8-2023

Simulating the Eccentricity Evolution of Accreting Equal-Mass Binaries: Numerical Sensitivity to the Computational Domain Size and Grid Resolution

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A Master's Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Philosophy
Physics

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by
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August 2023

Accepted by:
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Abstract

With high resolution hydrodynamics simulations, we show that the optimal values of domain radius and grid resolution for the software *Sailfish* when simulating time-based eccentricity evolution of equal mass, non-circular accreting binaries in a circumbinary disk to be $r_{\rm out} \leq 15a$ and $\delta x/a \leq 0.01$. These values provide a useful guideline for optimizing the performance of simulation runs while maintaining scientific accuracy. Each artificial parameter is probed with 15 runs of 2000 orbits each.

Keywords: hydrodynamics, accretion, binaries.

Acknowledgments

I would like to thank my advisor Dr. Jonathan Zrake for his committed support and guidance throughout this journal. I would also like to thank Dr. Marco Ajello and Dr. Dieter Hartmann for their professional guidance.

I would like to thank my parents for their never-fading love and support. I would also like to thank my significant other, Nicole Maric, for her continuous support and care throughout. I woud like to thank Madeline Clyburn and Antonio Circiello for their friendship and support.

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Chapter 1

Introduction

Circumbinary disks have been a source of scientific interest due to their potential influence on the motions of objects embedded within or around them, such as the motion of binaries. Disks can be found in young stellar and protoplanetary systems, such as in HD 142527 (Garg et al. (2020), Boehler et al. (2017), and Hunziker et al. (2021)). Binary black holes are expected to form in the accretion disks of Active Galatic Nuclei, where they are driven to mergers through gas dynamics and multi-body encounters (Bartos et al. (2017), Antonini and Rasio (2016), Tagawa et al. (2020)). AGN themselves could also be the result of binary mergers from supermassive black holes (SMBHs) during galactic mergers (Begelman et al., 1980). Since gravitational interaction is the primary way in which the binary influences the disk and vice versa, such as through momentum and mass transfer via binary accretion, there has been numerous studies conducted on this topic using numerical simulations (e.g. Macfadyen and Milosavljevic (2006), Shi et al. (2012), and D'Orazio et al. (2012)). When looking at long term orbital evolution of eccentric binaries in circumbinary disks, Muñoz et al. (2019) has found that the orbits of eccentric binaries settles around $e_b \approx 0.5$ if their initial eccentricity is $e_b \geq 0.1$ and $e_b \approx 0$ if their initial eccentricity is $e_b \leq 0.1$, respectively. Zrake et al. (2021) confirmed the existence of the eccentricity "attraction" from Muñoz et al. (2019) and refined the values to $e_b \approx 0.45$ and the lower threshold e at $e_b \approx 0.08$ for equal mass eccentric binaries. D'Orazio and Duffell (2021) expanded on studies from Zrake et al. (2021) with results on binary semi-major axis evolution, finding that $e_b \leq 0.1$ binaries circularize with their semi-major axes expanding, while those that tend to $e_b \approx 0.5$ have seen their semi-major axes decay. Siwek et al. (2023) studied the e_b evolution in unequal mass eccentric binaries and found that all large mass-ratio binaries evolved to steady state with eccentricities attracted to $e_b \approx 0.5$, and binaries with $q \leq 0.3$ went to coalescence, while smaller mass-ratio binaries expanded. For a comprehensive review on circumbinary disk accretion, please see the review from Lai and Muñoz (2022).

The purpose of this study is to probe the simulation code *Sailfish*'s sensitivity to the artificial parameters *domain radius* and *grid effective resolution.* These parameters affect the performance of simulations since they determine the number of cells computers have to process. Finding optimal values that let the simulation give converging results while minimizing performance workload can help further improve the code as well as optimization.

