August 2021

Dynamical and Spectral Inverse Problem for the Wave Equation

Antonio Marcello Pierrottet
Clemson University, wc3rocft@hotmail.com

Dynamical and Spectral Inverse Problem for the Wave Equation

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mathematics

---

by
Antonio Marcello Pierrottet
August 2021

Accepted by:
Dr. Shitao Liu, Committee Chair
Dr. Mishko Mitkovski
Dr. Jeong-Rock Yoon

Abstract

In this thesis we derive and give methods to solve the Dynamical Inverse Problem and the Spectral Inverse Problem for the one dimensional wave equation. In these problems, we have a semi-infinite spatial axis with constant wave speed and an unknown potential for the system. Given information about the system may only pertain to the boundary of the spatial axis. In the Dynamical Inverse Problem, we recover the potential from the Response Operator which represents the boundary measurement. In the Spectral Inverse Problem, we recover the potential from the Spectral Data of the Schrödinger Operator. The solution methods to both problems rely on the exact controllability of the underlying wave equation.

Acknowledgments

Thank you to my advisor and committee chair, Dr. liu, for his helpful critiques and guiding me through the process. As well to my committee members Dr. Mitkovski and Dr. Yoon, for taking the time to review this thesis. Also a special thanks to my mother who put herself through the headache of proofreading this thesis without any formal knowledge of the topic.

Table of Contents

Chapter 1

Introduction and Problem Formulation

1.1 Introduction

In this thesis we present two related inverse problems for the wave equation: the Dynamical Inverse Problem and the Spectral Inverse Problem. In these problems, we are given a medium and must determine the potential at every point along the medium. The two problems differ in how we gather information about the system. Physically speaking, one of the most visual of such problems is a wave in water with the potential being some set of objects floating on top. The wave doesn't travel through these objects but work is done to lift them up. For the sake of simplicity, we only explore the one dimensional case. These problems can also be solved in greater dimensions, however, certain properties are lost and greater effort must be taken to compensate for the loss.

In both of these problems we have a black box filled with some medium and under the effects of a potential. We cannot see inside the box nor can we see the potential but we can see what goes in and what comes out. We do know that the potential does not change. We find information from the boundaries via inducing a wave and "listening" for the response from the medium. From this, we try to determine what is the potential along any point along the medium. In the one dimensional case we focus on, we only assume to see one side of the boundary and allow the other to be infinity. This is known as the semi-infinite wave.

Waves follow a particular rule when traveling through a medium. Generally this only entails the equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.1}$$

where $c > 0$ is the speed the wave propagates through the medium, usually referred to as the *wave speed*. In some texts the *density* of the medium, ρ, is used in place of c with the relationship $\rho = {}^1\!/_{c^2}$. Modifications can be made to this equation by imposing different challenges to the wave. The one of interest to us is

$$u_{tt} - c^2 u_{xx} + q(x)u = 0. \tag{1.2}$$

Here the *potential*, q, is induced on the wave. This can come in many forms.

To begin, we look into the properties we can derive from the general wave solution from (1.2). We find the general solution to the wave equation first through finding the *fundamental solution*. From the fundamental solution and Dirichlet boundary conditions we can create an integral equation for (1.2) that can be solved iteratively. Given this new formulation for a solution we can show that we can control what the interior is be via the correct choice in boundary conditions. We then have the tools needed to create waves and now can start creating tools to read the potential.

The solution to the wave equation (1.2) with Dirichlet boundary conditions is well known to be solvable by the D'Alembert formula and with Duhamel's principle. This is solved through a double integral which makes it more difficult to work with. Finding the fundamental solution to the wave equation allows us to simplify the calculation to a Volterra integral equation of second kind. In general, Volterra integral equation of second kind has the following form,

$$f(t) = g(t) + \int_0^t k(s,t) f(s) ds,$$

where $g(t)$ is a given function and k is referred to as the kernel. Then Volterra integral equation of second kind are usually solved via some iterative methods, Euler's Method, recursion, etc..

For the general wave equation (1.2) we are trying to solve, we take the medium to be starting in a state of neutrality. That is the interior space, or *Inner Space*, has no waves present i.e.

$u(x,0) = u_t(x,0) = 0$. The Inner Space and its inner product are denoted as

$$\mathcal{H}^{cT} := L^2_c[0, cT] \tag{1.3}$$

$$\langle \varphi, \psi \rangle_{\mathcal{H}^{cT}} := \int_0^{cT} \frac{1}{c^2} \varphi(x)\psi(x)dx \qquad \text{for } \varphi, \psi \in \mathcal{H}^{cT}. \tag{1.4}$$

Then we control the amplitude of the wave at the boundary through the Dirichlet boundary condition, $u(0,t) = f(t)$, which is also called the *control function*, since this is what we are controlling. The set of boundary controls in which we can perform is also known as the *Outer Space*, as it is on the outside and visible to us. The Outer Space and its inner product are denoted as

$$\mathcal{F}^T := L^2[0, T]. \tag{1.5}$$

$$\langle f, g \rangle_{\mathcal{F}^T} := \int_0^T f(t)g(t)dt \qquad \text{for } f, g \in \mathcal{F}^T. \tag{1.6}$$

We send a signal for a certain time T. At that moment the truncation of the special wave is formed. This time T is referred to as the *final moment*. With these initial conditions, boundary conditions and (1.2) we have a full formulation of the PDE we wish to solve,

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) + q(x)u(x,t) = 0 & x \in (0,\infty),\ t \in [0,T] \\ u(x,0) = 0 \qquad u_t(x,0) = 0 & x \in (0,\infty) \\ u(0,t) = f(t) & t \in [0,T]. \end{cases} \tag{1.7}$$

We show that we can take our choice of control and convolute it with the fundamental solution to get the wave solution resulting from this control. The resulting operation is called the *Control Operator*, which is an operator that takes a choice of control function and outputs the solution to the wave equation at some time t. In particular, when $t = T$, the Control Operator is defined as,

$$W^T : \mathcal{F}^T \to \mathcal{H}^{cT}, \qquad f \mapsto u(\cdot, T). \tag{1.8}$$

Once we have the Control Operator we can examine the *controllability* of the system (1.7). By using properties the fundamental solution's tail passes on to solutions in $\mathcal{H}^{cT}$ and how the solution changes as the control changes we find that we can create any shape of wave in the Inner

Space. Not only that but there is precisely one control function that does this. This uniqueness gives us an invertible relationship between the Inner Space and the Outer Space. This property does not hold for higher dimensions as a wave may travel in a different axis then travel back towards the original axis ahead or directly affecting the original wave front.

With controllability we look to create a special kind of wave in the Inner Space. This wave has some properties that allow us to solve for the potential. The problem is that we cannot see the Inner Space directly. As such, we need a way to see what this wave is doing. Thus we introduce the *Truncation Operator*. For $\tau \in [0, T]$, we define

$$P^{c\tau} : \mathcal{H}^{cT} \to \mathcal{H}^{c\tau}, \qquad \varphi \mapsto \begin{cases} \varphi(x) & x \in [0, c\tau] \\ 0 & \text{else.} \end{cases} \tag{1.9}$$

This operator takes a function and sets it to be zero after a given point $c\tau$. We use τ as the time it takes for a signal to reach the point of truncation $c\tau$. The reason for doing this is to have a control doing only what is necessary in order to create this special wave up till the point of truncation. Then it takes a time of T for information about the wave front of the truncated special wave to return to us. That information is what we then use to solve for the potential, q, at the wave front at time T.

1.2 Dynamical Inverse Problem

In Chapter 3, we look into solving the Dynamical Inverse Problem. This section is dedicated to a general description of the problem and how to solve it. To set the stage, we can pick any control function and then receive the response from the medium. From that response we find an alternative way of measuring waves in the Inner Space without using the Inner Space directly. Then a basis of the Outer Space can be chosen such that it creates solutions in the Outer Space that are all orthonormal under the Outer Space's inner product. Then focus is shifted onto developing a relation giving us direct insight into the amplitude of a wave front. Lastly we impose a condition on the wave equation and find that it greatly simplifies any calculation it is involved in. From there we can relatively easily solve for the potential q.

In the Dynamical Inverse Problem, the way in which we gather information about the Inner Space is through how it is felt at the boundary. A signal is sent into the medium through the control function, interacts with the potential and comes back to the boundary. Since we are controlling the amplitude at the boundary, when the signal comes back it tries to move the amplitude up or down towards its position. How the medium reacts to our control function is called the *response function*, $u_x(0,t)$. We can derive a functional form of this operator steaming from the Control Operator, this is called the *Response Operator*. As the Response Operator is our source of information, it has a role to play in finding the solution to the potential. To precisely define the Response Operator,

$$R^T : \mathcal{F}^T \to \mathcal{F}^T, \qquad f \mapsto u_x^f(\cdot, T), \tag{1.10}$$

where u^f is the solution to (1.7) when $u(0,t) = f(t)$.

With the Response Operator we are in a position to formulate the Dynamical Inverse Problem. Given a constant wave speed, c, for any control function from the Outer Space we may know the corresponding response from the Response Operator. Thus the Dynamical Inverse Problem is to find the potential q from the knowledge gained by the Response Operator. Note that with the wave speed being c it takes T time to reach the point cT and then another T time to return to the Outer Space. Evidently we have to extend R^T to R^{2T} in order to gather the required knowledge to recover q on $[0, cT]$.

At this point we need to take two controls from the Outer Space, map them to their solution waves in the Inner Space then apply the inner product of the Inner Space, (1.4). With the Response Operator we can take our first steps away from the Inner Space. We create what is known as the *Connecting Operator*, which is defined as

$$C^T : \mathcal{F}^T \to \mathcal{F}^T, \qquad f \mapsto (W^T)^* W^T f \tag{1.11}$$

Where $(W^T)^*$ is the adjoint operator of W^T. This creates the connection between a control function and how it is measured in the inner product of the Inner Space. Through some clever representations we can find that the Connecting Operator can be represented by a manipulation of the Response Operator. Thus we can take elements of the Outer Space and interpret what the inner product of their solutions would be without the need to solve for the solutions themselves.

The Connecting Operator then allows us to take a basis of the Outer Space and look at it through the lens of the Inner Space. This allows us to take an arbitrary basis in the Outer Space and manipulate it as if we had the solutions. The key idea in this step is to have these functions be an orthonormal basis in the Inner Space. While this step is not explicitly necessary it does simplify any calculations going forward.

With that taken care of we can look towards some properties of the wave front. We can see that whenever our control function has a jump discontinuity it causes an equal and opposite jump in the wave's solution. Notably, when this jump happens at the start, $t = 0$, we get the same jump in the wave front. We call this relation the *Amplitude Formula*. Looking into how this comes about let us run a thought experiment with just one such jump. Just before the jump the control is sending a signal into the medium at one level. This signal has had some time to travel as a wave down the $x-$axis. Then occurs the jump in the control and the control continues sending a signal from this new second level. The wave sees this jump and then start to take on the signal from the second level of the control. Note that since the first level has more time to traverse the medium it is further in, farther down the $x-$axis, and the second level is closer to the boundary. As such, the wave shows the second level first then the first level second where as the control shows the first level first and second level second. Where one steps down the other steps up. This is the physical interpretation on the amplitude formula. The most useful aspect of this is when we start our control. The jump a control makes at the start is an equal and opposite jump happening at the wave front. Combine this with the Truncation Operator, mentioned previously, and we have a powerful tool to find the amplitude of any wave at any point. One may note that for an initial jump in the control, the wave front always has this equal and opposite jump. In other words, the wave front never changes in magnitude.

We have what we need to calculate the potential but there is some room to adjust our control functions in order to simplify our calculations. We first we would like to introduce what is known as the *Static Schrödinger Operator*,

$$\mathcal{A}\varphi(x) = -c^2\varphi_{xx}(x) + q(x)\varphi(x). \tag{1.12}$$

We pick our wave to be in the null space of (1.12). Using this with (1.2) we find that the wave solution has no acceleration at time T, i.e. $u_{tt}(x, T) = 0$. With the ability to eliminate this term we can greatly simplify the Connecting Operator. This is important as when we truncate to make our point of interest the wave front, taking our control and applying the Connecting Operator simplifies to a fixed linear function minus the response of this linear function. We even have some room to adjust the magnitudes of the two terms. Thus when we attempt to write this in the basis mentioned above, we can evaluate it relatively quickly. Not only that, but since $u_{tt}(x,T) = 0$ the wave equation itself simplifies and creates an excellent opportunity to find the value of the potential.

Finally, we are able to put all of this together to find the potential, q. Starting with an unknown potential, q, we probe the medium with any choice of control function. This allows us to find the formula for the Response Operator up to any given time T, specifically the values of the fundamental solution's tail for this medium. With this being known, we can use it to create the formulas for the Connecting Operator and the adjoint of the Response Operator. Now that the Connecting Operator is known for a given time, less then T, we can find a basis of functions for the Outer Space that maps to an orthonormal basis in the Inner Space. Next we know the form of Control Operators that maps to a solution in $\mathcal{H}^{cT}$ such that $u_{tt}(x,T) = 0$. The basis representation of this control, evaluated under a limit of t going to zero, gives us the initial jump. By the amplitude formula, this jump is the amplitude of the wave evaluated where our basis was truncated to. Repeat this until there is a satisfactory number of points known about $u(x,T)$. From that we may determine what is u_{xx} and use the simplified PDE to find that

$$q(x) = c^2 \frac{u_{xx}(x,T)}{u(x,T)}. \tag{1.13}$$

Thus the potential may be solved for any point x along the $x-$axis.

1.3 Spectral Inverse Problem

In Chapter 4 we examine finding the solution to the Spectral Inverse Problem. Here we are not allowed to apply any control to the medium but rather we are given some boundary data for eigenfunctions as well as eigenvalues of the Schrödinger Operator (1.12).

The Spectral Data for the Spectral Inverse Problem is given at the onset of the problem. To describe this we look to the Schrödinger Operator. This operator has eigenfunctions $\{\varphi_k\}$ and corresponding eigenvalues $\{\lambda_k\}$ for the fixed Final Time, T. The eigenfunctions are the functions that when passed through the Schrödinger Operator come out as the same function multiplied by a constant. Those functions are the eigenfunctions and the constant is known as its eigenvalue.

Relating this to the wave equation, these are going to be functions that the wave at time T is the negative of its second time derivative at time T. Having the Spectral Data is to know the response and eigenvalue of each eigenfunction but not the functions themselves.

It is well known from the Sturm-Liouville Theorem that the Schrödinger Operator (1.12), with zero boundary conditions $\varphi(0) = \varphi(cT) = 0$, has eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{\varphi_k\}$ satisfying the following properties

1. $\lambda_1 < \lambda_2 < \ldots$ with $\lim_{k\to\infty} \lambda_k = \infty$.

2. Each eigenfunction corresponding to the eigenvalue, λ_k is a unique (up to constant multiple). That eigenfunction φ_k has exactly $k-1$ zeros in $(0, cT)$.

3. The normalized eigenfunctions form an orthonormal basis under $\mathcal{H}^{cT}$.

The Spectral Inverse Problem is the task of recovering the value of $q(x)$ along every point in $[0, cT]$ from the *Spectral Data*,

$$\{\lambda_k,\ \varphi_k'(0)\}_{k=1}^{\infty}. \tag{1.14}$$

Also note, the eigenfunctions are different for different values of T so the information we start with would also be different. Since our information is static from the onset, so too is our Spectral Data. In the dynamical problem we were allowed to change the final moment and solve for q at any position $x < cT$, the distance the wave may travel before the final moment.

We show that from the Spectral Data we are able to compute the inner product of an eigenfunction with any other function in the Inner Space. We then form an orthonormal basis for the Inner Space using the Outer Space and the Spectral Data. From there we can find a formulation of how to solve for the value of an eigenfunction at any point, $\tau \in [0, cT]$. That allows us to find the value of the potential at any point in $[0, cT]$.

