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Augmented Solvability of Nth Degree Polynomials

Joseph Swanson
Clemson University, joeswans44@gmail.com

AUGMENTED SOLVABILITY OF NTH DEGREE POLYNOMIALS

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Joseph Swanson
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Accepted by:
Dr. James Coykendall, Committee Chair
Dr. Kevin James
Dr. Sean Sather-Wagstaff

Abstract

A group is called solvable if its derived series descends to the identity element. Galois discovered that a polynomial is solvable by radicals if and only if its Galois group is solvable. In 1824, Niels Abel published a paper proving the insolvability of a general quintic polynomial. In this paper, we provide two augmented strategies to solve all quintics, and discuss methods for how to make all nth degree polynomials solvable.

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Chapter 1

Introduction

1.1 Historical Background

Given $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ where $a_i \in \mathbb{R}$ the fundamental theorem of algebra says that $f(x)$ has n, not necessarily distinct, roots $r_1, ..., r_n \in \mathbb{C}$. The existence of roots does not mean, unfortunately, that they are easily computed. Mathematicians sought a formula to solve for the roots of a polynomial for centuries as access to the roots would solve both practical and abstract problems. For polynomials of degree no more than four, mathematicians have been successful in writing out the roots explicitly in terms of the four basic operations and radicals. For instance when $n = 2$, the well known quadratic formula gives the desired roots. For $n = 3$, Cardano's formula will give you the roots.

During the 19th century mathematicians attempted to find an analogous formula for fifth degree polynomials. Niels Abel published a theorem in 1824 which stated that in general a polynomial of degree greater than four could not be solved in terms of radicals. Despite Abel's proof, George Jerrard believed he would find the answer as he was able to reduce the general quintic polynomial

$$f(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

to

$$\bar{f}(x) = x^5 + px + q$$

using a Tschirnhaus transformation [5]. Although he was able to generalize his result to show that the $x^{n-1}, x^{n-2}, x^{n-3}$ terms could be deleted from a general nth degree polynomial, William Hamilton in 1836 showed that this was not sufficient to solve a quinitic polynomial and Abel's proof still held.

1.2 Mathematical Background

In this section we produce some background necessary for this study. We will only be working with finite field extensions of $\mathbb{Q}$.

Definition 1.2.1. Let $F \subseteq L$ be a field extension and $\alpha \in L$. Then α is said to be algebraic if there exists a monic polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$.

Example 1.2.2. Consider $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ and $\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. Then $\sqrt{2}$ is algebraic over $\mathbb{Q}$. Choosing $f(x) = x^2 - 2 \in \mathbb{Q}[x]$ shows $f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.

Example 1.2.3. Consider $\mathbb{Q} \subset \mathbb{R}$. Notice that i, $\sqrt[3]{2}$ are algebraic since they are roots of $x^2 + 1$ and $x^3 - 2$ respectively. However, it is well known that $e, \pi \in \mathbb{R}$ are transcendental and thus not algebraic over $\mathbb{Q}$.

We note that there are infinitely many polynomials which have $\sqrt{2}$ as a root. This leads us to our next definition which specifies a single polynomial for an algebraic element.

Definition 1.2.4. Given a field extension $F \subset L$ and an algebraic element α of L then the minimal polynomial of α is the unique nonconstant monic polynomial $p(x) \in F[x]$ such that the following hold:

1) α is a root of p, i.e., $p(\alpha) = 0$.

2) If $f(\alpha) = 0$ for any other polynomial $f \in F[x]$ then $p|f$.

It can be shown that a minimal polynomial always exists for an algebraic element. To reiterate, the minimal polynomial is unique. Therefore when given an algebraic element α and discussing properties of the polynomials which have α as a root, we will always choose the minimal polynomial of α to eliminate any ambiguity.

Definition 1.2.5. A field extension $F \subseteq L$ is an algebraic extension if every $\alpha \in L$ is algebraic over F.

Referring back to the above, $\mathbb{Q}(\sqrt{2})$ is algebraic over $\mathbb{Q}$ but the reals are not. Now we define the degree of a field extension.

Definition 1.2.6. A field extension $F \subset L$ is finite if L is a finite dimensional vector space over F. We denote the degree by $[L : F]$ and define the degree of L over F by

$$[L : F] = \begin{cases} \dim_F L & \text{if } L \text{ is a finite field extension of } F \\ \infty & \text{otherwise} \end{cases}$$

All of the field extensions we work with will be finite. Because of this, the next theorem tells us that all of our extensions are algebraic.

Theorem 1.2.7. Given a field extension $F \subset L$ and $\alpha \in L$. Then α is algebraic over F if and only if $[F(\alpha) : F] < \infty$. Also any extension of finite degree is algebraic.

Corollary 1.2.8. As $\mathbb{Q} \subseteq \mathbb{R}$ is not algebraic then $[\mathbb{R} : \mathbb{Q}] = \infty$.

The converse to the second statement of Theorem 1.2.7 that finite degree extensions are algebraic is false. There exists infinite dimensional algebraic extensions. For example the algebraic closure of $\mathbb{Q}$, denoted $\bar{\mathbb{Q}}$, is infinite dimensional over $\mathbb{Q}$.

Definition 1.2.9. Let $f(x) \in F[x]$ be a polynomial of degree n. Then a field extension $F \subset L$ is the splitting field of f if the following hold:
1) $f(x) = a(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$ where $a \in F,\ \alpha_i \in L$
2) For some field extension $F \subseteq K$, if $f(x) = a(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$ where $a \in F,\ \alpha_i \in K$ then $L \subset K$.

In other words the splitting field is the smallest field extension such that f splits into linear factors.

Definition 1.2.10. An algebraic extension is normal if every irreducible polynomial in $F[x]$ that has a root in L splits over L.

Note that being a normal extension is a strong condition. For an irreducible polynomial f there are many algebraic extensions which contain a root or a couple of roots of f without containing all the roots. Here is an example of an extension that is not normal.

Example 1.2.11. Consider the polynomial $f(x) = x^3 - 2$. The three roots of f are $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$ where ω is a primitive third root of unity. We see that $\mathbb{Q}(\sqrt[3]{2})$ is an algebraic field extension of $\mathbb{Q}$.

Observe that $\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2})$ but $\omega\sqrt[3]{2}, \omega^2\sqrt[3]{2} \notin \mathbb{Q}(\sqrt[3]{2})$ since this is a subfield of the reals, but the two elements are not real. Therefore $\mathbb{Q}(\sqrt[3]{2})$ is not a normal extension. We will see later that $\mathbb{Q}(\omega, \sqrt[3]{2})$ is a normal extension of $\mathbb{Q}$.

