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On Complete Integral Closure of Integral Domains

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Abstract

Given an integral domain D with quotient field K, an element x in K is called integral over D if x is a root of a monic polynomial with coefficients in D. The notion of integrality has roots in Dedekind's work with algebraic integers, and was later developed more rigorously by Emmy Noether. Different variations or generalizations of integrality have since been studied, including almost integrality and pseudo-integrality. In this work we give a brief history of integrality and almost integrality before developing the basic theory of these two notions. We will continue the theory of almost integrality further by examining anchor ideals of almost integral elements and by presenting a domain which sheds light on iterations of complete integral closure. Some time is also spent on developing pseudo-integrality and other generalizations.

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Chapter 1

Introduction

For an integral domain D with quotient field K, we say an element $x \in K$ is *integral* over D if x satisfies a monic polynomial with coefficients in D. The set of all elements in the quotient field of D which are integral over D forms a ring $\bar{D}$ and is called the *integral closure* of D. If $\bar{D} = D$, we say that D is *integrally closed.* Furthermore, we say that $x \in K$ is *almost integral* over D if there exists some $d \in D$ such that $dx^k \in D$ for all $k \in \mathbb{N}$. The set of all elements in K which are almost integral over D forms a ring D', called the *complete integral closure* of D. If $D' = D$, we say that D is *completely integrally closed*. If $x \in K$ is integral over D, then x is almost integral over D. Hence we have that $D \subseteq \bar{D} \subseteq D' \subseteq K$. The notions of integrality and almost integrality were first developed rigorously by Emmy Noether [39] and Wolfgang Krull [28] in the late 1920s and early 1930s. Integral closure has played a role in number theory and algebraic geometry. For example, the algebraic integers are exactly the complex number which are integral over $\mathbb{Z}$.

The majority of this work will focus on almost integrality, but we will lay a foundation with a discussion of integrality. Chapter 2 contains a brief history of the integral and complete integral closure. This history has given the author a greater appreciation for his current work by understanding the work of those before him. The history also gives insight into how integrality and almost integrality are related. Chapter 3 contains a brief overview of integrality and integral closure. Chapter 4 is the focus of this work and, as such, contains foundational results about almost integrality, as well as newer results concerning anchor ideals. A significant portion of this content is devoted to presenting an example which sheds light on iterations of complete integral closure. Chapter 5 contains basic results for other generalizations of integrality such as pseudo-integrality and $w-$integrality. Lastly, the Appendix contains some definitions and insightful results for concepts that may not be as well known. Content in the appendix is listed in the order that it is referenced to in the main body of work.

With this organization of content in mind, let us consider briefly part of the motivation behind this project. In Chapter 3 we will show that the integral closure of a domain D is equal to the intersection of all valuation overrings between D and its quotient field K (see Theorem 24). This is due in part to the fact that a valuation domain is integrally closed. On the other hand, a valuation domain is completely integrally closed if and only if it is of rank 1 (see Theorem 40). Moreover, the complete integral closure of D is not generally equal to the intersection of all rank 1 valuation overrings between D and K. But we also claim in Chapter 4 that the complete integral closure is integrally closed, and is thus the intersection of all its valuation overrings in K. One eventual goal is to characterize exactly which valuation overrings these are. With this motivation in mind, we now set out to lay a foundation for some of the results presented in Chapters 3 and 4. In particular, we give an introduction to valuation domains.

Definition 1. A *valuation domain* V is an integral domain with the property that for any $a, b \in V$ either $a|b$ or $b|a$.

Examples of valuation domains include any field as well as the localization of $\mathbb{Z}$ at a prime ideal. The following are some basic properties of valuation domains.

Lemma 2. *A valuation domain V with quotient field K has the following properties:*

1. *the ideals of V are totally ordered by inclusion;*
2. *any overring T of V is a valuation ring, moreover, if M is the unique maximal ideal of T, then $T = V_{M \cap V}$;*
3. *for any nonzero $u \in K$ either $u \in V$ or $u^{-1} \in V$;*
4. *V is integrally closed.*

Proof. We prove property 1 to give insight into the workings of valuation domains. Let I, J be two ideals in V. Suppose that $I \not\subseteq J$. We will show that $J \subseteq I$. Pick $a \in I \setminus J$ and note that $a \neq 0$. Now let $b \in J$. If $b = 0$, then $b \in I$ and we are done. So assume $b \neq 0$. Note that we must have $a|b$. For if not, then $b|a$ which implies that $a \in J$, which is a contradiction. Thus, $ac = b$ for some $c \in V$, so $b \in I$ and we are done. $\square$

We leave the other properties without proof, although we do give some reasoning as to why a valuation domain is integrally closed (see Corollary 19). Note that property 2 implies that every valuation overring of V can be realized as V_S for a multiplicatively closed set S. That is, every overring of V is a ring of quotients of V. When working with valuation domains, we often consider their rank. In order to define what we mean by rank, we must first define the Krull dimension of a ring.

Definition 3. We say that a chain of prime ideals of the form $P_0 \subset P_1 \subset \cdots \subset P_n$ has *length* n. The *Krull dimension* of a ring R is the supremum of the lengths of all chains of prime ideals in R.

Definition 4. The *rank* of a valuation domain V is the Krull dimension of V.

Of particular importance are valuation domains of rank 1, which contain a unique nonzero prime (maximal) ideal. A valuation domain V is completely integrally closed if and only if V has rank 1.

Now in order to verify our claim that the integral closure of a domain D is the intersection of all valuations between D and K, we will make use of the following result from Kaplansky (see [25], Theorem 56).

Theorem 5. *Let K be a field, R a subring of K, and I a proper ideal of R. Then there exists a valuation domain V with $R \subseteq V \subseteq K$, such that K is quotient field of V and I survives in V, that is, $IV \neq V$.*

The proof of this result uses property 3 from Lemma 2 and the fact that for any nonzero $u \in K$ and ideal $I \subset R$, I survives in either $R[u]$ or $R[u^{-1}]$ (see [25], Theorem 55).

These results concerning valuation domains will be useful for us in Chapters 3 and 4. Let us now dive into the history of integrality and almost integrality, which will hopefully give the reader a greater appreciation for these concepts.

Chapter 2

A History

2.1 Integral Closure

The notions of integral elements and integral closure have their beginnings in the work of Richard Dedekind. In particular, Dedekind generalized the concept of the integers to that of the algebraic integers (see [10], pg. 95). The algebraic integers make up the integral closure of $\mathbb{Z}$ in some finite extension field of $\mathbb{Q}$. His first written definition of algebraic integers appeared in the year 1871 in his Supplement X, §160, to the second edition of Dirichlet's *Vorlesungen über Zahlentheorie* (*Lectures on Number Theory*) [14]. His definition was

> Wir wollen nun... eine Zahl α eine *ganze algebraische Zahl* nennen, wenn sie die Wurzel einer Gleichung ist, deren Coefficienten rationale ganze Zahlen sind...

which is translated to

> Now we want...to call a number α an *algebraic integer* if it is the root of an equation whose coefficients are rational integers...

where the term "rational integers" refers to the integers $\mathbb{Z}$ ([13], pg 53). Dedekind also includes requirements for these integer coefficients, which seem to be equivalent to the requirement that the leading coefficient be 1. This can be understood from his remark directly following this definition that a rational number is an algebraic integer if and only if it is an integer. (The original German text with these requirements is slightly unclear so we did not include it here.) The algebraic integers are a generalization of the integers, moreover, integral elements are essentially a generalization of the algebraic integers.

Now Dedekind also in some sense developed the notion of integral closure. That is, he noted that a root of any monic polynomial with algebraic integer coefficients was again an algebraic integer. Although he did not use the modern day terminology of integral closedness, he relied on this fact about the algebraic integers when proving results about factorization.

Indeed this link between integral closedness and factorization is essentially what led to the modern study of the concept. But first, a true general definition of abstract rings needed to be developed. An axiomatic definition similar to the modern definition wasn't given until 1917 by Masazo Sono in his work "On Congruences" ([26] p. 59) He essentially extended Fraenkel's 1914 list of axioms (see [15], [6]). It was around this time that Emmy Noether also began using the notion of integral elements and integral closure. In a note in his book *Idealtheorie* (*Ideal Theory*) [29], Wolgang Krull calls it "historically remarkable" that Noether used the notion of integral closure as early as 1916 [38]. In her seminal 1927 paper "Abstrakter Aufbau der Idealtheorie in algebraischen Zahlund Funktionenkorpen" ("Abstract Construction of Ideal Theory in Algebraic Number Fields and Function Fields"), Noether refers to the "usual" definition of an integral element which is indeed the standard modern definition [39]. Indeed, in *Idealtheorie* Krull gives the modern definition of an integral element "after E. Noether" [29]. Gilmer also attributes the develop of integrality to Noether (see [16] pg. 83). But one should note that the property of integral closure as presented by Noether and Krull in the aforementioned papers was mainly studied in the connection to the following result [23]:

Proposition 6. *Let R be an integral domain, then every ideal of R is a product of prime ideals if and only if the following three properties are satisfied:*

1. *the ascending chain condition on ideals of R holds,*
2. *all nonzero proper prime ideals of R are maximal, and*
3. *R is integrally closed.*

An example of a ring satisfying Proposition 6 is a principal ideal domain. Now the fact that integral elements were often studied in rings that also satisfied the ascending chain condition (i.e., Noetherian) played a critical role in the development of the notion of almost integrality, as we shall see later.

Of course, Noether published other results between these 1915 and 1927 papers, but it is unclear to the author exactly when the "usual" definition of an integral element in the general setting of commutative ring was developed. Indeed, most citations concerning integral elements only go back as far as Emmy Noether's 1927 work. Due to Krull's recognition of Emmy Noether's definition, and given the fact that he was one of her students, it seems reasonable to suggest that Emmy Noether is responsible for the modern definition of an integral element.

2.2 Complete Integral Closure

We now turn to the development of complete integral closure. Recall that the complete integral closure is comprised of almost integral elements. It seems that Wolfgang Krull is the first to introduce the notion of almost integral elements. The definition that we commonly use is found in his 1928 work "Zur Theorie der allgemeinen Zahlringe" ("The Theory of General Number Rings") [27]. However, it seems that Krull was actually just giving an alternative definition for integral elements, since he was working with Noetherian rings. On page 60 we find the definition:

> Ein Element α aus dem Quotientenkörper $\mathfrak{K}$ heißt "von $\mathfrak{R}$ ganz abhängig," wenn es in $\mathfrak{R}$ ein Element $a \neq 0$ gibt, dessen Produkt mit einer beliebigen Potenz von α stets zu $\mathfrak{R}$ gehört

This is roughly translated to

> An element α from the quotient field $\mathfrak{K}$ is "integrally dependent on $\mathfrak{R}$" if there exists an element $a \neq 0$ in $\mathfrak{R}$ whose product with any power of α always belongs to $\mathfrak{R}$.

Krull provides a note explaining that his definition is equivalent to the standard definition for an integral element if $\mathfrak{R}$ satisfies the ascending chain condition.

Krull later makes it very clear that his definition of "ganz abhängig" elements can extend beyond Noetherian domains and retain interesting properties. In his 1932 article "Allgemeine Bewertungstheorie" ("General Valuation Theory") [28], Krull says the following:

> Das Element p aus $\mathfrak{K}$ soll von $\mathfrak{R}$ "fast ganz abhängig" heißen, wenn $\mathfrak{R}[p]$ zwar nicht notwendig über $\mathfrak{R}$ eine endliche Modulbasis besitzt, wenn sich aber wenigstens alle Elemente aus $\mathfrak{R}[p]$ als Quotienten von Elementen aus $\mathfrak{R}$ mit festem Nenner Darstellen lassen, d.h. wenn in $\mathfrak{R}$ ein festes Element $n \neq 0$ existiert, für das die Produkte $n \cdot p^i$ $(i = 1, 2, \dots)$ sämtlich zu $\mathfrak{R}$ gehören. Der Ring

$\mathfrak{R}$ soll "voll und ganz abgeschlossen" heißen, wenn alle von $\mathfrak{R}$ fast ganz abhänngigen Elemente aus $\mathfrak{K}$ bereits in $\mathfrak{R}$ vorkommen.

Which is roughly translated:

The element p of $\mathfrak{K}$ [the quotient field of $\mathfrak{R}$] is called "almost integrally dependent" on $\mathfrak{R}$, if $\mathfrak{R}[p]$ does not necessarily have a finite module basis over $\mathfrak{R}$, but if at least all elements of $\mathfrak{R}[p]$ can be represented as the quotients of elements with fixed denominators, i.e. if in $\mathfrak{R}$ there is a fixed element $n \neq 0$ for which the products $n \cdot p^i$ $(i = 1, 2, \ldots)$ all belong to $\mathfrak{R}$. The ring is called "completely closed" when all elements almost integrally dependent on $\mathfrak{K}$ already exist in $\mathfrak{R}$.

Again Krull makes a note that this definition which he uses for almost integral elements was used previously by others, such as Noether and van der Waerden in connection with integral elements. For example, in [44] (an English translation of van der Waerden's *Modern Algebra*), van der Waerden uses the definition that for rings $S \subseteq R$, and element $r \in R$ is integral over S if all powers of r belong to a finite S-module. This is precisely the definition of almost integral. We should add that while Krull's terminology *almost integral* has become standard, some have also used the term *quasi-integral* (see [42], [40]). However, such instances of varied terminology are not common.

Chapter 3

Integral Closure

In this chapter we explore the notion of integral closure and lay a foundation for our work in later chapters. The goal of this chapter is to list important definitions, theorems, and examples. Proofs that provide useful insight for our purposes will be included, but for other results we will merely include a citation to a work containing the proof. All rings are assumed to be commutative with identity unless otherwise stated. We begin by defining integral elements over a ring.

Definition 7. Let $R \subseteq T$ be commutative rings with identity. An element $u \in T$ is *integral* over R if u is the root of a monic polynomial with coefficients in R.

The set of all elements of T which are integral over R, denoted $\overline{R_T}$, is the integral closure of R in T. If $\overline{R_T} = T$, we say that T is integral over R. Note that R is always contained in $\overline{R_T}$ (an element $u \in R$ satisfies the monic polynomial $f(x) = x - u$). If $\overline{R_T} = R$, we say that R is integrally closed in T. In the special case that T is the total quotient ring of R, we have the following definition.

Definition 8. Let T be the total quotient ring of R, then the set of elements in T which are integral over R is the *integral closure* of R, which we denote by $\bar{R}$. If $\bar{R} = R$, then we say R is *integrally closed*.

We now provide an equivalent characterization of integrality, which we will mainly make use of when studying generalizations of integrality.

Theorem 9. *Let D be an integral domain with quotient field K. Then $u \in K$ is integral over D if and only if $uI \subseteq I$ for some nonzero finitely generated ideal I of D*

Proof. Suppose that $u \in K$ is integral over D. Then there is some n such that

$$u^n + r_{n-1}u^{n-1} + \cdots + r_0 \quad = \quad 0 \tag{3.1}$$

with $r_j \in D$. Set $u = \frac{a}{b}$ with $a, b \in D$. Then take $I = (ub^n, u^2b^n, \ldots, u^nb^n)$. Note that we can use (3.1) to represent u^{n+1} as a combination of $u, u^2, \ldots, u^n$. It follows that $uI \subseteq I$ as desired.

On the other hand, suppose $uI \subseteq I$ for some nonzero finitely generated ideal I of D. Say $I = (r_1, \ldots, r_n)$. Notice that we have the system of equations $ur_i = \sum_{j=1}^{n} a_{ij}r_j$ for $a_{ij} \in R$. It follows that the following determinant is 0.

$$\begin{vmatrix} u - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & u - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & u - a_{nn} \end{vmatrix}$$

This determinant will yield a monic polynomial with u as a root. That is, u is integral over R. $\square$

Example 10. The integral closure of $\mathbb{Z}$ in $\mathbb{Q}$ is $\mathbb{Z}$. That is, $\mathbb{Z}$ is integrally closed.

The previous example is a special case of the next proposition.

Proposition 11. *A UFD is integrally closed.*

Proof. Let D be an integral domain with quotient field K. Let $\alpha = \frac{a}{b} \in \bar{D}$ with $a \neq 0$. Since D is a UFD, we can factor a and b into primes and so assume that a and b have no common prime factors. Since $\alpha \in \bar{D}$, there exists $a_0, a_1, \ldots, a_{n-1} \in D$ such that

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0.$$

Multiplying both sides of this equation by b^n, we have

$$a^n + a_{n-1}a^{n-1}b + \cdots + a_0b^n = 0$$

which implies

$$b(a_{n-1}a^{n-1} + \cdots + a_0 b^{n-1}) = -a^n.$$

Hence, if p is a prime such that $p|b$, then $p|a^n$. Of course this implies $p|a$. But this contradicts the facts that a and b have no common prime factors. Therefore, b has no prime factors and must be a unit in D. Thus $\alpha = \frac{a}{b} \in D$. Since $\alpha \in \bar{D}$ was arbitrary, we have $\bar{D} = D$ as desired. □

This next theorem provides an alternative characterization of integral elements which we will use to show that the integral closure of D is a ring. This is Theorem 12 in [25]. The proof of this theorem is similar to that of Theorem 9, so we omit the proof here.

Theorem 12. *Let R be a commutative ring with identity, T a commutative R-algebra, and $\alpha \in T$. Then α is integral over R if and only if there exists a finitely generated R-submodule A of T such that $\alpha A \subseteq A$, the annihilator of A in T is 0, and $1 \in A$.*

Theorem 13. *Let T be an overring of R, and let $\alpha, \beta \in T$ be integral over R. Then $\alpha + \beta$ and $\alpha\beta$ are integral over R.*

Proof. By Theorem 12, we can take appropriate modules A and B for α and β, respectively, such that $1 \in A, B$. Then the product AB is also finitely generated, contains 1 and satisfies both $(\alpha+\beta)AB \subseteq AB$ and $(\alpha\beta)AB \subseteq AB$. Then again by Theorem 12, we have that $\alpha + \beta$ and $\alpha\beta$ are integral over R. □

The following corollary is immediate:

Corollary 14. *The integral closure of R in T is a subring of T. Moreover, if R and T have the same quotient field, then the integral closure of R in T is an overring of R.*

The following theorem resembles the definition of integrality given on page 86 of [16].

Theorem 15. *Let $R \subseteq T$ be rings. Then $u \in T$ is integral over R if and only if $R[u]$ is a finitely generated R-submodule of T.*

Proof. If $u \in T$ is integral over R, then $u^n + a_{n-1}u^{n-1} + \cdots + a_0 = 0$ for some $a_i \in R$. Then, u^m for $m \geq n$ can be expressed as a linear combination over R of $1, u, \ldots, u^{n-1}$. Hence any element of $R[u]$ can be expressed as polynomial in u with only the powers $1, u, \ldots, u^{n-1}$. That is, $R[u]$ is generated by $1, u, \ldots, u^{n-1}$.

On the other hand, if $R[u]$ is finitely generated, say by $1, u, \dots, u^{n-1}$. Then we have $u^n = a_0 + a_1 u + \cdots + a_{n-1}u^{n-1}$ for $a_i \in R$. Hence u is integral over R. □

As a consequence of Theorem 15, we can find even more characterizations of integral elements. For the following result, recall that a R-module A is called *faithful* if there is no nonzero element $r \in R$ such that $rA = 0$.

Theorem 16. *Let $R \subseteq T$ be rings, and let $u \in T$. The following conditions are equivalent:*

1. *u is integral over R.*

2. *$R[u]$ is contained in a subring of T which is a finitely generated R-module.*

3. *There is a finitely generated R-module S contained in T such that S is a faithful $R[u]$-module.*

The equivalences in the above theorem are relatively straightforward, but a proof can be found on page 85 of [16].

The following theorem is a statement of what we call the *transitivity* of integral closure. During our investigation of almost integral closure in the next chapter, we will see that transitivity is an important property which distinguished integral closure and almost integral closure.

Theorem 17. *Let $R \subseteq S \subseteq T$ be rings. If S is integral over R, and T is integral over S, then T is integral over R.*

Proof. Let $u \in T$. Since u is integral over R, we have

$$u^n + s_{n-1}u^{n-1} + \cdots + s_0 = 0$$

for some $s_i \in S$. Then u is integral over $R[s_0, s_1, \dots, s_{n-1}]$. Hence, $R[s_0, s_1, \dots, s_{n-1}, u]$ is finitely generated as an $R[s_0, s_1, \dots, s_{n-1}]$ module. Moreover, $R[s_0, \dots, s_{n-1}]$ is finitely generated R-module since each s_i is integral over R. Thus $R[s_0, \dots, s_{n-1}, u]$ is a finitely generated R-module. Hence u is integral over R, which implies that T is integral over R. □

A consequence of Theorem 17 is that the integral closure is integrally closed. For if an element u is integral over the integral closure of a ring R, then u is integral over R, and hence in the integral closure of R.

We next investigate when a ring is integrally closed. Of course we saw in the Proposition 11 that any UFD is integrally closed. There are however, many other cases to consider.

One large class of integrally closed domains is the class of GCD-domains (see Definition 98 in the appendix). We state this result as a theorem below; however, the proof is similar to that given for Proposition 11, so we do not include it here.

Theorem 18. *A GCD-domain is integrally closed.*

Note that a UFD is a GCD-domain, so Proposition 11 could be viewed as a consequence of Theorem 18. Moreover, every valuation domain is a GCD-domain and is hence integrally closed. This is a substantial result, so we record it as a corollary below.

Corollary 19. *A valuation domain is integrally closed.*

We next give what proves to be an alternate, although perhaps more complicated, approach to showing Corollary 19. However, this approach provides other useful insights into the behavior of integral closure.

Theorem 20. *If $R \subseteq T$ and T is integral over R, then $U(T) \cap R = U(R)$.*

Proof. Of course, if $u \in U(R)$, then $u \in U(T) \cap R$. So suppose that $u \in U(T) \cap R$. Then note $1/u \in T$ and is thus integral over R. So $1/u$ satisfies a monic polynomial with coefficients in R, say $(1/u)^n + a_{n-1}(1/u)^{n-1} + \cdots + a_0 = 0$ with $a_i \in R$. Hence, $(1/u)^n = -a_{n-1}(1/u)^{n-1} - \cdots - a_0$. Then multiplying by u^{n-1} gives us

$$\frac{1}{u} = -a_{n-1} - a_{n-2}u - \cdots - a_0 u^{n-1} \in R$$

since $u \in R$. That is, $u \in U(R)$. Hence we have shown that $U(T) \cap R = U(R)$. □

As a result of Theorem 20, we have the following corollary (see [12], Lemma 1).

Corollary 21. *If the integral closure of the integral domain R is a ring of quotients of R, then R is integrally closed.*

As noted in the introduction, every overring of a valuation domain V can be realized as V_S for some multiplicatively closed set S, so we could also arrive at Corollary 19 as a consequence of Corollary 21.

The following lemma (see [25] Theorem 52) is obvious but also useful.

Lemma 22. *An intersection of integrally closed domains is integrally closed.*

We now give a useful lemma which will be used to prove Theorem 24 which was instrumental in providing the motivation behind this project.

Lemma 23. *Let R be a ring and u an invertible element of an overring of R. Then u^{-1} is integral over R if and only if $u^{-1} \in R[u]$.*

Proof. Note u^{-1} is integral over R if and only if

$$u^{-n} + a_{n-1}u^{1-n} + \cdots + a_1 = 0 \ (a_i \in R)$$

if and only if

$$u(a_{n-1} + \cdots + a_1 u^{n-1}) = -1$$

if and only if $u^{-1} \in R[u]$. □

The following theorem provides a remarkable connection between the integral closure of a domain R and the valuation domains between R and its quotient field K (see [25], Theorem 57).

Theorem 24. *Let R be a domain with quotient field K. Then $\bar{R} = \cap V_\alpha$ where the V_α's are the valuation domains between R and K.*

Proof. ($\subseteq$) If u is integral over R then u is integral over each V_α. Hence, by Corollary 19, u is in each V_α. Thus $u \in \cap V_\alpha$ and $\bar{R} \subseteq \cap V_\alpha$.