We can then show that the coefficients of the representation of a function in the Inner Space are described by the Spectral Data and the control that maps to that Inner Space function. Density arguments let us extend this definition to all of $\mathcal{H}^{cT}$. With any basis for the Outer Space, $\mathcal{F}^{\tau}$ with $\tau \in [0, T]$, we can find the value of the eigenfunction at the position $c\tau$. Then this can solve for the value of the potential at $c\tau$.

When we take the inner product of an eigenfunction with any other function in the Inner Space, $\mathcal{H}^{cT}$, some interesting properties emerge. We find that the inner product can be represented as a function of time. Not only that but this function of time can be written, not dependent on the eigenfunction, but rather of the eigenvalue. We find that the inner product of any function in the Inner Space with any eigenfunction can be written as a function of time, the corresponding Spectral Data $\{\lambda_k,\ \varphi_k'(0)\}$ and the control that produces the wave in question.

We then take the set of orthonormal eigenfunctions as our basis for the Outer Space, $\mathcal{H}^{cT}$. This makes it so that we may be able to write it as the sum of separable functions. Combined with the above representation of the inner product, we can show that any element of the Inner Space can be represented using all of the Spectral Data and the control used to generate the wave.

Next we evaluate one eigenfunction with another. In particular we focus on an eigenfunction with a truncated version of itself. We cannot find the control that maps to the eigenfunction as we do not have the eigenfunction itself. As such, we need to use a different way. We instead can use an intermediary. First we must pick the point, τ, to find the value of $\varphi(c\tau)$. We pick any basis of the Outer Space $\mathcal{F}^{\tau}$ and manipulate its elements to create an orthonormal basis of $\mathcal{H}^{c\tau}$. Using this basis as an intermediary we can represent the inner product of the eigenfunction with the truncated version of that eigenfunction to be represented by the Spectral Data and the basis for Outer Space $\mathcal{F}^{\tau}$. When we take the τ derivative of the inner product we find that we get the value of the eigenfunction at the point $c\tau$ divided by c.

The final stage is to put this all together to solve for the potential. First we would pick a time τ. Then we find a basis for the Outer Space $\mathcal{F}^{\tau}$. We turn this basis into one that maps into an orthonormal basis of $\mathcal{H}^{c\tau}$. We can then represent the point $c\tau$ of the first eigenfunction with the Spectral Data and the basis of the Inner Space. Then we can evaluate the Schrödinger Operator at the position $c\tau$ and solve for the potential at that point.

1.4 Literature and Notation

Both the dynamical and spectral inverse problems considered in thesis are standard formulations of inverse problems for the wave equation. The specific methods we use, which rely on the controllability of the wave equation, is inspired by the well known Boundary Control method that was first introduced by Belishev [1]. In particular, our treatment for the one dimensional situations are based on [2] and [5].

Also notice our dynamical inverse problems require essentially infinitely many boundary measurements as we are given the Response Operator. Another classical formulation of such dynamical inverse problem is to use only a single measurement from the boundary. However, such formulation typically requires non-vanishing initial conditions which are not in favor of practical applications. The main methodology used in such single measurement inverse problems is based on Carleman type estimates. The technique was originated in [3] and has been developed tremendously since then. For more details about the single measurement formulation, we refer to [4, 6] and the references therein.

Notationally, throughout this thesis we use T or cT when look at a fixed time or distance. It should be noted that we can vary T in most circumstances of the Dynamical Inverse Problem, before any calculations, to essentially make T into the variable t for the wave function. After T is selected and calculations are underway we use τ as the value of time we wish to examine.

In this thesis we take u^f to be the solution to the PDE with the BC $u(0,t) = f(t)$. A control function, $f \in \mathcal{F}^T$, is considered to only be supported on $[0,T]$ so for any $t \notin [0,T]$, $f(t) = 0$. Then a wave function, u^f, is considered to be supported on $\{(x,t) | 0 < t,\ 0 \leqslant x \leqslant ct\}$ and possibly $(0,0)$. Outside of that domain is beyond the influence of a control function due to the finite wave speed c.

All functions in this thesis are considered to be real valued functions. As such the complex conjugate is left off of the inner product. Unless otherwise stated, all function extensions have value zero on their extended values.

Chapter 2

Properties of the Wave Equation

In this chapter we work out some properties of the wave solution of (1.7) and a couple of operators that greatly assist in both the Dynamical and Spectral Inverse Problem. For this thesis the potential, $q(x)$, is assumed to be a continuous function on $[0, \infty)$, i.e. $q \in C[0, \infty)$. We go through the process of finding the fundamental solution and derive some crucial proprieties such as continuity and boundedness.

Once we know we have this solution to the PDE we can turn our attention to the aforementioned *Control Operator* (2.6), $W^T : \mathcal{F}^T \to \mathcal{H}^{cT}$, mapping a control function to its wave solution in the Inner Space. The fundamental solution allows us to use convolution to turn solving a PDE into an integral equation. We show that the Control Operator is a linear and continuous mapping. Later, through controllability, this mapping is also shown to be invertible.

Controllability, specifically *Exact Controllability*, says that for any function in the Inner Space, $\mathcal{H}^{cT}$, there is a function in the Outer Space, $\mathcal{F}^T$, that maps to that wave. In one dimension, this has the consequence of making the Control Operator invertible.

Lastly, we introduce the *Truncation Operator*. This operator is what makes it possible for us to read the value of the wave at the point which we truncate. Then we can subsequently find the value of the potential at that point. That is getting ahead of what we are doing in this chapter, but we do use this in the end phases of both the Dynamical and the Spectral Inverse Problem.

2.1 Wave Solution

To start off, we be show the derivation of solving the wave equation (1.7). For this chapter we consider the potential, $q \in C[0,\infty)$, to be a known function. It is only in the later chapters where we take the potential to be unknown but fixed.

Starting with the wave equation in (1.7), we consider splitting the wave solution, u, into the sum of two wave equations as the parts are much easier to solve than to find the solution directly. We set $u = v + w$. The first wave, v, we set the potential to zero, $q(x) = 0$. The second wave, w, is there to pick up the remainder of the wave, also known as the *tail* of the fundamental solution. The PDE of v is then described as below from (1.1),

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = 0 & x \in (0,\infty)\,, t \in [0,T] \\ v(x,0) = 0 \qquad v_t(x,0) = 0 & x \in (0,\infty) \\ v(0,t) = f(t) & t \in [0,T] \end{cases} \tag{2.1}$$

and it has the solution (denoted as v^f)

$$v^f(x,t) = f\left(t - \frac{x}{c}\right). \tag{2.2}$$

The solution to the PDE of v is well known to be supported on $A = \{(x,t) | 0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant ct\}$ due to the finite wave speed c. We can then take $u = v^f + w$, and using (2.2) we can rewrite (1.7) and simplify using (2.1) to find that w satisfies the following equation,

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = -q(x)\left(w(x,t) + v^f(x,t)\right) & x \in (0,\infty)\,, t \in [0,T] \\ w(x,0) = 0 \qquad w_t(x,0) = 0 & x \in (0,\infty) \\ w(0,t) = 0 & t \in [0,T]. \end{cases} \tag{2.3}$$

Applying the D'Alembert formula and Duhamel's principle we can find a Volterra integral equation of second kind to find w from,

$$w(x,t) = -\frac{1}{2c} \int\int_{K_c(x,t)} q(s)(w(s,\tau) + v^f(s,\tau))\,ds\,d\tau, \tag{2.4}$$

where

$$K_c(x,t) = \{(s,\tau)|t - \frac{x}{c} \leqslant \tau - \frac{s}{c},\ t - \frac{x}{c} \leqslant \tau + \frac{s}{c} \leqslant t + \frac{x}{c},\ 0 \leqslant \tau\},$$

a trapezium. It is well known that this equation has a unique solution [7] when the potential is a continuous function. Similarly to v^f we denote the solution to this as w^f.

We now know $u = w^f + v^f$ so we also denote the wave solution as u^f. We denote it in this fashion as the formulation of v^f, (2.2), depends on f, q and c. Then the formulation of w^f, (2.3), depends on these three and v^f. So the only choices we can make that change the solution is the choice of control f. Then we know that Volterra integral equation of second kind (2.3) gives us a unique solution, w^f. As such the right hand side of $u^f = w^f + v^f$ and thus is a unique solution to (1.7).

2.2 The Fundamental Solution

Here we derive the fundamental solution of (1.7) and some of its properties. We start by how we define the fundamental solution via limits. We can then evaluate the wave front v^δ and then the tail w^δ of the fundamental solution. Then we can show that the tail is continuous on a compact space which leads to it being bounded. Then using the support of w^δ we can get that there may be a jump at the front of the tail, $\left(x, \frac{x}{c}\right)$.

First we define the δ distribution in terms of a sequence of functions δ_n

$$f(t) = \delta(t) := \lim_{n\to\infty} \delta_n(t) := \lim_{n\to\infty} \begin{cases} n & t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{otherwise.} \end{cases}$$

When we find u^δ, we find it in terms of a limit of solutions, $u^\delta = \lim_{n\to\infty} u^{\delta_n}$. Splitting $u^{\delta_n} = w^{\delta_n} + v^{\delta_n}$ with (2.2) and (2.4) and taking the limit we can find what is known as the *fundamental solution*, u^δ. Using the limit definition on (2.2) we find that $v^\delta(x,t) = \delta\left(t - \frac{x}{c}\right)$ from

$$\lim_{n\to\infty} v^{\delta_n} = \lim_{n\to\infty} \delta_n \left(t - \frac{x}{c}\right) = \delta\left(t - \frac{x}{c}\right).$$

The components of u^δ are thought of as v^δ being the probing singularity at its wave front and w^δ as a tail resulting from the interaction of the singularity with the potential, $q(x)$.

A bit more work is needed to find more about the tail, w^δ. In the following theorem we need to make use of a property of Volterra equations of second kind; if $q(x)$ is continuous and w^δ is zero when $x = 0$ then w^δ is continuous. We are ready to derive our first property of the fundamental solution's tail.

Theorem 2.2.1. *w^δ is in the space $C^1(A)$ where $A = \{(x,t)|0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant ct\}$*

Proof. We can take δ_n and use it in (2.4). We know that q is continuous so it is bounded on K_c by some $M \geqslant 0$. Then $v^{\delta_n} = n\mathcal{X}_{E_n}$, where $E_n = \left\{(x,t)|0 < x,\ t \leqslant \frac{x}{c} \leqslant t + \frac{1}{n}\right\}$. With some algebra we are able to find that the measure of the space $K_c \bigcap E_n$ is less then or equal to $\frac{2x}{n}$. Then for any n we have that

$$\begin{aligned}\frac{1}{2c}\int\int_{K_c(x,t)} q(s)v^{\delta_n}(s,\tau)dsd\tau \leqslant & \frac{1}{2c}\int\int_{K_c(x,t)\bigcap E_n} Mndsd\tau \\ \leqslant & \frac{1}{2c}Mn\frac{2x}{n} \leqslant M\frac{cT}{c} = MT.\end{aligned}$$

So we have that the integral of $q \cdot v^{\delta_n}$ over K_c is uniformly bounded. Thus by the dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2c}\int\int_{K_c(x,t)} q(s)v^{\delta_n}(s,\tau)dsd\tau =& \frac{1}{2c}\int\int_{K_c(x,t)} q(s)\lim_{n\to\infty} v^{\delta_n}(s,\tau)dsd\tau \\ =& \frac{1}{2c}\int\int_{K_c(x,t)} q(s)\delta\left(\tau - \frac{s}{c}\right)dsd\tau \\ =& \frac{1}{2c}\int_{(ct-x)/2}^{(ct+x)/2} q(s) < \infty.\end{aligned}$$

Getting to the last line comes from looking at where K_c intersects with the line $\tau - \frac{s}{c} = 0$. By continuity of the integral, for every n, we have that $\frac{1}{2c}\int\int_{K_c} q(s)v^{\delta_n}(s,\tau)dsd\tau$ is a continuous function for $(x,t) \in (0,\infty) \times [0,T]$.

Now we can use this in (2.4) by moving the integral component involving w^{δ_n} to the left hand side and evaluating the limit. The right hand side converges by above. Thus the right hand side must also converge.

$$w^{\delta_n}(x,t) + \frac{1}{2c}\int\int_{K_c(x,t)} q(s)w^{\delta_n}(s,\tau)dsd\tau = -\frac{1}{2c}\int\int_{K_c(x,t)} q(s)v^{\delta_n}(s,\tau)dsd\tau$$
$$\lim_{n\to\infty} w^{\delta_n}(x,t) + \frac{1}{2c}\int\int_{K_c(x,t)} q(s)w^{\delta_n}(s,\tau)dsd\tau = -\frac{1}{2c}\int_{(ct-x)/2}^{(ct+x)/2} q(s)ds$$

Note that at $w^{\delta_n}(0,0) = 0$ by the bounds of integration on the right hand side. Since the right hand side is in $C^1(A)$ by Volterra integral equation theory, w^δ is also in $C^1(A)$. □

This Next step is vital in showing several of our operators are bounded and thus continuous. This follows almost directly from w^δ being continuous.

Corollary 2.2.1.1. *w^δ is a bounded function on K_c.*

Proof. From Theorem 2.2.1, w^δ is continuous on $A = \{(x,t)|0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant ct\}$. Because this is a compact domain, w^δ is bounded on A. Since K_c is a subset of A then w^δ is bounded on K_c. □

2.3 Control Operator

Now that we know how to solve the PDE we can turn our attention to the *Control Operator* which maps a control function to its wave solution in the Inner Space through the equation (1.7).

Definition 2.3.1. The Control Operator mapping an Outer Space to an Inner Space is given by

$$W^T : \mathcal{F}^T \to \mathcal{H}^{cT}$$
$$W^T f(x) := u^f(x,T)$$

When we use the Control Operator on a control f we use the same notation as we did for solving the wave solution, $W^T f = u^f$. We can use convolution with the fundamental solution and a control to turn the Control Operator from the act of solving a PDE into an integration problem. We show that this operator is a linear and continuous mapping. We also show the effects of a delay in the control function change the output range. Though not in this section, we use controllability to show this mapping is invertible.

Our next step is to find the explicit formula for this operator. In order to do that, we find

u^f for any $f \in \mathcal{F}^T$, using a well known superposition principal involving convolution,

$$u^f = u^{f*\delta} = u^\delta * f = (v^\delta + w^\delta) * f = v^\delta * f + w^\delta * f, \tag{2.5}$$

which then implies

$$\begin{aligned} u^f(x,t) =& \int_0^t \left(\delta\left(t - \frac{x}{c} - s\right) + w^\delta(x, t-s)\right) f(s) ds \\ =& f\left(t - \frac{x}{c}\right) + \int_0^{t-x/c} w^\delta(x, t-s) f(s) ds \end{aligned} \tag{2.6}$$

The bounds on the second line change since the w^δ is a piece-wise function with $w^\delta(x,t) = 0$ when $ct < x$ and the solution to (2.4) everywhere else. Thus we can rewrite the Control Operator, with $x \in [0, cT]$, as

$$\begin{aligned} W^T f(x) :=& u^f(x,T) = \left(v^\delta + u^\delta\right) * f(x) \\ =& f\left(T - \frac{x}{c}\right) + \int_0^{T-\frac{x}{c}} w^\delta(x, T-s) f(s) ds, \end{aligned} \tag{2.7}$$

essentially giving us the wave solution $u(x,T)$ for any finite time T.

The next few proofs are needed throughout the rest of this thesis and are relatively simple to show. Showing that W^T is a linear operator is especially important for having a control f and mapping it to u^f. Then u^f can be rewritten in terms of a basis there. Linearity allow us to use those same coefficients for the set of Outer Space controls that map to that basis in the Inner Space.