Definition 1.2.12. A nonconstant polynomial $f(x) \in F[x]$ is separable if its roots in a splitting field have distinct linear factors.
A field extension $F \subseteq L$ is separable if for every $\alpha \in L$ the minimal polynomial of α is separable over L.

With these definitions we define a Galois extension.

Definition 1.2.13. The field extension $F \subseteq L$ is a Galois extension if L is normal and separable.

We now introduce the notion of the Galois group of an extension of fields. Galois was able to connect the solvability of a polynomial by radicals with the solvability of its Galois group.

Definition 1.2.14. Let $F \subset L$ be a finite extension. We define the Galois group to be the set

$$\mathrm{Gal}(L/F) := \{\sigma | \sigma : L \to L \text{ is an automorphism that fixes } F \text{ pointwise}\}$$

In other words the Galois group consists of all isomorphisms $\sigma : L \to L$ such that $\sigma(\alpha) = \alpha$ for all $\alpha \in F$.

Remark 1.2.15. Let $F \subseteq L$ be a Galois extension and $f(x) \in F[x]$ be an irreducible polynomial of degree n with roots of $\alpha_1, ..., \alpha_n$. If $\sigma \in \mathrm{Gal}(L/F)$ then $f(\sigma(\alpha_1)) = \sigma(f(\alpha_1)) = \sigma(0) = 0$ for all $1 \leq k \leq n$. In other words, σ permutes the roots of f. Therefore, the Galois group can be viewed as of a subgroup of the symmetric group on n points, i.e., $\mathrm{Gal}(L/F) \cong H$ where $H \leq S_n$ is transitive. We will always regard the Galois group as a transitive subgroup of S_n. Later we will establish a connection between subgroups of the Galois group and intermediate field extensions.

Example 1.2.16. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$. The minimal polynomial of $\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ is $f(x) = x^2 - 2$. Let $G = \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. By our previous remark if $\sigma \in G$ then $\sigma(\sqrt{2})$ is another root of f. We know that the two roots of f are $\pm\sqrt{2}$. Thus the only possibilities are $\sigma(\sqrt{2}) = \sqrt{2}$ or $\sigma(\sqrt{2}) = -\sqrt{2}$. Note that every element of $\mathbb{Q}(\sqrt{2})$ has the form

$$q_0 + q_1\sqrt{2}$$

where $q_0, q_1 \in \mathbb{Q}$. Let $\sigma_1(q_0 + q_1\sqrt{2}) = q_0 + q_1\sqrt{2}$ and $\sigma_2(q_0 + q_1\sqrt{2}) = q_0 - q_1\sqrt{2}$. Clearly σ_1, σ_2 are automorphisms and observe that for $q_0 \in \mathbb{Q}$,

$$\sigma_1(q_0) = \sigma_1(q_0 + 0(\sqrt{2})) = q_0 + 0(\sqrt{2}) = q_0$$

Similarly $\sigma_2(q_0) = q_0$. Therefore σ_1, σ_2 are the only elements in G. Thus $G \cong \mathbb{Z}_2$.

Definition 1.2.17. Assume F is a field and let $f(x) \in F[x]$ be an irreducible polynomial. Denote the splitting field of $f(x)$ by L. Then $\mathrm{Gal}(f) = \mathrm{Gal}(L/F)$.

We are almost ready to state the fundamental theorem of Galois theory. In order to understand the connection between Galois groups and intermediate fields we need the definition of a fixed field.

Definition 1.2.18. Given a finite extension $F \subset L$ and subgroup $H \in \mathrm{Gal}(L/F)$ then we define the fixed field of H to be

$$L_H = \{h \in L | \sigma(h) = h \text{ for all } \sigma \in H\}$$

Theorem 1.2.19. Given a finite extension $F \subset L$ and subgroup $H \in \mathrm{Gal}(L/F)$ then the fixed field is a subfield of L containing F.

We now state the first part of the fundamental theorem of Galois Theory which relates subfields of the splitting field, denoted L, with subgroups of $\mathrm{Gal}(L/F)$.

Theorem 1.2.20. Let $F \subset L$ be a finite Galois extension. Then the following hold:
1) For an intermediate field $F \subset K \subset L$, the Galois group $\mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/F)$ and $L_{\mathrm{Gal}(L/K)} = K$.

Furthermore, $|\mathrm{Gal}(L/K)| = [L : K]$ and $[\mathrm{Gal}(L/F) : \mathrm{Gal}(L/K)] = [K : F]$.

2) For a subgroup $H \subset \mathrm{Gal}(L/F)$, its fixed field $F \subset L_H \subset L$ has Galois group $\mathrm{Gal}(L/L_H) = H$.

Furthermore, $[L : L_H] = |H|$ and $[L_H : F] = [\mathrm{Gal}(L/F) : H]$.

Here is the second part of the fundamental theorem of Galois theory.

Theorem 1.2.21. Let $F \subset L$ be a finite Galois extension. Then the maps between intermediate fields $F \subset K \subset L$ and subgroups $H \subset \mathrm{Gal}(L/F)$ given by

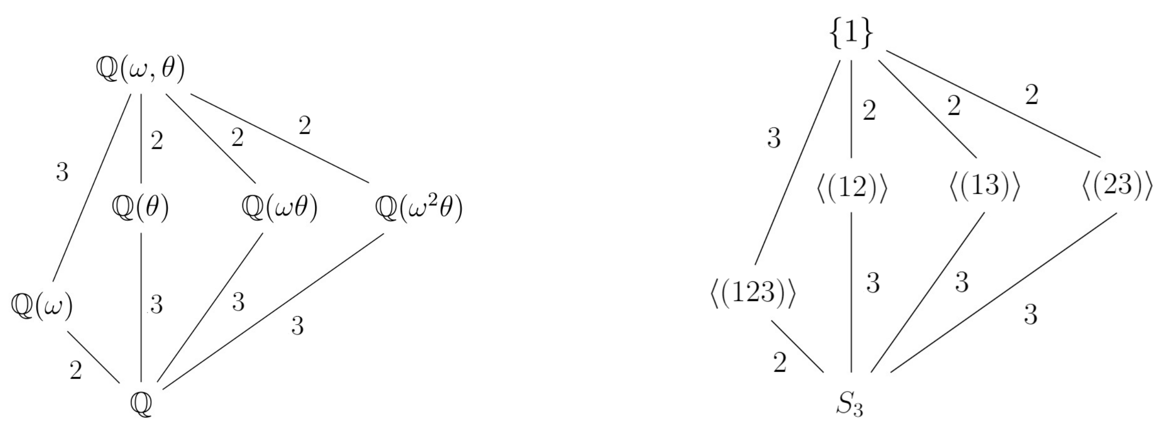
$$K \mapsto \mathrm{Gal}(L/K)$$

$$H \mapsto L_H$$

are inverses of each other. Furthermore, if K is an intermediate field corresponding to a subgroup H under these maps then K is Galois over F if and only if H is a normal subgroup of $\mathrm{Gal}(L/F)$.