($\supseteq$) Suppose that $u \in \cap V_\alpha$. We want to show that $u \in \bar{R}$. Assume $u \notin \bar{R}$. Then $u \notin R[u^{-1}]$ by Lemma 23. Thus u^{-1} is not invertible in $R[u^{-1}]$, that is, (u^{-1}) survives in $R[u^{-1}]$. Hence, by Theorem 5, we can enlarge $R[u^{-1}]$ to a valuation domain $V \subseteq K$ such that (u^{-1}) also survives in V. However, $u \in V$ by construction. This is a contradiction, so we must have $u \in \bar{R}$. Hence $\bar{R} = \cap V_\alpha$. □

It follows from the combination of Theorem 24, Corollary 19, and Lemma 22 that the integral closure of a ring is integrally closed.

Given two rings $R \subseteq T$ it is natural to study the relations between the prime ideals of R and prime ideals of T. In particular, we consider the case when T is integral over R. We have the following result, which is given in [31] and [25]. This result involves the concepts of *going up* and *going down*; see Definition 99 in the Appendix for more details.

Theorem 25. *Let $R \subseteq T$ be rings with T integral over R. Then the pair R, T satisfies GU and INC.*

The proof given here models that given in [25].

Proof. To show that GU holds, it suffices to show that if $P \subseteq R$ is prime, and S is the complement of P in R, and Q is a (prime) ideal in T maximal with respect to the exclusion of S, then $Q \cap R = P$ (see Theorem 100 in the Appendix). Now $Q \cap R \subseteq P$ is clear, since $Q \cap S = \emptyset$. If equality does not hold, then there exists some $u \in P \setminus (Q \cap R)$. Hence, the ideal (Q, u) properly contains Q. By the maximality of Q with respect to exclusion of S, we must have $(Q, u) \cap S \neq \emptyset$. So there exists some $s = q + au \in (Q, u) \cap S$ with $q \in Q, a \in T$. Now then a is integral over R, so we have

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$$

with each $c_i \in R$. Multiplying this equation by u^n, we have

$$(au)^n + c_{n-1}u(au)^{n-1} + \cdots + c_0u^n = 0$$

Now $au = s - q$, so we have $au \equiv s \pmod{Q}$ which implies

$$s^n + c_{n-1}us^{n-1} + \cdots + c_0u^n \equiv 0 \pmod{Q}$$

Note that the left side of the above equation is in R, as $s, u, c_i \in R$, and hence in $Q \cap R \subseteq P$. Since $u \in P$, we get that $s^n \in P$ which implies $s \in P$. This is a contradiction. Therefore, we have equality $Q \cap R = P$, and GU holds.

Now we show INC. It suffices to show that if $P \subseteq R$ is prime and $Q \in T$ is a prime ideal which contracts to P in R, then Q is maximal with respect to the exclusion of $S = R \backslash P$ (see Theorem 101 in the Appendix). So suppose that $Q \cap R = P$, and assume that Q is properly contained in an ideal J with $J \cap S = \emptyset$. Take $u \in J \setminus Q$. Since $u \in J \subseteq T$, u is integral over R. Now among all monic polynomials f with coefficients in R such that $f(u) \in Q$, we pick one of least degree, say

$$u^n + a_{n-1}u^{n-1} + \cdots + a_0.$$

Note that $n \geq 1$. Now the above expression is in $Q \subseteq J$, so since $u \in J$ it follows that $a_0 \in J$. Hence,

$a_0 \in (J \cap R) \subseteq P \subseteq Q$. Thus we have that

$$u(u^{n-1} + a_{n-1}n^{n-1} + \cdots + a_1) \in Q,$$

but neither factor is in Q. But Q is prime, so this is a contradiction. Hence, it must be that Q is maximal with respect to the exclusion of S, and INC holds. □

A corollary of Theorem 25 is that if T is integral over R, then the dimension of T equals that of R.

Given a ring R it is natural to investigate the properties of the integral closure of $R[x]$ in relation to those of R. The method we will use to do this follows that found in [17]. We begin with the following theorem; a proof can be found in [17].

Theorem 26. *Let $R \subseteq T$ be rings and x an indeterminate over T. If $f(x) \in R[x]$ and if in $T[x]$, $f(x) = g(x)h(x)$ where $h(x)$ is monic, then the coefficients of $g(x)$ are integral over R.*

Theorem 26 has two important corollaries.

Corollary 27. *Let $R \subseteq T$ be rings and let x be an indeterminate over T. Then R is integrally closed in T if and only if $R[x]$ is integrally closed in $T[x]$.*

Proof. That $R[x]$ is integrally closed in $T[x]$ implies R is integrally closed in T is clear.

On the other hand, suppose that R is integrally closed in T. If $f(x) = \sum_{i=0}^{k} a_i x^i \in T[x]$ is integral over $R[x]$, then $x^{k+1} + f(x)$ is also integral over $R[x]$. Hence, in showing that the coefficients of $f(x)$ are integral over R, there is no loss of generality in assuming $f(x)$ is monic.

Now let $f(x)$ satisfy a monic polynomial $t(y)$ in $R[x][y]$ of degree $k \geq 1$. To show that $f(x) \in R[x]$, we will use induction on k. For $k = 1$, it is clear that $f(x) \in R[x]$. Now, assuming that $f(x) \in R[x]$ for $k \leq n$, we also assume that $\sum_{i=0}^{n+1} d_i(x) f^i(x) = 0$ where $d_i(x) \in R[x]$ and $d_{n+1}(x) = 1$. Then in $T[x]$, we have $d_0(x) = f(x)\left(-\sum_{i=0}^{n} d_{i+1}(x) f^i(x)\right)$ where $f(x)$ is monic. By Theorem 26, the coefficients of $-\sum_{i=0}^{n} d_{i+1}(x) f^i(x) = q_0(x)$ are integral over R and $q_0(x) \in T[x]$. Hence $q_0(x) \in R[x]$ and we have

$$f^n(x) + \cdots + d_2(x)f(x) + d_1(x) + q_0(x) = 0$$

The induction hypothesis then yields the desired conclusion that $f(x) \in R[x]$. Hence $R[x]$ is integrally closed in $T[x]$. $\square$

The following corollary is essentially a more general statement of Corollary 27.

Corollary 28. *Let $R \subseteq T$ be rings, let x be an indeterminate over T, let $\overline{R_T}$ be the integral closure of R in T. Then $\overline{R_T}[x]$ is the integral closure of $R[x]$ in $T[x]$.*

Corollary 28 can be extended polynomial rings with any finite number of indeterminates.

We are also interested in studying the integral closure of $R[[x]]$ in relation to that of R. In general the integral closure of $R[[x]]$ is not as well behaved as that of $R[x]$ when viewed in relation to R. This is shown by the following theorem (an equivalent theorem can be found in [42]).

Theorem 29. *Let D be an integrally closed domain. Then $D[[x]]$ integrally closed, implies $\cap_{i=0}^{\infty} a^i D = 0$ for every nonunit $a \in D$.*

Corollary 30. *D is integrally closed does not imply $D[[x]]$ is integrally closed.*

We will prove Corollary 30 by constructing an integrally closed domain D such that $D[[x]]$ is not integrally closed. However, our construction relies on results concerning complete integral closure. The desired example is Example 53, which is found in the following chapter.

Chapter 4

Complete Integral Closure

As alluded to in the brief history presented at the beginning of this work, the definition of almost integral elements takes different forms depending on the context in which it is presented. Our context is commutative rings in general and integral domains in particular. For commutative rings we use the following definition:

Definition 31. Let $R \subseteq T$ be rings. An element of $u \in T$ is almost integral over R if all powers of u belong to a finite $R-$submodule of T.

We call the set of all elements of T which are almost integral over R the *complete integral closure of* R *in* T. Of course every element in R is almost integral over R. However, if every element in T which is almost integral over R is in R, we say that R is *completely integrally closed in* T. In the special case that T is the total quotient ring of R, we have the following definition.

Definition 32. Let T be the total quotient ring of R, then the set of elements in T which are almost integral over R is the *complete integral closure* of R, which we denote by R'. If $R' = R$, then we say R is *completely integrally closed*.

Note from the definition of almost integral, it is clear that R' is a ring.

Now, in the case that R is a domain and T is the quotient field of R, we have the following useful characterization of almost integrality.

Proposition 33. *Let R be a domain with quotient field T. An element $u \in T$ is almost integral over R if there exists a nonzero element r of R such that $ru^k \in R$ for each $k \in \mathbb{N}$.*

Proof. Suppose that $u \in T$ is almost integral over R. Then all powers of u are in a finite $R-$submodule of T with generators $t_1, \ldots, t_n \in T$ So $u^k = r_{k_1}t_1 + \cdots r_{k_n}t_n$ for some $r_{k_j} \in R$. Now, each $t_j = \frac{a_j}{b_j}$ for some $a_j, b_j \in R$. Hence, we have

$$u^k = r_{k_1}\frac{a_1}{b_1} + \cdots + r_{k_n}\frac{a_n}{b_n}.$$

Thus, $b_1 \cdots b_n u^k \in R$ for all $k \in \mathbb{N}$.

On the other hand, let $u \in T$ and suppose there exists some nonzero element $r \in R$ such that $ru^k \in R$ for each $k \in \mathbb{R}$. That is, there exists $a_k \in R$, such that $u^k = a_k/r \in T$ for each k. Thus, each power of u is in the finite $R-$submodule of T generated by $1/r$. □

We have another equivalent characterization

Theorem 34. *Let R be a domain with quotient field T. An element is $u \in T$ is almost integral over R if and only if there is a nonzero ideal $I \subseteq R$ for which $uI \subseteq I$.*

Proof. Suppose that $u \in T$ is almost integral over R. Then there exists some $r \in R$ such that $ru^k \in R$. Hence, we take $I = (ur, u^2r, u^3r, \ldots)$.

On the other hand, if $uI \subseteq I$, then note that $u^2I = u(uI) \subseteq uI \subseteq I$. It is clear then that $u^kI \subseteq I$ for all $k \in \mathbb{N}$. Hence, any nonzero element $r \in I$ satisfies $ru^k \in I \subseteq R$ for all k. □

It should be clear by comparing Definition 31 and Theorem 15 that almost integrality is essentially a relaxation of integrality. That is, integral elements are almost integral. We record this result below.

Theorem 35. *Let $R \subseteq T$ be rings. An element $u \in T$ is almost integral over R if it is integral over R.*

Proof. From Theorem 15, u is integral over R implies that $R[u]$ is a finitely generated R-submodule of T. Since every power of u belongs to $R[u]$, u is almost integral over R by Definition 31. □

Thus, in general we have $R \subseteq \bar{R} \subseteq R'$ for a ring R. In particular, a completely integrally closed ring is integrally closed. We now provide an example showing that almost integrality does not imply integrality.

Example 36. Consider the ring $R = \mathbb{Z} + x\mathbb{Q}[x]$ with quotient field T. Take $\frac{1}{2} \in T$. Then $\frac{1}{2}$ is almost integral over R as $x(1/2)^k \in R$ for all $k \in \mathbb{N}$. However, if $\frac{1}{2}$ is integral over R, then it is the root of some monic polynomial with coefficients in R, say

$$y^n + f_{n-1}(x)y^{n-1} + \cdots + f_0(x) = 0$$

for some n and $f_j \in R$. However, by simplifying this expression, we note that the constant term is a monic polynomial in y with coefficients in $\mathbb{Z}$. It follows that $\frac{1}{2}$ must be the root of a monic polynomial with coefficients in $\mathbb{Z}$, i.e., $\frac{1}{2}$ is integral over $\mathbb{Z}$. But this is impossible as $\mathbb{Z}$ is integrally closed. Thus $\frac{1}{2}$ is almost integral over R but not integral over R.

There is, however, one notable class of rings for which almost integrality and integrality coincide. Indeed, we are referring to Noetherian domains. Recall from our short history lesson in Chapter 2 that early definitions of integral elements were often identical to the definition of almost integral elements because only Noetherian domains were being considered.

Theorem 37. *Let D be an Noetherian domain, then u is almost integral over D if and only if u is integral over D.*

Proof. Since D is Noetherian, every ideal of D is finitely generated. Hence in this case Theorem 34 (characterization of almost integral elements) is identical to Theorem 9 (characterization of integral elements). □

It can also be shown that UFD's behave nicely with almost integrality just as they do with integrality. The following theorem is recorded in [16].

Theorem 38. *A UFD is completely integrally closed.*

Recall that GCD-domains are integrally closed. We are not so fortunate when dealing with complete integral closure as the following result shows (see [9] for the statement and [1] for a generalized proof).

Theorem 39. *Let D be a GCD domain. Then the complete integral closure of D is D_S where $S = \{a \in D | \cap_{n=1}^{\infty} (a^n) \neq 0\}$.*

Applying Theorem 39 to valuation domains, we have the following result.

Theorem 40. *Let V be a valuation domain of rank k with quotient field K. If V has a height 1 prime ideal P, then $V' = V_P$, otherwise, $V' = K$. Hence, a valuation domain is completely integrally closed if and only if it has rank ≤ 1.*

We now give an example of a valuation domain which has no height 1 prime ideal.

Example 41. Let F be a field and set $D = F[\{x_i, \frac{x_{j+1}}{x_j^k} | i, j, k \in \mathbb{N}\}]$. Let $\mathcal{M}$ be the maximal ideal of D generated by $\{x_i, \frac{x_{j+1}}{x_j^k} | i, j, k \in \mathbb{N}\}$. Then $V := D_{\mathcal{M}}$ is a valuation domain with no height 1 prime ideal. For suppose $P \subseteq V$ is a height 1 prime ideal. Then V_P is a rank 1 valuation domain with prime ideal PV_P. So if $\alpha \in V_P$ is a nonzero nonunit, then $PV_P = \sqrt{(\alpha)}$. For $\sqrt{(\alpha)}$ is the intersection of all prime ideals containing (α), and PV_P is the only such prime ideal. Now, up to a unit, α is a multivariate polynomial in $x_1, \ldots, x_n$, say $f(x_1, \ldots, x_n)$. Note that in V, $(x_{n+1}) \subset \sqrt{(f(x_1, \ldots, x_n))}$. Therefore, in V_P, $\sqrt{(x_{n+1})} \subset \sqrt{(f(x_1, \ldots, x_n))} = \sqrt{(\alpha)} = PV_P$, which is a contradiction.

More examples of (valuation) domains whose complete integral closure is their quotient field can be found in [2]. The next theorem shows that the intersection of completely integrally closed domains is again completely integrally closed.

Theorem 42. *If $\{D_i\}$ is a family of completely integrally closed domains with the same quotient field K, then $D = \cap_i D_i$ is completely integrally closed.*

Proof. If $u \in K$ is such that there exists some $d \in D$ with $du^k \in D$ for all $k > 0$, then $du^k \in D_i$ for each k. Since each D_i is completely integrally closed, we have $u \in D_i$ for each i. Hence $u \in D$. $\square$

Results like Theorem 40 and Theorem 42 led Krull to conjecture that every completely integrally closed integral domain can be expressed in its quotient field as an intersection of rank 1 valuation domains (see [28], [30]). However, Nakayama found a counterexample to Krull's conjecture ([35],[36],[37]). His original counterexample is given in [36], and a stronger version is found in [37]. Nakayama's counterexample is quite involved, as it spans both [35] and [36]. We give a brief explanation of his counterexample here.

Nakayama begins with a complete Boolean algebra A containing a countable set of non-atomic non-zero elements $v_1, v_2, \ldots, v_i, \ldots$ such that any $a > 0$ in A satisfies $a \geq v_i$ for a suitable i. He denotes its representation space by $\Omega = \Omega(A)$. He then considers a field K and variables $x(p)$ which are in one-to-one correspondence with the points p in Ω. For a finite set of distinct points $\{p_1, p_2, \ldots, p_s\}$ in Ω, a polynomial of the variables $x(p_1), x(p_2), \ldots, x(p_s)$ is called a $p_1p_2\ldots p_s$-polynomial. Then we let P be a set of first category in Ω and suppose that for each finite system $\{p_1, p_2, \ldots, p_s\}$ of points in Ω not belonging to P there is given a $p_1 \ldots p_s$-polynomial $F(x(p_1)\ldots x(p_s)) = F(p_1 \ldots p_s)$. Now, symbolically we say $F(p_1 \ldots p_s) \to F(p_1 \ldots p_t)$ if $F(p_1, \ldots, p_s)$ becomes $F(p_1, \ldots, p_t)$ by setting $x(p_{t+1}) = \cdots = x(p_s) = 1$. Then if $F(p_1 \ldots p_s) \to F(p_1 \ldots p_t)$ whenever $\{p_1, \ldots, p_s\} \supseteq \{p_1, \ldots, p_t\}$, the whole scheme is called a polynomial series on Ω. The totality of polynomial series (i.e., the totality of classes of equivalent polynomial series) forms an integral domain R_Ω. This domain R_Ω is completely integrally closed, but is not an intersection of rank 1 valuation domains in its quotient field.

Other simpler counterexamples have been given by Ohm (recorded by Gilmer in [16], p. 232) and Sheldon [43].

As a side note, we add that Krull made another conjecture in [30], namely that every completely integrally closed integral domain with at most one proper prime ideal is a valuation domain. In 1952, Nagata [32] found a counterexample to this conjecture. However, his original proof was incorrect, so he published corrections three years later [34].

Now let us return to comparing almost integrality with integrality. There are two important properties that integrality has, but almost integrality does not necessarily have. Differences in results on integrality and almost integrality are often consequences of the presence (or lack there of) of these properties. These properties are

1. Almost integrality is not necessarily transitive. That is, if T is almost integral over S and S is almost integral over R, then it is not necessarily the case that T is almost integral over R.

2. If $s \in S$ and $S \subseteq T$, then s may be almost integral over R as element of T but not as an element of S. (This property is relevant when consider Definition 31, see Example 46.)

The classic example below by Gilmer and Heinzer is found in [17]. It gives a case when almost integrality is not transitive by providing a domain whose complete integral closure is not completely integrally closed.

Example 43. $D = F[\{x^{2n+1}y^{n(2n+1)}\}_{n=0}^{\infty}]$ has quotient field $F(x, y)$, and the complete integral closure is $D' = F[\{xy^n\}_{n=0}^{\infty}]$. Note that y is almost integral over D', but not almost integral over D. Hence, $y \notin D'$. In fact, $D'' = F[x, y]$. Therefore D' is not completely integrally closed.

One should note that in this example D is not integrally closed. However, Heinzer provided an example of a Prüfer domain (hence integrally closed) whose complete integral closure was not completely integrally closed [20]. To make matters worse, Hill gives an example of a domain D which yields the strictly ascending infinite chain $D \subset D' \subset D'' \subset D''' \subset \cdots$ where D' is the complete integral closure of D [21].

It is clear then that it is not true in general that the complete integral closure is completely integrally closed; however, we give some cases where this does hold. If the complete integral closure of R in T is contained in a finite R-submodule of T, then the complete integral closure is completely integrally closed. This is a consequence of the following lemma found in [17].

Lemma 44. *If R, R_1, and T are rings with $R \subseteq R_1 \subseteq T$ and if R_1 is contained in a finite $R-$module contained in T, then R and R_1 have the same complete integral closure in T.*

While the complete integral closure need not be completely integrally closed, the complete integral closure is always integrally closed [44]. This is actually a key observation, as this implies the complete integral closure is the intersection of the valuation domains which contain it. Our eventual goal is to determine exactly which valuation overrings of the original domain contain the complete integral closure.

We mentioned earlier that if $u \in S$ and $S \subseteq T$, then u may be almost integral over R as element of T but not as an element of S. We now look at some results related to this observation. The following proposition is from Gilmer and Heinzer (see [17], Proposition 2).

Proposition 45. *Suppose R is a subring of S_1 and S_1 is a subring of S_2. If R_i is the complete integral closure of R in S_i, then $R_1 \subseteq R_2 \cap S_1$. And $R_1 = R_2 \cap S_1$ if either*

(a) S_2 is a submodule of some S_1-module S_3 such that S_1 is a direct summand of S_2, or

(b) each finite S_1-module contained in S_2 and containing S_1 is a submodule of an S_1-module of which S_1 is a direct summand.

The following is an example in which $R_1 \subset R_2 \cap S_1$ (see [17], Example 2).

Example 46. Set $R = F[xy, xy^2, xy^3, \dots]$, $S_1 = R[y]$, and $S_2 = S_1[1/x]$. Note that $y \in (R_2 \cap S_1) - R_1$. Note that $y \notin R_1$ because all powers of y cannot be contained in a finite $R-$submodule of S_1. However, $y \in R_2$ because all powers of y belong to the finite $R-$submodule of S_1 generated by $1/x$. Moreover, note that R, S_1, and S_2 are domains with a common quotient field $F(x, y)$.

In [30] Krull proves the following

Lemma 47. *Let D be a completely integrally closed domain with quotient field K, and let L be an algebraic extension field of K. If $\bar{D}$ is the integral closure of D in L, then $\bar{D}$ is completely integrally closed.*

This result can be extended in as the following theorem [17]:

Theorem 48. *Let D be a complete integrally closed domain with quotient field K, and let L be an extension field of K. If D' is the complete integral closure of D in L, then the following hold:*

1. *D' is a completely integrally closed domain*
2. *D' is the integral closure of D in L*
3. *D' is completely integrally closed in L.*

Corollary 49. *If the complete integral closure D' of the domain D is completely integrally closed, then the complete integral closure of D in any domain L containing the quotient field of D is again completely integrally closed in L.*

We know that the complete integral closure is integrally closed but not necessarily completely integrally closed. A natural question then is to ask which domains R have the property that the complete integral closure R' is completely integrally closed.

If R is Noetherian, then R' is completely integrally closed (see [33], Theorem 33.10 and [46]). If R is a Prüfer domain (see Definition 102) in which each principal ideal has only finitely many minimal prime ideals, then R' is again completely integrally closed (see [17]).

We now record a useful result giving a condition on an element equivalent to its inverse being almost integral.

Lemma 50. *Let D be an integral domain with quotient field K. If $d \in D$ is nonzero, then $1/d$ is almost integral over D if and only if $\cap_{i=1}^{\infty}(d^i) \neq (0)$.*

Proof. If $x \in D$ and k is a positive integer, then $x(1/d)^k \in D$ if and only if $x \in (d^k)$. □

It follows immediately that if R is completely integrally closed, then for each nonunit d of R, $\cap_{i=1}^{\infty}(d^i) = (0)$. However, the converse of this statement is not true, that is, the condition is not sufficient for being completely integrally closed. For example, consider any Noetherian domain which is not integrally closed. For instance, $\mathbb{Z} + 2i\mathbb{Z}$ is such a domain.

One might wonder when the above condition is sufficient for R to be completely integrally closed. As we have seen, for any Noetherian domain which is not integrally closed the condition does not suffice. Gilmer and Ohm [18] do give a class of domains where the condition is sufficient. For this, they defined the following property:

Definition 51. A domain D has the *QR-property* if each domain between D and its quotient field is of the form D_S for some multiplicative system S.

The following properties of a domain satisfying the the QR-property are given in [17].

Proposition 52. *Let D be a domain satisfying the QR-property. Then following are equivalent:*

1. *D is completely integrally closed*

2. *if d is a nonunit of D, then $\cap_{i=1}^{\infty}(d^i) = (0)$*

3. *if A is a finitely generated proper ideal of D, then $\cap_{i=1}^{\infty}A^i = (0)$.*

We finished the previous chapter on integral closure by highlighting the relationship between the integral closures of $R, R[x]$, and $R[[x]]$. In particular, we finished by alluding to an example which would be used to prove Corollary 30. We are now in a position to give this example.

Example 53. Consider any valuation ring V of rank greater than 1. Now V is of course integrally closed. We will show that $V[[x]]$ is not integrally closed. Assume for sake of contradiction that $V[[x]]$ is integrally closed, then by Theorem 29 we must have $\cap_{i=0}^{\infty}a^iV = 0$ for every nonunit $a \in V$. However, by Theorem 40 we have V is not completely integrally closed. Hence by Proposition 52 there must be some nonunit $a \in V$ such that $\cap_{i=1}^{\infty}a^iV \neq 0$. But this is a contradiction. Therefore, $V[[x]]$ cannot be integrally closed. (See [22], Example 2, for an example of a valuation domain of rank 2.)

We now discuss the relationship between the complete integral closures of $R, R[x]$, and $R[[x]]$. We first have a result that is analogous to Proposition 28. A proof can be found in [17].

Proposition 54. *Suppose R is a subring of T, and R' is the complete integral closure of R in T. Then $R'[x]$ is the complete integral closure of $R[x]$ in $T[x]$.*

It follows immediately that R is completely integrally closed in T if and only if $R[x]$ is completely integrally closed in $T[x]$.

