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Improvements to MLE Algorithm for Localizing Radiation Sources with a Distributed Detector Network

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IMPROVEMENTS TO MLE ALGORITHM FOR LOCALIZING RADIATION SOURCES WITH A DISTRIBUTED DETECTOR NETWORK

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Electrical Engineering

by
Guthrie Adam Cordone
May 2019

Accepted by:
Dr. Richard Brooks, Committee Chair
Dr. Harlan Russell
Dr. Adam Hoover
Dr. Satyabrata Sen
Abstract

Maximum Likelihood Estimation (MLE) is a widely used method for the localization of radiation sources using distributed detector networks. While robust, MLE is computationally intensive, requiring an exhaustive search over parameter space. To mitigate the computational load of MLE, many techniques have been presented, including iterative and multi-resolution methods.

In this work, we present two ways to improve the MLE localization of radiation sources. First, we present a method to mitigate the pitfalls of a standard multi-resolution algorithm. Our method expands the search region of each layer before performing the MLE search. Doing so allows the multi-resolution algorithm to correct an incorrect selection made in a prior layer. We test our proposed method against single-resolution MLE and standard multi-resolution MLE algorithms, and find that the use of grid expansion incurs a general decrease in localization error and a negligible increase in computation time over the standard multi-resolution algorithm.

Second, we present a method to perform the MLE localization without prior knowledge of the background radiation intensity. We estimate the source and background intensities using linear regression (LR) and then use these estimates to initialize the intensity parameter search for MLE. We test this method using single-resolution, multi-resolution, and multi-resolution with grid expansion MLE algorithms and compare performance to MLE algorithms that don’t use the LR initialization method. We found that using the LR estimates to initialize the intensity parameter search caused a marginal increase in both localization error and computation time for the tested algorithms. The technique is only beneficial in the case of an unknown background intensity.
Acknowledgments

This work has been partially supported by the U.S. Department of Homeland Security, Domestic Nuclear Detection Office, under competitively awarded contract No. IAA HSHQDC-13-X-B0002. This support does not constitute an express or implied endorsement on the part of the Government.
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Chapter 1

Introduction

It is important to be proactive against the threat of radioactive weapons to our national population centers. A nuclear detonation or the dispersal of radioactive material in a city would be catastrophic. One of the primary ways to defend against such events is the use of detection and localization systems [3].

There are many challenges to radiation detection and localization. Background radiation due to naturally occurring isotopes, cosmic rays, and human sources (x-rays, nuclear reactors, cat litter, etc.) often introduce non-negligible noise in detector measurements. Furthermore, gamma radiation measurements follow a Poisson distribution [4]. Therefore, measurements with higher values will also have a higher variance which could make it difficult to distinguish the presence of a dangerous source from background noise. In addition, the level of radiation intensity is attenuated by the distance to the source, obstacles between the source and the detector, and the propagation medium of the signal.

Radiation detection involves the deployment of a single sensor in the form of a portal monitor located at choke points along the road [3, 5]. The use of a single sensor allows for simple detection algorithms such as moving average and exponential smoothing filters [6]. While reliable for areas with a small number of entry points, the use of portal monitors is not practical to secure areas with more complex road structures [7]. In these scenarios, it is necessary to implement distributed networks of radiation detectors over the area. With distributed networks, the radiation detection problem becomes one that requires the fusion of independent sets of stochastic measurement data [8–10].
In this work we are interested in localization. Localization algorithms use a network of detectors with known locations to estimate the physical location of the source. There are two types of localization: cooperative and uncooperative. In cooperative localization the source actively helps in the localization. In uncooperative localization, the source does not help. For national security, the localization of radiation sources strictly falls into the uncooperative category. There are three major categories of uncooperative localization methods that have been used for radiation sources: geometric, Bayesian, and maximum likelihood.

Geometric localization methods such as Time Difference of Arrival (TDOA) and Ratio of Squared Distances (RoSD) use known detector locations and estimated distances from each detector to the source to compute the relative location of the source. In TDOA localization systems [11, 12], the distance from each sensor to the source is estimated. The distances are compared between detectors to form a hyperbolic locus. Source location estimates are placed at the intersection of two loci. If the hyperbola has imaginary roots then the loci won’t intersect, causing a failed estimation. In RoSD localization systems [13], a ratio is computed between the square distances to the source from three different detectors. There are two possible locations generated for each set of three sensors. The test is performed for multiple sets of sensors and a source estimation can be made if there is a clustering of solutions.

Bayesian estimation localization methods assume that parameters are stochastic with known prior distributions. Two common Bayesian localization methods are particle filters [14, 15] and Kalman Filters [16]. While theoretically functional, both of these techniques have been shown to be unreliable in real-life scenarios.

One of the most common radiation localization methods is Maximum Likelihood Estimation (MLE) [13, 16–18]. MLE is a statistical estimation technique that is used to determine most likely parameters of some objective function [19]. Unfortunately, in the case of radiation localization, the MLE objective function has no closed form solution. Thus, a grid search is required to determine the maximizing parameters. Depending on the bounds of the grid search, an exhaustive optimization over parameter space would be costly. To mitigate such costs, several papers introduce the use of multi-resolution [20, 21] or other iterative [13, 22–24] techniques. These methods aim to maximize the likelihood function without conducting an exhaustive search over parameter space. Both methods have drawbacks, however. Because they eliminate areas of search space, they are liable to perform poorly given a bad starting point or mistake in the algorithm.
Improving the MLE localization algorithm for radiation sources is the major focus of this work. Specifically, we attempt to solve two problems:

1. Can we mitigate the drawbacks of a multi-resolution MLE localization while keeping the computational benefits?

2. Can we develop an MLE localization algorithm that does not require prior knowledge of background intensity?

The rest of the document is as follows: In Chapter 2 we provide background information on necessary topics; in Chapter 3 we develop our mathematical model for the detector network, MLE estimate, and MLE grid-search; in Chapter 4 we explore the use of grid expansion to mitigate localization error in multi-resolution MLE algorithms; in Chapter 5 we present a linear regression model that can be used to initialize parameter search ranges for an MLE algorithm in the case of unknown background; and in Chapter 6 we provide a summary and present some prospects for future work.
Chapter 2

Background

2.1 Scintillation Counters

Scintillation counters are among the most common types of detectors used for radiation detection and localization applications. They consist of two major components: a scintillation material and a counting circuit. The scintillation material is a precisely shaped piece of organic or inorganic material that illuminates when exposed to ionizing radiation. The efficiency of the illumination depends on the type of material [25]. Two scintillation materials that are used in scintillation counters are Sodium Iodide (NaI) and Cesium Iodide (CsI).

In the counting circuit portion of the scintillation counter, photons released by the scintillation material are amplified and converted into electrical impulses that are compiled over a set amount of time. Since the number of photons released is proportional to the level of radiation, a higher pulse count over a time step indicates a higher radiation intensity at the location of the detector.

The counts for each time step are stored electronically at either the location of the detector or at an external node. Figure 2.1 provides a simplified model of a scintillation detector typically used for radiation detection and localization applications.
2.2 Radiation Measurements and Propagation

The measured counts of the scintillation detector can be modeled as a Poisson random variable, \( X \). The Poisson parameter, \( \lambda \), is proportional to the intensity of the radiation at the location of the detector [4]. Thus, the probability of recording \( c \) counts over time duration, \( t \), due to radiation intensity, \( \lambda \), is given by the Poisson probability mass function

\[
P(X = c) = \frac{e^{-\lambda} \lambda^c}{c!}.
\]  

(2.1)

Given a single radiation-generating point source with intensity \( A \) counts/second, the resulting count-rate at a nearby scintillation counter is given by [20]

\[
\lambda = \frac{\mu RA}{4\pi d^2} e^{-\rho d} + \frac{\mu RB}{4\pi}
\]  

(2.2)

where \( d \) is the distance between the detector and source, \( R \) is the cross-sectional area of the detector scintillation crystal, \( \mu \) is the detector photo-peak efficiency, \( \rho \) is the attenuation coefficient of the air, and \( B \) is the intensity of background radiation at the detector’s location. A discussion of issues related to modeling of background noise and environmental factors can be found in [26]. Note that this propagation equation does not take into account attenuating obstacles between the detector and source, which is an assumption that will be kept throughout.

Within the context of localization, a common assumption is made that all detectors in the network have the same cross-sectional area and efficiency, and the air attenuation coefficient is arbitrarily close to zero. These assumptions allow (2.2) to simplify down to

\[
\lambda = \zeta \left( \frac{A}{d^2} + B \right)
\]  

(2.3)
where $\zeta = \frac{\mu A}{4\pi}$ is an efficiency factor for the specified detector. An extensive discussion of detector efficiency is provided in [4].

Figure 2.2 provides a visual representation of the implications of the Poisson process (2.1) and radiation propagation model (2.3). Observe that the measurements of the detector located one meter from the source has a much larger mean counts value and a much larger variance in counts values than the measurements of the detector located six meters from the source.

Figure 2.2: Measurements of a 35$\mu$Ci Cs-137 source at two different distances using a 2”x2” NaI detector [1]

2.3 MLE for Radiation Source Localization

Maximum Likelihood Estimation (MLE) is a common statistical estimation technique [19]. Given a set of unknown parameters, $\theta = [\theta_1, \theta_2, \ldots, \theta_m]^T$, and a set of measurements parameterized by $\theta$, $x = [x_1, x_2, \ldots, x_n]$, the maximum likelihood estimate is the set of values that maximize the joint probability, $P(x; \theta)$:

$$\hat{\theta}_{ml} = \arg \max_\theta P(x; \theta).$$

(2.4)

The work required by (2.4) is dependent on the joint probability function. For example, if all measurements are independent and identically distributed, and each measurement follows a known distribution with parameters $\theta$, the the maximizing values are simply found by taking the derivative of the probability function and setting it equal to zero. In many cases, however, there is no closed-form solution to (2.4) and a numerical optimization is required. This is the unfortunate case for the localization of radiation sources, where $P(x; \theta)$ is the joint probability of detector measurements.
given by a product of (2.1) for each detector in the network. An in-depth derivation of the ML estimator for radiation source localization is provided in Chapter 3.

Finally, it is important to mention that the MLE estimate has the consistency property, which states that \( \hat{\theta}_{ml} \) will approach the true parameter values as the number of measurements, \( n \), increases. This property is generally useful for localization applications because it indicates that the localization result will approach the actual location given a large enough set of measurements.

## 2.3.1 Related Works in MLE Localization

There is an extensive body of work on using MLE for localization of a non-radioactive diffusive point source using a sensor network. Chen et al. [27] used MLE for the localization of wide-band radio signals. Vijaykumaran et al. [28] created an MLE algorithm for the localization of a diffusive gas source. Sheng and Hu [29] compared an MLE localization algorithm with other algorithms to localize multiple acoustic energy sources using a wireless ad-hoc sensor network. Vijaykumaran et al. Guantilaka et al. [30] combined MLE with Bayesian estimation methods to estimate the location and strength of a chemical plume source.

More recently, these techniques have been implemented for use in radiation source localization. In this context, the unknowns of interest are source intensity and source coordinates. The likelihood function of interest does not have a closed-form solution. Therefore, the maximization must be performed numerically, which introduces several implementation challenges, to be discussed in later chapters.