We now show a simple example which encapsulates these results.

Example 1.2.22. Again consider $f(x) = x^3 - 2$. Then the splitting field of f is $\mathbb{Q}(\omega, \theta)$ where $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\theta = \sqrt[3]{2}$. It is not difficult to show that the Galois group of f is isomorphic to S_3. Here is a subgroup lattice of both the splitting field with its intermediate fields and the subgroup lattice of S_3.



(a) Subgroup Lattice of the Splitting Field

(b) Subgroup Lattice of S_3

Figure 1.1: Fundamental Theorem of Galois Theory Correspondence

Notice that there is a 1-1 correspondence between intermediate fields of $\mathbb{Q}$ and $\mathbb{Q}(\omega, \theta)$ and subgroups of the Galois group. Also the indexes are preserved. Note that by Theorem 1.2.19 $\mathbb{Q} \subset \mathbb{Q}(\theta)$ is not a Galois extension as $\langle(12)\rangle$ is not a normal subgroup of S_3 while $\mathbb{Q} \subset \mathbb{Q}(\omega)$ is a Galois extension as $A_3 \trianglelefteq S_3$.

1.3 Solvability

Recall the definition of a group being solvable.

Definition 1.3.1. A group G is solvable if given its derived series

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq ...$$

then $G^{(k)} = 1$ for some $k \in \mathbb{N}$.

Theorem 1.3.2. [**3**] A group G is solvable if and only if it has a subnormal series

$$G = G_0 \geq G_1 \geq ... \geq G_m = 1$$

such that G_i/G_{i+1} is abelian for $i = 1, ..., m$.

We now state a result of Galois which connects a function $f(x)$ being solvable by radicals and $\text{Gal}(f)$ being solvable.

Theorem 1.3.3. [**1**] **(Galois).** Suppose that F is a field with $\text{char}(F) = 0$. Then $f(x) \in F[x]$ is solvable by radicals if and only if $\text{Gal}(f)$ is a solvable group.

As we discussed previously, the Galois group can be viewed as a transitive subgroup of S_n. As S_n is solvable for n less than or equal to 4 then by Galois' theorem and as any subgroup of a solvable group is solvable, any polynomial of degree less than or equal to 4 is solvable by radicals. Hence we have explicit formulas for the roots of a polynomial of degree less than or equal to 4. Since S_n for n greater than 4 is not solvable then a polynomial of degree n is not solvable by radicals in general. That is not to say that any polynomial of degree n is not solvable by radicals, but there are not explicit equations for the roots of a general nth degree polynomial.

Example 1.3.4. We provide the explicit formulas for the roots of a nth degree polynomial over $\mathbb{Q}$ for $n \leq 4$. When $n = 2$, S_2 has two subgroups and the roots of $f(x) = x^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

the Galois group is $\mathbb{Z}_2$ if $b^2 - 4c$ is not a square in $\mathbb{Q}$ and trivial otherwise.

When $n = 3$, S_3 has 6 subgroups. Given any cubic polynomial,

$$f(x) = x^3 + bx^2 + cx + d$$

The roots of f are given by

$$x = -\frac{b}{3} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

where $p = c - \frac{b^2}{3}$ and $q = d - \frac{cb}{3} + \frac{2b^3}{27}$ we choose the cube roots such that

$$\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right)\left(\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right) = -\frac{p}{3}$$

Finally for $n = 4$, S_4 has 30 subgroups. We first need to provide the definition of the discriminant of a polynomial.

Definition 1.3.5. Given a polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

denote the roots of f by $r_1, ..., r_n$. Then we define

$$\Delta = \prod_{i<j}(r_i - r_j)$$

The discriminant of f is $\text{Disc}(f) = \Delta^2$.

Now the roots of the polynomial

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are

$$r_{1,2} = -\frac{b}{4a} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$$
$$r_{3,4} = -\frac{b}{4a} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$$

where

$$p = \frac{8ac - 3b^2}{8a^2} \qquad q = \frac{b^3 - 4abc + 8a^2 d}{8a^3}$$
$$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}(Q + \frac{\Delta_0}{Q})} \qquad Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$
$$\Delta_1 = 2c^3 - 9bcd + 27b^2 e + 27ad^2 - 72ace \qquad \Delta_0 = c^2 - 3bd + 12ae$$
$$\Delta_1^2 - 4\Delta_0^3 = -27\text{Disc}(f)$$

As seen from above, we note that the formula for the roots of an nth degree polynomial grow in complexity as n increases. Similarly computing the Galois group of an nth degree polynomial grows in complexity. Let $f(x)$ be an irreducible polynomial with $\deg(f) = n$. Recall that the Galois group can be viewed as a transitive subgroup of S_n. When $n = 2$ the only Galois group is $S_2 \cong \mathbb{Z}/2\mathbb{Z}$. When $n = 3$ then there are two transitive subgroups of S_3, namely S_3 and A_3. Finally when $n = 4$ there are five transitive subgroups of S_4 up to isomorphisms. They are C_4, V, D_4, A_4, and S_4. For degrees $n = 3$ there are methods for computing the Galois group using the discriminant of a polynomial. For degree $n = 4$ there are methods for computing the Galois group using the resolvant which we will define later.

We note that the resultant can also be written in terms of the coefficients of a polynomial. For our purposes we only need the discriminant of two polynomials.

$$\text{Disc}(x^3 + px + q) = -4p^3 - 27q^2 \tag{1.1}$$
$$\text{Disc}(x^3 + ax^2 + bx + c) = -4a^3 c + a^2 b^2 + 18abc - 4b^3 - 27c^2 \tag{1.2}$$

Theorem 1.3.6. For an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree n. We have that $D, \Delta \in \mathbb{Q} \Leftrightarrow \mathrm{Gal}(f) \leq A_n$

Notice that this theorem completely determines the Galois group of an irreducible cubic polynomial since the only nontrivial Galois groups are S_3 and A_3.

Example 1.3.7. We claimed in Example 1.2.21 that the Galois group of $f(x) = x^3 - 2$ is S_3. Notice that

$$\Delta = \sqrt{\mathrm{Disc}(x^3 - 2)} = \sqrt{-27(-2)^3} = 6\sqrt{6} \notin \mathbb{Q}$$

Therefore by Theorem 1.3.6 $\mathrm{Gal}(f) = S_3$.

Definition 1.3.8. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ then the resolvant cubic is the polynomial $r = x^3 - bx^2 + (ac - 4d)x - a^2d + 4bd - c^2$

Theorem 1.3.9. Let f be an irreducible quartic with distict roots and let $G = \mathrm{Gal}(f)$. Let α, β, γ be roots of the resolvant cubic of f and $m = [\mathbb{Q}(\alpha, \beta, \gamma) : \mathbb{Q}]$. Then the following holds.