Moreover, the behavior of $R[[x]]$ with respect to complete integral closure is closely related to that of R. This can be seen in the following theorem, which can be found in [40].

Theorem 55. *If R is completely integrally closed, then $R[[x]]$ is completely integrally closed.*

Gilmer and Heinzer also related complete integral closure to the notion of conductors; the next three results can be found in [17]. For more on conductors, see the comments beginning with Definition 106 in the Appendix.

Lemma 56. *If D is a domain with quotient field K and if $u \in K$, then u is almost integral over D if and only if the conductor of D in $D[u]$ is nonzero.*

Lemma 57. *If D_1 and D_2 are domains having a common quotient field, if $D_1 \subseteq D_2$, and if the conductor of D_1 in D_2 is nonzero, then D_1 and D_2 have the same complete integral closure.*

Corollary 58. *If D' is such that the conductor of D in D' is nonzero, then D' is completely integrally closed. In particular, if $D' = D[t_1, \ldots, t_n]$ is a finite ring extension of D, then D' is completely integrally closed.*

We now include a few results specifically concerning Prüfer domains because results concerning complete integral closure are often easier to show in this context. One hint at this can be seen from the following result given by Butts and Smith in [7].

Theorem 59. *Every valuation ring of K lying over D is rank one if and only if $\bar{D}$ is a one dimensional Prüfer domain.*

Recall that if a valuation ring has rank one then it is completely integrally closed (Theorem 40).

Gilmer and Heinzer also give a number of results concerning Prüfer domains. In particular, they set out to show when the complete integral closure of a Prüfer domain is completely integrally closed. One way this can happen is if the complete integral closure is an intersection of valuation rings of rank at most 1. Gilmer and Heinzer imposed certain finiteness conditions on the domain to ensure this is the case. The first result we record has a helpful corollary when applied to Prüfer domains. The result was proved independently by Gilmer and Heinzer [17] and by Butts and Smith [7].

Proposition 60. *Let K be a field, let $\{V_\lambda\}$ be a family of valuation rings with K as a quotient field, and let $T = \cap V_\lambda$. Suppose for each λ, v_λ is a valuation associated with the valuation ring V_λ. If T has quotient field K and if the family $\{v_\lambda\}$ has finite character in the sense that for any nonzero element x of K, $v_\lambda(x) \neq 0$ for only finitely many λ's, then the complete integral closure of T is $\cap V'_\lambda$ where for any λ, V'_λ is K if there is no rank one valuation ring between V_λ and K, and V'_λ is the unique rank one valuation ring between V_λ and K otherwise*

Note that every valuation domain between a Prüfer domain D and its quotient field K is of the form D_P for some prime ideal P of D. Moreover, if D is Prüfer , then D_P is a valuation domain for each prime ideal P of D (see [25], Theorem 64). This observations leads us to this corollary of the above proposition.

Corollary 61. *Suppose D is a Prüfer domain containing a family $\{P_\lambda\}$ of prime ideals such that $D = \cap D_{P_\lambda}$ and such that each nonzero element of D belongs to only finitely many P_λ's. Then the complete integral closure of D is the intersection of all valuation rings of rank ≤ 1 lying between D and its quotient field, and is therefore completely integrally closed.*

The following result and corollaries are also from Gilmer and Heinzer [17].

Proposition 62. *Let P be a non-minimal (height ≥ 2) prime ideal of the Prüfer domain D satisfying this condition: there exists a nonzero prime ideal Q contained in P and an element x of $P \setminus Q$ such that (x) has only finitely many minimal prime ideals. Then D_P does not contain the complete integral closure of D*

Corollary 63. *Let D be a Prüfer domain such that each principal ideal of D has only finitely many minimal prime ideals. Then the complete integral closure D' of D has dimension ≤ 1 and is completely integrally closed.*

Corollary 64. *If D is a Prüfer domain in which each nonzero element belongs to only finitely many maximal ideals, the complete integral closure of D has dimension ≤ 1 and is completely integrally closed.*

It is however, not that case (as shown by Nakayama [35], [36], [37]) that the complete integral closure of a Prüfer domain is the intersection of valuation rings of rank ≤ 1. However, determining whether the complete integral closure of a Prüfer domain is completely integrally closed is by no means easy. As noted by Gilmer and Heinzer, even assuming the QR-property leads to inconclusive results. As of the writing of this paper, the author believes this problem is still unsolved.

In the next chapter we will define and discuss two generalizations: pseudo-integral closure and w-integral closure. Common to both of these generalizations is that they can be expressed in terms of fractional ideals. It turns out this property is also inherent to complete integral closure. Thus, as a prelude to the upcoming chapter, we present the following work - most of which comes from Barucci in [4].

Adopting the notation in [4], for a nonzero fractional ideal I of a domain D, we have $I^{-1} = (D : I) = \{x \in K | xI \subseteq D\}$, where K is the quotient field of D. We denote $(I^{-1})^{-1}$ by I_v and say that I is *divisorial* or a *v-ideal* if $I = I_v$. We say that I is a *strong ideal* if $II^{-1} = I$, which is equivalent to $(I : I) = (D : I)$ [4]. A *strongly divisorial* ideal is one which is strong and divisorial.

We now record three results from [4]. The second result is the one of most significance, but we need the first to prove the second, and the third is corollary of the second.

Proposition 65. *Let I be a nonzero fractional ideal of D. Then I is strongly divisorial if and only if I is the conductor of D in some overring R of D.*

Theorem 66. *Let $\boldsymbol{D}_s(D)$ be the set of strongly divisorial ideals of D. Then $D' = \bigcup\{I^{-1} | I \in \boldsymbol{D}_s(D)\}$.*

The proof below is also recorded in [4].

Proof. We note that $D' = \bigcup\{(F : F) | F \in \mathbf{F}(D)\}$ where $\mathbf{F}(D)$ is the set of fractional ideals of D. It follows then that $D' \supseteq \bigcup\{(I : I) | I \in \mathbf{D}_s(D)\} = \bigcup\{I^{-1} | I \in \mathbf{D}_s(D)\}$. On the other hand, if $x \in D'$, then $x \in (F : F)$ for some fractional ideal F of D. But $(F : F)$ is an overring of D, hence $I = (D : (F : F))$ is a strongly divisorial ideal of D by Proposition 65. So $x \in (F : F) \subseteq (D : (D : (F : F))) = I^{-1}$ with $I \in \mathbf{D}_s(D)$. Thus $x \in \bigcup\{I^{-1} | I \in \mathbf{D}_s(D)\}$ and we have $D' = \bigcup\{I^{-1} | I \in \mathbf{D}_s(D)\}$. □

Corollary 67. *D is completely integrally closed if and only if D is the unique strongly divisorial ideal of D.*

Proof. It follows immediately from Theorem 66 that if D is the unique strongly divisorial ideal of D, then $D' = D$. On the other hand, suppose that $D' = D$. Assume that a strongly divisorial ideal $I \subset D$ exists. Then by Theorem 66, $D \subset I^{-1} \subseteq D'$ which is a contradiction. Hence, D must be the unique strongly divisorial ideal of D. □

4.1 Anchor Ideals

Given ω an almost integral element over a domain R, there is a nonzero $x \in R$ such that $x\omega^n \in R$ for all $n \in \mathbb{N}$. All such elements x form an ideal I_ω. We have the following.

Proposition 68. *Let ω be almost integral over R and $I_\omega = \{x \in R | x\omega^n \in R$ for all $n \in \mathbb{N}\}$.*

1. *$I_\omega = R$ if and only if $\omega \in R$.*

2. *$I_\omega \neq 0$ if and only if ω is almost integral.*

3. *$I_{\omega_1} I_{\omega_2} \subseteq I_{\omega_1\omega_2}$*

4. *$I_{\omega_1} I_{\omega_2} \subseteq I_{\omega_1+\omega_2}$*

5. *If $r \in R$ then $I_{r+\omega} = I_\omega$*

6. *If $r \in R$, then $I_\omega \subseteq I_{r\omega}$*

Proof. 1. If $I_\omega = R$, then $1 \in I_\omega$, and $1\omega = \omega \in R$. If $\omega \in R$, then $1 \in I_\omega$, and thus $I_\omega = R$.

2. $I_\omega \neq 0$ if and only if there exists some nonzero $r \in I_\omega$, if and only if $r\omega^n \in R$ for all $n \in \mathbb{N}$ if and only if $\omega \in R'$.

3. If $r \in I_{\omega_1} I_{\omega_2}$, then $r = \sum_{i=1}^m a_i b_i$ where $a_i \in I_{\omega_1}$ and $b_i \in I_{\omega_2}$. Thus

$$r(\omega_1\omega_2)^n = \left(\sum_{i=1}^m a_i b_i\right)(\omega_1\omega_2)^n = \sum_{i=1}^m (a_i b_i (\omega_1\omega_2)^n) = \sum_{i=1}^m (a_i\omega_1^n)(b_i\omega_2^n) \in R$$

4. If $r \in I_{\omega_1} I_{\omega_2}$, then $r = \sum_{i=1}^m a_i b_i$ where $a_i \in I_{\omega_1}$ and $b_i \in I_{\omega_2}$. Thus

$$r(\omega_1 + \omega_2)^n = \left(\sum_{i=1}^m a_i b_i\right)(\omega_1 + \omega_2)^n = \sum_{i=1}^m (a_i b_i (\omega_1 + \omega_2)^n)$$

$$= \sum_{i=1}^m a_i b_i \left(\sum_{k=0}^n \binom{n}{k} \omega_1^n \omega_2^{n-k}\right) = \sum_{i=1}^m \sum_{k=0}^n \binom{n}{k} (a_i\omega_1^n)(b_i\omega_2^{n-k}) \in R$$

5. Let $a \in I_{r+\omega}$, then $a(r+\omega)^n \in R$ for all $n \in \mathbb{N}$. In particular, when $n = 1$, we have $ar + a\omega \in R$, which implies $a\omega \in R$. When $n = 2$, we have $ar^2 + r(a\omega) + a\omega^2 \in R$, which implies that $a\omega^2 \in R$.

Proceed by induction to show that $a\omega^n \in R$ for all $n \in \mathbb{N}$ and $a \in I_\omega$.

On the other hand, if $a \in I_\omega$, the $a\omega^n \in R$ and clearly

$$a(r+\omega)^n = \sum_{k=0}^{n} \binom{n}{k} r^n (a\omega^{n-k}) \in R$$

for all n.

6. If $a\omega^n \in R$ for all n, then $a(r\omega)^n = r^n(a\omega^n) \in R$ for all n.

□

Note that the inclusions in Properties 3 and 4 cannot be reversed in general.

For property 3, consider $R = \mathbb{Z}$. Then $I_{1/2} = 0$ and $I_2 = \mathbb{Z}$. But then $I_1 = \mathbb{Z}$ and $I_{1/2}I_2 = (0)$. Thus $I_1 = R \nsubseteq (0) = I_{1/2}I_2$.

For property 4, consider again $R = \mathbb{Z}$. Note that $I_{1/2+1/2} = I_1 = R$, but $I_{1/2}I_{1/2} = (0)$.

Note, we expect property 6 to be strict in general. For instance, if $\omega \in R' \setminus R$, and $r \in I_\omega$, then $r\omega \in R$, and $I_\omega \subset I_{r\omega} = R$.

It is interesting to also consider that we have $I_{\omega_1}I_{\omega_2} \subseteq I_{\omega_1} \bigcap I_{\omega_2}$, but we cannot make a comparison between $I_{\omega_1} \bigcap I_{\omega_2}$, $I_{\omega_1\omega_2}$, and $I_{\omega_1+\omega_2}$ in general.

Consider the following examples:

Example: Let $R = \mathbb{F}[xy^n, xz^n]$. Note that $x \in I_y \cap I_z$, but $x \notin I_{y+z}$. Note that $x(y+z) = xy + xz \in R$, but $x(y+z)^2 \notin R$ because $x(yz) \notin R$. (Note that x^2 anchors $(y+z)$ and yz, but x does not).

On the other hand, consider the domain $R = \mathbb{Z}$ and $\omega_1 = \omega_2 = 1/2$. Consider that $I_{1/2} \cap I_{1/2} = 0 \in I_1$.

So there's no comparison between $I_{\omega_1} \bigcap I_{\omega_2}$ and $I_{\omega_1+\omega_2}$.

For the same example, consider that $I_{1/4} = 0 \subset I_1$. But then take $\omega_1 = 2$ and $\omega_2 = 1/2$. Then $I_{2+1/2} = 0 \subset I_1$.

So then we can have $I_{\omega_1\omega_2} \subset I_{\omega_1+\omega_2}$ and vice versa.

Example 69. If $R = \mathbb{Z} + 2x\mathbb{Z}[x]$ then $I_x = (2, 2x\mathbb{Z}[x])$ is a maximal ideal but R is integrally closed. If we let $T = \mathbb{Z} + 2x\mathbb{Z} + x^2\mathbb{Z}[x]$ then I_x is again maximal and x is now integral.

Example 70. Let $R = \mathbb{Q} + x\mathbb{R}[x]$. Note that $I_{\sqrt{3}} = I_\pi = x\mathbb{R}[x]$. Both of these elements are 1-almost integral.

Theorem 71. *Let R be a domain with quotient field K. Let $\alpha \in K \setminus R$ be almost integral over R. If I_α is finitely generated, then α is integral over R.*

Proof. Recall Theorem 9 which states that $\omega \in K$ is integral over R if and only if $\omega I \subseteq I$ for some nonzero finitely generated ideal of R. In this case, we have $\omega I_\omega \subseteq I_\omega$, and I_ω is nonzero and finitely generated by assumption. □

The converse to the theorem above is not true. For example, consider
$R := \mathbb{F} + x\mathbb{F}[x^q | q \in \mathbb{R}^+]$. For a given $0 < \alpha < 1$, the element x^α is integral over R. However,
$I_\alpha = x\mathbb{F}[x^q | q \in \mathbb{R}^+]$ is not finitely generated.

We have two immediate corollaries from the previous theorem.

Corollary 72. *Suppose that R is integrally closed domain with quotient field K, and let $\omega \in K$. Then precisely one of the following holds.*

1. *If $I_\omega = 0$ then ω is not almost integral over R.*
2. *If $I_\omega = R$ then $\omega \in R$.*
3. *If I_ω is a proper nontrivial ideal of R then I_ω is not finitely generated.*

Proof. The first two statements are clear. For the third, consider that if I_ω is a proper nontrivial ideal of R and is finitely generated, then $\omega \in \bar{R}$ by the previous theorem. But since R is integrally closed, we have $\omega \in R$ and $I_\omega = R$. □

Corollary 73. *If R is Noetherian and integrally closed, then it is completely integrally closed.*

Proof. Since R is Noetherian, each ideal of R is finitely generated. Thus the third statement in the previous corollary is not possible. Hence for each $\omega \in K$ that is almost integral over R, we must have $I_\omega = R$, and thus $\omega \in R$. Therefore, R is completely integrally closed. □

It should be noted that the previous corollary is simply proven using the fact that integrality is the same as almost integrality over a Noetherian domain. However, the argument given above shows how the same conclusion can be reached without knowledge of this fact.

Now we consider how the anchor ideals behave in the valuation overrings of R.

Notation 74. Let R be a domain with quotient field K. If $\omega \in K$, we denote the anchor ideal of ω to V by I_ω^V

Example 75. $R = \mathbb{F}[x^{2n+1}y^{n(2n+1)} | n \in \mathbb{N}_0]$.

Note that $R' = \bar{R} = \mathbb{F}[x, xy^n]$. In particular, since $xy^n \in \bar{R}$, we must have $xy^n \in V$ for each valuation overring V and all n. Thus, for each V, we have $x \in I_y^V$. However, consider that $I_y = (0)$ since $y \notin R'$.

This example provides a counterexample for the general converse of the following theorem:

Theorem 76. *Let R be a domain and ω a nonzero element of K. For the following conditions, we have* $(1) \implies (2)$, *and if R is integrally closed, then these conditions are equivalent.*

1. *ω is almost integral over R.*

2. *$\bigcap_V I_\omega^V \neq 0$ where the intersection is taken over the valuation overrings of R.*

What is more, if $\bigcap_V I_\omega^V \neq 0$ and is finitely generated then ω is integral over R.

Proof. Suppose that $\omega \in R'$. Then $I_\omega \neq 0$. Moreover, note that $I_\omega \subseteq I_\omega^V$ for each V, so $I_\omega \subseteq \bigcap_V I_\omega^V$. Thus $\bigcap_V I_\omega^V \neq 0$.

Now consider when R is integrally closed. Suppose that $x \in \bigcap_V I_\omega^V$. It suffices to show that $x\omega^n \in R$ for all $n \in \mathbb{N}_0$. Note that $x\omega^n \in V$ for every valuation overring of R and for all $n \in \mathbb{N}_0$. Since R is integrally closed, $x\omega^n \in \overline{R} = R$ for all $n \in \mathbb{N}_0$.

□

Example 77. Consider the elements $\frac{1}{2}, e, \sqrt{2}$ in the quotient field of $\mathbb{Z} + x\mathbb{C}[x]$. Note that the first two are almost integral and the last is integral. Yet they all have the same anchor ideal $x\mathbb{C}[x]$.

On the other hand, in $\mathbb{Z} + x\mathbb{Q}[x]$, each element in $\mathbb{Q} \setminus \mathbb{Z}$ is almost integral (but not integral) and has the same anchor ideal, namely $x\mathbb{Q}[x]$.

As another example, consider $\mathbb{Z} + x\overline{\mathbb{Q}}[x]$ (where $\overline{\mathbb{Q}}$ is the set of algebraic numbers). Again elements in the quotient field that are almost integral all have anchor ideal $x\overline{\mathbb{Q}}[x]$.

In general, if $S \subseteq T$ and $R := S + xT[x]$ where S is completely integrally closed, then $R' = T[x]$, and if $\omega \in R' \setminus R$, then $I_\omega = xT[x]$.

Consider the domain $R = \mathbb{Q}+x\mathbb{C}[x]$ where the integral closure is $\overline{\mathbb{Q}}+x\mathbb{C}[x]$ and the complete integral closure is $\mathbb{C}[x]$.

So the element i is integral over R, and the anchor ideal is $I_i = x\mathbb{C}[x]$. This is similar to the last example, since $\mathbb{Q}$ is completely integrally closed. So there is no distinction here in terms of the anchor ideals

Theorem 78. *Let ω be almost integral over R with anchor ideal $I \subsetneq R$. If V is a valuation overring of R and $\omega \notin V$ then either $V = VI$ (that is, I blows up in V) or $V \supsetneq (\omega^{-1}) \supsetneq VI$.*

Proof. Since $\omega \notin V$, the containment $V \supsetneq (\omega^{-1})$ is clear. For the other containment, we suppose $V \neq VI$ and that $(\omega^{-1}) \subseteq VI$. Then $V \subseteq VI\omega = VI$ which is a contradiction. □

Theorem 79. *Suppose that $0 \neq I \subsetneq R$ is the anchor ideal for the almost integral element ω. Then if I blows up in the valuation overring V, then $\omega \in V$.*

Proof. Suppose that $\omega \notin V$, then (ω^{-1}) is a proper ideal of V and as I is the anchor ideal of ω, $I\omega \subseteq I$. So $VI\omega \subseteq VI$ and hence $VI \subseteq VI\omega^{-1}$. So if $VI = V$ then $V \subseteq V\omega^{-1}$ which is a contradiction. Hence I must be proper in V and this completes thte proof. □

Theorem 80. *If V is a valuation overring of R with a height one prime ideal, then ω is almost integral over V if and only if ω^{-1} is not contained in the height one prime of V.*

Proof. Assume that V has a height one prime ideal, say P_1. Then $V' = V_{P_1}$. Moreover, $\omega \in V_{P_1}$ if and only if $\omega^{-1} \notin P_1$. □

4.2 The Example

In this section we construct a collection of domains D_n, where the complete integral closure must be applied n time before reaching a completely integrally closed domain. The motivation and inspiration for this idea comes from a number of examples already in the literature.

In 1966 Gilmer and Heinzer gave an example of a domain where the complete integral closure is not completely integrally closed.

$$R = \mathbb{F}[x^{2n+1}y^{n(2n+1)}|n \in \mathbb{N}_0]$$

In 1972, Hill constructed a collection of domains D_n, where the complete integral closure must be applied n time before reaching a completely integrally closed domain. Hill also included a domain where the complete integral closure could be taken indefinitely. The example given by Hill relied on the construction of a group of divisibility of the desired domain.

We provide a collection of domains with the properties of Hill but the relative simplicity of Gilmer and Heinzer.

Example 81. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}_0^n$, define $\lambda^{(j)} = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ for $j = 1, 2, \ldots, n$.

Let $\mathbb{F}$ be a field of characteristic 0. We define the integral domain

$$D_n := \mathbb{F}\left[x_{\lambda^{(n-1)}}\left(x_{\lambda^{(n-2)}}\left(\cdots\left(x_{\lambda^{(2)}}\left(x_{\lambda^{(1)}}\left(xy^{\lambda_1}\right)^{\lambda_2}\right)^{\lambda_3}\right)^{\lambda_4}\cdots\right)^{\lambda_{n-1}}\right)^{m} |\lambda \in \mathbb{N}_0^{n-1}, m \in \mathbb{N}_0\right]$$

We claim that

$$D_n \subset D_n' \subset D_n'' \subset D_n''' \subset D_n^{(4)} \subset \cdots \subset D_n^{(n-1)} \subset D_n^{(n)} = D_n^{(n+1)}$$

where $D_n^{(4)}$ represents the complete integral closure of D_n''', and $D_n^{(j)}$ represents the complete inegral closure of $D_n^{(j-1)}$ for $j \in \{5, 6, \ldots, n+1\}$.

To demonstrate the claims made about this domain, we will work out proofs for the cases $n = 2, 3$, and 4 before proving the general case.

4.2.1 Case: $n = 2$

When $n = 2$, we have the domain

$$D_2 = \mathbb{F}\left[x_{\lambda^{(1)}}(xy^{\lambda_1})^n | \lambda \in \mathbb{N}_0^1, n \in \mathbb{N}_0\right] = \mathbb{F}\left[x_k(xy^k)^n | k, n \in \mathbb{N}_0\right]$$

First we verify that the quotient field is

$$K = \mathbb{F}\left(x_k, x, y | k \in \mathbb{N}_0\right).$$

Set $n = 0$, and we have $x_k \in K$ for each $k \in \mathbb{N}_0$. Then, in the quotient field, we also have

$$\frac{x_0 x}{x_0} = x \qquad \text{and} \qquad \frac{x_1 xy}{x_1 x} = y$$

So we have the desired quotient field. Next we determine D_2'.

Since n can vary independently of k, we see that $xy^k \in D_2'$ for each $k \in \mathbb{N}_0$.

Moreover, we claim that $y \notin D'$. Consider an arbitrary monomial d in D_2, which is of the form

$$d = \prod_{j=0}^{N} x_j^{\alpha_j} \left(xy^j\right)^{\beta_j}$$

Note that the ratio between the exponents of y and x is at most N. In particular, for a given element $d \in D_2$, the ratio between the exponents of y and x is at most the maximum j such that x_j is a factor of d. Now suppose that $dy^n \in D_2$ for each n. For each n, the x_j's in the product dy^n are fixed by the anchor d. Hence, the max ratio between the exponents of y and x is determined, independent of n. Thus the exponent on y cannot increase without bound without increasing the exponent on x as well. But the exponent of x is fixed in the anchor d, independent of n. Thus $y \notin D_2'$.

Therefore, we have

$$D_2' = \mathbb{F}\left[x_k, xy^k | k \in \mathbb{N}_0\right]$$

and, it follows that

$$D_2'' = \mathbb{F}\left[x_k, x, y | k \in \mathbb{N}_0\right].$$

4.2.2 Case: $n = 3$

When $n = 3$, we have

$$D_3 = \mathbb{F}\left[x_{\lambda^{(2)}}\left(x_{\lambda^{(1)}}\left(xy^{\lambda_1}\right)^{\lambda_2}\right)^n | \lambda \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right]$$

$$= \mathbb{F}\left[x_{a,b}(x_a(xy^a)^b)^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right]$$

We first verify that the quotient field is

$$K = \mathbb{F}\left(x_{a,b}, x_a, x, y | (a,b) \in \mathbb{N}_0^2\right).$$

Set $n = 0$, and we have $x_{a,b} \in D_3 \subset K$ for each $(a,b) \in \mathbb{N}_0^2$.

