<td>Physical Parameters for Small Aperture Solution Data</td>
<td>131</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>Array Layout for Cases 1 and 2</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>Array Layout for Case 3</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>Array Layout for Case 4</td>
<td>29</td>
</tr>
<tr>
<td>4</td>
<td>Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 2 array.</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole</td>
<td></td>
</tr>
<tr>
<td></td>
<td>moments with no coupling. Excitation is a normalized plane wave.</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 2 array.</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole</td>
<td></td>
</tr>
<tr>
<td></td>
<td>moments with no coupling. Excitation is a normalized plane wave.</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 4 array.</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole</td>
<td></td>
</tr>
<tr>
<td></td>
<td>moments with no coupling. Excitation is a normalized plane wave.</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 4 array.</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole</td>
<td></td>
</tr>
<tr>
<td></td>
<td>moments with no coupling. Excitation is a normalized plane wave.</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>A sketch of two wires separated by a slotted ground plane.</td>
<td>50</td>
</tr>
<tr>
<td>9</td>
<td>A sketch of the upper side equivalent model for two wires separated by a slotted ground</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>plane.</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Real Part of $Y_{11}$ for the Physical Parameters of Table 6: Measurement/Computation Plots.</td>
<td>51</td>
</tr>
<tr>
<td>11</td>
<td>Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 6: Measurement/Computation</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>Plots</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Real Part of $Y_{21}$ for the Physical Parameters of Table 6: Measurement/Computation Plots.</td>
<td>52</td>
</tr>
<tr>
<td>13</td>
<td>Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 6: Measurement/Computation</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>Plots</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Real Part of $Y_{11}$ for the Physical Parameters of Table 7: Integral Equation Solution,</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>Parametric Study Plots</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 7: Integral Equation Solution,</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>Parametric Study Plots</td>
<td></td>
</tr>
</tbody>
</table>
16 Real Part of $Y_{21}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots ........................................ 54
17 Real Part of $Y_{21}$ for the Physical Parameters of Table 7: Transmission Line Solution, Parametric Study Plots ........................................ 54
18 Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots ........................................ 55
19 Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 7: Transmission Line Solution, Parametric Study Plots ........................................ 55
20 Real Part of $Y_{11}$ for the Physical Parameters of Table 8: Simultaneous Solution ................................................................. 56
21 Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 8: Simultaneous Solution ................................................................. 56
22 Real Part of $Y_{21}$ for the Physical Parameters of Table 8: Simultaneous Solution ................................................................. 57
23 Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 8: Simultaneous Solution ................................................................. 57
24 Sketch of the Wire near a conducting Wedge ......................................................... 73
25 Sketch of the equivalent model for the Wire near a conducting Wedge .................. 73
26 The value of $k_z$ versus the index used in figures 27-30 .................................. 74
27 Values of Integrand Terms for the $k_z$ integral: $f = 300 MHz$,
$\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m = z_n$ .......................................................... 74
28 Value of the Complete Integrand for the $k_z$ integral: $f = 300 MHz$,
$\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m = z_n$ .......................................................... 75
29 Values of Integrand Terms for the $k_z$ integral: $f = 300 MHz$,
$\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m - z_n = 3\Delta$ .......................................................... 75
30 Value of the Complete Integrand for the $k_z$ integral: $k = 2\pi$,
$\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m - z_n = 3\Delta$ .......................................................... 76
31 Current Distribution on a Wire Protruding from a Backplane Near a Wedge: $f = 950 MHz$, $\Phi = \frac{3\pi}{2}$, Length= 0.0617 m, $\rho = 0.0414$ m, $\phi = 1.349$ radians, Wire Radius= 0.0004 m ........... 76
32 Current Distribution on a Wire Protruding from a Backplane Near a Wedge: $f = 1250 MHz$, $\Phi = \frac{3\pi}{2}$, Length= 0.0617 m, $\rho = 0.0414$ m, $\phi = 1.349$ radians, Wire Radius= 0.0004 m ........... 77
33 Input admittance versus frequency: $\Phi = \frac{3\pi}{2}$, Length= 0.0617 m, $\rho = 0.0414$ m, $\phi = 1.349$ radians, Wire Radius= 0.0004 m ........... 77
34 Input admittance versus frequency: $\Phi = \frac{\pi}{2}$, Length= 0.06 m, $\rho = 0.112$ m, $\phi = 0.464$ radians, Wire Radius= 0.0004 m ........... 78
Current Distribution on a Wire Protruding from a Backplane
Near a Wedge: \( f = 2000 \text{MHz}, \Phi = \frac{\pi}{2}, \text{Length}= 0.06 \text{m}, \rho = 0.112 \text{ m}, \phi = 0.464 \text{ radians}, \text{Wire Radius}= 0.0004 \text{ m} \)  

Input admittance versus frequency: \( \Phi = 3.8 \text{ radians}, \text{Length}= 0.06 \text{ m}, \rho = 0.0112 \text{ m}, \phi = 1.107 \text{ radians}, \text{Wire Radius}= 0.0004 \text{ m} \)  

Current Distribution on a Wire Protruding from a Backplane
Near a Wedge: \( f = 1200 \text{MHz}, \Phi = 3.8 \text{ radians}, \text{Length}= 0.06 \text{ m}, \rho = 0.0112 \text{ m}, \phi = 1.107 \text{ radians}, \text{Wire Radius}= 0.0004 \text{ m} \)  

Sketch of the two wires near a slotted wedge geometry. Interior region wire shown through slot and behind non-physical, semi-transparent region.  

Sketch of the interior equivalent problem.  

Sketch of the outer equivalent problem.  

The value of \( k_z \) versus the index used in figures 42-44.  

Values of Integrand Terms for the \( TM_z k_z \) integral: \( f = 300 \text{MHz}, \rho_m = \rho_n = 0.2 \text{ m}, \text{Slot Width}= 0.001 \text{ m}, \Delta_s = 0.05 \text{ m}, z_m = z_n = 0.15 \text{ m} \)  

Values of Integrand Terms for the \( TE_z k_z \) integral: \( f = 300 \text{MHz}, \rho_m = \rho_n = 0.2 \text{ m}, \text{Slot Width}= 0.001 \text{ m}, \Delta_s = 0.05 \text{ m}, z_m = z_n = 0.15 \text{ m} \)  

Values of the Integrands for the \( k_z \) integrals: \( f = 300 \text{MHz}, \rho_m = \rho_n = 0.2 \text{ m}, \text{Slot Width}= 0.001 \text{ m}, \Delta_s = 0.05 \text{ m}, z_m = z_n = 0.15 \text{ m} \)  

Y_{11}: Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions \((x,y)=(6.7,\pm 1.1) \text{ cm}, \text{Slot Length}=10.1 \text{ cm}, \text{Slot Width}=2.25 \text{ mm}, \text{Slot Center Position} (x,z)=(6.4,1.6) \text{ cm} \)  

Y_{21} (or Y_{12}): Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions \((x,y)=(6.7,\pm 1.1) \text{ cm}, \text{Slot Length}=10.1 \text{ cm}, \text{Slot Width}=2.25 \text{ mm}, \text{Slot Center Position} (x,z)=(6.4,1.6) \text{ cm} \)  

Y_{22}: Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions \((x,y)=(6.7,\pm 1.1) \text{ cm}, \text{Slot Length}=10.1 \text{ cm}, \text{Slot Width}=2.25 \text{ mm}, \text{Slot Center Position} (x,z)=(6.4,1.6) \text{ cm} \)  

Equivalent Magnetic Current on the Slot: Frequency: 1350 MHz, Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions \((x,y)=(6.7,\pm 1.1) \text{ cm}, \text{Slot Length}=10.1 \text{ cm}, \text{Slot Width}=2.25 \text{ mm}, \text{Slot Center Position} (x,z)=(6.4,1.6) \text{ cm}, V^+ = 1. \)
Current on the Wire in the Outer Region: Frequency: 1350 MHz, Wire Length=6.2 cm, Wire Radii=0.4 mm, Wire Positions (x,y)=(6.7,±1.1) cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position (x,z)=(6.4,1.6) cm, V⁺ = 1.

The signal flow graph, including definitions of the incident and reflected voltages and electric fields[4].

Problem Sketch

A Spectral Plot of the Voltage at Port 1 under the Parameters of Table 9

A Spectral Plot of the Voltage at Port 2 under the Parameters of Table 9

A Spectral Plot of the Electric Field at Port 3 under the Parameters of Table 9

Time Variation of the Equivalent Dipole Moment of the Exciting Slot for Time Domain Results

Spectrum of the Time Domain Signal in Figure 55, The Dipole Moment Variation for the Exciting Slot

A Time Domain Plot of the Voltage at Port 1 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55

A Time Domain Plot of the Voltage at Port 2 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55

A Time Domain Plot of the Electric Field at Port 3 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55
While the problem of slots in conducting bodies is not a new one, it does deserve further attention. Slotted metallic surfaces appear in a whole variety of shielding configurations as well as antennas. The accurate modeling of the effects of those slots on field penetration and scattering is critical in many applications. Having said this, the computation time for the accurate modeling of slots can be prohibitive. In this document, we present a variety work with the major goal of providing an improved ability to analyze the effects of narrow slots. In chapter one, we analyze the use of Bethe hole theory for modeling slots and describe an innovative technique utilizing Bethe hole theory in a hybrid formulation with integral equation techniques to effect a fast, accurate solution for the slot array problem. In chapter two, we examine transmission line and integral equation models for the problem of two wires coupling through a slotted ground plane. For this problem, we present a new, hybrid formulation which shows good results. Chapter three demonstrates a means of adding and subtracting a free space term to and from the Green’s function for a wedge space in order to provide, for the first time, a reasonably efficient means of performing a method of moments solution for axial wires. In the fourth chapter, we extend the wedge discussion to include the method of moments solution for transverse equivalent currents in the presence of a wedge. To our knowledge, this is the first time a Green’s function for the transverse dipole in the presence of a wedge has been presented. In chapter five, we discuss a novel modification of the Baum-Liu-Tesche equation to incorporate field effects and provide an example derivation for incorporating the effects of a short slot on a transmission line network.
ALTERNATE METHODS OF CALCULATING THE FIELDS FROM ARRAYS OF NARROW SLOTS

A specialization of Bethe hole theory to narrow slots is derived and then utilized in a new formulation of a coupled slot form of the Bethe hole theory equations. A means of calculating the relevant polarizabilities is presented. Integral equations are developed for the short, narrow slot and used for both comparison purposes and for mixed-formulation solutions to the slot array problem. Data is presented for representative cases.

Introduction

The accurate, fast characterization of field penetration through apertures in conducting surfaces has been of growing interest for some time. The effectiveness of various forms of shielding is of particular concern to the technical communities involved in electromagnetic compatibility and wireless communications. Aperture penetration issues are also of importance to the intelligence community, whether they be defending friendly electronic systems, disrupting hostile systems, or performing electronic eavesdropping. A challenge is to find a balance between fast, approximate techniques, such as those based on Bethe hole theory[1], and more accurate but numerically intensive techniques, such as the method of moments[2].

Field penetration through small-aperture arrays have been considered by a number of researchers. Two early papers documenting techniques based on integral equations are those by Chen[3] and by Kieburtz and Ishimaru[4]. Bhattacharyya and Sen deal
with arrays of apertures via Bethe hole theory, neglecting the coupling between the apertures[5]. Robinson, et al, like others, use Bethe hole theory to model a single aperture and extend these methods to estimate the shielding effectiveness of an array of apertures[6]. For periodic small-aperture structures, techniques exist for estimating the effect of the overall array based on an individual aperture’s polarizability as discussed in the EMP Interaction Handbook edited by Lee[7]. The field coupling effects of multiple apertures have been studied in the context of penetration through braided shielding on cables, and such techniques are reported by Tesche, Ianoz, and Karlsson[8] and in their references. In this chapter, we describe for the first time how to include the effects of inter-slot coupling into the Bethe hole theory formulation as well as how to effect a novel, hybrid integral equation/Bethe hole theory method for determining the penetration through a slot array. The perturbations in the scattered field due to the presence of the apertures is considered simultaneously with the penetrating fields.

Bethe hole theory states that the fields penetrating through an electrically small aperture as well as the scattered field perturbations can be approximated by the fields induced by two geometry- and excitation-dependent dipoles. The dipoles are placed with appropriate sign modifications on either side of the shorted aperture at the center. The first of these dipoles is an electric dipole normal to the surface in which the aperture exists, and the second is a magnetic dipole in the plane of the surface. For every small aperture geometry, a set of polarizabilities can be defined. These polarizabilities incorporate the geometry dependence of the equivalent dipoles and need only be scaled by the excitation (as described below) to arrive at moments of the equivalent dipoles. Bethe hole theory serves as a basis for methods which are fast and sufficiently accurate for many applications. Though the computational efficiency of a Bethe hole theory technique is attractive in aperture array problems, it is not
always practical due to required accuracies. Failure to account for inter-aperture
coupling can cause significant error (on the order of 10%) in the fields near aperture
arrays such as the narrow slot arrays treated in this chapter, but the principles of
Bethe hole theory can be modified to include the inter-aperture coupling.

The standard first step in the development of a Bethe hole theory method for
computing the penetration through an array of narrow slots is the calculation of the
polarizabilities. In the early 1950s, Cohn[9] performed measurements in an electrolytic
tank to obtain the polarizabilities for small apertures of a variety of shapes. De Meu-
lenaere and Van Bladel compared the values of polarizabilities obtained by numerical
techniques with those presented by Cohn and those obtained by analytic means[10].
In the mid-to-late eighties, McDonald derived a closed form approximation for the po-
larizabilities for rectangular apertures[11][12][13]. Related information can be found
in the review paper on apertures by Butler, Rahmat-Samii, and Mittra[14]. Cohn
and McDonald both present equations for the polarizabilities of rectangles, but their
results are not valid for very narrow rectangles and so are of minimal applicability to
the apertures considered in this chapter. Using the results of Butler and Van Bladel,
one can calculate the polarizabilities for short, very narrow slots via integral equation
techniques.

We discuss both individual slots and slot arrays. In both cases the slots are located
in an infinite ground plane coexisting with the xy-plane, straight, and axially aligned
in the y-direction. The slots are 2h in length and 2w in width, and are very short
with respect to wavelength and very narrow with respect to length (λ ≫ h ≫ w).
The slots in an array are identical. When only individual slots are considered, they
are centered at the origin for simplicity. It is assumed that the excitation varies time
harmonically (e^{jωt}).
Integral Equation Formulation

The integral equation for a narrow slot is well-understood[15]. In the development of the integral equation, one postulates an electric field in the slot, determines the magnetic field in both regions, and applies the condition,

\[ H^+ \times z = H^- \times z, \text{ on the slot,} \quad (1) \]

where \( H^+ \) refers to the total magnetic field above the slotted ground plane, and \( H^- \) is the total magnetic field below. By recognizing the contributions to the total magnetic field and stating the fields due to the presence of the slot in terms of potentials, one arrives at

\[ j\omega F^s_y + \frac{\partial}{\partial y} \Psi^s = \frac{1}{2}(H^{sc+}_y - H^{sc-}_y), \text{ on the slot axis,} \quad (2) \]

where

\[ F^s_y = \frac{1}{2\pi\varepsilon} \int_{-h}^{h} K(y')G(y - y')dy', \text{ on the slot axis,} \quad (3) \]

and

\[ \Psi^s = j\frac{1}{2\pi\eta k} \int_{-h}^{h} \left[ \frac{\partial}{\partial y} K(y') \right] G(y - y')dy', \text{ on the slot axis.} \quad (4) \]

\( H^{sc+}_y \) and \( H^{sc-}_y \) are \( y \)-directed portions of the short-circuit magnetic field, the field due to the independent sources which would exist with the slot shorted, in the upper and lower half-spaces, respectively. \( K(y) \) is the axial component of the equivalent magnetic current on the shorted slot \( M_s(r) = M_s(x, y)\hat{y} = T(x)K(y)\hat{y} \) in the upper half space, \( \varepsilon \) is the permittivity, \( \eta \) is the intrinsic impedance, \( k \) is the wave number, and \( \omega \) is the angular frequency. The super/subscript “s” emphasizes that these terms are related to the slot. The transverse variation \( T(x) \) of the equivalent magnetic
current is included in the well-known narrow slot kernel[15]

\[ G(y - y') = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jkR} \frac{R}{R} d\alpha \]  

where

\[ R = \sqrt{(y - y')^2 + 4a^2 \sin^2(\frac{\alpha}{2})} \]  

and \( a = \frac{w}{2} \). Because the slot is very short relative to wavelength, the vector potential term in (2) is negligible compared to the scalar potential term (the relationship is approximately proportional to \( (\frac{2h}{\lambda})^2 \)) and so can be suppressed[16]. Additionally, since \( R \) is very small with respect to wavelength for all field and source points on the slot, advantage is taken of the approximation \( e^{-jkR} \approx 1 \). The end result of these approximations applied to (2) is a quasi-static equation which can be written

\[ \frac{\partial}{\partial y} \Psi^s = \frac{1}{2} (H_{sc}^{+} - H_{sc}^{-}), \text{ on the slot axis,} \]  

where \( \Psi^s \) is as above with the quasi-static kernel,

\[ g(y - y') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{R} d\alpha. \]  

The integro-differential equation of (7) can be solved in the usual way[2].

\underline{Reduction to Polarizabilities}

For the short, narrow rectangular slot, the equivalent magnetic current obtained from the solution of (7) can be used to calculate the equivalent dipole moments and polarizabilities suggested in Bethe hole theory. In this chapter, all dipole moments
and polarizabilities are defined in the presence of a short-circuited conducting screen or ground plane.

Electric Polarizability

Bethe hole theory enables one to estimate the penetration through an aperture with the fields excited by two equivalent dipoles: a normally-directed electric dipole and a tangentially directed magnetic dipole. The fields excited by these dipoles also provide an approximation for the perturbations to the reflected field due to the presence of the aperture. The equivalent electric dipole moment $p_e$ is related to the equivalent magnetic surface current in an aperture centered at the origin according to

$$p_e = -\varepsilon \frac{1}{2} \int \nabla \times M_s(r) dS$$

(9)

where $R_s$ is the planar region of the aperture and $r$ is a position vector. Under the restrictions outlined in the introduction, the double integral of (9) becomes

$$p_e = -\varepsilon \frac{\hat{z}}{2} \int_{-h}^{h} \int_{-w}^{w} x M_s(x, y) dx dy.$$

(10)

Further simplification of the integral can be obtained by utilizing the narrow slot properties to separate the transverse and axial variation of the slot current. The current's transverse variation is known to be

$$T(x) = \frac{1/\pi}{\sqrt{w^2 - x^2}}.$$

(11)
leading to
\[ M_s(x, y) = \frac{1/\pi}{\sqrt{w^2 - x^2}} K(y), \quad (12) \]
where \( K(y) \) is again the axial current variation of the equivalent current. Note that the \( 1/\pi \) is a normalization term forcing the integral of the transverse current variation \( T(x) \) to be equal to 1. Substituting (12) into (10), one obtains
\[ \mathbf{p}_e = -\frac{\varepsilon}{2} \int_{-h}^{h} K(y) \int_{-w}^{w} \frac{x/\pi}{\sqrt{w^2 - x^2}} dx \, dy. \quad (13) \]
The \( x \)-varying portion of the integrand in (13) is an odd function implying
\[ \mathbf{p}_e = 0. \quad (14) \]
Therefore, the electric polarizability of a narrow slot (at any location) is zero. This is consistent with the limit for the electric polarizability of small rectangular apertures[9][10][11].

**Magnetic Polarizability**

The equivalent, tangential magnetic dipole moment \( \mathbf{p}_m \) is related to the equivalent magnetic surface current for an aperture centered at the origin according to[14]
\[ \mathbf{p}_m = -\frac{j}{\omega \mu} \oint_{R_s} \mathbf{M}_s(r) dS \quad (15) \]
where \( \mu \) is the permeability of the medium. Subject to the restrictions outlined in the introduction \( \mathbf{p}_m \) simplifies to
\[ \mathbf{p}_m = -\frac{j}{\omega \mu} \hat{y} \int_{-h}^{h} \int_{-w}^{w} M_s(x, y) dx \, dy, \quad (16) \]
which implies that the component of the magnetic dipole moment transverse to the slot axis is zero which is consistent with the limit of the transverse component of magnetic polarizability for rectangular apertures[9][10][12].

In the interest of simplifying the expression for $p_m$, we substitute (12) into (16) to arrive at

$$p_m = -\frac{j}{\omega \mu} \hat{y} \int_{-h}^{h} K(y) dy. \tag{17}$$

Letting

$$K = \int_{-h}^{h} K(y) dy, \tag{18}$$

one can write the equivalent magnetic dipole moment for the upper and lower sides of the shorted conducting plane as $p_m^\pm = \pm p_{m,y} \hat{y}$ where

$$p_{m,y} = -\frac{jK}{k\eta}. \tag{19}$$

It is also true that[14]

$$p_{m,y} = \alpha_{m,y} (H_{y}^{sc+}(0) - H_{y}^{sc-}(0)). \tag{20}$$

For a narrow slot located at the origin, one can solve (7) with an excitation such that

$$H_{y}^{sc+}(r_0) - H_{y}^{sc-}(r_0) = 1. \tag{21}$$

The resulting $K(y)$ can be substituted into (18) to solve for $K$ which can in turn be substituted with (21) into (20) and (19) to solve for the polarizability $\alpha_{m,y}$ of the specific, short and narrow slot. While the solution for the polarizability centers the slot at the origin, the polarizability is general for a slot of the same dimensions in the $xy$-plane, axially directed along the $y$-axis. Equation (20) applies generally for a slot.
at any location if the null vector is replaced with a vector locating the center of the aperture.