1. $m = 6 \Leftrightarrow G \cong S_4$

2. $m = 3 \Leftrightarrow G \cong A_4$

3. $m = 1 \Leftrightarrow G \cong V$

4. $m = 2 \Leftrightarrow G \cong C_4$ or $G \cong D_4$. We have that $G \cong D_4 \Leftrightarrow f$ is irreducible over $K(\alpha, \beta, \gamma)$.

Example 1.3.10. Consider $f(x) = x^4 + 4x^2 + 2$. The resolvant of this polynomial is $r(x) = x^3 - 4x^2 - 8x + 32$. Observe that $r(4) = 0$ so we have that

$$r(x) = (x - 4)(x^2 - 8)$$

Thus the roots of r are $x = 4, 2\sqrt{2}, -2\sqrt{2}$. So we have $\mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\sqrt{2})$ and $m = [\mathbb{Q}(\sqrt{2}), \mathbb{Q}] = 2$. So $\mathrm{Gal}(f) \cong C_4$ or D_4. Considering f over $\mathbb{Q}(\sqrt{2})$ we have that

$$f(x) = (x^2 - (-2 + \sqrt{2}))(x^2 - (-2 - \sqrt{2})) \in (\mathbb{Q}(\sqrt{2}))[x]$$

So f is reducible therefore by Theorem 1.3.9 $\mathrm{Gal}(f) \cong C_4$.

Chapter 2

Tschirnhaus Transformation

2.1 Definition

In 1683, Ehrenfried Walther von Tschirnhaus published a transformation which allowed one to remove intermediate terms in a general polynomial [5]. We will let k be the base field of characteristic zero. For our purposes k will always be an algebraic extension of $\mathbb{Q}$. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$$

be a general irreducible nth degree polynomial. Then let

$$L = k[x]/(f(x))$$

We choose an element of L,

$$y = h(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + ... + b_1x + b_0$$

where $b_i \neq 0$ for some $1 \leq i \leq n-1$. The minimal polynomial, g, of y over K is the Tschirnhaus transformation of f. We choose y such that coefficients of our original polynomial equal zero. We can calculate g directly as the resultant of $f(x)$ and $h(x) - y$ with respect to x.

Denote the roots of f by $x_1, ..., x_n$. Simply put, we send

$$x_i \mapsto x_i^m + \alpha_{m-1}x_i^{m-1} + ... + \alpha_1 x_i + \alpha_0$$

So that

$$f(x) = (x-x_1)...(x-x_n) \mapsto g(y) = (y-(x_1^m+\alpha_{m-1}x_1^{m-1}+...+\alpha_0))...(y-(x_n^m+\alpha_{m-1}x_n^{m-1}+...+\alpha_0))$$

where α_i's are chosen so that some coefficients of g are equal to zero. We provide two examples to illustrate.

Example 2.1.1. One well known Tschirnhaus transformation was used by by Cardano to take a general 3rd degree polynomial to a depressed 3rd degree polynomial to establish Cardano's formula. In detail, when given a general 3rd degree polynomial,

$$f(x) = x^3 + a_2x^2 + a_1x + a_0$$

then letting $y = x + \frac{a_2}{3}$ we find the minimal polynomial g of y takes the form,

$$g(y) = y^3 + py + q$$

where $p = a_1 - \frac{a_2^2}{3}$ and $q = a_0 - \frac{a_1a_2}{3} + \frac{2a_2^3}{27}$.

Example 2.1.2. In order to make two coefficients equal to zero then we need to use a quadratic Tschirnhaus transformation. Consider

$$f(x) = x^3 + x^2 + 4x - 1$$

where the roots of f are denoted by x_i. Then we let

$$y_i = x_i^2 + \alpha x_i + \beta.$$

So we have

$$f(x) = (x - x_1)(x - x_2)(x - x_3) \mapsto (y - (x_1^2 + \alpha x_1 + \beta))(y - (x_2^2 + \alpha x_2 + \beta))(y - (x_3^2 + \alpha x_3 + \beta)).$$

Note that

$$a_0 = x_1 x_2 x_3 = -1$$

$$a_1 = x_1 x_2 + x_1 x_3 + x_2 x_3 = 4$$

$$a_2 = x_1 + x_2 + x_3 = 1$$

Multiplying out and collecting terms yields

$$y^3 + y^2(7 + \alpha - 3\beta) + y(18 - 7\alpha + 4\alpha^2 - 14\beta - 2\alpha\beta + 3\beta^2)$$
$$+(-1 - 4\alpha + \alpha^2 - \alpha^3 - 18\beta + 7\alpha\beta - 4\alpha^2\beta + 7\beta^2 + \alpha\beta^2 - \beta^3)$$

Setting the coefficient on y^2 equal to zero we obtain that

$$\beta = \frac{7 + \alpha}{3}$$

The coefficient on y is now

$$\frac{-1}{27}(-45 + 315\alpha - 99\alpha^2)$$

Setting this coefficient equal to zero gives us that

$$\alpha = \frac{(35 \pm \sqrt{1005})}{22}$$

We are free to choose either one. Letting $\alpha = \frac{(35+\sqrt{1005})}{22}$ we obtain the

$$g(y) = y^3 + \frac{12120300 + 245220\sqrt{1005}}{287496}$$

2.2 Jerrard's and Bring's Work

George Jerrard and Erland Bring were independently able to eliminate three the coefficients on $x^{n-1}, x^{n-2}, x^{n-3}$ using a Tschirnhaus transformation. Here we address when $n = 5$. A general 5^{th} degree polynomial takes the form

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

Since we are working over a field then we can assume f is always monic by dividing through by the leading coefficient.

$$f(x) = x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$$

We use a Tschirnhaus transformation to make the coefficients of x^4, x^3, x^2 equal to zero. Let x_i denote the roots of f where $i = 1, ..., 5$. Then Jerrad and Bring both showed that using the relation

$$y_i = x_i^4 + \alpha x_i^3 + \beta x_i^2 + \gamma x_i + \delta$$

where $i = 1, ..., 5$ and $\alpha, \beta, \gamma, \delta \in \mathbb{A}$ can be choosen such that the minimal polynomial g of y takes the form

$$g(y) = y^5 + ay + b$$

We showed in Example 2.1.2 that you can use a quadratic Tschirnhaus transformation to eliminate the coefficients on x^2 and x. This can be generalized to remove the coefficients on x^{n-1} and x^{n-2}. We will show how to compute $\alpha, \beta, \gamma,$ and δ when we begin with