Then we have

$$\frac{x_{a,0}x_a}{x_{a,0}} = x_a \text{ for each } a \in \mathbb{N}_0$$

and

$$\frac{x_{0,1}x_0x}{x_{0,1}x_0} = x \qquad \text{and} \qquad \frac{x_{1,1}x_1xy}{x_{1,1}x_1x} = y$$

Now that we have the desired quotient field, we set out to determine D_3'.

Note that n varies independently of a and b, so we have $x_a(xy^a)^b \in D_3'$ for each $(a,b) \in \mathbb{N}_0^2$ (with anchor element $x_{a,b}$). In particular, $x_a \in D_3'$ for each $a \in \mathbb{N}_0$, with anchor $x_{a,0}$, and $x_0x^b \in D_3'$ for all $b \geq 0$, with anchor $x_{0,b}$.

Now consider an abritrary monomial in D_3.

$$d = \prod_{(a_i,b_i)\in\Lambda} x_{a_i,b_i}^{\alpha_i} \left(x_{a_i}(xy^{a_i})^{b_i}\right)^{\beta_i}$$

where Λ is a finite subset of $\mathbb{N}_0^2$.

Note that the exponent on x is $\sum b_i\beta_i$ and the exponent on y is $\sum a_ib_i\beta_i$. Both exponents are determined by the collection of (a_i, b_i) and β_i. In particular, once the collection of (a_i, b_i)'s are determined, then the only way to increase the exponent on x or y is to increase some of the β_i's.

Note that the ratio between the exponent on y and the exponent on x is at most $\max\{a_i\}$. Thus, once the collection of (a_i, b_i) is chosen, the max ratio of y to x is determined. Therefore, we cannot have the exponent on y increase without bound, if the exponent on x is fixed. Thus, if an element in D_3' is a multiple of y, it must also be a multiple of x.

Moreover, having fixed the collection of (a_i, b_i), the exponents on x and y can only increase without bound by increasing at least one of the β_i's. This in turn increases the exponent on at least one of the x_{a_i}'s. Thus, for an element in D_3' to be a multiple of x (or y), it must also be a multiple of x_a for some $a \in \mathbb{N}_0$.

In summary, we have shown the following so far:

1. $x_{a,b} \in D_3'$ for all $(a, b) \in \mathbb{N}_0^2$
2. $x_a(xy^a)^b \in D_3'$ for all $(a, b) \in \mathbb{N}_0^2$ (anchor $x_{a,b}$)

   Special case: $x_a \in D_3'$ for all $a \in \mathbb{N}_0$ (anchor $x_{a,0}$)

   Special case: $x_0x^b \in D_3'$ for all $b \in \mathbb{N}_0$ (anchor $x_{0,b}$). Hence $x \in D_3''$ with anchor $x_0 \in D_3'$.
3. If $z \in D_3'$ and y is a factor of z, then x is also a factor of z.
4. If $z \in D_3'$ and x is a factor of z, then x_a is also a factor of z for some $a \in \mathbb{N}_0$.

From items 3 and 4, we know that an arbitrary monomial in D_3' is of the form

$$z = \left(\prod_{(a,b)\in\Lambda^{(z)}} x_{a,b}^{r_{a,b}}\right)\left(\prod_{a\in K} x_a^{s_a}\right) x^t y^u$$

where $\Lambda^{(z)}$ is a finite subset of $\mathbb{N}_0^2$, and K is a finite subset of $\mathbb{N}_0$. We use the notation $\Lambda^{(z)}$ to indicate that the (a, b) pairs in $\Lambda^{(z)}$ occur in the element z (the use of this notation will be clear when we write the anchor d for z below). Now note from observations 3 and 4 above, if $u \neq 0$, then $t \neq 0$. Moreover, if $u \neq 0$, then K is non-empty (we will assume that $s_a \neq 0$ for each $a \in K$).

We claim that $u \leq t \max\{s_a\}_{a \in K}$.

Now since $z \in D_3'$ there exists an anchor $d \in D_3$ such that $dz^n \in D_3$ for all n. We can write d as

$$d = \prod_{(a,b) \in \Lambda^{(d)}} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}$$

where $\Lambda^{(d)}$ is a finite subset of $\mathbb{N}_0^2$, $\alpha_{a,b} \in \mathbb{N}$ and $\beta_{a,b} \in \mathbb{N}_0$ for each $(a,b) \in \Lambda^{(d)}$.

We now define some notation that will be useful to us. Let S be a subset of $\mathbb{N}_0^2$, and let T be a subset of $\mathbb{N}_0$.

$$S[\boldsymbol{a}] := \{a | (a,b) \in S\} \qquad S[\boldsymbol{b}] := \{b | (a,b) \in S\}$$

$$S_k := \{(a,b) \in S | a = k\} \qquad S_T := \{(a,b) \in S | a \in T\}$$

$$(S_T)' := \{(a,b) \in S | a \notin T\} = S \setminus S_T$$

Now let us make some requirements on the factors of d. Consider that $dz \in D_3$. Thus for each $a \in K$, we must have $\Lambda_a^{(d)} \neq \emptyset$. That is, if $a \in K$, there must be a factor $x_{a,b}$ of d for some b.

Thus, we can factor d as

$$d = \left(\prod_{(a,b) \in \Lambda_K^{(d)}} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}\right) \left(\prod_{(a,b) \in (\Lambda_K^{(d)})'} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}\right)$$

$$= \left(\prod_{(a,b) \in \Lambda_{a_1}^{(d)}} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}\right) \cdots \left(\prod_{(a,b) \in \Lambda_{a_m}^{(d)}} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}\right) \left(\prod_{(a,b) \in (\Lambda_K^{(d)})'} x_{a,b}^{\alpha_{a,b}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}}\right)$$

where $K = \{a_1, \ldots, a_m\}$.

Now once we determine an anchor d, we have fixed $\Lambda^{(d)}$. Thus, the only difference in the factorizations of d and dz^n are be the values of the $\alpha_{a,b}$'s and the $\beta_{a,b}$'s (note we may assume $\Lambda^{(z)} \subseteq \Lambda^{(d)}$). So we can factor dz^n as

$$dz^n = \left[\prod_{(a,b)\in\Lambda_{a_1}^{(d)}} x_{a,b}^{\alpha_{a,b}^{(n)}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}^{(n)}}\right]\cdots\left[\prod_{(a,b)\in\Lambda_{a_m}^{(d)}} x_{a,b}^{\alpha_{a,b}^{(n)}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}^{(n)}}\right]\left[\prod_{(a,b)\in(\Lambda_K^{(d)})'} x_{a,b}^{\alpha_{a,b}^{(n)}} \left(x_a(xy^a)^b\right)^{\beta_{a,b}^{(n)}}\right]$$

We define $\Delta\beta_{a,b}^{(n)} := \beta_{a,b}^{(n)} - \beta_{a,b}^{(n-1)}$ (where $\beta_{a,b}^{(0)} = \beta_{a,b}$). For a fixed a, note that $\sum_{(a,b)\in\Lambda_a^{(d)}} \Delta\beta_{a,b}^{(n)}$ represents the change in the exponent on x_a from dz^{n-1} to dz^n. Then we have the following equations which must hold for each n.

$$s_{a_1} = \sum_{(a_1,b)\in\Lambda_{a_1}^{(d)}} \Delta\beta_{a_1,b}^{(n)}, \qquad s_{a_2} = \sum_{(a_2,b)\in\Lambda_{a_2}^{(d)}} \Delta\beta_{a_2,b}^{(n)}, \qquad \cdots \qquad s_m = \sum_{(a_m,b)\in\Lambda_{a_m}^{(d)}} \Delta\beta_{a_m,b}^{(n)}$$

Also, for each $a \in (\Lambda_K^{(d)})'[\boldsymbol{a}] = \Lambda^{(d)}[\boldsymbol{a}] \setminus K$, we have

$$0 = \sum_{(a,b)\in\Lambda_a^{(d)}} \Delta\beta_{a,b}^{(n)} \qquad \text{and collectively, } 0 = \sum_{(a,b)\in(\Lambda_K^{(d)})'} \Delta\beta_{a,b}^{(n)}$$

Before we list the next two equations that must be satisfied, we introduce some notation to save space. For each $a \in \Lambda^{(d)}[\boldsymbol{a}]$ we define

$$\Delta B_a^{(n)} := \sum_{(a,b)\in\Lambda_a^{(d)}} b\Delta\beta_{a,b}^{(n)}$$

That is, $\Delta B_a^{(n)}$ is the change in the exponent on x as paired with x_a in the factorizations of dz^{n-1} to dz^n.

Then we must have that

$$t = \sum_{(a,b)\in\Lambda^{(d)}} b\Delta\beta_{a,b}^{(n)} = \sum_{(a,b)\in\Lambda_K^{(d)}} b\Delta\beta_{a,b}^{(n)} + \sum_{(a,b)\in(\Lambda_K^{(d)})'} b\Delta\beta_{a,b}^{(n)}$$

$$= \sum_{a\in\Lambda_K^{(d)}[\boldsymbol{a}]} \left(\sum_{(a,b)\in\Lambda_a^{(d)}} b\Delta\beta_{a,b}^{(n)} \right) + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \left(\sum_{(a,b)\in\Lambda_a^{(d)}} b\Delta\beta_{a,b}^{(n)} \right)$$

$$= \sum_{a\in K} \Delta B_a^{(n)} + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \Delta B_a^{(n)}$$

and similarly, we have

$$u = \sum_{a\in K} a\Delta B_a^{(n)} + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} a\Delta B_a^{(n)}$$

Now consider that the exponent on x as paired with x_a must always be nonnegative in the factorization of dz^n for each n. Thus, for any $N \in \mathbb{N}$, consider that $\sum_{n=1}^{N} \Delta B_a^{(n)}$ is bounded below by $-\sum_{(a,b)\in\Lambda_a^{(d)}} b\beta_{a,b}$ (the negative of the exponent on x as paired with x_a in d).

Thus, we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \geq 0$. We will use this limit often, so we use the notation

$$\overline{\Delta B_a} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \geq 0.$$

Moreover, if x_a is not a factor of z, but is a factor of d (i.e., $a \in (\Lambda_K^{(d)})'[\boldsymbol{a}]$), then the exponent on x_a remains fixed in dz^n for each n. Thus, the maximum exponent on x as paired with x_a in the factorization of dz^n for any n is

$$\left(\max\{b|(a,b) \in \Lambda_a^{(d)}\}\right) \left(\sum_{(a,b)\in\Lambda_a^{(d)}} \beta_{a,b} \right)$$

It follows that

$$\sum_{n=1}^{N} \Delta B_a^{(n)} \leq \left(\max\{b|(a,b) \in \Lambda_a^{(d)}\}\right) \left(\sum_{(a,b)\in\Lambda_a^{(d)}} \beta_{a,b} \right).$$

Thus, if $a \in (\Lambda_K^{(d)})'[\boldsymbol{a}]$ then we can conclude that

$$\overline{\Delta B_a} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} = 0$$

Now consider that

$$Nt = \sum_{a\in K} \left(\sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \left(\sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

Then divide through by N to and let N increase without bound to find

$$t = \sum_{a\in K} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \lim_{N\to\infty} \sum_{a\in K} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \lim_{N\to\infty} \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \sum_{a\in K} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \sum_{a\in K} \overline{\Delta B_a} + \sum_{a\in(\Lambda_K^{(d)})'[\boldsymbol{a}]} \overline{\Delta B_a}$$

$$= \sum_{a\in K} \overline{\Delta B_a}$$

Likewise, we have

$$u = \sum_{a\in K} a\overline{\Delta B_a}$$

Lastly, we note that

$$u = \sum_{a\in K} a\overline{\Delta B_a} \leq (\max\{a|a \in K\}) \sum_{a\in K} \overline{\Delta B_a} = t \cdot \max\{a|a \in K\}$$

Therefore, for any element $z \in D_3'$, the exponent on y is bounded above by the product of the exponent on x and the maximum value of a such that x_a is a factor of z.

This observation is actually quite significant in helping to determine D_3''. From this claim, we can see that $y \notin D_3''$. For given an element $d \in D_3'$, the exponent on y is bounded above. Thus we cannot have $dy^n \in D_3'$ for all $n \in \mathbb{N}$.

However, we do have $xy^n \in D_3''$, which implies that $y \in D_3'''$.

Thus, we can conclude that

$$D_3 \subset D_3' \subset D_3'' \subset D_3''' = D_3''''$$

So starting with D_3, we can apply the complete integral closure three times before reaching a completely integraly closed domain.

We have

$$D_3 = \mathbb{F}\left[x_{a,b}(x_a(xy^a)^b)^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right]$$

$$D_3' \supseteq \mathbb{F}\left[x_{a,b}, x_a(xy^a)^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right]$$

$$D_3'' \supseteq \mathbb{F}\left[x_{a,b}, x_a, xy^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right]$$

$$D_3''' = \mathbb{F}\left[x_{a,b}, x_a, x, y | (a,b) \in \mathbb{N}_0^2\right]$$

and, more precisely,

$$\begin{array}{lcl} D_3 & = & \mathbb{F}\left[x_{a,b}(x_a(xy^a)^b)^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right] \\ \cap & & \\ D_3' & \supseteq & \mathbb{F}\left[x_{a,b}, x_a(xy^a)^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right] \\ \cap & & \\ D_3'' & \supseteq & \mathbb{F}\left[x_{a,b}, x_a, xy^n | (a,b) \in \mathbb{N}_0^2, n \in \mathbb{N}_0\right] \\ \cap & & \\ D_3''' & = & \mathbb{F}\left[x_{a,b}, x_a, x, y | (a,b) \in \mathbb{N}_0^2\right] \end{array}$$

Now the astute reader will notice that throughout this proof our elements have been strictly monomials. We now present a brief argument that we need only consider monomials as anchors.

Consider that an arbitrary element of D_3 is a polynomial of the form

$$d = \sum_{j=1}^{N} \left(\prod_{i=1}^{M_j} x_{a_{j,i},b_{j,i}}^{\alpha_{j,i}} \left(x_{a_{j,i}} (xy^{a_{j,i}})^{b_{j,i}} \right)^{\beta_{j,i}} \right).$$

Suppose that d is the anchor for some $z \in D_3'$. Then we must have $dz^n \in D_3$ for each n. In particular, we must have

$$\sum_{j=1}^{N} \left(\prod_{i=1}^{M_j} x_{a_{j,i},b_{j,i}}^{\alpha_{j,i}} \left(x_{a_{j,i}} (xy^{a_{j,i}})^{b_{j,i}} \right)^{\beta_{j,i}} \right) z^n \in D_3$$

But this is just another polynomial in D_3, so each term of the polynomial must be a monomial in D_3. In particular, take $j = 1$, then we must have that

$$\prod_{i=1}^{M_1} x_{a_{1,i},b_{1,i}}^{\alpha_{1,i}} \left(x_{a_{1,i}} (xy^{a_{1,i}})^{b_{1,i}} \right)^{\beta_{1,i}} z^n \in D_3$$

But this is equivalent to saying that

$$\prod_{i=1}^{M_1} x_{a_{1,i},b_{1,i}}^{\alpha_{1,i}} \left(x_{a_{1,i}} (xy^{a_{1,i}})^{b_{1,i}} \right)^{\beta_{1,i}}$$

is an anchor of z.

Thus, to show that an element is not in D_3', it suffices to show that no monomial in D_3 can be an anchor.

This concludes the proof that our example D_n behaves as desired when $n = 3$. Before moving on to the case when $n = 4$, we note a couple intriguing examples of elements that are in D_3'. Determining D_3' exactly is no trivial task, and these examples highlight the significance of the bound on the exponent of y that we found in the preceeding proof.

For the first example let $z = x_2 x_4 x_8 x^2 y^{14}$. Note that z does not factor into a product of elements of the form $x_2(xy^2)^b$, $x_4(xy^4)^b$, and $x_8(xy^8)^b$ (and so is not trivially in D_3').

However, we will show that $d = x_{2,0}(x_{2,1}x_2xy^2)(x_{4,0}x_4)x_{4,1}x_{8,1}x_{8,2}$ is indeed an anchor for z. Using the notation in the proof above for an arbitrary anchor d, the main equations to solve are

$$2 = \sum_{(2,b)\in\Lambda_2^{(d)}} b\Delta\beta_{2,b}^{(n)} + \sum_{(4,b)\in\Lambda_4^{(d)}} b\Delta\beta_{4,b}^{(n)} + \sum_{(8,b)\in\Lambda_8^{(d)}} b\Delta\beta_{8,b}^{(n)}$$

$$= 0\cdot\Delta\beta_{2,0}^{(n)} + 1\cdot\Delta\beta_{2,1}^{(n)} + 0\cdot\Delta\beta_{4,0}^{(n)} + 1\cdot\Delta\beta_{4,1}^{(n)} + 1\cdot\Delta\beta_{8,1}^{(n)} + 2\cdot\Delta\beta_{8,2}^{(n)}$$

$$= \Delta\beta_{2,1}^{(n)} + \Delta\beta_{4,1}^{(n)} + \Delta\beta_{8,1}^{(n)} + 2\cdot\Delta\beta_{8,2}^{(n)}$$

and

$$14 = 2\sum_{(2,b)\in\Lambda_2^{(d)}} b\Delta\beta_{2,b}^{(n)} + 4\sum_{(4,b)\in\Lambda_4^{(d)}} b\Delta\beta_{4,b}^{(n)} + 8\sum_{(8,b)\in\Lambda_8^{(d)}} b\Delta\beta_{8,b}^{(n)}$$

$$= 0\cdot\Delta\beta_{2,0}^{(n)} + 2\cdot\Delta\beta_{2,1}^{(n)} + 0\cdot\Delta\beta_{4,0}^{(n)} + 4\cdot\Delta\beta_{4,1}^{(n)} + 8\cdot\Delta\beta_{8,1}^{(n)} + 16\cdot\Delta\beta_{8,2}^{(n)}$$

$$= 2\cdot\Delta\beta_{2,1}^{(n)} + 4\cdot\Delta\beta_{4,1}^{(n)} + 8\cdot\Delta\beta_{8,1}^{(n)} + 16\cdot\Delta\beta_{8,2}^{(n)}$$

Consider two solutions to this system:

$$(1): \Delta\beta_{2,0}^{(n)} = 2, \Delta\beta_{2,1}^{(n)} = -1, \Delta\beta_{4,0}^{(n)} = -1, \Delta\beta_{4,1}^{(n)} = 2, \Delta\beta_{8,1}^{(n)} = 1, \Delta\beta_{8,2}^{(n)} = 0$$

$$(2): \Delta\beta_{2,0}^{(n)} = 0, \Delta\beta_{2,1}^{(n)} = 1, \Delta\beta_{4,0}^{(n)} = 2, \Delta\beta_{4,1}^{(n)} = -1, \Delta\beta_{8,1}^{(n)} = 0, \Delta\beta_{8,2}^{(n)} = 1$$

To show that $z \in D_3'$, we use solution (1) to factor dz^n when n is odd, and we use solution (2) to factor dz^n when n is even.

We show the first few instances:

$$d = (x_{2,0})(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4)(x_{4,1})(x_{8,1})(x_{8,2})$$

$$dz = (x_{2,0}x_2^2)(x_{2,1})(x_{4,0})(x_{4,1}(x_4(xy^4))^2)(x_{8,1}(x_8(xy^8)))(x_{8,2})$$

$$dz^2 = (x_{2,0}x_2^2)(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4^2)(x_{4,1}(x_4(xy^4)))(x_{8,1}(x_8(xy^8)))(x_{8,2}(x_8(xy^8)^2))$$

$$dz^3 = (x_{2,0}x_2^4)(x_{2,1})(x_{4,0}x_4)(x_{4,1}(x_4(xy^4))^3)(x_{8,1}(x_8(xy^8))^2)(x_{8,2}(x_8(xy^8)^2))$$

$$dz^4 = (x_{2,0}x_2^4)(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4^3)(x_{4,1}(x_4(xy^4))^2)(x_{8,1}(x_8(xy^8))^2)(x_{8,2}(x_8(xy^8)^2)^2)$$

Moreover, note that d is also an anchor for $z = x_4 x_8 x^2 y^{14}$. The only modification to make, is to set $\Delta\beta_{2,0}^{(n)} = 1$ in solution (1), and set $\Delta\beta_{2,0}^{(n)} = -1$ in solution (2). Then the first few factorizations are

$$d = (x_{2,0})(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4)(x_{4,1})(x_{8,1})(x_{8,2})$$

$$dz = (x_{2,0}x_2)(x_{2,1})(x_{4,0})(x_{4,1}(x_4(xy^4))^2)(x_{8,1}(x_8(xy^8)))(x_{8,2})$$

$$dz^2 = (x_{2,0})(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4^2)(x_{4,1}(x_4(xy^4)))(x_{8,1}(x_8(xy^8)))(x_{8,2}(x_8(xy^8)^2))$$

$$dz^3 = (x_{2,0}x_2)(x_{2,1})(x_{4,0}x_4)(x_{4,1}(x_4(xy^4))^3)(x_{8,1}(x_8(xy^8))^2)(x_{8,2}(x_8(xy^8)^2))$$

$$dz^4 = (x_{2,0})(x_{2,1}(x_2(xy^2)))(x_{4,0}x_4^3)(x_{4,1}(x_4(xy^4))^2)(x_{8,1}(x_8(xy^8))^2)(x_{8,2}(x_8(xy^8)^2)^2)$$

This example highlights the importance influence of the presence of a given x_a (and subsequent xy^a) in d where x_a is not a factor of z. Note that while the average change on the exponent of x_2 is 0, at each step the presence (or absence) of xy^2 is crucial to the factorization. That is, z would be not in D_3' if all factors of the anchor d were of the form $x_4(xy^4)^b$ and $x_8(xy^8)^b$.

4.2.3 Case: $n = 4$

Consider the domain

$$D_4 = \mathbb{F}\left[x_{\lambda^{(3)}}\left(x_{\lambda^{(2)}}\left(x_{\lambda^{(1)}}\left(xy^{\lambda_1}\right)^{\lambda_2}\right)^{\lambda_3}\right)^n |\lambda \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

We claim that

$$D_4 \subset D_4' \subset D_4'' \subset D_4''' \subset D_4''''$$

Note we have the following containments:

$$D_4' \supset \mathbb{F}\left[x_{\lambda^{(3)}}, x_{\lambda^{(2)}}\left(x_{\lambda^{(1)}}\left(xy^{\lambda_1}\right)^{\lambda_2}\right)^n |\lambda \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4'' \supset \mathbb{F}\left[x_{\lambda^{(3)}}, x_{\lambda^{(2)}}, x_{\lambda^{(1)}}\left(xy^{\lambda_1}\right)^n |\lambda \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4''' \supset \mathbb{F}\left[x_{\lambda^{(3)}}, x_{\lambda^{(2)}}, x_{\lambda^{(1)}}, xy^n|\lambda \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4'''' = \mathbb{F}\left[x_{\lambda^{(3)}}, x_{\lambda^{(2)}}, x_{\lambda^{(1)}}, x, y|\lambda \in \mathbb{N}_0^3\right]$$

Equivalently, we have

$$D_4 = \mathbb{F}\left[x_{a,b,c}(x_{a,b}(x_a(xy^a)^b)^c)^n|(a,b,c) \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4' \supset \mathbb{F}\left[x_{a,b,c}, x_{a,b}(x_a(xy^a)^b)^n|(a,b,c) \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4'' \supset \mathbb{F}\left[x_{a,b,c}, x_{a,b}, x_a(xy^a)^n|(a,b,c) \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4''' \supset \mathbb{F}\left[x_{a,b,c}, x_{a,b}, x_a, xy^n|(a,b,c) \in \mathbb{N}_0^3, n \in \mathbb{N}_0\right]$$

$$D_4'''' = \mathbb{F}\left[x_{a,b,c}, x_{a,b}, x_a, x, y|(a,b,c) \in \mathbb{N}_0^3\right]$$

We first show that the quotient field is

$$K = \mathbb{F}\left(x_{a,b,c}, x_{a,b}, x_a, x, y|(a,b,c) \in \mathbb{N}_0^3\right)$$

First note that taking $n = 0$, gives $x_{a,b,c} \in D_4 \subset K$.

