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The Scarf Complex of Weighted Edge Ideals

James McKay Visser
Clemson University, mckay.visser@gmail.com

THE SCARF COMPLEX OF WEIGHTED EDGE IDEALS

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J. McKay Visser
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Dr. Keri Sather-Wagstaff, Committee Chair
Dr. James Coykendall
Dr. Kevin James

Abstract

Let R be a commutative ring with identity. A free resolution is a sequence of R-module homomorphisms of the form

$$X = \cdots \xrightarrow{\partial^{\beta_{i+1}}} R^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\partial_0} 0.$$

where R^{β_i} is a free R-module and $\ker(\partial_i^X) = \operatorname{im}(\partial_{i+1}^X)$ for all $i > 0$. When we only have the containment $\operatorname{im} \partial_{i+1}^X \subseteq \ker(\partial_i^X)$ we call X a chain complex. Computing these resolutions in a traditional fashion tends to be rather expensive, though some classes of resolutions can be constructed explicitly.

In 1998, Bayer, Peeva and Sturmfels proved that every labeled simplicial complex has an associated chain complex and gave a combinatorial/topological criterion for the chain complex to be a resolution [1]. One important example is the Scarf complex. We will explore when the Scarf complex is a resolution.

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Chapter 1

Introduction

Throughout this thesis, we will assume K is a field and $R = K[x_1, \ldots, x_d]$ is a polynomial ring in d variables, unless stated otherwise. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$. For all $n \in \mathbb{N}+$, define $[n] = \{1, 2, \ldots, n\}$.

Combinatorial commutative algebra was created by Hochster and Stanley in the mid-seventies [3, p.207]. This exciting branch of mathematics studies the interplay between commutative algebra and combinatorics. The idea is that one can study combinatorial objects via algebraic constructions and vice versa. Some of the interplay between these two fields can be visualized via the following diagram:

$$\begin{array}{ccc} \text{Algebra Problem} & \longrightarrow & \text{Combinatorial Problem} \\ \downarrow & & \downarrow \\ \text{Algebra Solution} & \longleftarrow & \text{Combinatorial Solution.} \end{array}$$

Alternatively, one can study combinatorics using algebra. For example, one can associate to each finite simple graph its edge ideal.

In this chapter we introduce some basic facts about monomial ideals, homological algebra, and combinatorial constructions for use in subsequent chapters.

1.1 Monomial Ideals

In the body of this thesis we will restrict our attention to resolutions of monomial ideals. Monomial ideals are particularly nice for two reasons. First, they arise naturally in combinatorics, geometry and topology; second, they are particularly simple. For example, every monomial ideal is finitely generated (see Theorem 1.3.1 (Dickson's Lemma) in [10], a precursor to Hilbert's Basis Theorem), and the ideal membership problem for monomial ideals is straightforward to solve.

Generators and Basic Properties

We begin with some definitions.

Definition 1.1.1 ([10, Definition A.1.5])**.** A monomial in R is a product of indeterminants, i.e., it is an element of the form $x_1^{n_1}\cdots x_d^{n_d}\in R$ where $n_1,\ldots,n_d\in\mathbb{N}$. For short, we write $\mathbf{n}=(n_1,\ldots,n_d)\in\mathbb{N}^d$ and $\mathbf{x}^{\mathbf{n}}=x_1^{n_1}\cdots x_d^{n_d}$.

Definition 1.1.2 ([10, Definition A.9.5])**.** A monomial ideal in R is an ideal of R that can be generated by monomials in $x_1,\ldots,x_d$.

Half of the aforementioned solution to the ideal membership problem for monomial ideals is described in the next result. One only needs to check if a given monomial is a multiple of one of the monomial generators as opposed to the more difficult test of checking if it is a linear combination of the generators.

Theorem 1.1.3 ([10, Theorem 1.1.9])**.** *Let $f,f_1,\ldots,f_m$ be monomials in R. Then, $f\in\langle f_1,\ldots,f_m\rangle$ if and only if $f\in\langle f_i\rangle$ for some i.*

From a computational perspective, it is advantageous to represent a monomial ideal with the least amount of data possible. We define this notion next.

Definition 1.1.4 ([10, Definition 1.3.4])**.** Let I be a monomial ideal of R. Let $f_1,\ldots,f_m$ be monomials such that $I=\langle f_1,\ldots,f_m\rangle$. The list $f_1,\ldots,f_m$ is an irredundant monomial generating sequence for I if each $i\in\{i,\ldots,m\}$ satisfies $\langle f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_m\rangle\neq I$.

The next result gives a criterion for checking if a generating sequence is irredundant.

Proposition 1.1.5 ([10, Proposition 1.3.5])**.** *Let I be a monomial ideal of R, and let $f_1,\ldots,f_m$ be monomials in I such that $I=\langle f_1,\ldots,f_m\rangle$. The following conditions are equivalent:*

(i) f_i is not a monomial multiple of f_j whenever $i\neq j$;

(ii) each $i\in\{1,\ldots,m\}$ satisfies $f_i\notin\langle f_1,\ldots,f_{i-1},f_{i+1},\ldots f_m\rangle$; and

(iii) the generating sequence $f_1, \ldots, f_m$ is irredundant.

First applying Dickson's Lemma and then iteratively throwing out redundant generators using Proposition 1.1.5, one obtains an irredundant generating sequence; moreover, such a sequence is unique. This process will terminate in finitely many steps because the original generating sequence is finite. For the remainder of this thesis, we can safely assume we are working with an irredundant generating sequence.

Decompositions of Monomial Ideals

Decomposing a monomial ideal is analogous to factorization in a commutative ring setting. Essentially, we express a monomial ideal as an intersection of the simplest monomial ideals in the sense that they themselves cannot be written as nontrivial intersections of monomial ideals. We will use these decompositions later when computing the Krull dimension of R/I in Subsection 4.3.

Definition 1.1.6 ([10, Definition 3.1.1])**.** A monomial ideal $J < R$ is m-reducible if there are monomial ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. A monomial ideal $J < R$ is m-irreducible if it is not m-reducible.

Definition 1.1.7 ([10, Definition 3.2.1])**.** Let A be a nonzero commutative ring with identity. An ideal $J < A$ is reducible if there are ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. An ideal $J < A$ is irreducible if it is not reducible.

Note, irreducible implies m-irreducible. However, when K is a field, then m-irreducible and irreducible are equivalent for monomial ideals. Furthermore, m-irreducible ideals are generated by "pure powers". These facts are summarized in the next result.

Theorem 1.1.8 ([10, Theorems 3.1.4 and 3.2.4])**.** *Let J be a nonzero monomial ideal of R. The following conditions are equivalent:*

(i) J is irreducible,

(ii) J is m-irreducible; and

(iii) there exists positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leq t_1 < \cdots < t_k \leq d$ and $J = \langle x_{t_1}^{e_1}, \ldots, x_{t_k}^{e_k} \rangle$.

Definition 1.1.9 ([10, Definition 3.3.1])**.** Let $J < R$ be a monomial ideal. An m-irreducible decomposition of J is an expression $J = \bigcap_{i=1}^{n} J_i$ with $n \geq 1$, where each J_i is m-irreducible.

The next result shows that every proper monomial ideal admits an m-irreducible decomposition. The proof is non-constructive and is essentially due to Emmy Noether. We refer the reader to Theorem 1.1.16 for a constructive description of a decomposition.

Theorem 1.1.10 ([10, Theorem 3.3.3])**.** *Every monomial ideal $J < R$ has an m-irreducible decomposition.*

We say that an m-irreducible decomposition $\bigcap_{i=1}^{n} J_i$ is irredundant if every ideal J_i "contributes" to the intersection. This is formalized as follows.

Definition 1.1.11 ([10, Definition 3.3.4])**.** Let $J < R$ be a monomial ideal. An m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is redundant if there is an index j such that $J = \bigcap_{i \neq j} J_i$, where the intersection is taken over $i = 1, \ldots, n$ such that $i \neq j$. An m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is irredundant if it is not redundant, that is, if every $j \in \{1, \ldots, n\}$ satisfies $J \neq \bigcap_{i \neq j} J_i$. As $J = \bigcap_{i=1}^{n} J_i \subseteq \bigcap_{i \neq j} J_i$ holds automatically, the given decomposition is irredundant if and only if every $j \in \{1, \ldots, n\}$ satisfies $J < \bigcap_{i \neq j} J_i$.

The next result gives a criterion for checking if a decomposition is irredundant.

Proposition 1.1.12 ([10, Proposition 3.3.6])**.** *Let J be a monomial ideal in R with m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. The following conditions are equivalent:*

(i) the decomposition $J = \bigcap_{i=1}^{n} J_i$ is redundant; and

(ii) there are indices $j \neq j'$ such that $J_{j'} \subseteq J_j$.

The following algorithm shows how to turn any given m-irreducible decomposition into an irredundant one.

Algorithm 1.1.13 ([10, Algorithm 3.3.7])**.** Let J be a monomial ideal with m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. Note that $n \geq 1$.

Step 1. Check whether the intersection $J = \bigcap_{i=1}^{n} J_i$ is irredundant, using Propositions 1.1.5 and 1.1.12.

Step 1a. If all distinct indices j and j' satisfy $J_j \not\subseteq J_{j'}$, then the intersection is irredundant; in this case, the algorithm terminates.

Step 1b. If there exist distinct indices j and j' and $J_j \subseteq J_{j'}$, then the intersection is redundant; in this case, continue to Step 2.

Step 2. Remove an ideal that causes a redundancy in the intersection. By assumption, there exist indices j and j' such that $j \neq j'$ and $J_j \subseteq J_{j'}$. Remove $J_{j'}$ from the list, and apply Step 1 to the new decomposition $J = \bigcap_{i \neq j} J_i$.

Corollary 1.1.14 ([10, Corollary 3.3.8])**.** *Every monomial ideal $J < R$ has an irredundant m-irreducible decomposition.*

In general m-irreducible decompositions are not unique, but irredundant m-irreducible decompositions are unique up to reordering the terms.

Theorem 1.1.15 ([10, Theorem 3.3.9])**.** *Let J be a monomial ideal in R with irredundant m-irreducible decompositions $J = \bigcap_{i=1}^{n} J_i = \bigcap_{j=1}^{m} I_j$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $J_t = I_{\sigma(t)}$ for $t = 1, \ldots, n$.*

The next result is the explicit m-irreducible decomposition mentioned above. One can pair it with Algorithm 1.1.13 to create irredundant m-irreducible decompositions. An example of this is discussed in the proof of Proposition 4.3.3.

Theorem 1.1.16 ([10, Theorem 7.5.1])**.** *Let I be a monomial ideal of R with monomial generating sequence $f_1, \ldots, f_t$. For $i = 1, \ldots, t$ write $f_i = \mathbf{x}^{\mathbf{a}_i}$ where $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}^d$. Then*

$$I = \bigcap_{i_1=1}^{d} \cdots \bigcap_{i_t=1}^{d} \left\langle x_{i_1}^{a_{1,i_1}}, \ldots, x_{i_t}^{a_{t,i_t}} \right\rangle .$$

1.2 Homological Algebra

Throughout this section, we will assume that R is a noetherian commutative ring with identity.

We review the basics of homological algebra here. For a more in-depth treatment, we refer the reader to [12] or Appendix 3 in [6].

Linear Algebra

Free R-modules are the simplest modules to understand because they are direct sums of copies of R. These are the modules that have a basis, which we define next.

Definition 1.2.1. Let M be an R-module.

(a) A sequence $e_1, \ldots, e_n \in M$ is a finite basis for M if it generates M as an R-module and it is linearly independent over R, i.e., for every $m \in M$ there exists unique $r_1, \ldots, r_n \in R$ such that $m = \sum_{i=1}^{n} r_i e_i$.

(b) A module M is a finite rank free R-module if it has a finite basis.

An R-module homomorphism from a free module can be defined easily by specifying where the basis elements get sent, as the next result explains. We use this fact when we define differentials in R-complexes and resolutions.

Fact 1.2.2 (Universal Mapping Property)**.**

(a) *Let F be a free R-module with basis $e_1, \ldots, e_n \in F$. For every R-module M and any collection of elements $m_1, \ldots, m_n \in M$, there exists a unique R-module homomorphism $\phi : F \to M$ such that $\phi(e_i) = m_i$ for $i = 1, \ldots, n$.*

$$\begin{array}{ccc} \{e_1, \ldots, e_n\} & \xrightarrow{\iota} & F \\ & \searrow & \downarrow \\ & & M \end{array}$$

(b) *If F and G are finite rank free R-modules with bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively, then $F \cong G$ as R-modules*

Exact Sequences

Exact sequences are ubiquitous in homological algebra. We only give the definition and some basic properties.

Definition 1.2.3 ([4, Definition 6.1.1])**.** Consider a sequence of R-modules and homomorphisms

$$A = \cdots \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \cdots$$

(a) We say that the sequence is exact at A_i if $\operatorname{im}(f_{i+1}) = \ker(f_i)$.

(b) The entire sequence is said to be exact if it is exact at each A_i, that is, if $\operatorname{im}(f_{i+1}) = \ker(f_i)$ for all $i \in \mathbb{Z}$.

Many important properties of homomorphisms can be expressed by saying that a particular sequence is exact. For example, we can phrase what it means for an R-module homomorphism $\phi : M \to N$ to be injective, surjective, or an isomorphism in terms of exact sequences.

Fact 1.2.4.

(a) $0 \longrightarrow A \xrightarrow{\phi} B$ *is exact if and only if* ϕ *is injective.*

(b) $B \xrightarrow{\phi} C \longrightarrow 0$ *is exact if and only if* ϕ *is surjective.*

(c) $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$ *is exact if and only if* ϕ *is an isomorphism.*

(d) $0 \longrightarrow A \longrightarrow 0$ *is exact if and only if* $A = 0$*.*

The last item is useful in showing that an R-module is the zero R-module.

Chain Complexes

Homological algebra arises when we relax the conditions defining exactness as follows.

Definition 1.2.5. A sequence of R-module homomorphisms

$$A = \cdots \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} A_{i-1} \xrightarrow{\partial^A_{i-1}} \cdots$$

is chain complex over R (or R-complex) if $\partial^A_i \circ \partial^A_{i+1} = 0$ for all $i \in \mathbb{Z}$. We say that the elements $a \in A_n$ have homological degree $|a| = n$.

Fact 1.2.6. *Given R-module homorphisms* $A \xrightarrow{f} B \xrightarrow{g} C$ *we have* $\operatorname{im} f \subseteq \ker g$ *if and only if* $g \circ f = 0$*.*

Definition 1.2.7. Let A be an R-complex. For all $n \in \mathbb{Z}$, denote by $Z_n(A) = Z_n$ the set of cycles of homological degree n and denote by $B_n(A) = B_n$ the set of boundaries of homological degree n, i.e.,

$$Z_n = \ker \partial^A_n \subseteq A_n$$

$$B_n = \operatorname{im} \partial^A_{n+1} \subseteq Z_n,$$

where the containment on the right are as submodules. The n^{th} homology module of A is

$$H_n(A) = \frac{Z_n(A)}{B_n(A)}.$$

The module $H_i(A)$ measures the exactness at A_i. If $B_n(A) = Z_n(A)$, then $H_n(A) = 0$ (and conversely), and we say that A has no homology in homological degree n. On the other hand, if $B_n(A) \subsetneq Z_n(A)$, then $H_n(A) \neq 0$ and we say that A has homology in degree n.

Definition 1.2.8. Given an R-complex A, the homology complex of A is the R-complex

$$H(A) = \cdots \xrightarrow{0} H_i(A) \xrightarrow{0} H_{i-1}(A) \xrightarrow{0} \cdots .$$

Free Resolutions

We introduce free resolutions somewhat intuitively to help us understand the question of how to understand an R-module M with generating sequence $f_1, \ldots, f_t$.

The generators do not adequately describe M because there are usually relations among them. In fact, the set of all relations

$$\mathrm{Syz}(f_1, \ldots, f_t) = \left\{ (a_1, \ldots, a_t) : \sum_{i=1}^{t} a_i f_i = 0 \right\}$$

is an R-submodule of R^t called the first syzygy module of $f_1, \ldots, f_t$. To understand $\mathrm{Syz}(f_1, \ldots, f_t)$ we need a generating sequence $g_1, \ldots, g_s$ and the set of relations $\mathrm{Syz}(g_1, \ldots, g_s)$, the so-called second syzygy module of $f_1, \ldots, f_t$. Continuing in this fashion we obtain an exact sequence called an augmented free resolution of M:

$$F = \cdots \longrightarrow F_2 \xrightarrow{\partial_2^F = \begin{bmatrix} \text{relations} \\ \text{on the} \\ \text{columns} \\ \text{of } \partial_1^F \end{bmatrix}} F_1 \xrightarrow{\partial_1^F = \begin{bmatrix} \text{relations} \\ \text{on the} \\ \text{generators} \\ \text{of } M \end{bmatrix}} F_0 \xrightarrow{\partial_0^F = \begin{bmatrix} \text{generators} \\ \text{of } M \end{bmatrix}} M \longrightarrow 0.$$

Among other things, a free resolution F encodes information about the module M it resolves. For example, when F is minimal the ranks of the free R-modules of F are the Betti numbers of M. In general, one uses chain complexes to compare an arbitrary module with nicer ones—namely, free, projective, or injective modules. Now, we give the precise definition in the special case $M = R/I$ that we care about.

Definition 1.2.9. Let $I \leq R$. A free resolution of R/I over R is a sequence of maps

$$X = \cdots \xrightarrow{\partial^{\beta_{i+1}}} R^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\partial_0} 0.$$

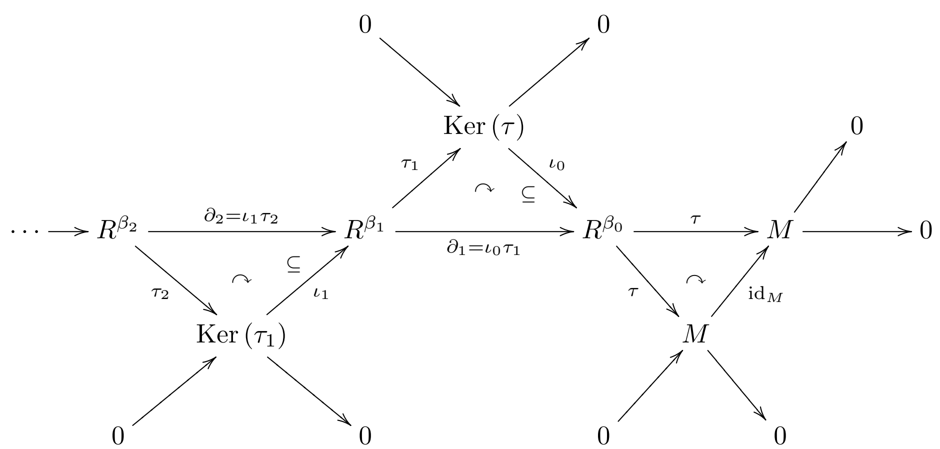
such that $H_0(X) \cong R/I$ and $H_i(X) = 0$ for all $i \neq 0$. An augmented free resolution of R/I over R is an exact sequence

$$X^+ = \cdots \xrightarrow{\partial^{\beta_{i+1}}} R^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\tau} R/I \longrightarrow 0.$$

where τ is the natural surjection and we say that X resolves R/I.