Theorem 2.3.2. *The Control Operator, W^T, is a linear operator.*

Proof. Let $f, g \in \mathcal{F}^T$ and $\alpha,\ \beta \in \mathbb{R}$. Then using (2.6),

$$\begin{aligned} W^T(\alpha f + \beta g)(t) =& \left(\alpha f\left(t - \frac{x}{c}\right) + \beta g\left(t - \frac{x}{c}\right)\right) + \int_0^{t-x/c} w^\delta(x, t-s)\left(\alpha f(s) + \beta g(s)\right) ds \\ =& \alpha\left(f\left(t - \frac{x}{c}\right) + \int_0^{t-x/c} w^\delta(x, t-s) f(s) ds\right) + \beta\left(g\left(t - \frac{x}{c}\right) + \int_0^{t-x/c} w^\delta(x, t-s) g(s) ds\right) \\ =& \alpha W^T f(t) + \beta W^T g(t). \end{aligned}$$

Hence we can see that W^T is a linear operator. □

The following theorem is used later in the proving the Amplitude Formula and in extending a representation in the Spectral Problem.

Theorem 2.3.3. *The Control Operator, W^T, is a continuous operator.*

Proof. From Corollary 2.2.1.1 we have that w^δ is bounded by some $M \geqslant 0$. Let $f \in \mathcal{F}^T$ then we have,

$$\begin{aligned}
\|W^T f(x)\|^2_{\mathcal{H}^{cT}} =& \left\| f\left(T - \frac{x}{c}\right) + \int_0^{T-x/c} w^\delta(x, T-\tau) f(\tau) d\tau \right\|^2_{\mathcal{H}^{cT}} \\
\leqslant& 2\left\| f\left(T - \frac{x}{c}\right)\right\|^2_{\mathcal{H}^{cT}} + 2\left\| \int_0^{T-x/c} |w^\delta(x, T-\tau)||f(\tau)| d\tau \right\|^2_{\mathcal{H}^{cT}} \\
\leqslant& 2\int_0^{cT} f^2\left(T - \frac{x}{c}\right)\frac{1}{c^2} ds + 2M \left\| \int_0^{T-x/c} |f(\tau)| d\tau \right\|^2_{\mathcal{H}^{cT}} \\
\leqslant& 2\int_T^0 -f^2(\tau)\frac{1}{c} d\tau + 2M \left\| \int_0^{T} |f(\tau)| d\tau \right\|^2_{\mathcal{H}^{cT}} \\
=& \frac{2}{c}\int_0^T f^2(\tau) d\tau + 2M \left\| \int_0^{T} |f(\tau)| * 1 d\tau \right\|^2_{\mathcal{H}^{cT}} \\
\leqslant& \frac{2}{c}\|f\|^2_{\mathcal{F}^T} + 2M \left\| \ \|f\|_{\mathcal{F}^T}\|1\|_{\mathcal{F}^T} \right\|^2_{\mathcal{H}^{cT}} \\
=& 2\|f\|^2_{\mathcal{F}^T}\left(\frac{1}{c} + 2M\|1\|_{\mathcal{F}^T}\ \|1\|^2_{\mathcal{H}^{cT}}\right) \\
=& \|f\|^2_{\mathcal{F}^T}\left(\frac{2}{c} + 2MT\frac{cT}{c^2}\right) = \|f\|^2_{\mathcal{F}^T}\frac{2+2MT^2}{c}
\end{aligned}$$

Thus W^T is a bounded operator so we get that W^T is a continuous operator from $\mathcal{F}^T$ to $\mathcal{H}^{cT}$. $\square$

Later we use the following theorem to create an equality in the following corollary. From that we create a useful basis for the domain. One that is made of delayed control operators.

Theorem 2.3.4. $W^{T-\tau} f = W^T f_\tau$ *where* $f_\tau(t) := f(t-\tau)$.

Proof. Fix $\tau \in (0, T)$ and $f \in \mathcal{F}^T$ with support on $[\tau, T]$. With $W^T f = u^f$ and $f_\tau(t) := f(t-\tau)$ used together with (2.7) we have

$$\begin{aligned}
W^T f_\tau(x) =& f_\tau\left(T - \frac{x}{c}\right) + \int_0^{T-\frac{x}{c}} w^\delta(x, T-s) f_\tau(s) ds \\
=& f\left(T - \frac{x}{c} - \tau\right) + \int_0^{T-\frac{x}{c}} w^\delta(x, T-s) f(s-\tau) ds \\
=& f\left(T - \frac{x}{c} - \tau\right) + \int_{-\tau}^{T-\frac{x}{c}-\tau} w^\delta(x, T-y-\tau) f(y) dy \\
=& f\left([T-\tau] - \frac{x}{c}\right) + \int_0^{[T-\tau]-\frac{x}{c}} w^\delta(x, [T-\tau]-y) f(y) dy \\
=& W^{T-\tau} f(x).
\end{aligned}$$

□

As we continue to use the Truncation Operator it behooves us to define particular space of truncated of control functions.

Definition 2.3.5. The τ *Semitruncated Outer Space* of $\mathcal{F}^T$ is defined by

$$\mathcal{F}^{T,\tau} = \{f \in \mathcal{F}^T | f \text{ is supported on } [T-\tau, T].$$

Corollary 2.3.5.1. *$W^T \mathcal{F}^{T,\tau} \subset \mathcal{H}^{c\tau}$ where $\mathcal{F}^{T,\tau} := \{f \in \mathcal{F}^T |\ f$ is supported on $[T-\tau, T]\}$.*

Proof. Let $f \in \mathcal{F}^{t,\tau}$. Since f is supported on $[T-\tau, T]$ then we may denote $f = g_{T-\tau}$. Then let $g \in \mathcal{F}^\tau$ such that $g = g_{T-\tau}$. By Theorem 2.3.4 $W^T f = W^T g_{T-\tau}(x) = W^{T-\tau} g = u(x, T-\tau) \in \mathcal{H}^{c(T-\tau)}$. Hence we have that $W^T \mathcal{F}^{T,\tau} \subset \mathcal{H}^{c\tau}$. □

2.4 Exact Controllability

Here we show that we have *Exact Controllability* for the wave equation (1.7). Exact Controllability says that for a given final moment, T, and any function, φ, in the Inner Space, $\mathcal{H}^{cT}$, there is a function, f, in the Outer Space, $\mathcal{F}^T$, that maps to that wave, $W^T f = \varphi$. In one dimension, this has the consequence of making the Control Operator invertible.

This is the most used important property in solving these inverse problems. It allows us to take the Outer Space $\mathcal{H}^{cT}$ and represent it with the Inner Space $\mathcal{F}^T$. It is a key element in writing the inner product of two waves in the Inner Space as the inner product of two controls in the Outer Space. It also helps in seeing where the Truncation Operator maps to.

At time T, for any $x > cT$, any wave u is be beyond the influence of the control function, $f(t)$. So the only influence on u comes from the boundary conditions, $u(x,0) = u_t(x,0) = 0$ for $x > 0$. We know that $u(\cdot,T)$ is supported on $x \in [0,cT]$, i.e. $u(\cdot,T) \in \mathcal{H}^{cT}$. Naturally, we choose to look in $\mathcal{F}^T$ to find a control mapping to this wave $u \in \mathcal{H}^{cT}$ as an initial influence can only reach as far as cT at time T and the final influences of f are at the start of u.

Theorem 2.4.1 (Exact Controllability)**.** *For all φ in the Inner Space $\mathcal{H}^{cT}$ there exists a control function, f in the Outer Space $\mathcal{F}^T$ such that $W^T f(x) = u^f(x,T) = \varphi(x)$.*

Proof. Fix $\varphi \in \mathcal{H}^{cT}$. Consider the control function $\delta \in \mathcal{F}$. Since (2.4) gives us a unique solution for any $f \in \mathcal{F}^T$, then there exists a unique $u^\delta \in \mathcal{H}^{cT}$ *s.t.* $W^T\delta = u^\delta$.We then can use (2.7) and a change of variables $x = c(T-t)$ to rewrite the Control Operator as a function of t,

$$u^f(c(T-t),T) = f(t) + \int_0^t w^\delta(c(T-t),T-s)f(s)ds.$$

This is also a Volterra equation of second kind when solving for f. It is known that such an equation has a unique solution, f, since w^δ is a unique solution to (2.3) and is continuous by (2.2.1), as shown in [7]. Placing φ in for u^f we find a unique $f \in \mathcal{F}^T$ such that the equation is satisfied. Hence f is the unique function such that $W^T f(x) = u(x,T) = \varphi(x)$. □

From controllability we have that W^T maps $\mathcal{F}^T$ onto $\mathcal{H}^{cT}$. Not only that, but because of uniqueness from the Volterra equation of second kind, we also have that the f that maps to $u \in \mathcal{H}^{cT}$ uniquely. That allows us to say that W^T is also one to one. Thus W^T is bijective and hence invertible for the one dimensional case.

2.5 Truncation Operator

We create the *Truncation Operator* to set the front $c\tau$ of a wave to zero. This is important as we can gather information from the wave front. The major issue is that to find information about a point $x < cT$, where x may not be at the front of the wave.

Definition 2.5.1. The Truncation Operator for an Inner Space is given by

$$\mathcal{H}^{cT} \to \mathcal{H}^{c\tau}$$
$$\left(P^{c\tau}u\right)(x) := \begin{cases} u(x,T) & x \in [0, c\tau] \\ 0 & x \in (c\tau, cT] \end{cases}$$

To do this we project $\mathcal{H}^{cT}$ to $\mathcal{H}^{c\tau}$ where $\tau \in (0, cT)$. τ now becomes the new wave front.

The Truncation Operator is an orthogonal projection of $\mathcal{H}^{cT}$ to $\mathcal{H}^{c\tau}$. The wave front of this projection is τ. The null space of this operator is supported on $(c\tau, cT]$ and the range on $[0, c\tau]$. So the $\mathcal{H}^{cT}$ inner product of the two spaces are zero. It can easily be seen that their direct sum is $\mathcal{H}^{cT}$.

Now we look into what control function maps to the range $\mathcal{H}^{c\tau}$.

Theorem 2.5.2. $W^T \mathcal{F}^{T,\tau} = \mathcal{H}^{c\tau}$.

Proof. Let $u \in \mathcal{H}^{c\tau}$. Then Theorem 2.4.1 gives us that there exists a unique $f \in \mathcal{F}^\tau$ such that $W^\tau f = u(\cdot, \tau)$. By Theorem 2.3.4 $W^T f = W^\tau f_{T-\tau}$ where $f_{T-\tau}(t) := f(t - T + \tau)$. Thus $\mathcal{H}^{c\tau} \subset W^\tau \mathcal{F}^T$. By Corollary 2.3.5.1, $W^T \mathcal{F}^{T,\tau} \subset \mathcal{H}^{c\tau}$ and hence $W^T \mathcal{F}^{T,\tau} = \mathcal{H}^{c\tau}$. □

In Theorem 2.4.1 we mention that the mapping is invertible. As such we can write a *Semi-Truncation Operator* for the Outer Space,

Definition 2.5.3. The Semi-Truncation Operator for an Outer Space is given by

$$\mathcal{P}^\tau : \mathcal{F}^T \to \mathcal{F}^{T,\tau} \mathcal{P}^\tau f := \left(W^T\right)^{-1} P^{c\tau} W^T f.$$

Chapter 3

Dynamical Inverse Problem

In this chapter we dig into solving the Dynamical Inverse Problem. There is an unknown potential, q, acting on a medium that we would like to find. We can set any Dirichlet boundary condition $f \in \mathcal{F}^T$ for $u(0,t) = f(t)$. We then receive feedback from the medium by through the Response Operator $R^{2T} f(t) = u_x(0,t)$ for any $t \in [0, 2T]$. From this information we must reconstruct the potential q.

Before going into the details, we first give an interview of how the entire process works. We begin by writing a form for the response of the medium. We can use the control operator to calculate a specific form for the Response Operator. Then we derive some properties of the Response Operator, R^T, the extension of the R^T to R^{2T} and its adjoint.

Next we look at the *Connecting Operator*. The purpose of the Connecting Operator is to find the inner product of two waves in the Inner Space (1.4) by using the inner product of controls that map to them in the Outer Space (1.6). It is defined from the Control Operator. The main focus we have is to represent it in terms of the Response Operator, R^{2T}, that we know.

With all of these operators ready to use, we move to the tools needed for solving the Dynamical Inverse Problem. The first thing we need to do is to represent an orthonormal basis of the Inner Space with elements of the Outer Space. We also need to develop the Amplitude Formula. The concept of this formula is that we can induce a jump on the wave via the Truncation Operator and the Amplitude Formula tells us how to evaluate that jump using the control mapping to the truncated wave. We also impose the restriction on the waves we create to be in the Null Space of the Static Schrödinger Operator (1.12). This restriction on the inner space allows us to simplify the

Connecting Operator, of which we use several times throughout the process.

Lastly, we combine all these to create a method of solving the Dynamical Inverse Problem. To reiterate, we are to find an unknown potential, q, acting on the wave. Starting by probing the medium with a control we receive the response. With these two we can use the Response Operator to determine the x derivative of the fundamental solution's tail, w_x^δ. This is then used to complete the formulation of the Connecting Operator and the adjoint of the Response Operator. Next the Connecting Operator can create an orthonormal basis of the Inner Space $\mathcal{H}^{c\tau}$ by using a basis from $\mathcal{F}^{T,\tau}$. Then we can rewrite our specially chosen Inner Space wave, from the imposed restriction, truncated at τ with the previously mentioned basis. Using the Amplitude Formula we can evaluate the infinite sum as t goes to 0^+ to find the value of the wave at $x = c\tau$. Repeating the process at several points allows us to find or approximate $u_{xx}(c\tau, T)$ for some $\tau \in (0, T)$. Then we can use the imposed restriction on the wave to find q at the point $c\tau$. The details are explained further in Section 3.6.

3.1 Response Operator

The response of the medium is defined to be $u_x(0,t)$. When a control is applied to the medium a response is given in turn. As such we strive to find a formulation of the Response Operator from the control f. We show this operator is continuous. Then derive a formulation for its adjoint. Also showing that it is a continuous operator.

Since we receive the response function from f we are motivated to rewrite the response of the wave function, $u_x^f(0,t)$, as a functional of f

Definition 3.1.1. The Response Operator is a mapping from the Outer Space to the Outer Space given by

$$R^T f(t) : \mathcal{F}^T \to \mathcal{F}^T$$
$$R^T f(t) := u_x^f(0,t), \qquad t \in [0,T]$$

Our first step is to put this in terms of the control function. This relation becomes invaluable in the initial steps of finding the potential.

Theorem 3.1.2.

$$R^T f(t) = -\frac{1}{c}f'(t) + \int_0^t w_x^\delta(0,t-s)f(s)ds$$

Proof.

$$\begin{aligned}
R^T f(t) :=& u_x^f(0,t) = u_x^f(x,t)\big|_{x=0} = \left(\frac{d}{dx}u^f(x,t)\right)\bigg|_{x=0} = \left(\frac{d}{dx}W^t f(x)\right)\bigg|_{x=0} \\
=& \left(\frac{d}{dx}\left(f\left(t-\frac{x}{c}\right) + \int_0^{t-\frac{x}{c}} w^\delta(x,t-s)f(s)ds\right)\right)\bigg|_{x=0} \qquad (2.7) \\
=& \left(-\frac{1}{c}\frac{d}{dx}f\left(t-\frac{x}{c}\right) + w^\delta\left(x,t-t+\frac{x}{c}\right)f\left(t-\frac{x}{c}\right) + \int_0^{t-\frac{x}{c}} w_x^\delta(x,t-s)f(s)ds\right)\bigg|_{x=0} \\
=& -\frac{1}{c}f'(t) + w^\delta(0,0)f(t) + \int_0^t w_x^\delta(0,t-s)f(s)ds \\
=& -\frac{1}{c}f'(t) + \int_0^t w_x^\delta(0,t-s)f(s)ds \qquad w^\delta(0,0) = 0 \text{ by BC}
\end{aligned}$$

The third line follows from the equality

$$\frac{d}{dx}\int_0^{g(x)} f(x,t)dt = f(x,g(x^+)) + \int_0^{g(x)} f_x(x,t)dt.$$

Thus we have the desired form of the Response Operator. □

Next we show that the Response Operator is a continuous operator for the purposes of expanding the use of the Connecting Operator through a density argument. It relies on the tail of the fundamental solution being a continuous function.