Then we have

$$\frac{x_{a,b,0}x_{a,b}}{x_{a,b,0}} = x_{a,b} \in K \text{ for each } (a,b) \in \mathbb{N}_0^2$$

and

$$\frac{x_{a,0,1}x_{a,0}x_a}{x_{a,0,1}x_{a,0}} = x_a \in K \text{ for all } a \in \mathbb{N}_0$$

and

$$\frac{x_{0,1,1}x_{0,1}x_0x}{x_{0,1,1}x_{0,1}x_0} = x \in K \qquad \text{and} \qquad \frac{x_{1,1,1}x_{1,1}x_1xy}{x_{1,1,1}x_{1,1}x_1x} = y \in K$$

Thus we have the desired quotient field. Now we will determine D_4'.

First note that by setting $n = 0$, we have $x_{a,b,c} \in D_4 \subseteq D_4'$. Also note that n varies independently of a, b, and c, so we have $x_{a,b}(x_a(xy^a)^b)^c \in D_4'$ for each $(a,b,c) \in \mathbb{N}_0^3$ (with anchor element $x_{a,b,c}$). In particular, $x_{a,b} \in D_4'$ for each $(a,b) \in \mathbb{N}_0^2$, with anchor $x_{a,b,0}$, and $x_{a,0}x_a^c \in D_4'$ for all $c \geq 0$, with anchor $x_{a,0,c}$.

Now consider an abritray monomial in D_4.

$$d = \prod_{(a_i,b_i,c_i)\in\Lambda} x_{a_i,b_i,c_i}^{\alpha_i} \left(x_{a_i,b_i} \left(x_{a_i}(xy^{a_i})^{b_i} \right)^{c_i} \right)^{\beta_i}$$

where Λ is a finite subset of $\mathbb{N}_0^3$.

Note that the exponent on y is $\sum a_ib_ic_i\beta_i$, the exponent on x is $\sum b_ic_i\beta_i$, and the exponent on a given x_a is $\sum c_i\beta_i$, where the sum is taken over c_i such that x_{a,b_i,c_i} is a factor of d. Each exponent is determined by the collection of (a_i, b_i, c_i) and the β_i's. In particular, once the collection of (a_i, b_i, c_i)'s are determined, then the only way to increase the exponent on x, y, or x_{a_i} is to increase some of the β_i's.

Consider that the ratio between the exponent on y and the exponent on x is at most $\max\{a_i\}$. Thus, once the collection of (a_i, b_i, c_i) is chosen, the max ratio of y to x is determined. Therefore, we cannot have the exponent on y increase without bound, if the exponent on x is fixed. Thus, if an element in D_4' has y as a factor, then x must also be a factor.

Moreover, the ratio between the exponent on x and the exponent on any x_{a_i} is at most $\max\{b_i\}$. Thus, once the collection of (a_i, b_i, c_i) is chosen, the max ratio of x to each x_{a_i} is determined.

Therefore, we cannot have the exponent on x increase without bound, if the exponent on each x_{a_i} is fixed. Thus, if an element in D_4' has x as a factor, then x_a must also be a factor for some $a \in \mathbb{N}_0$.

Lastly, consider that the ratio between the exponent on a given x_{a_i} and a x_{a_i,b_i} is at most $\max\{c_i\}$. Thus, once the collection of (a_i, b_i, c_i) is chosen, the max ratio of x_{a_i} to x_{a_i,b_i} is determined. Therefore, we cannot have the exponent on x_{a_i} increase without bound, if the exponent on each x_{a_i,b_i} is fixed. Thus, if an element in D_4' has x_a as a factor, then $x_{a,b}$ must also be a factor for some $b \in \mathbb{N}_0$.

In summary, we have shown the following so far:

1. $x_{a,b,c} \in D_4'$ for all $(a,b,c) \in \mathbb{N}_0^3$

2. $x_{a,b}(x_a(xy^a)^b)^c \in D_4'$ for all $(a,b,c) \in \mathbb{N}_0^3$ (anchor $x_{a,b,c}$)

   Special case: $x_{a,b} \in D_4'$ for all $(a,b) \in \mathbb{N}_0^2$ (anchor $x_{a,b,0}$)

   Special case: $x_{a,0}x_a^c \in D_4'$ for all $c \in \mathbb{N}_0$ (anchor $x_{a,0,c}$). Hence $x_a \in D_4''$ for each a with anchor $x_{a,0} \in D_4'$.

   Special case: $x_{0,b}(x_0x^b)^c \in D_4'$ for all $c \in \mathbb{N}_0$ (anchor $x_{0,b,c}$). Hence $x \in D_4'''$ with anchor $x_0 \in D_4''$.

3. If $z \in D_4'$ and y is a factor of z, then x is also a factor of z.

4. If $z \in D_4'$ and x is a factor of z, then x_a is also a factor of z for some $a \in \mathbb{N}_0$.

5. If $z \in D_4'$ and x_a is a factor of z, then $x_{a,b}$ is also a factor of z for some $b \in \mathbb{N}_0$.

Note that we will use much of the same notation as in the proof for the case $n = 3$. Only new notation will be defined in this proof.

With this in mind, an arbitrary monomial $z \in D_4'$ has the form

$$z = \left(\prod_{(a,b,c)\in\Gamma^{(z)}} x_{a,b,c}^{q_{a,b,c}} \right) \left(\prod_{(a,b)\in\Lambda} x_{a,b}^{r_{a,b}} \right) \left(\prod_{a\in K} x_a^{s_a} \right) x^t y^u$$

where $\Gamma^{(z)}$ is a finite subset of $\mathbb{N}_0^3$, Λ is a finite subset of $\mathbb{N}_0^2$, and K is a finite subset of $\mathbb{N}_0$. From observations 3, we know that $u \neq 0$ implies $t \neq 0$, and from observation 4, if $t \neq 0$, then K is non-empty. From observation 5, we know for each $a \in K$, there exists $(a,b) \in \Lambda$ (i.e., $\Lambda_a[\boldsymbol{b}] \neq \emptyset$).

We set out to show that $u \leq t \cdot \max\{a | a \in K\}$. So we assume that $u \neq 0$, and hence observations $3, 4$, and 5 are in effect.

To that end, we let $d \in D_4$ be an anchor for z. Then d is of the form

$$d = \prod_{(a,b,c) \in \Gamma^{(d)}} x_{a,b,c}^{\alpha_{a,b,c}} \left(x_{a,b} \left(x_a \left(xy^a \right)^b \right)^c \right)^{\beta_{a,b,c}}$$

For simplicity, since each $x_{a,b,c}$ that is a factor of z is already in D_3, we assume that $x_{a,b,c}$ is a factor of d as well (i.e., $\Gamma^{(z)} \subseteq \Gamma^{(d)}$).

Now, for each factor $x_{a,b}$ of z, there must be a factor $x_{a,b,c}$ of d for some c. Thus, the only difference in the factorization of d and dz^n are the $\alpha_{a,b,c}$'s and the $\beta_{a,b,c}$'s.

So we can write

$$dz^n = \prod_{(a,b,c) \in \Gamma^{(d)}} x_{a,b,c}^{\alpha_{a,b,c}^{(n)}} \left(x_{a,b} \left(x_a \left(xy^a \right)^b \right)^c \right)^{\beta_{a,b,c}^{(n)}}$$

Now consider the following equations which must be satisfied since $z \in D_3'$.

For each $(a, b) \in \Lambda$, we have

$$r_{a,b} = \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} \Delta\beta_{a,b,c}^{(n)}$$

For each $(a, b) \in (\Gamma_\Lambda^{(d)})'[\boldsymbol{a}, \boldsymbol{b}] = \Gamma^{(d)}[\boldsymbol{a}, \boldsymbol{b}] \setminus \Lambda$, we have

$$0 = \sum_{(a,b,c) \in \Gamma_{(a,b)}^{(d)}} \Delta\beta_{a,b,c}^{(n)}$$

For each $a \in K$, we have

$$s_a = \sum_{(a,b,c) \in \Gamma_a^{(d)}} c\Delta\beta_{a,b,c}^{(n)}$$

For each $a \in (\Gamma_K^{(d)})'[\boldsymbol{a}] = \Gamma^{(d)}[\boldsymbol{a}] \setminus \Lambda$, we have

$$0 = \sum_{(a,b,c) \in \Gamma_a^{(d)}} c\Delta\beta_{a,b,c}^{(n)}$$

Moreover, we have

$$t = \sum_{(a,b,c)\in\Gamma^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} = \sum_{(a,b,c)\in\Gamma_K^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b,c)\in(\Gamma_K^{(d)})'} bc\Delta\beta_{a,b,c}^{(n)}$$

$$= \sum_{a\in\Gamma_K^{(d)}[\boldsymbol{a}]} \left(\sum_{(a,b,c)\in\Gamma_a^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \right) + \sum_{a\in(\Gamma_K^{(d)})'[\boldsymbol{a}]} \left(\sum_{(a,b,c)\in\Lambda_a^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \right)$$

$$= \sum_{a\in K} \Delta B_a^{(n)} + \sum_{a\in(\Gamma_K^{(d)})'[\boldsymbol{a}]} \Delta B_a^{(n)}$$

and similarly, we have

$$u = \sum_{a\in K} a\Delta B_a^{(n)} + \sum_{a\in(\Gamma_K^{(d)})'[\boldsymbol{a}]} a\Delta B_a^{(n)}$$

Now for any $N \in \mathbb{N}$, and a given $a \in \Gamma^{(d)}[\boldsymbol{a}]$, consider that $\sum_{n=1}^{N} \Delta B_a^{(n)}$ gives the change in the exponent on x as paired with x_a from the factorization of d to dz^N. Note that this sum is bounded below by $-\sum_{(a,b,c)\in\Gamma_a^{(d)}} bc\beta_{a,b,c}$ (the negative of the exponent of x as paired with x_a in the factorization of d).

Therefore, we have that

$$\overline{\Delta B_a} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \geq 0.$$

Now, if $a \in \Gamma^{(d)}[\boldsymbol{a}] \setminus K = (\Gamma_K^{(d)})'[\boldsymbol{a}]$, then the exponent on x_a remains fixed in dz^n for each n. Note that this exponent is $\sum_{(a,b,c)\in\Gamma_a^{(d)}} c\beta_{a,b,c}$. Thus, the maximum exponent on x as paired with x_a in the factorization of dz^n for any n is

$$\left(\max\{b|(a,b,c)\in\Gamma_a^{(d)}\}\right) \left(\sum_{(a,b,c)\in\Gamma_a^{(d)}} c\beta_{a,b,c} \right).$$

It follows that

$$\sum_{n=1}^{N} \Delta B_a^{(n)} \leq \left(\max\{b|(a,b,c)\in\Gamma_a^{(d)}\}\right) \left(\sum_{(a,b,c)\in\Gamma_a^{(d)}} c\beta_{a,b,c} \right)$$

and

$$\overline{\Delta B_a} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} = 0.$$

Now consider that

$$Nt = \sum_{a\in K} \left(\sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in (\Gamma_K^{(d)})'[\boldsymbol{a}]} \left(\sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

Then divide through by N and let N increase without bound to find

$$t = \sum_{a\in K} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in (\Gamma_K^{(d)})'[\boldsymbol{a}]} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \lim_{N\to\infty} \sum_{a\in K} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \lim_{N\to\infty} \sum_{a\in (\Gamma_K^{(d)})'[\boldsymbol{a}]} \left(\frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \sum_{a\in K} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) + \sum_{a\in (\Gamma_K^{(d)})'[\boldsymbol{a}]} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right)$$

$$= \sum_{a\in K} \overline{\Delta B_a} + \sum_{a\in (\Gamma_K^{(d)})'[\boldsymbol{a}]} \overline{\Delta B_a}$$

$$= \sum_{a\in K} \overline{\Delta B_a}$$

Likewise, we have

$$u = \sum_{a\in K} a\overline{\Delta B_a}$$

Lastly, we note that

$$u = \sum_{a\in K} a\overline{\Delta B_a} \leq (\max\{a|a\in K\}) \sum_{a\in K} \overline{\Delta B_a} = t \cdot \max\{a|a\in K\}$$

Therefore, for any element $z \in D_4'$, the exponent on y is bounded above by the product of the exponent on x and the maximum value of a such that x_a is a factor of z. Thus, we can conclude that $y \notin D_4''$.

We now construct a similar argument to show that the exponent on x in z is also bounded, and hence $x \notin D_4''$.

Consider for a given pair $(a,b) \in \Gamma^{(d)}[\boldsymbol{a},\boldsymbol{b}]$ the exponent on x_a as paired with $x_{a,b}$ in the factorization of d is given by

$$\sum_{(a,b,c)\in\Gamma^{(d)}_{a,b}} c\beta_{a,b,c}.$$

We define the change of this exponent from the factorization of dz^{n-1} to dz^n as

$$\Delta C^{(n)}_{a,b} = \sum_{(a,b,c)\in\Gamma^{(d)}_{a,b}} c\Delta\beta^{(n)}_{a,b,c}$$

Note then that $\sum_{n=1}^{N} \Delta C^{(n)}_{a,b}$ gives the change on the exponent of x_a as paired with $x_{a,b}$ in the factorization of d to dz^N. So this sum is bounded below by $-\sum_{(a,b,c)\in\Gamma^{(d)}_{a,b}} c\beta_{a,b,c}$. Thus we have that

$$\overline{\Delta C_{a,b}} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta C^{(n)}_{a,b} \geq 0.$$

Also, if $x_{a,b}$ is a factor of d but not a factor of z (i.e., $(a,b) \in (\Gamma^{(d)}_{\Lambda})'[\boldsymbol{a},\boldsymbol{b}]$), then the exponent on $x_{a,b}$ is fixed in dz^n for each n. Thus the exponent on x_a as paired with $x_{a,b}$ is bounded above by

$$\left(\max\{c|(a,b,c) \in \Gamma^{(d)}_{a,b}\}\right) \left(\sum_{(a,b,c)\in\Gamma^{(d)}_{a,b}} \beta_{a,b,c} \right).$$

In this case, we have

$$\overline{\Delta C_{a,b}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta C^{(n)}_{a,b} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta C^{(n)}_{a,b} = 0.$$

Moreover, if x_a is a factor of d, but not a factor of z, (i.e., $a \in (\Gamma^{(d)}_{K})'[\boldsymbol{a}]$), then we have

$$0 = \sum_{(a,b,c)\in\Gamma^{(d)}_{a}} c\Delta\beta^{(n)}_{a,b,c} = \sum_{b\in\Gamma^{(d)}_{a}[\boldsymbol{b}]} \sum_{(a,b,c)\in\Gamma^{(d)}_{a,b}} c\Delta\beta^{(n)}_{a,b,c}.$$

Which implies that

$$0 = \sum_{b \in \Gamma_a^{(d)}[\boldsymbol{b}]} \overline{\Delta C_{a,b}}$$

Likewise, if x_a is a factor of z, then we have

$$s_a = \sum_{b \in \Gamma_a^{(d)}[\boldsymbol{b}]} \overline{\Delta C_{a,b}} = \sum_{b \in (\Gamma_\Lambda^{(d)})_a[\boldsymbol{b}]} \overline{\Delta C_{a,b}} = \sum_{b \in \Lambda_a[\boldsymbol{b}]} \overline{\Delta C_{a,b}}$$

Now then consider that

$$t = \sum_{(a,b,c) \in \Gamma^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} = \sum_{(a,b,c) \in \Gamma_\Lambda^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b,c) \in (\Gamma_\Lambda^{(d)})'} bc\Delta\beta_{a,b,c}^{(n)}$$

$$= \sum_{(a,b) \in \Gamma_\Lambda^{(d)}[\boldsymbol{a},\boldsymbol{b}]} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b) \in (\Gamma_\Lambda^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} \sum_{(a,b,c) \in (\Gamma_\Lambda^{(d)})'} bc\Delta\beta_{a,b,c}^{(n)}$$

$$= \sum_{(a,b) \in \Lambda} b \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} c\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b) \in (\Gamma_\Lambda^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} b \sum_{(a,b,c) \in (\Gamma_\Lambda^{(d)})'} c\Delta\beta_{a,b,c}^{(n)}$$

Moreover, we have

$$
\begin{aligned}
t &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in\Lambda} b \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} c\Delta\beta_{a,b,c}^{(n)} + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in(\Gamma_{\Lambda}^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} b \sum_{(a,b,c)\in(\Gamma_{\Lambda}^{(d)})'} c\Delta\beta_{a,b,c}^{(n)} \\
&= \sum_{(a,b)\in\Lambda} b \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} c\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b)\in(\Gamma_{\Lambda}^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} b \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b,c)\in(\Gamma_{\Lambda}^{(d)})'} c\Delta\beta_{a,b,c}^{(n)} \\
&= \sum_{(a,b)\in\Lambda} b\overline{\Delta C_{a,b}} + \sum_{(a,b)\in(\Gamma_{\Lambda}^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} b\overline{\Delta C_{a,b}} \\
&= \sum_{(a,b)\in\Lambda} b\overline{\Delta C_{a,b}} \\
&= \sum_{a\in K} \sum_{b\in\Lambda_a[\boldsymbol{b}]} b\overline{\Delta C_{a,b}} + \sum_{a\in(\Lambda_K)'[\boldsymbol{a}]} \sum_{b\in(\Lambda_K)'_a[\boldsymbol{b}]} b\overline{\Delta C_{a,b}} \\
&\leq (\max\{b|(a,b)\in\Lambda_K\}) \sum_{a\in K} \sum_{b\in\Lambda_a[\boldsymbol{b}]} \overline{\Delta C_{a,b}} \\
&= (\max\{b|(a,b)\in\Lambda_K\}) \sum_{a\in K} s_a
\end{aligned}
$$

Thus, the exponent on x in z is bounded above by the product of the maximum b such that $x_{a,b}$ is a factor z for some a (where x_a is also a factor of z) and the sum of the exponents on each x_a that is a factor of z.

Note that we can now conclude that $D_4 \subset D_4' \subset D_4'' \subset D_4'''$, since we know that $x \notin D_4, D_4', D_4''$, but $x \in D_4'''$. It now suffices to show that $y \notin D_4'''$, and then we will have shown that D_4 behaves as desired with respect to complete integral closure. But first, we make a couple more helpful observations about D_4'.

Recall that

$$t = \sum_{a\in K} \overline{\Delta B_a} = \sum_{a\in K} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Delta B_a^{(n)} \right) = \sum_{a\in K} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{(a,b,c)\in\Gamma_a^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \right) \right)$$

Consider if $a \in K$, but $x_{a,b}$ is a factor of d and not a factor of z (i.e., $(a,b) \in (\Gamma_\Lambda^{(d)})'[\boldsymbol{a}, \boldsymbol{b}]$). Then note that the change in the exponent on x as paired with $x_{a,b}$ in the factorization of dz^{n-1} to dz^n is given by

$$\sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)}.$$

Since $x_{a,b}$ is not a factor of z, the exponent on $x_{a,b}$ is fixed in dz^n for each n. So then, the maximum value of the exponent on x as paired with $x_{a,b}$ in the factorization of dz^n for any n is

$$b \left(\max\{c | (a,b,c) \in \Gamma_{a,b}^{(d)}\} \right) \left(\sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} \beta_{a,b,c} \right).$$

Thus, in this case, we can conclude that

$$\sum_{n=1}^{N} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \leq b \left(\max\{c | (a,b,c) \in \Gamma_{a,b}^{(d)}\} \right) \left(\sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} \beta_{a,b,c} \right).$$

It follows that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} = 0$$

Therefore, we have

$$\overline{\Delta B_a} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{(a,b,c)\in\Gamma_a^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \right)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{(a,b)\in\Lambda} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b)\in(\Gamma_\Lambda^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} \right)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in\Lambda} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in(\Gamma_\Lambda^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in\Lambda} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)} + \sum_{(a,b)\in(\Gamma_\Lambda^{(d)})'[\boldsymbol{a},\boldsymbol{b}]} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{(a,b)\in\Lambda} \sum_{(a,b,c)\in\Gamma_{a,b}^{(d)}} bc\Delta\beta_{a,b,c}^{(n)}$$

Thus, when determining $\overline{\Delta B_a}$, it is enough to look at the change in the exponent of x as paired with each $x_{a,b}$ where $x_{a,b}$ is a factor of z (i.e., $(a,b) \in \Lambda$).

Moreover, since

$$t = \sum_{a\in K} \overline{\Delta B_a} \qquad \text{and} \qquad u = \sum_{a\in K} a\overline{\Delta B_a}$$

we conclude that t and u can be determined by only considering the change in the exponent on x as paired with each $x_{a,b}$ where $x_{a,b}$ is factor of z.

With this observation in mind, we will now argue that $y \notin D_4'''$ either. However, since y is clearly in D_4'''', we will have shown that D_4 requires four iterations of complete integral closure before reaching a completely integrally closed domain.

We claim that for a given element $z \in D_4''$, the exponent on y in z is bounded above by the product of the exponent on t in z and the maximum a such that x_a is a factor of z. We will start with an example argument, considering one specific case, and then move to a general argument.

For the example, we will show that $z = x_1 x_2 x y^3$ is not in D''_4. Note here that the ratio between the exponent of y and the exponent of x is 3, but the maximum a such that x_a is a factor of z is 2. For sake of demonstration, we will work out this case using our notation. So we can say

$$z = \left(\prod_{a \in K^{(z)}} x_a^{s_a^{(z)}} \right) x^{t^{(z)}} y^{u^{(z)}}$$

Here we have $K^{(z)} = \{1, 2\}$, $s_1^{(z)} = s_2^{(z)} = 1$, $t^{(z)} = 1$, and $u^{(z)} = 3$.

Suppose that $z \in D''_4$. Then, there is an anchor $d \in D'_4$, such that $dz^n \in D'_4$ for all n. That is, $d(x_1 x_2 x y^3)^n \in D'_4$ for all n. Now, if $dz^n \in D'_4$, we know that the ratio of the exponent of y to that of x in dz^n is bounded by the maximum a such that x_a is a factor of dz^n. Since this ratio approaches 3 as n increases without bound, we must have x_a as a factor of d with $a \geq 3$. Let's say we have x_3 and x_5 as factors of d. Now recall that if x_a is a factor of an element in D'_4, then $x_{a,b}$ must also be a factor for some b. We may also have $x_{a,b}$ be a factor even if x_a is not a factor. So let's say then that

$$d = x_{1,0} x_{1,1} x_{1,2} x_{2,0} x_{2,1} x_{2,2} x_{3,0} x_{3,1} x_{3,2} x_{5,0} x_{5,1} x_{5,3} x_{6,0} x_{6,1} x_{6,2} x_3 x_5 x^3 y^{10}$$

or, with our notation:

$$d = \left(\prod_{(a,b) \in \Lambda^{(d)}} x_{a,b}^{r_{a,b}^{(d)}} \right) \left(\prod_{a \in K^{(d)}} x_a^{s_a^{(d)}} \right) x^{t^{(d)}} y^{u^{(d)}}$$

where we have

$$\Lambda^{(d)} = \{(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(5,0),(5,1),(5,3),(6,0),(6,1),(6,2)\}$$

$$K^{(d)} = \{3, 5\}, \qquad r_{a,b}^{(d)} = 1 \text{ for each } (a,b) \in \Lambda^{(d)}$$

$$s_a^{(d)} = 1 \text{ for each } a \in K^{(d)}, \qquad t^{(d)} = 3, \qquad u^{(d)} = 10$$

Again, in terms of our notation, we can say that for each $a \in K^{(z)} \cup K^{(d)}$, we must have $a \in \Lambda^{(d)}[\boldsymbol{a}]$.