A standard method of constructing a resolution is represented diagrammatically as follows:

(1.2.9.1)



where ι_i and τ_i are the natural inclusion and surjection, respectively. In other words, constructing a free resolutions boils down to repeatedly solving systems of linear equations over R. A systematic approach to accomplish this when $R = K[x_1, \ldots, x_d]$ is to use Gröbner bases and Schreyer's Theorem (Theorem 6.2.1 in [4]).

A natural question to ask is, "do finite resolutions always exist?" In general, the answer is "no". For example, if $R = K[x]/\langle x^2 \rangle$ and $M = \langle x \rangle \leq R$, then

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} M \longrightarrow 0 \tag{1.2.9.2}$$

is an infinite free resolution of M over R, where x denotes the map multiplication by x. In fact, every free resolution of M over R must contain the resolution (1.2.9.2) (See Exercise 6.2.11, in [4]).

In the case that R is a polynomial ring, Hilbert proved the existence of finite resolutions.

Theorem 1.2.10 (Hilbert's Syzygy Theorem [4, Theorem 6.2.1])**.** *Let $R = K[x_1, \ldots, x_d]$. Then every finitely generated R-module has a finite free resolution of length at most d.*

We are interested in resolutions for which optimal choices were made at every step in constructing (1.2.9.1). These are minimal resolutions, which we define next.

Definition 1.2.11. Let $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle \leq R = K[x_1, \ldots, x_d]$ be the irrelevant maximal ideal. A graded complex A of a finitely generated free R-module is minimal if every entry of ∂_i^A lies in $\mathfrak{m}$ for all i.

Related to the notion of a minimal resolution is the idea of the length of an R-complex, defined next.

Definition 1.2.12. Define the supremum of an R-complex X as $\sup X = \sup\{i \mid X_i \neq 0\}$. When $\sup X < \infty$, it is not uncommon to call this the length of X.

Note 1.2.13. Let $R = K[x_1, \ldots, x_d]$. Every finitely generated R-module has a minimal free resolution. Minimal free resolutions are the shortest resolutions with smallest ranks, that is, if

$$F = \cdots \longrightarrow R^{\beta_s} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow 0$$

is a minimal resolution of M and

$$G = \cdots \longrightarrow R^{\beta'_t} \longrightarrow \cdots \longrightarrow R^{\beta'_0} \longrightarrow 0$$

is any other resolution of M, then $\beta_i \leq \beta'_i$ for all i, and $\sup F \leq \sup G \leq d$ where the last inequality is due to Hilbert's Basis Theorem. Furthermore, every free resolution contains a minimal one as a summand, that is, there exists an exact complex F' such that $G \cong F \oplus F'$.

The following theorem provides a necessary condition for a complex to be a resolution. Frequently, this is our first stop when determining if an R-complex is a resolution.

Theorem 1.2.14. *Let $I \leq R$ be a non-zero ideal and let*

$$X = (0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow 0)$$

be a resolution of R/I. Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

1.3 Combinatorial Constructions

In this section, we introduce simplicial complexes and reduced simplicial homology, and we show how every simplicial complex gives rise to a complex of vector spaces. See [2], [8], and [9] for a more in-depth treatment of these subjects.

Simplicial Complexes

Simplicial complexes, defined next, can be thought of as higher dimensional graphs.

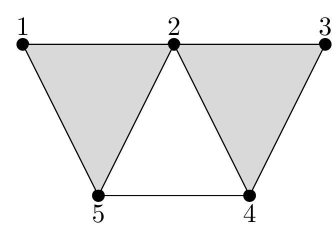
Definition 1.3.1 ([10, Definition 4.4.1]). Let $V = \{v_1, \ldots, v_d\}$ be a finite set. A simplicial complex on V is a nonempty collection Δ of subsets of V that is closed under subsets, that is, such that for all subsets $F, G \subseteq V$, if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$. An element of Δ is called a face of Δ. A faces of the form $\{v_i\}$ and $\{v_i, v_j\}$ are called a vertices and edges of Δ, respectively. A maximal element of Δ with respect to containment is called a facet of Δ. The (d-1)-simplex consists of all the subsets of V and is denoted Δ_{d-1}. We define the empty simplicial complex to be $\{\emptyset\}$.

Since V is finite, every face of Δ is contained in a facet of Δ. As simplicial complexes are closed under taking subsets, one can conveniently describe them by only listing their facets.

We will often label the vertices of our simplicial complexes with integers rather than v_i's.

Definition 1.3.2 ([10, Definition 4.4.10]). Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$. The dimension of a face $F \in \Delta$ is $|F|-1$. The dimension of Δ, denoted $\dim(\Delta)$, is the maximal dimension of a face of Δ. The simplicial complex Δ is *pure* if all facets of Δ have the same dimension.

Example 1.3.3. Set $R = A[X_1, \ldots, X_5]$, and let Δ be the simplicial complex with the following geometric realization, i.e., representation as a subset of Euclidean space.



The facets of Δ are the two shaded triangles $\{1, 2, 5\}$ and $\{2, 3, 4\}$, and the edge $\{4, 5\}$. Therefore, the dimension of Δ is $\dim(\Delta) = \max\{2, 2, 1\} = 2$ and Δ is not pure.

Notation 1.3.4 ([9, p.9])**.** Let Δ be a simplicial complex on $[n]$. For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ, and let $K^{F_i(\Delta)}$ be a vector space over K whose basis elements e_Λ correspond to i-faces of $\Lambda \in F_i(\Delta)$.

Every simplicial complex is naturally associated a chain complex by the following standard construction from algebraic topology. See Example 1.3.8 below for a sample computation.

Definition 1.3.5 ([9, Definition 1.16])**.** The (reduced) chain complex associated to Δ over K is the following complex $\widetilde{\mathcal{C}}(\Delta; K)$:

$$0 \longrightarrow K^{F_{n-1}(\Delta)} \xrightarrow{\partial^\Delta_{n-1}} \cdots \longrightarrow K^{F_i(\Delta)} \xrightarrow{\partial^\Delta_i} K^{F_{i-1}(\Delta)} \longrightarrow \cdots \xrightarrow{\partial^\Delta_0} K^{F_{-1}(\Delta)} \longrightarrow 0.$$

The boundary maps ∂^Δ_i are defined by

$$\partial^\Delta_i(e_\Lambda) = \sum_{\lambda \in \Lambda} \operatorname{sign}(\lambda, \Lambda) e_{\Lambda \setminus \{\lambda\}}$$

where $\operatorname{sign}(\lambda, \Lambda) = (-1)^{r-1}$ if λ is the r^{th} element of the set $\Lambda \subseteq [n]$, written in increasing order.

Remark 1.3.6. If $i < -1$ or $i > n-1$, then $K^{F_i(\Delta)} = 0$ and $\partial_i = 0$ by definition. A routine check shows $\partial^\Delta_i \circ \partial^\Delta_{i+1} = 0$ for all $i \in \mathbb{Z}$.

We simplify the notation of e_Λ by expressing Λ as a codeword. For example, if $\Lambda = \{1, 2, 4\} \in F_2(\Delta)$, then we will express e_Λ as $e_{124} \in K^{F_2(\Delta)}$.

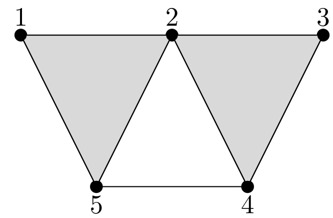
Definition 1.3.7 ([9, Definition 1.17])**.** For each integer i, the K-vector space

$$\widetilde{H}_i\left(\Delta; K\right) = H_i\left(\widetilde{\mathcal{C}}\left(\Delta; K\right)\right)$$

in homological degree i is the i^{th} reduced homology of Δ over K.

Just as in Definition 1.2.7, we call elements of $\ker(\partial_i)$ i-cycles and elements of $\operatorname{im}(\partial_{i+1})$ i-boundaries.

Example 1.3.8. For Δ as in Example 1.3.3



we have

$$
\begin{aligned}
F_2(\Delta) &= \{\{1,2,5\},\{2,3,4\}\} \\
F_1(\Delta) &= \{\{1,2\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{4,5\}\} \\
F_0(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\} \\
F_{-1}(\Delta) &= \{\emptyset\}.
\end{aligned}
$$

Ordering the basis as suggested above gives us $\widetilde{\mathcal{C}}(\Delta; K)$:

$$
0 \longrightarrow K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} K^7 \xrightarrow{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}} K^5 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}} K \longrightarrow 0.
$$

For example, in the first columns of ∂_1 and ∂_2 we have

$$
\begin{aligned}
e_{12} &\mapsto e_2 - e_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \end{bmatrix}^T \\
e_{125} &\mapsto e_{25} - e_{15} + e_{12} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T.
\end{aligned}
$$

In general one can compute $\widetilde{H}_i(\Delta; K)$ using basic linear algebra (i.e., Gaussian elimination).

For this example, since Δ is relatively small and uncomplicated, we can analyze reduced simplicial homology by examining the geometry of Δ. For instance, consider the boundary of the "missing triangle" $\{2, 4, 5\}$ in Δ:

$$\mathbf{x} = \partial^{\Delta_4}(e_{245}) = e_{45} - e_{25} + e_{24}$$

which corresponds to $\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}^T$ where Δ_4 is the 4-simplex. Since

$$\partial_2^{\Delta}(\mathbf{x}) = \partial_2^{\Delta}(e_{45} - e_{25} + e_{24}) = (e_5 - e_4) - (e_5 - e_2) + (e_4 - e_2) = 0$$

then $\mathbf{x} \in \ker(\partial_2^{\Delta})$. By inspection the vector $\mathbf{x}$ is not in $\operatorname{im}(\partial_3^{\Delta})$. Therefore, $\widetilde{H}_1(\Delta; K) \neq 0$.

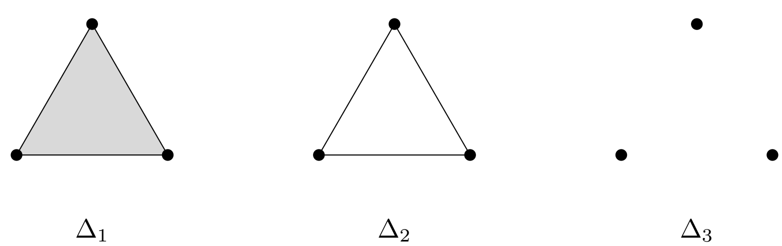
We frequently analyze the reduced simplicial homology using the following result.

Fact 1.3.9.

(a) Let Δ be a simplicial complex. If Δ is contractible (i.e., can be continuously deformed) to a point, then $\widetilde{H}_i(\Delta; K) = 0$ for all i [3, Example 5.3.3(a)].

(b) The vector space dimension of $\widetilde{H}_0(\Delta; K)$ is 1 less than the number of connected components of Δ. In particular, Δ is disconnected if and only if $\widetilde{H}_0(\Delta; K) \neq 0$ [9, p.9].

Example 1.3.10. Consider the following three simplicial complexes:



The first simplicial complex Δ_1 is contractible, the second Δ_2 is not contactible, and the third Δ_3 is disconnected.

In subsequent chapters, we see how these notions fit together, especially how reduced simplicial homology gives information about free resolutions.

Chapter 2

Scarf and Taylor Constructions

In this chapter we introduce the Taylor resolution and the Scarf complex. Each of these R-complexes has an associated labeled simplicial complex. Essentially, the labeled simplicial complex encodes all the information about the associated chain complex. This approach was first used by Bayer, Peeva and Sturmfels who proved that every labeled simplicial complex has an associated chain complex and gave a combinatorial/topological criterion for the chain complex to be a resolution [1]. One main advantage of doing this is that it allows one to easily describe a minimal free resolution.

2.1 Notational Conventions

We introduce some convenient notation for dealing with the bases and differentials of the Taylor resolution and Scarf complex.

Definition 2.1.1 ([10, Definition 2.1.3])**.** Set $R = K[x_1, \ldots, x_d]$. Let $f = \mathbf{x}^{\mathbf{m}}$ and $g = \mathbf{x}^{\mathbf{n}}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$. For $i = 1, \ldots, d$, set $p_i = \max\{m_i, n_i\}$. Define the least common multiple or lcm of f and g to be the monomial $\mathrm{lcm}(f, g) = \mathbf{x}^{\mathbf{p}}$. We define $\mathrm{lcm}(f) := \mathrm{lcm}(f, f) = f$ and $\mathrm{lcm}(\emptyset) := 1$. The generalized lcm is defined as one might expect; i.e., if $f_1 = \mathbf{x}^{\mathbf{n}_1}, \ldots, f_m = \mathbf{x}^{\mathbf{n}_m}$ where $\mathbf{n}_i = (n_{i,1}, \ldots, n_{i,d})$. Set $q_i = \max\{n_{1,i} \ldots, n_{m,i}\}$. Then $\mathrm{lcm}(f_1, \ldots, f_m) = \mathbf{x}^{\mathbf{q}}$.

Notation 2.1.2. Given $1 \leq m \leq n$ and $\Lambda = \{\lambda_1 < \cdots < \lambda_m\} \subseteq [n]$, let $f_\Lambda = \mathrm{lcm}(f_{\lambda_1}, \cdots, f_{\lambda_m})$ and let

$$\{\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_m\} = \{\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m\} = \Lambda \backslash \{\lambda_i\}.$$

2.2 The Taylor Resolution

This section is essentially due to the dissertation of Diana Taylor [14].

Definition 2.2.1. Let I be a monomial ideal in R with irredundant generating sequence $\mathbf{f} = f_1, \ldots, f_n$. The Taylor resolution of $\mathbf{f}$ is the following sequence of R-modules

$$T = T^R(\mathbf{f}) = (0 \longrightarrow R \xrightarrow{\partial_n^T} R^n \xrightarrow{\partial_{n-1}^T} \cdots \xrightarrow{\partial_{i+1}^T} R^{\binom{n}{i}} \xrightarrow{\partial_i^T} \cdots \xrightarrow{\partial_2^T} R^n \xrightarrow{\partial_1^T} R \longrightarrow 0)$$

with basis

$$\mathcal{B} = \{e_\Lambda \mid \Lambda = \{1 \leq \lambda_1 < \cdots < \lambda_i \leq n\}\} \subseteq R^{\binom{n}{i}} = T_i$$

where e_Λ is a standard basis vector in homological degree $|\Lambda|$ and $e_\emptyset := 1$. This basis is sometimes called the "exterior basis." The differentials are given by

$$\begin{aligned}
\partial_i^T(e_\Lambda) = \partial_i^T(e_{\lambda_1,\ldots,\lambda_i}) &= \sum_{p=1}^{i} (-1)^{p-1} \frac{\operatorname{lcm}(f_{\lambda_1}, \ldots, f_{\lambda_i})}{\operatorname{lcm}(f_{\lambda_1}, \ldots, \widehat{\lambda_p}, \ldots, f_{\lambda_i})} e_{\lambda_1,\ldots,\widehat{\lambda_p},\ldots,\lambda_i} \\
&= \sum_{p=1}^{i} (-1)^{p-1} \frac{f_\Lambda}{f_{\Lambda \setminus \{\lambda_p\}}} e_{\Lambda \setminus \{\lambda_p\}}.
\end{aligned}$$

Note 2.2.2. The map ∂_i^T is well-defined by Fact 1.2.2.

Next we give a concrete example of the Taylor resolution.

Example 2.2.3. Let $I \leq R = K[x, y, z]$ be the monomial ideal with irredundant generating sequence $\mathbf{f} = f_1, f_2, f_3 = xy, xz, yz$. Since I has 3 generators, then the Taylor resolution has the following shape and ordered basis:

$$\begin{array}{llllll}
T = T^R(xy, xz, yz) = & (0 \longrightarrow R & \xrightarrow{\partial_3^T} R^3 & \xrightarrow{\partial_2^T} R^3 & \xrightarrow{\partial_1^T} R & \longrightarrow 0). \\
 & \quad e_{123} & \quad e_{12} & \quad e_1 & \quad e_\emptyset & \\
 & & \quad e_{13} & \quad e_2 & & \\
 & & \quad e_{23} & \quad e_3 & &
\end{array}$$

Next we compute the differentials. Since

$$e_i \mapsto (-1)^{1-1} \frac{\operatorname{lcm}(f_i)}{\operatorname{lcm}()} e_\emptyset = 1 \cdot \frac{f_i}{1} \cdot 1 = f_i$$

then $\partial_1^T = \begin{bmatrix} xy & xz & yz \end{bmatrix}$. For ∂_2^X, we compute

$$e_{12} \mapsto \frac{f_{12}}{f_2}e_2 - \frac{f_{12}}{f_1}e_1 = \frac{xyz}{xz}e_2 - \frac{xyz}{xy}e_1 = ye_2 - ze_1 = \begin{bmatrix} -z \\ y \\ 0 \end{bmatrix}$$

$$e_{13} \mapsto \frac{f_{13}}{f_3}e_3 - \frac{f_{13}}{f_1}e_1 = \frac{xyz}{yz}e_3 - \frac{xyz}{xy}e_1 = xe_3 - ze_1 = \begin{bmatrix} -z \\ 0 \\ x \end{bmatrix}$$

$$e_{23} \mapsto \frac{f_{23}}{f_3}e_3 - \frac{f_{23}}{f_2}e_2 = \frac{xyz}{yz}e_2 - \frac{xyz}{xz}e_1 = xe_3 - ye_2 = \begin{bmatrix} 0 \\ -y \\ x \end{bmatrix}.$$

Therefore, we have

$$\partial_2^T = \begin{bmatrix} -z & -z & 0 \\ y & 0 & -y \\ 0 & x & x \end{bmatrix}.$$

For ∂_3^T, we have

$$e_{123} \mapsto \frac{f_{123}}{f_{23}}e_{23} - \frac{f_{123}}{f_{13}}e_{13} + \frac{f_{123}}{f_{12}}e_{12} = \frac{xyz}{xyz}e_{23} - \frac{xyz}{xyz}e_{13} + \frac{xyz}{xyz}e_{12} = e_{23} - e_{13} + e_{12}$$

which implies

$$\partial_3^T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Remark 2.2.4. Let $I = \langle f_1, \dots, f_n \rangle$ be a monomial ideal in R. For all our computations by hand, we order the basis of T lexicographically letting $1 <_{\text{LEX}} 2 <_{\text{LEX}} 3 <_{\text{LEX}} \cdots <_{\text{LEX}} n$. For example, if $n = 4$, then $123 <_{\text{LEX}} 124 <_{\text{LEX}} 234 <_{\text{LEX}}$. So, ∂_2^X can be represented by the matrix $\left[\; \partial_2^X(e_{123}) \; \middle| \; \partial_2^X(e_{124}) \; \middle| \; \partial_2^X(e_{234}) \; \right]$. In other words, this ordering fixes the rows and columns of the Scarf differential. This protocol rarely coincides with `Macaulay2`, since the default monomial order in `Macaulay2` is `GRevLex`.