$$f(x) = x^5 + a_2x^2 + a_1x + a_0$$

We have

$$y = h(x) = x_i^4 + \alpha x_i^3 + \beta x_i^2 + \gamma x_i + \delta$$

Let $P(y)$ equal the resultant of $f(x)$ and $h(x) - y$. We will display the coefficient on the powers of

y^2, y^3, y^4. We compute:

$$y^4 : -5\delta + 4a_1 + 3\alpha a_2$$

$$y^3 : 2a_1\beta^2 - 3a_2^2\beta + 5\alpha a_0\beta + 3\gamma a_2\beta + 10\delta^2 + 6a_1^2 + 3\alpha^2 a_2^2 + 5\gamma a_0 + 4\alpha\gamma a_1 - 16\delta a_1 - 12\alpha\delta a_2 - 4a_0a_2 + 5\alpha a_1 a_2$$

$$\begin{aligned} y^2 : &- a_2^4 + \alpha^3 a_2^3 - 3\alpha\beta a_2^3 + 3\gamma a_2^3 - \beta^3 a_2^2 - 3\gamma^2 a_2^2 + 3\alpha\beta\gamma a_2^2 - 9\alpha^2\delta a_2^2 + 9\beta\delta a_2^2 - \alpha a_0 a_2^2 + \alpha^2 a_1 a_2^2 - 2\beta a_1 a_2^2 + \gamma^3 a_2 \\ &+ 18\alpha\delta^2 a_2 + \alpha a_1^2 a_2 - 9\beta\gamma\delta a_2 - 8\beta^2 a_0 a_2 + 7\alpha^2\beta a_0 a_2 - \alpha\gamma a_0 a_2 + 12\delta a_0 a_2 - \alpha\beta^2 a_1 a_2 + 5\alpha^2\gamma a_1 a_2 - 2\beta\gamma a_1 a_2 \\ &- 15\alpha\delta a_1 a_2 - 8a_0 a_1 a_2 - 10\delta^3 + 4a_1^3 - 5\alpha^2 a_0^2 - 5\beta a_0^2 + 4\beta^2 a_1^2 - 4\alpha^2\beta a_1^2 + 8\alpha\gamma a_1^2 - 18\delta a_1^2 + 5\alpha\gamma^2 a_0 + 5\beta^2\gamma a_0 \\ &- 15\alpha\beta\delta a_0 - 15\gamma\delta a_0 + 4\beta\gamma^2 a_1 + 24\delta^2 a_1 - 6\beta^2\delta a_1 - 12\alpha\gamma\delta a_1 - 3\alpha^3 a_0 a_1 + 2\alpha\beta a_0 a_1 + 11\gamma a_0 a_1 \end{aligned}$$

We want $y^4 = y^3 = y^2 = 0$. Setting each of the coefficients equal to zero gives us a system of equations. To solve the system we first set the coefficient on y^4 equal to zero. This yields

$$3\alpha a_2 + 4a_1 - 5\delta = 0 \Rightarrow \delta = \frac{1}{5}\left(3\alpha a_2 + 4a_1\right)$$

Setting the coefficient on y^3 equal to zero with this substitution gives

$$\begin{aligned} &3\alpha^2 a_2^2 + 5\alpha a_0\beta + 4\alpha a_1\gamma + \frac{2}{5}\left(3\alpha a_2 + 4a_1\right)^2 + 5\alpha a_1 a_2 - \frac{16}{5}a_1\left(3\alpha a_2 + 4a_1\right) - \frac{12}{5}\alpha a_2\left(3\alpha a_2 + 4a_1\right) + 2a_1\beta^2 \\ &+ 3a_2\beta\gamma - 3a_2^2\beta + 5a_0\gamma + 6a_1^2 - 4a_0a_2 \\ &= \ 3\alpha^2 a_2^2 + \gamma\left(4\alpha a_1 + 3a_2\beta + 5a_0\right) + 5\alpha a_0\beta + \frac{2}{5}\left(3\alpha a_2 + 4a_1\right)^2 + 5\alpha a_1 a_2 - \frac{16}{5}a_1\left(3\alpha a_2 + 4a_1\right) \\ &- \frac{12}{5}\alpha a_2\left(3\alpha a_2 + 4a_1\right) + 2a_1\beta^2 - 3a_2^2\beta + 6a_1^2 - 4a_0a_2 = 0 \end{aligned}$$

We choose β such that the coefficient on γ equals to zero. So $\beta = \frac{-4\alpha a_1 - 5a_0}{3a_2}$. Now we are left with

$$-\frac{27\alpha^2 a_2^4 - 160\alpha^2 a_1^3 + 5a_0\left(60\alpha^2 a_1 a_2 - 80\alpha a_1^2 - 9a_2^3\right) + 27\alpha a_1 a_2^3 - 125a_0^2\left(2a_1 - 3\alpha a_2\right) + 18a_1^2 a_2^2}{45a_2^2} = 0.$$

Observe that this is a quadratic equation in terms of α and hence can be solved. Let α_1, α_2 be the solutions. We can use either root so we will use α_1.

We now turn out attention to the coefficient on y^2. Using the previous substitutions and setting it equal to zero yields:

$$a_2\gamma^3 - \gamma^2 \frac{\left(-3375\alpha_1 a_0 a_2^2 + 3600\alpha_1 a_1^2 a_2 + 2025 a_2^4 + 4500 a_0 a_1 a_2\right)}{675 a_2^2}$$
$$- \gamma \frac{\left(-675\alpha_1^2 a_1 a_2^3 - 6000\alpha_1^2 a_0 a_1^2 + 4050\alpha_1 a_0 a_2^3 - 7200\alpha_1 a_1^2 a_2^2 - 15000\alpha_1 a_0^2 a_1 - 2025 a_2^5 - 9675 a_0 a_1 a_2^2 - 9375 a_0^3\right)}{675 a_2^2}$$
$$- \Big(54\alpha_1^3 a_2^5 + 225\alpha_1^3 a_0 a_1 a_2^2 + 4 a_1^3 a_2 \left(80\alpha_1^3 + 27 a_2\right) + 756\alpha_1^2 a_1 a_2^4 + 1125\alpha_1^2 a_0^2 a_2^2 + 3900\alpha_1^2 a_0 a_1^2 a_2 + 960\alpha_1^2 a_1^4$$
$$- 1485\alpha_1 a_0 a_2^4 + 3843\alpha_1 a_1^2 a_2^3 + 9375\alpha_1 a_0^2 a_1 a_2 + 2400\alpha_1 a_0 a_1^3 + 675 a_2^6 + 4770 a_0 a_1 a_2^3 + 6250 a_0^3 a_2 + 1500 a_0^2 a_1^2\Big)/675 a_2^2$$
$$= 0.$$

Observe that this is a cubic in γ which is solvable. Any root will work so we will pick the real root, γ_1. Now the coefficients on y^4, y^3, y^2 equal zero and $g(y)$ takes the form

$$g(y) = y^5 + py + q$$

as desired.