Now since $dz^n \in D_4'$ for all n, there is a collection of anchors $d_n \in D_4$, such that for each n, we have $d_n(dz^n)^m \in D_4$ for all m. That is we have

$$d_n(dz^n)^m = d_n(x_{1,0}x_{1,1}x_{1,2}x_{2,0}x_{2,1}x_{2,2}x_{3,0}x_{3,1}x_{3,2}x_{5,0}x_{5,1}x_{5,2}x_{6,0}x_{6,1}x_{6,2}x_3x_5x^3y^{10}(x_1x_2xy^3)^n)^m$$

$$= d_n(x_{1,0}x_{1,1}x_{1,2}x_{2,0}x_{2,1}x_{2,2}x_{3,0}x_{3,1}x_{3,2}x_{5,0}x_{5,1}x_{5,2}x_{6,0}x_{6,1}x_{6,2}x_3x_5x_1^nx_2^nx^{3+n}y^{10+3n})^m \in D_4$$

Now, consider that for each $x_{a,b}$ that is a factor of $d_n(dz^n)$, we must have $x_{a,b,c}$ as a factor of d_n as well. Moreover, the value(s) of c may change for each d_n, so we should say that $x_{a,b,c(n)}$ is a factor of d_n. Moreover, we may have $x_{a,b,c}$ as a factor of d_n where either $x_{a,b}$ or x_a is not a factor of dz^n. Let's say that $x_{8,b(n),c(n)}$ is a factor of d_n. Then we have

$$d_n = \prod_{(a,b,c)\in\Gamma^{(d_n)}} x_{a,b,c}^{\alpha(n)_{a,b,c}} \left(x_{a,b}\left(x_a\left(xy^a\right)^b\right)^c\right)^{\beta(n)_{a,b,c}}$$

where for each $(a,b) \in \Lambda^{(d)}$, we have $(a,b) \in \Gamma^{(d_n)}[\boldsymbol{a},\boldsymbol{b}]$. And we have also chosen that $8 \in \Gamma^{(d_n)}[\boldsymbol{a}]$ for each n. Also note that the exponents $\alpha(n)_{a,b,c}, \beta(n)_{a,b,c}$ may vary for each d_n, and thus are written as functions of n.

Now consider that from $d_n(dz^n)^{m-1}$ to $d_n(dz^n)^m$, the change in the exponent of x is given by

$$t^{(d)} + nt^{(z)} = 3 + n = \sum_{(a,b,c)\in\Gamma^{(d_n)}} bc\beta(n)_{a,b,c}^{(m)}$$

and the change in the exponent of y is

$$u^{(d)} + nu^{(z)} = 10 + 3n = \sum_{(a,b,c)\in\Gamma^{(d_n)}} abc\beta(n)_{a,b,c}^{(m)}.$$

Moreover, from the observation made earlier, for each n, we know that the change in the exponent of x (y) can be determined as the sum of the average change in the exponent of x (y) as paired with each x_a that is a factor dz^n. Moreover, the average change of x (y) as paired with x_a can be determined only considering the the change in the exponent of x (y) on each $x_{a,b}$ that is a factor of dz^n as well.

Thus we have

$$t^{(d)} + nt^{(z)} = \sum_{a \in K^{(d)} \cup K^{(z)}} \overline{\Delta B_a}$$

$$= \sum_{a \in K^{(d)} \cup K^{(z)}} \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} bc\Delta\beta(n)_{a,b,c}^{(m)}$$

and

$$u^{(d)} + nu^{(z)} = \sum_{a \in K^{(d)} \cup K^{(z)}} a\overline{\Delta B_a}$$

$$= \sum_{a \in K^{(d)} \cup K^{(z)}} \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} abc\Delta\beta(n)_{a,b,c}^{(m)}$$

Now note that for each $a \in K^{(d)} \cup K^{(z)}$ and each n, we have

$$\begin{aligned}
\overline{\Delta B_a} &= \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d)}} bc\Delta\beta(n)_{a,b,c}^{(m)} \\
&= \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d_n)}} bc\Delta\beta(n)_{a,b,c}^{(m)} \\
&= \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} b \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d_n)}} c\Delta\beta(n)_{a,b,c}^{(m)} \\
&\leq \max\{b | b \in \Lambda_a^{(d)}[\boldsymbol{b}] \cup \Lambda_a^{(z)}[\boldsymbol{b}]\} \sum_{(a,b) \in \Lambda_a^{(d)} \cup \Lambda_a^{(z)}} \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \sum_{(a,b,c) \in \Gamma_{a,b}^{(d_n)}} c\Delta\beta(n)_{a,b,c}^{(m)} \\
&= (s_a^{(d)} + ns_a^{(z)}) \max\{b | b \in \Lambda_a^{(d)}[\boldsymbol{b}] \cup \Lambda_a^{(z)}[\boldsymbol{b}]\}
\end{aligned}$$

Now, note that if $a \in K^{(d)} \setminus K^{(z)}$, (i.e., $s_a^{(z)} = 0$), then $\overline{\Delta B_a}$ is bounded above by $(s_a^{(d)}) \max\{b | b \in \Lambda_a^{(d)}[\boldsymbol{b}] \cup \Lambda_a^{(z)}[\boldsymbol{b}]\}$, which is fixed independent of n.

In our example then, we have

$$\overline{\Delta B_1} \leq 2n, \qquad \overline{\Delta B_2} \leq 2n, \qquad \overline{\Delta B_3} \leq 2, \qquad \overline{\Delta B_5} \leq 3$$

So consider that

$$\frac{10+3n}{3+n} = \frac{10/n+3}{3/n+1} = \frac{\frac{1}{n}\left(\overline{\Delta B_1} + 2\overline{\Delta B_2} + 3\overline{\Delta B_3} + 5\overline{\Delta B_5}\right)}{\frac{1}{n}\left(\overline{\Delta B_1} + \overline{\Delta B_2} + \overline{\Delta B_3} + \overline{\Delta B_5}\right)}$$

Then as n increases without bound, we have

$$3 = \frac{\frac{1}{n}\left(\overline{\Delta B_1} + 2\overline{\Delta B_2}\right)}{\frac{1}{n}\left(\overline{\Delta B_1} + \overline{\Delta B_2}\right)} \leq \frac{\frac{2}{n}\left(\overline{\Delta B_1} + \overline{\Delta B_2}\right)}{\frac{1}{n}\left(\overline{\Delta B_1} + \overline{\Delta B_2}\right)} = 2$$

In terms of our notation, we have

$$\frac{u^{(z)}}{t^{(z)}} = \frac{\frac{1}{n}\left(\sum_{a\in K^{(z)}} a\overline{\Delta B_a}\right)}{\frac{1}{n}\left(\sum_{a\in K^{(z)}} \overline{\Delta B_a}\right)} \leq \frac{\frac{1}{n}\left(\max\{a|a\in K\}\right)\left(\sum_{a\in K^{(z)}} \overline{\Delta B_a}\right)}{\frac{1}{n}\left(\sum_{a\in K^{(z)}} \overline{\Delta B_a}\right)} = \max\{a|a\in K\}$$

This is a contradiction, so we cannot have $x_1x_2xy^3$ in D_4''.

We now present a general argument to show that y is not in D_4'''. Compared to the specific example that we worked through above, all that is really left is to generalize the form of z. Given the arguments presented for the specific example above, we will move through this general argument quickly.

Consider that an arbitrary element $z \in D_4''$ is of the form

$$z = \left(\prod_{(a,b,c)\in\Gamma^{(z)}} x_{a,b,c}^{q_{a,b,c}^{(z)}}\right)\left(\prod_{(a,b)\in\Lambda^{(z)}} x_{a,b}^{r_{a,b}^{(z)}}\right)\left(\prod_{a\in K^{(z)}} x_a^{s_a^{(z)}}\right) x^{t^{(z)}} y^{u^{(z)}}$$

We claim that if $z \in D_4''$, then $u^{(z)} \leq t^{(z)} \cdot \max\{a|a \in K^{(z)}\}$, and thus there is no anchor $d \in D_4''$ such that $dy^n \in D_4''$ for all n, and $y \notin D_4'''$.

Now if $z \in D_4''$, then there exists an anchor $d \in D_4'$ such that $dz^n \in D_4'$ for all n. Say we have

$$d = \left(\prod_{(a,b,c)\in\Gamma^{(d)}} x_{a,b,c}^{q_{a,b,c}^{(d)}}\right)\left(\prod_{(a,b)\in\Lambda^{(d)}} x_{a,b}^{r_{a,b}^{(d)}}\right)\left(\prod_{a\in K^{(d)}} x_a^{s_a^{(d)}}\right) x^{t^{(d)}} y^{u^{(d)}}$$

where for each $a \in K^{(z)} \cup K^{(d)}$ we must have $a \in \Lambda^{(d)}[\boldsymbol{a}]$.

First note that the exponent on y in dz^n is $u^{(d)} + nu^{(z)}$, and the exponent on x is $t^{(d)} + nt^{(z)}$. Since $dz^n \in D_4'$ for all n, then for each n, we must have

$$u^{(d)} + nu^{(z)} \leq \left(t^{(d)} + nt^{(z)}\right) \max\{a | a \in K^{(d)} \cup K^{(z)}\}.$$

Thus, we note that $t^{(z)} \neq 0$ if $u^{(z)} \neq 0$. That is, if $z \in D_4''$ has y has a factor, then x must also be a factor of z.

Now since $dz^n \in D_4'$ for all n, there is a collection of anchors $d_n \in D_4$ such that for each n, we have $d_n(dz^n)^m \in D_4$ for all m. But now we have a stricter form for d_n, since d_n must be a product of elements that are in D_4. In particular, we can write

$$d_n = \prod_{(a,b,c) \in \Gamma^{(d_n)}} x_{a,b,c}^{\alpha(n)_{a,b,c}} \left(x_{a,b}\left(x_a\left(xy^a\right)^b\right)^c\right)^{\beta(n)_{a,b,c}}$$

where for each $(a,b) \in \Lambda^{(d)}$, we have $(a,b) \in \Gamma^{(d_n)}[\boldsymbol{a}, \boldsymbol{b}]$. Also note that the exponents $\alpha(n)_{a,b,c}, \beta(n)_{a,b,c}$ may vary for each d_n, and thus are written as functions of n.

Now consider that from $d_n(dz^n)^{m-1}$ to $d_n(dz^n)^m$, the change in the exponent of x is given by

$$t^{(d)} + nt^{(z)} = \sum_{(a,b,c) \in \Gamma^{(d_n)}} bc\beta(n)_{a,b,c}^{(m)}$$

and the change in the exponent of y is

$$u^{(d)} + nu^{(z)} = \sum_{(a,b,c) \in \Gamma^{(d_n)}} abc\beta(n)_{a,b,c}^{(m)}.$$

Moreover, from the observation made earlier, for each n, we know that the change in the exponent of x (y) can be determined as the sum of the average change in the exponent of x (y) as paired with each x_a that is a factor dz^n. Moreover, the average change of x (y) as paired with x_a can be determined by only considering the change in the exponent of x (y) as paired with each $x_{a,b}$ that is a factor of dz^n as well.

Recall that for each $a \in K^{(d)} \cup K^{(z)}$ and each n, we have

$$\overline{\Delta B_a} \leq (s_a^{(d)} + n s_a^{(z)}) \max\{b | b \in \Lambda_a^{(d)}[\boldsymbol{b}] \cup \Lambda_a^{(z)}[\boldsymbol{b}]\}$$

Moreover, if $a \in K^{(d)} \setminus K^{(z)}$, (i.e., $s_a^{(z)} = 0$), then $\overline{\Delta B_a}$ is bounded above by $(s_a^{(d)}) \max\{b | b \in \Lambda_a^{(d)}[\boldsymbol{b}] \cup \Lambda_a^{(z)}[\boldsymbol{b}]\}$, which is fixed independent of n.

Note then that we have

$$\frac{u^{(d)} + nu^{(z)}}{t^{(d)} + nt^{(z)}} = \frac{u^{(d)}/n + u^{(z)}}{t^{(d)}/n + t^{(z)}} = \frac{\frac{1}{n}\left(\sum_{a \in K^{(z)}} a\overline{\Delta B_a} + \sum_{a \in K^{(d)} \setminus K^{(z)}} a\overline{\Delta B_a}\right)}{\frac{1}{n}\left(\sum_{a \in K^{(z)}} \overline{\Delta B_a} + \sum_{a \in K^{(d)} \setminus K^{(z)}} \overline{\Delta B_a}\right)}$$

Then as n increases without bound, we have

$$\begin{aligned}
\frac{u^{(z)}}{t^{(z)}} &= \frac{\frac{1}{n}\left(\sum_{a \in K^{(z)}} a\overline{\Delta B_a}\right)}{\frac{1}{n}\left(\sum_{a \in K^{(z)}} \overline{\Delta B_a}\right)} \\
&\leq \frac{\frac{1}{n}\left(\max\{a | a \in K\}\right)\left(\sum_{a \in K^{(z)}} \overline{\Delta B_a}\right)}{\frac{1}{n}\left(\sum_{a \in K^{(z)}} \overline{\Delta B_a}\right)} \\
&= \max\{a | a \in K\}
\end{aligned}$$

Thus, if $z \in D_4''$, then we must have $u^{(z)} \leq t^{(z)} \max\{a | a \in K^{(z)}\}$. Therefore, $y \notin D_4'''$, and we can conclude that

$$D_4 \subset D_4' \subset D_4'' \subset D_4''' \subset D_4''''.$$

4.2.4 Proof of General Case

We prove the general case of our example with induction, using the proofs that D_2, D_3, and D_4 behave as desired. Suppose then that D_n behaves as desired for some $n \leq 4$. We will argue that D_{n+1} requires $n+1$ iterations of complete integral closure before reaching a completely integrally closed domain.

Given

$$D_{n+1} = \mathbb{F}\left[x_\lambda \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1}\right)^{\lambda_2}\right)^{\lambda_3}\right)^{\lambda_4} \cdots\right)^{\lambda_n}\right)^m | \lambda \in \mathbb{N}_0^n, m \in \mathbb{N}_0\right]$$

we claim that

$$D''_{n+1} = D'_n[x_\lambda | \lambda \in \mathbb{N}_0^n].$$

Hence, since $D'_n[x_\lambda | \lambda \in \mathbb{N}_0^n]$ requires $n-1$ iterations of complete integral to reach a completely integrally closed domain (by induction assumption), we conclude that D_{n+1} requires $2+(n-1) = n+1$ such iterations.

The first observation is that m varies independently of λ, thus we have

$$x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \in D'_{n+1}$$

for each $\lambda \in \mathbb{N}_0^n$, with anchor x_λ.

In particular, for $\gamma \in \mathbb{N}_0^{n-1}$, we note that $x_{(\gamma,0)}$ is anchor to x_γ. Thus, $x_\gamma \in D'_{n+1}$ for each $\gamma \in \mathbb{N}_0^{n-1}$.

Also, for $\gamma \in \mathbb{N}_0^{n-2}$, we note that $x_{(\gamma,0,m)}$ is anchor to $x_{(\gamma,0)}(x_\gamma)^m$ for each m. Thus, we see that $x_\gamma \in D''_{n+1}$ for each $\gamma \in \mathbb{N}_0^{n-2}$ with anchor $x_{(\gamma,0)}$ in D'_{n+1}.

We first consider an arbitrary monomial in D_{n+1}, say

$$d = \prod_{\lambda \in \Lambda} x_\lambda^{\alpha_\lambda} \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \right)^{\beta_\lambda}$$

where Λ is a finite subset of $\mathbb{N}_0^n$.

Consider in this monomial that the exponent on y is equal to $\sum_{\lambda \in \Lambda} \lambda_1 \lambda_2 \cdots \lambda_n \beta_\lambda$. Consider that the exponent of x is equal to $\sum_{\lambda \in \Lambda} \lambda_2 \lambda_3 \cdots \lambda_n \beta_\lambda$. The exponent on a given $x_{\lambda^{(1)}}$ is equal to $\sum_{\lambda \in \Lambda_{\lambda^{(1)}}} \lambda_3 \lambda_4 \cdots \lambda_n \beta_\lambda$. And in general, the exponent on a given $x_{\lambda^{(j)}}$ for $j \le n-1$ is given by $\sum_{\lambda \in \Lambda_{\lambda^{(j)}}} \lambda_{j+2} \lambda_{j+3} \cdots \lambda_n \beta_\lambda$.

In particular, once the set Λ has been chosen, the only way to increase the exponent on each of the terms $y, x, x_{\lambda^{(j)}}$ is to increase some of the β_λ's.

Moreover, the ratio of the exponent on y to the exponent of x is at most $\max\{\lambda_1 | \lambda \in \Lambda\}$.

Thus, the exponent of y cannot increase without bound on an element in D_{n+1} unless the exponent on x is also allowed to increase. Thus, if y is a factor of an element in $z \in D'_{n+1}$, then x must also be a factor of z as well.

Also note that the ratio of the exponent on x to the exponent of a given $x_{\lambda^{(1)}}$ is at most $\max\{\lambda_2 | \lambda \in \Lambda\}$. Thus, the exponent of y cannot increase without bound on an element of D_{n+1} unless the the exonent on at least one $x_{\lambda^{(1)}}$ is allowed to increase. Thus, if x is a factor of an element in $z \in D'_{n+1}$, then there exists some $\lambda \in \mathbb{N}_0^n$ such that λ_1 is a factor of z as well (or more simply, there exists some $a \in \mathbb{N}_0$ such that x_a is a factor of z).

In general, for a given $x_{\lambda^{(j)}}$ with $j \leq n-2$, the ratio of the exponent on $x_{\lambda^{(j)}}$ to the exponent on a given $x_{\gamma^{(j+1)}}$ where $\lambda^{(j)} = \gamma^{(j)}$ is at most $\max\{\lambda_{j+2} | \lambda \in \Lambda_{\lambda^{(j)}}\}$. Therefore, if $x_{\lambda^{(j)}}$ is a factor of some element $z \in D'_{n+1}$, then there exists some $\gamma \in \mathbb{N}_0^n$ such that $\gamma^{(j)} = \lambda^{(j)}$ and $x_{\gamma^{(j+1)}}$ is a factor of z as well.

Thus we have the following observations about D'_{n+1}.

1. $x_\lambda \in D'_{n+1}$ for each $\lambda \in \mathbb{N}_0^n$.
2. $x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \in D'_{n+1}$ for each $\lambda \in \mathbb{N}_0^n$.
   (a) Special Case: $x_\gamma \in D'_{n+1}$ for each $\gamma \in \mathbb{N}_0^{n-1}$ with anchor $x_{(\gamma,0)}$.
   (b) Special Case: $x_{(\gamma,0)}(x_\gamma)^m \in D'_{n+1}$ for each $\gamma \in \mathbb{N}_0^{n-2}$ with anchor $x_{(\gamma,0,m)}$. Hence $x_\gamma \in D''_{n+1}$ with anchor $x_{(\gamma,0)} \in D'_{n+1}$.
3. If $z \in D'_{n+1}$, and y is a factor of z, then x is also a factor of z.
4. If $z \in D'_{n+1}$, and x is a factor of z, then x_a is also a factor of z for some $a \in \mathbb{N}_0$.
5. If $z \in D'_{n+1}$, and x_γ is a factor of z for some $\gamma \in \mathbb{N}_0^m$ with $m \leq n-2$, then there exists some $\lambda \in \mathbb{N}_0^{m+1}$ such that $\lambda^{(m)} = \gamma$ and x_λ is a factor of z as well.

We now set out to show that for a given element in D'_{n+1}, the exponents on y, x, and $x_{\lambda^{(m)}}$ are bounded for each $\lambda^{(m)} \in \mathbb{N}_0^m$ where $m \leq n-3$. To this end, consider an arbitrary monomial

$z \in D'_{n+1}$.

$$z = \left(\prod_{\gamma \in \Gamma_n} x_\gamma^{e_\gamma} \right) \left(\prod_{\gamma \in \Gamma_{n-1}} x_\gamma^{e_\gamma} \right) \cdots \left(\prod_{\gamma \in \Gamma_1} x_\gamma^{e_\gamma} \right) x^{e_x} y^{e_y}$$

where Γ_m is a finite subset of $\mathbb{N}_0^m$ and we assume each exponent is nonzero. Note from observations (3) above, we can specify that if $e_y \neq 0$ then $e_x \neq 0$. From observation (4), we know that if $e_x \neq 0$, then $\Gamma_1 \neq \emptyset$. Lastly, from observation (5), we know that if there is some $\gamma \in \Gamma_m$ (with $m \leq n-2$), then there exists some $\lambda \in \Gamma_{m+1}$, with $\lambda^{(m)} = \gamma$.

Let us pause here to introduce some notation. For a set $S \subseteq \mathbb{N}_0^n$, and a set $T \subseteq \mathbb{N}_0^m$ where $m < n$, and and element $\tau \in T$, we define:

$$S_\tau = \{\sigma \in S | \sigma^{(m)} = \tau\}, \qquad S_T = \{\sigma \in S | \sigma^{(m)} \in T\}$$

$$S^{(\boldsymbol{m})} = \{\sigma^{(m)} | \sigma \in S\}$$

We also use S' to denote the set complement of S.

In terms of this notation, we could state result from observation (5), that if there is some $\gamma \in \Gamma_m$ (with $m \leq n-2$), then $(\Gamma_{m+1})_\gamma \neq \emptyset$.

Now since $z \in D'_{n+1}$, there exists an anchor $d \in D_{n+1}$ such that $dz^m \in D_{n+1}$ for all m. Since $d \in D_{n+1}$, we can write

$$d = \prod_{\lambda \in \Lambda} x_\lambda^{\alpha_\lambda} \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \right)^{\beta_\lambda}$$

Here we assume that for each $j \in \{1, \ldots, n\}$, and each $\gamma \in \Gamma_m$, (i.e., each γ such that x_γ is a factor of z), there exists some $\lambda \in \Lambda$ such that $\lambda^{(j)} = \gamma$ (i.e., $\Lambda_\gamma \neq \emptyset$).

Thus, the only change in the factorizations of d and dz^m are the α_λ's and the β_λ's. So we write

$$dz^m = \prod_{\lambda\in\Lambda} x_\lambda^{\alpha_\lambda^{(m)}} \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \right)^{\beta_\lambda^{(m)}}$$

We define $\Delta\beta_\lambda^{(m)} := \beta_\lambda^{(m)} - \beta_\lambda^{(m-1)}$, where $\beta_\lambda^0 := \beta_\lambda$. Then we have the following equations which must be satisfied. The change in the exponent on y from dz^{m-1} to dz^m is

$$e_y = \sum_{\lambda\in\Lambda} \lambda_1\lambda_2\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

Likewise the change in the exponent on x from dz^{m-1} to dz^m is

$$e_x = \sum_{\lambda\in\Lambda} \lambda_2\lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

Now for $\gamma \in \Gamma_1$, the change on the exponent of x_γ is given by

$$e_\gamma = \sum_{\lambda\in\Lambda_\gamma} \lambda_3\lambda_4\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

and if $\gamma \in \Lambda^{(\mathbf{1})} \setminus \Gamma_1$, then

$$0 = \sum_{\lambda\in\Lambda_\gamma} \lambda_3\lambda_4\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

In general, for $\gamma \in \Gamma_k$, the change on the exponent of x_γ is given by

$$e_\gamma = \sum_{\lambda\in\Lambda_\gamma} \lambda_{k+2}\lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

and if $\gamma \in \Lambda^{(\boldsymbol{k})} \setminus \Gamma_k$, then

$$0 = \sum_{\lambda\in\Lambda_\gamma} \lambda_{k+2}\lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

Now for $\tau \in \mathbb{N}_0^k$, and $k \in \{1,\ldots,n\}$, we define

$$\Delta\boldsymbol{X}_\tau^{(m)} := \sum_{\lambda\in\Lambda_\tau} \lambda_2\lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)}$$

which is the change in the exponent of x from dz^{m-1} to dz^m as paired with x_τ. In particular, we can say

$$e_x = \sum_{a \in \Gamma_1} \Delta \boldsymbol{X}_a^{(m)} + \sum_{a \in \Lambda^{(1)} \setminus \Gamma_1} \Delta \boldsymbol{X}_a^{(m)}$$

Similarly, we have

$$e_y = \sum_{a \in \Gamma_1} a \Delta \boldsymbol{X}_a^{(m)} + \sum_{a \in \Lambda^{(1)} \setminus \Gamma_1} a \Delta \boldsymbol{X}_a^{(m)}$$

Now we also define

$$\overline{\Delta \boldsymbol{X}_\tau^{(m)}} := \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \Delta \boldsymbol{X}_\tau^{(m)}$$

For $a \in \Lambda^{(1)}$, consider that $\sum_{m=1}^{M} \Delta \boldsymbol{X}_a^{(m)}$ gives the total change in the exponent of x as paired with x_a in the factorization of d to dz^m. Note that this sum is bounded below by $-\sum_{\lambda \in \Lambda_a} \lambda_2 \lambda_3 \cdots \lambda_n \beta_\lambda$ (i.e., the negative of the exponent of x as paired with x_a in the factorization of d).