As the name suggests, the Taylor resolution is a resolution. This was proved by Diana Taylor in her dissertation [14].

Theorem 2.2.5 ([14])**.** *Let $I \leq R$ be a monomial ideal with irredundant generating sequence $\mathbf{f}$. The Taylor resolution $T^R(\mathbf{f})$ is a resolution of $R/\langle \mathbf{f} \rangle$.*

2.3 The Scarf Complex

The Scarf complex is named after Herbert Scarf. The combinatorial construction defined in Definition 2.4.4 is due to him in the context of mathematical economics [13]. The algebraic construction is due to Bayer, Peeva and Sturmfels in [1]. To avoid any possible confusion, we will refer to the algebraic construction as the Scarf complex and the combinatorial one as the Scarf simplicial complex.

Definition 2.3.1. Given a monomial ideal I in R with irredundant generating sequence $\mathbf{f} = f_1, \ldots, f_n$, the Scarf complex of $\mathbf{f}$, denoted $X = X^R(\mathbf{f})$, is a subcomplex of $T^R(\mathbf{f})$ with a basis $\mathcal{B}$ parameterized by the subsets of $\{f_1, \ldots, f_n\}$ "each of whose least common multiple is unique":

$$\mathcal{B} = \{e_\Lambda \mid \text{ for all } \Gamma, \text{ if } f_\Lambda = f_\Gamma \text{ then } \Lambda = \Gamma\}$$

and differentials $\partial_i^X(e_\Lambda) = \partial_i^T(e_\Lambda)$.

Example 2.3.2. Let I be the monomial ideal in $R = K[x, y, z]$ with irredundant generating sequence $\mathbf{f} = xy, xz, yz$ as in Example 2.2.3, and let $X = X^R(\mathbf{f})$. Since $\mathbf{f}$ is an irredundant generating sequence then f_1, f_2, f_3 are unique. Next, note that $f_{12} = f_{13} = f_{23} = f_{123} = xyz$, so $\mathcal{B}_X = \{e_\emptyset, e_1, e_2, e_3\}$. Therefore, the Scarf complex is

$$X = (0 \longrightarrow R^3 \xrightarrow{\partial_1^X} R \longrightarrow 0) \ .$$

We note that X is not a resolution. Indeed, since $\partial_1^X(\mathbf{x}) = \begin{bmatrix} xy & xz & yz \end{bmatrix} \begin{bmatrix} z \\ -y \\ 0 \end{bmatrix} = 0$ then ∂_1^X is not injective. Therefore, $H_1(X) \neq 0$ by Fact 1.2.4. Alternatively, we can simply invoke Theorem 1.2.14.

Next we show that the Scarf differential is well-defined.

Proposition 2.3.3. *Let I be a monomial ideal with irredundant generating sequence* $\mathbf{f}$*, and let* $\mathcal{B}$ *be a basis for* $X^R(\mathbf{f})$*. If* $e_\Lambda \in \mathcal{B}$ *then* $e_{\Lambda\backslash\{\lambda\}} \in \mathcal{B}$*.*

Proof. Suppose f_Λ is unique where $\Lambda = \{\lambda_1 < \cdots < \lambda < \cdots < \lambda_m\}$. We need to show that $f_{\Lambda\backslash\{\lambda\}}$ is unique. By way of contradiction suppose $f_{\Lambda\backslash\{\lambda\}}$ is not unique, that is, suppose there exists $\Gamma = \{\gamma_1, \ldots, \gamma_n\} \neq \Lambda\backslash\{\lambda\}$ such that $f_\Gamma = f_{\Lambda\backslash\{\lambda\}}$. Since

$$
\begin{aligned}
f_\Lambda = f_{(\Lambda\backslash\{\lambda\})\cup\{\lambda\}} &= \operatorname{lcm}(f_{\lambda_1}, \ldots, \hat{f_\lambda}, \ldots, f_{\lambda_m}, f_\lambda) \\
&= \operatorname{lcm}(\operatorname{lcm}(f_{\lambda_1}, \ldots, \hat{f_\lambda}, \ldots, f_{\lambda_m}), f_\lambda) \\
&= \operatorname{lcm}(f_{\Lambda\backslash\{\lambda\}}, f_\lambda) \\
&= \operatorname{lcm}(f_\Gamma, f_\lambda) \\
&= \operatorname{lcm}(\operatorname{lcm}(f_{\gamma_1}, \ldots, f_{\gamma_n}), f_\lambda) \\
&= \operatorname{lcm}(f_{\gamma_1}, \ldots, f_{\gamma_n}, f_\lambda) \\
&= f_{\Gamma\cup\{\lambda\}}
\end{aligned}
$$

and f_Λ is unique, then $\Lambda = \Gamma \cup \{\lambda\}$. Throwing out $\{\lambda\}$ from both sides, we get

$$
\Lambda\backslash\{\lambda\} = (\Gamma \cup \{\lambda\})\backslash\{\lambda\} = \begin{cases} \Gamma & \text{if } \lambda \notin \Gamma \\ \Gamma\backslash\{\lambda\} & \text{if } \lambda \in \Gamma. \end{cases}
$$

If $\lambda \notin \Gamma$, then $\Lambda\backslash\{\lambda\} = \Gamma$ which is a contradiction. If $\lambda \in \Gamma$, then we can rewrite $\Lambda = \Gamma \cup \{\lambda\} = \Gamma$. This implies $f_\Gamma = f_\Lambda$, but $f_\Gamma = f_{\Lambda\backslash\{\lambda\}}$. So we have $f_{\Lambda\backslash\{\lambda\}} = f_\Lambda$ with $\Lambda\backslash\{\lambda\} \neq \Lambda$. By definition this means f_Λ is not unique which is another contradiction. Therefore, $f_{\Lambda\backslash\{\lambda\}}$ is unique. □

The differentials of the Scarf complex can be obtained from the Taylor resolution by deleting rows and columns corresponding to the Λ such that f_Λ is not unique as we show next.

Example 2.3.4. Let $I \leq R = K[x, y, z, w]$ be the monomial ideal with irredundant generating sequence $\mathbf{f} = xy, zw, x^2z^2, y^2w^2$. Since $f_{34} = f_{1234} = x^2y^2z^2w^2$, then every subset of $\{1, 2, 3, 4\}$ containing $\{3, 4\}$ has the same lcm; namely, f_{34}, f_{134}, f_{234}, and f_{1234}. One checks directly that these are the only non-unique f_Λ. Therefore, the Scarf complex has the shape

$$X = X^R(\mathbf{f}) = \ (0 \longrightarrow R^{\overset{0}{\not 1}} \xrightarrow{\partial_3} R^{\overset{2}{\not 4}} \xrightarrow{\partial_3} R^{\overset{5}{\not 6}} \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R \longrightarrow 0)$$

$$\begin{array}{ccccc} \cancel{e_{1234}} & e_{123} & e_{12} & e_1 & e_\emptyset \\ & e_{124} & e_{13} & e_2 & \\ & \cancel{e_{134}} & e_{14} & e_3 & \\ & \cancel{e_{234}} & e_{23} & e_4 & \\ & & e_{24} & & \\ & & \cancel{e_{34}} & & \end{array}$$

and differentials

$$\partial_1^X = \begin{bmatrix} xy & zw & x^2z^2 & y^2w^2 \end{bmatrix}$$

$$\partial_2^X = \begin{bmatrix} -zw & -xz^2 & -yw^2 & 0 & 0 & \cancel{0} \\ xy & 0 & 0 & -x^2 & -y^2w & \cancel{0} \\ 0 & y & 0 & w & 0 & \cancel{-y^2w^2} \\ 0 & 0 & x & 0 & z & \cancel{x^2z^2} \end{bmatrix} = \begin{bmatrix} -zw & -xz^2 & -yw^2 & 0 & 0 \\ xy & 0 & 0 & -x^2 & -y^2w \\ 0 & y & 0 & w & 0 \\ 0 & 0 & x & 0 & z \end{bmatrix}$$

$$\partial_3^X = \begin{bmatrix} xz & yw & \cancel{0} & \cancel{0} \\ -w & 0 & \cancel{yw^2} & \cancel{0} \\ 0 & -z & \cancel{-xz^2} & \cancel{0} \\ y & 0 & \cancel{0} & \cancel{y^2w} \\ 0 & x & \cancel{0} & \cancel{-x^2z} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} \end{bmatrix} = \begin{bmatrix} xz & yw \\ -w & 0 \\ 0 & -z \\ y & 0 \\ 0 & x \end{bmatrix}$$

$$\partial_4^X = \begin{bmatrix} \cancel{-yw} \\ \cancel{xz} \\ \cancel{-1} \\ \cancel{-1} \end{bmatrix} = 0.$$

For example, removing e_{34} from the basis corresponds to deleting column 6 in ∂_2^X and row 6 in ∂_3^X. See Examples 2.3.7 and 3.1.7 below for further computations.

A distinguishing property of the Scarf complex is that it is always minimal.

Proposition 2.3.5. *Let* $\mathbf{f}$ *be an irredundant monomial generating sequence. The Scarf complex* $X^R(\mathbf{f})$ *is minimal.*

Proof. Let $(-1)^{p-1}\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_p\}}}$ be an arbitrary entry in ∂_i^X. Since $\Lambda \neq \Lambda\setminus\{\lambda_p\}$, then by definition $f_\Lambda \neq f_{\Lambda\setminus\{\lambda_p\}}$. Therefore, $(-1)^{p-1}\frac{f_\Lambda}{f_{\Lambda\setminus\{\lambda_p\}}} \neq \pm 1$ which implies that it lies in the irrelevant maximal ideal. □

Note 2.3.6. The Taylor resolution and Scarf complex have advantages and disadvantages. One major advantage of the Taylor resolution is that it is, in fact, a resolution. One disadvantage of the Taylor resolution is that often it is highly non-minimal. One non-minimal Taylor resolution is in Example 2.2.3. For a family of non-minimal Taylor resolutions, let $I = \langle x_1, \ldots, x_d \rangle$ be the irrelevant maximal ideal in $R = K[x_1, \ldots, x_d]]$. It is straightforward to show that $\{x_{\lambda_1} \cdots x_{\lambda_n} \mid \lambda_i \in \{1, \ldots, m\}\}$ is an irredundant generating set for I^n which implies that the Taylor resolution $T^R(I^n)$ has length nd. We know that a minimal resolution of R/I^n has length at most d by Hilbert's Syzygy Theorem (Theorem 1.2.10). In fact, the minimal resolution of R/I^n has length exactly d, e.g., by Theorem 4.1.9.

By Proposition 2.3.5, the Scarf complex avoids the problem of not being minimal, and sometimes it is even a minimal resolution, as demonstrated in the following example. However, Example 2.3.2 shows that the Scarf complex is not a resolution in general.

Example 2.3.7. Let $I \leq K[x, y]$ be monomial ideal with irredundant generating sequence $\mathbf{f} = x^2, xy, y^2$. The Taylor resolution $T^R(\mathbf{f})$ and the Scarf complex $X^R(\mathbf{f})$ are as follows:

$$T = T^R(\mathbf{f}) = (0 \longrightarrow R \xrightarrow{\begin{bmatrix} y \\ -1 \\ x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} R \longrightarrow 0)$$

$$X = X^R(\mathbf{f}) = \qquad (0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} R \longrightarrow 0).$$

We will show that X is a resolution of $R/\langle \mathbf{f} \rangle$.

$\underline{H_1(X)}$: We need to show $\ker(\partial_1^X) \subseteq \operatorname{im}(\partial_2^X)$. Notice that since the generating sequence $\mathbf{f}$ is irredundant (as we always assume), then the Taylor resolution and the Scarf complex always begin the same way, i.e.,

$$(R^n \xrightarrow{\partial_1^T} R \longrightarrow 0) = (R^n \xrightarrow{\partial_1^X} R \longrightarrow 0).$$

In particular, we have $\partial_1^T = \partial_1^X = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$ and therefore, $\ker(\partial_1^T) = \ker(\partial_1^X)$. Since the Taylor resolution is always exact in homological degrees $i \neq 0$, then $\ker(\partial_1^T) = \operatorname{im}(\partial_2^T)$. Putting it all together, $\ker(\partial_1^X) \subseteq \operatorname{im}(\partial_2^X)$ if and only if $\operatorname{im}(\partial_2^T) = \ker(\partial_1^T) = \ker(\partial_1^X) \subseteq \operatorname{im}(\partial_2^X)$. So, it suffices to show $\operatorname{im}(\partial_2^T) \subseteq \operatorname{im}(\partial_2^X)$. For this it suffices to show that $\partial_2^T(e_{ij}) \in \operatorname{im}(\partial_2^X)$ for all $1 \leq i < j \leq 3$. By definition $\partial_2^T(e_{12}) = \partial_2^X(e_{12})$ and $\partial_2^T(e_{23}) = \partial_2^X(e_{23})$. Therefore, we only need to show that $\partial_2^T(e_{13}) = \begin{bmatrix} -y^2 & 0 & x^2 \end{bmatrix}^T$ is in $\operatorname{im}(\partial_2^X)$. To this end, one checks readily that $\mathbf{a} = \begin{bmatrix} y & x \end{bmatrix}^T$ satisfies $\partial_2^X(\mathbf{a}) = \partial_2^T(e_{13}) \in \operatorname{im}(\partial_2^X)$.

$\underline{H_2(X)}$: By Fact 1.2.4, $\ker(\partial_2^X) \subseteq \operatorname{im}(\partial_3^X)$ if and only if ∂_3^X is injective. If

$$\partial_2^X(\mathbf{a}) = \begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 y \\ a_1 x - a_2 y \\ a_2 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

then the first row implies $a_1 = 0$, and then third row implies $a_2 = 0$.

Therefore, $X^R(\mathbf{f})$ is a (minimal) resolution of $R/\langle \mathbf{f} \rangle$.

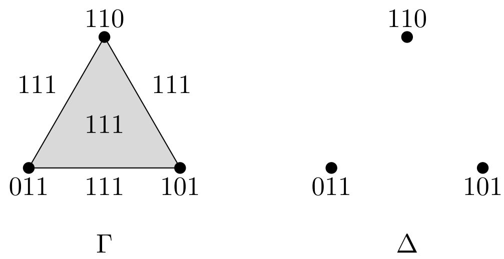
2.4 Labeled Simplicial Complexes

In this section, we introduce labeled simplicial complexes and show how they can be associated to R-complexes. In particular, we will focus on the combinatorial constructions associated to the Taylor resolution and Scarf simplicial complex. Last, we demonstrate how one gleans information about the algebraic construction from the combinatorial one.

Definition 2.4.1 ([9, Definition 4.2])**.** Suppose Δ is a labeled simplicial complex, by which we mean that its n vertices have labels that are vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{N}^d$. The label on an arbitrary face F of Δ is the exponent $\mathbf{a}_F$ on the least common multiple $\operatorname{lcm}(\mathbf{x}^{\mathbf{a}_i} \mid i \in F)$ of the monomial labels $\mathbf{x}^{\mathbf{a}_i}$ on vertices in F.

We will label exponent vectors as codewords. For example, the label for $x^2y^0z^1 \in K[x, y, z]$ is 201.

Example 2.4.2. Here are two labeled simplicial complexes.



Next we define two labeled simplicial complexes which are associated to the Taylor resolution and Scarf complex, respectively.

Definition 2.4.3. The Taylor simplicial complex of a monomial ideal I with irredundant generating sequence $\mathbf{f} = f_1, \ldots, f_n$ is the full $(n-1)$-dimensional simplex $\Gamma(\mathbf{f}) = \Delta_n$ whose faces are labeled by the exponent vectors of the f_Λ where $\Lambda \in [n]$.

Definition 2.4.4 ([10, Definition 5.1.2])**.** Let $I \leq R$ be a monomial ideal with an irredundant generating sequence $\mathbf{f} = f_1, \ldots, f_n$. The Scarf simplicial complex $\Delta(\mathbf{f})$ is the collection of all subsets of $[n]$ corresponding to subsets of $\{\mathbf{f}\}$ whose least common multiple is unique:

$$\Delta(\mathbf{f}) = \{\Lambda \subseteq \{1, \ldots, n\} \mid \text{for all } \Gamma, \text{ if } f_\Lambda = f_\Gamma \text{ then } \Lambda = \Gamma\}.$$

In other words, the faces of $\Delta(\mathbf{f})$ correspond to the basis vectors of $X^R(\mathbf{f})$. Like the Taylor simplicial complex, the faces of the Scarf simplicial complex are labeled by the exponent vectors of the f_Λ.

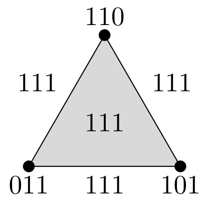
Note 2.4.5. As with the Scarf complex, the basis vectors of the Taylor resolution are in bijection with the faces of the Taylor simplicial complex.

Example 2.4.6.