Theorem 3.1.3. *R^T is a continuous operator on $H^1[0,cT]$.*

Proof. Let $f \in H^1[0,cT]$. By Corollary 2.2.1.1 w_x^δ is bounded by some $M \in \mathbb{R}$ on the set $\{(x,t)|0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant ct\}$. Starting with the formula stated in Theorem 3.1.2,

$$\begin{aligned}
\|R^T f(t)\|_{\mathcal{F}^T} &= \left\|-\frac{1}{c}f'(t) + \int_0^t w_x^\delta(0,t-s)f(s)ds\right\|_{\mathcal{F}^T} \\
&\leqslant \left\|-\frac{1}{c}f'(t)\right\|_{\mathcal{F}^T} + \left\|\int_0^t |w_x^\delta(0,t-s)||f(s)|ds\right\|_{\mathcal{F}^T} \leqslant \frac{1}{c}\left\|f'\right\|_{\mathcal{F}^T} + M\left\|\int_0^t |f(s)|ds\right\|_{\mathcal{F}^T}
\end{aligned}$$

$$\leqslant \frac{1}{c}\left\|f'\right\|_{\mathcal{F}^T} + M\left\| \ \|f\|_{\mathcal{F}^T}\right\|_{\mathcal{F}^T} = \frac{1}{c}\left\|f'\right\|_{\mathcal{F}^T} + MT\|f\|_{\mathcal{F}^T}$$

$$\leqslant \frac{1}{c}\left\|f\right\|_{H^1} + MT\|f\|_{H^1} = \left(\frac{1}{c} + MT\right)\|f\|_{H^1}$$

As such, R^T is a bounded operator on $H^1[0, cT]$. Hence R^T is a continuous operator on $H^1[0, cT]$.

□

Continuing to with this operator, we would also like to find its adjoint operator $(R^T)^*$. By definition of an adjoint operator for all $f, g \in \mathcal{F}^T$,

$$\begin{aligned}
\langle (R^T)^* f, g\rangle_{\mathcal{F}^T} :=& \langle f, R^T g\rangle_{\mathcal{F}^T} = \int_0^T f(\tau)\left[-\frac{1}{c}g'(\tau) + \int_0^\tau w_x^\delta(0, \tau - s)g(s)ds\right]d\tau \\
=& \int_0^T f(\tau)\left[-\frac{1}{c}g'(\tau) + \int_0^\tau w_x^\delta(0, \tau - s)g(s)ds\right]d\tau \\
=& -\frac{1}{c}\int_0^T f(\tau)g'(\tau)d\tau + \int_0^T\int_0^\tau w_x^\delta(0, \tau - s)f(\tau)g(s)dsd\tau \\
=& -\frac{1}{c}\left[f(\tau)g(\tau)|_0^T - \int_0^T f'(\tau)g(\tau)d\tau\right] + \int_0^T\int_s^T w_x^\delta(0, \tau - s)f(\tau)g(s)d\tau ds \\
=& -\frac{1}{c}\left(f(T)g(T) - f(0)g(0)\right) + \int_0^T \frac{1}{c}f'(\tau)g(\tau)d\tau + \int_0^T g(s)\int_s^T w_x^\delta(0, \tau - s)f(\tau)d\tau ds \\
=& \int_0^T\left[\frac{1}{c}f'(\tau) + \int_\tau^T w_x^\delta(0, s - \tau)f(s)ds\right]g(\tau)d\tau
\end{aligned}$$

For the last line, f and g vanish at the boundary so the first term goes away. Thus we have a formula for the adjoint operator of R^T given by

$$(R^T)^* f = \frac{1}{c}f'(\tau) + \int_\tau^T w_x^\delta(0, s - \tau)f(s)ds \tag{3.1}$$

Then, similarly to showing that the Response Operator was continuous in Theorem 3.1.3, we show that its adjoint is also continuous.

Theorem 3.1.4. $(R^T)^*$ *is a continuous operator on* $H^1[0, cT]$.

Proof. Let $f \in H^1[0, cT]$. By Theorem 2.2.1 for all $(x, t) \in [0, ct) \times [0, T]$, $\|w_x^\delta\| \leqslant M \in \mathbb{R}$. Then from the formulation of the Response Adjoint Operator (3.1),

$$\begin{aligned}\|(R^T)^* f\|_{\mathcal{F}^T} &= \left\| \frac{1}{c} f'(t) + \int_t^T w_x^\delta(0, t-s) f(s) ds \right\|_{\mathcal{F}^T} \\ &\leqslant \left\| \frac{1}{c} f'(t) \right\|_{\mathcal{F}^T} + \left\| \int_t^T |w_x^\delta(0, t-s)||f(s)| ds \right\|_{\mathcal{F}^T} \leqslant \frac{1}{c} \|f'\|_{\mathcal{F}^T} + M \left\| \int_t^T |f(s)| ds \right\|_{\mathcal{F}^T} \\ &\leqslant \frac{1}{c} \|f'\|_{\mathcal{F}^T} + M \left\| \, \|f\|_{\mathcal{F}^T} \right\|_{\mathcal{F}^T} = \frac{1}{c} \|f'\|_{\mathcal{F}^T} + MT \|f\|_{\mathcal{F}^T} \\ &\leqslant \frac{1}{c} \|f\|_{H^1} + MT \|f\|_{H^1} = \left(\frac{1}{c} + MT \right) \|f\|_{H^1}\end{aligned}$$

As such, $(R^T)^*$ is a bounded operator on $H^1[0, cT]$. Hence $(R^T)^*$ is a continuous operator on $H^1[0, cT]$. □

As noted in Chapter 1, we need to extend the range of t from $[0, T]$ to be $[0, 2T]$. $f(t)$ is defined for all $t \in [0, T]$. To extend it to all of $[0, 2T]$ we give it value zero from $[T, 2T]$ and apply the *Odd Extension Operator* to it. Whenever we extend a control function with the Odd Extension Operator it is written with the Odd Extension Operator. All other extensions are the zero extension.

Definition 3.1.5. The Odd Extension Operator is defined by

$$S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$$

$$S^T f(t) := \begin{cases} f(t) & t \in (0, T) \\ -f(2T - t) & t \in (T, 2T) \end{cases}$$

If we take (1.7) with the t domain is doubled to $[0, 2T]$ then we get the Response Operator on $[0, 2T]$ denoted by R^{2T}. We can use the same results we have already achieved for R^T and apply them here. While we are still using R^{2T} with $S^T f$ its domain is $\mathcal{F}^{2T}$. R^{2T} is a continuous operator and has the formula defined by

$$R^{2T} f(t) = -\frac{1}{c} f'(t) + \int_0^t w_x^\delta(0, t-s) f(s) ds \qquad t \in (0, 2T). \tag{3.2}$$

While we are on the topic, it is simple to show the adjoint of the Odd Extension Operator is

$$\left(S^T\right)^* f(t) = f(t) - f(2T - t), \qquad t \in [0, T]. \tag{3.3}$$

We still have one last thing that needs to be done with the reply functional. That is, we need to show that it commutes with the following *Integration Operator*.

Definition 3.1.6. The Integration Operator is defined by

$$\begin{aligned} &J^T : \mathcal{F}^T \to \mathcal{F}^T \\ &J^T f(t) := \int_0^t f(\tau) d\tau, \qquad t \in [0, T]. \end{aligned}$$

Lemma 3.1.7. *For all f in the Outer Space $H^1[0, cT]$ such that $f(0) = 0$, $R^T J^T f(t) = R^T J^T f(t)$.*

Proof. Let $f \in H^1[0, cT]$. Then we find

$$\begin{aligned} R^T J^T f(t) =& R^T \int_0^t f(y) dy \\ =& -\frac{1}{c}\frac{d}{dt}\int_0^t f(y)dy + \int_0^t w_x^\delta(0, t-s)\int_0^s f(y)dyds \\ =& -\frac{1}{c} f(y)\bigg|_0^t + \int_0^s \int_0^t w_x^\delta(0, t-s) f(y) dsdy \\ =& -\frac{1}{c} f(t) + \frac{1}{c} f(0) + \int_0^s \int_0^t w_x^\delta(0, t-s) f(y) dsdy \\ =& \int_0^t -\frac{1}{c} f'(s) ds + \int_0^t \int_0^s w_x^\delta(0, t-s) f(y) dyds \\ =& \int_0^t \left[-\frac{1}{c} f'(s) ds + \int_0^s w_x^\delta(0, t-s) f(y) dy\right] ds \\ =& \int_0^t R^T f(s) ds = J^T R^T f(t) \end{aligned}$$

□

3.2 Connecting Operator

Next we look at the *Connecting Operator*. The purpose of the Connecting Operator is to find the inner product of two waves in the Inner Space (1.4) by using the inner product of controls that map to them in the Outer Space (1.6). It is defined from the Control Operator. The main goal of this section is to get this operator in terms of the Response Operator. Most of what we need has already been proven for us in Section 3.1. We only need to find the relation.

Definition 3.2.1. The Connecting Operator is defined by

$$C^T : \mathcal{F}^T \to \mathcal{F}^T \qquad\qquad C^T f = (W^T)^* W^T f. \tag{3.4}$$

We can easily get this in terms of the Control Operator,

$$\langle C^T f, g\rangle_{\mathcal{F}^T} = \langle W^T f, W^T g\rangle_{\mathcal{H}^{cT}} = \langle u^f, u^g\rangle_{\mathcal{H}^{cT}}.$$

So we have

$$\langle C^T f, g\rangle_{\mathcal{F}^T} = \langle u^f, u^g\rangle_{\mathcal{H}^{cT}},$$

for all $f, g \in \mathcal{F}^T$.

There are two methods of finding a formula for C^T. The first is that if we can show what $(W^T)^*$ is then we have a formula for $C^T = (W^T)^* W^T$. The second is to find C^T in terms of other operators. We use the latter as we can put it in terms of $(R^T)^*$. As such, we demonstrate the second method here.

To do this we need to use the three operators mentioned at the end of Section 3.1; the *Integration Operator* (3.1.6) the *Odd Extension Operator* (3.1.5) and the adjoint of the Odd Extension Operator (3.3). We also use some of the properties proven about these operators too.

Theorem 3.2.2. *$\forall f \in \mathcal{F}^T$ with $f(0) = 0$ then*

$$C^T f(t) = -\frac{1}{2}(S^T)^* R^{2T} J^{2T} S^T f(t)$$

Proof. Fix $f, g \in \mathcal{F}^T$ such that $f(T) = 0 = f(0) = g(0)$. We define $f_-(t) := S^T f(t)$. So $W^{2T} f_-(x) = u^{f_-}(x, 2T)$ and $W^T g(x) = u^g(x, T)$. Now to help illustrate the relation we create the function, $b(s, t)$, defined on the domain $B^{2T} := \{(s,t)|0 \leqslant t \leqslant T,\ t \leqslant s \leqslant 2T-t\}$. For all $(s,t) \in B^{2T}$, $b(s,t)$ is defined as

$$b(s,t) := \int_0^\infty u^{f_-}(x,s)u^g(x,t)\frac{1}{c^2}dx.$$

Consider the following relation,

$$\begin{aligned}
b_{tt}(s,t) - b_{ss}(s,t) =& \frac{d^2}{dt^2}\int_0^\infty u^{f_-}(x,s)u^g(x,t)\frac{1}{c^2}dx - \frac{d^2}{ds^2}\int_0^\infty u^{f_-}(x,s)u^g(x,t)\frac{1}{c^2}dx \\
=& \int_0^\infty u^{f_-}(x,s)\frac{d^2}{dt^2}u^g(x,t)\frac{1}{c^2} - u^g(x,t)\frac{d^2}{ds^2}u^{f_-}(x,s)\frac{1}{c^2}dx \\
=& \int_0^\infty u^{f_-}(x,s)\frac{1}{c^2}u^g_{tt}(x,t) - u^g(x,t)\frac{1}{c^2}u^{f_-}_{ss}(x,s)dx \\
=& \int_0^\infty u^{f_-}(x,s)\left[u^g_{xx}(x,t) - q(x)u^g(x,t)\right] - u^g(x,t)\left[u^{f_-}_{xx}(x,s) - q(x)u^{f_-}(x,s)\right]dx \\
=& \int_0^\infty u^{f_-}(x,s)u^g_{xx}(x,t) - u^g(x,t)u^{f_-}_{xx}(x,s)dx \\
=& \left. u^{f_-}(x,s)u^g_x(x,t) - u^g(x,t)u^{f_-}_x(x,s)\right|_{x=0}^\infty - \int_0^\infty u^{f_-}_x(x,s)u^g_x(x,t) - u^g_x(x,t)u^{f_-}_x(x,s)dx \\
=& u^{f_-}(\infty,s)u^g_x(\infty,t) - u^g(\infty,t)u^{f_-}_x(\infty,s) - u^{f_-}(0,t)u^g_x(0,s) + u^g(0,t)u^{f_-}_x(0,s) \\
=& -f_-(s)\left(R^Tg\right)(t) + g(t)\left(R^{2T}f_-\right)(s)
\end{aligned}$$

u^{f_-} and u^g have finite support for a given time so they evaluate to be zero as $x \to \infty$. By the boundary conditions in (1.7), $u^g(x,0) = 0 = u^g_t(x,0)$ we have

$$\begin{aligned}
b(s,0) &= \int_0^\infty u^f(x,s)u^g(x,0)\frac{1}{c^2}dx = \int_0^\infty u^f(x,t) * 0\frac{1}{c^2}dx = 0 \\
b_t(s,0) &= \int_0^\infty u^f(x,s)u^g_t(x,0)\frac{1}{c^2}dx = \int_0^\infty u^f(x,t) * 0\frac{1}{c^2}dx = 0
\end{aligned}$$

The formulation of $b_{tt}(s,t) - b_{ss}(s,t)$ gives us a relation and the above gives us boundary conditions. Together these give us the following PDE with $(s,t) \in B^{2T}$,

$$\begin{cases} b_{ss}(s,t) - b_{tt}(s,t) = g(t)\left(R^{2T}f_-\right)(s) - f_-(s)\left(R^Tg\right)(t) & (s,t) \in B^{2T} \\ b(s,0) = 0 \qquad b_t(s,0) = 0 & s \in [0,2T] \end{cases}$$

We can then use the D'Alembert formula to find the solution to be

$$b(s,t) = \frac{1}{2}\int_0^t \int_{s-t+\tau}^{s+t-\tau} g(\tau)\left(R^{2T}f_-\right)(y) - f_-(y)\left(R^T g\right)(\tau)dyd\tau$$

Keep in mind that by the oddness of $f_-(t)$ about T we know the following

$$\int_\tau^{2T-\tau} f_-(y)dy = 0 \qquad \forall\tau \in [0,T].$$

We look to evaluating $b(s,t)$ at (T,T).

$$\begin{aligned}
b(T,T) =& \frac{1}{2}\int_0^T \int_\tau^{2T-\tau} g(\tau)\left(R^{2T}f_-\right)(y) - f_-(y)\left(R^T g\right)(\tau)dyd\tau \\
=& \frac{1}{2}\int_0^T g(\tau)\int_\tau^{2T-\tau}\left(R^{2T}f_-\right)(y)dyd\tau - \int_0^T \frac{1}{2}\left(R^T g\right)(\tau)\int_\tau^{2T-\tau} f_-(y)dyd\tau \\
=& \frac{1}{2}\int_0^T g(\tau)\int_\tau^{2T-\tau}\left(R^{2T}f_-\right)(y)dyd\tau - \int_0^T \frac{1}{2}\left(R^T g\right)(\tau) * 0d\tau \\
=& \frac{1}{2}\int_0^T g(\tau)\left[\int_0^{2T-\tau}\left(R^{2T}f_-\right)(y)dy - \int_0^\tau \left(R^{2T}f_-\right)(y)dy\right]d\tau \\
=& \frac{1}{2}\int_0^T g(\tau)(S^T)^*\left[\int_0^\tau \left(R^{2T}f_-\right)(y)dy\right]d\tau = \int_0^T g(\tau)\frac{1}{2}(S^T)^* J^{2T}\left[\left(R^{2T}f_-\right)(\tau)\right]d\tau \\
=& \int_0^T \left[\frac{1}{2}(S^T)^* J^{2T}R^{2T}S^T f(\tau)\right]g(\tau)d\tau = \left\langle \frac{1}{2}(S^T)^* J^{2T}R^{2T}S^T f, g\right\rangle_{\mathcal{F}^T}
\end{aligned}$$