The *shape* of the axial distribution $K(y)$ of the equivalent magnetic current of a very short, narrow slot is essentially independent of the excitation (short-circuit magnetic field) unless the excitation is predominantly an odd function of axial displacement $y$. Under such uncommon circumstances, the distribution $K(y)$ possesses a noticeable odd-function component. This odd-function portion of the magnetic current distribution radiates weakly and is negligible in almost all cases. The exception is when the observation point of interest is within 3-5 slot lengths of the slot. Moreover, an exciting field with a high odd-function component is unlikely in view of the typical sources that might illuminate a slotted plane including excitations approximating plane waves or even dipole-style sources as close as 3-5 slot lengths from the slot. It is worth noting that asymmetric coupling with nearby structures, including other slots, can cause odd-function equivalent magnetic current components in slots\[18\], but this asymmetric coupling is naturally very weak in the very short, narrow slots. Thus, except in cases of slots which are excited by pathological sources, the shape of $K(y)$ subject to various excitations differs very little from its shape when the very short, narrow slot is excited by a simple plane wave. This assertion is critical to the validity of Bethe hole theory.

### Slot Arrays

The extension from a single slot to an array of slots such as might be used in vents for electronic devices necessitates the calculation of coupling between the slots. It is of interest to identify the equations for the effect of a presence of one slot on another for both integral equation and Bethe hole techniques. Once one has these equations,
which we call coupling terms, one can solve for the fields that penetrate an array in 
a variety of ways. Several options include

1. solving the fully-coupled integral equations,

2. solving the coupled Bethe hole equations, or

3. solving a mixed set of Bethe hole/integral equations.

The advantage of the fully-coupled integral equation solution is accuracy while the 
coupled Bethe hole solution requires far less computer memory and time. A hybrid 
solution provides the potential for a balance among speed, accuracy, and the memory 
requirements. Once the coupling terms are specified, constructing a matrix equation 
from the fully-coupled integral equations and the coupled Bethe hole approximations 
is simple.

Integral Equation Coupling

Before identifying the coupling terms in the multi-slot integral equation, it is useful 
to define some notation. Let $F^p_y \hat{y}$ and $\Psi^p_y$ represent the electric vector potential 
and the magnetic scalar potential, respectively, arising from the equivalent magnetic 
current and its image in the $p^{th}$ slot.

The multi-slot integral equation is based on the boundary condition in (1). Let 
there exist $P$ slots. Following the same technique used for the single slot, one can 
develop the equation

$$\sum_{p=1}^{P} \left[ j\omega F^p_y + \frac{\partial}{\partial y} \Psi^p_y \right] = \frac{1}{2}(H^{sc+}_y - H^{sc-}_y), \text{ on slot } a. \quad (22)$$
Separating out the term related to a slot’s effect on itself, or self-slot term, one writes (22) as
\[
j \omega F_y^a + \frac{\partial}{\partial y} \Psi^a + \sum_{p \in S_p} [j \omega F_y^p + \frac{\partial}{\partial y} \Psi^p] = \frac{1}{2} (H_{y}^{sc+} - H_{y}^{sc-}), \text{ on slot } a, \tag{23}
\]
where
\[
\sum_{p \in S_p} [j \omega F_y^p + \frac{\partial}{\partial y} \Psi^p] \tag{24}
\]
is the array-to-slot coupling term that embodies the effect of the other slots on slot \( a \), and \( S_p \) is the set of all slots other than slot \( a \).

The small slot approximations allow the same simplifications for the self-slot term in (23) outlined for the isolated slot in integral equation formulation section. For the coupling terms of (24), the quasi-static approximation is not valid, and the following equations must be used:
\[
F_y^p = \frac{1}{2\pi \varepsilon} \int_{-h}^{h} K(y') \times g_2(x - x_p, y - (y_p + y')) dy', \text{ on the slot } a \text{ axis} \tag{25}
\]
and
\[
\Psi^p = j \frac{1}{2\pi \eta k} \int_{-h}^{h} \left[ \frac{\partial}{\partial y'} K(y') \right] \times g_2(x - x_p, y - (y_p + y')) dy', \text{ on the slot } a \text{ axis}, \tag{26}
\]
where
\[
g_2(x, y) = \frac{e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}}. \tag{27}
\]
x\(_p\) and y\(_p\) are the \( x\)- and \( y\)- displacement from the origin, respectively, of the center of slot \( p \).
Bethe Hole Theory Coupling

The equivalent magnetic dipole moment for slot $a$ can be written in terms of the polarizability and the short circuit magnetic field at slot $a$ such that

$$p_a = \alpha_{m,y} \left[ H_y^{sc+t}(r_a) - H_y^{sc-t}(r_a) \right]$$

(28)

where $r_a$ is a position vector locating the center of slot $a$, and $p_a$ is the $y$-directed magnetic dipole moment $p_{m,y}$ of the $a^{th}$ slot. $H_y^{sc+t}$ and $H_y^{sc-t}$ represent the total, $y$-directed short circuit magnetic fields in the upper and lower half-spaces, respectively, at the center of slot $a$. $H_y^{sc\pm,t}$ can be recognized as the sum of the $y$-directed short circuit magnetic fields generated by the independent sources $H_y^{sc\pm}$ and the $y$-directed short circuit magnetic fields penetrating through the other $P - 1$ slots $H_y^{sc\pm,s}$. In equation form, we write this as

$$H_y^{sc\pm,t}(r_a) = H_y^{sc\pm}(r_a) + H_y^{sc\pm,s}(r_a)$$

(29)

Substituting (29) into (28) leads to

$$p_a - \sum_{p \in S_P} \alpha_{m,y} \left( H_y^{sc+p}(r_a) - H_y^{sc-p}(r_a) \right) = \alpha_{m,y} \left( H_y^{sc+}(r_a) - H_y^{sc-}(r_a) \right).$$

(30)

$H_y^{sc+p}$ and $H_y^{sc-p}$ are the $y$-directed, short circuit (with respect to slot $a$) magnetic fields resulting from penetration through and reflected field perturbation from the $p^{th}$ slot

$$H_y^{sc\pm,s} = \sum_{p \in S_P} H_y^{sc\pm,p}.$$  

(31)
The magnetic field due to a $y$-directed magnetic dipole (and its image in an infinite $xy$-ground plane) is

$$H = -\frac{1}{2\pi} \nabla \times (p_{m,y} \hat{y} \times \nabla g(r, r_p))$$  \hfill (32)

where

$$g(r, r_p) = \frac{e^{-jk|r - r_p|}}{|r - r_p|},$$  \hfill (33)

$r$ locates the observation point, and $r_p$ locates the dipole (at the center point of the $p^{th}$ slot). From this, we can see that

$$H_{sc}^{\pm,y}(r_a) = \mp p_p \left. \left( \frac{\partial^2}{\partial z^2} g(r, r_p) + \frac{\partial^2}{\partial x^2} g(r, r_p) \right) \right|_{r=r_a},$$  \hfill (34)

where $p_p$ is the equivalent dipole moment $p_{m,y}$ of slot $p$. Substituting (34) into (30), it follows that

$$p_a + \frac{1}{\pi} \alpha_{m,y} \sum_{p \in S_p} p_p \left( \frac{\partial^2}{\partial z^2} g(r, r_p) + \frac{\partial^2}{\partial x^2} g(r, r_p) \right) \bigg|_{r=r_a} = \alpha_{m,y} \left( H_{sc}^{+}(r_a) - H_{sc}^{-}(r_a) \right).$$  \hfill (35)

The summation on the left side of (35) contains the coupling terms for the Bethe hole theory formulation.

By enforcing (35) on all $P$ slots, one can solve for the unknown dipole moments using standard linear equations techniques.

Hybrid Formulation Coupling

The computer resources - storage ($N^2$ doubles for $N$ unknowns) and time (minutes to days) - required to solve the fully-coupled integral equations give ample cause to
examine the use of Bethe hole theory to simplify the problem. Unfortunately, the separation between the slots must be significant with respect to the slot length, or the Bethe hole theory assumptions are violated. As a result, with an array of close-by slots, it is of interest to pursue a hybrid solution which allows one to choose which coupling effects to account for with integral equations and which to account for with Bethe hole theory.

As an initial step in this new formulation, consider a modified version of (23),

\[ j\omega F_a^y + \frac{\partial}{\partial y} \Psi_a + \sum_{p \in S_a} \left[ j\omega F_p^y + \frac{\partial}{\partial y} \Psi_p \right] = \frac{1}{2}(H_{y}^{sc+,t} - H_{y}^{sc-,t}), \text{ on slot } a \text{ axis,} \quad (36) \]

where we have restricted the coupling term summation to the slots in \( S_a \): the set of slots that, with respect to separation from slot \( a \), violate Bethe hole theory assumptions. In practice, one could define any number of rules by which to choose slots to be in \( S_a \). One reasonable rule would be to include all slots with a center-to-center distance from slot \( a \) of less than twice the slot length, and another would be to include all slots that might be characterized as “nearest neighbors.” \( H_{y}^{sc\pm,t} \) represents the \( y \)-directed short circuit magnetic fields emanating from the independent sources and the short circuit fields resulting from the penetration through and scattered field perturbations from slots that are neither slot \( a \) nor in \( S_a \). We appeal to (29) to extract the excitation due to the independent sources from \( H_{y}^{sc\pm,t} \) in (36) to arrive at

\[ j\omega F_a^y + \frac{\partial}{\partial y} \Psi_a + \sum_{p \in S_a} \left[ j\omega F_p^y + \frac{\partial}{\partial y} \Psi_p \right] - \sum_{p \in S_a'} \frac{1}{2}(H_{y}^{sc+,p} - H_{y}^{sc-,p}) = \frac{1}{2}(H_{y}^{sc+} - H_{y}^{sc-}), \text{ on slot } a \text{ axis,} \quad (37) \]

where \( S_a' \) is the set of all slots not in \( S_a \) (and not slot \( a \)). Under the assumption that
the slots exist in an infinite ground plane, (37) becomes

\[ j\omega F_y^a + \frac{\partial}{\partial y}\Psi^a + \sum_{p \in S_a} \left[ j\omega F_y^p + \frac{\partial}{\partial y}\Psi^p \right] + \sum_{p \in S_a'} \frac{p_p}{2\pi} \left( \frac{\partial^2}{\partial z^2} g(r, r_p) + \frac{\partial^2}{\partial x^2} g(r, r_p) \right) \]

\[ = \frac{1}{2}(H_y^{sc+} - H_y^{sc-}), \text{ on slot } a \text{ axis.} \quad (38) \]

The sum of the first two terms on the left side of (38) is the \( y \)-directed short-circuit magnetic field on the axis of the \( a \)th slot created by its equivalent magnetic current (or transverse electric field), the third and fourth terms are the \( y \)-directed short-circuit magnetic fields induced by the presence of the slots in \( S_a \) and \( S_a' \), respectively, observed on the \( a \)th slot axis. The right side of (38) consists of the \( y \)-directed short-circuit magnetic fields on the \( a \)th slot axis due to the independent sources.

With slots being modeled via an equivalent magnetic current and an equivalent magnetic dipole simultaneously, we require a set of equations to relate the magnetic currents on the slots to their equivalent dipole moments. Recall equations (18) and (19). Under the Bethe hole theory assumptions of current \textit{shape} discussed previously, one can scale the equivalent dipole moment of a slot so that it is proportional to the slot’s excitation. Consider that \textit{a priori} one calculates the polarizability \( \alpha_{m,y} \) for the slot, saving the following items from that calculation: the center current on the slot \( K_s' \) and the wave number \( k_a \). The equivalent dipole moment for slot \( a \) can then be written

\[ p_a = \alpha_{m,y} \frac{k_a K_a'}{k K_s'} \quad (39) \]

where \( K_a' \) is the center current on slot \( a \) given the excitation of interest. Note that
this equation can be rewritten in a form that is convenient for a matrix solution,

\[ p_a - \alpha_{m,y} \frac{k_a K'_k}{k K'_s} = 0. \]  \hspace{1cm} (40)

One can expand the magnetic current \( K \) in (38) with triangle basis functions and then perform testing of (38) with pulse functions in a manner consistent with the Method of Moments[2]. The resulting equation along with the equations found by enforcing (40) on every slot form a set of linear equations that can be solved in the usual way. One way to write the set of linear equations is

\[
\begin{bmatrix}
[I^P] & [C] \\
[B] & [S]
\end{bmatrix}
\begin{bmatrix}
[p] \\
[K]
\end{bmatrix} =
\begin{bmatrix}
[0] \\
[H]
\end{bmatrix}.
\]  \hspace{1cm} (41)

For the purpose of description, let there exist \( P \) slots and \( N \) basis functions per slot. The submatrix \([I^P]\) is an \( P \times P \) identity matrix. The submatrix \([B]\) can be written

\[
[B] =
\begin{bmatrix}
[B^{11}] & [B^{12}] & \cdots & [B^{1P}] \\
[B^{21}] & [B^{22}] & \cdots & [B^{2P}] \\
\vdots & \vdots & \ddots & \vdots \\
[B^{P1}] & [B^{P2}] & \cdots & [B^{PP}]
\end{bmatrix}
\]  \hspace{1cm} (42)

where \([B^{pq}]\) is a \( N \times 1 \) column vector whose elements are defined

\[
[B^{pq}]_n = \begin{cases} 
0, \text{ slot } q \in S_p \text{ or } p = q \\
\frac{1}{2\pi} \frac{\alpha_{m,y}}{r_{pq}} \Delta \left( \frac{\partial^2}{\partial z^2} g(r, r_q) + \frac{\partial^2}{\partial x^2} g(r, r_q) \right) \bigg|_{r=r_{pq}^n}, \text{ otherwise}
\end{cases}
\]  \hspace{1cm} (43)

and \( \Delta \) is the subdomain length. The position vector \( r_{pq}^n \) locates the center of the \( n^{th} \)
current element of slot \( p \), and \( r_q \) locates the center of slot \( q \). In the submatrix,

\[
[S] = \begin{bmatrix}
[S^{11}] & [S^{12}] & \cdots & [S^{1P}] \\
[S^{21}] & [S^{22}] & \cdots & [S^{2P}] \\
\vdots & \vdots & \ddots & \vdots \\
[S^{P1}] & [S^{P2}] & \cdots & [S^{PP}]
\end{bmatrix},
\]

(44)

\([S^{pp}]\) is a \( N \times N \) array whose elements are defined

\[
[S^{pp}_{ni}] = j \left( \frac{1}{2\pi \Delta} \right) \frac{1}{\Delta} \left[ \int_{-\Delta}^{0} g(y_n^p + \Delta/2 - y_i^p - y') dy' \\
- \int_{-\Delta}^{0} g(y_n^p - \Delta/2 - y_i^p - y') dy' \\
- \int_{0}^{\Delta} g(y_n^p + \Delta/2 - y_i^p - y') dy' \\
+ \int_{0}^{\Delta} g(y_n^p - \Delta/2 - y_i^p - y') dy' \right]
\]

(45)

where \( y_n^p \) and \( y_i^p \) are the \( y \)-positions of the \( n^{th} \) and \( i^{th} \) current elements, respectively, of slot \( p \). \([S^{pq}] \) \((p \neq q)\) is a \( N \times N \) array whose elements are defined

\[
[S^{pq}_{ni}] = j \left( \frac{1}{2\pi \Delta} \right) \frac{1}{\Delta} \left[ \int_{-\Delta}^{0} g_2(x_n^p - x_i^q, y_n^p + \Delta/2 - y_i^q - y') dy' \\
- \int_{-\Delta}^{0} g_2(x_n^p - x_i^q, y_n^p - \Delta/2 - y_i^q - y') dy' \\
- \int_{0}^{\Delta} g_2(x_n^p - x_i^q, y_n^p + \Delta/2 - y_i^q - y') dy' \\
+ \int_{0}^{\Delta} g_2(x_n^p - x_i^q, y_n^p - \Delta/2 - y_i^q - y') dy' \right], \text{ slot } q \in S_p
\]

(46)

where \( x_n^p \) and \( x_i^p \) are the \( x \)-positions of the \( n^{th} \) and \( i^{th} \) current elements, respectively,
of slot $p$. All elements of $S^{pq}$ are 0 when slot $q$ is not a member of $S_p$. The submatrix

$$[C] = \begin{bmatrix} [C^{11}] & 0 & \cdots & 0 \\ 0 & [C^{22}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [C^{PP}] \end{bmatrix} \tag{47}$$

where $[C^{pp}]$ is a $1 \times N$ row vector whose elements are defined

$$C^{pp}_n = \begin{cases} -k_{a} \frac{\sqrt{\alpha_{m,y}}}{k K_s}, & n \text{ is the center current element of the } p^{th} \text{ slot} \\ 0, & \text{otherwise} \end{cases} \tag{48}$$

The submatrix $[p]$ is an $P \times 1$ array of $\frac{1}{\sqrt{\alpha_{m,y}}}p_p$, the scaled unknown dipole moments. The submatrix $[K]$ is an $(P \cdot N) \times 1$ array containing the unknown magnetic current elements on the slots such that $[K] = [K^1_1, K^1_2, \cdots, K^1_N, K^2_1, \cdots, K^P_N]^T$ where $K^p_n$ is the coefficient of the basis function for the magnetic current on the $n^{th}$ current element of the $p^{th}$ slot. The submatrix $[H]$ is an $(P \cdot N) \times 1$ array whose elements are

$$H^n_p = \frac{\Delta}{2}(H^{sc+,t}(r^n_p) - H^{sc-,t}(r^n_p)). \tag{49}$$

Equation (41) can be solved in the usual way.

**Results**

In this section, numerical data are presented for the dipole moments for several slot array configurations, and the radiated fields produced by the various techniques discussed above. We also discuss the computation time advantage of the hybrid
Dipole Moments for Array Configurations

We present results from arrays of parallel slots, an array of axially aligned slots, and a two-dimensional array of slots. For comparison purposes, we consider the percentage differences in the effective dipole moments in formulations based on Bethe hole theory approximations when compared to truth. Truth is found by enforcing (23) on each slot and applying the method of moments with triangle basis functions and pulse testing functions[2]. The currents resulting from this solution are used to find the truth effective dipole moments via (19). The formulations under consideration include the coupled Bethe hole method (CBH) and two hybrid solutions. The CBH results are found by enforcing (35) on every slot and solving the ensuing system of linear equations for the equivalent dipole moments. The hybrid solution dipole moments result from the solution of (41). In the first hybrid solution (H1), $S_p$ is defined to include the nearest neighbor slots. In the second hybrid solution (H2), $S_p$ includes slots whose centers are closer to slot $p$ than $3h$ where $2h$ is the slot length. In both, the quasi-static simplifications (vector potential suppression and electrostatic kernel) are used to calculate a slot’s effect on itself while no simplifications are used for the coupling terms.

Case 1: A 1×4 Array of Parallel, Small Slots

In this case, the slots are arranged parallel to each other with their axial centers on the $x$-axis as shown in Figure 1. The general parameters are
• Slot Length: 10 mm
• Slot Width: 0.1 mm
• Slot Separation: 20 mm
• Frequency: 300 MHz

The separation is such that CBH provides reasonable results, with differences from the full integral equation technique being less than 1% as shown in Table 1. One can compare the difference generated by CBH with the no-coupling solution where the difference is around 0.9%. In saying ”no-coupling solution” we refer to the dipole moments derived from a traditional Bethe hole theory approach where the equivalent dipole moments are calculated neglecting the presence of other small apertures. The improvement provided by the hybrid solutions is significant. With a slot separation of twice the slot length, H2 did not use integral equations to account for any coupling in this case. As a result, H2 is slightly less accurate that H1. H2 shows such a significant improvement over CBH because of the renormalizing effect of equation (39).

Case 2: Another 1×4 Array of Parallel, Small Slots

The slots in this case are arranged the same as in Case 1 (again shown in Figure 1) except that the slot separation has been decreased from twice the slot length to half the slot length.

• Slot Length: 10 mm
• Slot Width: 0.1 mm
• Slot Separation: 5 mm
Table 2 shows the results for this array. As expected, the CBH solution breaks down due to violating the Bethe hole theory assumptions. The no-coupling solution introduces an error of approximately 8%. H1 and H2 still show significant improvement over CBH, with H2 better than H1. In this case H2 includes more of the coupling effects via the rigorous integral equation based coupling terms than does H1.

Case 3: A 4×1 Array of Axially-aligned, Small Slots

In this case, we have aligned the slots along their axis with the center to center separation being twice the slot length as shown in Figure 2.

- Slot Length: 10 mm
- Slot Width: 0.1 mm
- Slot Separation: 20 mm
- Frequency: 600 MHz

Table 3 contains the results. The errors produced here are similar to those of Case 1. There is no integral equation coupling in H2. Both hybrid methods still provide only a nominal error when compared to the results of the fully coupled integral equation technique.