Guantilaka et al. [16] compared the performance of an MLE algorithm, an unscented Kalman Filter, and an extended Kalman Filter with the theoretical Cramer-Rao bound on the localization of a single radioactive point source. Using simulations, they found that the MLE algorithm approached the Cramer-Rao bound as the number of measurements increased. They also found that MLE localization performed better than both Kalman Filter variants.

Chin et al. [13] proposed an iterative pruning (ITP) algorithm based on the RoSD technique for the localization of radiation point sources within a detector network. They tested their algorithm with other methods including MLE and determined that the MLE algorithm is more accurate while their ITP algorithm is more computationally efficient.

Vilim et al. [17] developed the RadTrac software system, which localizes a single radiation source by minimizing the negative log-likelihood function. They tested their method for a single
source of varying strengths within a field of four detectors. In [31], they used their RadTrac system to track a weak radioactive source for different cases of a crowded indoor scenario. In [18], Vilim et al. developed a probabilistic localization method using MLE to localize a single radiation source using directional and isotropic detectors.

Several papers tackle the specific challenge of determining proper initialization points for the MLE grid search. Deb [20] uses MLE to localize the parameters of multiple static and moving 1 mCi radiation sources located within a distributed detector network. They initialize the MLE using an iterative Expectation Maximization (EM) algorithm, which provides rough estimates of the location, number, and intensity of the sources within the search field. They use Fisher’s scoring iterations, a modification of Newton’s method, to maximize the likelihood function over parameter space, initialized with each of the points found by the EM algorithm. Furthermore, they estimate the trajectory of a moving source within the field using a Weighted Least Squares (WLS) optimization, which is shown to be equivalent to the MLE.

Wan et al. [23] use the geographic weight center of high-measurement detectors as an initial source location estimate for an MLE localization algorithm. They optimize the likelihood function at this location estimate to determine initial estimates for source and background intensities. The three estimates are then used as initialization parameters for a Newton-Raphson iterative optimization for the MLE, which they successfully tested with simulated data of a single radiation source located within a 10x10 grid of 100 detectors.

Zhao et al. [24] proposes Kernel Density Estimation (KDE) to localize a radiation source. This technique estimates the source location by maximizing the sum of a counts-weighted Gaussian kernel. They find that the method performs poorer in general than a standard MLE grid search, but may be suitable for the initialization of an iterative MLE. The major drawback to the KDE method is that it requires a large window of measurements to become accurate in comparison to MLE.

Bai et al. [22] attempt to find an initial estimate “within the basin of attraction” of an iterative MLE search for the localization of a single radiation source among a highly fluctuating background. To achieve this goal, they employ separable least squares to reduce the dimensionality of the parameter search and then minimizes the resulting function over the source parameters. The resulting estimates are used to initialize a standard Newton’s method iterative search for the MLE.
2.4 IRSS Datasets

The data used in this work were generated at the Low Scatter Irradiator (LSI) facility at the Savannah River National Laboratory as part of the Domestic Nuclear Detection Office (DNDO) Intelligent Radiation Sensing System (IRSS) program [32]. Several indoor and outdoor tests were performed using various source strengths and types. These canonical datasets were packaged [33] and uploaded to Github for public use [2].

There are seven datasets, all containing a single radiation source within a field of 2”x2” NaI scintillation counter detectors. All datasets use a field layout of 18 stationary detectors laid out in two concentric circles and a spiral pattern across an 8m × 8m indoor field. The strength and location of the source within the field varies between the different tests. Table 2.1 provides specific information for each dataset and Figure 2.3 provides the explicit field layout for each dataset. Note that the D-01 dataset contains a moving source while all other datasets contain a non-moving (stationary) source.

Table 2.1: Indoor IRSS Datasets

<table>
<thead>
<tr>
<th>Name</th>
<th>Isotope</th>
<th>Strength, ( \mu \text{Ci} )</th>
<th>#Dets</th>
<th>Source Location/Trajectory</th>
<th>Runs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>Cs-137</td>
<td>35.0</td>
<td>18</td>
<td>Non-Moving - Center of field</td>
<td>9</td>
</tr>
<tr>
<td>A-07</td>
<td>Co-57</td>
<td>45.0</td>
<td>18</td>
<td>Non-Moving - Center of field</td>
<td>9</td>
</tr>
<tr>
<td>C-01</td>
<td>Cs-137</td>
<td>7.6</td>
<td>18</td>
<td>Non-Moving - Center of field</td>
<td>7</td>
</tr>
<tr>
<td>C-02</td>
<td>Cs-137</td>
<td>7.6</td>
<td>18</td>
<td>Non-Moving - 1m NE of Center</td>
<td>7</td>
</tr>
<tr>
<td>C-03</td>
<td>Cs-137</td>
<td>7.6</td>
<td>18</td>
<td>Non-Moving - 2m NE of Center</td>
<td>8</td>
</tr>
<tr>
<td>C-04</td>
<td>Cs-137</td>
<td>7.6</td>
<td>18</td>
<td>Non-Moving - 4m NE of Center</td>
<td>11</td>
</tr>
<tr>
<td>D-01</td>
<td>Cs-137</td>
<td>16.0</td>
<td>18</td>
<td>Moving - Diagonal along NE of Center</td>
<td>6</td>
</tr>
</tbody>
</table>

Each dataset contains the synced counts-per-second readings of the individual detectors over a pre-defined set of spectral bins. Figure 2.4 provides the complete list of bins and their corresponding isotopes. Note that this figure was pulled directly from the description files of the IRSS data. To use this data for a detection or localization algorithm, one can extract the counts data from the bins of the isotope of interest, and ignore the rest. This method of bin usage requires that the isotope of interest in known beforehand, which is something assumed for the duration of this work. Observe in Table 2.1 that The source type is Cs-137 for all data sets except A-07, which uses a Co-57 source. These two isotopes correspond to bins 12 and 3, respectively. Note that if the isotope is not known beforehand, then the user would need to perform some statistical consolidation of all bins before performing their algorithm, or perform their algorithm on each bin individually.
and then consolidate results.

