Examining the Different Snap-Through Characteristics of Bistable CFRP Composite Laminates

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EXAMINING THE DIFFERENT SNAP-THROUGH CHARACTERISTICS OF BISTABLE CFRP COMPOSITE LAMINATES

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mechanical Engineering

by
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Accepted by:
Dr. Suyi Li, Committee Chair
Dr. Oliver Myers
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Abstract

Bistable carbon fiber composites, whose bistability arises from having asymmetric fiber layouts in different layers, have shown immense potential for using in shape morphing and adaptive structure applications. While many studies in this field focus on these composite laminates’ external shapes at the two stable states, their snap-through behavior of shifting from one stable shape to the other remains a critical aspect to be investigated in complete detail. Moreover, symmetric loading conditions have been extensively studied based on the classical lamination theory, but the asymmetric loading conditions received far less attention. Therefore, this study examines an asymmetric, localized point load on a \([0^\circ/90^\circ]\) bistable laminate and its complex transient deformation during the snap-through. Finite element simulation and experiment results reveal three uniquely different snap-through behaviors — two-step snap, one-step snap, and no snap — depending on the point load location. The localized initiation and propagation of a “curvature inversion zone,” calculated from finite element and digital image correlation results, are directly related to these snap-through characteristics. This study also explored the feasibility of using an extended analytical model of classical lamination theory. The study first establishes the working of analytical model in MATLAB for two-ply laminates with user-defined geometrical and material properties. This established model is later used to qualitatively reproduce the findings for the asymmetric loading case by introducing a novel way of constrain-
ing the equations. The model compares three polynomial functions of different orders to approximate the out-of-plane laminate displacement field. This study’s results can offer valuable insights into the fundamental mechanics of snap-through behaviors and the actuation designs for the bistable composites for different loading scenarios.
Dedication

I would like to dedicate the work to my parents and sister. Also, to my late Grandma, I love you always.
Acknowledgments

Here, I take the opportunity to appreciate all the people without who this thesis would not have been possible. Firstly, I would like to thank my advisor Dr. Suyi Li for his constant support and timely guidance. I also like to express my fullest gratitude to my committee members – Dr. Georges Fadel and Dr. Oliver Myers for their valuable insights and guidance. I would also like to extend my gratitude to my peers – Jebin Biju and Akshay Balasubrahmanian for their timely help and valuable inputs. It has been a pleasure working with you all in our Kirigami Composite group. Also, I would like to acknowledge National Science Foundation (NSF Grant: CMMI-1760943) for their funding resources. Lastly, I would like to thank my department, Mechanical Engineering, Clemson for providing world-class facilities that made sure smooth completion of my research experiments.
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Chapter 1

Composite Materials

Various domains of living seek betterment, improvement, a sense of ease of operation as the day goes by, and this is realized with the help of smart devices/structures; may it be as small as home-assistant such as Alexa, or as big as morphing aircraft. Our need for next-generation structures and materials with novel functionalities has increased significantly with continuously evolving engineering technology. One possible answer lies in well-known composite materials whose modern-day functionalities have progressed through the years.

A combination of two or more dissimilar materials at a macroscopic scale with a specific configuration and composition yields a novel material called composite material. This technique combines the best properties of its constituent materials. For example in civil structures, concrete is a widely used material that is a mixture of Portland cement, aggregates(gravel, stones, sand), and water. Cement provides an excellent binding base for all the ingredients to mix and come together; the gravel and stones boast good compression resistance and strength; sand provides good compaction and flowability of the mixture. Leveraging the capabilities and possibilities of composite materials, nowadays, most structures are products of them such as airplane body[15],
wind turbine blades. Hence, the research in composite materials continues to remain a hot topic.

1.1 General Terminologies associated with Composite Materials

Composite Materials mainly consists of two important ingredients:

Figure 1.1: Constituents of general composite material. Figure Source: https://mechanicalbase.com/explanation-of-components-and-classification-of-composite-materials/

(a) Matrix: is a binding material (resin) that holds the reinforcement materials and deforms upon application of load to transfer its major portion to reinforcements particles/fibres.

(b) Reinforcement: materials which enhance the physical properties like load bearing, strength, stiffness, toughness, electrical conductivity or resistance, etc. They could be added in different forms such as particles, short fibres or continuous fibres.

The composite materials could be classified on the basis of matrix type used. On the basis of matrix the composite materials could be classified under Ceramic Matrix composites (CMCs) which use ceramic materials such as silicon carbide and aluminium
oxide (Alumina), Metal Matrix Composites (MMCs) such as Al and Ti, Polymer Matrix Composites (PMCs) such as Epoxy and Polyesters, and Carbon Matrix composites (CAMCs). Most of the composites could be termed under the listed classes. All the types of composites have some important applications but the most widely used composites are PMCs. Being readily available and easy to manufacture with moderate to rich structural properties, these PMCs are optimum for a large variety of the industrial applications such as aircraft structures, medical artificial limbs, sports equipment, automotive parts, etc.

Composites could also be classified on the basis of reinforcements as Fibre-reinforced like Glass-fibre (GFR) and Carbon-fibre (CFR), Particulate-reinforced such as concrete and Structural-reinforced where two or more materials are either used in sandwich or laminated layup.
1.2 Carbon Fibre-Reinforced Polymer composites

Fibre-reinforced composites are an important class of materials which find application in a lot of domains such as civil structures (bridge structure members), energy production (wind technology, off-shore rigs), sports products (skateboards, road bikes, sports car parts such as spoilers, etc.), aircraft and boat repairs, etc. It could be owed to the tailoring ability of the material based on the properties required such as high stiffness to weight ratio, chemical resistance, light-weight, and other directional-properties.

Figure 1.3: Different Types of Weaves: (a) Plain Weave – highest stability (b) Satin Weave – great draping qualities (c) Twill Weave: Mid-range between plain and satin weaves (d) Uni-directional Weave: highly directional properties and lowest stability. The figure is adapted from the source: https://www.elevatedmaterials.com/carbon-fiber-weaves-what-they-are-and-why-to-use-them/

We here limit our study to a very specific type of composites – *Carbon Fibre-reinforced Polymers*. This material could be manufactured through different methods depend-
ing on the construction of the composite and initial states of the constituents but general procedure involves a *Curing* step where the matrix infused with the fibres is heated under pressure and followed by controlled cooling to working temperature. The carbon fibres come in different weaves such as fabric sheets with plain weave, satin weave, twill weave, and unidirectional fibres as shown in the figure 1.3. Depending on the weave types the mobility and stability of the fabric varies. Satin weave gives the most mobility i.e. could be easily draped around complex contours but the plain weave gives good stability of fibres in the structure.

![Asymmetric Ply Layout](image1.png) ![Symmetric Ply Layout](image2.png)

**Figure 1.4:** Different Ply Layouts: (a) Asymmetric ply layout where the distribution of plies about the mid-plane is in non-symmetric pattern (b) Symmetric ply Layout where the distribution of plies about the mid-plane is symmetric.

Uni-directional (UD) weaves have fibres laid in just one direction making their structural properties highly directional such as modulus of elasticity, thermal expansion coefficients. These uni-directional fibres usually are available in market with pre-impregnated resins such as epoxy which are just peel and apply – called as *prepregs*. The prepregs could be layered one over the other in specific pattern/configuration that yields laminates upon curing with required shape and properties. The ply orientation is described from the angle of fibres in the layer with respect to global coordinate sys-
tem. The layers could be laid in symmetrical or asymmetrical fashion, both having its own uses. Symmetric laminates have symmetrical distribution of the UD layers about the mid-plane whereas the asymmetric laminates have different ply orientations which are not symmetric about the mid-plane (refer figure 1.4).

1.3 Fabrication Process for UD-CFRP laminates

The process of fabricating laminates is referred through the University suppliers namely Rockwest Composites, Adhesive Prepregs for Composite Manufacturers (APCM) LLC. The steps involved are referred from our previous papers[24]:

- **Geometry Preparation**: the shape of the laminate to be prepared is marked on the prepreg sheet using visible marker and then cut into shape using the help of special scissors and roller cutter. According to the requirement layers are added on top of each other with required ply configurations.

- **Vacuum bagging Preparation**: the prepared samples are then placed in between two peel ply sheets, which are placed in between two breather sheets. This arrangement is then covered by two aluminium plates from either side which provide equal heating from both sides. This is an improvement suggested over the previous fabrication process; uneven heating on both sides led to samples loosing bistability (to be explained later) sooner. The resulting arrangement is then placed inside a vacuum sealing cover which is sealed using the yellow sealant tape. A nozzle is placed inside the packet which attaches to the suction pump outside.