For $t \in [0,T]$ we have that $f(t) = f_-(t)$ by construction. Thus $u^{f_-}(x,t) = u^f(x,t)$ for all $t \in [0,T]$. Using this with the above relation in with the definition of $b(s,t)$ we can find the following,

$$\begin{aligned}
&\left\langle \frac{1}{2}(S^T)^* J^{2T}R^{2T}S^T f, g\right\rangle_{L^2(0,T)} = b(T,T) \\
=& \int_0^\infty u^{f_-}(x,T)u^g(x,T)\frac{1}{c^2}dx = \int_0^\infty u^f(x,T)u^g(x,T)\frac{1}{c^2}dx \\
=& \langle u^f, u^g\rangle_{\mathcal{H}^{cT}} = \langle C^T f, g\rangle_{\mathcal{F}^T}.
\end{aligned}$$

Thus $C^T f$ is weakly equal to $\frac{1}{2}(S^T)^* J^{2T}R^{2T}S^T f$. Since we are in the one dimensional case we have strong equality. As shown earlier by Lemma 3.1.7 we have that C^T has the form,

$$C^T f = \frac{1}{2}(S^T)^* R^{2T} J^{2T} S^T f, \qquad f \in \mathcal{F}^T. \tag{3.5}$$

□

Note that f is supported on $(0, T)$. Elaborating on what this composition of operators is we can write out that the exact form of the Connecting Operator is

$$C^T f(t) = \frac{1}{c} f(t) + \frac{1}{2}\int_0^T f(s) \int_{|t-s|}^{2T-t-s} w_x^\delta(0,\tau) d\tau ds \tag{3.6}$$

Corollary 3.2.2.1. *The Connecting Operator is a linear operator.*

Proof. Since W^T and $(W^T)^*$ are linear operators their composition $(W^T)^* W^T = C^T$ is a linear operator. □

3.3 Dynamical Wave Basis

The first tool we have is the ability to be able to have a basis in $\mathcal{F}^T$ that maps to an orthonormal basis in $\mathcal{H}^{cT}$. To do this, first we pick a basis for $\mathcal{F}^T$, then we map it to $\mathcal{H}^{cT}$. Theorem 2.4.1, the Exact Controllability of (1.7), allows us to claim that this is a basis for $\mathcal{H}^{cT}$. We can then apply The Gram-Shmidt process to the basis in the Inner Space. We can then write the inner products as the Control Operator acting on the controls. Moving W^T to the left via its adjoint we have the process written in terms of the Connecting Operator. Since the Connecting Operator is linear we can rewrite this as a basis in $\mathcal{F}^T$. This basis maps to the orthonormal basis in $\mathcal{H}^{cT}$.

First, we start with the Outer Space, $\mathcal{F}^T$. We know that this is a Hilbert space. As such, there exists an orthonormal basis for $\mathcal{F}^T$, call it $\{f_n\}_{n=1}^\infty$. We can then use the Control operator to map this basis to $\{u^{f_n}\}_{n=1}^\infty$.

Lemma 3.3.1. *The image of a basis in the outer space, $\mathcal{F}^T$, forms a basis in the Inner Space, $\mathcal{H}^{cT}$.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a basis for $\mathcal{F}^T$. Now fix some $u \in \mathcal{H}^{cT}$. By Theorem 2.4.1 there exists a unique $f \in \mathcal{F}^{\mathcal{T}}$ such that $W^T f(x) = u(x,T)$. Since $\{f_n\}_{n=1}^\infty$ is a basis for $f \in \mathcal{F}^{\mathcal{T}}$, $f = \sum_{n=1}^\infty \langle f, f_n\rangle_{\mathcal{F}^{\mathcal{T}}} f_n$.

Since the Control Operator is continuous, Theorem 2.3.2,

$$u = W^T \left[\sum_{n=1}^{\infty} \langle f, f_n \rangle_{\mathcal{F}^T} f_n \right] = \sum_{n=1}^{\infty} \langle f, f_n \rangle_{\mathcal{F}^T} W^T f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle_{\mathcal{F}^T} u^{f_n}.$$

Hence we can say that $\{u^{f_n}\}_{n=1}^{\infty}$ is a basis for $\mathcal{H}^{cT}$. □

Theorem 3.3.2. *We can construct a C^T orthonormal basis of $\mathcal{H}^{cT}$ from any basis of $\mathcal{F}^T$.*

Proof. Let $\{u, \tau^{f_n}\}_{n=1}^{\infty}$ be a basis for $\mathcal{F}^{\mathcal{T}}$. By Lemma 3.3.1 we have that $\{u^{f_n}\}_{n=1}^{\infty}$ forms a basis for $\mathcal{H}^{cT}$. We can then use the Gram-Shmidt method to orthonormalize them. Thus we define a new sequence, $\{g_n\}_{n=1}^{\infty}$ inductively. Now we move to define g_k in terms of the Connecting Operator and $\{u^{f_n}\}_{n=1}^{k}$. Starting with $v_1(t)$,

$$v_1 := a_{11} u^{f_1} = a_{11} W^T f_1 = W^T(a_{11} f_1),$$

where $a_{11} = 1/\langle u^{f_1}, u^{f_1} \rangle_{\mathcal{H}^{cT}} = 1/\langle C^T f_1, f_1 \rangle_{\mathcal{F}^T}$. We choose $g_1 = a_{11} f_1$. Then, through induction, we define g_k by orthonormalizing v_k,

$$\hat{v}_k := u^{f_n} - \sum_{j=1}^{k-1} a_{kj} v_j = W^T f_n - \sum_{j=1}^{k-1} a_{kj} W^T g_j = W^T \left[f_n - \sum_{j=1}^{k-1} a_{kj} g_j \right],$$

where $a_{kj} = \langle u^{f_k}, u^{f_j} \rangle_{\mathcal{H}^{cT}} = \langle C^T f_k, f_j \rangle_{\mathcal{F}^T}$. We choose $\hat{g}_k = f_n - \sum_{j=1}^{k-1} a_{kj} g_j$. Continuing,

$$v_k(t) := a_{kk} \hat{v}_k(t) = a_{kk} W^T \hat{g}_k(t) = W^T(a_{kk} \hat{g}_k(t)),$$

where $a_{kk} = 1/\langle \hat{v}_k, v_k \rangle_{\mathcal{H}^{cT}} = 1/\langle C^T g_k, g_k \rangle_{\mathcal{F}^T}$. We then choose $g_k = a_{kk} \hat{g}_k$. Thus we have that $W^T g_k = v_k$ and $\{v_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}^{CT}$. So by definition $\{g_n\}_{n=1}^{\infty}$ (a basis of $\mathcal{F}^T$) is a C^T orthonormal basis of $\mathcal{H}^{cT}$. □

One small thing is that we have to show is that this process also works for the Outer Space $\mathcal{F}^{t,\tau}$.

Corollary 3.3.2.1. *We can construct a C^T orthonormal basis of $\mathcal{H}^{c\tau}$ from any basis of $\mathcal{F}^{T,\tau}$ (2.3.5).*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a basis for $\mathcal{F}^{T,\tau}$. Then we can write $f_k = f_{t-\tau}^{(k)} \in \mathcal{F}$ as a delayed control. Theorem 2.3.4 $W^\tau f^{(k)} = W^T f_{T-\tau}^{(k)}$ where $f^{(k)} \in \mathcal{F}^\tau$. By Theorem 2.5.2 $W^T \mathcal{F}^{T,\tau} = \mathcal{H}^{c\tau} = W^\tau \mathcal{F}^\tau$.

By invertability of W^T, $\{f^{(n)}\}_{n=1}^{\infty}$ forms a basis of $\mathcal{F}^{\tau}$ by Theorem 2.4.1.

By Theorem 3.3.2 we construct a C^T orthonormal basis of $\mathcal{H}^{c\tau}$, $\{g^{(n)}\}_{n=1}^{\infty}$, from $\{f^{(n)}\}_{n=1}^{\infty}$. Then again by Theorem 2.5.2 and Theorem 2.4.1 we have that $\{g_{T-\tau}^{(n)}\}_{n=1}^{\infty} \subset \mathcal{F}^{T,\tau}$ forms a C^T orthonormal basis of $\mathcal{H}^{c\tau}$. □

This allows us to create a representation of the truncated Inner Space, $\mathcal{H}^{c\tau}$, with the basis of the semi truncated Outer Space, $\mathcal{F}^{T,\tau}$, and the original f that maps to u under the Control Operator.

Theorem 3.3.3.

$$\mathcal{P}^{\tau} f = \sum_{n=1}^{\infty} \langle C^T f, g_n^{\tau} \rangle_{\mathcal{F}^T} g_n^{\tau}$$

Where $\{g_n^{\tau}\}_{n=1}^{\infty}$ *is the* C^T *orthonormal basis of* $\mathcal{F}^{T,\tau}$.

Proof. Fix $u \in \mathcal{H}^{cT}$. By Theorem 2.4.1 there exists a $f \in \mathcal{F}^T$ such that $W^T f = u(\cdot, T)$. Then $P^{c\tau} W^T f = P^{c\tau} u \in \mathcal{H}^{c\tau}$. Fix $\{f_n\}_{n=1}^{\infty}$ as some basis for $\mathcal{F}^{T,\tau}$. By Corollary 3.3.2.1 we can derive $\{g_n^{\tau}\}_{n=1}^{\infty}$ that is a C^T orthonormal basis in $\mathcal{H}^{c\tau}$. Since $P^{c\tau} W^T f \in \mathcal{H}^{c\tau}$ we can represent it by the basis $\{W^T g_n^{\tau}\}_{n=1}^{\infty}$. Then using linearity of W^T, Theorem 2.3.3, on the representation we have,

$$\begin{aligned} P^{c\tau} u = & P^{c\tau} W^T f = \sum_{k=1}^{\infty} \langle W^T f, W^T g_n^{\tau} \rangle_{\mathcal{H}^{c\tau}} W^T g_n^{\tau} \\ & = \sum_{k=1}^{\infty} \langle C^T f, g_n^{\tau} \rangle_{\mathcal{F}^{\tau}} W^T g_n^{\tau} = W^T \sum_{k=1}^{\infty} \langle C^T f, g_n^{\tau} \rangle_{\mathcal{F}^{\tau}} g_n^{\tau} \end{aligned}$$

By the invertability of the Control Operator, $\sum_{k=1}^{\infty} \langle C^T f, g_n^{\tau} \rangle_{\mathcal{F}^{\tau}} g_n^{\tau} = \left(W^T\right)^{-1} P^{c\tau} W^T f = \mathcal{P}^{\tau} f$ by definition of $\mathcal{P}^{\tau}$ 2.5.3. □

3.4 Amplitude Formula

Now we develop the Amplitude Formula. This stems from jumps in f being reflected as jumps in u^f. We find that there has to be a corresponding jump in f. So we can use the Truncation Operator (2.5.1) on the wave then look to the jump at the truncation. Theorem 2.4.1, Exact Controllability of (1.7) gives us a function that maps to this wave and the jump at the truncation point must show up as a jump in this control. The *Amplitude Formula* gives us the relation between these two jumps.

To begin we use the Control Operator (2.6) to observe the effects of a jump in the Outer Space on the Inner Space.

$$\begin{aligned}
\lim_{h\to 0^+} u^f(x,t)\Big|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} &= \lim_{h\to 0^+} f\left(t-\frac{x}{c}\right) + \int_0^{t-\frac{x}{c}} w^\delta(x,t-s)f(s)ds\Bigg|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} \\
&= \lim_{h\to 0^+} f\left(t-\frac{x}{c}\right)\Big|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} + \lim_{h\to 0^+} \int_0^{t-\frac{x}{c}} w^\delta(x,t-s)f(s)ds\Bigg|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} \\
&= \lim_{h\to 0^+} f\left(t-\frac{x}{c}\right)\Big|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h}
\end{aligned}$$

The second line uses the boundedness of w^δ and f to split the limit. The third line is by continuity of the integral. So when we have a jump the following equation (3.7) tells us that there is an equal and opposite jump in u^f,

$$\lim_{h\to 0^+} u^f(x,t)\Big|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} = \lim_{h\to 0^+} f\left(t-\frac{x}{c}\right)\Big|_{x=c(t-\tau)-h}^{x=c(t-\tau)+h} \tag{3.7}$$

With this we can create a very important theorem in solving for $q(x)$ known as the *Amplitude Formula*.

Theorem 3.4.1. *The Amplitude Formula holds for any fixed $\tau \in [0,T]$,*

$$u^f(c\tau,T) = \lim_{h\to 0^+} \sum_{k=1}^{\infty} \left\langle C^T f, g_k^\tau \right\rangle_{\mathcal{F}^T} g_k^\tau(t)\Bigg|_{t=T-\tau+h}.$$

Where $\{g_n^\tau\}_{n=1}^\infty$ is the basis for $\mathcal{F}^\tau$ in Corollary 3.3.2.1.

Proof. Let $u^f \in \mathcal{H}^{cT}$ and $\tau \in [0,T]$ where $f \in \mathcal{F}^T$ satisfies $W^T f = u^f(\cdot,T)$. This can be done by Exact Controllability, Theorem 2.4.1. Let $\{f_n^\tau\}_{n=1}^\infty$ be a basis of $\mathcal{F}^{T,\tau}$. By Corollary 3.3.2.1 we can create $\{g_n^\tau\}_{n=1}^\infty$, a basis for $\mathcal{F}^{T,\tau}$, as a C^T orthonormal basis of $\mathcal{H}^{c\tau}$. Then by 3.3.3 $\mathcal{P}^\tau f = \sum_{n=1}^\infty \langle f, g_n^\tau\rangle_{\mathcal{F}^T} g_n^\tau$. Then by using Theorem 2.3.3 and (3.7) we find,

$$\begin{aligned}
u^f(c\tau,T) =& W^T f(c\tau) = \lim_{h\to 0^+} W^T f(c\tau-h) = \lim_{h\to 0^+} P^{c\tau} W^T f(c\tau-h) \\
&= \lim_{h\to 0^+} -P^{c\tau} W^T f(x)\big|_{x=c\tau-h}^{c\tau+h} = \lim_{h\to 0^+} \mathcal{P}^\tau f(t)\big|_{t=T-\tau-h}^{T-\tau+h} \\
&= \lim_{h\to 0^+} \mathcal{P}^\tau f(T-\tau+h) = \lim_{h\to 0^+} \sum_{n=1}^\infty \langle C^T f, g_n^\tau\rangle_{\mathcal{F}^T} g_n^\tau(T-\tau+h)
\end{aligned}$$

Hence the amplitude formula is satisfied. □

3.5 ODE constraint on Wave at time T

We only need one more tool at our disposal in order to use this method to solve for the potential, q. We impose a restriction on the wave structure at time T. The wave must be in the Null Space of the Static Schrödinger Operator (1.12) at the Final Moment. This is a significant condition as according to (1.2) we then have $u_{tt}(x,T) = 0$. This reduces the requirements to substitute different parts into an equation and make it possible to find the potential if we can find u_{xx}. Here we focus on the restriction applied to the Connecting Operator.