Figure 2.3: Field layouts for IRSS datasets recorded from indoor LSI facility
Table 2.4: List of spectral bins and corresponding isotopes for IRSS datasets [2]

<table>
<thead>
<tr>
<th>Bin #</th>
<th>Lower Bound (keV)</th>
<th>Upper Bound (keV)</th>
<th>ISOTOPE(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>42</td>
<td>86</td>
<td>Am-241 (Americium 241)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ti-210 (Thallium 210)</td>
</tr>
<tr>
<td>02</td>
<td>64</td>
<td>103</td>
<td>Ba-133 (Barium 133)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cd-109 (Cadmium 109)</td>
</tr>
<tr>
<td>03</td>
<td>105</td>
<td>145</td>
<td>Co-57 (Cobalt 57)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Pu-239 (Plutonium 239)</td>
</tr>
<tr>
<td>04</td>
<td>123</td>
<td>160</td>
<td>Tc-99m (Technetium 99m)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>U-235 (Uranium 235)</td>
</tr>
<tr>
<td>05</td>
<td>166</td>
<td>203</td>
<td>U-235 (Uranium 235)</td>
</tr>
<tr>
<td>06</td>
<td>189</td>
<td>227</td>
<td>Am-241 (Americium 241)</td>
</tr>
<tr>
<td>07</td>
<td>227</td>
<td>330</td>
<td>NORM</td>
</tr>
<tr>
<td>08</td>
<td>330</td>
<td>391</td>
<td>Ba-133 (Barium 133)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>I-131 (Iodine 131)</td>
</tr>
<tr>
<td>09</td>
<td>384</td>
<td>442</td>
<td>Pu-239 (Plutonium 239)</td>
</tr>
<tr>
<td>10</td>
<td>476</td>
<td>545</td>
<td>F-18 (Fluorine 18)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Na-22 (Sodium 22)</td>
</tr>
<tr>
<td>11</td>
<td>545</td>
<td>621</td>
<td>U-232 (Uranium 232)</td>
</tr>
<tr>
<td>12</td>
<td>621</td>
<td>704</td>
<td>Cs-137 (Cesium 137)</td>
</tr>
<tr>
<td>13</td>
<td>704</td>
<td>857</td>
<td>NORM</td>
</tr>
<tr>
<td>14</td>
<td>857</td>
<td>963</td>
<td>Ac-228 (Actinium 228)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>NORM</td>
</tr>
<tr>
<td>15</td>
<td>944</td>
<td>1098</td>
<td>U-238 (Uranium 238)</td>
</tr>
<tr>
<td>16</td>
<td>1108</td>
<td>1239</td>
<td>Co-60 (Cobalt 60)</td>
</tr>
<tr>
<td>17</td>
<td>1205</td>
<td>1345</td>
<td>Na-22 (Sodium 22)</td>
</tr>
<tr>
<td>18</td>
<td>1260</td>
<td>1406</td>
<td>Co-60 (Cobalt 60)</td>
</tr>
<tr>
<td>19</td>
<td>1381</td>
<td>1581</td>
<td>K-40 (Potassium 40)</td>
</tr>
<tr>
<td>20</td>
<td>1581</td>
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<td>NORM</td>
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<td>2452</td>
<td>2776</td>
<td>Ti-208 (Thallium 208)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>U-232 (Uranium 232)</td>
</tr>
</tbody>
</table>

Figure 2.4: List of spectral bins and corresponding isotopes for IRSS datasets [2]
Chapter 3

Mathematical Model and MLE Algorithm

3.1 Radiation and Sensor Network Model

A radioactive point source characterized by parameters, $\theta_g = [x_g, y_g, A_g]^T$, is located in a field of $N$ detectors. The source has intensity $A_g$ in units of gamma disintegration events per second and is located at coordinates $(x_g, y_g)$. In this notation, the subscript 'g' denotes the parameters as ground truth. Each individual detector, $m_i \in \mathcal{M} = \{m_1, m_2, \ldots, m_N\}$, has coordinates $(x_i, y_i)$ and records a sequence of counts-per-second values, $c_i = [c_{i1}, c_{i2}, \ldots, c_{iw}]^T$, over time window $w$. The individual coordinates of each detector are represented by vectors $x = [x_1, x_2, \ldots, x_N]^T$ and $y = [y_1, y_2, \ldots, y_N]^T$. Each counts value $c_{ij}$ for $j \in \{1, 2, \ldots, w\}$ is the number of gamma particles detected over given second of operation, and within the photo-peak region of the isotope considered.

Assuming a uniform propagation medium with a negligible attenuation coefficient, a model for the effective count-rate at detector $m_i$ is

$$\lambda_i(\theta_g) = \frac{A_g}{(x_g - x_i)^2 + (y_g - y_i)^2} + B.$$ (3.1)

The denominator of the first term is the square of the Euclidean distance between the radiation source and $m_i$. The background radiation intensity at $m_i$ is $B$ and known beforehand. If detector locations and background intensity are known prior to localization, then (3.1) is solely a function of
the source parameters $\theta_g$.

### 3.2 MLE for Specified Model

The assumptions of our propagation radiation model (3.1) can be distilled in the following list:

1. the background intensity, $B$, is constant and known,

2. there is only one radiation source within detection range of all detectors,

3. there are no significant obstacles between the detectors and the ground truth,

4. the ground truth and network of detectors all lie within the same 2-dimensional plane.