Case 4: A 4×4 Array of Small Slots

In this case, we consider a two-dimensional array of identical slots as shown in Figure 3. For the sake of simplicity, the CBH solution has been left out of this data.
This array is a square with the parameters shown below:

- Slot Length: 10 mm
- Slot Width: 0.1 mm
- Slot Separation: 12 mm
- Frequency: 600 MHz

We have expanded the distance at which we use integral equations to account for the coupling in H2 so as to include the slots at the diagonal. Table 4 contains the results for this case. The reader will note that despite including more integral equation coupling in H2 than H1, the percentage difference in the results from H1 is smaller. The competing inaccuracies of the current shape assumption and the lack of rigorous coupling calculations happen to partially cancel. In a two dimensional array with these separations, the current shape distortion is such as to make the assumption of a constant current shape less than perfect. The results in Table 5 demonstrate this argument. In result of Table 5, the assumption of a constant current shape has been removed, also removing the \textit{a priori} calculation. To be specific, we replace (48) in the solution of (41) with

\[ C_{nn}^{pp} = \frac{j\Delta}{k\eta\sqrt{\alpha_{m,y}}} . \]  

An improvement in the calculation time is still obtained since the dipole moment based coupling terms (43) are easier and faster to calculate than the integral equation coupling terms (46). The reader will note that both errors have decreased, but the results from H2 are now more accurate.
Radiated Fields

The fields produced by the hybrid techniques, as verified by full integral equation techniques, confirm the advantage of including the coupling effects over the basic Bethe hole theory method. Figure 4 displays a plot of the radiated $E$ field as produced by integral equation techniques and the basic Bethe hole theory method (using the single element dipole moments with no inter-slot coupling accounted for) for Case 2 with the slot locations symmetric about the $y$-axis. The impinging field is a plane wave normally incident (from the positive $z$ half-space) on the slotted ground plane with the magnetic field aligned along the slots ($y$-directed). The observation points for the scattered field in figure 4 are located along $x^2 + z^2 = .05^2$ in the positive $z$ half plane with $x$ plotted as the independent variable. In a similar fashion, the observation points for the scattered field in figure 5 are such that $y^2 + z^2 = .05^2$ with $y$ as the independent variable. The two hybrid solution techniques produce results indistinguishable from the full integral equation technique and, as a result, are not included. Figures 6 and 7 depict the parallel plots for the array in Case 4.

Computation Time Improvement

It is difficult to quantify computation time improvement from the hybrid solution methods as compared to full integral equation techniques without problem specific information. A general idea of the improvement can be seen by examining the number of coupling terms that do not need to be calculated when using the hybrid forms. Begin by assuming $N$ basis functions per slot in (41). If there are $M$ sets of integral equation coupling terms (46) that do not need to be calculated due to the use of hybrid formulations then there are $2M \times N^2$ potential integral equation coupling
terms not calculated. There are $2M \times N$ additional Bethe hole theory coupling terms (43) required. The result is a net savings of $2M \times (N^2 - N)$ terms. Additionally, the computation of (43) is much less computationally intensive than the computation of (46). It is apparent that for a set distance between the slots, the net savings increases with array size.

Conclusion

Choosing a technique for modeling arrays of small slots is a matter of understanding requirements. When the distance between the observation point and the slots is several times the slot length, traditional Bethe hole theory approximations can provide significant computation time improvement over fully coupled integral equation techniques at a minimal loss of accuracy. Traditional Bethe hole theory techniques, however, are only reasonable when the distance between the slots is also several times the slot length. The inter-slot coupling effects can be included into a solution utilizing Bethe hole theory in a variety of ways depending on the users requirements and resources. While the inter-slot coupling effects can be included in a purely Bethe hole theory formulation, a balance between accuracy and computation time can be obtained with the novel, hybrid solution presented herein consisting of Bethe hole theory and integral equation techniques. The computation time advantage of a solution technique at least partially based on Bethe hole theory when compared to full integral equation techniques only increases with array size.

It is also worth noting that this new hybrid technique can be used even when the equivalent dipole moments are not the desired end-result. One obvious extension of the hybrid Bethe hole theory/integral equation technique would be to use Bethe hole theory approximations in the formulation of a wire near an array of small apertures.
References Cited


Figure 1: Array Layout for Cases 1 and 2

Figure 2: Array Layout for Case 3
Figure 3: Array Layout for Case 4

Figure 4: Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 2 array. IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole moments with no coupling. Excitation is a normalized plane wave.
Figure 5: Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 2 array. IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole moments with no coupling. Excitation is a normalized plane wave.

Figure 6: Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 4 array. IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole moments with no coupling. Excitation is a normalized plane wave.
Figure 7: Radiated E-field at a radius of 0.05m or 5 slot lengths from the center of the Case 4 array. IE is the integral equation solution. BHT is the field from the basic Bethe hole theory dipole moments with no coupling. Excitation is a normalized plane wave.

Table 1: Case 1: Percentage Difference in Effective Dipole Moments between Solution Techniques

<table>
<thead>
<tr>
<th>Slot Position</th>
<th>CBH</th>
<th>H1</th>
<th>H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.64%</td>
<td>0.01%</td>
<td>0.03%</td>
</tr>
<tr>
<td>2</td>
<td>0.63%</td>
<td>0.02%</td>
<td>0.04%</td>
</tr>
<tr>
<td>3</td>
<td>0.63%</td>
<td>0.02%</td>
<td>0.04%</td>
</tr>
<tr>
<td>4</td>
<td>0.64%</td>
<td>0.01%</td>
<td>0.03%</td>
</tr>
</tbody>
</table>
### Table 2: Case 2: Percentage Difference in Effective Dipole Moments between Solution Techniques

<table>
<thead>
<tr>
<th>Slot Position</th>
<th>CBH</th>
<th>H1</th>
<th>H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.62%</td>
<td>0.66%</td>
<td>0.24%</td>
</tr>
<tr>
<td>2</td>
<td>8.39%</td>
<td>0.25%</td>
<td>0.40%</td>
</tr>
<tr>
<td>3</td>
<td>8.39%</td>
<td>0.25%</td>
<td>0.40%</td>
</tr>
<tr>
<td>4</td>
<td>3.62%</td>
<td>0.66%</td>
<td>0.24%</td>
</tr>
</tbody>
</table>

### Table 3: Case 3: Percentage Difference in Effective Dipole Moments between Solution Techniques

<table>
<thead>
<tr>
<th>Slot Position</th>
<th>CBH</th>
<th>H1</th>
<th>H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.62%</td>
<td>0.03%</td>
<td>0.05%</td>
</tr>
<tr>
<td>2</td>
<td>0.58%</td>
<td>0.03%</td>
<td>0.08%</td>
</tr>
<tr>
<td>3</td>
<td>0.58%</td>
<td>0.03%</td>
<td>0.08%</td>
</tr>
<tr>
<td>4</td>
<td>0.62%</td>
<td>0.03%</td>
<td>0.05%</td>
</tr>
</tbody>
</table>
Table 4: Case 4: Percentage Difference in Effective Dipole Moments between Solution Techniques

<table>
<thead>
<tr>
<th>Slot Position</th>
<th>H1</th>
<th>H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>Transverse</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.11%</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.21%</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.21%</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.11%</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.09%</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.17%</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.17%</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.09%</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.09%</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.17%</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.17%</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.09%</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.11%</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.21%</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.21%</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.11%</td>
</tr>
</tbody>
</table>
Table 5: Modified Case 4: Percentage Difference in Effective Dipole Moments between Solution Techniques

<table>
<thead>
<tr>
<th>Slot Position</th>
<th>H1</th>
<th>H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 Axial</td>
<td>1</td>
<td>0.07%</td>
</tr>
<tr>
<td>2 Axial</td>
<td>1</td>
<td>0.10%</td>
</tr>
<tr>
<td>3 Axial</td>
<td>1</td>
<td>0.10%</td>
</tr>
<tr>
<td>4 Axial</td>
<td>1</td>
<td>0.07%</td>
</tr>
<tr>
<td>1 Transverse</td>
<td>2</td>
<td>0.09%</td>
</tr>
<tr>
<td>2 Transverse</td>
<td>2</td>
<td>0.14%</td>
</tr>
<tr>
<td>3 Transverse</td>
<td>2</td>
<td>0.14%</td>
</tr>
<tr>
<td>4 Transverse</td>
<td>2</td>
<td>0.09%</td>
</tr>
<tr>
<td>1 Transverse</td>
<td>3</td>
<td>0.09%</td>
</tr>
<tr>
<td>2 Transverse</td>
<td>3</td>
<td>0.14%</td>
</tr>
<tr>
<td>3 Transverse</td>
<td>3</td>
<td>0.14%</td>
</tr>
<tr>
<td>4 Transverse</td>
<td>3</td>
<td>0.09%</td>
</tr>
<tr>
<td>1 Transverse</td>
<td>4</td>
<td>0.07%</td>
</tr>
<tr>
<td>2 Transverse</td>
<td>4</td>
<td>0.10%</td>
</tr>
<tr>
<td>3 Transverse</td>
<td>4</td>
<td>0.10%</td>
</tr>
<tr>
<td>4 Transverse</td>
<td>4</td>
<td>0.07%</td>
</tr>
</tbody>
</table>
SOLUTION TECHNIQUES FOR WIRES COUPLED THROUGH SLOTS IN CONDUCTING GROUND PLANES

Several models for the problem of two wires separated by a slotted ground plane are presented. It is shown through data that a basic transmission line model for the calculation of line current or port parameters in this problem type is insufficient. A new, modified transmission line model useful for the case of small apertures and specialized to small slots is presented.

Introduction

Wires coupled through apertures in ground planes constitute a problem type of interest to a variety of electrical engineering communities, including those of electromagnetic compatibility, intelligence/counter-intelligence, and wireless communications. In the determination of such coupling, it is often desirable to avoid the computational intensity of the method of moments[1] and other such time-consuming solution techniques. In many cases, a transmission line based solution may be utilized[2]. The approximations inherent in the transmission line solution do not naturally include the effects of an aperture, though several sources in the literature discuss adding them[3][4]. In this chapter, we consider a simple transmission line model in order to assess the effect of the basic transmission line approximations on the accuracy of solution. For comparison purposes, both experimental data and results from an integral equation solution are presented. The wires are required to be thin and the slots narrow. Though few published results are available for the case of two separate wires
coupled through slots, the equations and solution techniques are a simple progression from the case of a single wire near slot in a conducting ground plane, a case for which extensive data are available[5][6][7][8][9]. When the slots are sufficiently short, additional efficiency may be obtained by modeling their effects with Bethe hole theory rather than the more intensive integral equation techniques[10][11] as discussed in the first chapter. In the case of very short, narrow slots, this chapter utilizes such Bethe hole theory techniques. Note that in this chapter it is assumed that the excitation varies time harmonically ($e^{j\omega t}$).

The errors found in the simple transmission line solution suggest a modified transmission line model is warranted. Such a modified model is presented and specialized to the case of a short, narrow slot as a logical extension of development performed in the first chapter.

Structure Description

The problem under discussion consists of two wires separated by a ground plane in which there are one or more slots. In order to facilitate comparison with measurements from a test fixture, we further specialize the problem such that the wires above and below the ground plane extend out from a backplane. A diagram is shown in Figure 8. From the equivalence principle and image theory, we can separate this structure and the sources into two equivalent models that are coupled through the slot. Figure 9 contains a sketch of the equivalent model which is valid for $z > 0$ (the upper side equivalent model). The equivalent model valid for $z < 0$ (the lower side equivalent model) is very similar to the upper side equivalent model. The following changes must be effected to figure 9 to obtain a sketch of the lower side equivalent model: the signs of the currents and voltages must be made negative (or, equivalently, the
polarizations and directions changed), and \( z_0^+ \) must be replaced with \( z_0^- \). In the upper side equivalent model, the magnetic currents are directed into the page (positive \( y \)). Conversely, the magnetic currents in the lower side equivalent model are directed out of the page (negative \( y \)). The ground planes are assumed to be of negligible thickness. As depicted in the figures, the wires and slot are assumed straight and are \( x \)- and \( y \)-directed, respectively. In general, superscripts + and − identify a symbol associated with the upper and lower side equivalent models, respectively.

In order to facilitate a comparison of the data from different solution techniques and measurements, it is necessary to characterize the coupling between the two wires. In this chapter, the coupling is characterized by the “Y” or short-circuit port parameters where the ports are defined at the interfaces between the wires and the ground plane (across the generator and generator impedance as depicted in figure 9). Port excitation was provided in the measurements by extending the center conductor of a coaxial transmission line through the ground plane and connecting it to the wire. Measurements were performed with a network analyzer (Agilent 8720). The S-parameters from the network analyzer are then converted to short circuit parameters by standard techniques[13]. A short circuited connector was used to establish port reference. In the numerical techniques, we sequentially excite a port and solve for the port currents with the non-excited port shorted, consistent with the definition of Y-parameters. Short circuit parameters can be found with the knowledge of this set of currents and the excitation voltage via standard techniques[13]. When integral equation techniques are used to model the wires, voltage sources are modeled as delta gaps.
Integral Equation Formulations

The fully coupled integro-differential equations for thin wires in the presence of narrow slots in conducting ground planes are well-known[5][6][7][8][9]. In this chapter, we utilize the mixed potential formulation and the exact kernel.

In the case of electrically short slots, it is possible to use Bethe hole theory instead of integral equation techniques in the determination of the scattering and penetration caused by slots[10] of which further discussion can be found in the first chapter. As shown previously, in the case of a short, narrow slot, the equivalent electric dipole is zero, and the magnetic dipole reduces to a single axially directed equivalent magnetic dipole of moment \( \mathbf{p}_m \) whose polarizability \( \alpha \) can be calculated \textit{a priori}. The equivalent dipole moment can be written in terms of its polarizability in the form discussed in the first chapter

\[
\mathbf{p}_m = \alpha (\hat{n} \times \hat{x}) \left( H_y^{sc+}(\mathbf{r}_0) - H_y^{sc-}(\mathbf{r}_0) \right),
\]

where \( H_y^{sc\pm}(\mathbf{r}_0) \) is the \( y \)-directed short circuit magnetic field on the top (+) or bottom (−) side of the shorted slot, as appropriate; \( \hat{n} \) is either \( \hat{z} \) or \( -\hat{z} \) as appropriate for an outward normal vector in the top (+) or bottom (−) half-space; and \( \mathbf{r}_0 = x_0\hat{x} + y_0\hat{y} \) is a position vector locating the center of the slot. The equivalent dipoles reside at the slot center on the top and bottom sides of the shorted slots. By applying the boundary condition of \( \mathbf{E} \cdot \hat{x} = 0 \) on the wires, one can write the equation

\[
- j \frac{\eta}{4\pi k} \left[ k^2 \int_{-L_z^\pm/2}^{L_z^\pm/2} I^{\pm}(x') G^{\pm}(x - x') dx' 
  + \frac{\partial}{\partial x} \int_{-L_z^\pm/2}^{L_z^\pm/2} \left[ \frac{d}{dx'} I^{\pm}(x') \right] G^{\pm}(x - x') dx' \right] 
+ j \frac{k\eta}{2\pi \alpha_m} \frac{\partial}{\partial z} \left[ g(|\mathbf{r} - \mathbf{r}_0|) + g(|\mathbf{r} - \mathbf{r}_0 + 2x_0\hat{x}|) \right] 
= -2V_0^{\pm} \delta(x), \quad \mathbf{r} = x\hat{x} + z_0^{\pm}\hat{z} \text{ on the wire(±) (52)}
\]
where
\[ G^\pm(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-jk\sqrt{\zeta^2 + 4a^2 \sin^2(\phi'/2)}} - e^{-jk\sqrt{\zeta^2 + (2z_o^2)}}}{\sqrt{\zeta^2 + (2z_o^2)}} \, d\phi' , \quad (53) \]

and
\[ g(|r - r'|) = \frac{e^{-jk|r-r'|}}{|r - r'|} . \quad (54) \]

$L^\pm_w$ and $a^\pm$ are the length and radius, respectively, of the wire from the upper or lower equivalent model as indicated by the superscript; $V^\pm_0$ is the voltage of the delta gap source; and $p_m$ is defined with $\check{n} = \hat{z}$. $k$ and $\eta$ are defined in the usual way where $k$ is the wave number and $\eta$ is the intrinsic impedance of the medium. Once the slot is specified, the dipole moment $p_m$ is known within a constant. This constant is defined in (51). Equation (51) can be specialized to arrive at

\[
p_m - \left[ \frac{\alpha}{2\pi} \int_{-L^+_w/2}^{L^+_w/2} I^+(x') \frac{\partial}{\partial z} g(|r - x'\hat{x} - z^+_o\hat{z}|) \, dx' \right. \\
+ \left. \frac{\alpha}{2\pi} \int_{-L^-_w/2}^{L^-_w/2} I^-(x') \frac{\partial}{\partial z} g(|r - x'\hat{x} - z^-_o\hat{z}|) \, dx' \right]_{r=r_0} = 0 . \quad (55) \]

Triangle basis functions for representing the currents in (52) and (55) and pulse functions for testing (52) provide a set of $N^+ + N^- + 1$ linear equations and $N^+ + N^- + 1$ unknowns. This system can be solved in the usual way.

**Simple Transmission Line Formulation**

There are several possible choices for a first-order transmission line solution to the problem of interest. The technique utilized here follows these five steps:

1. Excite one transmission line as if the slots were not present and calculate the current distribution;
2. Allow the current on the transmission line to radiate, and calculate the short circuit magnetic fields on the slot;

3. Calculate the equivalent magnetic current or dipole moment on the slot under the excitation from the previously calculated short circuit magnetic fields;

4. Allow the equivalent magnetic current or dipole moment to radiate on the opposite side from the locally excited transmission line; and

5. Calculate the current on the non-locally excited transmission line due to the radiated fields from the equivalent magnetic current or dipole moment.

In solving this problem, the end effects of the transmission line are neglected as are copper losses. The reader will note several other assumptions inherent in this solution technique. First, the effect of the slot on the line impedance is neglected. Second, the backscatter from the slot onto the exciting transmission line is assumed to cause a negligible change in the current distribution on the line. Finally, it is assumed that the TEM mode is still dominant despite the presence of the slot.

This solution technique is employed for both small and significant slots. For the case of a narrow slot of general length, integral equation techniques are exploited to solve for the equivalent magnetic current on the slot. As with the fully coupled integral equation-based solution, the method of solving for said current is well understood[12]. In the case of short, narrow slots, it is possible to take advantage of Bethe hole theory to solve for the penetration and scattering using the short circuit magnetic fields on either side of the slot as described in (51).

For the transmission line solution, it is necessary to calculate the current on a transmission line due to a radiating source. A variety of models are available for the excitation of a transmission line due to a radiating source[14][15][16]. In this problem, the model of Taylor, Satterwhite, and Harrison is used[15].
Modified Transmission Line Formulation

In this section we present an alternate transmission line-based solution for the problem of two wires separated by a slotted ground plane. We propose a simultaneous solution for the slot effects and the currents on the transmission lines. Though the formulation is specialized to the case of a very short, narrow slot so that we can exploit Bethe hole theory, it is a simple matter to extend this formulation for use with a general length slot whose equivalent magnetic current is solved for by means of integral equation techniques. To facilitate a matrix solution form, we discretize the transmission lines into $N^\pm$ pulse-shaped current elements. Using the Telegrapher’s equations from the Taylor method, one can show the $n^{th}$ current element’s magnitude to be

$$I_n^\pm = I_h(x_n) + I_p(x_n)$$

(56)

where

$$I_h(x_n) = \frac{2V_0^\pm}{Z_c^\pm} \left[ \left( \frac{1 - e^{-jkL_w^\pm}}{e^{-jkL_w^\pm} - e^{jkl_w^\pm}} \right) e^{-jkx_n} \right. $$

$$- \left( \frac{e^{jkl_w^\pm} - 1}{e^{-jkL_w^\pm} - e^{jkl_w^\pm}} e^{jkx_n} \right), \quad -L_w^\pm/2 < x < 0, \quad (57)$$

$$I_h(x_n) = \frac{2V_0^\pm}{Z_c^\pm} \left[ \left( \frac{1 - e^{jkl_w^\pm}}{e^{-jkL_w^\pm} - e^{jkl_w^\pm}} \right) e^{-jkx_n} \right. $$

$$- \left( \frac{e^{-jkL_w^\pm} - 1}{e^{-jkL_w^\pm} - e^{jkl_w^\pm}} e^{jkx_n} \right), \quad 0 \leq x \leq L_w^\pm/2, \quad (58)$$
and

\[
I_p(x_n) = \frac{p_m}{2Z_c} \left[ \frac{e^{-jkL_{w}^{\pm}/2}}{1 - e^{-j2kL_{w}^{\pm}/2}} \right. \\
\left. \left( e^{-jkx_n} \int_{-L_{w}^{\pm}/2}^{L_{w}^{\pm}/2} (e^{jk(x' + L_{w}^{\pm}/2)} - e^{-jk(x' + L_{w}^{\pm}/2)}) K_t(x') dx' ight) - e^{jkx_n} \int_{x_n}^{x_n} e^{jkx'} K_t(x') dx' \right. \\
\left. - e^{-jkx_n} \int_{-L_{w}^{\pm}/2}^{L_{w}^{\pm}/2} (e^{jk(x' - L_{w}^{\pm}/2)} - e^{-jk(x' - L_{w}^{\pm}/2)}) K_t(x') dx' \right].
\]

(59)

In these equations, \( Z_c^{\pm} \) is the characteristic impedance of the two-wire transmission line in the indicated equivalent model, and

\[
K_t(x) = \frac{j k \eta}{2 \pi} [g_z(r_{+h}, r_0) - g_z(r_{-h}, r_0) + g_z(r_{+h}, r_i) - g_z(r_{-h}, r_i)].
\]

(60)

\( r_{\pm h} \) is a position vector locating the transmission line (subscript \( +h \)) or its image (subscript \( -h \)) at \( x \); \( r_0 \) and \( r_i \) are position vectors locating the center of the slot and the center of the slot image, respectively; and \( x_n \) is the \( x \)-location of the \( n \)th current element. \( g_z \) is the \( z \)-derivative of the kernel

\[
g_z(r, r') = \frac{\partial}{\partial z} \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}.
\]

(61)

In addition to the current segments, the dipole moment \( p_m \) is unknown. It is related to the currents on the transmission lines by the \( y \)-directed magnetic field excited by
said currents on the short circuited slot by (51),

\[
p_m + \sum_{n=1}^{N^+} I_n^+ \left[ \frac{\alpha}{2\pi} \int_{x_n-\Delta^+ / 2}^{x_n+\Delta^+ / 2} g_z(r_0, r') dx' \right]
+ \sum_{n=1}^{N^-} I_n^- \left[ \frac{\alpha}{2\pi} \int_{x_n-\Delta^- / 2}^{x_n+\Delta^- / 2} g_z(r_0, r') dx' \right] = 0, \tag{62}
\]

where \( \Delta^\pm \) is the appropriate subdomain length, and \( r' \) locates the appropriate transmission line at \( x' \). Equations (62) and (56) comprise a system of linear equations with \( N^+ + N^- + 1 \) equations and \( N^+ + N^- + 1 \) unknowns. The system can be solved in the usual way.