By Exact Controllability, Theorem 2.4.1, we can make the wave equation into any shape we would like at time T. If take a wave, φ in Null($\mathcal{A}$) then $\varphi \in \mathcal{H}^{cT}$. As such we write $\varphi(x) = u^f(x,T)$, where f is the control that maps to $u^f(x,T)$ through the Control Operator (2.6). The ODE on φ from the (1.12) must also be satisfied by $u^f(x,T)$. So the remainder of (1.7), $u_{tt} = 0$, is be zero. In other words the wave equation has zero acceleration at time T. Since we want to characterize all possible waves in Null($\mathcal{A}$), we put arbitrary initial conditions $\varphi(0) = \alpha$, and $\varphi_x(0) = \beta$ with $\alpha^2 + \beta^2 > 0$ to remove the zero wave. As a consequence of $u(\cdot,T) = \varphi$ being in the null space of (1.12) we have the ODE,

$$\begin{cases} 0 = u_{xx}(x,T) - q(x)u(x,T) \qquad x \in (0,T) \\ u(0,T) = \alpha, \qquad u_x(0,T) = \beta. \end{cases} \tag{3.8}$$

This allows us to solve for the potential as $q(x) = \dfrac{u_{xx}(x,T)}{u(x,T)}$. Note that this means we are fixing $f(T) = \alpha$ and $R^T f(T) = \beta$ before we solve this ODE.

Theorem 3.5.1. *If $u \in \mathcal{H}^{cT}$ satisfies the ODE (3.8) then*

$$u(c\tau,T) = \lim_{h\to 0^+} \sum_{n=1}^{\infty} \left\langle \beta\gamma - \alpha(R^T)^*\gamma, g_n^\tau \right\rangle_{\mathcal{F}^T} g_n^\tau(T-\tau+h)$$

Proof. Let $u(\cdot,T) \in \mathcal{H}^{cT}$ and $\tau \in [0,T]$. By Exact Controllability, Theorem 2.4.1, there is a unique $f \in \mathcal{F}^T$ such that $W^T f = u(\cdot,T)$. Then using the PDE, $u_{tt} - c^2 u_{xx} + qu = 0$, and initial data,

$u(x,0)=0$ and $u_t(x,0)=0$,

$$\begin{aligned}
\langle u, u^{g_n^\tau}\rangle_{\mathcal{H}^{cT}} &= \int_0^{cT} \frac{1}{c^2} u(y,T) u^{g_n^\tau}(y,T) dy = \int_0^{cT} \frac{1}{c^2} u(y,T) \left(u^{g_n^\tau}(y,T) - u^{g_n^\tau}(y,0) \right) dy \\
&= \int_0^{cT} \frac{1}{c^2} u(y,T) \int_0^T u^{g_n^\tau}(y,\tau) d\tau dy = \int_0^{cT} \frac{1}{c^2} u(y,T) \left(-\int_0^T -u_t^{g_n^\tau}(y,\tau) d\tau \right) dy \\
&= \int_0^{cT} \frac{1}{c^2} u(y,T) \left((T-T) u_t^{g_n^\tau}(y,T) - (T-0) u_t^{g_n^\tau}(y,0) - \int_0^T -u_t^{g_n^\tau}(y,\tau) d\tau \right) dy \\
&= \int_0^{cT} \frac{1}{c^2} u(y,T) \left((T-\tau) u_t^{g_n^\tau}(y,\tau) \Big|_{\tau=0}^T - u_t^{g_n^\tau}(y,0) - \int_0^T (T-t)' u_t^{g_n^\tau}(y,\tau) d\tau \right) dy \\
&= \int_0^{cT} \frac{1}{c^2} u(y,T) \int_0^T (T-\tau) u_{tt}^{g_n^\tau}(y,\tau) d\tau dy = \int_0^T (T-\tau) \int_0^{cT} \frac{1}{c^2} u(y,T) u_{tt}^{g_n^\tau}(y,\tau) dy d\tau \\
&= \int_0^T (T-\tau) \int_0^{cT} \frac{1}{c^2} u(y,T) \left(c^2 u_{xx}^{g_n^\tau}(y,\tau) - q(y) u^{g_n^\tau}(y,\tau) \right) dy d\tau \\
&= \int_0^T (T-\tau) \left(\int_0^{cT} u(y,T) u_{xx}^{g_n^\tau}(y,\tau) - \frac{1}{c^2} q(y) u(y,T) u^{g_n^\tau}(y,\tau) dy \right) d\tau \\
&= \int_0^T (T-\tau) \left(u(y,T) u_x^{g_n^\tau}(y,\tau) - u_x(y,T) u^{g_n^\tau}(y,\tau) \Big|_0^{cT} \right. \\
&\qquad\qquad \left. + \int_0^{cT} u_{xx}(y,T) u^{g_n^\tau}(y,\tau) - \frac{1}{c^2} q(y) u(y,T) u^{g_n^\tau}(y,\tau) dy \right) d\tau \\
&= \int_0^T (T-\tau) \left(u(cT,T)0 - u_x(cT,T)0 - u(0,T) u_x^{g_n^\tau}(0,\tau) + u_x(0,T) u^{g_n^\tau}(0,\tau) \right. \\
&\qquad\qquad \left. + \int_0^{cT} \left(u_{xx}(y,T) - \frac{1}{c^2} q(y) u(y,T) \right) u^{g_n^\tau}(y,\tau) dy \right) d\tau \\
&= \int_0^T (T-\tau) \left(-u(0,T) R^T g_n^\tau(\tau) + u_x(0,T) g_n^\tau(\tau) + \int_0^{cT} 0 u^{g_n^\tau}(y,\tau) dy \right) d\tau \\
&= \int_0^T (T-\tau) \left[\beta g_n^\tau(\tau) - \alpha R^T g_n^\tau(\tau) \right] d\tau = \left\langle \gamma, \beta g_n^\tau - \alpha R^T g_n^\tau \right\rangle_{\mathcal{F}^T} \\
&= \beta \langle \gamma, g_n^\tau \rangle_{\mathcal{F}^T} - \alpha \left\langle \gamma, R^T g_n^\tau \right\rangle_{\mathcal{F}^T} = \langle \beta\gamma, g_n^\tau \rangle_{\mathcal{F}^T} + \left\langle -\alpha (R^T)^* \gamma, g_n^\tau \right\rangle_{\mathcal{F}^T} \\
&= \left\langle \beta\gamma - \alpha (R^T)^* \gamma, g_n^\tau \right\rangle_{\mathcal{F}^T}
\end{aligned}$$

Here $\gamma(t) := T - t$. So we have that $\langle C^T f, g_n^\tau \rangle_{\mathcal{F}^T} = \left\langle \beta\gamma - \alpha (R^T)^* \gamma, g_n^\tau \right\rangle_{\mathcal{F}^T}$. By using

Theorem 3.4.1 we can represent this element of the Inner Space with elements of the Outer Space,

$$\begin{aligned} u(c\tau, T) &= \lim_{h\to 0^+} \sum_{n=1}^{\infty} \langle C^T f, g_n^\tau \rangle_{\mathcal{F}^T} g_n^\tau (T-\tau+h) \\ &= \lim_{h\to 0^+} \sum_{n=1}^{\infty} \left\langle \beta\gamma - \alpha(R^T)^*\gamma, g_n^\tau \right\rangle_{\mathcal{F}^T} g_n^\tau(T-\tau+h) \end{aligned}$$

□

As long as u satisfies the ODE in (3.8) we can write $u(c\tau, T)$ in terms of its boundary conditions and the C^T ONB of $\mathcal{F}^{T,\tau}$.

3.6 Recovering the Potential

From here we can devise our method of finding the potential at $c\tau \in (0, cT)$. This creates a six step process. There is an alternative for step four indicated at the end.

1. Find $w_x^\delta(0,t)$ for $t \in (0, 2T)$ from (3.2) and the given value of c. We may choose any $f \in \mathcal{F}^{2T}$ to make an attempt at solving for the variables.

$$R^{2T} f(t) = -\frac{1}{c} f'(t) + \int_0^t w_x^\delta(0, t-s) f(s) ds \qquad \forall t \in (0, 2T)$$

2. With $w_x^\delta(0,t)$ found for $t \in [0, 2T]$ we can find C^T with (3.6) and $(R^T)^*$ with (3.1)

$$\begin{aligned} C^T f(t) &= \frac{1}{c} f(t) + \frac{1}{2} \int_0^T f(s) \int_{|t-s|}^{2T-t-s} w_x^\delta(0,\tau) d\tau ds \\ (R^T)^* f(t) &= \frac{1}{c} f'(t) + \int_t^T w_x^\delta(0, s-t) f(s) ds \end{aligned}$$

3. Fix $\tau \in (0,T)$ and construct the C^T ONB of $\mathcal{F}^{T,\tau}$, $\{g_n^\tau\}_{n=1}^\infty$. The method is described in Corollary 3.3.2.1.

4. Choose $\alpha,\ \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$ and use Theorem 3.5.1 to find the value of $u(c\tau, T)$.

$$u(c\tau, T) = \lim_{h\to 0^+} \sum_{n=1}^{\infty} \left\langle \beta\gamma - \alpha(R^T)^*\gamma, g_n^\tau \right\rangle g_n^\tau(T-\tau+h)$$

5. Repeat steps three and four with varying $\tau \in (0,T]$ to approximate $u(x,T)$ over $[0,cT]$. Note that choosing τ solves for $x = c\tau$. Generally, you must find enough points to be able to approximate u_{xx} within tolerance as well.

6. Derived from the ODE on u, (3.8), we can find $q(x)$. In the case that $u(x,T) = 0$ we must go back to step four and pick different values for α and β

$$q(x) = c^2 \frac{u_{xx}(x,T)}{u(x,T)}$$

The alternative approach is to change step four by allowing a more general control f and then solve with the following,

4*. Determine $\mathcal{P}^\tau$ from Theorem 3.3.3 then use it to solve for $u(c\tau,T)$ with Theorem 3.4.1.

$$\mathcal{P}^\tau = \sum_{n=1}^{\infty} \langle C^T f, g_n^\tau \rangle_{\mathcal{F}^T} g_n^\tau$$

$$u(c\tau,T) = \lim_{h \to 0^+} \sum_{k=1}^{\infty} \left\langle C^T f, g_k^\tau \right\rangle_{\mathcal{F}^T} g_k^\tau(t) \Bigg|_{t=T-\tau+h}$$

Chapter 4

Spectral Inverse Problem

In this final Chapter, we derive a solution to the Spectral Inverse Problem. This problem stems directly from the Static Schrödinger Operator (1.12). Recall in Chapter 1, the problem is stated as such: for a predetermined Final Moment, T, we are given the Spectral Data (1.14) of the Static Schrödinger Operator (1.12), from that, determine the value of the potential $q(x)$ for $x \in [0, cT]$. We cannot gather additional information as in the Dynamical Problem since the Final Moment is fixed by the Spectral Data given at the onset.

First we show that from this data we are able to compute the inner product of any function in the Inner Space with any other function in the Inner Space by using their controls in the Outer Space. This is very helpful in taking a basis of the Outer Space and writing it as a C^T orthonormal basis. It is worth mentioning that although we mention the Connecting Operator here it is in no way used to create this basis.

The last step before discussing the method of solving the Spectral Problem is in finding a property of the basis representation of a truncated eigenfunction. We use the Truncation Operator (2.5.1) on an eigenfunction to truncate it at some position $c\tau < cT$. Then the basis of the Outer Space $\mathcal{F}^\tau$ is able to create an orthonormal basis for the Inner Space $\mathcal{H}^{c\tau}$ without knowing what any of the component waves are. That allows us to represent the truncated eigenfunction. Then we can manipulate the inner product to give us the value of the eigenfunction at the point of truncation.

Finally we are able to utilize the method for solving the Spectral Inverse Problem. First we calculate the formula for the time coefficients of the eigenfunction representation. Then we pick a position $c\tau$ to find the value of the eigenfunction. We create a basis of $\mathcal{F}^\tau$ which we use to make a basis for $\mathcal{H}^{c\tau}$. Then we can evaluate the eigenfunction at the point $c\tau$. After varying τ we can then evaluate the terms of the ODE in the Static Schrödinger Operator (1.12). Using that relation we can solve for the potential at $c\tau$.

4.1 Schrödinger Operator Properties

For the Spectral Problem we are given the Spectral Data (1.14) of which describes the eigenfunctions of the *Static Schrödinger Operator*.

Definition 4.1.1. The Static Schrödinger Operator, $\mathcal{A} : L^2[0, cT] \to L^2[0, cT]$, is defined by

$$\mathcal{A}\varphi(x) := -c^2\varphi_{xx}(x) + q(x)\varphi(x)$$

As such it makes sense that we need to derive some properties of this operator. First we show that this is a self-adjoint operator when we restrict the domain to $D(\mathcal{A}) = H^2[0, cT] \bigcap H_0^1[0, cT]$. Then we show that the ODE associated with this operator is *Regular*. That lets us apply the Sturm-Liouville Theorem. This gives us all the additional properties we need about our eigenfunctions and eigenvalues.

The Spectral Data comes in the form of $\{\lambda_k, \varphi_k'(0)\}_{k=1}^{\infty}$ where λ_k is an eigenvalue associated with the eigenfunction $\varphi_k(x)$ and $\varphi_k'(0)$ is the derivative of the eigenfunction evaluated at $x = 0$. Since these are eigenfunctions of the Static Schrödinger Operator we have the relation,

$$\mathcal{A}\varphi_k(x) = \lambda_k\varphi_k(x)$$

Our first task is to show that the Static Schrödinger Operator is self-adjoint. This is in the direction of satisfying the hypothesis for the Sturm-Liouville Theorem.

Theorem 4.1.2. *$\mathcal{A}$ is self-adjoint on the Inner Space $H_0^1[0, cT]$.*

Proof. Let $\varphi, \psi \in H_0^1[0, cT]$

$$\begin{aligned}
\langle (\mathcal{A})^* \varphi, \psi \rangle_{\mathcal{H}^{cT}} =& \langle \varphi, \mathcal{A}\psi \rangle_{\mathcal{H}^{cT}} = \int_0^{cT} \frac{1}{c^2}\varphi(y)\left(-c^2\psi_{xx}(y) + q(x)\psi(y)\right) dy \\
=& \int_0^{cT} -\frac{1}{c^2}c^2\varphi(y)\psi_{xx}(y)dy + \int_0^{cT} \frac{1}{c^2}\varphi(y)q(x)\psi(y)dy \\
=& \, \varphi(y)\psi_x(y) - \varphi_x(y)\psi(y)|_0^{cT} + \int_0^{cT} -\frac{1}{c^2}c^2\varphi_{xx}(y)\psi(y)dy \\
& + \int_0^{cT} \frac{1}{c^2}q(x)\varphi(y)\psi(y)dy \\
=& \varphi(cT)\psi_x(cT) - \varphi_x(cT)\psi(cT) - \varphi(0)\psi_x(0) + \varphi_x(0)\psi(0) \\
& + \int_0^{cT} \frac{1}{c^2}\left(-c^2\varphi_{xx}(y) + q(x)\varphi(y)\right)\psi(y)dy \\
=& 0 - 0 - 0 + 0 + \int_0^{cT} \frac{1}{c^2}\mathcal{A}\varphi(\cdot)(y)\psi(y)dy = \langle \mathcal{A}\varphi, \psi \rangle_{\mathcal{H}^{cT}}
\end{aligned}$$

Thus $(\mathcal{A})^* = \mathcal{A}$, for all $\varphi \in H_0^1[0, cT]$. □

We also have that each eigenfunction satisfies the Robin Boundary Condition, $\alpha_1\varphi_k(0) + \alpha_2\varphi_k'(0) = 0$ and $\beta_1\varphi_k(T)+\beta_2\varphi_k'(T) = 0$ for some $\alpha_1, \ \alpha_2, \ \beta_1, \beta_2 \in \mathbb{R}$ such that $\alpha_1^2+\alpha_2^2 > 0, \ \beta_1^2+\beta_2^2 > 0$. For any eigenfunction φ_k, choosing $\alpha_1 = 1 = \beta_1, \ \alpha_2 = 0 = \beta_2$ we have that the boundary conditions in (1.7) satisfy the Robin Boundary Condition. Also $\mathcal{A}$ has the form $\frac{d}{dx}\left[p(x)\frac{d}{dx}\varphi_k(x)\right] + q(x)\varphi_k(x) = -\lambda_k w(x)\varphi_k(x)$ with $p(x) = 1 = w(x)$ and $q(x)$ as our potential. Together with $\mathcal{A}$ being self-adjoint, this operator is *regular*. As such we may apply the Sturm-Liouville Theorem. The results are $\lim_{k\to\infty} \lambda_k = \infty$, $\{\phi_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $H_0^1[0, cT] \subset \mathcal{H}^{cT}$ and the eigenvalues have a corresponding order

$$\lambda_1 < \lambda_2 < ... < \lambda_k < ...,$$

Thus the Spectral Data has a partial ordering by the eigenvalues. The last result from the Sturm-Liouville Theorem is that in this ordering the k^{th} eigenfunction has exactly $k-1$ zeroes in $(0, cT)$.