In addition, we make the assumption that all detectors in the network have the same efficiency, such that the set of all detector measurements can be represented as a set of independent and identically distributed Poisson random variables. Note that in this model we are omitting the $\zeta$ term that is present in (2.3). This omission is done because we are not particularly interested in determining the source intensity, just the source location, and as long as all detectors have the same efficiency, the MLE localization result will be the same.

The given assumptions are common for radiation localization [13, 16]. MLE localization methods that optimize over the more advanced radiation model (such as (2.2)) can be found in [20, 22]. That said, given our assumptions, the more advanced models boil down to the one provided by (3.1).

#### 3.2.1 Derivation of the MLE

**Result:** The MLE estimates for the parameters, $\theta$, in our radiation and sensor model are given by

$$\hat{\theta}_{ml} = [\hat{x}_{ml}, \hat{y}_{ml}, \hat{A}_{ml}]^T = \arg \max_{\theta} \mathcal{L}(C; \theta) \quad (3.2)$$

where

$$\mathcal{L}(C; \theta) = \sum_{i=1}^{N} \sum_{j=1}^{w} (c_{ij} \ln \lambda_i - \lambda_i). \quad (3.3)$$
Derivation: The probability of detector $m_i$ registering $c_{ij}$ disintegration events due to a source characterized by parameters $\theta = [x, y, A]^T$ over time period $j - 1$ to $j$ is given by Poisson probability mass function:

$$P(c_{ij}; \theta) = \frac{e^{-\lambda_i} \lambda_i^{c_{ij}}}{c_{ij}!}. \quad (3.4)$$

Since detector measurements are independent and identically distributed (i.i.d.), the probability of observing the vector of counts, $c_i$, at each detector is given by joint likelihood:

$$P(C; \theta) = \prod_{i=1}^{N} \prod_{j=1}^{w} e^{-\lambda_i} \frac{\lambda_i^{c_{ij}}}{c_{ij}!}. \quad (3.5)$$

where $C = [c_1, c_2, \ldots, c_N]$. Putting Equation (3.5) in log space gives:

$$\ln P(C; \theta) = \sum_{i=1}^{N} \sum_{j=1}^{w} (c_{ij} \ln \lambda_i - \lambda_i - \ln(c_{ij}!)). \quad (3.6)$$

The $\ln(c_{ij}!)$ term in Equation (3.6) is constant for all possible input parameters. We remove it, since it is redundant for the maximization of (3.6). This gives simplified log-likelihood function:

$$L(C; \theta) = \sum_{i=1}^{N} \sum_{j=1}^{w} (c_{ij} \ln \lambda_i - \lambda_i). \quad (3.7)$$

The maximum likelihood estimate of the ground truth parameters, $\theta_g$, will be the parameter set that maximizes Equation (3.7):

$$\hat{\theta}_{ml} = [\hat{x}_{ml}, \hat{y}_{ml}, \hat{A}_{ml}]^T = \arg \max_{\theta} L(C; \theta). \quad (3.8)$$

### 3.3 Mid-Point Grid Search Algorithm for MLE

There is no closed-form solution to (3.7), since it is a function of $\lambda$, which is non-linear with respect to search parameters, $\theta$. However, since we have the count values $c_{ij}$ and known detector locations $(x_i, y_i)$, it is possible to directly calculate the likelihood that a given parameter set $\theta$ generated those observations. Thus, to find the best solution, we calculate likelihoods over a grid search within ranges $[x_{min}, x_{max}]$, $[y_{min}, y_{max}]$, and $[A_{min}, A_{max}]$. Within those bounds, we take
$N_x$, $N_y$, and $N_A$ samples, respectively. This produces a program with three nested loops:

\[
\begin{align*}
    x_m &= x_{\text{min}} + \left( m - \frac{1}{2} \right) \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \\
    y_n &= y_{\text{min}} + \left( n - \frac{1}{2} \right) \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \\
    A_p &= A_{\text{min}} + \left( p - \frac{1}{2} \right) \frac{A_{\text{max}} - A_{\text{min}}}{N_A},
\end{align*}
\]

where $m \in \{1, \ldots, N_x\}$, $n \in \{1, \ldots, N_y\}$, $p \in \{1, \ldots, N_A\}$, and the current search parameters are $\theta = [x_m, y_n, A_p]^T$. Note that each individual equation in (3.9) contains a one-half shift to the right of the index values. This technique effectively shifts the search points over to the center of the grid region defined by the specified min/max bounds. Thus, we refer to this grid-search method as a Mid-Point grid search. The general process for MLE localization using the Mid-Point grid search is provided in Algorithm 1.

**Algorithm 1** MLE Localization using Mid-Point Grid Search

1: $L_{\text{max}} =$ arbitrarily low value
2: $\hat{B} =$ known background estimate
3: for $m = 1$ to $N_x$ do
4:     for $n = 1$ to $N_y$ do
5:         for $p = 1$ to $N_A$ do
6:             $x_m = x_{\text{min}} + \left( m - \frac{1}{2} \right) \frac{x_{\text{max}} - x_{\text{min}}}{N_x}$
7:             $y_n = y_{\text{min}} + \left( n - \frac{1}{2} \right) \frac{y_{\text{max}} - y_{\text{min}}}{N_y}$
8:             $A_p = A_{\text{min}} + \left( p - \frac{1}{2} \right) \frac{A_{\text{max}} - A_{\text{min}}}{N_A}$
9:         for each $i$th detector do
10:            $\lambda_i = \frac{A_p}{(x_i-x_m)^2+(y_i-y_n)^2} + \hat{B}$
11:         end for
12:         $L = \sum_{i=1}^{N} \sum_{j=1}^{w} (c_{ij} \ln \lambda_i - \lambda_i)$
13:         if ($L > L_{\text{max}}$) then
14:             $L_{\text{max}} \leftarrow L$
15:             $\hat{x}_{\text{ml}} \leftarrow x_m$
16:             $\hat{y}_{\text{ml}} \leftarrow y_n$
17:             $\hat{A}_{\text{ml}} \leftarrow A_p$
18:         end if
19:     end for
20: end for
21: return $\hat{\theta}_{\text{ml}} = [\hat{x}_{\text{ml}}, \hat{y}_{\text{ml}}, \hat{A}_{\text{ml}}]^T$