Chapter 2

Background

2.1 The orbital mechanics of binary eccentricity evolution

The definition of the orbital eccentricity of a gravitational binary system is:

$$e_b = \sqrt{1 + \frac{2\epsilon h^2}{\mu^2}}, \tag{2.1}$$

where ϵ is the specific orbital energy; h^2 is the magnitude squared of the specific orbital angular momentum; μ is the standard gravitational parameter GM, with M being the reduced mass of the binary. It is a dimensionless parameter that characterizes how circular the orbit is: $e = 0$ means the orbit is a perfect circle; $0 < e < 1$ means the orbit is an ellipse, and so on. To turn this equation into a more useful form, we first examine the total energy of the binary without the disk when the binary components are at pericenters or apocenters:

$$E = -\frac{\mu}{r} + \frac{h^2}{2Mr^2}, \tag{2.2}$$

where it can be rewritten as:

$$E + \frac{\mu}{r} - \frac{h^2}{2Mr^2} = 0$$

or

$$r^2 + \frac{\mu}{E} r - \frac{h^2}{2mE} = 0.$$

The negative sum of the solutions to this quadratic equation is the coefficient of the $\frac{\mu}{E} r$ term, hence we can define the semi-major axis of the orbit as:

$$a = -\frac{h}{2E}. \tag{2.3}$$

The eccentricity of the orbit thus can be rewritten with the definition of semi-major axis as:

$$e = \sqrt{1 - \frac{h^2}{\mu a}}. \tag{2.4}$$

To study the eccentricity evolution of the binary in a disk, the eccentricity vector associated with the binary orbit must be used, and we take into account the gravitational interaction between the binary and all the gas parcels present. To derive the eccentricity vector, we first consider a gravitational central force as the time derivative of the total momentum of the system:

$$\dot{\mathbf{p}} = -\frac{\mu}{r^2} \hat{r}. \tag{2.5}$$

Knowing that the eccentricity of an orbit can be seen as how oblate the orbit is, we want to find the vector that is parallel to the semi-major axis:

$$\dot{\mathbf{p}} \times \mathbf{L} = -\mu(\hat{r} \times (\mathbf{r} \times \dot{\mathbf{r}})). \tag{2.6}$$

With a bit of vector manipulation, the equation becomes:

$$\begin{aligned} \dot{\mathbf{p}} \times \mathbf{L} &= -\frac{\mu}{r^3}(\mathbf{r} r \dot{r} - r^2 \dot{\mathbf{r}}) \\ &= -\mu\left(\frac{\mathbf{r}\dot{r}}{r^2} - \frac{\dot{\mathbf{r}}}{r}\right). \end{aligned} \tag{2.7}$$

Noting the conservation of angular momentum, we can rewrite the LHS as taking the time derivative of $\mathbf{p} \times \mathbf{L}$:

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}\frac{\mu \mathbf{r}}{r}, \tag{2.8}$$

we immediately arrive at the defintion of the Laplace-Runge-Lenz vector by integrating both sides with respect to time:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu\frac{\mathbf{r}}{r}. \tag{2.9}$$

Dividing both sides by μ, the eccentricity vector of the binary orbit is thus:

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}. \tag{2.10}$$

where $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, and $\mathbf{v} = \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1}$, where $\mathbf{r_1}$ and $\mathbf{r_2}$ are the respective positions of the binary components with respect to the origin. In a conserved orbit, the eccentricity vector has a fixed direction pointing along the semi-major axis, with its magnitude being the eccentricity of the orbit. In this study, however, the circumbinary disk takes angular momentum away from the binary, which results in a rate of change on the eccentricity vector's direction and magnitude. For this project, the change in magnitude is explored.

The rate of change of the binaries orbital eccentricity can be determined by taking the time derivative of the eccentricity vector we just found:

$$\begin{aligned}\dot{\mathbf{e}} &= \frac{d}{dt}(\frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}) \\ &= \frac{\dot{\mathbf{v}} \times \mathbf{h} + \mathbf{v} \times \dot{\mathbf{h}}}{\mu} - (\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}}{r^2}\dot{\mathbf{r}}) \\ &= \frac{\dot{\mathbf{v}} \times \mathbf{h} + \mathbf{v} \times \dot{\mathbf{h}}}{\mu} - \frac{\mathbf{v}_\perp}{r}\end{aligned} \tag{2.11}$$

where $\mathbf{v}_\perp$ is the radial velocity. By rewriting the force terms as $\dot{\mathbf{v}} = -\frac{\mu}{r^2}\hat{r} + \mathbf{f}_{\rm gas}$ and $\mu = GM$, the radial velocity cancels out with the gravitational force term, and we are left with:

$$GM\dot{\mathbf{e}} = \mathbf{f}_{\rm gas} \times (\mathbf{r} \times \mathbf{v}) + \mathbf{v} \times (\mathbf{r} \times \mathbf{f}_{\rm gas}) - \frac{\dot{M}}{M}[\mathbf{v} \times (\mathbf{r} \times \mathbf{v})] \tag{2.12}$$

where $M = M_1 + M_2$ is the total mass of the binary; $\mathbf{f}_{\rm gas}$ is the gravitational force of the gas in the circumbinary disk on the binary; $\dot{M}$ is the accretion of mass from the disk by the binary; $\mathbf{r} \times \mathbf{v}$ is the specific angular momentum, where $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$ and $\mathbf{v} = \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1}$. We use this form to calculate the $\dot{\mathbf{e}}$ evolution.