- **Curing Cycle**: The vacuum sealed packet is then placed inside an oven which is then heated to the 280°F (recommended curing temp for 8552 resin system)
and maintained for 90 minutes. After completion, the oven and suction pump are switched off, and samples are allowed to cool inside the oven until the temperature reaches the room temperature (assumed 20°C).

1.4 Research Opportunity and Objective

Asymmetric Fibre layup has been found to create warpage and twist in the laminates. In the earlier stages of composite world, the symmetric laminates were favoured for their stability in shape for immobile/rigid structures. The twist and warping of laminates was treated as a failure/discontinuity in structure caused by local asymmetries in the fibre layups. But now, this nature of asymmetric laminates is looked upon as a potential to develop advanced structures which show morphing capabilities.

Due to the uneven thermal coefficients of expansion in two perpendicular directions for a UD-prepreg layer, the expansion happens unevenly in different directions under asymmetric ply layup. This results in instabilities of stress across the laminate layers, and forces the laminate to bend and twist in a peculiar way which is a function of the geometrical and material properties of individual layers.

On controlling the geometrical parameters such as the shape, ply orientations, thickness of ply and aspect ratio, the laminates could be precisely manufactured for the desired shape after curing. It is also observed that for some configurations, the shape after curing turns out cylindrical which is reversible i.e. curvature inversion in different directions is achieved upon application of external stimuli on the laminate. This nature of having two stable equilibria is well known as bistability. And the process of morphing from one shape to the other is termed as Snap-through process. The laminates bistability has been studied quite thoroughly in the past but the snap-through process studies received far less attention.
Hence, we base our study in finding different characteristics of snap-through process which depend on the boundary conditions applied to the laminate. To address the fundamentals of the problem, we restrict ourselves to a simple $[0^\circ/90^\circ]$ laminate with square geometry. And to apply the minimum boundary conditions, the laminate is held at the centre while using a single point to apply sufficient displacement to snap. This arrangement of snapping the laminates is asymmetric in nature, which is unique in comparison to the previous studies about the snap-through of bistable composite laminates. Previously symmetric loading conditions were considered where sharp snap-through behaviour (single-step) is observed. But our research questions bases on the asymmetric loading case –

- Does the laminate always follow the same path for snap-through from one stable state to the other?
- If not, then how are these snap-through different from each other and what are the possible reasons?
- Are there any analytical models to explain the snap-through behaviours?

Such questions are answered in the following study where load-deformation characteristics and curvature analysis are used as tools to justify the findings.

### 1.5 Thesis Outline

In what follows, Chapter 2 discusses first Macromechanics of Laminates listing all the assumptions and equations related to CFRP prepregs. Later the chapter broadly lists all the relevant literature explaining their experiments and mathematical modelling pertaining to our study. Chapter 3 presents the Analytical Model adopted in our
study, listing all the necessary polynomial estimations and equations to predict the stable shapes of the composite laminate. A two-ply bistable composite laminate shape predictor MATLAB application is presented. The Chapter 4 focuses on the experiment of our study, discussing the setup for both the lab experiments and FEA simulations. The results our followed with discussion explaining our findings. Also, the analytical model is used to recreate the results observed in our experiment; novel way of constraining the equations is explained. Chapter 5 concludes this study with a summary and discussion. The MATLAB code used for implementing the analytical model is attached in the Appendix.
Chapter 2

Literature Review

2.1 Macromechanics of CFRP Laminates

The CFRP prepregs as we know have continuous straight fibres infused between an uncured epoxy matrix which shows transversely isotropic material properties i.e. the directions perpendicular to the fibre are isotropic which is matrix dominant properties. From the Hooke’s Law, the relation between stress and strain is given by,

\[ \sigma_i = C_{ij} \epsilon_j, \quad \text{where the } i, j = 1 \text{ to } 6 \]  

(2.1)

here, the tensor notation is contracted to simplify the stress-strain state definition (refer the table 2.1) of a 3D-stress element as shown in the fig 2.1.

The \( C_{ij} \) matrix comprises of the material constants whose value depends on the

<table>
<thead>
<tr>
<th>Stresses Notation</th>
<th>( \sigma_{11} )</th>
<th>( \sigma_{22} )</th>
<th>( \sigma_{33} )</th>
<th>( \sigma_{23} )</th>
<th>( \sigma_{31} )</th>
<th>( \sigma_{12} )</th>
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<tr>
<td>( \sigma_1 )</td>
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<td>( \sigma_4 )</td>
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<td>Strains Notations</td>
<td>( \epsilon_{11} )</td>
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<td>( \epsilon_1 )</td>
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<td>( \epsilon_4 )</td>
<td>( \epsilon_5 )</td>
<td>( \epsilon_6 )</td>
</tr>
</tbody>
</table>

Table 2.1: Contracted tensor notation for 3D-stress element for composite laminae
material type such as anisotropic, orthotropic, isotropic, etc. This matrix is called the stiffness matrix. For a transversely isotropic material (in our case the CFRP prepregs) this material constants matrix has only 5 independent constants and is given by,

$$
C_{ij} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{22}-\frac{C_{23}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
$$

(2.2)

Now, the CFRP prepreg laminae are modeled using the plane stress formulation where the inter-laminar stresses such as the $\sigma_4$ and $\sigma_5$ (refer the fig 2.1) are neglected. And assumed that there is no extension/compression of laminae in the out-of-plane
direction. Hence, the \( \sigma_3 = 0 \). Therefore, the stress-strain relation reduces to,

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{66}
\end{bmatrix} \begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix}
\]

(2.3)

Here, the reduced stiffness matrix \( (C_{ij}) \) is also termed as \( Q_{ij} \) where, the individual coefficients could be given as a function of material parameters as,

\[
C_{11} = Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad C_{22} = Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}; \\
C_{12} = Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}; \quad C_{66} = Q_{66} = G_{12}
\]

(2.4)

where, \( E_i \) are the modulus of elasticity in perpendicular directions, \( \nu_{ij} \) poisson’s ratios and \( G_{12} \) is the modulus of rigidity.

\[\text{Figure 2.2: CFRP prepreg lamina with fibres not aligned with the global coordinate system. The angle } \theta \text{ with respect to } x\text{-axis refers to the angle with which the UD fibres are aligned.}\]

The above stress-strain equation are written in material coordinate system where
the 1-direction is the direction in which the fibres are laid and the 2-direction is perpendicular to the fibre-direction. If the material coordinate system is not aligned with the global coordinate system then the relation needs to be transformed. After transforming the stress and the strain vector, the relation 2.3 changes to,

\[ \sigma_i = Q_{ij} \cdot \varepsilon_j \rightarrow \mathbf{\sigma} = \mathbf{Q} \cdot \mathbf{\varepsilon} \]

\[ \mathbf{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T; \quad \mathbf{\varepsilon} = \begin{bmatrix} \epsilon_x & \epsilon_y & \gamma_{xy} \end{bmatrix}^T \]

From \( Q \), one can calculate \( Q^{(k)} \) of every layer \((k)\) in the laminate by \( Q^{(k)} = T^{-1}Q^{(k)}T^{-\top} \). The transformation matrix \( T \) with \( \theta \) being the angle at which the fibres of the prepreg lamina are aligned in the global coordinate system is given by,

\[
T = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & \sin 2\theta \\
\sin^2 \theta & \cos^2 \theta & -\sin 2\theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & \cos 2\theta
\end{bmatrix}.
\]

\[(2.6)\]

### 2.1.1 Classical Lamination Theory

The composite laminates have various layers or laminae stacked together, where each lamina is assumed to behave according to the Kirchoff’s plate theory which has following assumptions –

The first assumption states that the cross-section area normal to the mid-plane of the laminate remains straight even after deformation. This could be understood from figure 2.3, the line AD is perpendicular to the mid-plane initially and remains flat after deformation (normal direction given by the red arrow). By this assumption, through-the-thickness deformation is ignored.

The second assumption states that the slope of the laminate’s mid-plane is equal
Figure 2.3: Kirchoff’s Plate theory

to the rotation angle of the corresponding cross-section area. As observed in the figure, the angle $\beta$ represents the rotation of the mid-plane axis. Following the above assumption, the cross-section also has a slope equal to $\beta$ with respect to the vertical reference axis. Therefore, this angle is typically considered small — the in-plane deformations due to this rotation in different laminate layers follow the equation $z \ast \beta$, where $z$ is the distance between this layer and the mid-plane.