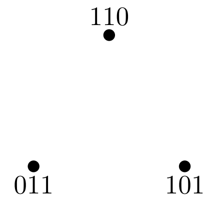
(a) Let $\mathbf{f} = xy, yz, xz$ be an irredundant monomial generating sequence as in Example 2.3.2. Geometric realizations of $\Gamma(\mathbf{f})$ and $\Delta(\mathbf{f})$ are given below (Compare to Example 2.4.2):

(b) Let $\mathbf{g} = x^2, xy, y^2$ be an irredundant monomial generating sequence as in Example 2.3.7. Geometric realizations of $\Gamma(\mathbf{g})$ and $\Delta(\mathbf{g})$ are given below:

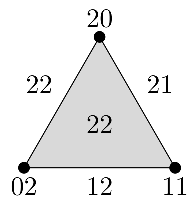
At a first glance, it is not obvious that the Scarf simplicial complex is a simplicial complex. However, proving this is equivalent to proving that the differential of the Scarf complex is well-defined.



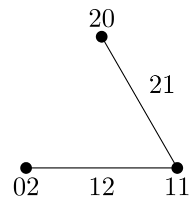
$\Gamma(\mathbf{f})$



$\Delta(\mathbf{f})$



$\Gamma(\mathbf{g})$



$\Delta(\mathbf{g})$

Proposition 2.4.7. *The Scarf simplicial complex is closed under subsets, i.e., it is a simplicial complex.*

Proof. Apply Proposition 2.3.3. □

As alluded to earlier, it turns out that every labeled simplicial complex gives rise to an R-complex.

Definition 2.4.8 ([8, Construction 4.1])**.** Let I be a monomial ideal in R with irredundant generating sequence $\mathbf{m} = m_1, \ldots, m_n$. Let Δ be a simplicial complex on $\mathbf{m}$ (recall that this means that the vertices of Δ are the monomials in $\mathbf{m}$). We assign a multidegree to each face $G \in \Delta$ by the rule $\mathbf{m}_G = \operatorname{lcm}(m_i = \mathbf{x}^{\mathbf{n}_i} \in \mathbf{m} \mid \mathbf{n}_i \in G\}$ Let H_s be the free module with basis $\{e_G : |G| = s\}$, and let the differential $\partial_{s-1} : H_s \to H_{s-1}$ be given by

$$\partial_i(e_G) = \sum_{g \in G} \operatorname{sign}(g, G) \frac{\mathbf{m}_G}{\mathbf{m}_{G \setminus \{g\}}} e_{G \setminus \{g\}}$$

where $\operatorname{sign}(g, G) = (-1)^{r-1}$ if g is the r^{th} element of the set $G \subseteq [n]$, written in increasing order. The <u>R-complex associated to Δ over K</u> is the R-complex

$$F = F_\Delta = (0 \longrightarrow F_r \xrightarrow{\partial_r^F} \cdots \longrightarrow F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\partial_0^F} R/I \longrightarrow 0).$$

When this complex is exact, we call it a simplicial resolution or the (simplicial) resolution supported on Δ.

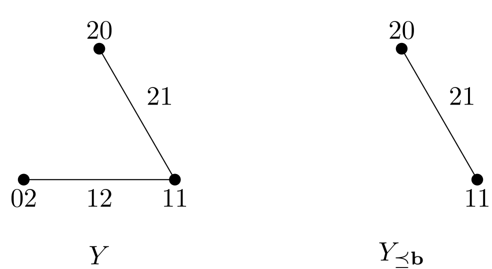
Note 2.4.9 ([8, p.6])**.** Definition 2.4.8 differs from Definition 1.3.5 because the former is a complex of R-modules and the latter is a complex of vector spaces. The boundary maps are identical except for the monomial coefficients, which are necessary to make the R-complex homogeneous.

Not surprisingly, the Taylor resolution and the Scarf complex are the R-complexes associated to the Taylor simplicial complex and Scarf simplicial complex, respectively. In other words, if $\Gamma = \Gamma(\mathbf{f})$ is the Taylor simplicial complex and $\Delta = \Delta(\mathbf{f})$ is the Scarf simplicial complex, then $F_\Gamma = T^R(\mathbf{f})$ and $F_\Delta = X^R(\mathbf{f})$. Furthermore, the Taylor resolution is supported on the Taylor simplicial complex.

The length of these R-complexes are related to the dimension of their associated simplicial complexes. In fact, $\sup X^R(\mathbf{f})$ and is finite and is precisely the dimension of the Scarf simplicial complex $\Delta(\mathbf{f})$ and the same is true for the Taylor resolution and Taylor simplicial complex.

Notation 2.4.10 ([9, p.64])**.** Let Y be a labeled simplicial complex. Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$, we write $\mathbf{a} \preceq \mathbf{b}$ and say that $\mathbf{a}$ precedes $\mathbf{b}$, if $\mathbf{b} - \mathbf{a} \in \mathbb{N}^n$. For each $\mathbf{b} \in \mathbb{N}^n$ let $Y_{\preceq \mathbf{b}}$ denote the subcomplex of Y consisting of all faces with labels coordinatewise at most $\mathbf{b}$. Similiarly, let $Y_{\prec \mathbf{b}}$ denote the subcomplex of Y consisting of all faces with labels $\prec \mathbf{b}$, where $\mathbf{b}' \prec \mathbf{b}$ if $\mathbf{b}' \preceq \mathbf{b}$ and $\mathbf{b}' \neq \mathbf{b}$.

Example 2.4.11. Let $I \leq K[x, y]$ be the monomial ideal with irredundant generating sequence $\mathbf{g} = x^2, xy, y^2$ as in Example 2.4.6(b). Let $Y = \Delta(\mathbf{g})$ and $\mathbf{b} = (3, 1)$. Then $(2, 0) \preceq (3, 1)$ because $2 \leq 3$ and $0 \leq 1$ but, $(1, 2) \npreceq (3, 1)$ because in the second coordinate $2 \nleq 1$. Therefore the vertex with label 20 is in $Y_{\preceq \mathbf{b}}$ and the edge with label 12 is not in $Y_{\preceq \mathbf{b}}$. Continue in this fashion, to obtain the following geometric realization of $Y_{\preceq \mathbf{b}}$.



The next result is a topological condition describing whether a simplicial complex Δ supports a resolution.

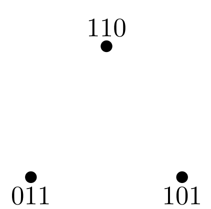
Proposition 2.4.12 ([2, Proposition 1.1])**.** *Let Δ be a simplicial complex supported on M, and set $I = \langle M \rangle$. Then Δ supports a resolution of R/I, i.e., F_Δ is a resolution, if and only if for all $\mathbf{b} \in \mathbb{N}^d$, the simplicial complex $\Delta_{\preceq \mathbf{b}}$ has no homology over K.*

Coupling Proposition 2.4.12 and Fact 1.3.9(a), one can prove that the Taylor resolution is a resolution with relative ease. Originally proved by Diana Taylor, the proof sketched here is from Mermin in [8].

Alternate proof of Theorem 2.2.5 (sketch). We apply Proposition 2.4.12 and Fact 1.3.9(a). Let I be a monomial ideal with irredundant generating sequence $\mathbf{f} = f_1, \ldots, f_n = \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$. Let $\Gamma = \Gamma(\mathbf{f})$ be the Taylor simplicial complex and let $\mathbf{b} \in \mathbb{N}^d$ be arbitrary. Then $\Gamma_{\preceq \mathbf{b}}$ is the simplex with vertices $\{f_i : \mathbf{a}_i \preceq \mathbf{b}\}$, which is either the empty simplicial complex or it is contractible. $\square$

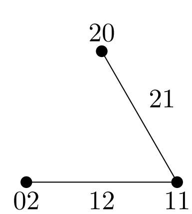
Example 2.4.13.

(a) Let $\Delta(\mathbf{f})$ be the Scarf simplicial complex



as in Example 2.4.6(a). Since $\Delta(\mathbf{f})$ is disconnected then $\widetilde{C}\left(\Delta\left(\mathbf{f}\right); K\right)$ is not exact by Fact 1.3.9(b) and therefore, $X^R(\mathbf{f})$ is not a resolution by Proposition 2.4.12.

(b) Let $\Delta = \Delta(\mathbf{g})$ be the Scarf simplicial complex



as in Example 2.4.6(b). One can verify that $\Delta_{\preceq \mathbf{b}}$ is contractible or it is the empty simplicial complex for all $\mathbf{b} \in \mathbb{N}^2$. So, $X^R(\mathbf{g})$ is a resolution.

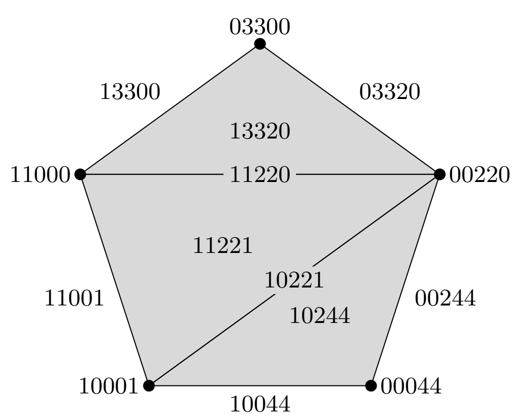
We will show that computationally, one only needs to consider a finite number of subcomplexes of Δ in order to invoke Proposition 2.4.12.

Proposition 2.4.14. *Suppose Δ is a simplicial complex with s faces. For $1 \leq i \leq s$, let the i^{th} face have label $\mathbf{f}_i = (f_{i,1}, \ldots, f_{i,d})$. For $1 \leq j \leq d$, set $m_j = \max\{f_{i,j} \mid 1 \leq i \leq s\}$ and $\mathbf{m} = (m_1, \ldots, m_d)$. Notice that $\{\mathbf{b} \mid \mathbf{b} \preceq \mathbf{m}\}$ is a finite set. Then for every exponent vector $\mathbf{a} \in \mathbb{N}^d$ there exists $\mathbf{b} \in \mathbb{N}^d$ such that $\mathbf{b} \preceq \mathbf{m}$ and $\Delta_{\preceq \mathbf{a}} = \Delta_{\preceq \mathbf{b}}$.*

Proof. Let $\mathbf{a} \in \mathbb{N}^d$ be given. For $1 \leq j \leq d$, set $b_j = \min\{a_j, m_j\}$. Notice that $\mathbf{b} \preceq \mathbf{m}$. We claim that $\Delta_{\preceq \mathbf{a}} = \Delta_{\preceq \mathbf{b}}$. It suffices to show that $F = \{f_{i,1}, \ldots, f_{i,d}\}$ is a face of the left-hand side if and only if it is a face of the right-hand side.
$(\implies)$: Suppose for this implication that $F \in \Delta_{\preceq \mathbf{a}}$, that is, $f_{i,j} \leq a_j$ for all $j \in [d]$. By definition of $\mathbf{m}$, we have $f_{i,j} \leq m_j$, hence $f_{i,j} \leq b_j$, so $F \in \Delta_{\preceq \mathbf{b}}$, as desired.
$(\impliedby)$: Now suppose that $F \in \Delta_{\preceq \mathbf{b}}$, that is, $f_{i,j} \leq b_j$ for all $j \in [d]$. Let $j \in [d]$ be arbitrary. If $b_j = a_j$, then $f_{i,j} \leq b_j = a_j$. Now suppose $b_j = m_j$. Since $b_j = \min\{a_j, m_j\}$, then $m_j \leq a_j$. Thus, we have $f_{i,j} \leq m_j \leq a_j$, so $F \in \Delta_{\preceq \mathbf{a}}$, as desired. □

Next we exhibit an example where we prove $X^R(\mathbf{f})$ is a resolution by applying Propositions 2.4.12 and 2.4.14 (this verifies the conclusion of Theorem 4.3.6 for a specific example).

Example 2.4.15. Let $I \leq R = K[x_1, \ldots, x_5]$ be the monomial ideal with irredundant generating sequence $\mathbf{f} = x_1x_2, x_2^3x_3^3, x_3^2x_4^2, x_4^4x_5^4, x_1x_5$ and let $Y = \Delta(\mathbf{f})$ be the Scarf simplicial complex.



Y

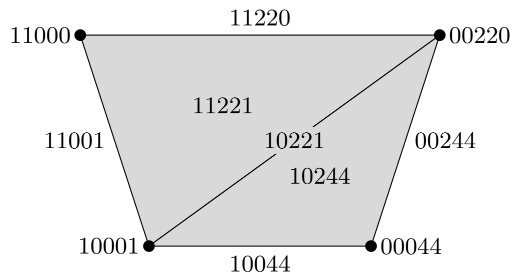
(We computed Y in `Macaulay2` via the commands below. `Macaulay2` is an open-source computer algebra system designed to aid research in algebraic geometry and commutative algebra [7]. See Appendix A for more details about these commands.)

```
R = QQ[x_1..x_5];
L = {1,3,2,4,1};
I = wCycle (L,R);
netList exps scarfBasisLabeled I
```

Notice that the facets of Y are shaded triangles with labels 13320, 11221, and 10244. By Proposition 2.4.14, we only need to consider $\mathbf{b} \preceq (1,3,3,4,4)$. There are 2 choices for the first digit (0 or 1), 4 choices for the second digit (0,1,2,3), etc. So, there are $2 \cdot 4 \cdot 4 \cdot 5 \cdot 5 = 800$ distinct $\mathbf{b}$'s to consider. Using excel, it is not too difficult to compute the subcomplex $Y_{\preceq \mathbf{b}}$ for each of the $\mathbf{b}$'s. It turns out that every subcomplex has one of the following geometric realizations:

- the empty simplicial complex.
- a single vertex.
- an edge.
- 3 vertices connected by 2 edges, i.e., a 2-path.
- a single shaded triangle.
- 2 shaded triangles that share an edge.
- The entire simplicial complex Y.

For example, for we will compute the subcomplex $Y_{\preceq \mathbf{b}}$ for $\mathbf{b} = (1,2,2,4,4)$. Since $(0,3,3,0,0) \not\preceq (1,2,2,4,4)$ then none of the faces containing the vertex with label 03300 are in $Y_{\preceq \mathbf{b}}$. On the other hand, since $Y_{\preceq \mathbf{b}}$ is a simplicial complex and we have $(1,1,2,2,1) \preceq (1,2,2,4,4)$ and $(1,0,2,4,4) \preceq (1,2,2,4,4)$, then all faces contained in these shaded triangles are in $Y_{\preceq \mathbf{b}}$. Hence, $Y_{\preceq \mathbf{b}}$ has geometric realization



$Y_{\preceq 12245}$

Therefore, for all $\mathbf{b} \in \mathbb{N}^5$ we have $Y_{\preceq \mathbf{b}}$ is the empty simplicial complex or it is contractible a point. Hence, $X^R(\mathbf{f})$ is a resolution by Proposition 2.4.12.

The last class of monomial ideals that we will consider in this chapter are generic monomial ideals. These are the monomial ideals for which the Scarf complex is always a resolution.

Definition 2.4.16. A monomial ideal I is generic if no variable x_i appears with the same nonzero exponent in two distinct minimal generators of I.

Example 2.4.17. The monomial ideal $I = \langle xy, xz, yz \rangle$ as in Example 2.4.6(a) is not generic because the indeterminant x appears with the same nonzero exponent in two distinct generators; namely xy and xz. On the other hand, $J = \left\langle x^2, xy, y^2 \right\rangle$ as in Example 2.4.6(b) is generic.

Theorem 2.4.18 ([1, Theorem 3.2])**.** *Let $I \leq R$ be a generic monomial ideal with irredundant generating sequence $\mathbf{f}$. Then the Scarf complex $X^R(\mathbf{f})$ is the minimal free resolution of $R/\langle \mathbf{f} \rangle$ over R.*

Remark 2.4.19. Applying Theorem 2.4.18 to $I = \langle \mathbf{f} \rangle = \left\langle x^2, xy, y^2 \right\rangle \leq R$ as in Example 2.3.7 gives an alternate proof that $X^R(\mathbf{f})$ is a resolution.

For the rest of this thesis we will investigate the behavior of the Scarf complex $X^R(\mathbf{f})$ when $I = \langle \mathbf{f} \rangle$ is not a generic monomial ideal.

Chapter 3

Edge Ideals

In this chapter we will explore how the Scarf complex behaves on finite simple graphs, that is, graphs with finite vertex sets with no multi-edges or loops.

3.1 Graphs and Edge Ideals

First we give a brief review the basics of graph theory, referring the reader to [5] for a more in-depth treatment.

Definition 3.1.1 ([10, Definition 4.2.1])**.** Let $V = \{v_1, \dots, v_d\}$ be a finite set. A graph with vertex set V is an ordered pair $G = (V, E)$, where E is a set of unordered pairs v_iv_j with $v_i \neq v_j$. (The pairs are unordered, so $v_iv_j = v_jv_i$.) An element $v_i \in V$ is a vertex of G. The set E is the edge set of G. Given an edge $e = v_iv_j$, the endpoints of e are the vertices v_i and v_j. Two vertices, $v_i, v_j \in V$ are adjacent if there is an edge $e \in E$ with endpoints v_i and v_j, that is, if $v_iv_j \in E$; in this case, we also say that the edge v_iv_j is incident to its endpoints v_i and v_j.

Definition 3.1.2 ([10, p.121])**.** For each integer $n \geq 1$, an n-path is the graph P_n with vertex set $\{v_1, v_2, \dots, v_{n+1}\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_nv_{n+1}\}$ consisting of n edges.

Definition 3.1.3 ([10, p.121])**.** For each integer $d \geq 3$, a d-cycle is the graph C_d with vertex set $\{v_1, v_2, \dots, v_d\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{d-1}v_d, v_dv_1\}$ consisting of d edges.

Definition 3.1.4 ([10, p.122])**.** For each $d \geq 2$, the complete graph on d vertices is the graph K_d with vertex set $\{v_1, \dots, v_d\}$ and edge set $\{v_iv_j \mid 1 \leq i < j \leq d\}$.

The following notion is our first connection between graph theory and algebra. It is due to Villarreal [15].

Definition 3.1.5 ([10, Definition A.1.5])**.** Let G be a graph with vertex set $V = \{v_1, \dots, v_d\}$. The edge ideal of G is the ideal $I(G) \leq R$ that is "generated by the edges of G":

$$I(G) = \langle x_i x_j \mid v_i v_j \text{ is an edge in } G \rangle.$$

Notation 3.1.6. In the special case of the edge ideal of a graph G, we simplify the notation of the Scarf complex and Scarf simplicial complex of $I(G)$ by writing $X^R(G)$ and $\Delta(G)$, respectively.