Figure 3.1a provides an example MLE localization on a run of the C-01 dataset using this approach. Observe that the search ranges $[x_{\text{min}}, x_{\text{max}}]$ and $[y_{\text{min}}, y_{\text{max}}]$ are constrained by the bounds of the detector field, such that $[x_{\text{min}}, x_{\text{max}}] = [\min x_i, \max x_i]$ and $[y_{\text{min}}, y_{\text{max}}] = [\min y_i, \max y_i]$. 
The constraints of the source range, \([A_{\min}, A_{\max}]\), will be examined in Chapters 4 and 5.

Figure 3.1b provides the a scaled map of the log-likelihood values computed at the center of each grid region in Figure 3.1a. These are the values computed with (3.7). Observe that the region with the highest likelihood value is the same region ML estimate coordinates for the localization in Figure 3.1a.

### 3.3.1 Concavity of Likelihood Function given Spatial Coordinates

**Claim:** Under the assumptions provided at the beginning of this chapter, given a fixed geographic coordinate pair, \((x_m, y_n)\), the log-likelihood function (3.7) is concave in both \(A\) and \(B\).

**Proof:** The log-likelihood function (3.7) can be re-written in terms of source and background intensity

\[
\mathcal{L}(C; \theta) = \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \ln \left( \frac{A}{d_i^2} + B \right) - \left( \frac{A}{d_i^2} + B \right) .
\]

(3.10)

where \(d_i\) is the Euclidean distance between the current search location, \((x_m, y_n)\), and the location of detector \(m_i\), \((x_i, y_i)\), given by the denominator of (3.1). Thus, given a specific set of search coordinates, \(d_i\) is constant, and the remaining variable in (3.10) are \(A\) and \(B\). Note that in our actual MLE we assume that \(B\) is known, but we are treating it as unknown in this proof for the sake of completeness.

A function is concave if its corresponding Hessian is negative definite or negative semidefinite. The Hessian for (3.10) with respect to \(A\) and \(B\) is defined

\[
H = \begin{bmatrix}
\frac{\delta^2 \mathcal{L}(C; \theta)}{\delta A^2} & \frac{\delta^2 \mathcal{L}(C; \theta)}{\delta AB} \\
\frac{\delta^2 \mathcal{L}(C; \theta)}{\delta B A} & \frac{\delta^2 \mathcal{L}(C; \theta)}{\delta B^2}
\end{bmatrix} .
\]

(3.11)

The first order partial derivatives of (3.10) are

\[
\frac{\delta \mathcal{L}(C; \theta)}{\delta A} = \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \left( \frac{1}{\frac{A}{d_i^2} + B} - 1 \right) .
\]

(3.12)

\[
\frac{\delta \mathcal{L}(C; \theta)}{\delta B} = \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \left( \frac{1}{\frac{A}{d_i^2} + B} - 1 \right) .
\]

(3.13)
The corresponding second order partial derivatives are

\[
\frac{\delta^2 L(C; \theta)}{\delta A^2} = \sum_{i=1}^{N} \sum_{j=1}^{w} -c_{ij} \left( \frac{1}{\frac{A}{d_i^2} + B} \right)^2,
\]

(3.14)

\[
\frac{\delta^2 L(C; \theta)}{\delta B^2} = \sum_{i=1}^{N} \sum_{j=1}^{w} -c_{ij} \left( \frac{1}{\frac{A}{d_i^2} + B} \right)^2,
\]

(3.15)

\[
\frac{\delta^2 L(C; \theta)}{\delta A \delta B} = \frac{\delta^2 L(C; \theta)}{\delta B \delta A} = \sum_{i=1}^{N} \sum_{j=1}^{w} -c_{ij} \left( \frac{1}{\frac{A}{d_i^2} + B} \right)^2.
\]

(3.16)

Define value, \( k_i \) for simpler formulation,

\[
k_i = \frac{\sum_{j=1}^{w} c_{ij}}{\left( \frac{A}{d_i^2} + B \right)^2}.
\]

(3.17)

Plugging the second order partial derivatives and substituting values for \( k_i \), the Hessian (3.11) becomes

\[
H = \begin{bmatrix}
\sum_{i=1}^{N} \frac{k_i}{d_i^2} & \sum_{i=1}^{N} \frac{k_i}{d_i^2} \\
\sum_{i=1}^{N} \frac{k_i}{d_i^2} & \sum_{i=1}^{N} \frac{k_i}{d_i^2}
\end{bmatrix}.
\]

(3.18)

The Hessian in (3.18) is real-valued and symmetric. Thus, it is negative definite if

\[
z^T H z < 0
\]

(3.19)

where \( z = [a, b]^T \) for arbitrary constants \( \{a, b\} \in \mathcal{R} \). Plugging (3.18) into the left side of (3.19),

\[
z^T H z = -\sum_{i=1}^{N} k_i \left( \frac{a}{d_i^2} + b \right)^2.
\]

(3.20)

Under the bounds of our model \((c_{ij} > 0, A \geq 0, B > 0)\) \( k_i \) is always positive. Therefore, (3.20) is negative for all values of \( \{a, b\} \), and (3.18) is a negative semidefinite matrix with respect to \( A \) and \( B \). Therefore, given search location \((x_m, y_n)\), the log-likelihood function (3.10) is concave over \( A \) and \( B \).