Chapter 3

Augmented Methods for Making 5th Degree Polynomials Solvable

3.1 Utilizing the Tschirnhaus Transformation

For the general quintic

$$f(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

we saw that we can use a Tschirnhaus Transformation to make f in the form

$$g(y) = y^5 + ay + b$$

Setting this polynomial equal to zero and using one more substitution namely, $y = a^{\frac{1}{4}}z$, yields

$$a^{\frac{5}{4}}z^5 + a(a^{\frac{1}{4}}z) + b = a^{\frac{5}{4}}z^5 + a^{\frac{5}{4}}z + b \Rightarrow z^5 + z + \bar{b} = 0$$

where $\bar{b} = \frac{b}{a^{\frac{5}{4}}}$. We have shown that every general 5^{th} degree polynomial can take the form

$$x^5 + x + a.$$

In order to make this polynomial solvable all we need to do is extract one root. Once a root is extracted, the polynomial is of degree 4 and hence solvable. To extract a root we consider the inverse function of $h(x) = x^5 + x$. Assuming h^{-1} is known then we can extract a root out as follows,

$$x^5 + x + a = 0 \Rightarrow x^5 + x = a \Rightarrow x = h^{-1}(a)$$

So we can factor $x^5 + x + a$ as

$$\Big(x - h^{-1}(a)\Big)\Big(x^4 + h^{-1}(a)x^3 + (h^{-1}(a))^2x^2 + (h^{-1}(a))^3x - \frac{a}{h^{-1}(a)}\Big)$$

Thus we only need to add one inverse function to make all 5^{th} degree polynomials solvable.

Remark 3.1.1. In the previous example, $h(x)$ is one to one as $h'(x) = x^4+1 > 0$ for all $x \in \mathbb{R}$. Most of the functions we will need will not be injective. We rely on the inverse function theorem to obtain our inverses within a neighborhood of a root of the original polynomial. Note that this theorem will always apply because our polynomials are separable and therefore the derivative evaluated at a root of the original polynomial will not be zero.

Example 3.1.2. Consider $f(x) = x^5 - 2x + 2$. We will first compute the Galois group of f. Note that f is irreducible by Einstein's criteria. Observe that the irreducible factorization of $f \pmod 3$ is

$$f(x) = (x+1)^2(x^3 - 2x^2 + 2) \pmod 3$$

So $\text{Gal}(f)$ must contain a 3-cycle. Since $3 \nmid |\mathbb{F}_{20}|, |D_5|, |C_5|$ then $\text{Gal}(f) = S_5$ or A_5.
The irreducible factorization of $f \pmod 5$ is

$$f(x) = (3 + x)(4 + 3x + 4x^2 + 2x^3 + x^4)$$

Therefore $\text{Gal}(f)$ contains a 4-cycle which is an odd permutation. Therefore $\text{Gal}(f) = S_5$.
Setting f equal to zero and making the substitution $z = 2^{\frac{1}{4}}x$ we get

$$z^5 + z + \frac{1}{2^{\frac{1}{4}}} = 0 \Rightarrow z = h^{-1}(2^{\frac{-1}{4}}).$$

Now we have that

$$g(z) = (x - \alpha)(x^4 + \alpha x^3 + \alpha^2x^2 + \alpha^3x - \frac{2}{\alpha})$$

where $\alpha = h^{-1}(2^{\frac{-1}{4}})$. As the degree of the second term is 4, it solvable and hence we can extract out all of the roots.

The question of how many inverse functions do we need for a general nth degree polynomial is unknown. We will now discuss other ways to make a polynomial solvable.

3.2 Adjoining Roots of Polynomials

3.2.1 Adding the General Polynomial

Let f be an irreducible polymomial of degree n. Let L be the splitting field of f and let H be the Galois group of f. By the fundamental theorem of Galois theory there is a 1-1 correspondence between subgroups of the Galois group and subfields of the splitting field which preserves indexes. Since $[L : \mathbb{Q}] = m < \infty$ then L is generated by a primitive element say, α [4]. So $L = \mathbb{Q}(\alpha)$. Now letting g be the minimal polynomial of α, we have that $\deg(g) = m$. Therefore, every root of f is of the form

$$\sum_{i=0}^{m-1} q_i \alpha^i$$

where $q_i \in \mathbb{Q}$.

Example 3.2.1. Let $f(x) = x^3 - a$ where $a \in \mathbb{Z}$ is not a perfect cube. It can be shown that $\text{Gal}(f) = S_3$. Let L denote the splitting field of f. Then $L = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{A}$. Let g be the minimal polynomial of α. For this method, we allow g^{-1} as an operation so that we can extract α and thus write the roots of f as linear combinations of powers of α.

In particular using $f(x) = x^3 - 2$ from Example 1.2.20 we know the $\text{Gal}(f) = S_3$. We know $L = \mathbb{Q}(\omega, \sqrt[3]{2}) = \mathbb{Q}(\omega + \sqrt[3]{2})$. The minimal polynomial of $\omega + \sqrt[3]{2}$ is

$$g(x) = x^6 + 3x^5 + 6x^4 + 3x^3 + 9x + 9$$

We want to allow g^{-1} as an operation. By the inverse function theorem we define g^{-1} in a neighborhood of a root of f. Thus adding our new g^{-1} allows us to extract a root, say $\omega + \sqrt[3]{2}$. Observe that we now have all of our roots:

$$\sqrt[3]{2} = 2 + (\omega + \sqrt[3]{2}) + \frac{6}{5}(\omega + \sqrt[3]{2})^2 + -\frac{2}{15}(\omega + \sqrt[3]{2})^3 + \frac{1}{15}(\omega + \sqrt[3]{2})^4 + -\frac{2}{15}(\omega + \sqrt[3]{2})^5$$
$$\omega\sqrt[3]{2} = -1 + 0(\omega + \sqrt[3]{2}) + \frac{2}{15}(\omega + \sqrt[3]{2})^2 + \frac{2}{15}(\omega + \sqrt[3]{2})^3 + -\frac{1}{15}(\omega + \sqrt[3]{2})^4 + \frac{1}{45}(\omega + \sqrt[3]{2})^5$$
$$\omega^2\sqrt[3]{2} = -1 + (\omega + \sqrt[3]{2}) + \frac{4}{3}(\omega + \sqrt[3]{2})^2 + 0(\omega + \sqrt[3]{2})^3 + 0(\omega + \sqrt[3]{2})^4 + -\frac{1}{9}(\omega + \sqrt[3]{2})^5$$