Chapter 3

Simulation Setup

3.1 Equations of motion and initial conditions

To probe the sensitivity of the binary eccentricities to artificial parameters, we use our hydrodynamics numerical simulation application named *Sailfish*, a high order GPU-powered Godunov code and the spiritual successor of *Mara3* (Zrake et al., 2021), to simulate a coupled, equal-mass binary that sits in a locally isothermal circumbinary disk in a two dimensional, Cartesian statically refined mesh. *Sailfish* numerically solves the 2D vertically averaged Navier-Stokes equations with constant kinematic viscosity ν:

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \mathbf{v}) = \dot{\Sigma}_{\mathrm{sink}} \tag{3.1}$$

$$\frac{\partial \Sigma \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \mathbf{v}\mathbf{v} + P\mathbf{I} - \mathbf{T}_{\mathrm{vis}}) = \dot{\Sigma}_{\mathrm{sink}} \mathbf{v} + \boldsymbol{F}_g \tag{3.2}$$

which are described in full in Tiede et al. (2020). The binaries accrete material from the disk by acting as sinks of mass and momentum, which are denoted by $\dot{\Sigma}$ in the equations; $\mathbf{v}$ is the gas velocity, and P is the vertically integrated gas pressure Tiede et al. (2020). The prescription for the sink is acceleration free, as seen in the $\dot{\Sigma}_{\mathrm{sink}}\mathbf{v}$ term in equation (2.1). We note from Dittmann and Ryan (2021) and Dempsey et al. (2020) that implementing a torque-free sink model can significantly decrease the dependence of density profiles around the sinks on the sink rate, but since only converging runs are conducted in this study, the results don't share major differences. The thickness of the disk is described by the orbital Mach number parameter $\mathcal{M} = (\frac{h}{r})^{-1}$, which comes from the vertically averaged thin disk approximation Tiede et al. (2020). For this study, the thickness of the disks in all simulations are kept at $\mathcal{M} = 10$.

3.2 Simulation parameters

The two parameters studied are the size r_d (in units of semi-major axis a) and the grid spacing Δx of the computational domain. We measure the binary eccentricity evolution as a function of the domain size r_d, holding fixed the grid spacing at $\Delta x = 0.01a$. When studying the grid resolution parameter, the domain radius is held constant at $r_d = 15a$ while the grid spacing is varied by selecting values of Δx which correspond to 50, 100, 150, 200, and 250 zones per semi-major axis a. The derivative $\dot{e}$ of the binary orbital eccentricity is measured as a function of e, the domain radius r_d, and the grid spacing Δx, in order to probe how sensitive $\dot{e}(e)$ is to the numerical parameters r_d and Δx. Each batch of runs with fixed Δx or fixed r_d consists of $\dot{e}$ measurements taken at three binary orbital eccentricities: $e = 0.2, 0.4, 0.6$. A total of 30 runs were conducted.

Chapter 4

Results

Figure 4.1 shows what the density profile of the inner region of the circumbinary disk looks like while the binary interacts with it. With non-circular binaries, material from the inner edge of the disk gets flung to the other side of the central cavity, creating this "seashell" density profile as shown in the figure. Figure 4.2 shows the results of the 30 runs conducted. Both domain radius ($r_{\rm out}$) varying runs and effective resolution ($\delta x/a$) varying runs are shown. The $r_{\rm out}$ runs seems to indicate that a domain radius of $15a$ is needed to resolve the binary's eccentricity evolution. The $\delta x/a$ runs show that a 0.01 $\frac{\Delta x}{a}$ grid resolution is required, but the general trend for each eccentricity is inconsistent, meaning eccentricity is sensitive to this parameter. The time evolution data selected for these runs begin at $t = 1500$ orbits and ending at $t = 2000$ orbits, which is when the system has evolved through 3 viscous relaxation times (Zrake et al., 2021). The following table contains the data from the two blocks of runs:

eccentricity \ domain radius a	5	10	15	20	25
0.2	7.70	8.06	7.92	8.61	13.44
0.4	4.12	-2.81	0.68	1.12	1.74
0.6	3.07	-2.39	-2.45	-2.68	-2.51

eccentricity \ effective resolution $\frac{\Delta x}{a}$	0.02	0.01	0.0067	0.005	0.004
0.2	11.67	10.22	12.10	8.92	9.38
0.4	5.01	-0.07	-0.72	-2.63	2.43
0.6	-3.20	-2.46	-2.25	-2.14	-2.06