Hence, the in-plane deformations of the laminate can be generalised into two displacement functions i.e. $U$ and $V$,

$$
U = U_0 - z \cdot \beta_y \\
V = V_0 - z \cdot \beta_x
$$

(2.7)

where, the $U_0$ is the mid-plane displacement in x-direction while $V_0$ is the displacement in y-direction. Both of them are functions of only in-plane spatial coordinates $x, y$. The rotations about the respective axes are functions of z-displacement $W_0$ which is also the function of in-plane spatial coordinates $x$ and $y$. Mathematically,

$$
\beta_y = \frac{\partial W_0}{\partial x} ; \quad \beta_x = \frac{\partial W_0}{\partial y}
$$

(2.8)
Now, as we know the strains are partial derivatives of the displacement functions
which are calculated as follows,

$$
\epsilon_x = \frac{\partial U}{\partial x}; \quad \epsilon_y = \frac{\partial V}{\partial y}; \quad \gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}
$$

(2.9)

The displacement functions from above 2.7 are substituted in the above equations.
The two separate terms could be differentiated separately which yields mid-plane
strains $\epsilon_0$ and curvatures $\kappa$ of the laminates. The strains and curvatures could be put
together in a matrix form as,

$$
\epsilon_0 = \begin{bmatrix}
\epsilon_{x0} \\
\epsilon_{y0} \\
\gamma_{xy0}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial U_0}{\partial x} \\
\frac{\partial V_0}{\partial y} \\
\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}
\end{bmatrix}; \quad \kappa = \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{-\partial^2 W_0}{\partial x^2} \\
\frac{-\partial^2 W_0}{\partial y^2} \\
-\frac{\partial^2 W_0}{\partial x \partial y}
\end{bmatrix}.
$$

(2.10)

Now, the in-plane forces and moments for the laminate could be calculated by adding
respective individual quantity experienced by each layer. The force resultant vector
is calculated using the expression,

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} dz
$$

(2.11)

And the moment resultant,

$$
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} z \, dz
$$

(2.12)
These forces and moments could be simplified to calculate by defining the following summations,

\[ A_{ij} = \sum_{k=1}^{k=n} Q_{ij}^{(k)} (z_k - z_{k-1}) ; \]
\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{k=n} Q_{ij}^{(k)} (z_k^2 - z_{k-1}^2) ; \quad (2.13) \]
\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{k=n} Q_{ij}^{(k)} (z_k^3 - z_{k-1}^3) \]

Hence, using the above matrices in the equations 2.11 and 2.12 we get a simplified form,

\[ \{N\} = [A] \cdot \{\epsilon_0\} + [B] \cdot \{\kappa\} \]
\[ \{M\} = [B] \cdot \{\epsilon_0\} + [D] \cdot \{\kappa\} \quad (2.14) \]

### 2.2 Relevant Studies

To this end, we have witnessed the development of many “smart” structures that exploit the nonlinear phenomenon of *bistability* — the co-existence of two distant stable equilibria. A bistable structure can settle into either of its stable equilibria (or stable states) without external aid. However, if an external or internal actuation deforms such a structure to a critical configuration, elastic instability would occur and rapidly switch the structure to a different stable state; this process is referred to as the *snap-through*. The applications of bistable structures span across mechanical, [26, 4], electronic [5, 38], and biochemical systems [3, 16]. In particular, there has been a surge of interest in integrating the bistability in meta-materials [33, 23, 36], robotics [7, 9, 39], origami structures [34, 18, 37] and energy harvesters [21, 29], which are amongst the emerging engineering technologies.
Figure 2.4: Stacking sequence of non-symmetric laminate of configuration $[0^\circ/90^\circ]$. The red lines indicate the direction in which the fibres are laid in the matrix. The ply-1 is of $0^\circ$ configuration while the top ply-2 is of $90^\circ$ configuration.

One promising example of bistable structures is the asymmetric Carbon Fibre Reinforced Polymers (CFRPs), pioneered in Hyer and Dano’s groundbreaking study in 1981 [22]. The reinforcing fibers in these structures are oriented differently in each layer, and their order is non-symmetric through the thickness direction (e.g., $[0^\circ/90^\circ]$ layup in Figure 2.4). As a result, bi-stability occurs due to the non-uniform residual stress developed in the curing process. These asymmetric composites can feature a higher strength to weight ratio and better conformability than other types of bistable structures, and many applications have seen benefits from utilizing them, such as automobiles [15], morphing airframes, and wind turbine blades [14, 35, 1]. Potential applications also extend in renewable energy infrastructure [19, 1, 10], and energy harvesting [17].

The promising potentials of these bistable composites sparked many research efforts,
and an important question is to predict the composite’s external shapes at different stable states theoretically. Hyer and Dano formulated a model using classical lamina-
tion theory in an early work [12], and they used cubic polynomials to approximate the strain fields and quadratic polynomials for the out-of-plane displacements of a simple $[0^\circ/90^\circ]$ laminate. As the curvatures are second-order derivatives of the out-of-plane displacements, this model predicted cylindrical-shaped composite laminate with a uniform curvature distribution. However, it also over-predicted the deformations for cross-ply laminates with more generic $[\theta/\theta - 90^\circ]$ ply layouts. In a more recent study, Weaver et al. presented a different method for estimating the laminate’s curvature, which correlated the neutral planes of fiber plies to their anisotropic elastic moduli, and assumed linear strain distribution through the thickness [32]. By minimizing the total potential energy with respect to curvature, this model calculated the laminate’s maximum deflection at a stable state with a satisfactory agreement with the experimental measurements.

Examinations of the external shapes continue to evolve, with new analytical methods [6, 30], the inclusion of embedded actuators like piezoelectric patches [6, 27, 20, 31], and extension to multi-patched structures [25, 35, 28].

As, the adaptive structures usually have complex design constraints, one cannot rely on single ply orientation structure. Blending of multiple patches of different ply orientations is seen as the way forward and hence, a lot of studies have branched with the same idea. Such as the study from Mattioni et al. [25] which is one of the pio-
neering studies who successfully utilised the Hyer’s model with interesting extensions to predict the shapes of two-patches connected through an edge. Additional terms to the out-of-plane displacement function were introduced that ensured the second-order curvature terms rather than constant values. This improved the overall accuracy of the results while listing the important patch connectivity constraints.
Chapter 3

Analytical Model

3.1 Investigation Using An Extended Hyer’s Model

We formulated a comprehensive analytical model by extending the Hyer’s model, a widely used method for analyzing bistable composite laminates [11, 35, 17, 25]. This model’s accuracy and capability hinge on the order of polynomial functions that approximate the laminate’s deformation. Hyer and Dano chose a second-order polynomial for the out-of-plane displacement field. Hence the curvatures of the composite laminate were constant and uniform [22]. Mattioni used a product of two parabolic equations in both $x$ and $y$–directions [25]. As a result, the corresponding curvature distribution was a reduced second-order polynomial.

From the Hyer’s model, the total strain energy of the bistable laminate is formulated using the following equation

$$\Pi = \int_{-L_x/2}^{L_x/2} \int_{-L_y/2}^{L_y/2} \frac{1}{2} [\epsilon_0 \ \kappa] \begin{bmatrix} A & B \\ B & D \end{bmatrix} [\epsilon_0 \ \kappa] - [\epsilon_0 \ \kappa] \begin{bmatrix} N_t \\ M_t \end{bmatrix} dy \, dx, \quad (3.1)$$
where the mid-plane strains $\epsilon_0$ and curvatures $\kappa$ of the laminates are referred from the equations 2.10. As we know, the internal thermal strains are pivotal in producing bistability in CFRP laminates and are major part of this model. The in-plane thermal resultant forces ($N_t$) and moments ($M_t$) induced due to the curing are calculated as,

$$N_t = \sum_{k=1}^{k=n} \bar{Q}_{ij}^{(k)} \alpha_t \nabla T(z_k - z_{k-1})$$

$$M_t = \frac{1}{2} \sum_{k=1}^{k=n} \bar{Q}_{ij}^{(k)} \alpha_t \nabla T(z_k^2 - z_{k-1}^2) \tag{3.2}$$

The vector of thermal expansion coefficient is expressed as $\alpha_t = [\alpha_x \alpha_y \alpha_{xy}]^T$, where this vector is a transformed thermal expansion coefficients matrix. In the above Equations 2.13 and 3.2, $\bar{Q}^{(k)}$ is the stiffness of the $k^{th}$ fiber prepreg layer, transformed from the material coordinate system to the geometry coordinate system. The matrix $Q$ in the material coordinate system is referred back from equation 2.13 and transformations are applied according to the change in ply orientation (refer equation 2.6).

The most critical part of this analytical approach is the estimation of strains, curvatures, and hence the displacements of the bistable composites. The in-plane tensile strains are estimated using complete second order polynomials in that

$$\epsilon_{x0} = c_1 + c_2 \frac{x}{L_x} + c_3 \frac{y}{L_y} + c_4 \frac{x^2}{L_x^2} + c_5 \frac{y^2}{L_y^2} + c_6 \frac{xy}{L_x L_y},$$

$$\epsilon_{y0} = c_7 + c_8 \frac{x}{L_x} + c_9 \frac{y}{L_y} + c_{10} \frac{x^2}{L_x^2} + c_{11} \frac{y^2}{L_y^2} + c_{12} \frac{xy}{L_x L_y}, \tag{3.3}$$

where $c_i$ are constants to be determined. It is worth highlighting that, unlike the previous studies [22, 25], the linear terms in the strain estimation must be included. They were considered insignificant and thus omitted previously. However, these terms’ influences are significant in this study due to the localized deformation in the laminate.