To further enhance the accuracy of the solution, one can include the end effects of the wires. In our formulation, we approximated the end effects by modeling the thin wire of radius \( a \) as a narrow strip of width \( 4a \) and employing well-known microstrip length modification for open ends[17].

**Results**

Three sets of results are in this chapter. The first presents data from the transmission line and integral equation solutions in contrast to data measured from a test fixture. The second contains examples of a parametric study in which the position of the slot was varied. In these first two sets, the size of the slot is allowed to be significant with respect to wavelength. In the last set, the slot is limited to being short so that Bethe hole theory approximations can be used.
A Comparison with Measured Data

The plots in this section are representative of the data obtained for a variety of wire and slot configurations. The physical parameters of the system for which data is shown are in Table 6. Wire heights are defined as the distance between the wire center and the ground plane to which the wire is parallel. Slot offset is the distance between the slot and the backplane, the ground plane to which the slot is parallel.

Figure 10 shows the real part of the port parameter $Y_{11}$ over a range of frequencies. The agreement between the integral equation calculations and the measured data is excellent, capturing all features and magnitudes with errors on the order of 5% of less when accounting for expected frequency shifts (less than 1.5%). The transmission line solution, of course, shows an ideal transmission line response for the real part of $Y_{11}$, a poor reflection of reality in this instance. In Figure 11, data are shown for the imaginary part of $Y_{11}$. Again, one can see that the integral equation solution shows excellent agreement with the measured data (error on the order of 5% or less). The transmission line solution shows poor agreement despite providing the general behavior of the resonances and reasonable accuracy well away from resonant behavior. At lower frequencies, well below the slot resonance, the agreement between the transmission line solution and the measured data is better. In Figures 12 and 13, one can find the plots of the real and imaginary parts of the transmission parameter, $Y_{21}$, for the same problem. Again, the agreement between the integral equation solution and the measured data demonstrates that the predictions are capturing the physical effects. It is apparent, however, that the transmission line solution technique has neglected significant effects in the problem. Having said this, there is an almost impressionistic agreement between the transmission line solution and the measured data, with agreement being better where the slot is small with respect to wavelength.
Port parameters for other wire and slot lengths were measured and calculated. The data obtained from those measurements and their respective computational plots show similar characteristics to the data shown for this case. The data support the claim that the integral equation solution results provide a reasonable baseline for comparison in more varied physical configurations than could be reasonably set up for measurement.

A Slot Position Parametric Study

In the interest of providing insight into the nature of the breakdown of the transmission line solution, we present a parametric study. In this section, representative data from a parametric study is shown where the slot offset is allowed to vary. The physical parameters for the example data set are in Table 7. Wire height is defined in the same manner as the previous section: the distance from wire center to the parallel ground plane. Slot offset, the parameter being varied, is again the distance between the slot center and the backplane, the ground plane to which it is parallel.

In Figures 14 and 15, the results of the integral equation solution for the port parameter $Y_{11}$ are shown. One can easily see that the integral equation solution shows a transmission line-like response for the cases where the slot offset is such that the slot is not yet underneath the wire. Once it is underneath the wire, the response ceases to behave in a manner which the transmission line solution would accurately model.

Figures 16 and 17 depict the results from the integral equation solution and transmission line solution, respectively, for the real part of the port parameter $Y_{21}$. The real part of the transmission line solution for $Y_{21}$ fails to show good agreement with the integral equation solution at any point except the zero crossings. Indeed, only
the magnitude of the results change with slot offset for the real part of $Y_{21}$.

Figures 18 and 19 contain the results from the integral equation solution and transmission line solution, respectively, for the imaginary part of $Y_{21}$. While the frequency sampling is not quite dense enough to show it, there is fair agreement between the two solution techniques in the case where the slot is not yet underneath the wire. When the slot is underneath the wire, the shape shows poor agreement except for the zero crossings. This again demonstrates the importance of the slot to wire coupling neglected in the transmission line solution. The transmission line solution shows only a change in magnitude with changes in the slot offset.

In this section we have shown that there is an improvement in agreement between the integral equation and transmission line solutions when the coupling between the slot and the wire is weak. Even in those cases, the failure of the transmission line model to account for losses on the transmission line due to the presence of the aperture still causes significant differences in both shape and magnitude of the frequency variations of the port parameters.

Small Aperture Solutions

In this section, we present results from the modified transmission line solution technique, referred to here as the simultaneous solution. Results are compared to those of the fully coupled integral equation solution technique and the first order transmission line solution. The additional complication of copper line losses has been added to the first order transmission line solution for this data. The physical parameters are found in Table 8.

Figures 20 and 21 show the results for port parameter $Y_{11}$. One can see that the simultaneous solution results show very good agreement with the integral equation
solution. Of course, the real part of the transmission line solution does not compare well with the other solutions in this case. One can observe from this data that the frequency breadth of the resonance response is greatly underestimated using the first order transmission line approximation while significant improvement is made when considering the additional complications reflected in the simultaneous solution.

Figures 22 and 23 show the results for port parameter $Y_{21}$. The results from the simultaneous solution and integral equation solution techniques show very good agreement. As with $Y_{11}$, the resonances of the real part show slightly less breadth for the simultaneous solution than the integral equation solution. The transmission line solution compares poorly. The shift in resonant frequency depicted in the transmission line data is again easily accounted for by the lack of end effect correction. The shape of the real part of $Y_{21}$, however, matches poorly with the integral equation results. Though the shape of the imaginary part is reasonable for the transmission line solution, the magnitude is off by a fair degree even when acknowledging the resonant frequency shift.

In this section we have shown that much of the inaccuracy of a basic transmission line solution for a wire near an aperture can be accounted for by means of a technique in which the slot parameters are solved for simultaneously with the transmission line current.

Conclusion

Though intensive numerical solution techniques provide a high degree of accuracy, it is desirable to develop faster, simpler techniques for solving problems related to wires near apertures. While a simple transmission line theory approach with the addition of radiated field coupling provides reasonable results in some problems, such
as when the coupling between wire and aperture is weak, it leaves something to be
desired in the case of a nearby aperture. A simultaneous solution of slot parameters
and transmission line current provides a significant decrease in computation time over
integral equation techniques while maintaining what would be a reasonable level of
accuracy for many applications. While the data and equations presented in the case
of the new, modified transmission line solution technique are for the case of a small
slot, it should be a logical progression to modify the equations to include a slot of
significant size or other small aperture shapes.

References Cited

1968


ground plane,” Interaction Notes, Note 317, February 1977

Aperture Coupling to a Wire,” IEEE Trans. on Nuclear Science, vol. 35, No. 6,
Dec. 1998

vol. 24, No. 4, pp. 456–462, July 1976

AP, vol. 33, No. 6, June 1985


Figure 8: A sketch of two wires separated by a slotted ground plane.

Figure 9: A sketch of the upper side equivalent model for two wires separated by a slotted ground plane.
Figure 10: Real Part of $Y_{11}$ for the Physical Parameters of Table 6: Measurement/-Computation Plots

Figure 11: Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 6: Measurement/Computation Plots
Figure 12: Real Part of $Y_{21}$ for the Physical Parameters of Table 6: Measurement/Computation Plots

Figure 13: Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 6: Measurement/Computation Plots
Figure 14: Real Part of $Y_{11}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots

Figure 15: Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots

53
Figure 16: Real Part of $Y_{21}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots

Figure 17: Real Part of $Y_{21}$ for the Physical Parameters of Table 7: Transmission Line Solution, Parametric Study Plots
Figure 18: Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 7: Integral Equation Solution, Parametric Study Plots

Figure 19: Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 7: Transmission Line Solution, Parametric Study Plots
Figure 20: Real Part of $Y_{11}$ for the Physical Parameters of Table 8: Simultaneous Solution

Figure 21: Imaginary Part of $Y_{11}$ for the Physical Parameters of Table 8: Simultaneous Solution
Figure 22: Real Part of $Y_{21}$ for the Physical Parameters of Table 8: Simultaneous Solution

Figure 23: Imaginary Part of $Y_{21}$ for the Physical Parameters of Table 8: Simultaneous Solution
Table 6: Physical Parameters for Measured Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Radius</td>
<td>0.39 mm</td>
</tr>
<tr>
<td>Slot Width</td>
<td>1.17 mm</td>
</tr>
<tr>
<td>Wire 1 Length</td>
<td>12.7 cm</td>
</tr>
<tr>
<td>Wire 2 Length</td>
<td>17.78 cm</td>
</tr>
<tr>
<td>Slot Length</td>
<td>10.2 cm</td>
</tr>
<tr>
<td>Wire 1 Height</td>
<td>3.5 cm</td>
</tr>
<tr>
<td>Wire 2 Height</td>
<td>3.3 cm</td>
</tr>
<tr>
<td>Slot Offset</td>
<td>11.3 cm</td>
</tr>
</tbody>
</table>

Table 7: Physical Parameters for Parametric Study Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Radius</td>
<td>0.1 mm</td>
</tr>
<tr>
<td>Slot Width</td>
<td>0.5 mm</td>
</tr>
<tr>
<td>Wire 1 Length</td>
<td>12.5 cm</td>
</tr>
<tr>
<td>Wire 2 Length</td>
<td>10 cm</td>
</tr>
<tr>
<td>Slot Length</td>
<td>22 cm</td>
</tr>
<tr>
<td>Wire 1 Height</td>
<td>1 cm</td>
</tr>
<tr>
<td>Wire 2 Height</td>
<td>1 cm</td>
</tr>
</tbody>
</table>
Table 8: Physical Parameters for Small Aperture Solution Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire Radius</td>
<td>0.1 mm</td>
</tr>
<tr>
<td>Slot Width</td>
<td>0.1 mm</td>
</tr>
<tr>
<td>Wire 1 Length</td>
<td>12.5 cm</td>
</tr>
<tr>
<td>Wire 2 Length</td>
<td>10 cm</td>
</tr>
<tr>
<td>Slot Length</td>
<td>1 cm</td>
</tr>
<tr>
<td>Wire 1 Height</td>
<td>2 cm</td>
</tr>
<tr>
<td>Wire 2 Height</td>
<td>2 cm</td>
</tr>
<tr>
<td>Slot Offset</td>
<td>2 cm</td>
</tr>
<tr>
<td>Polarizability</td>
<td>3.00E-8</td>
</tr>
</tbody>
</table>
ON THE INTEGRAL EQUATION SOLUTION OF A WIRE PARALLEL TO A BEND IN A CONDUCTING GROUND PLANE

The potential based Green’s function for the problem of a wire parallel to a bend in a conducting ground plane (a wedge) is discussed. In particular, convergence acceleration for near field and integral equation calculations is addressed. Results are presented for a variety of cases. Verification of the solution is performed using measurements and problem setups where image theory can be used to derive an equivalent problem.

Introduction

The Green’s function for the problem of an axial dipole (electric or magnetic) in the presence of a conducting wedge is well known. Discussions and derivations of various forms of this Green’s function can be found in a variety of works using an array of techniques[1][2][3][4]. Efforts have been focused primarily on the calculation of fields scattered from the wedge in the presence of various forms of excitation. The writers of this manuscript have not found an instance where this Green’s function has been used to perform the integral equation solution for the current on a wire near a conducting wedge. This is likely due to the time consuming nature of computing the Green’s function for observation points very near the source. In the problem of a wire near a conducting, right circular cylinder, the Green’s function has similar difficulties, but it is readily apparent how to extract a free space term[2], allowing the use of traditional thin wire techniques for the computation of the singular part of
the Green’s function[5]. In the case of the conducting wedge, it is not apparent how to extract a free space term from the Green’s function at most wedge angles. In this paper, we propose that a free space term may be both added to and subtracted from the wedge Green’s function to provide similar benefits to those of the mathematical extraction in the circular cylinder scatterer problem. Figure 24 contains a sketch of the problem discussed here. A conducting wedge at an arbitrary angle $\Phi$ protrudes from a conducting backplane. The backplane lies in the $xy$-plane, and the bend of the wedge lies along the $z$-axis. This configuration allows for both computational and measurement-based verification of the solution. A test fixture was built for the case of $\Phi = \frac{3\pi}{2}$ by soldering a bent piece of copper to a copper backplane. The center pin of a coaxial feed is extended through the ground plane to provide the wire. An equivalent model can be developed using image theory to remove the backplane. The equivalent model is depicted in Figure 25. Similar techniques to those used here could be applied to any $z$-directed wire with a local or $TM_z$ excitation in the presence of the wedge. It is assumed in this paper that the excitation varies time harmonically ($e^{j\omega t}$).

In this paper, measurements and solution techniques are compared at a port defined at the interface between the wire and the ground plane (across the generator and generator impedance as depicted in figure 25) and are characterized by the “Y” or short-circuit port parameters. Measurements were performed with a network analyzer (Agilent 8720). The S-parameters from the network analyzer are then converted to short circuit parameters by standard techniques[6]. A short circuited connector was used to establish port reference. In the numerical techniques, we excite the port (shorting the generator impedance) and solve for the port currents, consistent with the definition of Y-parameters. Short circuit parameters can be found with the knowledge of this set of currents and the excitation voltage via standard techniques[6]. Voltage
excitations are modeled as delta gap sources.

Green’s Function

A $z$-directed electric line current in the presence of a $z$-aligned bend produces fields that are $TM_z$. The $TM_z$ vector potential for a $z$-directed electric line current can be written \[2\]

\[
A_z = \mu \frac{1}{2\Phi} \int_{L} I(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} a_w \, d\alpha' dz'
\]

where

\[
a_w = -j2 \int_{0}^{\infty} \sum_{i=1}^{\infty} B_{\nu}(\beta \rho, \beta \rho') \sin(\nu \phi') \sin(\nu \phi) \cos(k_z(z - z'))dk_z.
\]

$\Phi$ is the angle of the bend in the conducting ground plane in radians and $\nu = \frac{i\pi}{\Phi}$. $\rho'$, $\phi'$, and $z'$ locate the surface of the wire in cylindrical coordinates such that $\rho'$ and $\phi'$ are $\alpha'$-dependent. $\rho$, $\phi$, and $z$ define the position of the observation point in cylindrical coordinates. $\beta$ is defined to be $\sqrt{k^2 - k_z^2}$ such that its imaginary part is non-positive. $\mu$ is the permeability of the medium, $k$ is the wave number, and

\[
B_{\chi}(\kappa, \kappa') = \begin{cases} 
J_{\chi}(\kappa')H_{\chi}^{(2)}(\kappa), & \kappa > \kappa' \\
H_{\chi}^{(2)}(\kappa')J_{\chi}(\kappa), & \kappa < \kappa'
\end{cases}.
\]

$J_{\chi}$ and $H_{\chi}^{(2)}$ are the Bessel function and Hankel function of the second kind, respectively, of order $\chi[11]$. 
Integral Equation

The development of the integral equation follows the usual technique for a thin wire with image theory being leveraged to remove the backplane. At \( z = 0 \), we insert a delta-gap source. By applying the boundary condition \( \mathbf{E} \cdot \hat{z} = 0 \) on the wire, one arrives at

\[
j \frac{\omega}{k^2} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) A_z = 2V_0 \delta(z), \quad \text{on the wire,} \tag{66}
\]

where \( \omega \) is the angular frequency. Applying the method of moments with triangle testing functions and pulse basis functions \( (N \text{ unknowns}) \) to (66) and utilizing the approximation

\[
\int_{z_m-\Delta}^{z_m+\Delta} \Lambda(z) f(z) \, dz \approx \Delta f(z_m) \tag{67}
\]

(where \( \Lambda(z) \) is the normalized triangle function with a center of \( z \) and a base of \( 2\Delta \) and \( \Delta \) is the subdomain length) one arrives at[7]

\[
j \frac{\omega}{k^2} \sum_{n=1}^{N} I_n \frac{1}{\Delta} \left[ A_{z,n}(z_m + \Delta; z_n) - 2 \left( 1 - \frac{(k\Delta)^2}{2} \right) A_{z,n}(z_m; z_n) + A_{z,n}(z_m - \Delta; z_n) \right]
\]

\[
= \begin{cases} 
2V_0, & r_m \text{ locates the region including the source} \\
0, & \text{otherwise} 
\end{cases} \tag{68}
\]

where \( r_m \) and \( r_n \) are position vectors locating the centers of the \( m^{th} \) and \( n^{th} \) current elements, respectively; \( z_m \) and \( z_n \) are the \( z \)-components of the position vectors; and

\[
A_{z,n}(\zeta; z_n) = \mu \frac{1}{2\Phi} \int_{z_n-\Delta/2}^{z_n+\Delta/2} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} a_w \bigg|_{z=\zeta} \, d\alpha' \right] \, dz'. \tag{69}
\]

Note that the delta gap source of (68) must be located at the center of a testing function to be accurate. Paralleling techniques reported for wires in the presence of
right circular cylinders, one can switch the order of integration and summation such that (69) can be written

\[ A_{z,n}(\zeta; z_n) = -j\mu \frac{1}{2\Phi} \int_0^\infty 2 \sum_{i=1}^\infty \frac{1}{2\pi} \int_{-\pi}^\pi B_\nu(\beta \rho_m, \beta \rho') \sin(\nu \phi') \sin(\nu \phi_m) d\alpha' h(k_z; \zeta - z_n) dk_z \]  

(70)

where

\[ h(k_z; \zeta) = \Delta \cos(k_z \zeta) \frac{\sin(k_z \Delta/2)}{k_z \Delta/2}. \]  

(71)

Define \( \rho_n \) (and \( \rho_m \)) as the distance from the z-axis to the closest edge of the wire and \( \phi_n \) (and \( \phi_m \)) as the \( \phi \) position of the center of the wire, and set

\[ \rho' = \sqrt{(\rho_n + a)^2 + a^2 - 2a(\rho_n - a) \cos \alpha'} \]  

(72)

and

\[ \phi' = \phi_n - \arctan\left(\frac{a \sin \alpha'}{\rho_n + a - a \cos \alpha'}\right) \]  

(73)

where \( a \) is the wire radius. It can then be shown that the \( \alpha' \) integral of (70) is

\[ \frac{1}{2\pi} \int_{-\pi}^\pi B_\chi(\beta \rho_m, \beta \rho') \sin(\chi \phi') \sin(\chi \phi_m) d\alpha' = \]

\[ \frac{1}{2\pi} \left( \int_{-\pi}^{0^+} + \int_{0^+}^{\pi} \right) J_\chi(\beta \rho_m) H^{(2)}(\beta \rho') \sin(\chi \phi') \sin(\chi \phi_m) d\alpha' \]

\[ = B'_\chi(\beta \rho_m, \beta \rho'). \]  

(74)

Equation (70) can be rewritten in terms of (74) as

\[ A_z(\zeta; z_n) = -j\mu \frac{1}{2\Phi} \int_0^\infty 2 \sum_{i=1}^\infty B'_\nu(\beta \rho_m, \beta \rho') h(k_z; \zeta - z_n) dk_z \]  

(75)
Convergence Acceleration

At wedge angles where it is not possible to recognize the portion of the potential related to the free space term, one can still remove it. Without approximation, one can both add and subtract the free space potential for a z-directed wire, effectively removing the free space term. The effect of doing so is to accelerate the rate of decay in the tail of the $k_z$ integrand in (64). For most problems of interest (wires with $\phi' < \frac{1}{2} \Phi$), we found it effective to add and subtract the potential for the wire and its image in a flat ground plane lying in the $xz$-plane. The potentials in this section are specialized to an observation point on the wire. We begin by presenting the free space potential for a z-directed wire and its image[2]:

$$\hat{A}_z = \mu \frac{1}{4\pi} \int L \int_{-\pi}^\pi a_{w,2} \, d\alpha' \, dz'$$  \hspace{1cm} (76)

where

$$a_{w,2} = K(z - z', \alpha') = \frac{e^{-jk\sqrt{(z-z')^2 + 4a^2 \sin^2(\alpha'/2)}}}{\sqrt{(z-z')^2 + 4a^2 \sin^2(\alpha'/2)}} \tag{77}$$

or

$$a_{w,2} = -j^4 \int_0^\infty \sum_{i=1}^\infty B_i(\beta \rho, \beta \rho') \sin(i\phi) \sin(i\phi') \cos(k_z(z - z')) \, dk_z. \tag{78}$$
By setting $A_z = \hat{A}_z + A_z - \hat{A}_z$ as described above, we can write

$$A_z = \mu \frac{1}{4\pi} \int_{L} I(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} K(z - z'; \alpha') d\alpha' dz'$$

$$+ \mu \frac{1}{2\Phi} \int_{L} I(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} [a_w - \frac{\Phi}{2\pi} a_{w,2}] d\alpha' dz'$$