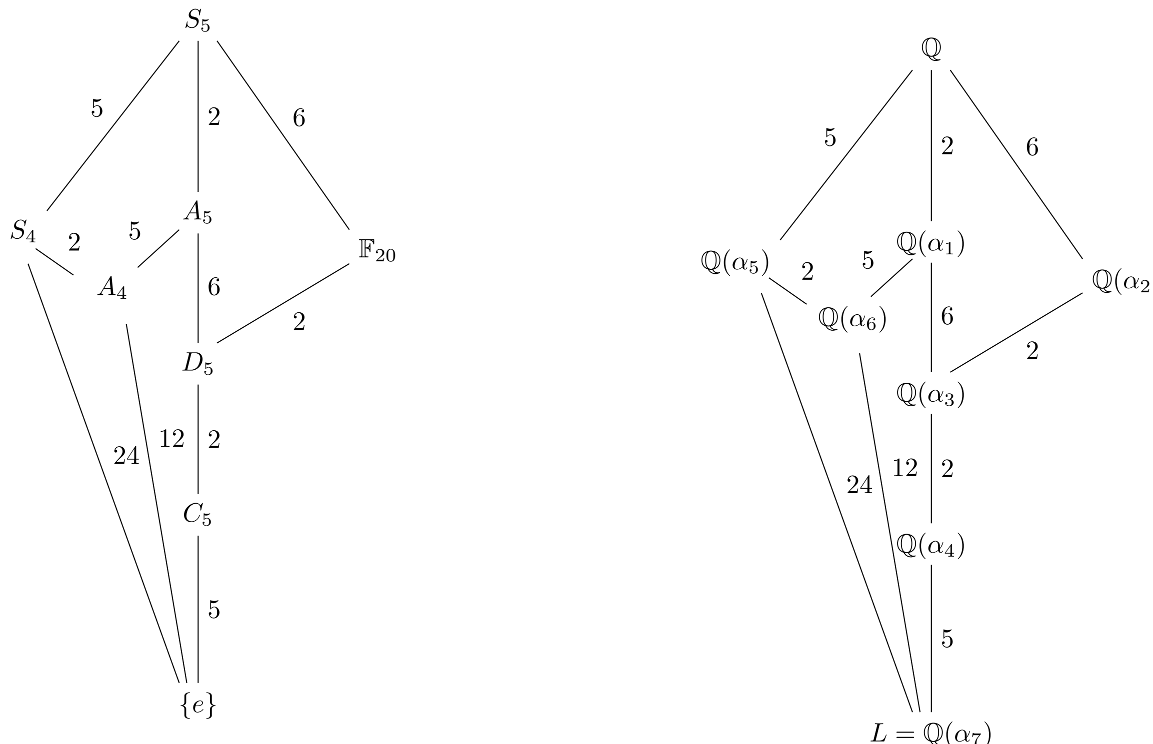
Therefore we only need to assume to know one inverse function, $g^{-1}(y)$. Since attaching any root of it yields the same splitting field as α. However finding the polynomial $g(y)$ poses serious difficulty as there is no known way to calculate it in general. The question arises can we add an inverse function that would make our original polynomial solvable such that $\deg(h) < \deg(g)$. We will discover the answer is always yes. Which leads to the next method.

3.2.2 Finding the Maximal Solvable Subgroup

Recall that there is not a general equations for the roots of a general 5^{th} degree polynomial. This does not mean that every Galois group is not solvable. If the Galois group is solvable then we are done. If it is not, then it must lie over a solvable group. Once we reach a solvable subgroup then the roots of our polynomial can be extracted.

In detail, let f be an irreducible polynomial and L its splitting field. Suppose K is the Galois group of f. Then K contains a maximal solvable subgroup $H \leq K$. By the fundamental theorem of Galois theory we can find $L_K = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{A}$. Let g be the minimal polynomial of α. We then add the inverse function of g, g^{-1}, so that we can consider f over $\mathbb{Q}(\alpha)$. Since $\mathrm{Gal}(L/\mathbb{Q}(\alpha)) = H$ which is solvable then f is solvable by radicals and g^{-1}.

Example 3.2.2. Consider S_5. The transitive subgroups up to isomorphisms are: $C_5, D_5, \mathbb{F}_{20}, A_5, S_5$. The subgroup lattice of the transitive subgroups and maximal solvable subgroups of S_5 and A_5 along with the corresponding subfields are shown on the next page:



(a) Subgroup Lattice of Transitive Subgroups and S_4

(b) Subfield Lattice of the Splitting Field L

Figure 3.1: Partial Subgroup Lattice of S_5

Therefore if $\text{Gal}(f) = S_5$ or $\text{Gal}(f) = A_5$ then we can add a fifth degree extension to take it to S_4 or A_4 respectively. Since both S_4 and A_4 are solvable we are able to extract the rest of the roots.

Example 3.2.3. Returning to our previous example where $f(x) = x^5 - 2x + 2$ we already have shown that $\text{Gal}(f) = S_5$. Notice that $[S_5 : S_4] = [\mathbb{Q}(\alpha_5) : \mathbb{Q}] = 5$. Letting g be the minimal polynomial of α_5 then we allow g^{-1} to be used. Then we consider f over $\mathbb{Q}(\alpha_5)$. Since $\text{Gal}(L/\mathbb{Q}(\alpha_5)) = S_4$ is solvable then f is solvable by radicals and g^{-1}.

Chapter 4

Applying our methods for Sixth and Seventh Degree Polynomials

4.1 Results for 6th Degree Polynomials

4.1.1 Tschirnhaus Transformation

The general sixth degree polynomial has the form

$$f(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Using a Tschirnhaus transformation we can get f in the form

$$x^6 + b_2x^2 + b_1x + b_0.$$

Consider the polynomial

$$h_{b_2,b_1}(x) := x^6 + b_2x^2 + b_1x + b_0.$$

If $b_2 = 0$ and $b_1 \neq 0$ then $h_{0,b_1}(x) := x^6 + b_1x + b_0$. Setting this function equal to zero and using the substitution $x = b_1^{1/5}y$ we get

$$(b_1^{1/5}y)^6 + b_1(b_1^{1/5}y) + b_0 = 0 \Rightarrow y^6 + y + b = 0$$

where $b = \frac{b_0}{b_2^{6/5}}$. So we add f^{-1} where $f(x) = x^6 + x$.
If $b_2 \neq 0$ and $b_1 = 0$ then $h_{b_2,0}(x) := x^6 + b_2x^2 + b_0$ is solvable. To see this, letting $u = x^2$ yields

$$u^3 + b_2u + b_0$$

which is solvable by radicals and then we can recover what x is from $u = x^2$.