Thus, we can say that

$$\overline{\Delta \boldsymbol{X}_a} = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \Delta \boldsymbol{X}_a^{(m)} \geq 0$$

Moreover, if $a \in \Lambda^{(1)} \setminus \Gamma_1$, then the exponent on x_a is fixed in dz^m for each m. Thus, the exponent of x as paired with x_a is at most

$$\max\{\lambda_2 | \lambda \in \Lambda_a\} \sum_{\lambda \in \Lambda_a} \lambda_3 \lambda_4 \cdots \lambda_n \beta_\lambda$$

Hence we have $\displaystyle\sum_{m=1}^{M} \Delta \boldsymbol{X}_a^{(m)} \leq \max\{\lambda_2 | \lambda \in \Lambda_a\} \sum_{\lambda \in \Lambda_a} \lambda_3 \lambda_4 \cdots \lambda_n \beta_\lambda$, which implies

$$\overline{\Delta \boldsymbol{X}_a} = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \Delta \boldsymbol{X}_a^{(m)} = 0$$

Therefore, we can say that

$$
\begin{aligned}
e_y &= \sum_{a\in\Gamma_1} a\Delta \boldsymbol{X}_a^{(m)} + \sum_{a\in\Lambda^{(1)}\setminus\Gamma_1} a\Delta \boldsymbol{X}_a^{(m)} \\
&= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{a\in\Gamma_1} a\Delta \boldsymbol{X}_a^{(m)} + \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{a\in\Lambda^{(1)}\setminus\Gamma_1} a\Delta \boldsymbol{X}_a^{(m)} \\
&= \sum_{a\in\Gamma_1} a \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\Delta \boldsymbol{X}_a^{(m)} + \sum_{a\in\Lambda^{(1)}\setminus\Gamma_1} a \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\Delta \boldsymbol{X}_a^{(m)} \\
&= \sum_{a\in\Gamma_1} a\overline{\Delta \boldsymbol{X}_a} + \sum_{a\in\Lambda^{(1)}\setminus\Gamma_1} a\overline{\Delta \boldsymbol{X}_a} \\
&= \sum_{a\in\Gamma_1} a\overline{\Delta \boldsymbol{X}_a} \\
&\le (\max\{a | a\in\Gamma_1\})\sum_{a\in\Gamma_1}\overline{\Delta \boldsymbol{X}_a} \\
&= e_x \max\{a | a\in\Gamma_1\}
\end{aligned}
$$

Thus, if $z \in D'_{n+1}$, then the exponent on y in z is at most the product of the exponent on x with the maximum a such that x_a is a factor of z. Therefore, we cannot have $y \in D''_{n+1}$.

With a similar argument in mind for $\gamma \in \mathbb{N}_0^{(j)}$ with $j \in \{1, 2, \ldots, n-2\}$ and $\tau \in \mathbb{N}_0^k$ with $k \in \{j+1, j+2, \ldots, n\}$, we define

$$
\Delta(\boldsymbol{X}_{\boldsymbol{\gamma}})_\tau^{(m)} := \sum_{\lambda\in\Lambda_\tau} \lambda_{j+2}\lambda_{j+3}\cdots\lambda_n \Delta\beta_\lambda^{(m)}
$$

which is the change in the exponent of x_γ from dz^{m-1} to dz^m as paired with x_τ. Note: we only consider this sum when $\tau^{(j)} = \gamma$.

In particular, fix $j \in \{1, 2, \ldots, n-2\}$ and $\gamma \in \Gamma_j$, then we can write

$$
e_\gamma = \sum_{\lambda\in(\Gamma_{j+1})_\gamma} \Delta(\boldsymbol{X}_{\boldsymbol{\gamma}})_\lambda^{(m)} + \sum_{\lambda\in\Lambda_\gamma^{(j+1)}\setminus(\Gamma_{j+1})_\gamma} \Delta(\boldsymbol{X}_{\boldsymbol{\gamma}})_\lambda^{(m)}
$$

Now we also define

$$
\overline{\Delta(\boldsymbol{X}_{\boldsymbol{\gamma}})_\tau} := \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\Delta(\boldsymbol{X}_{\boldsymbol{\gamma}})_\tau^{(m)}
$$

Now for $\tau \in \Lambda_\gamma^{\boldsymbol{(j+1)}}$, consider that $\sum_{m=1}^{M} \Delta(\boldsymbol{X_\gamma})_\tau^{(m)}$ gives the total change in the exponent of x_γ as paired with x_τ in the factorization of d to dz^m. Note that this sum is bounded below by $-\sum_{\lambda \in \Lambda_\tau} \lambda_{j+2}\lambda_{j+3} \cdots \lambda_n \beta_\lambda$ (i.e., the negative of the exponent on x as paired with x_τ in d).

Thus, we can say that

$$\overline{\Delta(\boldsymbol{X_\gamma})_\tau} = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \Delta(\boldsymbol{X_\gamma})_\tau^{(m)} \geq 0$$

Moreover, if $\tau \in \Lambda_\gamma^{\boldsymbol{(j+1)}} \setminus (\Gamma_{j+1})_\gamma$, the the exponent on x_τ is fixed in dz^m for each m. Thus, the exponent on x_γ as paired with x_τ is at most

$$\max\{\lambda_{j+2} | \lambda \in \Lambda_\tau\} \sum_{\lambda \in \Lambda_\tau} \lambda_{j+3}\lambda_{j+4} \cdots \lambda_n \beta_\lambda.$$

Thus we can concluse that

$$\overline{\Delta(\boldsymbol{X_\gamma})_\tau} = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \Delta(\boldsymbol{X_\gamma})_\tau^{(m)} = 0$$

Therefore we can say

$$
\begin{aligned}
e_x &= \sum_{\lambda\in\Lambda} \lambda_2\lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{(a,b)\in\Gamma_2}\sum_{\lambda\in\Lambda_{a,b}} b\lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \sum_{(a,b)\in\Lambda^{(2)}\setminus\Gamma_2}\sum_{\lambda\in\Lambda_{a,b}} b\lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{(a,b)\in\Gamma_2} b\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \sum_{(a,b)\in\Lambda^{(2)}\setminus\Gamma_2} b\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{(a,b)\in\Gamma_2} b\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{(a,b)\in\Lambda^{(2)}\setminus\Gamma_2} b\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{(a,b)\in\Gamma_2} b\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \sum_{(a,b)\in\Lambda^{(2)}\setminus\Gamma_2} b\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sum_{\lambda\in\Lambda_{a,b}} \lambda_3\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{(a,b)\in\Gamma_2} b\overline{\Delta(\boldsymbol{X_a})_{(a,b)}} + \sum_{(a,b)\in\Lambda^{(2)}\setminus\Gamma_2} b\overline{\Delta(\boldsymbol{X_a})_{(a,b)}} \\
&= \sum_{(a,b)\in\Gamma_2} b\overline{\Delta(\boldsymbol{X_a})_{(a,b)}} \\
&\le (\max\{b|(a,b)\in\Gamma_2\})\sum_{(a,b)\in\Gamma_2}\overline{\Delta(\boldsymbol{X_a})_{(a,b)}} \\
&= (\max\{b|(a,b)\in\Gamma_2\})\left(\sum_{a\in\Gamma_1}\sum_{(a,b)\in(\Gamma_2)_a}\overline{\Delta(\boldsymbol{X_a})_{(a,b)}} + \sum_{a\in\Gamma_2^{(1)}\setminus\Gamma_1}\sum_{(a,b)\in(\Gamma_2)_a}\overline{\Delta(\boldsymbol{X_a})_{(a,b)}}\right) \\
&= (\max\{b|(a,b)\in\Gamma_2\})\left(\sum_{a\in\Gamma_1}\sum_{(a,b)\in(\Gamma_2)_a}\overline{\Delta(\boldsymbol{X_a})_{(a,b)}}\right) \\
&= (\max\{b|(a,b)\in\Gamma_2\})\left(\sum_{a\in\Gamma_1} e_a\right)
\end{aligned}
$$

Thus, we have that if $z \in D'_{n+1}$, then the exponent on x is at most the product of the maximum b such that $(a,b)\in\Gamma_2$ with the sum of the exponents on x_a for each $a\in\Gamma_1$.

And in general, for a given $\gamma\in\Gamma_k$ with $k\in\{1,2,\ldots,n-3\}$, we have

$$
\begin{aligned}
e_\gamma &= \sum_{\lambda\in\Lambda_\gamma} \lambda_{k+2}\lambda_{k+3}\cdots\lambda_n \Delta\beta_\lambda^{(m)} \\
&= \sum_{\tau\in(\Gamma_{k+2})_\gamma} \sum_{\lambda\in\Lambda_\tau} \tau_{k+2}\lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \sum_{\tau\in\Lambda_\gamma^{(\boldsymbol{k+2})}\setminus(\Gamma_{k+2})_\gamma} \sum_{\lambda\in\Lambda_\tau} \tau_{k+2}\lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{\tau\in(\Gamma_{k+2})_\gamma} \tau_{k+2} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} + \sum_{\tau\in\Lambda_\gamma^{(\boldsymbol{k+2})}\setminus(\Gamma_{k+2})_\gamma} \tau_{k+2} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} \sum_{\tau\in(\Gamma_{k+2})_\gamma} \tau_{k+2} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&\qquad + \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} \sum_{\tau\in\Lambda_\gamma^{(\boldsymbol{k+2})}\setminus(\Gamma_{k+2})_\gamma} \tau_{k+2} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{\tau\in(\Gamma_{k+2})_\gamma} \tau_{k+2} \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&\qquad + \sum_{\tau\in\Lambda_\gamma^{(\boldsymbol{k+2})}\setminus(\Gamma_{k+2})_\gamma} \tau_{k+2} \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} \sum_{\lambda\in\Lambda_\tau} \lambda_{k+3}\cdots\lambda_n\Delta\beta_\lambda^{(m)} \\
&= \sum_{\tau\in(\Gamma_{k+2})_\gamma} \tau_{k+2}\overline{\Delta(\boldsymbol{X}_{\boldsymbol{\tau}^{(\boldsymbol{k+1})}})_\tau} + \sum_{\tau\in\Lambda_\gamma^{(\boldsymbol{k+2})}\setminus(\Gamma_{k+2})_\gamma} \tau_{k+2}\overline{\Delta(\boldsymbol{X}_{\boldsymbol{\tau}^{(\boldsymbol{k+1})}})_\tau} \\
&= \sum_{\tau\in(\Gamma_{k+2})_\gamma} \tau_{k+2}\overline{\Delta(\boldsymbol{X}_{\boldsymbol{\tau}^{(\boldsymbol{k+1})}})_\tau} \\
&\le (\max\{\tau_{k+2} | \tau\in(\Gamma_{k+2})_\gamma\}) \sum_{\tau\in(\Gamma_{k+2})_\gamma} \overline{\Delta(\boldsymbol{X}_{\boldsymbol{\tau}^{(\boldsymbol{k+1})}})_\tau} \\
&= (\max\{\tau_{k+2} | \tau\in(\Gamma_{k+2})_\gamma\}) \left(\sum_{\lambda\in(\Gamma_{k+1})_\gamma} \sum_{\tau\in(\Gamma_{k+2})_\lambda} \overline{\Delta(\boldsymbol{X}_{\boldsymbol{\lambda}})_\tau} + \sum_{\lambda\in(\Gamma_{k+2})_\gamma^{(k+1)}\setminus\Gamma_{k+1}} \sum_{\tau\in(\Gamma_{k+2})_\lambda} \overline{\Delta(\boldsymbol{X}_{\boldsymbol{\lambda}})_\tau} \right) \\
&= (\max\{\tau_{k+2} | \tau\in(\Gamma_{k+2})_\gamma\}) \left(\sum_{\lambda\in(\Gamma_{k+1})_\gamma} \sum_{\tau\in(\Gamma_{k+2})_\lambda} \overline{\Delta(\boldsymbol{X}_{\boldsymbol{\lambda}})_\tau} \right) \\
&= (\max\{\tau_{k+2} | \tau\in(\Gamma_{k+2})_\gamma\}) \left(\sum_{\lambda\in(\Gamma_{k+1})_\gamma} e_\lambda \right)
\end{aligned}
$$

Thus, if $z \in D'_{n+1}$ and $\gamma \in \Gamma_k$ with $k \in \{1, 2, \ldots, n-3\}$, then the exponent on x_γ is at most the product of the maximum λ_{k+2} such that $\lambda \in \Gamma_{k+2}$ and $\lambda^{(k)} = \gamma$ with the sum of the exponents on each x_λ such that $\lambda \in \Gamma_{k+1}$ and $\lambda^{(k)} = \gamma$.

We next set out to show that similar bounds still hold in D''_{n+1} for e_y, e_x, and each e_γ where $\gamma \in \mathbb{N}_0^k$ for $k \in \{1, 2, \ldots, n-4\}$. That is, as we jump from D'_{n+1} to D''_{n+1}, the bounds on exponents of terms still remain, except for e_γ where $\gamma \in \mathbb{N}_0^{n-3}$.

To this end, consider an arbitrary element $z \in D''_{n+1}$. We may factor z as

$$z = \left(\prod_{\kappa \in K_n} x_\kappa^{e_\kappa^{(z)}} \right) \left(\prod_{\kappa \in K_{n-1}} x_\kappa^{e_\kappa^{(z)}} \right) \cdots \left(\prod_{\kappa \in K_1} x_\kappa^{e_\kappa^{(z)}} \right) x^{e_x^{(z)}} y^{e_y^{(z)}}$$

where each K_j is a finite subset of $\mathbb{N}_0^j$.

Since $z \in D''_{n+1}$, there exists an anchor $d \in D_{n+1}$ such that $dz^m \in D'_{n+1}$ for all m. We can factor d as

$$d = \left(\prod_{\gamma \in \Gamma_n} x_\gamma^{e_\gamma^{(d)}} \right) \left(\prod_{\gamma \in \Gamma_{n-1}} x_\gamma^{e_\gamma^{(d)}} \right) \cdots \left(\prod_{\gamma \in \Gamma_1} x_\gamma^{e_\gamma^{(d)}} \right) x^{e_x^{(d)}} y^{e_y^{(d)}}.$$

Now since $d \in D'_{n+1}$, we know that if $e_y^{(d)} \neq 0$ then $e_x^{(d)} \neq 0$. We also know that if $e_x^{(d)} \neq 0$, then $\Gamma_1 \neq \emptyset$. Moreover we know that if there is some $\gamma \in \Gamma_j$ (with $j \leq n-2$), then there exists some $\lambda \in \Gamma_{j+1}$, with $\lambda^{(j)} = \gamma$.

Moreover, since $dz^m \in D'_{n+1}$ for all m, and we know that the exponents on the terms of elements in D'_{n+1} are bounded, we can make the following claims:

1. If y is a factor of z, then x is also a factor of z.

2. If x is a factor of z, then x_a is also a factor of z for some $a \in \mathbb{N}_0$.

3. If x_κ is a factor of z for some $\kappa \in \mathbb{N}_0^j$ with $j \leq n-3$, then there exists some $\gamma \in \mathbb{N}_0^{j+1}$ with $\gamma^{(j)} = \kappa$ such that x_γ is a factor of z as well.

Now since $dz^m \in D'_{n+1}$ for all m, there exists a collection of anchors $d_m \in D_{n+1}$ such that for each m, we have $d_m(dz^m)^k \in D_{n+1}$ for all k. Since $d_m \in D_{n+1}$, we can factor d_m as

$$d_m = \prod_{\lambda \in \Lambda(m)} x_\lambda^{\alpha(m)_\lambda} \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \right)^{\beta(m)_\lambda}$$

We can make the assumption that $K_j \cup \Gamma_j \subseteq \Lambda(d_m)^j$ for each j and m. Thus, the only difference in the factorization from $d_m(dz^m)^{k-1}$ to $d_m(dz^m)^k$ are the $\alpha(m)_\lambda$'s and the $\beta(m)_\lambda$'s.

So we factor $d_m(dz^m)^k$ as

$$d_m(dz^m)^k = \prod_{\lambda \in \Lambda(m)} x_\lambda^{\alpha(m)_\lambda^{(k)}} \left(x_{\lambda^{(n-1)}} \left(\cdots \left(x_{\lambda^{(2)}} \left(x_{\lambda^{(1)}} \left(xy^{\lambda_1} \right)^{\lambda_2} \right)^{\lambda_3} \right)^{\lambda_4} \cdots \right)^{\lambda_n} \right)^{\beta(m)_\lambda^{(k)}}$$

Set $\Delta\beta(m)_\lambda^{(k)} = \beta(m)_\lambda^{(k)} - \beta(m)_\lambda^{(k-1)}$ where $\beta(m)_\lambda^{(0)} = \beta(m)_\lambda$.

Then we have the following equations which must be satisfied.

The change in the exponent on y from $d_m(dz^m)^{k-1}$ to $d_m(dz^m)^k$ is

$$e_y^{(d)} + me_y^{(z)} = \sum_{\lambda \in \Lambda(m)} \lambda_1 \lambda_2 \cdots \lambda_n \Delta\beta(m)_\lambda^{(k)}$$

Likewise the change in the exponent on x from $d_m(dz^m)^{k-1}$ to $d_m(dz^m)^k$ is

$$e_x^{(d)} + me_x^{(d)} = \sum_{\lambda \in \Lambda(m)} \lambda_2 \lambda_3 \cdots \lambda_n \Delta\beta(m)_\lambda^{(k)}$$

Also for $\tau \in K_j \setminus \Gamma_j$, the change on the exponent of x_τ is given by

$$me_\tau^{(z)} = \sum_{\lambda \in \Lambda(m)_\tau} \lambda_{j+2} \lambda_{j+3} \cdots \lambda_n \Delta\beta_\lambda^{(m)}$$

and if $\tau \in \Gamma_j \setminus K_j$, the change on the exponent of x_τ is given by

$$e_\tau^{(d)} = \sum_{\lambda \in \Lambda(m)_\tau} \lambda_{j+2} \lambda_{j+3} \cdots \lambda_n \Delta\beta_\lambda^{(m)}$$

and if $\tau \in K_j \cap \Gamma_j$, the change on the exponent of x_τ is given by

$$e_\tau^{(d)} + me_\tau^{(z)} = \sum_{\lambda \in \Lambda(m)_\tau} \lambda_{j+2}\lambda_{j+3}\cdots\lambda_n \Delta\beta_\lambda^{(m)}$$

and if $\tau \in \Lambda(m)^{(\boldsymbol{j})} \setminus (K_j \cup \Gamma_j)$, then

$$0 = \sum_{\lambda \in \Lambda(m)_\gamma} \lambda_{j+2}\lambda_{j+3}\cdots\lambda_n \Delta\beta_\lambda^{(m)}$$

Now consider that the change in the exponent of x from $d_m(dz^m)^{k-1}$ to $d_m(dz^m)^k$ can be determined as the sum of the average change in the exponent of x as paired with each x_a that is a factor of dz^m. Moreover, we can determine this average change for each x_a by considering the average change in x as paired with each $x_{a,b}$ such that $x_{a,b}$ is a factor of dz^m.

Similarly, for $\tau \in K_j \cup \Gamma_j$, the change in the exponent of x_τ from $d_m(dz^m)^{k-1}$ to $d_m(dz^m)^k$ can be determined as the sum of the average change in the exponent of x_τ as paired with each x_λ that is a factor of dz^m where $\lambda \in (K_{j+1} \cup \Gamma_{j+1})_\tau$. Moreover, we can determine this average change for each x_λ by considering the average change in x_τ as paired with each x_γ such that $\gamma \in (K_{j+2} \cup \Gamma_{j+2})_\lambda$.

Now recall for each $a \in K_1 \cup \Gamma_1$ and each m, we have

$$\overline{\Delta \boldsymbol{X}_a} \leq (e_a^{(d)} + me_a^{(z)}) \max\{b | (a,b) \in (K_2 \cup (\Gamma_2)_a\}$$

And in general, for each $\tau \in K_j \cup \Gamma_j$ and $\lambda \in (K_{j+1} \cup \Gamma_{j+1})_\tau$ and each m, we have

$$\overline{\Delta(\boldsymbol{X_\tau})_\lambda} \leq (e_\lambda^{(d)} + me_\lambda^{(z)}) \max\{\gamma_{j+2} | \gamma \in (K_{j+2} \cup \Gamma_{j+2})_\lambda\}$$

Moreover, if $a \in \Gamma_1 \setminus K_1$, then we have $\overline{\Delta \boldsymbol{X}_a} \leq (e_a^{(d)}) \max\{b | (a,b) \in (K_2 \cup (\Gamma_2)_a\}$ which is fixed independent of m. And in general, for each $\lambda \in (\Gamma_{j+1} \setminus K_{j+1})_\tau$ we have

$$\overline{\Delta(\boldsymbol{X_\tau})_\lambda} \leq (e_\lambda^{(d)}) \max\{\gamma_{j+2} | \gamma \in (K_{j+2} \cup \Gamma_{j+2})_\lambda\}$$

which is fixed independent of m.

Note then that we have

$$\frac{e_y^{(d)} + me_y^{(z)}}{e_x^{(d)} + me_x^{(z)}} = \frac{e_y^{(d)}/m + e_y^{(z)}}{e_x^{(d)}/m + e_x^{(z)}} = \frac{\frac{1}{m}\left(\sum_{a\in K_1} a\overline{\Delta \boldsymbol{X}_a} + \sum_{a\in \Gamma_1\setminus K_1} a\overline{\Delta \boldsymbol{X}_a}\right)}{\frac{1}{m}\left(\sum_{a\in K_1} \overline{\Delta \boldsymbol{X}_a} + \sum_{a\in \Gamma_1\setminus K_1} \overline{\Delta \boldsymbol{X}_a}\right)}$$

Then as m increases without bound, we have

$$\begin{aligned}\frac{e_y^{(z)}}{e_x^{(z)}} &= \frac{\frac{1}{m}\sum_{a\in K_1} a\overline{\Delta \boldsymbol{X}_a}}{\frac{1}{m}\sum_{a\in K_1} \overline{\Delta \boldsymbol{X}_a}} \\ &\le \frac{\frac{1}{m}(\max\{a|a\in K_1\})\sum_{a\in K_1} \overline{\Delta \boldsymbol{X}_a}}{\frac{1}{m}\sum_{a\in K_1} \overline{\Delta \boldsymbol{X}_a}} \\ &= \max\{a|a\in K_1\}\end{aligned}$$

Thus, if $z \in D''_{n+1}$, then we must have $e_y^{(z)} \le (\max\{a|a\in K_1\})e_x^{(z)}$.

Similarly, consider

$$\frac{e_x^{(d)} + me_x^{(z)}}{\sum_{a\in K_1}(e_a^{(d)} + me_a^{(z)})} = \frac{e_x^{(d)}/m + e_x^{(z)}}{\sum_{a\in K_1}(e_a^{(d)}/m + e_a^{(z)})}$$

$$= \frac{\frac{1}{m}\left(\sum_{a\in K_1} \overline{\Delta \boldsymbol{X}_a} + \sum_{a\in \Gamma_1\setminus K_1} \overline{\Delta \boldsymbol{X}_a}\right)}{\frac{1}{m}\left(\sum_{a\in K_1}\left(\sum_{\tau\in (K_2)_a} \overline{\Delta(\boldsymbol{X_a})}_\tau + \sum_{\tau\in (\Gamma_2\setminus K_2)_a} \overline{\Delta(\boldsymbol{X_a})_\tau}\right)\right)}$$

$$= \frac{\frac{1}{m}\left(\sum_{a\in K_1}\left(\sum_{\tau\in (K_2\cup \Gamma_2)_a} \tau_2\overline{\Delta(\boldsymbol{X_a})_\tau}\right) + \sum_{a\in \Gamma_1\setminus K_1} \overline{\Delta \boldsymbol{X}_a}\right)}{\frac{1}{m}\left(\sum_{a\in K_1}\left(\sum_{\tau\in (K_2)_a} \overline{\Delta(\boldsymbol{X_a})}_\tau + \sum_{\tau\in (\Gamma_2\setminus K_2)_a} \overline{\Delta(\boldsymbol{X_a})_\tau}\right)\right)}$$

$$= \frac{\frac{1}{m}\left(\sum_{a\in K_1}\left(\sum_{\tau\in (K_2)_a} \tau_2\overline{\Delta(\boldsymbol{X_a})_\tau} + \sum_{\tau\in (\Gamma_2\setminus K_2)_a} \tau_2\overline{\Delta(\boldsymbol{X_a})_\tau}\right) + \sum_{a\in \Gamma_1\setminus K_1} \overline{\Delta \boldsymbol{X}_a}\right)}{\frac{1}{m}\left(\sum_{a\in K_1}\left(\sum_{\tau\in (K_2)_a} \overline{\Delta(\boldsymbol{X_a})}_\tau + \sum_{\tau\in (\Gamma_2\setminus K_2)_a} \overline{\Delta(\boldsymbol{X_a})_\tau}\right)\right)}$$

Thus, as m increases without bound, we have

$$\frac{e_x^{(z)}}{\sum_{a\in K_1} e_a^{(z)}} = \frac{\frac{1}{m}\sum_{a\in K_1}\left(\sum_{\tau\in(K_2)_a}\tau_2\overline{\Delta(\boldsymbol{X_a})_\tau}\right)}{\frac{1}{m}\sum_{a\in K_1}\sum_{\tau\in(K_2)_a}\overline{\Delta(\boldsymbol{X_a})}_\tau}$$

$$\leq \frac{\frac{1}{m}\left(\max\{\tau_2|\tau\in(K_2)_{K_1}\}\right)\sum_{a\in K_1}\sum_{\tau\in(K_2)_a}\overline{\Delta(\boldsymbol{X_a})_\tau}}{\frac{1}{m}\sum_{a\in K_1}\sum_{\tau\in(K_2)_a}\overline{\Delta(\boldsymbol{X_a})}_\tau}$$

$$= \max\{\tau_2|\tau\in(K_2)_{K_1}\}$$

Thus, if $z\in D'_{n+1}$, then we must have $e_x^{(z)} \leq \left(\max\{\tau_2|\tau\in(K_2)_{K_1}\}\right)\sum_{a\in K_1} e_a^{(z)}$.