4.2 Spectral Wave Basis

In this section we show that for any wave in the Inner Space, $\mathcal{H}^{cT}$, we can write the coefficients of the eigenfunction representation, c_k as a function of the Spectral Data and the wave's control function. We start with a wave in a subset of the Inner Space, $H^2[0, cT] \bigcap H_0^1[0, cT] = Y^{cT}$.We show that when we find the inner product of the wave with an eigenfunction we have a function of t.

For a dense subspace of waves and corresponding controls, we work to find an explicit formula for c_k. Since the eigenfunctions are a basis of this space the representation converges. Then by a density argument we can represent any wave in the Inner Space, $\mathcal{H}^{cT}$. Lastly we show that we can use this representation to express the inner product of the Inner Space in terms of the coefficients c_k.

We have to start simple. Let $u \in Y^{cT}$. From the Sturm-Liouville Theorem we have that $\{\varphi_k\}_{k=1}^{\infty}$ forms an orthonormal basis of Y^{cT}. Since $Y^{ct} \subset Y^{cT}$ it can also represent an element in Y^{ct}, $\forall t \in [0, T]$. So fixing $t \in [0, T]$, $u(\cdot, t) \in Y^{ct}$. Thus u can be represented by the orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ by $u(x, t) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x)$, where $c_k(t) = \langle u(\cdot, t), \varphi_k\rangle_{\mathcal{H}^{cT}}$ $\forall k \in \mathbb{N}$. Thus $c(t)$ is defined for all $t \in [0, T]$.

For this given $u \in Y^{cT}$, we have that $f \in H^2(0, T) \bigcap H_0^1(0, T) = X^T$ by using the fact that W^T is invertible. We can write out a relation for c_k by analysing the inner product it represents.

$$
\begin{aligned}
\frac{d^2}{dt^2} c_k(t) =& c_k''(t) = \langle u_{tt}(\cdot, t), \varphi_k\rangle_{\mathcal{H}^{cT}} \\
=& \langle c^2 u_{xx}(\cdot, t) - qu(\cdot, t), \varphi_k\rangle_{\mathcal{H}^{cT}} = \int_0^{cT} \frac{1}{c^2}\left(c^2 u_{xx}(y, t) - q(x)u(y, t)\right)\varphi_k(y)dy \\
=& \int_0^{cT} \frac{1}{c^2} c^2 u_{xx}(y, t)\varphi_k(y)dy + \int_0^{cT} -\frac{1}{c^2} q(x)u(y, t)\varphi_k(y)dy \\
=& \left(\frac{1}{c^2} c^2 u_x(y, t)\varphi_k(y) - \frac{1}{c^2} c^2 u(y, t)\varphi_k'(y)\right)\Bigg|_0^{cT} \\
& \quad + \int_0^{cT} \frac{1}{c^2} c^2 u(y, t)\varphi_k''(y)dy + \int_0^{cT} -\frac{1}{c^2} q(x)u(y, t)\varphi_k(y)dy \\
=& u_x(cT, t)\varphi_k(cT) - u(cT, t)\varphi_k'(cT) - u_x(0, t)\varphi_k(0) + u(0, t)\varphi_k'(0) \\
& \quad + \int_0^{cT} \frac{1}{c^2} u(y, t)\left(c^2\varphi_k''(y) - q(x)\varphi_k(y)\right)dy \\
=& 0\varphi_k(cT) - 0\varphi_k'(cT) - u_x(0, t)0 + f(t)\varphi_k'(0) \\
& \quad - \int_0^{cT} \frac{1}{c^2} u(y, t)\left(-c^2\varphi_k''(y) + q(x)\varphi_k(y)\right)dy \\
=& f(t)\varphi_k'(0) - \langle u(\cdot, t), \mathcal{A}\varphi_k\rangle_{\mathcal{H}^{cT}} = f(t)\varphi_k'(0) - \langle u(\cdot, t), \lambda_k\varphi_k\rangle_{\mathcal{H}^{cT}} \\
=& f(t)\varphi_k'(0) - \lambda_k c_k(t)
\end{aligned}
$$

From this we get the relation $c_k'' + \lambda_k c_k = f\varphi_k'(0)$. Now, looking to the boundary conditions of u we can also create conditions on c_k. Using $u(\cdot, 0) = 0$ and $u_t(\cdot, 0) = 0$ we have,

$$
\begin{aligned}
0 =& \langle 0, \varphi\rangle_{\mathcal{H}^{cT}} = \langle u(\cdot, 0), \varphi\rangle_{\mathcal{H}^{cT}} \qquad & 0 =& \langle 0, \varphi\rangle_{\mathcal{H}^{cT}} = \langle u_t(\cdot, 0), \varphi\rangle_{\mathcal{H}^{cT}} \\
=& c_k(0) & =& c_k'(0)
\end{aligned}
$$

By combining these two we have a formulation for an ODE of c_k,

$$
\begin{cases}
c_k''(t) + \lambda_k c_k(t) = f(t)\varphi_k'(0) & t \in [0, T] \\
c_k(0) = 0 \qquad c_k'(0) = 0.
\end{cases} \tag{4.1}
$$

The general solution to this is taken from the convolution of the right hand side of the formulation with what is called the *Kernel*, $c_k(t) = S_k * (f\varphi_k'(0))(t) = \int_{-\infty}^{\infty} S_k(t-s)f(s)\varphi_k'(0)ds$. We require that S_k is a locally integrable function that is twice differentiable. So solving for c_k is the same as solving for S_k. What we need now is to find the first and second time derivatives of c_k in terms of S_k and rewrite the ODE.

$$
\begin{aligned}
c_k'(t) =& S_k(t-t)f(t)\varphi_k'(0) + \int_{-\infty}^{\infty} S_k'(t-s)f(s)\varphi_k'(0)ds \\
=& S_k(0)\varphi_k'(0)f(t) + \int_{-\infty}^{\infty} S_k'(t-s)f(s)\varphi_k'(0)ds \\
c_k''(t) =& S_k(0)\varphi_k'(0)f'(t) + S_k'(t-t)f(t)\varphi_k'(0) + \int_{-\infty}^{\infty} S_k''(t-s)f(s)\varphi_k'(0)ds \\
=& S_k(0)\varphi_k'(0)f'(t) + S_k'(0)\varphi_k'(0)f(t) + \int_{-\infty}^{\infty} S_k''(t-s)f(s)\varphi_k'(0)ds \\
f(t)\varphi_k'(0) =& c_k''(t) + \lambda_k c_k(t) \\
=& S_k(0)\varphi_k'(0)f'(t) + S_k'(0)\varphi_k'(0)f(t) \\
&+ \int_{-\infty}^{\infty} S_k''(t-s)f(s)\varphi_k'(0)ds + \lambda_k \int_{-\infty}^{\infty} S_k(t-s)f(s)\varphi_k'(0)ds \\
=& S_k(0)\varphi_k'(0)f'(t) + S_k'(0)\varphi_k'(0)f(t) \\
&+ \int_{-\infty}^{\infty} \left(S_k''(t-s) + \lambda_k S_k(t-s)\right) f(s)\varphi_k'(0)ds
\end{aligned}
$$

We must ensure this equality holds for all $f \in X^T$. As such, due to the arbitrary nature from selecting u and hence f, we must set $S_k(0) = 0$, $S_k'(0) = 1$ and $S_k''(t) + \lambda_k S_k(t) = 0$ for all

$t \in [0,T]$. Thus we can formulate an ODE for S_k,

$$\begin{cases} S_k''(t) + \lambda_k S_k(t) = 0 & t \in [0,T] \\ S_k(0) = 0, \quad S_k'(0) = 1. \end{cases}$$

This is a well known ODE. Using the boundary condition it has the following solution,

$$S_k(t) = \begin{cases} c_1 e^{\sqrt{-\lambda_k}t} + c_2 e^{-\sqrt{-\lambda_k}t} = c_1 e^{\sqrt{-\lambda_k}t} - c_1 e^{-\sqrt{-\lambda_k}t} & \lambda_k \neq 0 \\ c_1 t + c_2 = c_1 t & \lambda_k = 0 \end{cases}$$

$$= \begin{cases} -2c_1 \sin\left(\sqrt{\lambda_k}t\right) = \frac{\sin\left(\sqrt{\lambda_k}t\right)}{\sqrt{\lambda_k}} & \lambda_k > 0 \\ 2c_1 \sinh\left(\sqrt{-\lambda_k}t\right) = \frac{\sinh\left(\sqrt{-\lambda_k}t\right)}{\sqrt{-\lambda_k}} & \lambda_k < 0 \\ t & \lambda_k = 0. \end{cases} \tag{4.2}$$

Thus we can now back substitute (4.2) in for $c_k(t)$ for an explicit equation,

$$c_k(t) = \int_{-\infty}^{\infty} S_k(t-s) f(s) \varphi_k'(0) ds = \begin{cases} \displaystyle\int_0^T \frac{\sin\left(\sqrt{\lambda_k}(t-s)\right)}{\sqrt{\lambda_k}} f(s)\varphi_k'(0) ds & \lambda_k > 0 \\ \displaystyle\int_0^T \frac{\sinh\left(\sqrt{-\lambda_k}(t-s)\right)}{\sqrt{-\lambda_k}} f(s)\varphi_k'(0) ds & \lambda_k < 0 \\ \displaystyle\int_0^T (t-s) f(s)\varphi_k'(0) ds & \lambda_k = 0. \end{cases} \tag{4.3}$$

Theorem 4.2.1. *For all $u \in \mathcal{H}^{cT}$, u can be represented as $u(x,t) = \sum_{k=1}^{\infty} c_k^f(t)\varphi_k(x)$ where for any $k \in \mathbb{N}$, $c_k^f(t) := \langle u(\cdot,t), \varphi_k \rangle_{\mathcal{H}^{cT}}$ and f is the control that maps to u through the Control Operator.*

Proof. Let $u \in \mathcal{H}^{cT}$. Then there exists an $f \in \mathcal{F}^T$ such that $W^T f = u$. Since X^T is dense in $\mathcal{F}^{cT}$, there exists $\{f_n\}_{n=1}^{\infty} \subset X^T$ such that $f_n \to f$ as $n \to \infty$.

Fix $k \in \mathbb{N}$. Since W^T is continuous, Theorem 2.3.3, there exists $\delta_k > 0$ such that for all $g \in \mathcal{F}^T$ such that $\|f - g\|_{\mathcal{F}^T} < \delta_k$, $\|W^T f - W^T g\|_{\mathcal{H}^{cT}} < \dfrac{\epsilon}{c^2 3 * 2^k}$. By convergence of $\{f_k\}_{k=1}^{\infty}$ there exists an $N_k \in \mathbb{N}$ such that for all $n \geqslant N_k$, $\|f - f_n\|_{\mathcal{F}^T} < \delta_k$.

Recall that $\{\varphi_k\}$ is an orthonormal basis of Y^{cT} so $\|\varphi_k\|_{\mathcal{H}^{cT}} = 1$. So for all $n \geqslant N_k$,

$$\begin{aligned}
\|c_k^{f_n} - c_k^f\|_{\mathcal{F}^{cT}} =& \|\langle u^{f_n}(\cdot,t), \varphi_k\rangle_{\mathcal{H}^{cT}} - \langle u(\cdot,t), \varphi_k\rangle_{\mathcal{H}^{cT}}\|_{\mathcal{F}^{cT}} = \|\langle u^{f_n}(\cdot,t) - u(\cdot,t), \varphi_k\rangle_{\mathcal{H}^{cT}}\|_{\mathcal{F}^{cT}} \\
=& \frac{1}{c^2}\|\langle u^{f_n}(\cdot,t) - u(\cdot,t), \varphi_k\rangle_{\mathcal{F}^{cT}}\|_{\mathcal{F}^{cT}} \leqslant \|u^{f_n}(\cdot,t) - u(\cdot,t)\|_{\mathcal{F}^{cT}} \frac{1}{c^2}\|\varphi_k\|_{\mathcal{F}^{cT}} \\
\leqslant & c^2\|u^{f_n}(\cdot,t) - u(\cdot,t)\|_{\mathcal{H}^{cT}}\|\varphi_k\|_{\mathcal{H}^{cT}} < c^2\frac{\epsilon}{c^2 3 * 2^k} = \frac{\epsilon}{3 * 2^k}.
\end{aligned}$$

With Bessel's inequality we have $\sum_{k=1}^{\infty}\|c_k^f - c_k^{f_n}\|_{\mathcal{F}^{cT}} \leqslant \|W^T f - W^T f_n\|_{\mathcal{H}^{cT}} < \infty$. Since the infinite sum is absolutely bounded we know that the infinite sum converges. As such there exists a $K \in \mathbb{N}$ such that $\sum_{k=K+1}^{\infty}\|c_k^f - c_k^{f_n}\|_{\mathcal{F}^{cT}} < \frac{\epsilon}{3}$ for all $n \geqslant N = \max\{N_k\}_{k=1}^{K}$. We then use these together in an ϵ over three argument,

$$\begin{aligned}
\left\|u(x,t) - \sum_{k=1}^{\infty} c_k^f(t)\varphi_k(x)\right\|_{\mathcal{H}}^2 \leqslant & \|u(x,t) - u^{f_n}(x,t)\|_{\mathcal{H}}^2 + \left\|u^{f_n}(x,t) - \sum_{k=1}^{\infty} c_k^f(t)\varphi_k(x)\right\|_{\mathcal{H}}^2 \\
=& \|W^t f(x) - W^t f_n(x)\|_{\mathcal{H}}^2 + \left\|\sum_{k=1}^{\infty} c_k^{f_n}(t)\varphi_k(x) - \sum_{k=1}^{\infty} c_k^f(t)\varphi_k(x)\right\|_{\mathcal{H}}^2 \\
=& \|W^t f(x) - W^t f_n(x)\|_{\mathcal{H}}^2 + \left\|\sum_{k=1}^{\infty}\left(c_k^{f_n}(t) - c_k^f(t)\right)\varphi_k(x)\right\|_{\mathcal{H}}^2 \\
=& \|W^t f(x) - W^t f_n(x)\|_{\mathcal{H}}^2 + \sum_{k=1}^{\infty}\left\|c_k^{f_n}(t) - c_k^f(t)\right\|_{\mathcal{H}}^2 \|\varphi_k(x)\|_{\mathcal{H}}^2 \\
=& \|W^t f(x) - W^t f_n(x)\|_{\mathcal{H}}^2 + \sum_{k=1}^{\infty}\left\|c_k^{f_n}(t) - c_k^f(t)\right\|_{\mathcal{H}}^2 \\
=& \|W^t f(x) - W^t f_n(x)\|_{\mathcal{H}}^2 + \sum_{k=1}^{K}\left\|c_k^{f_n}(t) - c_k^f(t)\right\|_{\mathcal{H}}^2 + \sum_{k=K+1}^{\infty}\left\|c_k^{f_n}(t) - c_k^f(t)\right\|_{\mathcal{H}}^2 \\
<& \frac{\epsilon}{3} + \sum_{k=1}^{K}\frac{\epsilon}{3}\frac{1}{2^k} + \frac{\epsilon}{3} = \frac{\epsilon}{3} + \frac{\epsilon}{3}\sum_{k=1}^{\infty}\frac{1}{2^k} + \frac{\epsilon}{3} \\
=& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Hence $W^t f(x) = u(x,t) = \sum_{k=1}^{\infty} c_k^f(t)\varphi_k(x)$ for all $(x,t) \in [0,ct) \times [0,T]$. □

Note that taking an arbitrary element $u \in \mathcal{H}^{cT}$ is the same as taking an arbitrary element that is in the range of W^T since the Control Operator is invertable.