If $b_2 \neq 0$ and $b_1 \neq 0$ then using the substitution $x = b_2^{1/4}y$ gives

$$(b_2^{1/4}y)^6 + b_2(b_2^{1/4}y)^2 + b_1(b_2^{1/4}y) + b_0 = 0 \Rightarrow y^6 + y^2 + \bar{b_1}y + \bar{b_0} = 0$$

where $\bar{b_1} = b_1/b_2^{3/2}, \bar{b_0} = b_0/b_2^{3/2}$

We add the family of inverse functions $\{h_{1,b_1}^{-1}\}_{b_1 \in \mathbb{A}}$ along with g^{-1} where $g = x^5 + x$ and f^{-1} defined above to make all sixth degree polynomials solvable.

Example 4.1.1. Let $f(x) = x^6 + 4x^2 - 3x + 2$. Then setting $f(x)$ equal to zero and solving yields:

$$f(x) = x^6 + 15x^2 - 6x - 3 = 0$$
$$x^6 + 15x^2 - 6x = 3$$
$$h_{15,-6}^{-1}(x^6 + 15x^2 - 6x) = h_{15,-6}^{-1}(3)$$
$$x = h_{15,-6}^{-1}(3)$$

Refactoring our original polynomial gives

$$f(x) = \left(x - h_{15,-6}^{-1}(3)\right)\left(x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\right)$$

The second term, $x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, is a quintic and hence solvable by our previous method.

4.1.2 Adjoining Roots

To add the general polynomial we consider an irreducible polynomial f with $\deg(f) = 6$. Let L be its splitting field. Then we consider $\mathrm{Gal}(f) \subset S_6$. As $|S_6| = 6! = 720$ then

$$[L : \mathbb{Q}] = [S_6 : e] = m \leq 720 < \infty$$

So we can find α such that $L = \mathbb{Q}(\alpha)$. Letting g be the minimal polynomial of α we add g^{-1} using the inverse function theorem to make f solvable in terms of g^{-1}.

For out last method we find the maximal solvable subgroup of our Galois group. Letting the setup be as above then we find $H \leq \mathrm{Gal}(f)$ such that H is a solvable subgroup. We first note that there are 16 transitive subgroups of S_6 on 6 points. Out of the 16 there are four that are not solvable. They are $S_6, A_6, PGL(2,5)$, and $PSL(2,5)$.

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K : H]$
S_6	$S_3 \wr 2$	10
A_6	$C_3 \times C_3 \wr C_4$	10
PGL(2,5)	S_4	5
PSL(2,5)	A_4	5

Therefore the highest degree polynomial we need to add the inverse of is 10. Which is significantly better than degree 720 as in the previous method.

4.2 Results for 7th Degree Polynomials

4.2.1 Tschirnhaus Transformation

Using the Tschirnhaus Transformation we can get a general 7th degree polynomial to the form

$$f(x) = x^7 + a_3x^3 + a_2x^2 + a_1x + a_0$$

So we add the family of inverse functions $\{h_{\overline{a}}^{-1}\}_{\overline{a} \in \mathbb{A}}$ where $\overline{a} = (a_1 \quad a_2 \quad a_3)$ along with the family of inverse functions $\{h_{b_2,b_1}^{-1}\}_{b_2,b_1 \in \mathbb{A}}$ and g^{-1}.

4.2.2 Adjoining Roots

We consider the last method of finding the maximal solvable subgroups of our Galois Group. For S_7 we have seven transitive subgroups on 7 points. Three of them are not solvable. They are $S_7, A_7,$ and $GL(3,2)$. We have the following Maximal Subgroups,

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K:H]$
S_7	$S_4 \times S_3$	35
A_7	$(C_3 \times A_4) \wr C_2$	35
GL(3,2)	S_4	7

Chapter 5

Maximal Solvable Subgroups Up to Order 11

5.1 Results Up to Degree 11

All results here were provided by GAP including the group description. The following code used to obtain the results is listed below:

```
n:= 5;; G:=Group([PermList(Concatenation([2..n],[1])),(1,2)]);; L:=[];; K:=[];;
for x in ConjugacyClassesSubgroups(G) do Add(L,Representative(x)); od;
for x in L do if not IsSolvableGroup(x) and IsTransitive(x,[1..n]) then Add(K,x);
else continue; fi; od;
S:=[];;T:=[];;

for x in K do for g in ConjugacyClassesSubgroups(x)
    do Add(T, Representative(g)); od;
    for y in Reversed(T) do if IsSolvableGroup(y) then
    Add(S,[x,Order(x), y, Order(y), Order(x)/Order(y)]); T:=[]; break; else continue;
    fi; od;
od;
S;
```

If the description of the maximal solvable subgroup is too long we simply list its order.

Degree 8:

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K:H]$
S_8	576	35
A_8	$((A_4 \times A_4) \wr C_2) \wr C_2$	35
PSL(3,2) $\wr C_2$	$C_7 \wr C_6$	8
PSL(3,2)	S_4	7

Degree 9:

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K:H]$
S_9	1296	280
A_9	648	280
PSL(2,8) $\wr C_3$	168	9
PSL(2,8)	$(C_2 \times C_2 \times C_2) \wr C_7$	9

Degree 10:

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K : H]$
S_{10}	$C_2 \times ((S_4 \times S_4) \wr C_2)$	1575
A_{10}	1152	1575
$(A_5 \times A_5) \wr D_8$	$(S_4 \times S_4) \wr C_2$	25
$(A_5 \times A_5) \wr C_4$	$(A4 \times A4) \wr C_4$	25
$(A_5 \times A_5) \wr (C_2 \times C_2)$	$((A4 \times A4) \wr C_2) \wr C_2$	25
$(A_5 \times A_5) \wr C_2$	288	25
$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \wr S5)$	288	25
$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \wr A5)$	384	5
$(C_2 \times C_2 \times C_2 \times C_2) \wr S5$	384	5
$(C_2 \times C_2 \times C_2 \times C_2) \wr S5$	384	5
$(A_6 \wr C_2) \wr C_2$	144	10
$(C_2 \times C_2 \times C_2 \times C_2) \wr A5$	192	5
$A_6 \wr C_2$	$(C_3 \times C_3) \wr Q8$	10
$A_6 \wr C_2$	$(C_3 \times C_3) \wr C8$	10
S_6	$(S3 \times S3) \wr A_5$	10
A_6	$(C_3 \times C_3) \wr C_4$	10
$C_2 \times S_5$	$A_5 \times S_4$	5
S_5	S_4	5
$C_2 \times A_5$	$A_5 \times A4$	5
S_5	S_4	5
A_5	A_4	5

Degree 11:

Transitive Subgroup: K	Maximal Solvable Subgroup: H	$[K : H]$
S_{11}	$((S_4 \times S_4) \wr C_2) \times S3$	5775
A_{11}	$((C_3 \times A_4 \times A_4) \wr C_4) \wr C_2$	5775
M_{11}	$(C_3 \times C_3) \wr QD_{16}$	55
PSL(2,11)	$C_{11} \wr C_5$	12

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