Likewise, for a given $\gamma\in K_j\cup\Gamma_j$ consider that

$$\frac{e_\gamma^{(d)} + me_\gamma^{(z)}}{\sum_{\lambda\in(K_{j+1})_\gamma}(e_\lambda^{(d)} + me_\lambda^{(z)})} = \frac{e_\gamma^{(d)}/m + e_\gamma^{(z)}}{\sum_{\lambda\in(K_{j+1})_\gamma}(e_\lambda^{(d)}/m + e_\lambda^{(z)})}$$

$$= \frac{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\overline{\Delta(\boldsymbol{X_\gamma})}_\lambda + \sum_{\lambda\in(\Gamma_{j+1}\setminus K_{j+1})_\gamma}\overline{\Delta(\boldsymbol{X_\gamma})}_\lambda\right)}{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\left(\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau + \sum_{\tau\in(\Gamma_{j+2}\setminus K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau\right)\right)}$$

$$= \frac{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\left(\sum_{\tau\in(K_{j+2}\cup\Gamma_{j+2})_\lambda}\tau_{j+2}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau\right) + \sum_{\lambda\in(\Gamma_{j+1}\setminus K_{j+1})_\tau}\overline{\Delta(\boldsymbol{X_\gamma})}_\lambda\right)}{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\left(\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau + \sum_{\tau\in(\Gamma_{j+2}\setminus K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau\right)\right)}$$

$$= \frac{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\left(\sum_{\tau\in(K_{j+2})_\lambda}\tau_{j+2}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau + \sum_{\tau\in(\Gamma_{j+2}\setminus K_{j+2})_\lambda}\tau_{j+2}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau\right) + \sum_{\lambda\in(\Gamma_{j+1}\setminus K_{j+1})_\tau}\overline{\Delta(\boldsymbol{X_\gamma})}_\lambda\right)}{\frac{1}{m}\left(\sum_{\lambda\in(K_{j+1})_\gamma}\left(\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau + \sum_{\tau\in(\Gamma_{j+2}\setminus K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau\right)\right)}$$

Thus, as m increases without bound, we have

$$\frac{e_\gamma^{(z)}}{\sum_{\lambda\in(K_{j+1})_\gamma} e_\lambda^{(z)}} = \frac{\frac{1}{m}\sum_{\lambda\in(K_{j+1})_\gamma}\sum_{\tau\in(K_{j+2})_\lambda}\tau_{j+2}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau}{\frac{1}{m}\sum_{\lambda\in(K_{j+1})_\gamma}\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau}$$

$$\leq \frac{\frac{1}{m}\left(\max\{\tau_{j+2}|\tau\in(K_{j+2})_\gamma\}\right)\sum_{\lambda\in(K_{j+1})_\gamma}\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau}{\frac{1}{m}\sum_{\lambda\in(K_{j+1})_\gamma}\sum_{\tau\in(K_{j+2})_\lambda}\overline{\Delta(\boldsymbol{X_\lambda})}_\tau}$$

$$= \max\{\tau_{j+2}|\tau\in(K_{j+2})_\gamma\}$$

Thus, if $z \in D''_{n+1}$, and x_γ is a factor of z for a given $\gamma \in \mathbb{N}_0^j$, then we must have $e_\gamma^{(z)} \leq (\max\{\tau_{j+2} | \tau \in (K_{j+2})_\gamma\}) \sum_{\lambda \in (K_{j+1})_\gamma} e_\lambda^{(z)}$.

Therefore we can see that elements in D''_{n+1} have the same structure as elements in D'_n except that D''_{n+1} contains elements of the form x_λ for $\lambda \in \mathbb{N}_0^n$. Thus we can say

$$D''_{n+1} = D'_n[x_\lambda | \lambda \in \mathbb{N}_0^n]$$

Since D'_n requires $n-1$ interations of complete integral closure to reach a completely integrally closed domain, then D_{n+1} requires $(n-1)+2 = n+1$ iterations.

Therefore, by induction, for every $n \in \mathbb{N}$, our example D_n requires n iterations of complete integral closure before reaching a completely integrally closed domain.

Chapter 5

Other Variations for Further Study

5.1 Pseudo-integral Closure

The notion of pseudo-integral closure was at one point studied under the name of *regular integral closure* (see [5], Ch. 7, Sect. 1, Exercise 30). Kang then studied special cases of pseudo-integral closure in his Ph.D. thesis [24]. A more systematic study of pseudo-integral closure was done shortly after this by Anderson, Houston, and Zafrullah in 1991 (see [3]).

Recall from the comments before Proposition 65 that for a nonzero fractional ideal I of a domain D, we denote $(I^{-1})^{-1}$ by I_v and say that I is *divisorial* or a *v-ideal* if $I = I_v$. Following [3], we now define pseudo-integral elements:

Definition 82. An element $a \in K$ is pseudo-integral over D if $aI_v \subseteq I_v$ for some nonzero finitely generated ideal $I \subseteq D$.

Note that the definition could be equivalently stated with I^{-1} replacing I_v, for $I^{-1} = ((I^{-1})^{-1})^{-1} = (I_v)^{-1}$ implies

$$aI_v \subseteq I_v \iff a^{-1}(I_v)^{-1} \supseteq (I_v)^{-1} \iff a^{-1}I^{-1} \supseteq I^{-1} \iff I^{-1} \supseteq aI^{-1}.$$

The set of all elements which are pseudo-integral over D is a ring, which we denote by $\tilde{D}$ after [3]. We should note that $\tilde{D}$ is equal to the directed union of $\{(I_v : I_v) | I \text{ is a nonzero finitely generated ideal of } D\}$.

Now recall from Theorem 9, that an element $a \in K$ is integral over D if $aI \subseteq I$ for some nonzero finitely generated ideal $I \subseteq D$. Similarly from Theorem 34 an element $a \in K$ is almost integral over D if $aI \subseteq I$ for some ideal $I \subseteq D$.

Note then that integral implies pseudo-integral implies almost integral. For $aI \subseteq I \implies a^{-1}I^{-1} \supseteq I^{-1} \implies aI_v \subseteq I_v$. It follows that $D \subseteq \bar{D} \subseteq \tilde{D} \subseteq D'$. In particular, if D is Noetherian, than $\bar{D} = \tilde{D} = D'$.

It is also important to note that $\tilde{D}$ is integrally closed. We state this as a theorem and record a proof based on the one given in [3].

Theorem 83. *If T is a pseudo-integral overring of D (i.e., $T \subseteq \tilde{D}$) and $x \in K$ is integral over T, then x is pseudo-integral over D. In particular, $\tilde{D}$ is integrally closed.*

Proof. From the equation of integrality satisfied by x, there are elements $u_1, \dots, u_n \in T$ with x integral over $S = D[u_1, \dots, u_n]$. Since each u_i is pseudo-integral over D, there is a nonzero finitely generated ideal J_i of D such that $u_i(J_i)_v \subseteq (J_i)_v$. Let $J = \prod J_i$; then $u_i J_v \subseteq J_v$ for each i. Since $(J_v : J_v)$ is a ring, it follows that $S \subseteq (J_v : J_v)$. Now, since x is integral over S, there is a nonzero finitely generated ideal $I = Sz_1 + \dots + Sz_m$ of S with $xI \subseteq I$. Let $A = Jz_1 + \dots + Jz_m$. Then A is a finitely generated ideal of D, and we shall complete the proof by showing that $x \in (A_v : A_v)$. First note that $JSz_i \subseteq J_v z_i$ for each i. Thus $JI \subseteq J_v z_1 + \dots + J_v z_m \subseteq (Jz_1 + \dots + Jz_q)_v = A_v$. Since $xz_i \in I$ for each i, we therefore have $xA = x(Jz_1 + \dots + Jz_m) \subseteq JI \subseteq A_v$. It follows that $x \in (A_v : A_v)$, as desired. □

We also include an example when the pseudo-integral closure is not pseudo-integrally closed. This example given in [3] for this purpose is actually the example used by Gilmer and Heinzer [17] to show that the complete integral closure is not completely integrally closed.

Example 84. Set $R = F[\{x^{2n+1}y^{n(2n+1)}\}_{n=0}^{\infty}]$, and set $T = F[\{xy^n\}_{n=0}^{\infty}]$. Then

1. $\bar{R} = \tilde{R} = R' = T$
2. $\tilde{T} = T' = F[x, y]$
3. y is pseudo-integral over T but not over R.

The pseudo-integral closure and complete integral closure of a domain D do share the property of being t-linked over D. Before showing this, let us define what we mean by t-linked.

Definition 85. Let $D \subseteq R$ be an extension of integral domains. Then R is said to be *t-linked over D* if, for each finitely generated ideal I of D, $I^{-1} = D$ implies $(IR)^{-1} = R$.

We then can verify our claim above, which is originally from [19].

Theorem 86. *The complete integral closure and pseudo-integral closure of a domain D are t-linked over D.*

To prove this theorem, we rely on a second result given in [19]. Note that a *t-ideal* is a fractional ideal such that $I = I_t := \bigcup\{J_v | (0) \neq J \subseteq I \text{ is a finitely generated ieal}\}$.

Proposition 87. *Let $(I_\lambda)_{\lambda\in\Lambda}$ be a family of fractional t-ideals of D satisfying*

1. *$D \subseteq I_\lambda$ for all $\lambda \in \Lambda$*

2. *For all $\mu, \nu \in \Lambda$, there exists some $\lambda \in \Lambda$ such that $I_\mu I_\nu \subseteq I_\lambda$*

Then $I = \cup_{\lambda\in\Lambda} I_\lambda$ is t-linked over of D.

We now prove Theorem 86.

Proof. The complete integral closure and pseudo-integral closure of D are given by

$$D' = \cup_{I\in F(D)}(I_v : I_v) \quad \text{and} \quad \tilde{D} = \cup_{I\in f(D)}(I_v : I_v)$$

respectively, where $F(D)$ is the set of all fractional ideals of D and $f(D)$ is the set of all finitely generated fractional ideals of D (see [19], Corollary 2). Our desired conclusion then follows from Proposition 87 by setting D' and $\tilde{D}$, respectively, equal to I in Proposition 87. □

The interested reader can find more detailed study of pseudo-integrality in [41],[19].

5.2 w-integral Closure

The variation of integral closure discussed here was first introduced by Fanggui Wang in 2004 (see [45]). The variation was developed further by Chang and Zafrullah in 2006 [8]. Let I be a fractional ideal of D and define $I_w := \{x \in K |$ there is a finitely generated ideal A such that $A^{-1} = D$ and $xA \subseteq I\}$. If $I = I_w$ then I is called a w-ideal. Now following [8] and [45], we now can define the notion of w-integral elements and w-integral closure.

Definition 88. An element $u \in K$ is $w-$integral over D if $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of D. The w-integral closure is $D^w = \{x \in K | x \text{ is } w - \text{integral over } D\}$.

Note that a w-integral element is pseudo-integral and hence almost integral. Therefore, $\bar{D} \subseteq D^w \subseteq \tilde{D} \subseteq D'$. In particular, if D is Noetherian, then $\bar{D} = D^w = \tilde{D} = D'$.

We first note that similarly to complete integral closure and pseudo-integral closure, the w-integral closure of D is t-linked over D.

Theorem 89. *The w-integral closure D^w of D is t-linked over D.*

We record the proof given in [8], but first we need a couple definitions. If K is the quotient field of D, then we define the *content*, A_f, of a polynomial $f \in K[x]$ to be the fractional ideal of D generated by the coefficients of f. We let $N_v(D) = \{f \in D[x] | (A_f)_v = D\}$ where $(A_f)_v = (A_f^{-1})^{-1}$. Lastly, we let $* - Max(D)$ denote the set of $*$-ideals of D which are maximal among proper integral $*$-ideals of D. We can now read the proof.

Proof. For convenience, we let $N_v = N_v(D)$. Let R be an overring of D, and note that R is t-linked over D if and only if $R = R[x]_{N_v} \cap K$. Thus it suffices to show that $D^w = D^w[x]_{N_v} \cap K$. The containment $D^w \subseteq D^w[x]_{N_v} \cap K$ is clear enough. On the other hand, let $u = \frac{f}{g} \in D^w[x]_{N_v} \cap K$, where $g \in N_v$ and $f \in D^w[x]$. Let $f = \sum_{i=0}^{n} a_i x^i \in D^w[x]$. Since each $a_i \in D^w$, there is a nonzero finitely generated ideal J_i of D such that $a_i(J_i)_w \subseteq (J_i)_w$. Let $i = J_1 \cdots J_n$. Then I is a nonzero finitely generated ideal such that $a_i I_w = (a_i J_i J_1 \cdots J_{i-1} J_{i+1} \cdots J_n)_w \subseteq (J_i J_1 \cdots J_{i-1} J_{i+1} \cdots J_n)_w = I_w$. So $A_f I_w \subseteq I_w$, and hence $uI_w = u((A_g)_w I)_w = u(A_g I)_w = (uA_g I)_w = (A_f I)_w = (A_f I_w)_w \subseteq (I_w)_w = I_w$ (note that $(A_g)_w = D$ is t-Max(D) $= w$-Max(D)). Thus $u \in D^w$ and we have shown that $D^w = D^w[x]_{N_v} \cap K$. □

The following result is intriguing as it shows that w-integral closure and the integral closure of a domain are w-integrally closed. It also provides us with a way to represent the w-integral closure as an intersection of certain valuation overrings. For the third property in the following theorem, recall that a fractional ideal I is a w-ideal if $I_w = I$.

Theorem 90. *Let D be an integral domain, then we have the following properties of w-integrality:*

1. *$(D^w)^w = D^w$ and $(\bar{D})^w = \bar{D}$*

2. *If R is a t-linked overring of D, the $D^w \subseteq R^w$*

3. *The pair D, D^w satisfies INC, GU, and LO for prime w-ideals of D*

4. *D^w is the intersection of t-linked valuation overrings of D*

As a result of property 1 above, we have that D is integrally closed if and only if D is w-integrally closed. Many of the remaining results in [8] concern UMT-domains, and we leave it to the interested reader to view these results in the original paper.

5.3 Ω-almost Integral Closure

In 2010, Coykendall and Dutta introduced a new generalization of integrality which was coined Ω-almost integrality [11]. Their definition is given below.

Definition 91. Let R be an integral domain with quotient field K. An element $u \in K$ is said to be Ω-almost integral if for all nonzero $b \in R$ such that $bu \in R$, then there is a nonnegative integer m_b such that $b^{m_b}u^n \in R$ for all $n \geq 1$.

A refined definition is also given in the case that m_b is independent of the choice of b.

Definition 92. Let R be an integral domain with quotient field K and m a nonnegative integer. An element $u \in K$ is m-almost integral if for all nonzero $b \in R$ such that $bu \in R$, then $b^m u^n \in R$ for all $n \geq 1$.

The set of all elements of the quotient field K which are m-almost integral over D is called the m-almost integral closure of D.

From the definitions, it is clear that integrality implies Ω-almost integrality which implies almost integrality. A similar result holds for m-almost integrality.

One may ask the question whether m-integral elements exist for all m. Such is the case as shown in the following example given in [11].

Example 93. Let $\mathbb{F}$ be a field, and consider $R = \mathbb{F}[y, \frac{y}{x}, \frac{y^m}{x}, \frac{y^m}{x^2}, \frac{y^m}{x^3}, \ldots]$ for indeterminates x and y. Note that $1/x$ is an element of the quotient field of R which is m-almost integral. Note that if $r \in R$ is such that $r\frac{1}{x} \in R$, then $r \in (y, \frac{y}{x}, \frac{y^m}{x}, \frac{y^m}{x^2}, \ldots)$. Thus $r^m(1/x)^n \in R$ for all $n \geq 1$. In particular, we have $y^m(1/x)^n \in R$ for all $n \geq 1$, and no smaller power of y will suffice.

One downfall of Ω-almost integral elements is they do not form a ring. Indeed it may not even be an D-submodule of the complete integral closure of D. This is illustrated by the following example (see [11], Example 6.2). However, in order to fully appreciate the example, we first note (similarly to integral extensions) that if $R \subseteq T$ is an Ω-almost integral extension, then units in T are units in R (see [11], Proposition 2.5).

Example 94. Consider the domain $R = \mathbb{Z}[\pi] + x\mathbb{R}[x]$. Let $\alpha \in \mathbb{R}$ is transcendental over $\mathbb{Q}(\pi)$, and note that α is Ω almost integral over R. For if $\alpha p(\pi) \in \mathbb{Z}[x]$ is in $\mathbb{Z}[\pi]$ for some $p(x) \in \mathbb{Z}[x]$, then $\alpha \in \mathbb{Q}(\pi)$. But this yields a contradiction unless $p(x) = 0$. So $p(x) = 0$, and it follows that α is Ω-almost integral. Furthermore, $1/\pi - \alpha$ is also transcendental over $\mathbb{Q}(\pi)$ and hence Ω-almost integral over R. However, $\alpha + (1/\pi - \alpha) = 1/\pi$ cannot be Ω-almost integral over R as $\pi \in R$ is a nonunit.

Now of course it is not all bad news with Ω-almost integrality as this next proposition shows. Recall that valuation domains are of great interest with regard to integral and almost integral closure, so the following result is of some importance (see [11], Proposition 3.1).

Proposition 95. *Any valuation domain is Ω-almost integrally closed.*

Proof. Let V be a valuation domain with quotient field K, and suppose that $\alpha \in K$ is Ω-almost integral over V. If $\alpha \in V$, we are done. So assume that $\alpha \notin V$ and derive and contradiction. Since V is a valuation domain and $\alpha \notin V$, we must have $\alpha^{-1} \in V$. But note that $\alpha^{-1}\alpha \in V$ so there must be some m such that $(\alpha^{-1})^m \alpha^n \in V$ for all $n \geq 1$. Hence, when $n = m + 1$, we have $(\alpha^{-1})^m \alpha^{m+1} = \alpha \in V$, and we have reached our desired contradiction. □

It is shown in [11] that Ω-almost integrality does not behave well with respect to intermediate extensions. This unfortunate property is combated with a stronger version of Ω-almost integrality given below.

Definition 96. The extension $R \subseteq T$ is *strongly Ω-almost integral* (resp. *strongly m-almost integral*) if every element $t \in T$ is Ω-almost integral (resp. m-almost integral) over every intermediate extension A that is integrally closed in T.

We find that this stronger version of Ω-almost integrality behaves more like integrality. For example, recall from Theorem 25 that integral extensions satisfy going-up (GU) and incomparability (INC). It is shown in [11] that for if $R \subseteq T$ is a strongly Ω-almost integral extension, then $R \subseteq T$ satisfies GU. However, even strongly Ω-almost integral and strongly m-almost integral extensions do not necessarily satisfy INC. Examples are given in [11] which show this failure for both types of extensions. We record the example given which shows that strongly m-almost integral extensions need not satisfy INC.

Example 97. The ring extension $\mathbb{Z} + 2x\mathbb{Z}[x] \subseteq \mathbb{Z}[x]$ is strongly 1-almost integral, yet the ideals $2\mathbb{Z}[x]$ and $(2, x)\mathbb{Z}[x]$ are comparable primes as they both lie over the prime $2x\mathbb{Z}[x]$.

For a more complete study of Ω-integrality, including the results given here and many others, the interested reader should see [11].

Appendices

This appendix contains definitions and remarks that are intended to supplement and clarify some of the material in Chapters 3 and 4. Definitions here are presented in the order that they are needed in Chapters 3 and 4.

The following definition of a GCD-domain is given in relation to Theorem 18.

Definition 98. An integral domain D is a *GCD-domain* if any two elements in D have a greatest common divisor.

For given rings $R \subseteq T$, we often interested in the relations between the prime ideals of R and the prime ideals of T. There are four properties of particular interest. The two of particular interest for Theorem 25 are GU and INC.

Definition 99. Let $R \subseteq T$ be rings. We define the following:

Lying over (LO). For any prime $P \in R$ there exists a prime $Q \in T$ with $Q \cap R = P$.

Going up (GU). Given primes $P \subseteq P_0 \in R$ and $Q \in T$ with $Q \cap R = P$, there exists $Q_0 \in T$ such that $Q \subseteq Q_O$, $Q_0 \cap R = P_0$.

Going down (GD). The same as GU but with $\subseteq$ replaced by $\supseteq$.

Incomparable (INC). Two different primes in T with the same contraction in R cannot be comparable.

One can show that for any pair of rings, GU implies LO (see [25], Theorem 42). The next two theorems are Theorems 41 and 43 in [25].

Theorem 100. *The following two statements are equivalent for rings $R \subseteq T$:*

1. *GU holds.*

2. *If P is a prime ideal in R, S is the complement of P in R, and Q is a (prime) ideal in T maximal with respect to the exclusion of S, then $Q \cap R = P$.*

Theorem 101. *The following two statements are equivalent for rings $R \subseteq T$:*

1. *INC holds.*

2. *If P is a prime ideal in R, and Q is a prime ideal in T contracting to P in R, then Q is maximal with respect to the exclusion of S, the complement of P in R.*

Chapter 4 contains a brief section on Prüfer domains. The following definitions are given here to supplement that subsection.

Definition 102. An integral domain R is a *Prüfer domain* if for each maximal ideal M, R_M is a valuation domain. A *Dedekind domain* is Noetherian Prüfer domain.

The restriction to maximal ideals in the above definition, can be extended equivalently to prime ideals. It is important to note that every overring of a Prüfer domain is another Prüfer domain. Below we record the fact that Prüfer domains are integrally closed.

Proposition 103. *A Prüfer domain is integrally closed.*

Proof. Let D be a Prüfer domain. Consider that $D = \cap D_M$ where the intersection is taken over all maximal ideal of D. Since D is Prüfer , each D_M is a valuation domain and hence integrally closed. So D is an intersection of integrally closed domains, and thus is integrally closed. □

The following is an equivalent definition for Prüfer and Dedekind domains [25].

Lemma 104. *An integral domain R is a Prüfer domain if every nonzero finitely generated ideal of R is invertible. R is Dedekind domain if every nonzero ideal of R is invertible.*

There are other characterizations of Dedekind domains

Lemma 105. *Let R be an integral domain. TFAE:*

- *R is Dedekind*
- *R is one-dimensional (or less), integrally closed, and Noetherian*
- *R is Noetherian and the localization of R at any prime is a Noetherian valuation domain*
- *Every nonzero ideal in R is a product of prime ideals*

The following definition for a conductor is given in relation to Lemmas 56 and 57.

Definition 106. For a ring S and subring R of S, the *conductor of R in S* is the set C of elements x of R such that $xS \subseteq R$.

Note that C is the largest ideal of R which is also an ideal of S.

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