Moreover, we normalize every term in the polynomials, i.e. divide by the laminate’s size in either $x$ or $y$—dimension with the same power as that particular term. This
normalization can increase the computational efficiency of numerical solver. The out-of-plane displacement $W_0$ is chosen as a specific third-order polynomial which is found sufficient to capture the stable shapes for most of the general laminates with good accuracy. The terms $x^2, y^2$ and $x,y$ are the important terms that were considered in the first revised model presented by Hyer and Dano [11]. The quadratic nature of stable shapes make these terms crucial to be included in our formulation.

$$W_0 = c_{13} \frac{x^2}{L_x^2} + c_{14} \frac{y^2}{L_y^2} + c_{15} \frac{xy}{L_x L_y} + c_{16} \frac{x^2 y}{L_x^2 L_y} + c_{17} \frac{xy^2}{L_x L_y^2} + c_{18} \frac{x^3}{L_x^3} + c_{19} \frac{y^3}{L_y^3}, \quad (3.4)$$

The in-plane displacements of the laminate $(U_0, V_0)$ are determined by using the strain formulations,

$$U_0 = \int \left[ \epsilon_x - \frac{1}{2} \left( \frac{\partial W_0}{\partial x} \right)^2 \right] dx + g(y) ; \quad V_0 = \int \left[ \epsilon_y - \frac{1}{2} \left( \frac{\partial W_0}{\partial y} \right)^2 \right] dy + h(x). \quad (3.5)$$

Here, $g(y)$ and $h(x)$ serve to complete the in-plane displacements and eliminate the rigid-body rotations:

$$g(y) = g_1 \frac{y}{L_y} + g_2 \frac{y^3}{L_y^3} + g_3 \frac{y^5}{L_y^5} ; \quad h(x) = h_1 \frac{x}{L_x} + h_2 \frac{x^3}{L_x^3} + h_3 \frac{x^5}{L_x^5}. \quad (3.6)$$

To make sure the rigid body rotation is eliminated, $g_1$ has to be equal to $h_1$. Finally, the in-plane shear strain is

$$\gamma_{xy0} = \frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} + \frac{\partial W_0}{\partial x} \frac{\partial W_0}{\partial y}. \quad (3.7)$$

The analytical model has twelve undetermined constants from the strain estimations, five from the in-plane displacement estimations, and more from the out-of-plane displacement estimation ($W_0$) depending on the order of the polynomials. The total
number of undetermined constants for third-order polynomial is 24, for fourth-order polynomial its 29, and fifth-order polynomial it is 35.

Here, we obtain the solution by minimizing the strain energy (Eq. 3.1) using the MATLAB `fmincon` function, which is a constrained multi-variable optimizer for highly non-linear problems. To reduce the computational time, we used the “interior-point” algorithm and explicitly derived the gradients of strain energy with respect to the undetermined constants.

### 3.2 Bistable Composite Shape Simulator

![Bistable composite Shape Generator (Two-Ply)](image)

Figure 3.1: Two stable shapes of composite laminate $[0^\circ/90^\circ]$ 

The above model is adapted in the form of a MATLAB project where the aim is to create a Graphical User Interface (GUI) to estimate the stable shapes of two-
ply composite laminates upon curing. The app allows the user to enter geometrical parameters as well as material properties for the composite ply.

As observed from the fig 3.1, the top four fields take into account the geometrical parameters such as the Patch length, ply orientations for the top ply and the bottom ply, and the cut size (irrelevant here). The material properties are listed under the box which include modulus of elasticity (E1 and E2), Modulus of rigidity (G12, G13, and G23) and thermal coefficients (α₁, α₂, and α₃). The calculate button is push-activated that creates the mathematical formulation in background using the input from our user. Here, examples of [0°/90°], [-45°/45°] (refer fig. 3.2), [0°/60°] (refer fig. 3.3) are shown.

Figure 3.2: Two stable shapes of composite laminate [45°/-45°]

The output shapes from the model were compared with the profilometer scans of the experiments; good agreement was shown. The cross-ply laminates upon physical...
inspection show a stronger bistablity nature (figured through series of snapping the laminates by hand and observing the force required and the amount of sound produced during the snapping) compared to the other angle-ply laminates. Also, the shapes from [-45°/45°] show higher deformation at the corners when compared to the [0°/60°] laminate.

![Bistable composite Shape Generator (Two-Ply)](image)

Figure 3.3: Two stable shapes of composite laminate [0°/60°]
Chapter 4

Different Snap-through Characteristics

4.1 Introduction

Besides the external shapes, the asymmetric composites’ snap-through deformation as they switch between the stable states also received attention. Dano et al. examined snap-through by using weights to apply symmetric bending moments from the opposite edges of a bistable laminate [11]. To predict the critical load required for snap-through, they also revised their mathematical model by including the virtual principle and approximating the strain field using quadratic polynomials rather than cubic. The predicted loads (moments) for snap-through showed decent agreement with the experiment results. Cantera et al. captured the load-displacement curves and intermediate laminate shapes during snap-through and snap-back, where the square laminates of varying side lengths were held at corners and loaded at the center [8]. Potter et al. provided another critical insight regarding the bifurcations during the snap-through of $[0^\circ/90^\circ]$ laminate [32]. They found that the snap-through occurs
via two closely coupled bifurcations rather than a single bifurcation. However, they
did not examine the nature of these two bifurcations in detail. Pirrera et al. compared
the performance of higher-order polynomials in predicting the snap-through behaviors
[30]. Their experiment involved four loads at the corners of a freely supported square
laminate to measure the force-displacement relationship during snap-through. They
then attempted to theoretically reproduce the experiment results by approximating
the laminate deformation using different polynomials functions, and concluded that
the polynomials had to be of a high order (eleventh in their study) to produce a
quantitative agreement with the FEA result.

All the studies of snap-through deformations mentioned above, however, only in-
volved symmetric loading to the laminate. Moreover, several studies pointed out
the importance of curvature change during the snap-through [32, 30], but none of
them examined its role in detail. Therefore, the objective of our experiment is to
1) examine the transient snap-through deformations of bistable composite laminate
under asymmetric and localized point load, and 2) investigate the role of curvature
changes. More specifically, we fix a $[0°/90°]$ square laminate at its center and apply
a transverse point load at different locations (either along the laminate edge or in
the interior). We measure the force-displacement relationships corresponding to dif-
ferent loading positions during snap-through and used the digital image correlation
(DIC) technique to track the laminate deformation and surface curvature evolution
in real-time. Finite element simulation is carried out in parallel to cross-validate the
experimental observations. Our results show that, as we change the loading positions,
the bistable laminate exhibits three uniquely different snap-through behaviors even
though its external shapes before and after snap-through remain the same. That is,
depending on its position, the point load can either complete the snap-through with
two consecutive steps or one step. However, at some locations, the point load cannot
induce a snap-through at all. The localized distribution and propagation of curvature changes (or curvature inversions) are directly related to these unique snap-through behaviors. Moreover, to assess the feasibility of theoretically predicting these complex deformations, we conduct a comparative study based on an extension of Hyer’s model with three different polynomial functions that approximate the laminate deformation. Our results show that a higher-order polynomial is necessary to reproduce the experiment and finite element results qualitatively, but there are trade-offs and limitations.

Since the bistable composites are increasingly popular in systems that involve dynamic loading conditions (e.g., morphing applications), it is crucial to understand their deformation characteristics under complex external stimuli. On the most fundamental level, this study establishes the behavioral trends and requirements for completing the snap-through between two stable states. Therefore, our results on the $[0^\circ/90^\circ]$ bistable laminate under localized load can provide a comprehensive insight and actuation design guidelines for many upcoming applications.

Table 4.1: Constituent material properties of DA409u 8552 carbon composite prepregs. $E_i$ and $G_{ij}$ are the elastic modulus (unit of GPA). $\nu$ is the Poisson’s ratio. $\alpha_{ij}$ are the thermal coefficients of expansion (unit of $^\circ$C$^{-1}$).