Corollary 4.2.1.1. *For all $u^f, u^g \in \mathcal{H}^{cT}$,*

$$\langle u^f(\cdot,t), u^g(\cdot,t)\rangle_{\mathcal{H}^{cT}} = \sum_{k=1}^{\infty} c_k^f(t) c_k^g(t)$$

where $W^T f(x) = u^f(x,T)$, $W^T g(x) = u^g(x,T)$ and $t \in [0,T]$.

Proof. Let $u^f, u^g \in \mathcal{H}^{cT}$ with $W^T f(x) = u^f(x,T)$ and $W^T g(x) = u^g(x,T)$. By using the representation in Theorem 4.2.1 and $\{\varphi_n\}_{n=1}^{\infty}$ are orthonormal,

$$\begin{aligned}
\langle u^f(\cdot,t), u^g(\cdot,t)\rangle_{\mathcal{H}^{cT}} &= \left\langle \sum_{k=1}^{\infty} \langle u^f(\cdot,t), \varphi_k\rangle_{\mathcal{H}^{cT}} \varphi_k(\cdot), \sum_{j=1}^{\infty} \langle u^g(\cdot,t), \varphi_j\rangle_{\mathcal{H}^{cT}} \varphi_j(\cdot) \right\rangle_{\mathcal{H}^{cT}} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \langle u^f(\cdot,t), \varphi_k\rangle_{\mathcal{H}^{cT}} \langle u^g(\cdot,t), \varphi_j\rangle_{\mathcal{H}^{cT}} \langle \varphi_k(\cdot), \varphi_j(\cdot)\rangle_{\mathcal{H}^{cT}} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_k^f(t) c_k^g(t) \delta_{jk} = \sum_{k=1}^{\infty} c_k^f(t) c_k^g(t).
\end{aligned}$$

□

4.3 Eigenfunction Representation

It is time to derive a property of the basis representation of a truncated eigenfunction. We are able to find the value of the eigenfunction at the position $c\tau$ using the eigenfunction basis representation of the truncated eigenfunction. First we use the Truncation Operator (2.5.1) on an eigenfunction to truncate it at some position $c\tau < cT$. Then Corollary 2.5.2 tells us that we can use a basis of the Outer Space $\mathcal{F}^{\tau}$ to form a basis for the Inner Space $\mathcal{H}^{c\tau}$. We can use Corollary 4.2.1.1 to make this a C^T orthonormal basis of $\mathcal{F}^{\tau}$. Representing the truncated eigenfunction in this basis allows us to rewrite its inner product with another eigenfunction in terms of c_k's. Then taking the τ derivative of the inner product gives us the magnitude of the eigenfunction at that point.

The first major step is to show that we can construct a C^T orthonormal basis for the Inner Space, $\mathcal{H}^{c\tau}$, from the Outer Space, $\mathcal{F}^{\tau}$. This is very similar to the Dynamical Inverse Problem where we let $\{f_n\}_{n=1}^{\infty}$ be a basis for $\mathcal{F}^{\tau}$. Then using the Control Operator and Theorem 2.5.2, we found $W^T f_n(x) = u^{f_n}(x,T)$. Finally using the Gram-Schmit process to orthonormalize them.

Lemma 4.3.1. *If $\{f_n\}_{n=1}^{\infty}$ is a basis of $\mathcal{F}^{\tau}$ then there exists $\{g_n^{\tau}\}_{n=1}^{\infty}$ that maps to an orthonormal basis of $\mathcal{H}^{c\tau}$ defined in terms of the basis $\{f_n\}_{n=1}^{\infty}$.*

Proof. Let $u \in \mathcal{H}^{c\tau}$. By Theorem 2.5.2, there exists $f \in \mathcal{F}^{\tau}$ such that $W^{\tau}f(x) = u(x,\tau)$. Since $\{f_n\}_{n=1}^{\infty}$ is a basis for $\mathcal{F}^{\tau}$, there exists $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $f = \sum_{n=1}^{\infty} a_n f_n$. Then by linearity of W^{τ},

$$\begin{aligned} u(x,\tau) =& W^{\tau}f(x) = W^{\tau}\left(\sum_{n=1}^{\infty} a_n f_n\right)(x) \\ =& \left(\sum_{n=1}^{\infty} a_n W^{\tau} f_n\right)(x) = \sum_{n=1}^{\infty} a_n u^{f_n}(x,\tau). \end{aligned}$$

Thus $\{u^{f_n}\}_{n=1}^{\infty}$ forms a basis for $\mathcal{H}^{c\tau}$. Then we can apply the Gram Schmidt process to $\{u^{f_n}\}_{n=1}^{\infty}$ and create an for $\mathcal{H}^{c\tau}$, $\{v_n\}_{n=1}^{\infty}$. Here $v_n = \sum_{j=1}^{n} b_{jn} u^{f_j}$ where b_{jn} is defined by the Gram Schmidt process. As done in the Dynamical Inverse Problem, Theorem 3.3.2, we can take each of the $\mathcal{H}^{c\tau}$ inner products and represent them as in Corollary 4.2.1.1.

It is also important to note that by the linearity of W^{τ} we can also find the $g_n^{\tau} \in \mathcal{F}^{\tau}$ that maps to v_n,

$$\begin{aligned} v_n(\cdot,\tau) =& \sum_{j=1}^{n} b_{jn} u^{f_j} = \sum_{j=1}^{n} b_{jn} W^{\tau} f_j \\ =& W^{\tau}\left(\sum_{j=1}^{n} b_{jn} f_j\right). \end{aligned}$$

Thus defining $g_n^{\tau} := \sum_{j=1}^{n} b_{jn} f_j$ we have $W^{\tau} g_n^{\tau} = v_n(\cdot,\tau)$ and $\{v_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}^{c\tau}$. So by definition $\{g_n^{\tau}\}_{n=1}^{\infty}$ maps to an orthonormal basis of $\mathcal{H}^{c\tau}$. $\square$

With this representation in hand we can now find a way to calculate the inner product of a truncated eigenvector with another eigenvector.

Theorem 4.3.2. *Let $\{f_n\}_{n=1}^{\infty}$ is a basis of $\mathcal{F}^{\tau}$ and $\{\lambda_n, \varphi_n'(0)\}_{n=1}^{\infty}$ be the Spectral Data of the Spectral Inverse Problem. For all $\tau \in [0,T]$, $\langle P^{c\tau}\varphi_k, \varphi_j\rangle_{\mathcal{H}^{c\tau}}$ can be found solely in terms of $\{\lambda_n, \varphi_n'(0)\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{F}^{\tau}$ and fix $j,k \in \mathbb{N}$, $\tau \in [0,T]$. By Lemma 4.3.1 we can create a basis that maps to an orthonormal basis of $\mathcal{H}^{c\tau}$, $\{v_n\}_{n=1}^{\infty}$, $W^{\tau}g_n^{\tau} = v_n(\cdot,\tau)$, $g_n^{\tau} = \sum_{j=1}^{n} b_{jn} f_j$ and $\{b_{jn}\}_{j=1}^{n}$ is found though the Gram-Schmidt process using $\{f_j\}_{j=1}^{n}$ in Corollary 4.2.1.1. Since $\varphi_k \in H_0^1[0,cT] \bigcap H^2[0,cT] \subset \mathcal{H}^{cT}$, then $P^{c\tau}\varphi_n \in \mathcal{H}^{c\tau}$.

Since $\{v_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}^{c\tau}$ we have that $P^{c\tau}\varphi_k = \sum_{n=1}^{\infty}\langle\varphi_k, v_n\rangle_{\mathcal{H}^{c\tau}} v_n$. Recalling that v_n is supported on $[0, c\tau]$ and using Corollary 4.2.1.1 we can achieve the following relation,

$$\begin{aligned}\langle P^{c\tau}\varphi_k, \varphi_j\rangle_{\mathcal{H}^{cT}} &= \left\langle \sum_{n=1}^{\infty}\langle\varphi_k, v_n\rangle_{\mathcal{H}^{c\tau}} v_n, \varphi_j \right\rangle_{\mathcal{H}^{cT}} = \sum_{n=1}^{\infty}\langle\varphi_k, v_n\rangle_{\mathcal{H}^{c\tau}} \langle v_n, \varphi_j\rangle_{\mathcal{H}^{cT}} \\ &= \sum_{n=1}^{\infty}\langle v_n, \varphi_k\rangle_{\mathcal{H}^{c\tau}}\langle v_n, \varphi_j\rangle_{\mathcal{H}^{c\tau}} = \sum_{n=1}^{\infty} c_k^{g_n^\tau} c_j^{g_n^\tau}\end{aligned}$$

From (4.3), the formulation of $c_i^{g_n^\tau}$, we have,

$$\begin{aligned}c_k^{g_n^\tau}(t) &= \int_0^\tau S_k(t-s) g_n^\tau(s)\varphi_k'(0)ds \\ &= \int_0^\tau S_k(t-s)\left(\sum_{j=1}^{n} b_{jn} f_j(s)\right)\varphi_k'(0)ds\end{aligned}$$

Since S_k is only dependent on λ_k we have a formulation for $c_k^{g_n^\tau}$ that is only dependent on τ, $\{\lambda_n, \varphi_n'(0)\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$. Similarly we can find a formulation of $c_j^{g_n^\tau}$. Thus we have the above formulation for $\langle P^{c\tau}\varphi_k, \varphi_j\rangle_{\mathcal{H}^{cT}}$ is only dependent on τ, $\{\lambda_n, \varphi_n'(0)\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$. □

Finally, with that, we can derive a formula to evaluate an eigenfunction at any given point $c\tau \in [0, cT]$. This brings us in reach of our goal of recovering the potential, q.

Theorem 4.3.3. *Let $\{f_n\}_{n=1}^{\infty}$ be a basis of $\mathcal{F}^\tau$, then*

$$\varphi_k(c\tau) = \pm\sqrt{c}\left(\sum_{n=1}^{\infty}\left|\frac{d}{d\tau}\int_{T-\tau}^{T} S_k(t-s)\left(\sum_{j=1}^{n} b_{jn} f_j\right)\varphi_k'(0)ds\right|^2\right)^{1/2}.$$

Proof. Fix $k \in \mathbb{N}$ and $\tau \in [0, T]$. We can let $\{f_n^\tau\}_{n=1}^{\infty}$ be a basis of $\mathcal{F}^\tau$ with $\{g_n^\tau\}_{n=1}^{\infty}$ as the basis that maps to an orthonormalized basis of $\mathcal{H}^{c\tau}$ as described in Lemma 4.3.1. Then using $j = k$ in

Theorem 4.3.2 with we have

$$\begin{aligned}\int_0^{c\tau} \frac{1}{c^2}|\varphi_k(x)|^2 dx =& \int_0^{cT} \frac{1}{c^2} P^{c\tau}\varphi_k(x)\varphi_k(x)dx \\ =&\langle P^{c\tau}\varphi_k, \varphi_k\rangle_{\mathcal{H}^{cT}} = \sum_{n=1}^{\infty} \left| c_k^{g_n^\tau} \right|^2 \\ =& \sum_{n=1}^{\infty} \left| \int_0^\tau S_k(t-s) \left(\sum_{j=1}^{n} b_{jn} f_j^\tau \right) \varphi_k'(0) ds \right|^2 .\end{aligned}$$

Then The derivative of this with respect to τ is

$$\frac{d}{d\tau} \int_0^{c\tau} \frac{1}{c^2}|\varphi_k(x)|^2 dx = \frac{1}{c^2}|\varphi_k(c\tau)|^2 (c\tau)' = \frac{1}{c}|\varphi_k(c\tau)|^2.$$

Therefore

$$\begin{aligned}\varphi_k(c\tau) =& \pm \sqrt{c \frac{d}{d\tau} \int_0^{c\tau} \frac{1}{c^2}|\varphi_k(x)|^2 dx} \\ =& \pm \sqrt{c} \left(\sum_{n=1}^{\infty} \left| \frac{d}{d\tau} \int_0^\tau S_k(t-s) \left(\sum_{j=1}^{n} b_{jn} f_j \right) \varphi_k'(0) ds \right|^2 \right)^{1/2}\end{aligned}$$

□

4.4 Recovering the Potential

Now we are ready to describe the method of recovering the potential, q. Though in this process we do not need to recover any operators and the form of the C^T inner products are explicitly given.

1. Find the operators $c_k f(t) := c_k^f(t)$ from their form described in (4.3)

$$c_k f(t) = \begin{cases} \int_0^T \frac{\sin\left(\sqrt{\lambda_k}(t-s)\right)}{\sqrt{\lambda_k}} f(s)\varphi_k'(0)ds & \lambda_k > 0 \\ \int_0^T \frac{\sinh\left(\sqrt{-\lambda_k}(t-s)\right)}{\sqrt{-\lambda_k}} f(s)\varphi_k'(0)ds & \lambda_k < 0 \\ \int_0^T (t-s)f(s)\varphi_k'(0)ds & \lambda_k = 0. \end{cases}$$

2. Fix $\tau \in (0,T)$ and construct the basis of $\mathcal{F}^\tau$ that maps to an ONB in $\mathcal{H}^{cT}$, $\{g_n^\tau\}_{n=1}^\infty$. The

method is described in Lemma 4.3.1.

3. Use the formulation for $\varphi_1(c\tau)$ in Theorem 4.3.3 to find the value of $\varphi_1(c\tau)$. Note that any eigenfunction can be used, however φ_1 has no zeros on the interval $[0, cT]$.

$$\varphi_1(c\tau) = \pm\sqrt{c}\left(\sum_{n=1}^{\infty}\left|\frac{d}{d\tau}\int_0^{\tau} S_1(t-s)\left(\sum_{j=1}^{n} b_{jn}f_j\right)\varphi_1'(0)ds\right|^2\right)^{1/2}$$

4. Repeat steps two and three with varying $\tau \in (0, T]$ to approximate $\varphi_1(x)$ over $[0, cT]$. Note that choosing τ solves for $x = c\tau$. Generally, you must find enough points to be able to approximate $(\varphi_1)_{xx}$ within tolerance.

5. Derived from the ODE on φ_1, $\lambda_1\varphi_1(x) = -c^2\varphi_1''(x) + q(x)\varphi_1(x)$ (1.12), we can find $q(x)$

$$q(x) = \lambda_1 + c^2\frac{\varphi_1''(x)}{\varphi_1(x)}. \tag{4.4}$$

By the second property provided by the Sturm-Liouville Theorem φ_1 has no zeroes on $(0, cT)$ so the denominator will never be zero with φ_1.

Bibliography

[1] M. Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk SSSR*, 297, 1987.

[2] M. Belishev. *Boundary Control Methods in Dynamical Inverse Problems–An Introductory Course. Chapter in "Dynamical Inverse Problems: Theory and Applicattion, 85–150.* Springer, 2011.

[3] A.L. Bukhgeim and M. Klibanov. Global uniqueness of class of multidimensional inverse problems. *Soviet Mathematics. Doklady*, 24, 01 1981.

[4] V. Isakov. *Inverse Problems for Partial Differential Equations.* Spplied Mathematical Series **127**, Springer, 1998.

[5] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse Boundary Spectral Problems.* Chapman & Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics **123**, 2001.

[6] R. Triggiani and S. Liu. Determining damping and potential coefficients of an inverse problem for a system of two coupled hyperbolic equations. part i: Global uniqueness. *Conference Publications*, 2011(Special):1001–1014, 2011.

[7] V.S. Vladimirov. *Equations of Mathematical Physics. Translated from the Russian by A. Littlewood, Ed. A.Jeffrey, Pure and Applied Mathematics.* New York, M. Dekker, 1971.