Example 3.1.7. Next, we consider some of our algebraic constructions associated to relatively small paths, cycles, and complete graphs. We order the rows and columns of the differentials according to `Macaulay2`'s protocol.

(a) The edge ideal associated to P_5 is $I(P_5) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6 \rangle$ in $K[x_1, \dots, x_6]$. Consider the Scarf complex

$$X = X^R(P_5) = (0 \longrightarrow R^2 \xrightarrow{\partial_3^X} R^7 \xrightarrow{\partial_2^X} R^5 \xrightarrow{\partial_1^X} R^1 \xrightarrow{\partial_0^X} 0)$$

with differentials

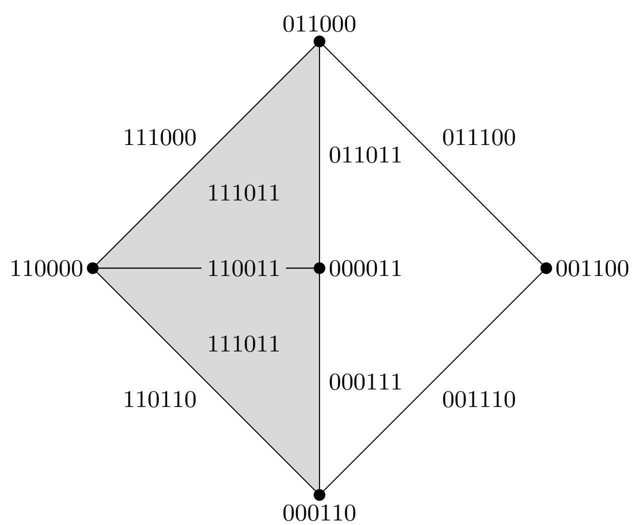
$$\partial_1^X = \begin{bmatrix} x_1x_2 & x_2x_3 & x_3x_4 & x_4x_5 & x_5x_6 \end{bmatrix}$$

$$\partial_2^X = \begin{bmatrix} -x_3 & 0 & -x_4x_5 & 0 & -x_5x_6 & 0 & 0 \\ x_1 & -x_4 & 0 & 0 & 0 & -x_5x_6 & 0 \\ 0 & x_2 & 0 & -x_5 & 0 & 0 & 0 \\ 0 & 0 & x_1x_2 & x_3 & 0 & 0 & -x_6 \\ 0 & 0 & 0 & 0 & x_1x_2 & x_2x_3 & x_4 \end{bmatrix}$$

$$\partial_3^X = \begin{bmatrix} x_5x_6 & 0 \\ 0 & 0 \\ 0 & x_6 \\ 0 & 0 \\ -x_3 & -x_4 \\ x_1 & 0 \\ 0 & x_1x_2 \end{bmatrix}.$$

The geometric realizations of P_5 and $\Delta(P_5)$ are as follows:

P_5



$\Delta(P_5)$

(b) The edge ideal associated to C_5 is $I(C_5) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1\rangle$ in $K[x_1, \ldots, x_5]$. Consider the Scarf complex

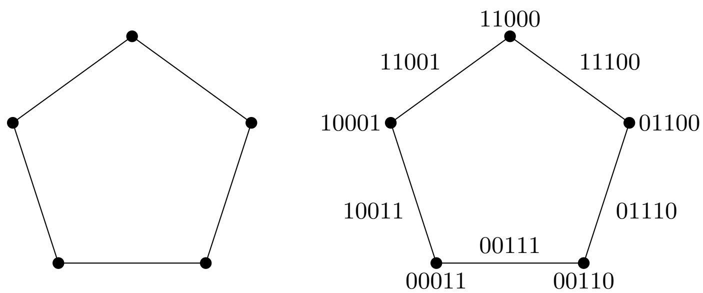
$$X = X^R(C_5) = \quad (0 \longrightarrow R^5 \xrightarrow{\partial_2^X} R^5 \xrightarrow{\partial_1^X} R^1 \xrightarrow{\partial_0^X} 0)$$

with differentials

$$\partial_1^X = \begin{bmatrix} x_1x_2 & x_2x_3 & x_3x_4 & x_1x_5 & x_4x_5 \end{bmatrix}$$

$$\partial_2^X = \begin{bmatrix} -x_3 & 0 & -x_5 & 0 & 0 \\ x_1 & -x_4 & 0 & 0 & 0 \\ 0 & x_2 & 0 & -x_5 & 0 \\ 0 & 0 & x_2 & 0 & -x_4 \\ 0 & 0 & 0 & x_3 & x_1 \end{bmatrix}.$$

The geometric realizations of C_5 and $\Delta(C_5)$ are actually the same and are sketched as follows:



C_5 $\Delta(C_5)$

(c) The edge ideal associated to K_4 is $I(K_4) = \langle x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4\rangle$ in $K[x_1, \ldots, x_4]$.

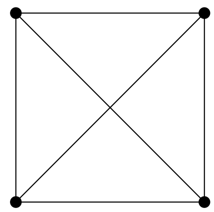
Consider the Scarf complex

$$X = X^R(K_4) = \ (0 \longrightarrow R^6 \xrightarrow{\partial_1^X} R^1 \xrightarrow{\partial_0^X} 0)$$

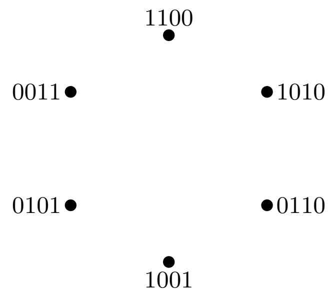
with differential

$$\partial_1^X = \begin{bmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x_1x_4 & x_2x_4 & x_3x_4 \end{bmatrix}.$$

The geometric realization of K_4 and $\Delta(K_4)$ are as follows:



K_4



$\Delta(K_4)$

In each of these examples we can see that $\Delta(G)$ is not contractible; moreover, $\Delta(K_4)$ is disconnected.

3.2 Complete Graphs

The Scarf complexes associated to complete graphs are easily understood.

Proposition 3.2.1. *If K_n is a complete graph on $n \geq 3$ vertices, then the Scarf complex $X^R(K_n)$ is of the form*

$$X = X^R(K_n) = \ (0 \longrightarrow R^{\binom{n}{2}} \xrightarrow{\partial_1^X} R \xrightarrow{\partial_0^X} 0).$$

Proof. It is straightforward to show that $X_0 = R$ and $X_1 = R^{\binom{n}{2}}$. All that remains to be shown is that f_{ij} is not unique for distinct generators $f_i = x_a x_b$ and $f_j = x_c x_d$. There are 2 cases.

Case 1. If the edges $v_a v_b$ and $v_c v_d$ share an endpoint, say, $b = d$, then $f_i = x_a x_b$ and $f_j = x_b x_c$ for distinct a, b, c. Since K_n is a complete graph, then $f_k = x_a x_c$ is a generator of $I(K_n)$ and we have

$$f_{ij} = \text{lcm}(f_i, f_j) = x_a x_b x_c = \text{lcm}(f_i, f_j, f_k) = f_{ijk}.$$

Case 2. Suppose the edges $v_a v_b$ and $v_c v_d$ do not share an endpoint. Then $f_i = x_a x_b$ and $f_j = x_c x_d$ for distinct a, b, c, d. Since $f_k = x_a x_c$ and $f_l = x_b x_d$ are also generators then we have

$$f_{ij} = x_a x_b x_c x_d = f_{kl}.$$

□

Corollary 3.2.2. *If K_n is a complete graph on $n \geq 3$ vertices, then the Scarf complex $X^R(K_n)$ is not a resolution of $R/I(K_n)$.*

Proof. Invoke Theorem 1.2.14. □

3.3 Periodic Vanishing

Next, we seek to understand the homological vanishing for $X^R(P_n)$ and $X^R(C_n)$ as n increases. To analyze the behavior of $X^R(P_n)$ and $X^R(C_n)$, we consider the following subsequent difference functions.

Notation 3.3.1. Let $I(P_{n-1})$ and $I(P_n)$ be edge ideals of an $(n-1)$-path and n-path, respectively. Let $\Delta \sup X^R(P_n) := \sup X^R(P_n) - \sup X^R(P_{n-1})$. Similarly, let $\Delta \sup H(X^R(P_n)) := \sup H(X^R(P_n)) - \sup H(X^R(P_{n-1}))$. Define $\Delta(\sup(X^R(C_n)))$ similarly.

Using Clemson University's supercomputing facility, the Palmetto cluster, we computed the vanishing and nonvanishing of $H_i(X^R(P_n))$ for $n = 1, \dots, 17$ via the following `Macaulay2` code.

```
needsPackage "ChainComplexExtras"; -- Scarf complex
load "main.m2"; -- load user-defined functions
L = for i from 1 to 18 list i;
netList apply(L, i -> {i, hhList scarfComplex nPath(i,QQ[x_1..x_(i+1)])})
```

Consult Appendix A for more details about this block of code.

In the following summary table, we let "0" indicate that there is not any homology in degree i (i.e., $\ker \partial_i^X = \operatorname{im} \partial_{i+1}^X$) and "1" otherwise. For example, when $X = X^R(P_5)$, the sequence 1 0 1 0 signifies that $H_0(X) \neq 0$, $H_1(X) = 0$, $H_2(X) \neq 0$ and $H_3(X) = 0$.

Remark 3.3.2. Even though 50 GB of memory was allocated to a node on the Palmetto cluster with a max run-time of 24 hours, `Macaulay2` version 1.16 failed to compute $X^R(P_{18})$.

Slightly modifying the last line of code above returns the Scarf homology for n-cycles for $n = 3, \dots, 18$:

```
netList apply(L, i -> {i, hhList scarfComplex nCycle(i,QQ[x_1..x_i])})
```

The output is summarized in Figure 3.2 below.

path length n											$\sup X$	$\Delta \sup X$	$\sup H(X)$	$\Delta \sup H(X)$
1	1	0									1		0	
2	1	0	0								2	1	0	0
3	1	0	0								2	0	0	0
4	1	0	1								2	0	2	0
5	1	0	1	0							3	1	2	0
6	1	0	1	0	0						4	1	2	0
7	1	0	1	0	0						4	0	2	0
8	1	0	1	0	0						4	0	2	0
9	1	0	1	0	0	0					5	1	2	0
10	1	0	1	0	1	0	0				6	1	4	2
11	1	0	1	0	1	0	0				6	0	4	0
12	1	0	1	0	1	0	0				6	0	4	0
13	1	0	1	0	1	0	0	0			7	1	4	0
14	1	0	1	0	1	0	0	0	0		8	1	4	0
15	1	0	1	0	1	0	0	0	0		8	0	4	0
16	1	0	1	0	1	0	1	0	0		8	0	6	2
17	1	0	1	0	1	0	1	0	0	0	9	1	6	0
degree i	0	1	2	3	4	5	6	7	8	9				

Figure 3.1: Homology of n-paths, $X = X^R(P_n)$

cycle length n										$\sup X$	$\Delta \sup X$	$\sup H(X)$	$\Delta \sup H(X)$
3	1	1								1		1	
4	1	0	1							2	1	2	1
5	1	0	1							2	0	2	0
6	1	0	1							2	0	2	0
7	1	0	1	0						3	1	2	0
8	1	0	1	0	0					4	1	2	0
9	1	0	1	1	0					4	0	3	1
10	1	0	1	0	1					4	0	4	1
11	1	0	1	0	1	0				5	1	4	0
12	1	0	1	0	1	0	0			6	1	4	0
13	1	0	1	0	1	0	0			6	0	4	0
14	1	0	1	0	1	0	0			6	0	4	0
15	1	0	1	0	1	1	0	0		7	1	5	1
16	1	0	1	0	1	0	1	0	0	8	1	6	1
17	1	0	1	0	1	0	1	0	0	8	0	6	0
18	1	0	1	0	1	0	1	0	0	8	0	6	0
degree i	0	1	2	3	4	5	6	7	8				

Figure 3.2: Homology of n-cycles, $X = X^R(C_n)$

These data give rise to the following interrelated questions about periodic vanishing.

Question 3.3.3. Is $H_i(X^R(P_n)) = 0$ if and only if $H_i(X^R(C_{n+2})) = 0$ for $n \geq 4$?

Question 3.3.4.

(a) Does $\Delta \sup X^R(P_n)$ have period 4 (as a function of n) for $n \geq 2$?

(b) Does $\Delta \sup(H(X^R(P_n)))$ has period 6 (as a function of n) for $n \geq 4$?

Question 3.3.5.

(a) Does $\Delta \sup X^R(C_n)$ have period 4 (as a function of n) for $n \geq 4$?

(b) Does $\Delta \sup(H(X^R(C_n)))$ have period 6 (as a function of n) for $n \geq 4$?

Question 3.3.6. For all $n \geq 4$, is $H_2(X^R(P_n)) \neq 0$ and $H_2(X^R(C_n)) \neq 0$?

Suffice it to say that Scarf complexes associated to n-paths, n-cycles and complete graphs on n vertices are not terribly well-behaved in the sense that they are almost never a resolution. The problem is that for $n \geq 3$ these monomial ideals tend to be highly nongeneric—meaning that every generator contains an x_i such that x_i appears in another generator. For example, if two edges in G are incident to the same vertex (i.e., v_iv_j and v_jv_k), then x_j occurs in the generators x_ix_j, x_jx_k of $I(G)$.

In the next chapter, we introduce the notion of a weighted graph which somewhat remedies the problem above. One can find classes of “almost” generic monomial ideals (ideals which only contain two distinct generators which share a variable with the same exponent) for which the Scarf complex is a resolution.

Chapter 4

The Scarf Complex of Weighted 5-Cycles

In this chapter we give a brief introduction to Cohen-Macaulay rings, weighted graphs and weighted edge ideals. The goal of this chapter is to prove that the Scarf complex is a resolution for a class of non-generic monomial ideals I for which R/I is Cohen-Macaulay.

4.1 Cohen-Macaulay Rings

Two fundamental invariants of a standard graded ring are its dimension and depth. The former is a geometric invariant and the latter is an algebraic invariant. When they are equal, we say that the ring is Cohen-Macaulay.

Krull Dimension

In this subsection let R be a commutative ring with identity.

Recall the definition of a prime ideal.

Definition 4.1.1. An ideal $I \leq R$ is prime if $I \neq R$ and the complement $R \backslash I$ is closed under multiplication, i.e., for all $a, b \in R$ if $ab \in I$, then either $a \in I$ or $b \in I$.

The Krull dimension gives a measure of the size of a ring.

Definition 4.1.2 ([3, p.413])**.** The Krull dimension (or dimension) of R, denoted $\dim(R)$, is the supremum of lengths of chains of prime ideals in R. In symbols:

$$\dim(R) = \sup\{n \geq 0 \mid \text{there is a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } R\}.$$

In the case that $R = K[x_1, \ldots, x_d]$ and I is a monomial ideal, then we can calculate the dimension of R/I using the irredundant m-irreducible decomposition of I defined in Section 1.1.

Theorem 4.1.3 ([10, Theorem 5.1.12])**.** *Set $R = K[x_1, \ldots, x_d]$, and let I be a monomial ideal in R with m-irreducible decomposition $I = \cap_{i=1}^m J_i$. Then $\dim(R/I) = d - n$ where n is the smallest number of generators needed for one of the J_i. In particular, $\dim(R) = d$.*

Example 4.1.4. Let I be a monomial ideal in $R = K[x_1, \ldots, x_4]$ with irredundant generating sequence $\mathbf{f} = x_1x_2, x_2^2x_3^2, x_3x_4$. It can be shown that I has an irredundant m-irreducible decomposition

$$\left\langle x_1, x_2^2, x_4 \right\rangle \cap \left\langle x_1, x_3 \right\rangle \cap \left\langle x_1, x_3^2, x_4 \right\rangle \cap \left\langle x_2, x_3 \right\rangle \cap \left\langle x_2, x_4 \right\rangle.$$

Therefore, $\dim(R/I) = 4 - 2 = 2$ by Theorem 4.1.3.

Remark 4.1.5. The irredundant m-irreducible decomposition of an edge ideal can be computed by first computing all minimal vertex covers of a graph (see Theorem 4.3.6 in [10]). This is an example of how combinatorial properties of a graph give insight into algebraic properties of its edge ideal.

Depth

Assume R is a non-zero commutative ring with identity, and let $\mathfrak{a} < R$.

Depth is defined in terms of R-regular sequences, which we define next.

Definition 4.1.6 ([3, Definition 1.1.1])**.** A sequence $r_n, r_{n-1}, \ldots, r_1 \in \mathfrak{a}$ is R-regular if:

(1) r_n is a non-zero-divisor on R.

(2) r_{n-1} is a non-zero-divisor on $R/\langle r_n \rangle$.

(3) r_{n-2} is a non-zero-divisor on $R/\langle r_n, x_{r-1} \rangle$.

$\vdots$

(i) r_{n-i+1} is a non-zero-divisor on $R/\langle r_n, \dots, r_{n-i+2}\rangle$.

$\vdots$

(n) r_1 is a non-zero-divisor on $R/\langle r_n, \dots, r_2\rangle$.

Example 4.1.7. In $R = K[x_1, \dots, x_d]$ and $\mathfrak{m} = \langle x_1, \dots, x_d\rangle$, the list $x_1^{a_1}, \dots, x_d^{a_d}$ is R-regular for any $\mathbf{a} \in \mathbb{N}_+^d$.

Definition 4.1.8. The depth of $\mathfrak{a}$ on R, denoted $\operatorname{depth}_{\mathfrak{a}}(R)$, is the length of a maximal R-regular sequence in $\mathfrak{a}$; this is sometimes also denoted $\operatorname{depth}(\mathfrak{a}, R)$ or $\operatorname{depth}(\mathfrak{a})$. If R is standard graded and $\mathfrak{m} = \bigoplus_{n\geq 1} R_n < R$ is the irrelevant ideal, then the depth of R is $\operatorname{depth}(R) := \operatorname{depth}_{\mathfrak{m}}(R)$.

Next we give a homological characterization for the depth of a ring which we often use to calculate depth.