(79)

implying that

$$A_{z,n}(\zeta; z_n) = \mu \frac{1}{4\pi} \int_{z_n-\Delta/2}^{z_n+\Delta/2} I(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\zeta - z'; \alpha') d\alpha' dz'$$

$$- j\mu \frac{1}{2\Phi} \int_{0}^{\infty} 2 \left[ \sum_{i=1}^{\infty} B_{\nu}'(\beta \rho_m, \beta \rho') - \frac{\Phi}{\pi} \sum_{i=1}^{\infty} B_{\nu}'(\beta \rho_m, \beta \rho') \right] h(k_z; \zeta - z_n) dk_z.$$ 

(80)

On a term by term basis the elements of the two series of (80) approach each other as $k_z$ increases. This can be seen by examining the large argument forms of the Bessel and Hankel functions. While this does not prove improved convergence, it does assist in convincing us that the intuitive premise of our "add and subtract" technique has basis when considering the regularity of the two Fourier series. Next, one can convert the $k_z$ integral to a sum by setting

$$s_0 = \mu \frac{1}{4\pi} \int_{z_n-\Delta/2}^{z_n+\Delta/2} I(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\zeta - z'; \alpha') d\alpha' dz'$$

(81)

$$s_1 = -j\mu \frac{1}{2\Phi} \int_{0}^{(1.1)k} 2 \left[ \sum_{i=1}^{\infty} B_{\nu}'(\beta \rho_m, \beta \rho')$$

$$- \frac{\Phi}{\pi} \sum_{i=1}^{\infty} B_{\nu}'(\beta \rho_m, \beta \rho') \right] d\alpha' h(k_z; \zeta - z_n) dk_z.$$ 

(82)
\[ s_2 = -j\mu \frac{1}{2\Phi} \int_{(1.1)k}^{2k} 2 \left[ \sum_{i=1}^{\infty} B'_\nu(\beta\rho_m, \beta\rho') \right. \\
\quad \left. - \frac{\Phi}{\pi} \sum_{i=1}^{\infty} B'_i(\beta\rho_m, \beta\rho') \right] d\alpha' h(k_z; \zeta - z_n) dk_z, \quad (83) \]

and

\[ s_p = -j\mu \frac{1}{2\Phi} \int_{(p-1)k}^{(p)k} 2 \left[ \sum_{i=1}^{\infty} B'_\nu(\beta\rho_m, \beta\rho') \right. \\
\quad \left. - \frac{\Phi}{\pi} \sum_{i=1}^{\infty} B'_i(\beta\rho_m, \beta\rho') \right] d\alpha' h(k_z; \zeta - z_n) dk_z, \quad p = 3, 4, ... \quad (84) \]

such that

\[ A_{z,n}(\zeta; z_n) = \sum_{p=0}^{\infty} s_p \quad (85) \]

Though it is possible to use a Shank’s transform on this summation, it did not prove necessary in our calculations. With the partial sums

\[ S_P = \sum_{p=0}^{P} s_p, \quad (86) \]

the summation can be considered converged when

\[ \left| \frac{S_{P+q} - S_P}{S_P} \right| \leq \text{Tol}, \quad q = 1, 2, 3 \quad (87) \]

where Tol is the desired tolerance. Typical values for Tol are on the order of \(10^{-3}[5]\).

It is possible to use techniques such as the steepest decent path to further improve the calculation of the \(k_z\) integral. We did not utilize such techniques but did perform contour deformation to make the computation easier. The integral from 0 to \((1.1)k\)
was calculated with the deformation defined by

\[
\int_0^{(1.1)k} = \int_0^{j(0.1)k} + \int_{(1.1+j0.1)k}^{(1.1)k} + \int_{(1.1+j0.1)k}^{(1.1)k}.
\]  

(88)

The deformation was performed to avoid values of \(k_z\) which would make the integrand highly oscillatory (\(0 \leq k_z \leq k\)). The integrals of (83)-(84) remain on the real axis.

Calculation speed can be further improved (weeks/months to days/hours for wire sizes on the order of a wavelength) by increasing the rate of convergence of the summation of \(B'_{\nu}\) and \(B'_{i}\). Kummer’s method and the Shank’s transform are two techniques that are often used for this purpose and can even be used simultaneously[5][8][9][10]. Using the large order approximations of the Bessel and Hankel functions and the series[11][12]

\[
\frac{j}{\pi} \sum_{i=1}^{\infty} \frac{1}{\chi} \left( \frac{z_1}{z_2} \right)^{\chi} \sin(\chi \phi') \sin(\chi \phi) =
\]

\[
j \frac{1}{2\pi b} \ln \left( \frac{1}{\sqrt{1 - 2(z_1/z_2)^b \cos(b(\phi - \phi')) + (z_1/z_2)^{2b}} \right)
\]

\[
- j \frac{1}{2\pi b} \ln \left( \frac{1}{\sqrt{1 - 2(z_1/z_2)^b \cos(b(\phi + \phi')) + (z_1/z_2)^{2b}} \right), \frac{z_1}{z_2} < 1 \quad (89)
\]

where \(\chi = ib\), one can rewrite the sum such that

\[
\sum_{i=1}^{\infty} \int_{\alpha^-}^{\alpha^+} J_{\chi}(\beta \rho_m) H_\chi^{(2)}(\beta \rho') \sin(\chi \phi') \sin(\chi \phi) d\alpha =
\]

\[
\int_{\alpha^-}^{\alpha^+} \left[ j \frac{1}{2\pi b} \ln \left( \frac{1}{\sqrt{1 - 2(\rho_m/\rho')^b \cos(b(\phi - \phi')) + (\rho_m/\rho')^{2b}} \right) \right] d\alpha
\]

\[
- j \frac{1}{2\pi b} \ln \left( \frac{1}{\sqrt{1 - 2(\rho_m/\rho')^b \cos(b(\phi + \phi')) + (\rho_m/\rho')^{2b}} \right) d\alpha
\]

\[
+ \sum_{i=1}^{\infty} \int_{\alpha^-}^{\alpha^+} \left[ J_{\chi}(\beta \rho_m) H_\chi^{(2)}(\beta \rho') - j \frac{1}{\pi} \frac{\rho_m}{\rho'}^{\chi} \right] \sin(\chi \phi') \sin(\chi \phi) d\alpha. \quad (90)
\]
Equation (90) may be used by itself or inside a Shank’s transform to calculate $B'_\nu$ and $B'_i$.

Results

In this section, we present a variety of data for the problem of the wire parallel to a bend in a conducting ground plane. In the first data set, we present example plots of the terms of the $k_z$ integrand in (80), providing insight into the effectiveness of the removal of the free space terms. The second and third sets include input admittance and current distribution results for $\Phi = \frac{3\pi}{2}$ and $\frac{\pi}{2}$, respectively, and contain independent data for comparison purposes. In the final set, we present data for $\Phi = 3.8$ radians.

Set 1: Integrand Plots

The removal of the free space, or image theory, term provides great improvement in the decay rate of the $k_z$ integrand. In order to facilitate the plots, we define an index. Figure 26 displays the relationship between the index of Figures 27-30 and the $k_z$ value. Figure 27 shows the variation in $k_z$ of the Green’s function and free space terms of the $k_z$ integrand in (80) for $z_m = z_n$. Figure 28 depicts the value of the full $k_z$ integrand of (80) for $z_m = z_n$. Similar plots are presented for a case of $z_m \neq z_n$ in Figures 29 and 30. With the support of this data, one can argue strongly for the benefit of removing the free space term. Results for other wire positions ($\rho', \phi'$), etc., are similar. The free space and Green’s function terms have a greater degree of discrepancy for geometries where the wire is strongly affected by the presence of the wedge when $k_z$ is small, as one might expect. In cases where the wire is very close to
the ground plane in comparison to distance from the bend, the integrand terms are almost identical.

\[ \Phi = \frac{3\pi}{2} \] Case

Figures 31 and 32 show current distributions on a wire parallel to the bend in the wedge when excited by a 1 \( (V_0 = 1) \) Volt delta gap source at the interface between the wire and the backplane at frequencies just above and below the wire resonance, respectively. For the purpose of verification, we considered several wire positions near a 270° wedge. Example inputs admittance results are shown in Figure 33. The agreement between the calculated and measured data sets is excellent. The discrepancies at higher frequencies are likely related to our limited ability to account for phase error in our \( S \)-parameter measurement. The discrepancies are equivalent to a very small phase difference between the calculated and measured reflection coefficients at those frequencies.

\[ \Phi = \frac{\pi}{2} \] Case

In this section, the angle of the conducting wedge is 90°. The most efficient way to solve this problem is to use image theory to create an equivalent problem for which there is no wedge present. We present data from both the image theory technique and the Green’s function approach described in this chapter. Figure 34 contains a plot of input admittances versus frequency as calculated by both techniques. Figure 35 consists of an example current distributions on the line as calculated by both techniques. Both figures show good agreement between the Green’s function and image theory solution techniques. In the current distribution plot (figure 35), the
imaginary parts from each solution method are indistinguishable.

Set 4: $\Phi = 3.8$ radians Case

The previous two data sets both include data for angles at a multiple of $\frac{\pi}{2}$. In this section, we present data for the angle of 3.8 radians. Figure 36 contains a plot of the calculated input admittance versus frequency. Figure 37 shows an example calculated current distribution on the line.

Conclusion

In this chapter, we have shown that it is possible to greatly improve the efficiency of computation for the Green’s function for the axially directed dipole in the presence of a conducting wedge by adding and subtracting the free space term. The improved efficiency is such as to make feasible the use of this Green’s function in integral equation solutions. Results from other solution techniques and measurements confirm the accuracy of the solution technique. In future work, the method of adding and subtracting the free space potential could be applied to the axial slot and to other physical configurations in which the free space term cannot be easily extracted from the Green’s function.

References Cited


Figure 24: Sketch of the Wire near a conducting Wedge

Figure 25: Sketch of the equivalent model for the Wire near a conducting Wedge
Figure 26: The value of $k_z$ versus the index used in figures 27-30

Figure 27: Values of Integrand Terms for the $k_z$ integral: $f = 300\,MHz$, $\Phi = \frac{3\pi}{2}$, $\rho = 0.2\,m$, $\phi = 1.728\,\text{radians}$, Wire Radius= 0.001 m, $\Delta = 0.05\,m$, $z_m = z_n$
Figure 28: Value of the Complete Integrand for the $k_z$ integral: $f = 300 MHz$, $\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m - z_n$

Figure 29: Values of Integrand Terms for the $k_z$ integral: $f = 300 MHz$, $\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m - z_n = 3\Delta$
Figure 30: Value of the Complete Integrand for the $k_z$ integral: $k = 2\pi$, $\Phi = \frac{3\pi}{2}$, $\rho = 0.2$ m, $\phi = 1.728$ radians, Wire Radius= 0.001 m, $\Delta = 0.05$ m, $z_m - z_n = 3\Delta$

Figure 31: Current Distribution on a Wire Protruding from a Backplane Near a Wedge: $f = 950 MHz$, $\Phi = \frac{3\pi}{2}$, Length= 0.0617 m, $\rho = 0.0414$ m, $\phi = 1.349$ radians, Wire Radius= 0.0004 m
Figure 32: Current Distribution on a Wire Protruding from a Backplane Near a Wedge: \( f = 1250 \text{MHz}, \Phi = \frac{3\pi}{2}, \text{Length} = 0.0617 \text{ m}, \rho = 0.0414 \text{ m}, \phi = 1.349 \text{ radians}, \text{Wire Radius} = 0.0004 \text{ m}\)

Figure 33: Input admittance versus frequency: \( \Phi = \frac{3\pi}{2}, \text{Length} = 0.0617 \text{ m}, \rho = 0.0414 \text{ m}, \phi = 1.349 \text{ radians}, \text{Wire Radius} = 0.0004 \text{ m}\)
Figure 34: Input admittance versus frequency: $\Phi = \frac{\pi}{2}$, Length= 0.06 m, $\rho = 0.112$ m, $\phi = 0.464$ radians, Wire Radius= 0.0004 m

Figure 35: Current Distribution on a Wire Protruding from a Backplane Near a Wedge : $f = 2000\, MHz$, $\Phi = \frac{\pi}{2}$, Length= 0.06 m, $\rho = 0.112$ m, $\phi = 0.464$ radians, Wire Radius= 0.0004 m
Figure 36: Input admittance versus frequency: $\Phi = 3.8$ radians, Length= 0.06 m, $\rho = 0.0112$ m, $\phi = 1.107$ radians, Wire Radius= 0.0004 m

Figure 37: Current Distribution on a Wire Protruding from a Backplane Near a Wedge : $f = 1200 MHz$, $\Phi = 3.8$ radians, Length= 0.06 m, $\rho = 0.0112$ m, $\phi = 1.107$ radians, Wire Radius= 0.0004 m
AN INTEGRAL EQUATION SOLUTION FOR TWO WIRES COUPLED THROUGH A SLOT IN A BENT GROUND PLANE

The potential-based Green’s functions for the problem of a magnetic dipole transverse to a bend in a conducting ground plane are derived. The fully coupled integral equation solution for two wires separated by an infinite, bent ground plane with narrow slots normal to the bend is presented along with methods of accelerating the convergence of the required sums and integrals. Results from measurements are then used to validate the results.

Introduction

The problem of a dipole near a wedge is not a new one. Extensive work on the calculation of fields generated by a dipole in the presence of a wedge has been done using a variety techniques[1][2][3]. For the specific case of a slot in a conducting wedge, various papers have documented solution techniques for an infinite array of slots[4], or infinite slots[5][6]. In this chapter, we consider the problem of thin wires coupling through finite, narrow slots in a conducting wedge. The wires are assumed to be parallel to the bend in the conducting ground plane, and the slots are assumed to be normal to it. To effect an integral equation solution, one desires to have a set of potential-based Green’s functions for the slot. Since the fields generated by the dipole in the presence of the wedge are neither TE nor TM, a dual potential approach is proposed. The resulting Green’s functions can then be used in conjunction with the Green’s function for a dipole parallel to the bend in a conducting ground plane[7]
to effect a fully-coupled integral equation solution. Acceleration techniques for the integrals and series of interest are also presented.

In this chapter, without loss of generality, we assume the bend in the conducting ground plane to be oriented along the $z$-axis. The wires are therefore also $z$-directed. The slot is assumed to be narrow and on the half of the ground plane lying in the $xz$-plane and is $x$-directed. A backplane is located at $z = 0$, and the angle of the bend is $\gamma$. A sketch is shown in figure 38. Note that figure 38 contains a non-physical, semi-transparent region of the bent ground plane to allow depiction of the interior region wire. This chapter often refers to position vectors $\mathbf{r}$ and $\mathbf{r}'$ and their components. $\mathbf{r}$ and $\mathbf{r}'$ are defined in the normal way locating the observation point and the source point, respectively. They are of the form \( \mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} = \rho\hat{\rho} + \phi\hat{\phi} + z\hat{z} \).

In order to facilitate a comparison of the data from different solution techniques and measurements, it is necessary to characterize the coupling between the two wires. In this chapter, the coupling is characterized by the “Y” or short-circuit port parameters where the ports are defined at the interfaces between the wires and the ground plane (across the generator and generator impedance as depicted in figures 39 and 40). Port excitation was provided in the measurements by extending the center conductor of a coaxial transmission line through the ground plane and connecting it to the wire. Measurements were performed with a network analyzer (Agilent 8720). The S-parameters from the network analyzer are then converted to short circuit parameters by standard techniques[8]. A short circuited connector was used to establish port reference. In the numerical techniques, we sequentially excite a port and solve for the port currents with the non-excited port shorted, consistent with the definition of Y-parameters. Short circuit parameters can be found with the knowledge of this set of currents and the excitation voltage via standard techniques[8]. When integral equation techniques are used to model the wires, voltage sources are modeled as delta
gaps. In the full integral equation solution, we specialize the bend to an angle of $\frac{3\pi}{2}$. In all cases, a time-harmonic excitation ($e^{j\omega t}$) is assumed.

**Potential-Based Green’s Function**

In the previous chapter we discussed the potential-based Green’s function for the wire parallel to a bend in a conducting ground plane and the acceleration of the convergence of the infinite integrals and sums inherent in the solution. To solve for the currents of the geometry described in this chapter, we require the potential-based Green’s functions for a magnetic current transverse to the bend in the conducting ground plane. Let the excitation be a general, outward-directed dipole such that the magnetic volume current density $\mathbf{M}_d$ can be defined

$$\mathbf{M}_d = K\frac{1}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z')\hat{\rho}'$$  \hspace{1cm} (91)

where $K$ is the magnetic current of the dipole and $\ell$ is the length of the dipole. To develop the Green’s functions, we use the dual potential approach, defining both $TE_z$ and $TM_z$ potentials $E^d_z$ and $A^d_z$, respectively, such that[7]

$$E^d_z = -j\omega A^d_z\hat{z} - j\frac{\omega}{k^2} \nabla \left( \frac{\partial A^d_z}{\partial z} \right) + \frac{1}{\varepsilon} (\hat{z} \times \nabla F^d_z) \hspace{1cm} (92)$$

and

$$H^d_z = -\frac{1}{\mu} (\hat{z} \times \nabla A^d_z) - j\omega F^d_z\hat{z} - j\frac{\omega}{k^2} \nabla \left( \frac{\partial F^d_z}{\partial z} \right) \hspace{1cm} (93)$$

where $E^d_z$ and $H^d_z$ are the electric and magnetic fields, respectively, excited by the equivalent magnetic current on the shorted slot, $k$ is the wave number, $\omega$ is the angular frequency, and $\varepsilon$ and $\mu$ are the permittivity and permeability of the medium,
respectively. The potentials $F_d^z$ and $A_d^z$ must both satisfy the homogeneous wave equation of the form

$$\left(\nabla^2_{\rho\phi} + \frac{\partial^2}{\partial z^2} + k^2\right)\Psi = 0, \ \rho \neq \rho'.$$  

(94)

One can use the Fourier transform method with the fourier transform pair

$$\tilde{f} = \int_{-\infty}^{\infty} f e^{jk_z(z-z')}dz$$  

(95)

and

$$f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f} e^{-jk_z(z-z')}dk_z$$  

(96)

to convert (94) to the transform domain wave equation

$$\left(\nabla^2_{\rho\phi} + \beta^2\right)\tilde{\Psi} = 0, \ \rho \neq \rho'$$  

(97)

where $\beta^2 = k^2 - k_z^2$ such that the imaginary part of $\beta$ is non-positive. Following the separation of variables technique outlined in Harrington[9] as applied to (97), one arrives at the differential equations

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR_a(\phi)}{d\rho}\right) + [(\beta\rho)^2 - \nu]R_a(\phi) = 0$$  

(98)

and

$$\frac{d^2\Phi_a(\phi)}{d\phi^2} + \nu^2\Phi_a(\phi) = 0$$  

(99)

where $A_d^z = R_a(\rho)\Phi_a(\phi)$ and $F_d^z = R_f(\rho)\Phi_f(\phi)$. By enforcing the boundary condition that $E_z$ and $E_p$ (and therefore $\tilde{E}_z$ and $\tilde{E}_p$) must be zero at $\phi = 0$ and $\phi = \gamma$, one
obtains the transform domain potentials,

\[
\tilde{A}_d^z = \begin{cases} 
\sum_{i=1}^{\infty} C_i J_\nu(\beta \rho') H^{(2)}_\nu(\beta \rho) \sin(\nu \phi), & \rho > \rho' \\
\sum_{i=1}^{\infty} c_i H^{(2)}_\nu(\beta \rho') J_\nu(\beta \rho) \sin(\nu \phi), & \rho < \rho'
\end{cases}
\] (100)

and

\[
\tilde{F}_d^z = \begin{cases} 
\sum_{i=0}^{\infty} D_i J'_\nu(\beta \rho') H^{(2)}_\nu(\beta \rho) \cos(\nu \phi), & \rho > \rho' \\
\sum_{i=0}^{\infty} d_i H^{(2)'}_\nu(\beta \rho') J_\nu(\beta \rho) \cos(\nu \phi), & \rho < \rho'
\end{cases}
\] (101)

where \( \nu = \frac{iz}{\gamma} \).