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Property</th>
<th>Value</th>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>135</td>
<td>$G_{12}$</td>
<td>5</td>
<td>$\alpha_{11}$</td>
<td>$-2 \times 10^{-8}$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>9.5</td>
<td>$G_{13}$</td>
<td>7.17</td>
<td>$\alpha_{22}$</td>
<td>$3 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.3</td>
<td>$G_{23}$</td>
<td>3.97</td>
<td>$\alpha_{33}$</td>
<td>$3 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
4.2 Methods and Materials

The bistable composite laminate samples in this study are 100 × 100 mm in size, made from DA409u 8552 unidirectional carbon fiber prepregs having ply thickness of 0.15 mm. A layer of 0° prepreg was placed on top of a 90° layer to develop the asymmetric [0°/90°] layout (Figure 4.1(a)), and then cured the assembly in an oven at 135°C to develop bi-stability. Vacuum bagging techniques were used in the fabrication, and interested readers can refer to the previous section 1.3 for more details. Table 4.1 summarizes the constituent material properties of the prepregs.

To measure the reaction force-deformation of the bistable patch under localized load, a small hole was drilled at the center of a bistable laminate and fixed it to a 3D-printed fixture with a bolt. This fixture has a tall tower with the laminate on its top, and four legs with adjustable screw holes for mounting on a universal testing machine (ADMET eXpert 5061) (Figure 4.1(c)). This test setup fully constraints the translational and rotational motions at the laminate center. To apply a localized load, an eyebolt was fastened at the selected node on the laminate, and connected it to the universal tester’s moving overhead with a nylon thread. This thread was changed for every run and was ensured to be taught before the start of snap-through experiment to minimise any slackness errors or length variation errors. A controlled overhead displacement of (0.5mm/sec) is applied at the selected node until snap-through completes, and the load cell (ADMET eXpert S-type 500 series) records the corresponding reaction force.

Besides measuring the reaction force-deformation relationship, the digital image correlation (DIC) was used to track the transient patch deformation throughout the snap-through process (Figure 4.1(d,e)). The DIC test involves two high-speed cameras that focus from top of the testing machine, adjusted to an optimum height in
order to capture the 3D displacement field of the complete laminate accurately. The camera frame rate was set at 30 fps, which turned out to be sufficient to capture the rapid composite deformation in snap-through. The nodal coordinates are extracted from the DIC data by Vic3D software (Correlated Solutions) to calculate the distribution of surface mean curvatures at different stages of loading.

The setup of finite element simulation (ABAQUS 6.14, Static Structural Solver) is the same as the experiment regarding the fiber laminate design, constituent material properties, boundary conditions, and localized loading. The laminate is modeled using the standard $S4R$ elements with a mesh density of $41 \times 41$ nodal points. The simulation involves two consecutive steps: The first step cures the laminate from its initially flat configuration to a stable state. The second step snaps the laminate to another stable state by applying sufficient displacement at the selected node, one at
a time.
During the first curing step, the laminate is fixed completely flat, as in the vacuum bagging techniques [24]. Then, we applied simulated heating at 135°C and cooling to room temperature at 20°C, allowing internal thermal stress to develop. Once cured, the laminate is released from its fixed condition and free to deform into one of the two stable shapes. In the second snapping step, the laminate is constrained at its center, as shown in Figure 4.1(b). The middle node is fixed in the $x - y$ reference plane with no rotation allowed about the $z$-axis (i.e, $U_1, U_2, UR3 = 0$). The four nodes adjacent to the middle node are constrained in the $z$-axis ($U3 = 0$). Then, we simulate a controlled displacement at the selected node until snap-through completes. The simulation records the reaction forces in all three directions ($RF1, RF2, RF3$) and displacements ($U1, U2, U3$) at the selected node of loading, and then calculate the total reaction force-nodal displacement relationship.

In this study, we focus on three different responses to examine the transient snap-through behaviors of the bistable laminate under asymmetric point load. They are the intermediate deformation, change in curvature distribution, and the reaction force during the snap-through. We discovered that as we move the point load location, the transient behavior of the composite laminate fundamentally changes during the snap-through irrespective of its two stable states before and after. The subsections below discuss three representative cases in detail.
Figure 4.2: The transient deformation during the snap-through when the point load is at the corner (Node #1). Different instants throughout this process are labeled from (i) to (vi). (a) The reaction force-deformation relationship from finite element simulations and experiments. (b) Finite element simulations show the progressive switches regarding the shape of the different laminate edges. (c) The mean curvature distribution shows a complicated evolution during the snap-through. These curvature results come from finite element simulation.

4.3 Results and Discussion

Point Load at Laminate Corner (Node #1)

When the point force is at node #1 (defined in Figure 4.1(b)), snap-through occurs in two consecutive steps, as indicated by the two sharp drops in the force-displacement
curves in Figure 4.2(a). The variation in the experimental response curves attributes to three different samples being tested, the unavoidable errors in the fabrication process, and discrepancy in the fixture holding torque. By carefully observing the patch deformation both in the experiment and finite element simulations (Figure 4.2(b)), we find that the first step occurs when the initially straight edge of the laminate (right edge in Figure 4.2(b)) acquires a curved shape. This relatively rapid change in the edge shape explains the first peak and then dip in the force-displacement curve and marks the onset of shifting from one state to the other. The second step occurs when the top, initially curved edge of the laminate becomes straight. It is worth noting that if we remove the point load before the second snap occurs, the laminate would return to its original stable state. On the other hand, after the second step occurs, the laminate would settle into a new stable state if the point load is removed.

Besides the reaction force, we further examine these transient snap-through behaviors based on the distribution of mean curvature in the bistable laminate. Figure 4.2(c) and the supplemental video illustrates the evolution of curvature distribution throughout the snap-through based on finite element simulation and the DIC measurements. Before loading, the curvature of the fiber laminate is relatively uniform and negative. This is consistent from the findings of Portela et. al. and reassures are results [31]. When the point load starts, we observe a region of inverted (positive) curvature that starts to initiate and grow from the loading point and the fixed center. We refer to this region as “curvature inversion zone” hereafter. As the node #1 displacement increases, the front of this curvature inversion — which corresponds to the front of zero mean curvature — begins to propagate. More importantly, when the curvature inversion reaches the bottom edge of the bistable laminate, the first step of the snap-through completes (iii). As the node #1 displacement continues to increase,
the curvature inversion zone progresses slowly along the top edge to the left. When it finally reaches the left edge, the second snap occurs and laminate rapidly deforms to the new stable state with a relatively uniform distribution of positive curvature. It is worth noting that the experiment results agree with finite element simulation reasonably well (regarding both the reaction force from the universal tester in Figure 4.2(a) and the curvature measurement from DIC in Figure 4.2(c)). Differences in the experimental results and FEA simulations are due to the fact that the material properties of the prototype might differ slightly from those used in the simulation. And the manual fabrication error also plays a role. Based on the simulation, the critical force required for the snap-through is 0.70 N (maximum reaction force), and the critical displacement is 29.0 mm (at the occurrence of the second step).

**Point Load at the Mid-Point of Initially Curve Edge (Node #2)**

If the point load moves to the mid-point of initially curved edge (node #2 in Figure 4.1(b)), the bistable laminate snap-through occurs in only one step. During the snap-through, the initially curved edge at which node #2 is located and the adjacent straight edges (right and left edges) change their shapes simultaneously. That is, the initially curved top edge becomes straight and initially straight left/right edges becomes curved at the same time. Therefore, the deformation occurs almost symmetrically about the longitudinal axis passing through the fixed center, as shown in Figure 4.3(b)). Such symmetry in snap-through is also evident in the distribution of mean curvature (Figure 4.3(c) and supplemental video). We observe that the fronts of curvature inversion zone progress symmetrically towards the top edge. When these fronts reach the respective right and left edges, snap-through occurs, and the laminate
Figure 4.3: The transient deformation during the snap-through when the point load is at the center of initially curved edge (Node #2). Different instants throughout this process are labeled from (i) to (vi) (a) The reaction force-deformation relationships show that, in this case, the snap-through consists of only one step. (b) Finite element simulations show the simultaneous switches regarding the shape of the different laminate edges. (c) The mean curvature distribution shows a symmetric evolution during the snap-through. At the beginning of snap-through (labeled by iii, iv), we also included the mean curvature based on DIC readings for validation. Here, the curvature distribution at the first stable state (i) and second stable state (vi) are omitted since they are the same as those in Figure 4.2(c).

deforms to the new stable state rapidly. The corresponding critical force required for completing the snap-through is 4.1 N based on finite element, and the corresponding critical displacement is 16.7 mm.
Figure 4.4: The transient deformation during the snap-through when the point load is at the center of initially straight edge (Node #5). (a) The reaction force-deformation relationships show that, in this case, no snap-through occurs. (b) Finite element simulations showing the deformed shape of the laminate. (c) The mean curvature distribution based on finite element and DIC data.