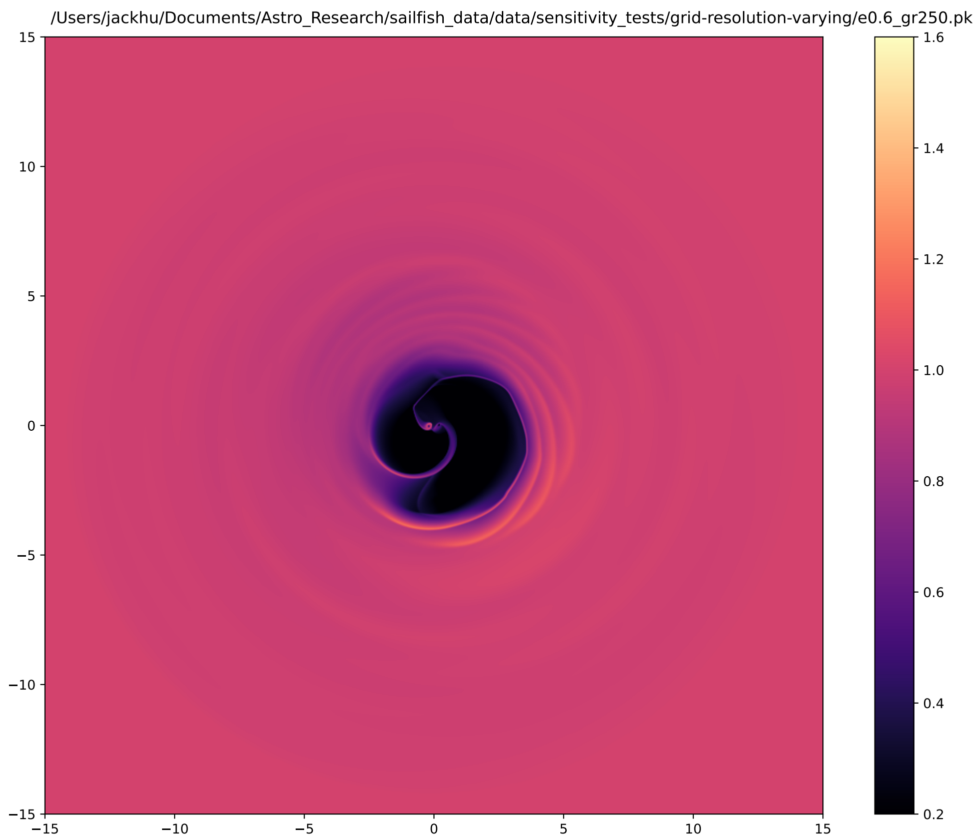

Figure 4.1: Example of simulation. X and Y axes are the domain axes, while the color bar indicates the density of material present at each cell.