Using

\[
\mathbf{J}_d = - \frac{j}{k \eta} \nabla \times \mathbf{M}_d,
\] (102)

the equation relating the magnetic volume current density to an equivalent electric volume current density, one can show that the the electric surface current can be written in the transform domain as

\[
\tilde{\mathbf{J}}_s = - \frac{k_z}{k \eta} K \ell \frac{1}{\rho'} \delta(\phi - \phi') \cos(\phi - \phi') \hat{\phi} \\
+ j \frac{1}{k \eta} K \ell \left[ \left( \frac{1}{\rho'} \delta(\phi - \phi') - \frac{1}{\rho'^2} \delta(\phi - \phi') \right) \sin(\phi - \phi') \\
+ \frac{1}{\rho'^2} \frac{\partial}{\partial \phi} \left[ \delta(\phi - \phi') \right] \cos(\phi - \phi') \right] \hat{z}
\] (103)

where \( \eta \) is the intrinsic impedance. From this equivalent surface current, we can see that the remaining conditions are that \( E_\phi \) and \( E_z \) must be continuous at \( \rho = \rho' \) and that \( H_\phi \) and \( H_z \) must obey the “jump” condition at \( \rho = \rho' \) defined by

\[
\hat{\rho} \times (\mathbf{H}|_{\rho=\rho'^+} - \mathbf{H}|_{\rho=\rho'^-}) = \mathbf{J}_s.
\] (104)
The two continuity conditions imply that \( C_i = c_i \) and \( D_i = d_i \). Application of the “jump” conditions requires the Fourier series expansions

\[
\cos(\phi - \phi')\delta(\phi - \phi') = \sum_{i=0}^{\infty} \varepsilon_i \frac{1}{\gamma} \cos(\nu\phi') \cos(\nu\phi)
\]

(105)

and

\[
\cos(\phi - \phi') \frac{d}{d\phi} \delta(\phi - \phi') = -\sum_{i=1}^{\infty} \varepsilon_i \nu \frac{1}{\gamma} \cos(\nu\phi') \sin(\nu\phi)
\]

(106)

where \( \varepsilon_i \) is Neumann’s number, i.e.

\[
\varepsilon_i = \begin{cases} 
1, & i = 0 \\
2, & \text{otherwise}
\end{cases}
\]

(107)

From these expansions, one can show that the transform domain potentials can be written

\[
\tilde{A}_d = -\varepsilon k\eta K\ell \frac{1}{\rho} \frac{\pi}{2\gamma} \sum_{i=1}^{\infty} \varepsilon_i \nu \frac{1}{\beta^2} B_\nu(\beta\rho, \beta\rho') \cos(\nu\phi') \sin(\nu\phi)
\]

(108)

and

\[
\tilde{F}_d = \varepsilon K\ell \frac{\pi}{2\gamma} \sum_{i=0}^{\infty} \varepsilon_i k_z \frac{\partial}{\partial \rho'} B_\nu(\beta\rho, \beta\rho') \cos(\nu\phi') \cos(\nu\phi)
\]

(109)

where

\[
B_\chi(\kappa, \kappa') = \begin{cases} 
J_\chi(\kappa') H_\chi^{(2)}(\kappa), & \kappa > \kappa' \\
H_\chi^{(2)}(\kappa') J_\chi(\kappa), & \kappa < \kappa'
\end{cases}
\]

(110)

\( J_\chi \) and \( H_\chi^{(2)} \) are the Bessel function and the Hankel function of the second kind, respectively[11]. By taking the inverse Fourier transform and utilizing the properties of even- and odd-functions, one can write the potentials as

\[
A_d = \varepsilon \frac{1}{2\gamma} K\ell a_s(\mathbf{r}, \mathbf{r'})
\]

(111)
and

\[ F_z^d = \varepsilon \frac{1}{2\gamma} K \ell f_s(r; r') \]  \hspace{1cm} (112)

where

\[ a_s(r; r') = -k\eta \frac{1}{\rho'} \int_0^\infty \sum_{i=1}^\infty \varepsilon \nu \frac{1}{\beta^2} B_i(\beta \rho, \beta \rho') \cos(\nu \phi') \sin(\nu \phi) \cos(k_z(z-z')) dk_z, \]  \hspace{1cm} (113)

\[ f_s(r; r') = -j \int_0^\infty \sum_{i=0}^\infty \varepsilon \frac{k_z}{\beta^2} \frac{\partial}{\partial \rho'} B_i(\beta \rho, \beta \rho') \cos(\nu \phi') \cos(\nu \phi) \sin(k_z(z-z')) dk_z. \]  \hspace{1cm} (114)

These potentials are valid in the limiting sense as \( \phi' \) approaches zero. The add/subtract method used in the next section requires the \( TE_z \) and \( TM_z \) potentials for a \( \rho' \)-directed magnetic dipole in free space. Following a parallel procedure to the one used above, one can write the potentials

\[ A_z^o = \varepsilon \frac{1}{4\pi} K \ell a_z^o(r, r') \]  \hspace{1cm} (115)

and

\[ F_z^o = \varepsilon \frac{1}{4\pi} K \ell f_z^o(r, r') \]  \hspace{1cm} (116)

where

\[ a_z^o(r, r') = -k\eta \frac{1}{\rho} \int_0^\infty \sum_{i=1}^\infty \varepsilon_i i \frac{1}{\beta^2} B_i(\beta \rho, \beta \rho') \sin(i(\phi - \phi')) \cos(k_z(z-z')) dk_z \]  \hspace{1cm} (117)

and

\[ f_z^o(r, r') = -j \int_0^\infty \sum_{i=0}^\infty \varepsilon_i \frac{k_z}{\beta^2} \frac{\partial}{\partial \rho'} B_i(\beta \rho, \beta \rho') \cos(i(\phi - \phi')) \sin(k_z(z-z')) dk_z. \]  \hspace{1cm} (118)
As stated in the introduction, we develop the integral equation for \( \gamma = \frac{3\pi}{2} \). Using the equivalence principle, one can separate the problem into two equivalent problems that are coupled by the effects of the slot. The reader will note that while the solution technique presented here could be applied to a general angle, the use of \( \frac{3\pi}{2} \) allows the use of image theory and the standard thin-wire Green’s function rather than the specialized Green’s functions for the inner equivalent problem. Sketches of the inner and outer equivalent problems are shown in figures 39 and 40, respectively. By applying the the boundary conditions

\[
\mathbf{E} \cdot \hat{z} = 0, \text{ on the wires,} \quad (119)
\]

and

\[
(\mathbf{H} \cdot \hat{x})_{y=0^+} = (\mathbf{H} \cdot \hat{x})_{y=0^-}, \text{ on the slots,} \quad (120)
\]

one can develop the integral equation solution in the usual way[12][13]. We can write the equations resulting from application of the boundary conditions as

\[
\frac{j \omega}{k^2} \left[ k^2 A^{w+}_z + \frac{\partial^2}{\partial z^2} A^{w+}_z \right] - \frac{1}{\epsilon} \frac{\partial}{\partial y} F^{s+}_x = 2V^+ \delta(z), \text{ on the inner wire,} \quad (121)
\]

\[
\frac{j \omega}{k^2} \left[ k^2 A^{w-}_z + \frac{\partial^2}{\partial z^2} A^{w-}_z + k^2 A^s_x + \frac{\partial^2}{\partial z^2} A^s_x \right] = 2V^- \delta(z), \text{ on the outer wire,} \quad (122)
\]
and

\[
- \frac{1}{\mu} \frac{\partial}{\partial y} A^w_+ + j \frac{\omega}{k^2} [k^2 F^s_+ + \frac{\partial^2}{\partial x^2} F^s_+] \\
- j \frac{\omega}{k^2} \frac{\partial^2}{\partial \rho \partial z} F^s_- + \frac{11}{\mu \rho} \frac{\partial}{\partial \phi} A^s_- \\
+ \frac{11}{\mu \rho} \frac{\partial}{\partial \phi} A^w_- = 0, \text{ on the slot. (123)}
\]

As previously, \( A \) and \( F \) represent magnetic and electric vector potentials, respectively. Superscripts \( s \) and \( w \) denote potentials arising from the magnetic current and the electric current, respectively; and superscripts \( + \) and \( - \) indicate potentials from the inner and outer equivalent problems, respectively. \( 2V^+ \) and \( 2V^- \) are the magnitudes of the delta gap sources on the inner and outer wires, respectively. The potentials from the inner problem use image theory to account for the presence of the bent ground plane. The potentials for (121) are[12][14]

\[
A^w_+ = \frac{\mu}{4\pi} \int_{-\ell^+}^{\ell^+} I^+(z') g_1^w(r; r') dz'
\]

and

\[
F^s_+ = \frac{\varepsilon}{2\pi} \int_{L_s} K(x') g_2^w(r; r') dx'
\]

where

\[
g_1^w(r; r') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-jk \sqrt{(z-z')^2 + 4a_+^2 \sin^2(\alpha/2)}}}{\sqrt{(z-z')^2 + 4a_+^2 \sin^2(\alpha/2)}} d\alpha \\
- \frac{e^{-jkR(r; r' + 2x'\hat{x})}}{R(r; r' + 2x'\hat{x})} - \frac{e^{-jkR(r; r' + 2y'\hat{y})}}{R(r; r' + 2y'\hat{y})} \\
+ \frac{e^{-jkR(r; r' + 2x'\hat{x} + 2y'\hat{y})}}{R(r; r' + 2x'\hat{x} + 2y'\hat{y})}, \quad (126)
\]
\[ g_z^w(r; r') = \frac{e^{-jkR(r; r')}}{R(r; r')} + \frac{e^{-jkR(r; r' + 2z' \hat{z})}}{R(r; r' + 2z' \hat{z})} - \frac{e^{-jkR(r; r' + 2x' \hat{x})}}{R(r; r' + 2x' \hat{x})} - \frac{e^{-jkR(r; r' + 2x' \hat{x} + 2z' \hat{z})}}{R(r; r' + 2x' \hat{x} + 2z' \hat{z})}, \]  
\[ R(r; r') = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \]

where \( a_+ \) is the wire radius of the inner wire, \( \ell^+ \) is the length of the inner wire (the length of the wire in the equivalent inner problem is \( 2\ell^+ \)), \( I^+ \) is the current on the outer wire, \( 2w \) is the slot width, \( L_s \) indicates the slot’s \( x' \) variation, and \( K \) is the \( x \) variation of the equivalent magnetic current on the slot. \( K \) is defined to be consistent with the narrow slot current definition \[14\]

\[ M_d(x, z) = \frac{1}{\pi} \sqrt{\frac{w}{w^2/4 - z^2}} K(x), \text{ on the slot.} \]  

The function \( R \) presented above is approximate. The variation in the distance between the source and the observation points due to the wire radius and slot width are neglected in terms utilizing \( R \). When the potentials are specialized for use in (123) rather than (121), the forms are not identical to those in (124) and (125) but rather

\[ A_z^{w+} = \frac{\mu}{4\pi} \int_{-\ell^+}^{\ell^+} I^+(z')g_1^s(r; r')dz' \]  

and

\[ F_x^{s+} = \frac{\varepsilon}{2\pi} \int K(x')g_2^s(r; r')dx' \]
where

\[
g_1^s(r; r') = \frac{e^{-jkR(r; r')}}{R(r; r')} - \frac{e^{-jkR(r; r' + 2x'\hat{x})}}{R(r; r' + 2x'\hat{x})} - \frac{e^{-jkR(r; r' + 2y'\hat{y})}}{R(r; r' + 2y'\hat{y})} + \frac{e^{-jkR(r; r' + 2x'\hat{x} + 2y'\hat{y})}}{R(r; r' + 2x'\hat{x} + 2y'\hat{y})}
\]

(132)

and

\[
g_2^s(r; r') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-jk\sqrt{(z-z')^2 + 4(w/2)^2}}}{\sqrt{(z-z')^2 + 4(w/2)^2}} d\alpha
\]

\[
+ \frac{e^{-jkR(r; r' + 2x'\hat{x})}}{R(r; r' + 2x'\hat{x})} - \frac{e^{-jkR(r; r' + 2z'\hat{z})}}{R(r; r' + 2z'\hat{z})} - \frac{e^{-jkR(r; r' + 2x'\hat{x} + 2z'\hat{z})}}{R(r; r' + 2x'\hat{x} + 2z'\hat{z})}
\]

(133)

The rest of the potentials are

\[
A_{z-}^s = -\varepsilon \frac{1}{3\pi} \int_{L_s} K(\rho') \frac{1}{\pi} \int_{h-w/2}^{h+w/2} \frac{1}{\sqrt{w^2/4 - (z' - h)^2}}
\]

\[
[a_s(r; r') + a_s(r; r' - 2z'\hat{z})]dz'd\rho',
\]

(134)

\[
F_{z-}^s = -\varepsilon \frac{1}{3\pi} \int_{L_s} K(\rho') \frac{1}{\pi} \int_{h-w/2}^{h+w/2} \frac{1}{\sqrt{w^2/4 - (z' - h)^2}}
\]

\[
[f_s(r; r') + f_s(r; r' - 2z'\hat{z})]dz'd\rho',
\]

(135)

and

\[
A_{z-}^w = \mu \frac{1}{3\pi} \int_{-\ell}^{\ell} I^-(z') \frac{1}{2\pi} \int_{-\pi}^{\pi} a_w(r; r') d\alpha' dz'
\]

(136)
where, as can be seen from the previous chapter,

\[
 a_w(r; r') = -j \int_0^\infty \sum_{i=1}^\infty \varepsilon_i B_\nu(\beta \rho, \beta \rho') \sin(\nu \phi') \sin(\nu \phi) \cos(k_z(z - z')) dk_z, \tag{137}
\]

\( w \) is the slot width, \( h \) is the \( z \)-position of the slot, \( I^- \) is the current on the outer wire, and \( \ell^- \) is the length of the outer wire (the length of the outer wire in the exterior equivalent problem is \( 2\ell^- \)). Note that in this formulation the effect of the slot images resulting from the backplane has been included in the slot kernels. The effect of the wire images in the backplane has not been included in the wire kernels. It is possible to include the effect of wire images in the backplane in the wire kernels and some advantage in computation time and computer memory resources exists in doing so.

Equations (121)-(123) can be solved by the moment method with triangle testing functions and pulse basis functions[15]. The resulting matrix equation can be written

\[
 \begin{bmatrix}
 [W^+] & [WS^+] & [0] \\
 [SW^+] & [S] & [SW^-] \\
 [0] & [WS^-] & [W^-]
\end{bmatrix}
 \begin{bmatrix}
 [I^+] \\
 [K] \\
 [I^-]
\end{bmatrix}
 =
 \begin{bmatrix}
 [E^+] \\
 [H] \\
 [E^-]
\end{bmatrix}. \tag{138}
\]

The first row is derived from the current expansions and testing as applied to (121). The second row is derived from the current expansions and testing as applied to (123). The third row is derived from (122). For convenience, the subdomain lengths on each wire and the slot are assumed to be uniform. In other words, the subdomain lengths on a wire are all the same, but not necessarily the same as the subdomain lengths on the other wire or the slot. The elements of the submatrices \([W^+], [WS^+], \) and \([SW^+]\) are generated by the moment method expansion and testing of

\[
 j \frac{\omega}{k^2} [k^2 A^w_+ + \frac{\partial^2}{\partial z^2} A^w_+],
\]
and

\[- \frac{1}{\epsilon} \frac{\partial}{\partial y} F_{s+},\]

respectively, and are well-known as are the terms of the submatrices on the right hand side of (138)[12][14]. The elements of the submatrices denoted “[0]” are identically zero for every term. The computation of the terms of \([W^-]\) from the expansion and testing of

\[j \frac{\omega}{k^2} [k^2 A_z^{w-} + \frac{\partial^2}{\partial z^2} A_z^{w-}]\]

on the outer wire was discussed in detail in a previous chapter. We therefore concentrate on the computation of the elements of \([S]\), \([WS^-]\), and \([SW^-]\). In this section, the position vectors \(\mathbf{r}_m\) and \(\mathbf{r}_n\) are defined as discussed in the introduction such that \(\mathbf{r}_m\) and \(\mathbf{r}_n\) locate the centers of the \(m^{th}\) and \(n^{th}\) subdomains, respectively. \(\mathbf{r}_m\) locates the testing subdomain for a matrix term, and \(\mathbf{r}_n\) locates the source subdomain for a matrix term.

We begin with the submatrix \([WS^-]\) whose elements come from the moment method expansion and testing of

\[j \frac{\omega}{k^2} [k^2 A_z^{s-} + \frac{\partial^2}{\partial z^2} A_z^{s-}]\]

on the outer wire and relate to the contribution of the equivalent magnetic current on the slot to the \(z\)-directed electric field on the wire in the exterior region equivalent problem. If the approximation

\[\int_{z_m-\Delta}^{z_m+\Delta} \Lambda_m(z) f(z) dz \approx \Delta f(z_m) \quad (139)\]

92
is used, the elements of this submatrix can be written

\[(WS^-)_{mn} = j \frac{1}{3\pi \eta k} \Delta_w A_w^s(z_m + \Delta_w; z_n) - 2 \left( 1 - \frac{(k\Delta_w)^2}{2} \right) A_w^s(z_m; z_n) + A_w^s(z_m - \Delta_w; z_n) \]  

(140)

where

\[
A_w^s(z_m; z_n) = - \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\pi} \int_{h-w/2}^{h+w/2} \frac{1}{\sqrt{w^2/4 - (z' - h)^2}} [a_s(r_m; r') + a_s(r_m; r' - 2z'\hat{z})] dz'd\rho',
\]

(141)

and \(\Delta_w\) and \(\Delta_s\) are the subdomain sizes on the outer wire and the slot, respectively.

\(\Lambda_m(z)\) is a triangle function centered at \(r_m\) such that

\[
\Lambda_m(z) = \begin{cases} 
1 - \frac{|z - z_m|}{\Delta_w}, & \rho \in (z_{m-1}, z_{m+1}) \\
0, & \text{otherwise}
\end{cases}.
\]

(142)

Recognizing the integral[16]

\[
\frac{1}{\pi} \int_{h-w/2}^{h+w/2} \frac{\cos(k_z(z - z'))}{\sqrt{w^2/4 - (z' - h)^2}} dz' = \cos(k_z(z - h)) J_0(k_z w/2)
\]

(143)
and changing the order of integration and summation, one can rewrite (141) as

\[ A_w^* (z_m; z_n) = \eta k \int_0^\infty \sum_{i=1}^{\infty} \left[ \varepsilon_i \frac{\nu}{\beta^2} \int_{\rho_n - \Delta s / 2}^{\rho_n + \Delta s / 2} \frac{1}{\rho'} B_{\nu} (\beta \rho_m, \beta \rho') d\rho' \sin (\nu \phi_m) \right] \left[ \cos (k_z (z_m - h)) + \cos (k_z (z_m + h)) \right] J_0 (k_z w) dk_z \quad (144) \]

One can use Kummer’s method[17] and the Shank’s transform[18][19][20] to accelerate the convergence of the summation. The \( k_z \) integral for this term converges rapidly. To avoid regions where the integral is more difficult to perform, a contour deformation may be used such as the one described for the problem of a wire parallel to bend in the preceding chapter. Using Kummer’s method, one can write the summation such that

\[ \sum_{i=1}^{\infty} \left[ \frac{\nu}{\beta^2} \int_{\rho_n - \Delta s / 2}^{\rho_n + \Delta s / 2} \frac{1}{\rho'} B_{\nu} (\beta \rho_m, \beta \rho') d\rho' \sin (\nu \phi_m) \right] = \]

\[ - j \frac{1}{\pi} \int_{\rho_n - \Delta s / 2}^{\rho_n + \Delta s / 2} \frac{1}{\rho'} \left[ \frac{(\rho^< / \rho^>)^b \sin (b \phi_m)}{\rho^3} - 2 (\rho^< / \rho^>)^b \cos (b \phi_m) + (\rho^< / \rho^>)^b \right] d\rho' \sin (\nu \phi_m) \quad (145) \]

where \( \nu = ib \) (in this problem, of course, \( b = \frac{2}{3} \)) and \( \rho^< \) and \( \rho^> \) are \( \rho_m \) and \( \rho' \) such that \( \rho^< < \rho^> \) at all times.