**Point Load at the Mid-Point of Initially Straight Edge (Node #5)**

If the point load is at the mid-point of the initially straight edge (node #5 in Figure 4.1(b)), snap-through never occurs regardless of the load amplitude (Figure 4.4).
This unique observation has never been reported by previous studies. By carefully observing the evolution of curvature distribution (Figure 4.4(c) and supplemental video), we discover that the curvature inversion zone never fully reaches the initially straight edge where the point force locates, even though the inversion zone can reach the initially curved edge (top and bottom edge in Figure 4.4(c)).

Figure 4.5: The force-displacement relationships based on other loading positions. At node #3, the snap-through involves two steps, although the distance between the two steps are very small. At node #4, snap-through involves only one step. At node #6, no snap-through occurs.

Table 4.2: Critical Displacement and Reaction forces for point loading at different nodes

<table>
<thead>
<tr>
<th>Node</th>
<th>Method</th>
<th>Critical Force</th>
<th>Critical Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Simulation</td>
<td>0.70N</td>
<td>29.0mm</td>
</tr>
<tr>
<td></td>
<td>Experiments</td>
<td>0.63N ± 14.3%</td>
<td>27.4mm ± 1.5%</td>
</tr>
<tr>
<td></td>
<td>% difference</td>
<td>12.8%</td>
<td>5.5%</td>
</tr>
<tr>
<td>2</td>
<td>Simulation</td>
<td>4.07N</td>
<td>16.7mm</td>
</tr>
<tr>
<td></td>
<td>Experiments</td>
<td>4.19N ± 3.8%</td>
<td>17.0mm ± 1.8%</td>
</tr>
<tr>
<td></td>
<td>% difference</td>
<td>2.9%</td>
<td>1.7%</td>
</tr>
<tr>
<td>3</td>
<td>Simulation</td>
<td>3.97N</td>
<td>11.7mm</td>
</tr>
<tr>
<td></td>
<td>Experiments</td>
<td>2.68N ± 4.5%</td>
<td>15.4mm ± 4.5%</td>
</tr>
<tr>
<td></td>
<td>% difference</td>
<td>32.4%</td>
<td>31.6%</td>
</tr>
<tr>
<td>4</td>
<td>Simulation</td>
<td>10.56N</td>
<td>9.9mm</td>
</tr>
<tr>
<td></td>
<td>Experiments</td>
<td>7.24N ± 11.7%</td>
<td>12.2mm ± 9%</td>
</tr>
<tr>
<td></td>
<td>% difference</td>
<td>31.4%</td>
<td>23.2%</td>
</tr>
</tbody>
</table>
Point Load at Other Positions

When the point load is at node #3, snap-through also occurs in two steps. After a careful observation of the finite element simulation and experiment result, we found that the transient deformation and curvature development are similar to those of node #1. However, unlike the case for node #1, the distance between these two steps in terms of point load displacement is minimal (Figure 4.5 shows the zoom-in view at the point of snap). When node #4 is under point load, the transient deformation is similar to that of node #2. That is, the snap-through occurs in only one step. However, the critical force requirement for snap-through increases significantly, whereas the critical displacement reduces. These differences indicate that the laminate gives more resistance as the point load moves closer to the fixed center. The curvature inversion development is similar to the node #2 case in that two inversion fronts originate from the center and progress towards the top and right edges. Finally, if the point load is at node #6, snap-through never occurs regardless of the load amplitude. For clarity, the critical force and displacement corresponding to the six different point load locations are summarized in Table 4.2.

Therefore, we deduce that if the loading point is on the axis defined by node #2 and laminate center, the corresponding snap-through involves only one step. Also, there would be two fronts of curvature inversion emerging out of the laminate centre and progressing towards respective left and right edges, thus snap-through happens symmetrically. If the point load is on the axis defined by node #5 and laminate center, no snap-through will occur. Thus, a full curvature inversion never occurs. Otherwise, the snap-through involves two steps and a single front of curvature inversion develops and progresses through the laminate asymmetrically. Moreover, we deduce that the snap-through is complete only when the curvature inversion zone reaches both of the
the initially straight edges. In the following subsection, we use the analytical model to test these observations.

4.4 Extending Analytical Model to recreate Experiment

The Hyer’s model and its variations are mostly used for predicting the external shape at the stable states, few used this model to estimate the transient deformation between stable states, let alone the asymmetric loading in this study. Therefore, we want to explore the feasibility of using the extension of Hyer’s model — by comparing the results from third, fourth, and fifth-order polynomials functions — for new insights.

To approximate the shape and curvature changes during the snap-through of the laminate and recreate the observations from above, we formulated a comprehensive analytical model by extending the Hyer’s model and applied the necessary boundary conditions to the model.

We estimate the out-of-plane displacement field of the laminate (in $z-$ axis) using complete third-order (refer equation 3.4, fourth-order, and fifth-order polynomial functions. Accordingly, the assumed curvatures (which are second-order differentials of out-of-plane displacement) are linear, second-order, and third-order polynomials. The $z-$displacement polynomials are listed below,

$$ W_0 = c_{13} \frac{x^2}{L_x^2} + c_{14} \frac{y^2}{L_y^2} + c_{15} \frac{xy}{L_x L_y} + c_{16} \frac{x^2 y}{L_x^2 L_y} + c_{17} \frac{x y^2}{L_x L_y^2} + c_{18} \frac{x^3}{L_x^3} + c_{19} \frac{y^3}{L_y^3} + c_{20} \frac{x^2 y^2}{L_x^2 L_y^2} + c_{21} \frac{x^3 y}{L_x^3 L_y} + c_{22} \frac{x y^3}{L_x L_y^3} + c_{23} \frac{x^4}{L_x^4} + c_{24} \frac{y^4}{L_y^4}, \tag{4.1} $$
Like the experiment and finite element, solving the analytical model involves two steps: curing and snapping. In the first step, we use the same temperature (i.e., $\nabla T = 20^\circ C - 135^\circ C$) to calculate the laminate shape right after curing. Moreover, no constraints are enforced in this step so that the undetermined constants can take any real values. In the second step of snapping the laminate from one stable state to another, we apply the localized loading (controlled displacement) by using the equality constraints from fmincon in a novel way. Denote $x = x_{load}$ and $y = y_{load}$ as the loading position and $d_{initial}$ as the calculated out-of-plane displacement of this position after the first curing step, we assign an equality constraint during optimization in that

$$W_0(x = x_{load}, y = y_{load}) = d_{initial} + \delta d,$$  \hspace{1cm} (4.3)$$

where $\delta d$ is the increment displacement loading. We increase $\delta d$ until the laminate shifts to the other state. In this way, we can observe the laminate’s intermediate shapes during loading and calculate the strain energy corresponding to every $\delta d$ increment. This could be understood as way of solving the problem using a quasi-static approach, in which every intermediate step is calculated to quasi-stable shape by minimising the potential function of the laminate when it is held at that increment displacement loading. Now, the reaction force becomes

$$F = -\frac{d\Pi}{d(\delta d)}.$$  \hspace{1cm} (4.4)
As we have three different polynomials to estimate the out-of-plane displacement field, we derive results from all of them for each case.

### 4.4.1 Results and Discussion

**Two-step snap at Node #1:**

![Graph](image)

Figure 4.6: Analytical prediction of the transient deformation during the snap-through when the point load is at Node #1. (a) The reaction force-deformation relationships based on three polynomial functions with different orders. (b) Simulated patch deformation based on the fifth order polynomial fit. (c) The mean curvature distribution based on fifth order polynomial fit. The colormap in this case is the same as Figure 4.2(c).

Figure 4.6 summarizes the analytical prediction when the point load is at Node #1. We observe that the third-order polynomial estimation (aka. linear mean curvature distribution) gives a clear indication that the snap-through process involves two different steps (Figure 4.6(a)). The response curve from the fourth order polynomial —
although also conveys that the overall snap-through has two-step — does not match experimental or FEA results well. The fifth order polynomial results (aka. cubic distribution of mean curvatures) successfully capture the trend qualitatively when compared to FEA and experimental results. We see that the reaction force increases and reaches its critical force at point (ii), where the initially straight edge at the right deforms into a curved shape ((ii)-(iii) in Figure 4.6(b)). Then the reaction force decreases to a local minimum and increases again, a trend similar to finite element and experiment results. The increase in the reaction force from this local minima is relatively gradual and, at the critical displacement, the laminate snaps completely to its second stable state in that the top edge changes from a curved shape to straight. Moreover, the reaction force at the occurrence of second step ((v) in Figure 4.6(a)) is less than the critical force at point (ii). This phenomenon of right edge snapping first and then top edge (hence the lag between two peaks) is captured satisfactorily by the fifth order polynomial model and correlates well to the FEA and experiment results. Here, its worth noting that some modifications were found necessary in-order to get the correct results. That is, $c_2$, $c_5$, $c_9$, and $c_{10}$ are set to zero. The physical principle underpinning this modification will be the subject of future study.