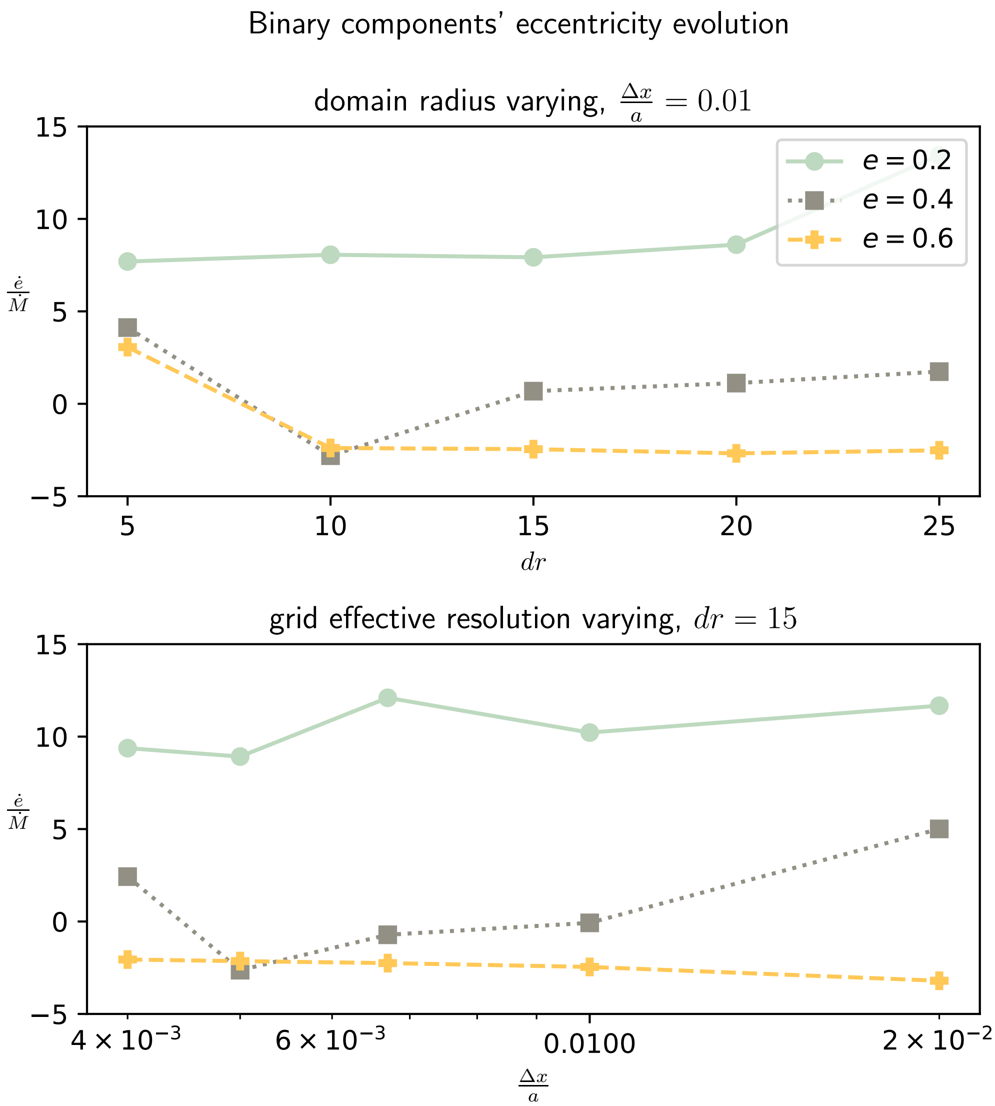

Figure 4.2: Results of e-dot over m-dot as a function of domain size and effective grid resolution.

Chapter 5

Conclusion and Discussion

We have probed the sensitivity of the hydrodynamics code *Sailfish* to the artificial parameters **domain radius** and **effective resolution** by examining the time-evolution signatures of eccentricity of equal mass, accreting binaries in different initial eccentricity configurations and comparing with known converging results from Zrake et al. (2021). The range of eccentricities were $e = [0.2, 0.4, 0.6]$, and, at each eccentricity, the domain radii and the grid resolution values probed were $r_{\rm out} = [5, 10, 15, 20, 25]$ and $\delta x/a = [0.02, 0.01, 0.0067, 0.005, 0.004]$, respectively. Each run was ran until $t = 2000$ orbits. The domain size runs appeared to have converged with a sufficiently large value. However, we found eccentricity evolution of the binary is sensitive to the effective grid resolution, especially for the case of $e = 0.4$ and $e = 0.6$. The sensitivity seems to introduce random error rather than a consistent trend, so we conclude that even the lowest resolution we performed, with $\delta x/a = 0.02$ yields a measurement in the right ballpark, however we caution that such measurements may carry an uncertainty on the other of $10 - 20\%$ associated with the random error arising from grid resolution effects.

The large domain size of $15a$, that seems to be required for a consistent measurement of the eccentricity time derivative, indicates that an imposed outer boundary condition can influence measurements of the orbital evolution of the binary even when the domain is large. We understand this is because the surface density in the disk controls the gravitational force exerted on the binary, and that the outer boundary condition can introduce spurious features in the mass distribution of the circumbinary disk when it is placed too close to the binary. This is because the boundary condition imposes an axial symmetry at a finite radius, so unless the disk extends far enough from the binary to erase the effects of the time-varying tidal field, the boundary condition can lead to suppression of real non-axisymmetric disk features that influence the orbital evolution. For these reasons, we recommend that the domain size of future simulations of binaries with disks be set to $15a$ or larger.

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