The elements of the submatrix \([SW^-]\) are from the expansion and testing of the term

\[ \frac{1}{\mu} \frac{\partial}{\partial \phi} A^w_- \]

on the slot and relate to the contribution of the electric current on the wire to the \( x \)-directed magnetic field on the slot in the exterior equivalent problem. We can write
the elements of this submatrix as

\[(SW^-)_{mn} = \frac{1}{3\pi} \frac{\partial}{\partial \phi} \int_{\rho - \Delta}^{\rho + \Delta} 1 \int_{z - \Delta}^{z + \Delta} 1 \int_{-\pi}^\pi a_w(r; r')d\alpha' dz'd\rho \]  

(146)

We do not utilize (139) in this submatrix to avoid convergence problems when \( \rho_m = \rho_n \). As long as distance between the wire and the slot is large compared to the wire radius, the \( \alpha' \) integral can be neglected such that

\[(SW^-)_{mn} = \frac{1}{3\pi} \frac{\partial}{\partial \phi} \int_{\rho - \Delta}^{\rho + \Delta} 1 \int_{z - \Delta}^{z + \Delta} a_w(r; r')dz' \]  

(147)

By rearranging the order of integration and differentiation and recognizing that

\[h(\zeta; \Delta) = \int_{-\Delta/2}^{\Delta/2} \cos(k_z(\zeta - z'))dz'\]  

(148)

\[= \Delta \cos(k_z\zeta) \frac{\sin(k_z\Delta/2)}{k_z\Delta/2},\]  

(149)

(147) becomes

\[(SW^-)_{mn} = -\frac{j}{3\pi} \int_0^\infty \sum_{i=1}^\infty [\gamma_i \nu \int_{\rho - \Delta}^{\rho + \Delta} 1 \int_{\rho - \Delta}^{\rho + \Delta} B_i(\beta\rho_m, \beta\rho_n) d\rho \sin(\nu\phi)] h(h - z_n; \Delta)dk_z. \]  

(150)

As with \((WS^-)_{mn}\), the convergence of the summation can be accelerated using the Shank’s transform and Kummer’s method. The form of the summation allows the use of (145) with obvious modifications. Again, the \( k_z \) integral converges rapidly, but still benefits from the contour deformation described in the preceding chapter.
[\mathcal{S}] contains the terms from the moment method expansion and testing of

\[ j \frac{\omega}{k^2} k^2 F_x^{s+} + \frac{\partial^2}{\partial x^2} F_x^{s+} \] \[ - j \frac{\omega}{k^2} \frac{\partial^2}{\partial \rho \partial z} F_z^{s-} + \frac{1}{\mu} \frac{\partial}{\partial \phi} A_z^{s-} \]

on the slot and relate to the contribution of the equivalent magnetic current on the slot to the \( x \)-directed magnetic field on the slot. The convergence of the \( k_z \) integral inherent in this problem is very slow. The convergence can be improved through a process of adding and subtracting from the Green’s function-based potential the free space potential for an equivalent magnetic current on the shorted slot and its images in \( xz \) and \( xy \) ground planes. To effect this approach, we find the terms of [\mathcal{S}] from the moment method expansion and testing of

\[ j \frac{\omega}{k^2} k^2 F_x^{s+} + \frac{\partial^2}{\partial x^2} F_x^{s+} \] \[ - j \frac{\omega}{k^2} \frac{\partial^2}{\partial \rho \partial z} F_z^{s-} + \frac{1}{\mu} \frac{\partial}{\partial \phi} A_z^{s-} \] \[ ^{(151)} \]

on the slot where the primed (’) potentials can be written

\[ F_x^{s+} = \frac{\varepsilon}{2\pi} \int_{L_s} K(x') g_3(r; r') dx', \] \[ ^{(152)} \]

\[ A_z^{s-} = -\varepsilon \frac{1}{3\pi} \int_{L_s} K(\rho') \frac{1}{\pi} \int_{h-w/2}^{h+w/2} \frac{1}{\sqrt{w^2/4 - (z' - h)^2}} \]

\[ \left[ a_s(r; r') + a_s(r; r' - 2z' \hat{z}) - \frac{3}{2} a_o^s(r; r') - \frac{3}{2} a_o^s(r; r' - 2z' \hat{z}) \right] \] \[ ^{(153)} \]
and

\[ F_{s}^{s-} = -\varepsilon \frac{1}{3\pi} \int_{L_s} K(\rho') \frac{1}{\pi} \int_{h+w/2}^{h-w/2} \frac{1}{\sqrt{w'^2/4 - (z' - h)^2}} \]

\[ \{ f_s(\mathbf{r}; \mathbf{r}') + f_s(\mathbf{r}; \mathbf{r}' - 2z' \hat{z}) - \frac{3}{2} f_s^\circ(\mathbf{r}; \mathbf{r}') - \frac{3}{2} f_s^\circ(\mathbf{r}; \mathbf{r}' - 2z' \hat{z}) \} dz'd\rho'. \quad (154) \]

The previously undefined kernel \( g_3 \) can be written

\[ g_3(\mathbf{r}; \mathbf{r}') = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{-jk\sqrt{(z-z')^2 + 4(w/2)^2 \sin^2(\alpha/2)}}}{\sqrt{(z-z')^2 + 4(w/2)^2 \sin^2(\alpha/2)}} d\alpha 
+ 2 \frac{e^{-jkR(\mathbf{r}; \mathbf{r}' + 2z' \hat{z})}}{R(\mathbf{r}; \mathbf{r}' + 2z' \hat{z})} - \frac{e^{-jkR(\mathbf{r}; \mathbf{r}' - 2z' \hat{z})}}{R(\mathbf{r}; \mathbf{r}' - 2z' \hat{z})}. \quad (155) \]

Applying moment method expansion and testing to (151) and leveraging the approximation (139) one can define the terms of \([S]\) to be

\[ S_{mn} = j \frac{1}{2\pi \eta k} \frac{1}{\Delta_s} \left[ F_{s}^{s+}(\rho_m + \Delta_s; \rho_n) - 2 \left( 1 - \frac{(k\Delta_s)^2}{2} \right) F_{s}^{s+}(\rho_m; \rho_n) 
+ F_{s}^{s+}(\rho_m - \Delta_s; \rho_n) \right] - j \frac{1}{3\pi \eta k} F_{s}^{s-} + \frac{1}{3\pi \eta^2} A_s \quad (156) \]

where

\[ F_{s}^{s+} = \int_{x_n - \Delta_s/2}^{x_n + \Delta_s/2} g_3(\mathbf{r}_m; \mathbf{r}_n + x' \hat{x}) dx', \quad (157) \]

\[ A_s = - \int_{\rho_m - \Delta_s/2}^{\rho_m + \Delta_s/2} \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\pi} \int_{z_n - w/2}^{z_n + w/2} \frac{1}{\sqrt{w'^2/4 - (z' - z_n)^2}} \]

\[ \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[ a_s(\mathbf{r}; \mathbf{r}') + a_s(\mathbf{r}; \mathbf{r}' - 2z' \hat{z}) - \frac{3}{2} a_s^\circ(\mathbf{r}; \mathbf{r}') - \frac{3}{2} a_s^\circ(\mathbf{r}; \mathbf{r}' - 2z' \hat{z}) \right] dz'd\rho'd\rho, \quad (158) \]
and

\[
F_s^- = - \int_{\rho_m - \Delta_s}^{\rho_m + \Delta_s} \Lambda_m(\rho) \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\pi} \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\sqrt{w^2/4 - (z' - z_n)^2}} \frac{\partial^2}{\partial \rho \partial z} \left[ f_s(\mathbf{r}, \mathbf{r}') + f_s(\mathbf{r}, \mathbf{r}' - 2z' \hat{z}) - \frac{3}{2} f_s(\mathbf{r}, \mathbf{r}') - \frac{3}{2} f_s(\mathbf{r}, \mathbf{r}' - 2z' \hat{z}) \right] dz' d\rho' d\rho.
\]

(159)

\[\Lambda_m(\rho)\] is defined in a fashion parallel to that for \[\Lambda_m(z)\] in (142). By changing the order of integration and derivation and applying (143), one can write (158) and (159) as

\[
A_s = k \eta \int_0^\infty \frac{1}{\beta^2} \sum_{i=1}^\infty \varepsilon_i \int_{\rho_m - \Delta_s/2}^{\rho_m + \Delta_s/2} \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\rho \rho'} B_v(\beta \rho, \beta \rho') d\rho' d\rho \\
- \frac{3}{2} \sum_{i=1}^\infty \varepsilon_i \int_{\rho_m - \Delta_s/2}^{\rho_m + \Delta_s/2} \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{1}{\rho \rho'} B_i(\beta \rho, \beta \rho') d\rho' d\rho \\
\left[ \cos(k_z(z_m - z_n)) + \cos(k_z(z_m + z_n)) \right] J_0\left( \frac{k_z w}{2} \right) dk_z,
\]

(160)

and

\[
F_s^- = j \int_0^\infty \frac{k_z^2}{\beta^2} \sum_{i=0}^\infty \varepsilon_i \int_{\rho_m - \Delta_s}^{\rho_m + \Delta_s} \Lambda_m(\rho) \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{\partial^2}{\partial \rho \partial \rho'} B_v(\beta \rho, \beta \rho') d\rho' d\rho \\
- \frac{3}{2} \sum_{i=0}^\infty \varepsilon_i \int_{\rho_m - \Delta_s}^{\rho_m + \Delta_s} \Lambda_m(\rho) \int_{\rho_n - \Delta_s/2}^{\rho_n + \Delta_s/2} \frac{\partial^2}{\partial \rho \partial \rho'} B_i(\beta \rho, \beta \rho') d\rho' d\rho \\
\left[ \cos(k_z(z_m - z_n)) + \cos(k_z(z_m + z_n)) \right] J_0\left( \frac{k_z w}{2} \right) dk_z.
\]

(161)

In (160) and (161) it is important to calculate the integrals over \(\rho\) and \(\rho'\) carefully. If approximations are used when \(\rho_m = \rho_n\) or \(\rho_m = \rho_n \pm 1\), it is possible to end up with a series which seems to diverge. The integrals over \(\rho\) and \(\rho'\) can be performed exactly
such that (161) becomes

\[
F_s^* = j \int_0^\infty \frac{k_z^2}{\beta^2 \Delta_s} \left[ \sum_{i=0}^{\infty} \varepsilon_i \left[ \int_{\rho_m}^{\rho_m + \Delta_s} - \int_{\rho_m - \Delta_s}^{\rho_m} \right] [B_\nu(\beta \rho, \beta(\rho_n + \Delta_s/2)) - B_\nu(\beta \rho, \beta(\rho_n - \Delta_s/2))]d\rho \right. \\
- \frac{3}{2} \sum_{i=0}^{\infty} \varepsilon_i \left[ \int_{\rho_m}^{\rho_m + \Delta_s} - \int_{\rho_m - \Delta_s}^{\rho_m} \right] \left[ B_i(\beta \rho, \beta(\rho_n + \Delta_s/2)) - B_i(\beta \rho, \beta(\rho_n - \Delta_s/2))]d\rho \right] \\
\left. \ast \left[ \cos(k_z(z_m - z_n)) + \cos(k_z(z_m + z_n)) \right] J_0(\frac{w}{2})dk_z \right) \quad (162)
\]

It is again worthwhile to use convergence acceleration techniques in this sum. Applying Kummer's method, one can write the summations of (160) and (162) as[11][16]

\[
\sum_{i=1}^{\infty} \chi^2 \int_{\rho^-}^{\rho^+} \int_{\rho'^-}^{\rho'^+} \frac{1}{\rho \rho'} B_\chi(\beta \rho, \beta \rho')d\rho' d\rho = \\
j \frac{b}{\pi} \int_{\rho^-}^{\rho^+} \int_{\rho'^-}^{\rho'^+} \frac{1}{\rho \rho'} \left( \frac{\rho^<}{\rho^>} \right)^b (1 - (\rho^</\rho^>)^b)^2 d\rho' d\rho \\
+ \sum_{i=1}^{\infty} \chi^2 \int_{\rho^-}^{\rho^+} \int_{\rho'^-}^{\rho'^+} \frac{1}{\rho \rho'} \left[ B_\chi(\beta \rho, \beta \rho') - j \frac{1}{\pi \chi} \left( \frac{\rho^<}{\rho^>} \right)^\chi \right] d\rho' d\rho \quad (163)
\]

and

\[
\sum_{i=1}^{\infty} \int_{\rho^-}^{\rho^+} B_\chi(\beta \rho, \beta \rho')d\rho = -j \frac{1}{b\pi} \int_{\rho^-}^{\rho^+} \ln \left[ 1 - \left( \frac{\rho^<}{\rho^>} \right)^b \right] d\rho \\
+ \sum_{i=1}^{\infty} \int_{\rho^-}^{\rho^+} \left[ B_\chi(\beta \rho, \beta \rho') - j \frac{1}{\pi \chi} \left( \frac{\rho^<}{\rho^>} \right)^\chi \right] d\rho \quad (164)
\]

where \(\rho^>\) and \(\rho^<\) are \(\rho'\) and \(\rho\) such that \(\rho^> > \rho^<\), and \(\chi = ib\). These summations can then be calculated with traditional techniques or a Shank's transform. The infinite integrals can be calculated by the same manner used for the inverse fourier transform in the problem of a wire near (and parallel to) a bend in a conducting ground plane.
Results

To demonstrate the effectiveness of the solution technique, we present two sets of data. The first consists of plots of the inverse fourier integrand over the contour, depicting the effect of the add/subtract method on convergence for (160) and (162). The second set includes port parameter and current plots for the complete problem of two wires coupling through slots transverse to a bend in an infinite ground plane. The port parameters of this second set are presented in contrast to measured data.

Integrand Plots

As for the case of the $z$-directed wire, the benefit gained from the add/subtract method is significant. In order to facilitate the plots, we define an index. Figure 41 displays the relationship between the index of figures 42-44 and the $k_z$ value. Figures 42 and 43 contain sample plots of the $k_z$-integrand terms for the $TM_z$ and $TE_z$ potentials, respectively. “Free Space term” refers to the terms of (160) and (162) which are due to the free space potential. “Green’s function term” refers to the terms of (160) and (162) due to the Green’s function potential for the outward-directed magnetic current in the presence of a wedge. Figure 44 shows the fully accelerated integrand values. It is easy to see that there is a significant improvement (approximately a factor of two) in the convergence of the integrands in $k_z$ using the add/subtract method.
Port Parameters and Current Distributions

Plots of the port parameters, admittance parameters in this case, for the problem corresponding to the test fixture are shown in figures 45-47 with port 1 being defined at the interface between the backplane and the wire in the outer region and port 2 being defined at the interface between the backplane and the wire in the inner region. The test fixture was created by taking a bent piece of copper with a slot in it and soldering it to a backplane. The $z$-position of the slot for the test fixture (and therefore our calculations) was slightly different in the inner and outer regions due to a sheet of copper, laying against the backplane, used on the inner side for bracing. The soldering process caused the slot to warp to a certain degree which is believed to be the root of the discrepancies between calculated and measured data shown in figures 45-47. Having said this, the measured and predicted data compare favorably (accounting for frequency shifts, errors are on the order of 10% or less) with all major features captured. Sample calculated current distributions for the equivalent magnetic current on the slot and the current on the wire in the outer region are shown in figures 48 and 49, respectively.

Conclusion

In this chapter we present a foundation for the solution of narrow slots in conducting, bent ground planes. Using the solution techniques discussed here along with the well-known potential-based Green’s function for a dipole parallel to a bend, one can develop an integral equation solution for the general geometry of thin wires the near narrow slots in a bent, conducting ground plane. The required computation time is highly dependent on the problem configuration and ranges from minutes to
weeks, but it is likely the computation time can be made reasonable for a variety of high accuracy applications given further effort to carefully choose a contour for the Sommerfeld-type integral (the inverse Fourier transform).

References Cited


Figure 38: Sketch of the two wires near a slotted wedge geometry. Interior region wire shown through slot and behind non-physical, semi-transparent region.
Figure 39: Sketch of interior equivalent problem.
Figure 40: Sketch of the outer equivalent problem.

Figure 41: The value of $k_z$ versus the index used in figures 42-44.
Figure 42: Values of Integrand Terms for the $TM_z$ $k_z$ integral: $f = 300 MHz$, $\rho_m = \rho_n = 0.2$ m, Slot Width= 0.001 m, $\Delta_s = 0.05$ m, $z_m = z_n = 0.15$ m

Figure 43: Values of Integrand Terms for the $TE_z$ $k_z$ integral: $f = 300 MHz$, $\rho_m = \rho_n = 0.2$ m, Slot Width= 0.001 m, $\Delta_s = 0.05$ m, $z_m = z_n = 0.15$ m
Figure 44: Values of the Integrands for the $k_z$ integrals: $f = 300 \text{MHz}$, $\rho_m = \rho_n = 0.2$ m, Slot Width= 0.001 m, $\Delta_s = 0.05$ m, $z_m = z_n = 0.15$ m

Figure 45: $Y_{11}$: Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions $(x,y)=(6.7, \pm 1.1)$ cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position $(x,z)=(6.4,1.6)$ cm
Figure 46: $Y_{21}$ (or $Y_{12}$): Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions $(x,y)=(6.7, \pm 1.1)$ cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position $(x,z)=(6.4,1.6)$ cm

Figure 47: $Y_{22}$: Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions $(x,y)=(6.7, \pm 1.1)$ cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position $(x,z)=(6.4,1.6)$ cm
Figure 48: Equivalent Magnetic Current on the Slot: Frequency: 1350 MHz, Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions (x,y)=(6.7,±1.1) cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position (x,z)=(6.4,1.6) cm, $V^+ = 1$.

Figure 49: Current on the Wire in the Outer Region: Frequency: 1350 MHz, Wire Lengths=6.2 cm, Wire Radii=0.4 mm, Wire Positions (x,y)=(6.7,±1.1) cm, Slot Length=10.1 cm, Slot Width=2.25 mm, Slot Center Position (x,z)=(6.4,1.6) cm, $V^+ = 1$. 
In this chapter, a Baum-Liu-Tesche (BLT) equation-based formulation that includes field coupling and propagation is presented for a “wire over a ground plane” transmission line. The general solution form is derived, and then that solution is specialized to an example geometry. Frequency and time domain results for the BLT solution are presented in contrast to integral equation results. Limitations of the solution technique are discussed.

Introduction

The Baum-Liu-Tesche (BLT) equation formulation provides a framework for the solution of transmission line networks[1][2]. Recently, Tesche and Butler proposed a modified BLT equation formulation which allows for the inclusion of field propagation in space[3]. To do this, in addition to the traditional transmission line “tubes”, they proposed the addition of a mathematical construct referred to as a field propagation “tube.” The resulting extended BLT equation allows one to solve for the voltages at the transmission line nodes and the fields at an observation point. Recent work has included an example use of the extended BLT equation, solving for the relevant voltages and fields in the simple geometry of a two-wire transmission line excited by an ideal dipole in the presence of a ground plane scatterer[4]. In this chapter, we consider an alternate form for the extended BLT equation for a “wire over a ground plane” transmission line. This alternate BLT equation is then used to solve
the problem of a “wire over a ground plane” transmission line in the presence of a
ground plane scatterer where the excitation is provided by fields entering the region
of interest through a very short, narrow slot. The fields penetrating through the slot
are approximated with those produced by an slot-axial magnetic dipole as suggested
by Bethe hole theory[5] and as discussed in the first chapter of this manuscript. While
this extended BLT technique provides an additional option for solving the problem
of a slot exciting a wire, there are many other means of doing so. As discussed
in a previous chapter, prominent techniques include integral equations[6][7][8][9][10]
and Bethe hole theory based techniques for coupling small apertures to transmission
lines[11][12]. The BLT approach presented here is useful and appropriate in problems
where the distance between the slot and the transmission line is significant with
respect to wavelength. Note that in this chapter it is assumed unless otherwise stated
that the excitation varies time harmonically ($e^{j\omega t}$). The $e^{j\omega t}$ is suppressed.

**BLT Development**

Let there exist a “wire over a ground plane” transmission line $2\ell$ in length and
with a radius $a$. For the sake of generality, the sources are defined as a general
radiated source and a general local source (on the line).

The appropriate signal flow graph is in Figure 50[4]. Tube 1 is a traditional,
transmission line tube. Tube 2 is a field propagation tube[3]. Node 3, in our problem,
is an observation node with no field reflections inherent to its existence.

Consider the incident/reflected voltage wave propagation relationships on tube 1

$$V_{inc}^{1,1} = V_{ref}^{1,2} e^{-j2k\ell} + V_{S,1}$$

(165)
and
\[ V_{1,2}^{\text{inc}} = V_{1,1}^{\text{ref}} e^{-j2k\ell} + V_{S,2} \]  \hspace{1cm}\text{(166)}

where \( V_{S,1} \) and \( V_{S,2} \) are the voltages incident on nodes 1 and 2, respectively, due to the source, and \( k \) is the wave number. \( V_{1,1}^{\text{inc/ref}} \) and \( V_{1,2}^{\text{inc/ref}} \) are as defined in figure 50.

Rearranging (165) and (166) in terms of the reflected fields, one arrives at

\[ V_{1,1}^{\text{ref}} = V_{1,2}^{\text{inc}} e^{j2k\ell} - V_{S,2} e^{j2k\ell} \]  \hspace{1cm}\text{(167)}

and

\[ V_{1,2}^{\text{ref}} = V_{1,1}^{\text{inc}} e^{j2k\ell} - V_{S,1} e^{j2k\ell}. \]  \hspace{1cm}\text{(168)}

It is important to note here that \( V_{S,1} \) and \( V_{S,2} \) must include the incident voltages on nodes 1 and 2 from all sources, sources on the line and radiating sources. For convenience, we separate out the incident voltages due to radiated sources for \( V_{S,1} \) and \( V_{S,2} \) such that

\[ V_{S,1} = g_1 E_{2,4}^{\text{inc}} + V'_{S,1} \]  \hspace{1cm}\text{(169)}

and

\[ V_{S,2} = g_2 E_{2,4}^{\text{inc}} + V'_{S,2} \]  \hspace{1cm}\text{(170)}

where \( V'_{S,1} \) and \( V'_{S,2} \) are the voltages incident on nodes 1 and 2, respectively, due to sources on the line. \( g_1 E_{2,4}^{\text{inc}} \) and \( g_2 E_{2,4}^{\text{inc}} \) are the incident voltages excited at nodes 1 and 2, respectively, due to the fields incident on the transmission line. Functions \( g_1 \) and \( g_2 \) along with the incident fields form geometry-dependent terms in the general BLT equation. Derivation of these terms for a specific geometry is included in a following section.