The fifth order polynomial approximation of the out-of-plane displacement give a cubic distribution of mean curvature, which gives a fair representation of the curvature inversion (CI) evolution (Figure 4.6(c)). We observe that the CI front develops at the top corner point where the load is applied and then progresses throughout the laminate. It reaches the bottom edge first, snapping the right edge to deformed curve shape. Then it progresses towards the top left corner. At the critical configuration (Figure 4.6(c)(v)), the curvature inversion reaches the left edge and hence, the laminate snaps rapidly to a new stable state.
No-snap at Node #5:

Similar to what we observe from the FEA and experimental results, the laminate does not snap if the load is applied at node #5 in the analytical model. This is true for all of the three polynomial estimations for out-of-plane displacement field. Figure 4.7(b) shows the sequence of intermediate shapes of laminate during the point load application. We observe that the right edge changes to a curved shape for a short while but quickly shifts back straight, indicated by the small dip in the force-reaction curve. This phenomenon was also observed in the experiment.

Figure 4.7: Analytical prediction of the transient deformation during the snap-through when the point load is at middle of initially straight edge (Node #5). (a) The reaction force-deformation relationships based on three polynomial functions with different orders. (b) Simulated patch deformation based on the fifth order polynomial fit. (c) The mean curvature distribution based on fifth order polynomial fit.

Similar to the curvature results we obtained from FEA and DIC, curvature inversion does not occur completely in the laminate. We observe that the initial negative
curvature of the laminate starts to invert from the centre towards the opposite top and bottom edges. And then slowly progresses in the laminate but never reaches the right or left edges. This is the reason due to which the laminate never snaps.

One-step snap at Node #2:

The analytical model used above can reproduce the results of two-step snap and no-snap behaviors gracefully; however, it appears to fail for the one-step snap when the point load is at node #2. As observed from the FEA and experiment results (Figure 4.3), there are two fronts of curvature inversion emerging from the center of the laminate. Even the fifth-order polynomial approximation of the out-of-plane displacement (4.2) is found incapable of capturing such complex mean curvature changes during the snap-through. To reproduce the results of the one-step snap-through, we would probably need a much higher-order polynomial, which would drastically increase the number of unknown constants, computation time, numerical error, and the overall analytical model’s complexity. This is against our analytical study’s original aim: to explore the feasibility of using the computationally inexpensive Hyer’s model to qualitatively predict the different snap-through behaviors based on point load positions with reasonable accuracy.

Therefore, the extended Hyer’s model with a fifth-order polynomial approximation has both success and limitations. It gives a good approximation of the transient deformation and curvature distribution in the two-step snap-through and no snap-through scenarios. However, it fails to capture the complexity in the one-step snap-through. A critical insight from this study is that, for any reduced-order analytical model to be successful in approximating the transient snap-through behavior, it must approximate the corresponding curvature changes with reasonable accuracy.
Chapter 5

Conclusions and Discussion

Via experimental measurements, finite element simulations, and analytical modeling, this study thoroughly examines the transient snap-through deformation of a $[0^\circ/90^\circ]$ bistable composite laminate subjected to an asymmetric point load. The results show three unique types of snap-through behaviors, depending on the point load location. When the point load is on the axis defined by the middle point of the initially curved edge (Node #2) and the fixed laminate center, the laminate snap-through occurs in a single step. If the point load on a line joining the middle of the initially straight edge (Node #5) and the laminate center, snap-through never occurs regardless of the point load magnitude. For all other point load locations, the laminate snap-through occurs in two consecutive steps.

Moreover, the evolution of laminate surface curvature distribution directly correlates to the occurrence of the snap-through. As the point load increases, a localized zone of inverted curvature would initiate from the fixed center and grow towards the point load position. The different steps of snap-through occur when the curvature inversion zone reaches different edges of the bistable laminate. In particular, the snap-through completes only when the curvature inversion reaches both of the initially straight...
edges (aka. left and right edge). In the two-step snap, the curvature inversion reaches these two edges consecutively; in the one-step snap, the inversion reaches two edges simultaneously; and finally in the no-snap scenario, the inversion never reaches the left edge.

Finally, this study explores the feasibility of using an extended Hyer’s model to predict the complicated snap-through behaviors qualitatively. This model uses a third, fourth, and fifth-order polynomial functions to approximate the laminate out-of-plane displacements. Using a novel way to constrain the potential function, we were able to solve the snap-through process by using quasi-static approach. The results show that, depending on the complexity of curvature inversion distribution, the extended model has a mix of success and limitation. In particular, the fifth-order polynomial approximation of the displacement field (aka. cubic curvature distribution) can satisfactorily predict the two-step snap (with some model modification) and no-snap scenarios. However, it fails to capture the complex curvature inversions in the one-step snap scenario. Predicting this one-step snap-through would require a much higher order polynomial function with a much higher computational cost.

Regardless, this study’s results can offer valuable insights into the fundamental mechanics of snap-through behaviors and the actuation locations associated with the bistable composite laminates. The out-of-plane actuation used in this study has unique advantages compared to in-plane actuators, such as embedded piezoelectric patches, because the latter could hinder bistability due to their additional stiffness [13]. And this study’s result could serve as a guide, allowing the user to choose the actuation force’s location based on the snap-through mechanics. One could first examine the characteristics of different snap-through processes and then select the actuator’s position according to its capability. For example, if the $[0^\circ/90^\circ]$ patch is selected and the actuator has a high block force but low stroke (e.g., piezoelectric
actuators), one can place the actuator at any point along the axis joining the Node #2 and center. This is because the corresponding snap-through involves a single step with minimal displacement requirement. On the other hand, if the actuator has a high stroke but low block force (e.g., shape memory polymers), Node #1 is the favorable actuating location as the critical force requirement is the least.
Chapter 6

Future Work

The research sees potential to be extended in the following domains.

Better Analytical Model

The Hyer’s model provides us with a good analytical model with reasonable accurate shape predictions when it comes to single patch laminates. But our groups recent efforts to use an extended Hyer’s model for multi-patch system, referred from Mattioni’s research [25] and Algmuni’s research [2], the accuracy of the model decreased significantly when considering angle-ply laminates (other than cross-ply laminates).

From the section 2.1.1 we see that the Hyer’s model adapts the Kirchoff’s Plate theory that has two important assumptions which may lead to inaccuracy in the results. The cross-section is assumed to remain straight and normal to the surface even after the deformation. And, the angle of rotation of the cross-section with vertical is equal to the tangent at that cross-section of the surface (refer to angle $\beta$ in the figure 2.3).

Due to the assumptions the inter-laminar stresses ($\sigma_4$ and $\sigma_5$ in the figure 2.1), the stresses which are present between the layers of the composite laminate, are neglected; this contributes to the inaccuracy too. Also, as the research community desires to
make the laminate smart, various actuators are studied to install on the laminate such as the piezoelectric patches. Hence, to accommodate the effects of actuators in the analytical model is difficult.

Advanced plate theories such as the Reissner-Mindlin plate theory may be helpful in formulating a better model in terms of accuracy. The complexity of solving such a model may be increased. Though, leveraging the better accuracy, multi-patch models could be improved for angle-ply laminates.

**Dual-Stiffness Structures**

Recently, a lot of stiffness adapting structures have surfaced creating an advanced framework that gauges the exterior conditions and adapts to optimise its performance. One such Dual-stiffness framework could be possible using these simple bistable CFRP laminates.

The uni-directional CFRP laminates have highly directional properties which show different stiffness in two perpendicular directions. For an example, consider a $[0^\circ/90^\circ]$ bistable laminate. Consider an in-plane load at the boundaries is applied to the laminate in one-direction for both the stable shapes; two different responses are observed – *Rigid* response when the load is applied at the curved edges, *Compliant* response when the load is applied at the straight edge. Just switching between the two shapes could achieve completely different responses. The current study may help in understanding the optimal snapping point for such structures depending on the input requirements such as limitations on either force application or maximum displacement allowance. Also, the mean curvature plots of these structures while in action may help in understanding the transient deformation characteristics better and avert any near-future damages. Such type of dual-stiffness structures could be utilised for
energy absorption tasks, variable load carrying duties, shape morphing with variable stiffness for airfoil structures, etc.

try involve transient behaviour references.