**Implication of Result:** It is not necessary to search over the entire specified range for the source
intensity parameter, \([A_{\text{min}}, A_{\text{max}}]\). The MLE grid search described in Algorithm 1 can be stopped early at each set of test coordinates by checking if the likelihood value goes down between source intensity values. If the value does go down, then the prior intensity parameter is the maximum likelihood intensity estimate for the given set of grid coordinates. We refer to this technique as 'Autostop'. A version of Algorithm 1 with this modification is provided in Algorithm 2. Observe that the only difference between the two algorithms is a conditional 'break' term at line 19. Note that we could also do this for a search over the background intensity if the background is not assumed to be known beforehand. This case will be examined in Chapter 5.

Algorithm 2 MLE Localization using Mid-Point Grid Search with Autostop

1: \(L_{\text{max}} = \) arbitrarily low value
2: \(\hat{B} = \) known background estimate
3: \(\text{for } m = 1 \text{ to } N_x \text{ do} \)
4: \(\quad \text{for } n = 1 \text{ to } N_y \text{ do} \)
5: \(\quad \quad \text{for } p = 1 \text{ to } N_A \text{ do} \)
6: \(\quad \quad \quad x_m = x_{\text{min}} + \left( m - \frac{1}{2} \right) \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \)
7: \(\quad \quad \quad y_n = y_{\text{min}} + \left( n - \frac{1}{2} \right) \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \)
8: \(\quad \quad \quad A_p = A_{\text{min}} + \left( p - \frac{1}{2} \right) \frac{A_{\text{max}} - A_{\text{min}}}{N_A} \)
9: \(\quad \quad \quad \text{for each } i\text{th detector do} \)
10: \(\quad \quad \quad \quad \lambda_i = \frac{A_p}{(x_i - x_m)^2 + (y_i - y_n)^2} + \hat{B} \)
11: \(\quad \quad \quad \end{for} \)
12: \(\quad \quad L = \sum_{i=1}^{N} \sum_{j=1}^{w} (c_{ij} \ln \lambda_i - \lambda_i) \)
13: \(\quad \text{if } (L > L_{\text{max}}) \text{ then} \)
14: \(\quad L_{\text{max}} \leftarrow L \)
15: \(\quad \hat{x}_{ml} \leftarrow x_m \)
16: \(\quad \hat{y}_{ml} \leftarrow y_n \)
17: \(\quad \hat{A}_{ml} \leftarrow A_p \)
18: \(\text{else} \)
19: \(\quad \text{break} \)
20: \(\text{end if} \)
21: \(\text{end for} \)
22: \(\text{end for} \)
23: \(\text{return } \hat{\theta}_{ml} = [\hat{x}_{ml}, \hat{y}_{ml}, \hat{A}_{ml}]^T \)

It is important to mention that one could use an initial search value with any hill-climbing algorithm such as Gradient Descent or the Nelder-Mead Simplex Method to optimize the likelihood function over the intensity parameter. In fact, almost all MLE implementations referenced in Section 2.3.1 use these iterative optimization algorithms to maximize the likelihood function. Instead, we are choosing to use the described autostop algorithm, which simply defines the bounds of the source search space, searches from the minimum to the maximum using a designated increment value, and
then stops once the likelihood value goes down. We decide to use the described method because it gives us easy and direct control of both the bounds of the search space as well as the maximum number of points searched. Thus, the computation time of the algorithm is much more predictable, which is necessary for real time implementation of these algorithms. The use of existing optimization algorithms to search over intensity space is left for a future endeavor.

### 3.4 Multi-Resolution MLE

The effective resolution of an MLE grid search can be gleaned from (3.9) as the total search range of the parameter values divided by the quantization values $N_x$, $N_y$, and $N_A$. For example, the width and height of a single grid region defined by a Mid-Point grid search is given by $\frac{x_{\text{max}} - x_{\text{min}}}{N_x}$ and $\frac{y_{\text{max}} - y_{\text{min}}}{N_y}$, respectively. In this context, a high resolution grid search uses large values for $N_x$, $N_y$, and $N_A$, and a low resolution search uses comparatively smaller values for $N_x$, $N_y$, and $N_A$. As consequence, the required number of solutions of the log-likelihood function (3.7) scales with the resolution of the search, quickly approaching unreasonable complexity as the resolution increases.

A modification to the standard MLE grid search, multi-resolution MLE, is a common method \[20, 34\] used to achieve a high resolution estimate without the typical drawback of large computational complexity. Multi-resolution MLE is a series of increasingly higher resolution grid searches that are each bounded by the estimate selected by the previous search. Such a process allows the grid search to focus in on a high-likelihood area of parameter space while ignoring low-likelihood areas, effectively reducing the total number of computations.

For the multi-resolution algorithm, define the search area of each consecutive layer of resolution as the grid region of the parameter coordinates selected by the previous search, $(\hat{x}_{ml}, \hat{y}_{ml})$. The area of this selected grid region is defined by ranges, $[x'_{\text{min}}, x'_{\text{max}}]$ and $[y'_{\text{min}}, y'_{\text{max}}]$, such that

\[
\begin{align*}
x'_{\text{min}} &= \hat{x}_{ml} - \frac{1}{2} \left( \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \right) \\
x'_{\text{max}} &= \hat{x}_{ml} + \frac{1}{2} \left( \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \right) \\
y'_{\text{min}} &= \hat{y}_{ml} - \frac{1}{2} \left( \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \right) \\
y'_{\text{max}} &= \hat{y}_{ml} + \frac{1}{2} \left( \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \right).
\end{align*}
\]

The general steps for a multi-resolution MLE grid search over the spatial parameters are given in
Algorithm 3.