Theorem 4.1.9 (Auslander-Buchsbaum [6, Theorem 19.9])**.** *Let $R = K[X_1, \dots, X_d]$ and $\mathbf{f} = f_1, \dots, f_n \in R$, where each f_i is a non-constant homogeneous polynomial. Let $I = \langle \mathbf{f}\rangle \lneq R$ and $\overline{R} = R/I$ and $\Delta = \operatorname{depth}(\overline{R})$.*

(a) *There exists a free resolution $0 \to F_{d-\Delta} \to F_{d-\Delta-1} \to \cdots \to F_0 \to \overline{R} \to 0$.*

(b) *If $0 \to G_n \to \cdots \to G_0 \to R \to 0$ is a free resolution over R, then $n \geq d - \Delta$. Furthermore, $0 \to \operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right) \to G_{d-\Delta-1} \to \cdots \to G_0 \to \overline{R} \to 0$ is exact and $\operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right)$ is projective. By a result of Serre, $\operatorname{Ker}\left(\partial^G_{d-\Delta-1}\right)$ is also free.*

The slogan here is that the "projective dimension" of $\overline{R}$ over R is $\dim(R) - \operatorname{depth}(\overline{R})$.

Example 4.1.10. Let I be the monomial ideal in $R = K[x_1, \dots, x_4]$ with irredundant generating sequence $\mathbf{f} = x_1x_2, x_2^2x_3^2, x_3x_4$ as in Example 4.1.4. It can be shown that f_Λ is unique for all Λ. Therefore,

$$X = X^R(\mathbf{f}) = \left(0 \longrightarrow R \xrightarrow{\partial_3^X} R^3 \xrightarrow{\partial_2^X} R^3 \xrightarrow{\partial_1^X} R \xrightarrow{\partial_0^X} 0\right).$$

Since I is generic, then $X^R(\mathbf{f})$ is a minimal free resolution by Theorem 2.4.18 (Alternatively, $X^R(\mathbf{f})$ is a free resolution because $X^R(\mathbf{f}) = T^R(\mathbf{f})$). Thus, $\operatorname{depth} R/I = \dim(R) - \sup X^R(\mathbf{f}) = 4 - 3 = 1$ by Theorem 4.1.9.

Cohen-Macaulayness

Throughout this subsection, let R be a standard graded ring.

Cohen-Macaulayness is a niceness condition. The definition is sufficiently broad to allow a wealth of examples in other fields such as algebraic geometry. However, it is sufficiently strict to admit a rich theory [3, p.57]. Two nice conditions of the Cohen-Macaulay property are as follows.

(a) Cohen-Macaulay rings are stable under localization. Localization is a powerful tool in commutative algebra.

(b) A problem of interest to algebraic geometers is to determining when a projective algebraic variety is unmixed. In the special case that $R = K[x_1, \ldots, x_d]$ and K is a field, then one can use Theorem 4.1.14 below to access information about the unmixedness of an monomial ideal I by studying the Cohen-Macaulay property of R/I.

Definition 4.1.11 ([3, Definition 2.1.1])**.** We say that a ring R is Cohen-Macaulay if $\operatorname{depth} R = \dim R$.

Example 4.1.12. Let $I = \langle x_1x_2, x_2^2x_3^2, x_3x_4 \rangle \leq R = K[x_1, \ldots, x_4]$ as in Example 4.1.10. In Examples 4.1.4 and 4.1.10 we calculated $\dim(R/I)$ and $\operatorname{depth}(R/I)$. Since

$$\dim(R/I) = 2 \neq 1 = \operatorname{depth}(R/I)$$

then R/I is not Cohen-Macaulay by Definition 4.1.11.

Definition 4.1.13 ([10, Definition 5.3.5])**.** Set $R = K[x_1, \ldots, x_d]$. Consider a monomial ideal $J \subsetneq R$ with irredundant m-irreducible decomposition $J = \bigcap_{i=1}^k Q_i$. Then J is m-unmixed if the number of generators of Q_i is equal to the number of generators of Q_j for all $i \neq j$. We say that J is m-mixed if it is not m-unmixed, that if, if there are indices $i \neq j$ such that the number of generators of Q_i is not equal to the number of generators of Q_j.

The relationship between Cohen-Macaulay rings and m-unmixed monomial ideals is given next.

Theorem 4.1.14 ([10, Theorem 5.3.16(a)])**.** *Set $R = K[x_1, \ldots, x_d]$ and let I be a monomial ideal in R. If R/I is Cohen-Macaulay, then I is m-unmixed.*

Example 4.1.15. Let $I = \langle x_1x_2, x_2^2x_3^2, x_3x_4 \rangle \leq R = K[x_1, \ldots, x_4]$ as in Example 4.1.4. Recall that I has an irredundant m-irreducible decomposition

$$\langle x_1, x_2^2, x_4 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_1, x_3^2, x_4 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle .$$

Since the decomposition admits monomial ideals with generating sequences of different sizes (2 and 3), then I is m-mixed. This gives another proof that R/I is not Cohen-Macaulay by Theorem 4.1.14.

4.2 Weighted Graphs and Weighted Edge Ideals

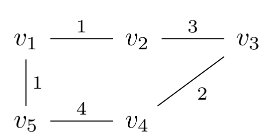
Next we define weighted graphs and weighted edge ideals, pioneered by Paulsen and Sather-Wagstaff in [11]. These can be thought of as a generalization of graphs and their associated edge ideals.

Definition 4.2.1 ([11, Definition 1.1])**.** A weight function on a graph G is a function $\omega : E \to N_+$ that assigns a weight to each edge. A weighted graph G_ω is a graph G equipped with a weight function ω. A weighted graph G_ω where each edge has the same weight is a trivally weighted graph.

Definition 4.2.2 ([11, Definition 3.1])**.** The weighed edge ideal associated to G_ω is the ideal $I(G_\omega) \leq R$ that is "generated by the weighted edges of G":

$$I(G_\omega) = \left\langle x_i^{\omega(e)} x_j^{\omega(e)} \mid e = v_iv_j \in E \right\rangle .$$

Example 4.2.3. Let G_ω be a weighted 5-cycle



Then the associated weighted edge ideal is $I(G_\omega) = \langle x_1x_2, x_2^3x_3^3, x_3^2x_4^2, x_4^4x_5^4, x_1x_5 \rangle$. (Compare to Example 2.4.15).

Notation 4.2.4. Consistent with Notation 3.1.6, if $I(G_\omega)$ is the weighted edge ideal associated to G_ω, then we simplify the notation of the Scarf complex and Scarf simplicial complex to $X^R(G_\omega)$ and $\Delta(G_\omega)$, respectively.

Definition 4.2.5 ([11, Definition 5.1]). The weighted graph G_ω is Cohen-Macaulay over K if the ring $R/I(G_\omega)$ is Cohen-Macaulay. If G_ω is Cohen-Macaulay over every field, we simply say that it is Cohen-Macaulay.

4.3 Cohen-Macaulay Weighted Graphs

In this section, we will compute some more examples and non-examples of Cohen-Macaulay rings. In the case that I is m-mixed, we will invoke Theorem 4.1.14 to conclude that R/I is not Cohen-Macaulay. When I is m-unmixed, we will measure the depth and dimension of R/I via Theorems 4.1.9 and 4.1.3, respectively.

Proposition 4.3.1. *Let $I = \left\langle X^aY^b, Y^cZ^d \right\rangle$ be a monomial ideal in $R = K[X, Y, Z]$ where $a, b, c, d > 0$. Then R/I is never Cohen-Macaulay.*

Proof. Consider the m-irreducible decomposition

$$I = \left\langle X^a, Y^cZ^d \right\rangle \cap \left\langle Y^b, Y^cZ^d \right\rangle = \langle X^a, Y^c \rangle \cap \left\langle X^a, Z^d \right\rangle \cap \left\langle Y^{\min(b,c)} \right\rangle.$$

Since $\left\langle Y^{\min(b,c)} \right\rangle$ and $\left\langle X^a, Z^d \right\rangle$ are irredundant in the decomposition (i.e., $\left\langle Y^{\min(b,c)} \right\rangle \cap \left\langle X^a, Z^d \right\rangle \neq \emptyset$) with generating sequences of different sizes (1 and 2), then I is m-mixed which implies R/I is not Cohen-Macaulay by Theorem 4.1.14. □

Remark 4.3.2. Letting $a = b$ and $c = d$ in Proposition 4.3.1 implies weighted 2-paths are never Cohen-Macaulay.

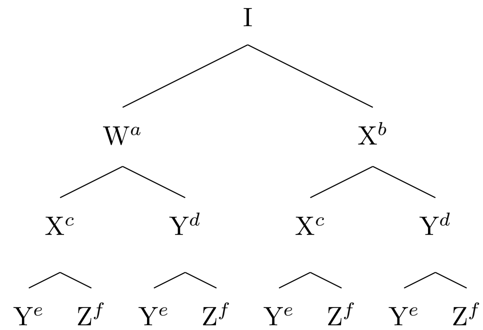
Proposition 4.3.3. *Let I be the monomial ideal in $K[W, X, Y, Z]$ with irredundant generating sequence $\mathbf{f} = W^aX^b, X^cY^d, Y^eZ^f$ where $a, \ldots, f > 0$. Then the following are equivalent:*

(i) R/I is Cohen-Macaulay,

(ii) I is m-unmixed,

(iii) $b \geq c$ and $d \leq e$.

Proof. First we compute the irredundant m-irreducible decomposition by first computing a decision tree and then removing the redundancies (See Theorem 1.1.16).



Therefore an m-irreducible decomposition is

$$\begin{aligned} I &= \langle W^a, X^c, Y^e \rangle \cap \left\langle W^a, X^c, Z^f \right\rangle \cap \left\langle W^a, Y^d, Y^e \right\rangle \cap \left\langle W^a, Y^d, Z^f \right\rangle \cap \left\langle X^b, X^c, Y^e \right\rangle \cap \left\langle X^b, X^c, Z^f \right\rangle \\ &\quad \cap \left\langle X^b, Y^d, Y^e \right\rangle \cap \left\langle X^b, Y^d, Z^f \right\rangle \\ &= \langle W^a, X^c, Y^e \rangle \cap \left\langle W^a, X^c, Z^f \right\rangle \cap \left\langle W^a, Y^{\min(d,e)} \right\rangle \cap \left\langle W^a, Y^d, Z^f \right\rangle \cap \left\langle X^{\min(b,c)}, Y^e \right\rangle \\ &\quad \cap \left\langle X^{\min(b,c)}, Z^f \right\rangle \cap \left\langle X^b, Y^{\min(d,e)} \right\rangle \cap \left\langle X^b, Y^d, Z^f \right\rangle. \end{aligned}$$

$(i) \implies (ii)$ This follows from Theorem 4.1.14.

$(ii) \implies (iii)$ Proof by contrapositive. Suppose $b < c$ or $e < d$. If $b < c$ then $J_6 \not\subseteq J_2$ and if $e < d$ then $J_3 \not\subseteq J_4$. In either case I is m-mixed by Definition 4.1.13.

$(iii) \implies (i)$ Suppose $b \geq c$ and $d \leq e$. Then the irredundant irreducible decomposition is

$$I = \left\langle x_1^a, x_3^d \right\rangle \cap \left\langle x_2^c, x_3^e \right\rangle \cap \left\langle x_2^c, x_4^f \right\rangle \cap \left\langle x_2^b, x_3^d \right\rangle$$

which implies $\dim(R/I) = 4 - 2 = 2$. We will compute $\operatorname{depth}(R/I)$ using Theorem 4.1.9 and by showing that $X^R(\mathbf{f})$ is a minimal resolution. Consider the Scarf complex

$$X = X^R(\mathbf{f}) = \left(0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} -Y^d & 0 \\ W^a X^{b-c} & -Y^{e-d} Z^f \\ 0 & X^c \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} W^a X^a & X^c Y^d & Y^e Z^f \end{bmatrix}} R \longrightarrow 0\right).$$

$\underline{H_1(X)}$: It suffices to show that $\partial_2^T(e_{13}) \in \operatorname{im}(\partial_2^X)$. To this end, one checks readily that

$$\partial_2^T(e_{13}) = \begin{bmatrix} -Y^eZ^f \\ 0 \\ W^aX^b \end{bmatrix} = \begin{bmatrix} -Y^d & 0 \\ W^aX^{b-c} & -Y^{e-d}Z^f \\ 0X^c & \end{bmatrix} \begin{bmatrix} Y^{e-d}Z^f \\ W^aX^{b-c} \end{bmatrix} = \partial_2^X\left(\begin{bmatrix} Y^{e-d}Z^f \\ W^aX^{b-c} \end{bmatrix}\right) \in \operatorname{im}(\partial_2^X).$$

$\underline{H_2(X)}$: It suffices to show that ∂_2^X is injective. If

$$\begin{bmatrix} -Y^d & 0 \\ W^aX^{b-c} & -Y^{e-d}Z^f \\ 0X^c & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

then the first row implies $f_1 = 0$ and the third row implies $f_2 = 0$, as desired. Therefore, $X^R(\mathbf{f})$ is a minimal resolution which implies $\operatorname{depth}(R/I) = 4 - 2 = 2$. Since $\dim(R/I) = 2 = \operatorname{depth}(R/I)$, then R/I is Cohen-Macaulay by Definition 4.1.11. □

Letting $a = b$, $c = d$, and $e = f$ in the Proposition 4.3.3 gives us the classification of all Cohen-Macaulay weighted 3-paths.

Corollary 4.3.4. *Let $R = K[W, X, Y, Z]$ and $J = \langle W^aX^a, X^bY^b, Y^cZ^c\rangle$ be the weighted edge ideal of a 3-path where $a, b, c > 0$. Then the following conditions are equivalent:*

(i) R/J is Cohen-Macaulay,

(ii) J is m-unmixed,

(iii) $b < \min(a, c)$.

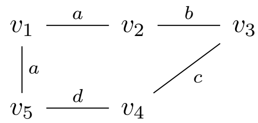
The following result is due to Paulsen and Sather-Wagstaff. They proved that the only two Cohen-Macaulay weighted cycles are weighted 3-cycles and weighted 5-cycles.

Theorem 4.3.5 ([11, Theorem A])**.** *Consider a weighted d-cycle C_ω^d.*

(a) *If C_ω^d is Cohen-Macaulay, then $d \in \{3, 5\}$.*

(b) *C_ω^3 is always Cohen-Macaulay.*

(c) *C_ω^5 is Cohen-Macaulay if and only if it can be written in the form*



(4.3.5.1)

such that $a \leq b \geq c \leq d \geq a$.

First, notice that if $J = I(C^5_\omega)$ is Cohen-Macaulay, then J is not generic! We wish to find weights such that $X^R(C^5_\omega)$ is a resolution. Not all choices of ω such that C^5_ω is Cohen-Macaulay result in $X^R(C^5_\omega)$ being a resolution. For example, if ω is trivially weighted, say $\omega = (1,1,1,1,1)$, then $C^5_\omega = C_5$ and we have

$$X^R(C_5) = \ (0 \longrightarrow R^5 \longrightarrow R^5 \longrightarrow R \longrightarrow 0),$$

which is not a resolution because $\sum_{i=0}^{2} -1^i \beta_i \neq 0$ (see Theorem 1.2.14). On the other hand, it turns out that if we assume strict inequalities $a < b > c < d > a$ then $I(C^5_\omega)$ is "almost generic" (i.e., only 2 particular generators share an indeterminant with the same power) and $X^R(C^5_\omega)$ is a resolution.

Theorem 4.3.6. *Let C^5_ω be a weighted 5-cycle as in* (4.3.5.1)*. Then the Scarf complex $X^R(C^5_\omega)$ is a resolution if $\omega = (a < b > c < d > a)$.*

Proof. Let $I(C^5_\omega) \leq R = K[x_1, \dots, x_5]$ be a weighted edge ideal with irredundant monomial generating sequence $\mathbf{f} = x_1^a x_2^a, x_2^b x_3^b, x_3^c x_4^c, x_4^d x_5^d, x_5^a x_1^a$. A routine computation shows that the Scarf complex has the shape

$$X = X^R(C^5_\omega) = \ (0 \longrightarrow R^3 \xrightarrow{\partial_3^X} R^7 \xrightarrow{\partial_2^X} R^5 \xrightarrow{\partial_1^X} R \xrightarrow{\partial_0^X} 0),$$

and differentials

$$\partial_1^X = \begin{bmatrix} x_1^a x_2^a & x_2^b x_3^b & x_3^c x_4^c & x_4^d x_5^d & x_5^a x_1^a \end{bmatrix}$$

$$\partial_2^X = \begin{bmatrix} -x_2^{b-a} x_3^b & -x_3^c x_4^c & -x_5^a & 0 & 0 & 0 & 0 \\ x_1^a & 0 & 0 & -x_4^c & 0 & 0 & 0 \\ 0 & x_1^a x_2^a & 0 & x_2^b x_3^{b-c} & -x_4^{d-c} x_5^d & -x_1^a x_5^a & 0 \\ 0 & 0 & 0 & 0 & x_3^c & 0 & -x_1^a \\ 0 & 0 & x_2^a & 0 & 0 & x_3^c x_4^c & x_4^d x_5^{d-a} \end{bmatrix}$$

$$\partial_3^X = \begin{bmatrix} x_4^c & 0 & 0 \\ -x_2^{b-a}x_3^{b-c} & x_5^a & 0 \\ 0 & -x_3^c x_4^c & 0 \\ x_1^a & 0 & 0 \\ 0 & 0 & x_1^a \\ 0 & x_2^a & -x_4^{d-c}x_5^{d-a} \\ 0 & 0 & x_3^c \end{bmatrix}.$$

Degree 1: It suffices to show that that the columns of ∂_2^T that are not columns of ∂_2^X are in the image of ∂_2^X. Since $f_{14} = \text{lcm}(f_1, f_4) = x_1^a x_2^a x_4^d x_5^d = \text{lcm}(f_{14}, f_5) = f_{145}$, then $\partial_2^T(e_{14})$ is a column of ∂_2^T but not ∂_2^X. Similarly for $\partial_2^T(e_{24})$ and $\partial_2^T(e_{25})$.