On the field propagation tube, equations parallel to (165) and (166) can be written
for the incident and reflected fields:

\[ E_{2,3}^{\text{inc}} = g_5 E_{2,4}^{\text{ref}} + g_3 S + g_7 V_{1,1}^{\text{inc}} + g_8 V_{1,2}^{\text{inc}} \]  
\begin{equation}
(171)
\end{equation}

and

\[ E_{2,4}^{\text{inc}} = g_6 E_{2,3}^{\text{ref}} + g_4 S \]  
\begin{equation}
(172)
\end{equation}

where \( E_{2,3}^{\text{inc}} \), \( E_{2,4}^{\text{ref}} \), and \( E_{2,3}^{\text{ref}} \) are as defined in Figure 50. \( g_3 S \) and \( g_4 S \) are the contributions to \( E_{2,3}^{\text{inc}} \) and \( E_{2,4}^{\text{inc}} \), respectively, due to the source. \( g_7 V_{1,1}^{\text{inc}} \) and \( g_8 V_{1,2}^{\text{inc}} \) are the contributions to \( E_{2,3}^{\text{inc}} \) due to the differential mode current on the transmission line in terms of the incident voltages at nodes 1 and 2, respectively. \( g_5 \) and \( g_6 \) are, in theory, propagation terms along the field propagation line. In this derivation, they are retained for form’s sake as they disappear in the final form of the equations. In the case of other transmission line geometries, they may be necessary. By rearranging (171) and (172) in terms of the reflected fields, one arrives at

\[ E_{2,3}^{\text{ref}} = \frac{1}{g_6} E_{2,4}^{\text{inc}} - \frac{g_4}{g_6} S \]  
\begin{equation}
(173)
\end{equation}

and

\[ E_{2,4}^{\text{ref}} = \frac{1}{g_5} E_{2,3}^{\text{inc}} - \frac{g_3}{g_5} S - \frac{g_7}{g_5} V_{1,1}^{\text{inc}} - \frac{g_8}{g_5} V_{1,2}^{\text{inc}}. \]  
\begin{equation}
(174)
\end{equation}

One can then combine (167)-(170), (173), and (174) into a single matrix equation such that

\[
\begin{bmatrix}
V_{\text{ref}}^{1,1} \\
V_{\text{ref}}^{1,2} \\
E_{\text{ref}}^{2,3} \\
E_{\text{ref}}^{2,4}
\end{bmatrix} =
\begin{bmatrix}
0 & e^{j2k\ell} & 0 & -g_2 e^{j2k\ell} \\
e^{j2k\ell} & 0 & 0 & -g_1 e^{j2k\ell} \\
0 & 0 & 0 & \frac{1}{g_6} \\
\frac{-g_7}{g_5} & \frac{-g_8}{g_5} & \frac{1}{g_5} & 0
\end{bmatrix}
\begin{bmatrix}
V_{\text{inc}}^{1,1} \\
V_{\text{inc}}^{1,2} \\
E_{\text{inc}}^{2,3} \\
E_{\text{inc}}^{2,4}
\end{bmatrix} -
\begin{bmatrix}
V_{S,2}^{1} e^{j2k\ell} \\
V_{S,1}^{1} e^{j2k\ell} \\
\frac{g_4}{g_5} S \\
\frac{g_3}{g_5} S
\end{bmatrix}.
\begin{equation}
(175)
\end{equation}
As with $g_1$ and $g_2$, functions $g_{3-8}$ are geometry-dependent terms in the general BLT equation. Derivation of these terms for a specific geometry is included in a following section.

To complete a BLT equation, one needs the node reflection relationships. The node reflection relationships at nodes 1 and 2 are defined in the normal way for a transmission line: $\rho_1$ and $\rho_2$ are the reflection coefficients at nodes 1 and 2 respectively. Since node 3 is an observation node, there is no inherent reflection at that node. By convention[3], the reflection at node 4 is only non-zero when a common mode current exists on the transmission line. Since there is no common mode on a “wire over a ground plane” transmission line, the term $E_{2,4}$ includes only the incident field at node 4. The node reflection relationship matrix can be written

\[
\begin{bmatrix}
V_{\text{ref}}^{1,1} \\
V_{\text{ref}}^{1,2} \\
E_{\text{ref}}^{2,3} \\
E_{\text{ref}}^{2,4}
\end{bmatrix} =
\begin{bmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
V_{\text{inc}}^{1,1} \\
V_{\text{inc}}^{1,2} \\
E_{\text{inc}}^{2,3} \\
E_{\text{inc}}^{2,4}
\end{bmatrix}.
\]  

(176)

Inserting (176) into (175), one arrives at

\[
\begin{bmatrix}
V_{\text{inc}}^{1,1} \\
V_{\text{inc}}^{1,2} \\
E_{\text{inc}}^{2,3} \\
E_{\text{inc}}^{2,4}
\end{bmatrix} =
\begin{bmatrix}
-\rho_1 e^{j2kt} & 0 & -g_2 e^{j2kt} \\
0 & -\rho_2 & 0 & -g_1 e^{j2kt} \\
0 & 0 & 0 & \frac{1}{g_6} \\
-g_7 & 0 & \frac{g_8}{g_5} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
V'_{S,2} e^{j2kt} \\
V'_{S,1} e^{j2kt} \\
V'_{S,2} \\
V'_{S,1}
\end{bmatrix}.
\]  

(177)
The total voltage and fields at the nodes can be written in the familiar BLT form as

\[
\begin{bmatrix}
V_{1,1} \\
V_{1,2} \\
E_{2,3} \\
E_{2,4}
\end{bmatrix} =
\begin{bmatrix}
1 + \rho_1 & 0 & 0 & 0 \\
0 & 1 + \rho_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-\rho_1 & e^{j2k\ell} & 0 & -g_2 e^{j2k\ell} \\
e^{j2k\ell} & -\rho_2 & 0 & -g_1 e^{j2k\ell} \\
0 & 0 & 0 & \frac{1}{g_6} \\
\frac{g_8}{g_5} & -\frac{g_6}{g_5} & \frac{1}{g_5} & 0
\end{bmatrix}
^{-1}
\begin{bmatrix}
V'_{S,2} e^{j2k\ell} \\
V'_{S,1} e^{j2k\ell} \\
\frac{g_4}{g_6} S \\
\frac{g_4}{g_5} S
\end{bmatrix}.
\]  

(178)

Simplifying the matrix results in the set of equations

\[
V_{1,1} = \frac{1 + \rho_1}{1 - \rho_1 \rho_2 e^{-j4k\ell}} [V'_{S,1} + \rho_2 V'_{S,2} e^{-j2k\ell} + g_4 g_2 S + \rho_2 g_2 g_4 S e^{-j2k\ell}],
\]  

(179)

\[
V_{1,2} = \frac{1 + \rho_2}{1 - \rho_1 \rho_2 e^{-j4k\ell}} [V'_{S,2} + \rho_1 V'_{S,1} e^{-j2k\ell} + g_2 g_4 S + \rho_1 g_4 g_2 S e^{-j2k\ell}],
\]  

(180)

\[
E_{2,3} = g_3 S + \frac{1}{1 - \rho_1 \rho_2 e^{-j4k\ell}} [(V'_{S,1} + g_4 g_1 S)(g_7 + \rho_1 g_8 e^{-j2k\ell})
+ (V'_{S,2} + g_4 g_2 S)(g_8 + \rho_2 g_7 e^{-j2k\ell})],
\]  

(181)

and

\[
E_{2,4} = g_4 S.
\]  

(182)

Notice that, as predicted, \(g_5\) and \(g_6\) do not affect these equations. Equations (179)-(182) provide a means of calculating the voltages and the fields of Figure 50 for the general geometry of a “wire over a ground plane” excited by both radiating sources and sources on the line. \(g_1-8\) must be derived for each a specific geometry. The problem of interest must also lead in defining \(E_{2,3}\) and \(E_{2,4}\).
Application Specialization

In this section we demonstrate the application of (179)-(182) to a specific problem. The “wire over a ground plane” transmission line has an $x, y$-ground plane and is centered, axially, at $y = 0$. It is centered, transversely, at $x = h$. The separation between the wire and the ground plane is $\frac{d}{2}$. The excitation is provided by fields penetrating into the region of interest through a $y$-directed, very short, narrow slot located at $(r_1 + h, 0, 0)$. The fields penetrating the slot are approximated with those induced by a $y$-directed magnetic dipole located at the center of the slot as suggested by Bethe hole theory[5] of which further discussion can be found in the first chapter of this document. The observation point (node 3) is located at $(r_0 + h, 0, 0)$ where $r_0 = r_1 + r_2$ where $r_0, r_1, r_2 > 0$. There is a ground plane scatterer located at $x = 0$. A sketch of this configuration is shown in Figure 51. This is a simplified form of geometries found in many real world problems from computer cases to military systems such as radars and aircraft.

Since this problem does not include sources on the line, $V_{S,1}' = V_{S,2}' = 0$. The geometry-dependent terms are then $g_1, g_2, g_3, g_4, g_7$, and $g_8$. It is assumed that $r_0, r_1, r_2$ are long with respect to both wavelength and the length of the transmission line.

The field terms $E_{2,3}$ and $E_{2,4}$ must be defined. $E_{2,3}$ and $E_{2,4}$ are intended to be the fields of interest at nodes 3 and 4, respectively. The chosen geometry suggests the $z$-directed electric fields as a good choice, and such shall we choose herein. In the case of node 4, one must also choose the location along the transmission line at which to evaluate the electric field. For convenience, we define that point to be the center of the transmission line. Recall from previous sections that $g_4 S$ is the contribution to $E_{2,4}$ due to the radiating source. In this case, $g_4 S$ is the $z$-directed electric field at the
center of the transmission line due to the source. At the center of the transmission line (considered to be in the far field), the electric field generated by the slot and its image in the y, z-ground plane is, as covered in detail in the first chapter and elsewhere[10][13],

\[
E_z \approx \frac{k^2 \eta}{2\pi} p_{m,y} \left( \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jk(r_1+2h)}}{r_1 + 2h} \right) = g_4 S \tag{183}
\]

where \( \eta \) is the intrinsic impedance of the medium, and \( p_{m,y} \) is the equivalent dipole moment of the slot. Note that \( S \) is the magnitude of the source and can be defined as appropriate to the problem. Here we let

\[
S = \frac{k^2 \eta}{2\pi} p_{m,y} \tag{184}
\]

and

\[
K_4(r) = \frac{e^{-jkr}}{r} - \frac{e^{-jk(r+2h)}}{r + 2h}, \tag{185}
\]

such that

\[
g_4 = K_4(r_1). \tag{186}
\]

Similarly, \( g_3 S \) is defined as previously to be contribution to \( E_{2,3} \) due to the source. At the observation point the \( z \)-directed electric field excited by the source and its image in the \( y, z \)-ground plane is

\[
E_z \approx -\frac{k^2 \eta}{2\pi} p_{m,y} \left( \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jk(r_2+2r_1+2h)}}{r_2 + 2r_1 + 2h} \right) = g_3 S. \tag{187}
\]

Letting

\[
K_3(r) = \frac{e^{-jkr}}{r} + \frac{e^{-jk(r+2r_1+2h)}}{r + 2r_1 + 2h} \tag{188}
\]
one can see that
\[ g_3 = -K_3(r_2). \]  

Recall that \( g_1 E_{2,4} \) is defined as the incident voltage on node 1 due to the incident fields exciting the transmission line. Similarly, \( g_2 E_{2,4} \) is defined as the incident voltage on node 2 due to the incident fields exciting the transmission line. To obtain \( g_1 \) and \( g_2 \) we use a Green’s function solution of Telegrapher’s equations from the Taylor method[14][2][15] and select the portions of the resulting voltages that are incident on the nodes:

\[
g_1 E_{2,4}^{\text{inc}} = e^{jk(h-\ell)} \frac{1}{2} \int_{h-\ell}^{h+\ell} e^{-jkx} \left[ -jk \int_{-d/2}^{d/2} [E_z^{\text{inc}}(x, z) - \eta H_y^{\text{inc}}(x, z)] dz \right] dx
\]  

and

\[
g_2 E_{2,4}^{\text{inc}} = e^{-jk(h+\ell)} \frac{1}{2} \times \int_{h-\ell}^{h+\ell} e^{jkx} \left[ -jk \int_{-d/2}^{d/2} [E_z^{\text{inc}}(x, z) + \eta H_y^{\text{inc}}(x, z)] dz \right] dx.
\]  

Similar to (183),

\[
E_z^{\text{inc}}(x, z) \approx S \left( \frac{e^{-jk|x-r_1-h|}}{|x - r_1 - h|} - \frac{e^{-jk|x+r_1+h|}}{|x + r_1 + h|} \right).
\]  

Using far field approximations, one can see that

\[
\eta H_y^{\text{inc}}(x, z) \approx S \left( \frac{e^{-jk|x-r_1-h|}}{|x - r_1 - h|} + \frac{e^{-jk|x+r_1+h|}}{|x + r_1 + h|} \right).
\]  

Inserting (192) and (193) into the integrands of (190) and (191), one obtains

\[
E_z^{\text{inc}}(x, z) - \eta H_y^{\text{inc}}(x, z) = -2S \frac{e^{-jk|x+r_1+h|}}{|x + r_1 + h|}.
\]
and
\[ E_{2}^{\text{inc}}(x, z) + \eta H_{y}^{\text{inc}}(x, z) = 2S \frac{e^{-jk|x-r_1-h|}}{|x-r_1-h|} \] (195)
implying that
\[ g_1E_{2,4}^{\text{inc}} = jkde^{jk(h-\ell)}S \int_{h-\ell}^{h+\ell} e^{-jkr_1+hx} \frac{e^{-jk(r_1+h-x)}}{r_1+h+x} \, dx \] (196)
and
\[ g_2E_{2,4}^{\text{inc}} = -jkde^{-jk(h+\ell)}S \int_{h-\ell}^{h+\ell} e^{jkx} \frac{e^{-jk(r_1+h-x)}}{r_1+h-x} \, dx. \] (197)
Simplifications can be made by substituting \( u = h - x \) into (196) and (197) and utilizing the approximation \( \frac{1}{r_1+2h\pm u} \approx \frac{1}{r_1+2h} \) implied by \( r_1 \gg \ell \). The resulting equations are
\[ g_1E_{2,4}^{\text{inc}} = jkde^{-jk\ell}SK_1(r_1) \int_{-\ell}^{\ell} e^{j2ku} \, du \] (198)
and
\[ g_2E_{2,4}^{\text{inc}} = -jkde^{-jk\ell}SK_2(r_1) \int_{-\ell}^{\ell} e^{-j2ku} \, du \] (199)
where
\[ K_1(r) = \frac{e^{-jk(r+2h)}}{r+2h} \] (200)
and
\[ K_2(r) = \frac{e^{-jkr}}{r}. \] (201)
Evaluating the integrals of (198) and (199) yields
\[ g_1E_{2,4}^{\text{inc}} = \frac{d}{2} e^{-jk\ell} SK_1(r_1)(e^{j2k\ell} - e^{-j2k\ell}) \] (202)
and
\[ g_2E_{2,4}^{\text{inc}} = -\frac{d}{2} e^{-jk\ell} SK_2(r_1)(e^{j2k\ell} - e^{-j2k\ell}). \] (203)
Recall that $E_{2,4}^{\text{ref}} = 0$ by convention for this structure. Substituting (182), (184), and (186) into (202) and (203) leads to

$$g_1 = \frac{d K_1(r_1)}{2 K_4(r_1)} e^{-jk\ell} (e^{j2k\ell} - e^{-j2k\ell})$$
\[(204)\]

and

$$g_2 = -\frac{d K_2(r_1)}{2 K_4(r_1)} e^{-jk\ell} (e^{j2k\ell} - e^{-j2k\ell}).$$
\[(205)\]

Recall that the contribution to $E_{2,3}$ due to the current on the transmission line is $g_7 V_{\text{inc}}^{1,1} + g_8 V_{\text{inc}}^{1,2}$. At the observation point, the $z$-directed electric field from the transmission line current is caused primarily by the current in the loads (and their images). The current in each load, in terms of the incident voltages, can be written

$$I_1 = -\frac{V_{\text{inc}}^{1,1}}{Z_c} + \frac{V_{\text{inc}}^{1,2}}{Z_c} e^{j2k\ell}$$
\[(206)\]

and

$$I_2 = \frac{V_{\text{inc}}^{1,1}}{Z_c} e^{j2k\ell} - \frac{V_{\text{inc}}^{1,2}}{Z_c}$$
\[(207)\]

where $I_1$ and $I_2$ are the currents in the loads at nodes 1 and 2, respectively. The sign reference has been chosen such that positive currents are traveling in the positive $z$-direction. The $z$-directed electric field due to those currents and their images at node 3 (the observation point) can be approximated

$$E_z = -\frac{jk\eta}{4\pi} \left[ (I_1 e^{-jk\ell} + I_2 e^{jk\ell}) \frac{e^{-jk\sigma}}{r_0} - (I_1 e^{jk\ell} + I_2 e^{-jk\ell}) \frac{e^{-jk(r_0+2h)}}{r_0+2h} \right].$$
\[(208)\]
Inserting (206) and (207) into (208) yields

\[ E_z = -\frac{jk}{2\pi Z_c} \left[ V_{1,1}^{\text{inc}} \frac{d}{2} e^{jkt}(e^{j2\kappa l} - e^{-j2\kappa l}) \frac{e^{-jkr_0}}{r_0} \right. \]

\[- V_{1,2}^{\text{inc}} \frac{d}{2} e^{jkt}(e^{j2\kappa l} - e^{-j2\kappa l}) \frac{e^{-jk(r_0+2h)}}{r_0 + 2h} \]. \quad (209)

From (209), one can see that

\[ g_7 = -\frac{jk}{2\pi Z_c} \frac{d}{2} e^{jkt}(e^{j2\kappa l} - e^{-j2\kappa l}) K_2(r_0) \quad (210) \]

and

\[ g_8 = \frac{jk}{2\pi Z_c} \frac{d}{2} e^{jkt}(e^{j2\kappa l} - e^{-j2\kappa l}) K_1(r_0). \quad (211) \]

Inserting (186), (189), (204), (205), (210), and (211) into (179)-(182) gives the application specific equations for the voltages and fields.

\underline{Results}

In this section we present example data for the problem presented in the previous section comparing the BLT formulation results with integral equation results. The specific physical parameters for the problem can be found in Table 9.

\underline{Frequency Domain Results}

Results in the frequency domain are presented for \( V_{1,1}, V_{1,2} \) and \( E_{2,3} \) in Figures 52, 53, and 54, respectively. In this data, the dipole moment \( p_{m,y} \) in (184) has been normalized to 1. In all cases, the BLT solution is compared to an integral equation
(IE) solution of the same problem. The largest percent errors in the peaks are resident in the $V_{1,1}$ data at approximately 15%. For the other two data sets, the errors in the peaks are on the order of 5% or less. The regions around the nulls in the BLT formulation cause much more significant errors. This inaccuracy is effectively the cost of the computational efficiency of the BLT technique. In the case of $E_{2,3}$, the dominant contribution is from the source with the effect of the transmission line being barely noticeable. As a result, in $E_{2,3}$ the BLT and IE solutions show no noticeable difference.

Time Domain Results

To obtain results in the time domain, we vary the value of the dipole moment $p_{m,y}$ as shown in Figure 55. The waveform is that of a sinusoid at a frequency of 100 MHz whose envelope has been restricted by a time shifted Gaussian. The resulting spectrum of this signal as calculated by a fast fourier transform can be seen in Figure 56. This excitation was chosen to insure easily discernible time domain characteristics.

Results in the time domain are presented for $V_{1,1}$, $V_{1,2}$, and $E_{2,3}$ in Figures 57, 58, and 59, respectively. As with the frequency domain data, the largest error is in the $V_{1,1}$ data. The zero crossings, peak locations, and general shape are captured, but there are significant differences in some of the peak magnitudes (50%). The other two data sets similarly capture shape and zeros, but the matching of the peak magnitudes is much better (errors of less than 15%). In the case of $E_{2,3}$, the difference between the solutions cannot be seen at the depicted scale. Excitation waveforms with a larger amount of high frequency content than that of Figure 56 have also been tested and result in differences comparable to those shown here for the BLT and integral equation solutions.
Conclusion

The BLT formulation approach to this problem provides good results when compared to proven techniques. In the formulation, there are approximations requiring the distance between the source and the transmission line to be large which make this technique unsuitable for some configurations. In future work it is desirable to maintain the efficient matrix solution form while removing the requirement for a significant distance between the transmission line and the source.

References Cited


Figure 50: The signal flow graph, including definitions of the incident and reflected voltages and electric fields[4].

Figure 51: Problem Sketch
Figure 52: A Spectral Plot of the Voltage at Port 1 under the Parameters of Table 9

Figure 53: A Spectral Plot of the Voltage at Port 2 under the Parameters of Table 9
Figure 54: A Spectral Plot of the Electric Field at Port 3 under the Parameters of Table 9

Figure 55: Time Variation of the Equivalent Dipole Moment of the Exciting Slot for Time Domain Results
Figure 56: Spectrum of the Time Domain Signal in Figure 55, The Dipole Moment Variation for the Exciting Slot

Figure 57: A Time Domain Plot of the Voltage at Port 1 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55
Figure 58: A Time Domain Plot of the Voltage at Port 2 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55

Figure 59: A Time Domain Plot of the Electric Field at Port 3 when the Equivalent Dipole Moment of the Exciting Slot varies as shown in Figure 55
Table 9: Physical Parameters for Small Aperture Solution Data

<table>
<thead>
<tr>
<th>Wire Radius</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>1 cm</td>
</tr>
<tr>
<td>$h$</td>
<td>3.5 m</td>
</tr>
<tr>
<td>$\ell$</td>
<td>0.5 m</td>
</tr>
<tr>
<td>$r_0$</td>
<td>15 m</td>
</tr>
<tr>
<td>$r_1$</td>
<td>9.5 m</td>
</tr>
<tr>
<td>$r_2$</td>
<td>5.5 m</td>
</tr>
<tr>
<td>Load Impedances</td>
<td>69.078 Ohms</td>
</tr>
</tbody>
</table>


CONCLUSIONS

Current industry techniques for the modeling of apertures are either woefully inaccurate or extremely expensive in terms of computational time and resources, requiring cluster computing environments for times ranging from hours to weeks. Further development of efficient and accurate means of calculating the effects of apertures has the potential to keep researchers working for many years. The tools presented here provide solutions for a finite problem set and building blocks for further aperture configurations as well as the foundation for further computational improvements. In particular, further effort is required to allow for a larger variety of aperture shapes and so as to permit the incorporation of large arrays of slots. It is also of interest to consider the use of Bethe hole theory for the modeling of small apertures in a conducting wedge, and the extended BLT formulation requires further effort to extend its applicability to the nearby aperture.