**Infusing Bistability in Kirigami/Origami Structures**

Origami/Kirigami has recently seen great adaptations owing to the immense potential to fold or deploy structures beyond its initial range. Various properties such as elasticity, stretchability, bending/buckling, twisting could be enhanced through these techniques. In some cases, these properties need external aids to hold the structures in different states. Such as a sheet with slit cut pattern is used to enhance stretching range of the sheet. But a constant external tensile load needs to be applied at the ends of the sheet to hold the sheet in this state. Now, infusing bistability in the sheet may help in locking the state of the sheet with no external aid. One could think of ample of possibilities where it could benefit to include bistability in these type of structures. Understanding the transient deformation characteristics using the mean curvature plots (as used in the current study) may help to identify different zones in the structure which experience the applied tensile force with separate intensities. Once these zones are identified, the bistability could be tailored for each zones of the structure to have uniform behaviour through out. Hence, such designs could benefit future applications and our study could provide a base to analyse and optimise the design.
Appendices
Appendix A  MATLAB code for shape prediction of Angle-ply Laminates

The following MATLAB code is used to predict the stable shapes upon curing of Angle-ply Laminates. Various comments in between are used to highlight certain sections and properties which maybe user-input based. Initially, the symbolic package is used to create the equations and later converted to the MATLAB function for the use of fmincon solver. The undetermined coefficients are denoted by the "c" variable which defined the strain and displacement fields.

```matlab
clc
clear
syms x y z
c = sym('c',[1 35]);

A = zeros(3);
B = zeros(3);
D = zeros(3);
Nt = zeros(3,1);
Mt = zeros(3,1);

%input the material properties
E1 = 135e9; %Pa
E2 = 9.5e9; %Pa
h = 0.15e-3; %m Thickness
G12 = 5e9; %Pa
```
lenx = 100e-3;
leny = 102e-3;%m side of the laminate
alpha11 = -2e-8;
alpha22 = 3e-5;
alpha33 = 0e-5;%k^-1
a = [alpha11,0,0; 0,alpha22,0; 0,0,alpha33];
delT = 20-135; %°C
v12 = 0.3;
v21 = v12*E2/E1;

%Assumed Polynomials
ex0 = c(1,24) + c(1,25)*x^2/lenx^2 + c(1,26)*y^2/leny^2
    + c(1,27)*x*y/lenx^2 + c(1,32)*y/leny + c(1,33)*x/lenx;
ey0 = c(1,28) + c(1,29)*x^2/lenx^2 + c(1,30)*y^2/leny^2
    + c(1,31)*x*y/lenx^2 + c(1,34)*x/lenx + c(1,35)*y/leny;
w0 = c(1,1)*x^2/lenx^2 + c(1,2)*y^2/leny^2 + c(1,3)*x*y/lenx^2
    + c(1,5)*y*x^2/lenx^3 + c(1,6)*x*y^2/leny^3 + c(1,7)*x^4/lenx^4
    + c(1,8)*y^4/leny^4 + c(1,9)*x*y^2/lenx^4 + c(1,10)*x*y^3/leny^4
    + c(1,11)*x^3/lenx^3 + c(1,12)*y^3/leny^3 + c(1,13)*y*x^2/lenx^5
    + c(1,14)*x*y^4/lenx^5 + c(1,15)*x*(x*y)^2/lenx^5
    + c(1,16)*y*(x*y)^2/leny^5
    + c(1,17)*x^5/lenx^5 + c(1,18)*y^5/leny^5;

%Derived Parameters
kx = -diff(diff(w0,x),x);
ky = -diff(diff(w0,y),y);
kxy = -2*diff(diff(w0,x),y);
K = [kx; ky; kxy];
u0 = vpa(int(ex0 - 0.5*(diff(w0,x))^2,x) + c(1,19)*y/leny
    + c(1,20)*y^3/leny^3 + c(1,21)*y^5/leny^5,7);
v0 = vpa(int(ey0 - 0.5*(diff(w0,y))^2,y) + c(1,19)*x/lenx
    + c(1,22)*x^3/lenx^3 + c(1,23)*x^5/lenx^5,7);
exy0 = diff(u0,y) + diff(v0,x) + diff(w0,x)*diff(w0,y);
e0 = [ex0;ey0;exy0];

%Stiffness matrix
Q11 = E1/(1-v12*v21);
Q22 = E2/(1-v12*v21);
Q12 = v12*E2/(1-v12*v21);
Q66 = G12;
Q = [Q11, Q12, 0; Q12, Q22, 0; 0, 0, Q66]; %reduced Stiffness matrix

tta = [90;0]; %Configuration
Z = -h:h:h;%in m

for i = 1:2
    %Transformation Matrix for Stiffness
    T = [cosd(tta(i))^2, sind(tta(i))^2, -sind(2*tta(i));
         sind(tta(i))^2, cosd(tta(i))^2, sind(2*tta(i));
         (sind(2*tta(i)))/2, -(sind(2*tta(i)))/2, cosd(2*tta(i))];
Qbar = T*Q*transpose(T);
A = A + Qbar*(Z(i+1)-Z(i));
B = B + 0.5*Qbar*((Z(i+1)^2)-(Z(i)^2));
D = D + (1/3)*Qbar*((Z(i+1)^3)-(Z(i)^3));

% Transformation matrix for Thermal Coefficients
T_a = [cosd(tta(i)), -sind(tta(i)), 0;
      sind(tta(i)),  cosd(tta(i)), 0;
      0,       0,               1]
att = T_a*a*transpose(T_a);
at = [att(1,1); att(2,2); att(1,2)];
Nt = Nt + Qbar*at*delT*(Z(i+1)-Z(i));
Mt = Mt + Qbar*at*delT*((Z(i+1)^2)-(Z(i)^2))/2;
end

%%% Potential function
V = simplify(vpa(0.5*transpose([e0; K])*[A, B, D]*[e0; K]
               - transpose([e0; K])*[Nt; Mt], 7));
en1 = int(int(V,x,[-lenx/2 lenx/2]),y,[-leny/2 leny/2]);
x_pos = lenx/2;
y_pos = leny/2;
exp = [];
for expi = 1:35
  q = diff(en1,c(1,expi));
exp = [exp; q];
end

%% Converting to matlab function for solving using fmincon method.

curvs = matlabFunction(K,'Vars',{c,[x,y]});
ux = matlabFunction(u0,'Vars',{c,[x,y]});
vy = matlabFunction(v0,'Vars',{c,[x,y]});
wz = matlabFunction(w0,'Vars',{c,[x,y]});

main_pot = totpot(en1,exp,c)

[pt,dfp] = main_pot()

% To solve the minimisation of Potential function
% using the fmincon method.

x0 = 0*ones(1,35);

count = 1;

Aeq = [];
beq = [];
constants = [];
A1 = [];
b1 = [];  
lb = [];  
ub = [];  
nonlcon = [];

options = optimoptions('fmincon','Algorithm','interior-point',  
'MaxFunctionEvaluations',80e3,'SpecifyObjectiveGradient',true);

[x_ans,v_val] =  
fmincon({pt,dfp},x0,A1,b1,Aeq,beq,lb,ub,nonlcon,options);

% Plotting the images of the laminate
x_d = -lenx/2:lenx/80:lenx/2;  \text{in m}
y_d = -leny/2:leny/80:leny/2;  \text{in m}
[x_dom,y_dom] = meshgrid(x_d,y_d);

for i = 1:size(x_dom,1)  
    for j = 1:size(y_dom,1)  
        x_co(i,j) = feval(ux,x_ans,[x_dom(i,j),y_dom(i,j)]);  
        y_co(i,j) = feval(vy,x_ans,[x_dom(i,j),y_dom(i,j)]);  
        wf(i,j) = feval(wz,x_ans,[x_dom(i,j),y_dom(i,j)]);  
    end
end

xf = x_dom + x_co;
%Calculation for all the fields
for i = 1:size(xf,1)
    for j = 1:size(yf,1)
        kval = feval(curvs,x_ans,[xf(i,j),yf(i,j)]);
        kx_co(i,j) = kval(1);
        ky_co(i,j) = kval(2);
        kxy_co(i,j) = kval(3);
        k1(i,j) = 0.5*(kx_co(i,j) + ky_co(i,j)) + sqrt((0.5*(kx_co(i,j) - ky_co(i,j)))^2 + (kxy_co(i,j)^2)/4);
        k2(i,j) = 0.5*(kx_co(i,j) + ky_co(i,j)) - sqrt((0.5*(kx_co(i,j) - ky_co(i,j)))^2 + (kxy_co(i,j)^2)/4);
        meank(i,j) = -0.5e-3*(k1(i,j) + k2(i,j)); % per m to per mm
    end
end

h = figure
surf(xf,yf>wf,'EdgeColor','none')
xlabel('x')
ylabel('y')
axis equal
title(sprintf("Shape of the [%d/%d] laminate",tta(1),tta(2)))
colorbar
axis equal

function func = totpot(pot,dif,c)
func = @fun;

    function [fu,dfy] = fun()
        fu = matlabFunction(pot,'Vars',{c});
        dfy = matlabFunction(dif,'Vars',{c});
    end
end