We begin by considering

$$\partial_2^T(e_{14}) = \begin{bmatrix} -x_4^d x_5^d & 0 & 0 & x_1^a x_2^a & 0 \end{bmatrix}^T = \partial_2^X(\mathbf{g}) \tag{4.3.6.1}$$

where $\mathbf{g}$ is a column vector $\begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \end{bmatrix}^T$ with entries in R. Using divisor arguments we obtain the solution

$$\mathbf{g} = \begin{bmatrix} -x_4^c & x_2^{b-a}x_3^{b-c} & x_4^d x_5^{d-a} & -x_1^a & 0 & 0 & -x_2^a \end{bmatrix}^T.$$

For example, in equation 4 of (4.3.6.1) we have $x_1^a x_2^a = g_5 x_3^c - g_7 x_1^a$. Since x_3 does not divide the left-hand side, then $g_5 = 0$ which implies $g_7 = -x_2^a$. Similarly,

$$\mathbf{g}' = \begin{bmatrix} 0 & x_5^a & -x_3^c x_4^c & x_4^{d-c}x_5^d & x_2^b x_3^{b-c} & x_2^a & 0 \end{bmatrix}^T$$
$$\mathbf{g}'' = \begin{bmatrix} -x_5^a & 0 & x_2^{b-a}x_3^b & 0 & x_1^a & -x_4^{d-c}x_5^{d-a} & x_3^c \end{bmatrix}^T$$

satisfy $\partial_2^T(e_{24}) = \begin{bmatrix} 0 & -x_4^d x_5^d & 0 & x_2^b x_3^b & 0 \end{bmatrix}^T = \partial_2^X(\mathbf{g}')$ and $\partial_2^T(e_{25}) = \begin{bmatrix} 0 & -x_1^a x_5^a & 0 & 0 & x_2^b x_3^b \end{bmatrix}^T = \partial_2^X(\mathbf{g}'')$, respectively.

Degree 2: We need to show $\ker(\partial_2^X) \subseteq \text{im}(\partial_3^X)$. Suppose $\partial_2^X(\mathbf{g}) = 0$, that is, suppose

$$-g_1 x_2^{b-a} x_3^b - g_2 x_3^c x_4^c - g_3 x_5^a = 0 \tag{4.3.6.2}$$

$$g_1x_1^a - g_4x_4^c = 0 \tag{4.3.6.3}$$

$$g_2x_1^ax_2^a + g_4x_2^bx_3^{b-c} - g_5x_4^{d-c}x_5^d - g_6x_1^ax_5^a = 0 \tag{4.3.6.4}$$

$$g_5x_3^c - g_7x_1^a = 0 \tag{4.3.6.5}$$

$$g_3x_2^a + g_6x_3^cx_4^c + g_7x_4^dx_5^{d-a} = 0. \tag{4.3.6.6}$$

To show $\mathbf{g} \in \operatorname{im}(\partial_3^X)$ we need to show there exists h_1, h_2, h_3 such that

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{bmatrix} = h_1 \begin{bmatrix} x_4^c \\ -x_2^{b-a}x_3^{b-c} \\ 0 \\ x_1^a \\ 0 \\ 0 \\ 0 \end{bmatrix} + h_2 \begin{bmatrix} 0 \\ x_5^a \\ -x_3^ax_4^c \\ 0 \\ 0 \\ x_2^a \\ 0 \end{bmatrix} + h_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_1^a \\ -x_4^{d-c}x_5^{d-a} \\ x_3^c \end{bmatrix}.$$

We will accomplish this using nonzero divisor arguments and substitution. First consider equation (4.3.6.3). Since x_4^c divides the second term and the right-hand side, then it divides the first term. So, $g_1 = h_1x_4^c$ for some $h_1 \in R$. Substitution and factoring gives $h_1x_4^cx_1^a - g_4x_4^c = x_4^c(h_1x_1^a - g_4) = 0$. Since x_4^c is a nonzero divisor, then the factor $h_1x_1^a - g_4$ is equal to 0. This implies $g_4 = h_1x_1^a$. Equation (4.3.6.5) implies $g_5 = h_3x_1^a$. So, we can rewrite this equation as $x_1^a(h_3x_3^c - g_7) = 0$ which implies $g_7 = h_3x_3^c$. In equation (4.3.6.2) we can factor x_3^c out of the first two terms which implies x_3^c divides $g_3x_5^a$. So, $g_3 = \alpha x_3^c$. Similarly, equation (4.3.6.6) implies $g_3 = \beta x_4^c$. Since $\gcd(x_3^c, x_4^c) = 1$, then $g_3 = -h_2x_3^cx_4^c$ (below it is immediately apparent why we express g_3 with a negative coefficient). Rewriting equation (4.3.6.6), we get $-h_2x_3^cx_4^cx_2^a + g_6x_3^cx_4^c + h_3x_3^cx_4^dx_5^{d-a} = x_3^cx_4^c(-h_2x_2^a + g_6 + h_3x_4^{d-c}x_5^{d-a}) = 0$ which implies $g_6 = h_2x_2^a - h_3x_4^{d-c}x_5^{d-a}$. All that remains to be shown is that $g_2 = -h_1x_2^{b-a}x_3^{b-c} + h_2x_5^a$. Putting it all together, equation (4.3.6.4) can be rewritten as

$$\begin{aligned} 0 &= g_2x_1^ax_2^a + h_1x_1^ax_2^bx_3^{b-c} - h_3x_1^ax_4^{d-c}x_5^d - (h_2x_2^a - h_3x_4^{d-c}x_5^{d-a})x_1^ax_5^a \\ &= g_2x_1^ax_2^a + h_1x_1^ax_2^bx_3^{b-c} - h_3x_1^ax_4^{d-c}x_5^d - h_2x_1^ax_2^ax_5^a + h_3x_1^ax_4^{d-c}x_5^d \\ &= g_2x_1^ax_2^a + h_1x_1^ax_2^bx_3^{b-c} - h_2x_1^ax_2^ax_5^a \end{aligned}$$

$$= x_1^a x_2^a (g_2 + h_1 x_2^{b-a} x_3^{b-c} - h_2 x_5^a).$$

Therefore, $g_2 = -h_1 x_2^{b-a} x_3^{b-c} + h_2 x_5^a$, as desired.

Degree 3: It suffices to show ∂_3^X is injective. Suppose

$$\partial_3^X(\mathbf{g}) = g_1 \begin{bmatrix} x_4^c \\ -x_2^{b-a} x_3^{b-c} \\ 0 \\ x_1^a \\ 0 \\ 0 \\ 0 \end{bmatrix} + g_2 \begin{bmatrix} 0 \\ x_5^a \\ -x_3^c x_4^c \\ 0 \\ 0 \\ x_2^a \\ 0 \end{bmatrix} + g_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_1^a \\ -x_4^{d-c} x_5^{d-a} \\ x_3^c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

In equation 1, $g_1 x_4^c = 0$ implies $g_1 = 0$. Then in equation 2, $g_1(-x_2^{b-a} + g_2 x_5^a = g_2 x_5^a = 0$ implies $g_2 = 0$, and last, in equation 5, $g_3 x_1 a = 0$ implies $g_3 = 0$.

Since $H_i(X) = 0$ for all $i \neq 0$, then $X^R(C_\omega^5)$ is a resolution. □

In the next chapter we give an alternate proof of Theorem 4.3.6.

Chapter 5

An Acyclicity Criterion

Let R be a noetherian commutative ring with identity in this chapter.

As we saw in the proof of Theorem 4.3.6, proving that an R-complex is a resolution by repeatedly solving systems of equations can be difficult and at the very minimum tedious. Buchsbaum and Eisenbud's acyclicity criterion turns this problem into a numerical and ideal theoretic one. In some cases this is easier to solve. Let us begin with the definition of an acyclic complex.

Definition 5.0.1. An R-complex

$$G = \cdots \longrightarrow G_m \xrightarrow{\phi_m} G_{m-1} \longrightarrow \cdots G_1 \xrightarrow{\phi_1} G_0 \longrightarrow 0$$

is called acyclic if $H_i(G) = 0$ for all $i > 0$.

Note 5.0.2. Free resolutions are acyclic complexes.

We proceed with some prerequisite background for the acyclicity criterion.

Definition 5.0.3. Let U be an $m \times n$ matrix over R $(m, n \geq 0)$. For $t = 1, \ldots, \min(m, n)$ we denote $I_t(U)$ the ideal generated by the t-minors of U (i.e. determinants of $t \times t$ submatrices). Set $I_t(U) = R$ for $t \leq 0$ and $I_t(U) = 0$ for $t > \min(m, n)$.

Example 5.0.4. Let $U = \begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}$. Consider the 2-minors of U:

$$\text{removing the first row: } \begin{vmatrix} x & -y \\ 0 & x \end{vmatrix} = x^2,$$

$$\text{removing the second row: } \begin{vmatrix} -y & 0 \\ 0 & x \end{vmatrix} = -xy,$$

$$\text{removing the third row: } \begin{vmatrix} -y & 0 \\ x & -y \end{vmatrix} = y^2.$$

Therefore, the ideal generated by the 2-minors of U is $I_2(U) = \langle x^2, xy, y^2 \rangle$.

Definition 5.0.5. If $I \leq R$ be an ideal. Define $\operatorname{grade} I = \operatorname{depth}(I; R)$.

Definition 5.0.6. Let $\mathfrak{p} \in \operatorname{Spec} R$. then <u>height</u> of $\mathfrak{p}$ is the supremum of lengths of t of strictly descending chains

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_t$$

of prime ideals. For an arbitrary ideal I one sets

$$\operatorname{height} I = \inf\{\operatorname{height} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \supset I\}.$$

It turns out that when R is Cohen-Macaulay, grade and height are precisely the same. Furthermore, in the local (or graded and homogeneous) setting, one can compute height using the Krull dimension. This is made precise in the following theorem.

Theorem 5.0.7 ([3, Corollary 2.1.4])**.** *Let R be a Cohen-Macaulay ring, and $I < R$ an ideal. Then* $\operatorname{grade} I = \operatorname{height} I$*, and if R is local (or R is graded and I is homogeneous) then* $\operatorname{height} I = \dim R - \dim R/I$.

Pairing Theorems 5.0.7 and 4.1.3, we can obtain the grade of a monomial ideal from the irredundant irreducible decomposition of I.

Corollary 5.0.8. *Let $R = K[x_1, \ldots, x_d]$ and $I \subsetneq R$ is a monomial ideal with irredundant irreducible decomposition $I = \bigcap_{i=1}^{m} J_i$. Recall by Theorem 4.1.3 that $\dim R = d$ and $\dim R/I = d - n$ where n is the minimum number of generators for J_i. Since R is a graded Cohen-Macaulay ring and I is a monomial ideal (and therefore homogeneous) then $\operatorname{grade} I = \operatorname{height} I = d - (d - n) = n$ by Theorem 5.0.7.*

Example 5.0.9. Let $J = I_2(U) = \langle x^2, xy, y^2 \rangle$ be the monomial ideal in R as in Example 5.0.4. Since J has irredundant irreducible decomposition

$$\langle x, y^2 \rangle \cap \langle x^2, y \rangle,$$

then $\operatorname{grade} J = 2$ by Corollary 5.0.8.

We are finally ready to state Buchsbaum and Eisenbud's acyclicity criterion and see it in action.

Theorem 5.0.10 ([3, Theorem 1.4.13])**.** *Let R be a Noetherian ring and*

$$F = \quad (0 \longrightarrow F_s \xrightarrow{\phi_s} F_{s-1} \longrightarrow \cdots F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0)$$

a complex of finite free R-modules. Set $r_i = \sum_{j=1}^{s} (-1)^{j-i} \operatorname{rank} F_j$. Then the following are equivalent:

(i) F is acyclic, i.e., a free resolution;

(ii) $\operatorname{grade} I_{r_i}(\phi_i) \geq i$ for $i = 1, \ldots, s$.

Example 5.0.11. Let I be the monomial ideal in R with irredundant generating sequence $\mathbf{f} = x^2, xy, y^2$ as in Example 2.3.7. Notice that $X^R(\mathbf{f})$ is a resolution because $\langle \mathbf{f} \rangle$ is generic. We will provide an alternate proof that $X^R(\mathbf{f})$ is a resolution using Theorem 5.0.10.

Recall that the Scarf complex $X^R(\mathbf{f})$ is as follows:

$$X = X^R(\mathbf{f}) = (0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} R \longrightarrow 0).$$

First we compute the r_i's:

$$r_1 = (-1)^{1-1}\operatorname{rank}(X_1) + (-1)^{2-1}\operatorname{rank}(X_2) = 3 - 2 = 1$$
$$r_2 = (-1)^{2-2}\operatorname{rank}(X_2) = 2.$$

So, we need to show $\operatorname{grade} I_{r_i}(\partial_i^X) \geq i$ for $i = 1, 2$. Showing that $\operatorname{grade} I_1(\partial_1^X) \geq 1$ is trivial. All that remains to be shown is that

$$I_2(\partial_2^X) \geq 2. \tag{5.0.11.1}$$

By Example 5.0.4, we know $I_2(\partial_2^X) = \langle x^2, xy, y^2 \rangle$ and by Example 5.0.9 we have $\operatorname{grade} I_2(\partial_2^X) = 2$ which satisfies (5.0.11.1). Therefore, $X^R(\mathbf{f})$ is a resolution by Theorem 5.0.10.

Without further ado, we provide an alternate proof of Theorem 4.3.6.

Alternate proof of Theorem 4.3.6. Let $X = X^R(C_\omega^5)$. For $i = 1, 2, 3$, we used Mathematica to verify that $I_{r_i}(\partial_i^X)$ are in fact monomial ideals. Therefore, we can use Corollary 5.0.8 to produce a lower bound for $\operatorname{grade} I_{r_i}(\partial_i^X)$. Recall that the Scarf complex X has the shape

$$X = (0 \longrightarrow R^3 \xrightarrow{\partial_3^X} R^7 \xrightarrow{\partial_2^X} R^5 \xrightarrow{\partial_1^X} R \xrightarrow{\partial_0^X} 0).$$

We need to show $I_{r_i}(\partial_i^X) \geq i$ for $i = 1, 2, 3$. First compute r_i:

$$r_1 = \sum_{j=1}^{3}(-1)^{j-1}\operatorname{rank} X_j = \operatorname{rank} X_1 - \operatorname{rank} X_2 + \operatorname{rank} X_3 = 5 - 7 + 3 = 1$$
$$r_2 = \sum_{j=2}^{3}(-1)^{j-2}\operatorname{rank} X_j = \operatorname{rank} X_2 - \operatorname{rank} X_3 = 7 - 3 = 4$$
$$r_3 = \sum_{j=3}^{3}(-1)^{j-3}\operatorname{rank} X_j = \operatorname{rank} X_3 = 3.$$

Since $I_{r_1}(\partial_1^X) = I_1(\partial_1^X) = \langle x_1^a x_2^a, x_2^b x_3^b, x_3^c x_4^c, x_4^d x_5^d, x_5^a x_1^a \rangle$ is a nonempty monomial ideal, then $\operatorname{grade} I_1(\partial_1^X) \geq 1$ because the minimum number of generators for J_i is greater than or equal to 1.

Let $I = I_{r_2}(\partial_2^X) = I_4(\partial_2^X)$. Instead of computing the irredundant irreducible decomposition of I to give us the exact numeric value of $\operatorname{grade} I$, we will show that any J_i with a single generator cannot occur in the irredundant irreducible decomposition, thus giving us a lower bound on $\operatorname{grade} I$.

By way of contradiction, suppose grade $I = 1$. Then there exists J_i in the irredundant irreducible decomposition of I such that $J_i = \langle x_j^\alpha \rangle$ where $j \in \{1, 2, \ldots, 5\}$ by Theorem 5.0.7. We know that J_i is generated by "pure powers" (as opposed to a product of indeterminates) by Theorem 1.1.8. Let $M_{\Lambda,\Gamma}$ denote the determinant of the 4-minor of ∂_2^X obtained after deleting rows Λ and columns Γ. Consider the minors

$$M_{3,137} = \begin{vmatrix} -x_2^{b-a}x_3^b & -x_3^c x_4^c & -x_5^a & 0 & 0 & 0 & 0 \\ x_1^a & 0 & 0 & -x_4^c & 0 & 0 & 0 \\ 0 & x_1^a x_2^a & 0 & x_2^b x_3^{b-c} & -x_4^{d-c}x_5^d & -x_1^a x_5^a & 0 \\ 0 & 0 & 0 & 0 & x_3^c & 0 & -x_1^a \\ 0 & 0 & x_2^a & 0 & 0 & x_3^c x_4^c & x_4^d x_5^{d-a} \end{vmatrix}$$

$$= \begin{vmatrix} -x_3^c x_4^c & 0 & 0 & 0 \\ 0 & -x_4^c & 0 & 0 \\ 0 & 0 & x_3^c & 0 \\ 0 & 0 & 0 & x_3^c x_4^c \end{vmatrix} = x_3^{3c} x_4^{3c}$$

and $M_{1,456} = -x_1^{3a}x_2^{2a}$. Since $x_1^{3a}x_2^{2a}, x_3^{3c}x_4^{3c}$ are generators of I, then they are in the decomposition $\bigcap_{\ell=1}^m J_\ell$. By definition $x_1^{3a}x_2^{2a}, x_3^{3c}x_4^{3c} \in J_\ell$ for all ℓ. In particular, $x_1^{3a}x_2^{2a}, x_3^{3c}x_4^{3c} \in J_i = \langle x_j^\alpha \rangle$. Since $x_1^{3a}x_2^{2a}$ is not an element of $\langle x_3^\alpha \rangle, \langle x_4^\alpha \rangle$, nor $\langle x_5^\alpha \rangle$ for any $\alpha \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then $J_i \neq \langle x_3^\alpha \rangle, \langle x_4^\alpha \rangle, \langle x_5^\alpha \rangle$. Similarly, $x_3^{3c}x_4^{3c} \notin \langle x_1^\alpha \rangle, \langle x_2^\alpha \rangle$ implies $J_i \neq \langle x_1^\alpha \rangle, \langle x_2^\alpha \rangle$. Since we have exhausted all possible J_i with 1 generator, then grade $I \neq 1$ which implies grade $I \geq 2$.

Now let $I = I_{r_3}(\partial_3^X) = I_3(\partial_3^X)$. By way of contradiction, suppose grade $I \leq 2$. Then the minimum number of generators of J_i in the irredundant irreducible decomposition of I is at most 2. So, $J_i = \left\langle x_j^\alpha, x_k^\beta \right\rangle$. Since $R = K[x_1, \ldots, x_d]$, then there are $\binom{5}{2} = 10$ distinct choices for j and k.

Let M_Λ denote the determinant of the 3-minor of ∂_3^X obtained after deleting rows Λ. Consider $M_{1237} = -x_1^{2a}x_2^a$. Since M_{1237} is not a multiple of x_3, nor x_5, then $M_{1237} \notin \left\langle x_3^\alpha, x_5^\beta \right\rangle$ by Theorem 1.1.3. Similarly, since

$$M_{3457} = -x_4^d x_5^d \notin \left\langle x_1^\alpha, x_2^\beta \right\rangle, \left\langle x_1^\alpha, x_3^\beta \right\rangle, \left\langle x_2^\alpha, x_3^\beta \right\rangle,$$

$$M_{1345} = -x_2^b x_3^b \notin \left\langle x_1^\alpha, x_4^\beta \right\rangle, \left\langle x_1^\alpha, x_5^\beta \right\rangle, \left\langle x_4^\alpha, x_5^\beta \right\rangle,$$

$$M_{1367} = -x_1^{2a} x_5^a \notin \left\langle x_2^\alpha, x_4^\beta \right\rangle, \left\langle x_3^\alpha, x_4^\beta \right\rangle,$$

$$M_{2456} = x_3^{2c} x_4^{2c} \notin \left\langle x_2^{\alpha}, x_5^{\beta} \right\rangle.$$

then we can conclude grade $I \neq 2$ which implies grade $I \neq 1$. Therefore, grade $I \geq 3$ and hence, X is acyclic (and therefore a resolution) by Theorem 5.0.10. $\square$

Chapter 6

Future Research

In addition to investigating Questions 3.3.3, 3.3.4, 3.3.5 and 3.3.6 regarding periodic vanishing, we wish to investigate the following questions.

Question 6.0.1. If the alternating sum of the ranks of the Scarf complex equals 0, does that necessarily imply that the Scarf complex is a resolution? In other words, if

$$X = X^R(\mathbf{f}) = \quad (0 \longrightarrow R^{\beta_n} \xrightarrow{\partial_n^X} \cdots \xrightarrow{\partial_2^X} R^{\beta_1} \xrightarrow{\partial_1^X} R^{\beta_0} \xrightarrow{\partial_0^X} 0)$$

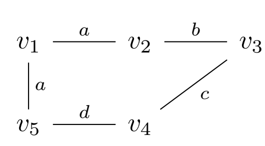
is the Scarf complex and and $\sum_{i=0}^{n}(-1)^i\beta_i = 0$, then is X a resolution?

Question 6.0.2. If $\Delta = \Delta(\mathbf{f})$ is contractible, then is $\Delta_{\preceq \mathbf{b}}$ contractible for all $\mathbf{b} \in \mathbb{N}^d$?

Question 6.0.3. Let $X = X^R(\mathbf{f})$ be the Scarf complex. Set $r_i = \sum_{j=1}^{\operatorname{length} X}(-1)^{j-i} \operatorname{rank} X_j$ and let $I_{r_i}(\partial_i^X)$ be the ideal generated by the r_i-minors of ∂_i^X as in Definition 5.0.3 and Theorem 5.0.10. When is $I_{r_i}(\partial_i^X)$ a monomial ideal?

We are curious if we can improve Theorem 4.3.6 using a contractibility argument.

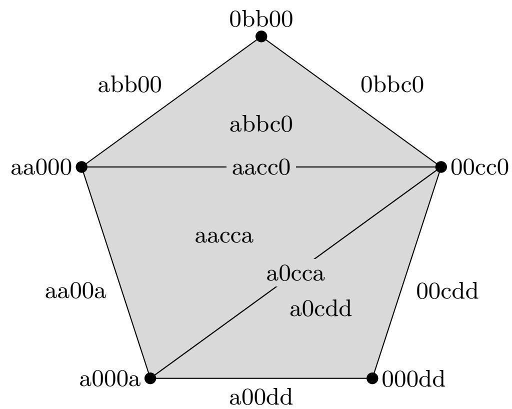
Question 6.0.4. Is C_ω^5 Cohen-Macaulay and $X^R(C_\omega^5)$ a resolution if and only if it can be written in the form



such that $a \leq b > c < d \geq a$ or $a < b \geq c \leq d > a$?

Our progress on Question 6.0.4 is as follows.

Remark 6.0.5. The support for $X^R(C_\omega^5)$ as in Question 6.0.4 is the Scarf simplicial complex $\Delta = \Delta(C_\omega^5)$ which can be visualized as follows:



$\Delta(C_\omega^5)$

We wish to show that $\Delta_{\preceq \mathbf{b}}$ is contractible or the empty simplicial complex for all $\mathbf{b} \in \mathbb{N}^5$. Essentially, there are two cases where the simplicial subcomplex $\Delta_{\preceq \mathbf{b}}$ is not contractible. First, if $\Delta_{\preceq \mathbf{b}}$ contains a triangle that does not bound a face of $\Delta_{\preceq \mathbf{b}}$; Second, if Δ is disconnected. The first case cannot occur because if $\Delta_{\preceq \mathbf{b}}$ is obtained by removing a single shaded in triangle $F = \{v_1, v_2, v_3\} \in \Delta$ then a vertex $\{v_i\} \subseteq F$ must also be removed by construction of the Scarf simplicial complex Δ and definition of "$\preceq$". So, all that remains to be shown is that $\Delta_{\preceq \mathbf{b}}$ is connected for all $\mathbf{b} \in \mathbb{N}^5$.

Appendices

Appendix A `Macaulay2` Functions

`Macaulay2` (`M2`) is an open-source computer algebra system designed to aid research in algebraic geometry and commutative algebra. We used `M2` to test our conjectures and compute examples. This appendix contains M2 functions that we wrote. We will demonstrate our code on a number of examples that closely resemble those from the body of the thesis, with the exception that the field K is set to $\mathbb{Q}$ in the computations. We refer the interested reader to `https://faculty.math.illinois.edu/Macaulay2/` to learn more about `M2` [7].

A.1 Edge Ideals

The following three functions are "shorthand" for defining edge ideals of graphs and weighted graphs of interest.

Paths

Name: `nPath`

Usage: `nPath(n,R)`

Input:

- `n`, an integer
- `R`, a polynomial ring

Output:

- an edge ideal of an n-path in `R`

Implementation:

```
nPath = (n,R) -> (
    -- error check
    if # (flatten entries vars R) <= n then error "expected # vars R > # edges";
    --mons := for i from 0 to n list x_i*x_(i+1);
    mons := for i from 0 to n-1 list (vars R)_(0,i)*(vars R)_(0,i+1);
    monomialIdeal mons
    );
```

Example:

We will define the edge ideal $I(P_5)$ in $R = \mathbb{Q}[x_1, \ldots, x_6]$ (compare to Example 3.1.7(a)).

```
i1 : load "main.m2"; -- loads function declarations


i2 : R = QQ[x_1..x_6];


i3 : I = nPath (5,R)


o3 = monomialIdeal (x x , x x , x x , x x , x x )
                     1 2   2 3   3 4   4 5   5 6


o3 : MonomialIdeal of R
```

Cycles

Name: `nCycle`

Usage: `nCycle(n,R)`

Input:

- `n`, an integer
- `R`, a polynomial ring

Output:

- an edge ideal of an n-cycle in `R`

Implementation:

```
nCycle = (n,R) -> (
    -- error check
    if #(flatten entries vars R) < n then error "expected # vars R >= # edges";
    mons := for i from 0 to n-1 list (vars R)_(0,i)*(vars R)_(0,(i+1)%n);
    monomialIdeal mons
    );
```

Example:

We will define the edge ideal $I(C_5)$ in $R = \mathbb{Q}[x_1, \ldots, x_5]$ (compare to Example 3.1.7(b)).

```
i4 : S = QQ[x_1..x_5];


i5 : J = nCycle (5,S)


o5 = monomialIdeal (x x , x x , x x , x x , x x )
                     1 2   2 3   3 4   1 5   4 5


o5 : MonomialIdeal of S
```

Weighted Cycles

Name: `wCycle`

Usage: `wCycle(L,R)`

Input:

- `omega`, weights of the edges
- `R`, a polynomial ring

Output:

- an edge ideal of a weighted n-Cycle in `R`

Implementation:

```
wCycle = (weights,R) -> (
    -- make sure the list has the same number of entries as the variables in R
     if #weights != #flatten entries vars R then error
     "expected exponent length to equal number of variables";
     -- generate monomials
     mons := for i from 0 to #weights-1 list ((vars R)_(0,i)*(vars R)_(0,(i+1)
     %(#weights)))^(weights_i);
     monomialIdeal mons
     )
```

Example:

Let $\omega = (1, 3, 2, 4, 1)$. We will define the edge ideal $I(C_\omega^5)$ of a weighted 5-cycle in $R = \mathbb{Q}[x_1, \ldots, x_5]$ (compare to Example 2.4.15).

```
i6 : omega = {1,3,2,4,1};


i7 : K = wCycle (omega,S)


                          3 3   2 2         4 4
o7 = monomialIdeal (x x , x x , x x , x x , x x )
                    1 2   2 3   3 4   1 5   4 5


o7 : MonomialIdeal of S
```

A.2 Exactness

The next function is "shorthand" for determining if an R-complex is a resolution.

Is Resolution

Name: `isRes`

Usage: `isRes X`

Input:

- `X`, a chain complex

Output:

- true, if X is a resolution

Implementation:

```
isRes = X -> (
    H := drop(apply(length X+1, i->HH_i X==0),1); -- nonzero homology
    if member(false,H) then false else true
    )
```

Example:

We will verify that $X^R(C^5_\omega)$ is a resolution. (compare to Example 2.4.15).

```
i8 : X = scarfComplex K


       1      5      7      3
o8 = S  <-- S  <-- S  <-- S  <-- 0


     0      1      2      3      4


o8 : ChainComplex


i9 : isRes X


o9 = true
```

Homology

Even when the Scarf complex fails to be a resolution, it is still valuable to know where it is exact. The command `HH_i X` returns the i^{th} homology $H_i(X)$. The function below is used extensively in Section 3.3.

Name: `hhList`

Usage: `hhList X`

Input:

- `X`, a chain complex

Output:

- `{H_0(X), H_1(X)..H_n(X)} = {0/1, 0/1..0/1}`, a sequence of 0's and 1's where `0` indicates no homology and `1` indicates homology

Implementation:

```
hhList = X -> (
    homList := apply(length X+1, i-> HH_i X == 0);
```

```
apply(homList, i-> if i == true then 0 else 1)
);
```

Example:

Next, we determine where $X^R(P_5)$ is exact (compare to Example 3.1.7(a) and line 5 in Figure 3.1)

```
i10 : Y = scarfComplex I


        1      5      7      2
o10 = R  <-- R  <-- R  <-- R  <-- 0


      0      1      2      3      4


o10 : ChainComplex


i11 : hhList Y


o11 = {1, 0, 1, 0}


o11 : List
```

A.3 Scarf Simplicial Complex

As we saw in Chapter 2, it can be extremely useful to sketch a geometric realization of the Scarf simplicial complex to give insight into the algebraic construction. The following command returns the faces and labels of the Scarf simplicial complex. For user convenience the vertices are labeled by integers. The faced are labeled by lcms (as usual).

Scarf Simplicial Complex

Name: `scarfSimplicialComplex`

Usage: `scarfSimplicialComplex I`

Input:

- `I`, a monomial ideal

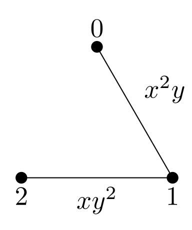
Output:

- Scarf simplicial complex with monomial labels, a list of lists

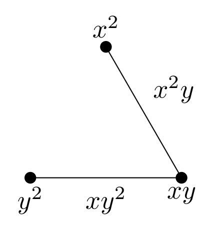
Implementation:

```
scarfSimplicialComplex = I -> (
    gensI = I_*; -- generating set for I
    nums = toList(0..(#(gensI)-1)); -- nums = list{0,1..n-1}
    -- compute list of {{f_(Lambda_1..f_(Lambda_n)},{Lambda_1..Lambda_n}
    labeledMonomials = apply(delete({},subsets(nums)), i ->
    {apply(i,j -> gensI_j), i});
    -- compute list of {f_Lambda,Lambda}
    fLambdaLambda = apply(labeledMonomials, i -> if #(i_1)>1 then
    {fold(lcm,i_0),i_1} else {(i_0)_0, i_1});
    fLambda = apply(fLambdaLambda, i->i_0); -- unlabeled list of f_Lambda
    -- return only distinct f_Lambda; i.e. if f_Lambda = f_Gamma then
    Lambda = Gamma;
    -- i.e. two monomials are the same iff they have the same label
    select(fLambdaLambda, i->number(fLambda,j->i_0==j)==1)
    )
```

Example: Let $I = \langle x^2, xy, y^2 \rangle \leq \mathbb{Q}[x, y]$ (compare to Example 2.4.6(b)). Sketching the geometric realization of the Scarf simplicial complex $\Delta(I)$ from the code below is typically a two-step process. First we label the vertices with integers and then replace those labels with the generators.



$\Delta(I)$ with integer vertex labels



$\Delta(I)$

To make the output more readable we will use the command `netList` which outputs a list of lists as a table.

```
i12 : SS = QQ[x,y];

i13 : II = monomialIdeal (x^2,x*y,y^2);

o13 : MonomialIdeal of SS

i14 : netList scarfSimplicialComplex II

      +----+------+
      | 2  |      |
o14 = |x   |{0}   |
      +----+------+
      |x*y |{1}   |
      +----+------+
      | 2  |      |
      |x y |{0, 1}|
      +----+------+
      | 2  |      |
      |y   |{2}   |
      +----+------+
      |   2|      |
      |x*y |{1, 2}|
      +----+------+
```

Exponent Vector

The next function is intended to be used in tandem with the previous command `scarfSimpicialComplex`. It replaces the monomial labels with their exponent vectors.

Name: `exps`

Usage: `exps Delta`

Input:

- `Delta`, Scarf simplicial complex (`scarfSimplicialComplex I`)

Output:

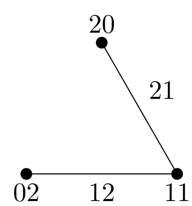
- Scarf simplicial complex with exponent vector labels.

Implementation:

```
exps = L -> (
    if #L_0>1 then apply(L, i-> {flatten exponents(i_0), i_1})
    else apply(L, i-> flatten exponents(i))
    )
```

Example:

Let $I = \langle x^2, xy, y^2 \rangle \leq \mathbb{Q}[x,y]$ (compare to Example 2.4.6(b)). In this example we sketch the geometric realization of the Scarf simplicial complex $\Delta(I)$ with exponent vector labels (compare to Example 2.4.6(b)).



$\Delta(I)$

```
i16 : netList exps scarfSimplicialComplex II

      +------+------+
o16 = |{2, 0}|{0}   |
      +------+------+
      |{1, 1}|{1}   |
      +------+------+
      |{2, 1}|{0, 1}|
      +------+------+
      |{0, 2}|{2}   |
      +------+------+
```

```
|{1, 2}|{1, 2}|
+------+------+
```

A.4 Tests

The first two tests were instrumental in producing Figures 3.1 and 3.2. That latter was helpful for testing our conjectures about weighted 5-cycles; in particular, see Theorem 4.3.6 and Question 6.0.4.

Path Test

Name: `pathTest`

Usage: `pathTest n`

Input:

- `n`, positive integer (number of iterations)

Output:

- A list of 2-tuples where the first component is the length of the n-path and the second component is a list $H_i(X^R(P_n)$ from 1 to n.

Implementation:

```
pathTest = n -> (
    if n < 1 then error "expected n >= 1";
    for i from 1 to n list(i,hhList scarfComplex nPath(i,QQ[x_1..x_(i+1)]))
    )
```

Example:

We will produce the first 5 lines of Figure 3.1; that is, we will compute where an n-path is exact for `n=1..5`.

```
i14 : netList pathTest 5


      +-----------------+
o14 = |(1, {1, 0})      |
```

```
+-----------------+
|(2, {1, 0, 0})   |
+-----------------+
|(3, {1, 0, 0})   |
+-----------------+
|(4, {1, 0, 1})   |
+-----------------+
|(5, {1, 0, 1, 0})|
+-----------------+
```

Cycle Test

Name: `cycleTest`

Usage: `cycleTest n`

Input:

- `n`, positive integer (number of iterations)

Output:

- A list of lists where the first component is the length of the n-cycle and the second component is a list $H_i(X^R(C_n)$ from 1 to n.

Implementation:

```
cycleTest = n -> (
    if n < 3 then error "expected n >= 3";
    for i from 3 to n list(i,hhList scarfComplex nCycle(i,QQ[x_1..x_i]))
    )
```

Example:

We will produce the first 3 lines of Figure 3.2; that is, we will compute where an n-cycle is exact for `n=3..5`.

```
i15 : netList cycleTest 5
```

```
      +--------------+
o15 = |(3, {1, 1})   |
      +--------------+
      |(4, {1, 0, 1})|
      +--------------+
      |(5, {1, 0, 1})|
      +--------------+
```

Weighted Cycle Test

Name: `wCycleTest`

Usage: `wCycleTest n`

Input:

- `n`, positive integer (number of iterations)

Output:

- $\{(\omega_1, \texttt{isRes}\ X^R(C^5_{\omega_1})), \ldots, (\omega_n, \texttt{isRes}\ X^R(C^5_{\omega_n}))\}$. The first component is a random weight ω such that C^5_ω is Cohen-Macaulay and the second component is true or false; that is, true if the Scarf complex is a resolution and false otherwise.

Implementation:

```
wCycleTest = n -> ( -- n = number of iterations
    R = QQ[x_1..x_5];
    L = {}; -- list of results
    for i from 0 to n-1 do (
    a = random(1,10);
    b = random(a,10);
    d = random(a,10);
    c = random(1,min(b,d));

    omega = {a,b,c,d,a}; -- weights
    I = wCycle(omega,R);
```

```
    L = append(L,(omega,isRes scarfComplex I)));
    net L
    );
```

Example:

Let ω be a random weight such that C^5_ω is Cohen-Macaulay (see Theorem 4.3.5). We will compute 3 examples to determine when $X^R(C^5_\omega)$ is a resolution.

```
i16 :
setRandomSeed(5)

o16 = 5

i17 : conjTest 3

o17 = {({9, 10, 9, 9, 9}, false), ({10, 10, 5, 10, 10}, true),
      ({2, 9, 3, 9, 2}, true)}
```

To see that the first example is not a resolution, consider the ranks of the Scarf complex.

```
i18 : load "main.m2"

i19 : R = QQ[x_1..x_5];

i20 : w = (9,10,9,9,9);

i21 : scarfComplex (wCycle(w,R))

       1      5      6      1
o22 = R  <-- R  <-- R  <-- R  <-- 0

      0      1      2      3      4

o22 : ChainComplex
```

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