Algorithm 3 Multi-Resolution MLE Localization using Mid-Point Grid Search with Autostop

1: for \( l = 1 \) to \( M \) do
2: Perform Algorithm 2
3: \( x_{\text{min}} \leftarrow \hat{x}_{ml} - \frac{1}{2} \left( \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \right) \)
4: \( x_{\text{max}} \leftarrow \hat{x}_{ml} + \frac{1}{2} \left( \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \right) \)
5: \( y_{\text{min}} \leftarrow \hat{y}_{ml} - \frac{1}{2} \left( \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \right) \)
6: \( y_{\text{max}} \leftarrow \hat{y}_{ml} + \frac{1}{2} \left( \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \right) \)
7: end for
8: return \( \hat{\theta}_{ml} = [\hat{x}_{ml}, \hat{y}_{ml}, \hat{A}_{ml}]^T \)

With the correct \( N_x, N_y, \) and \( M \) values, a multi-resolution MLE algorithm will run faster than a single-resolution MLE algorithm to the same final resolution. For example, a single-resolution MLE localization with \( N_x = 32 \) and \( N_y = 32 \) requires \( N_x N_y N_A = 1024 N_A \) computations of (3.7). A multi-resolution MLE localization with \( N_x = 2, N_y = 2, \) and \( M = 5, \) achieves the same final resolution as the single-resolution localization, but only requires \( N_x N_y N_A M = 20 N_A \) computations of (3.7). A visual comparison of this example is provided in Figure 3.2. Observe that each consecutive layer of the multi-resolution localization is nested within the boundary grid region of the estimate selected by the prior layer. Also note that the two algorithms do not return the same exact location.

The source intensity parameter, \( A, \) is not included in the multi-resolution algorithm. This exclusion is because (i) we already limit the search over \( A \) using the autostop method in Algorithm 2, and (ii) we are primarily interested in the location of the source; the source intensity is simply a means to improve the coordinate estimate. That said, an extension of the multi-resolution algorithm to the intensity parameter search is an interesting task and left for future work.
(a) The search coordinates determined by (3.9) are located at the center of each grid region. The grid region that contains the ML estimate is highlighted red.

(b) Color map representing the likelihood value computed at each respective search location

Figure 3.1: Example of an MLE localization on a run of the C-01 dataset using a mid-point grid search with $N_x = N_y = 15$
Figure 3.2: Single-resolution MLE localization vs. multi-resolution localization to an equivalent final resolution

(a) Single-Resolution MLE performed with a 32x32 grid ($N_x = N_y = 32$)

(b) Multi-Resolution MLE performed with a 2x2 grid for 5 layers of resolution ($N_x = N_y = 2, M = 5$)
Chapter 4

Improving Multi-Resolution MLE

4.1 Overview

Although it is robust, MLE is a computationally expensive algorithm. The time required by the exhaustive search may be impractical for many radiation localization scenarios. One flaw of MLE is that localization is based on calculations at isolated points distributed over the search field. This issue is prevalent when using multi-resolution MLE localization, which uses several iterations of a coarse grid to speed up the localization. In this case, an incorrect grid selection on the first iteration will cause following iterations to search within an area that does not contain the source, leading to a large localization error in comparison to a similar single-resolution MLE.

In this chapter, we present a modification to the multi-resolution MLE algorithm (Algorithm 3) that attempts to mitigate the pitfalls of the multi-resolution MLE without sacrificing too many of the benefits. Specifically, our modified algorithm is intended to achieve the localization accuracy of a single-resolution MLE algorithm but at the computational cost of a multi-resolution MLE. We compare our modified multi-resolution algorithm to a standard multi-resolution and single-resolution algorithm by measuring localization error and computation time over 10-second windows of the IRSS datasets. Initial publication of the method proposed in this chapter can be found in [35].
4.2 Correcting Grid Selection Errors in Multi-Resolution MLE

The use of multi-resolution MLE comes with performance trade-offs. Specifically, the multi-resolution MLE is liable to be mislead by local maxima within the search field [29]. For example, if an early search layer chooses an incorrect grid region, the following search layers get stuck localizing on an area that does not contain the source. Define such an event as a 'grid selection error'. Grid selection errors are liable to occur when the ground truth is located near a boundary between two or more grid regions. In this situation, the log-likelihood values computed for the two or more regions are close in value, leaving room for the signal variance to cause incorrect selection. Grid selection errors have the potential to add significant error to the multi-resolution MLE localization. Specifically, the localization error is likely to be much larger when using a small grid size (such as $N_x = N_y = 2$) than when using a large grid size (such as $32 \times 32$) since fewer points are compared over the same area and the distance between search points is greater.

To mitigate grid selection errors, we propose to expand the search area in each direction by a factor, $f$, after each layer, $l$, of the multi-resolution algorithm given in Algorithm 3. This expansion is done such that the ranges of the next layer of the multi-resolution search, $[x'_{\text{min}}, x'_{\text{max}}]$ and $[y'_{\text{min}}, y'_{\text{max}}]$, are now defined by

$$
x'_{\text{min}} = \hat{x}_{\text{ml}} - \left(\frac{1}{2} + f\right) \left(\frac{x_{\text{max}} - x_{\text{min}}}{N_x}\right)
$$

$$
x'_{\text{max}} = \hat{x}_{\text{ml}} + \left(\frac{1}{2} + f\right) \left(\frac{x_{\text{max}} - x_{\text{min}}}{N_x}\right)
$$

$$
y'_{\text{min}} = \hat{y}_{\text{ml}} - \left(\frac{1}{2} + f\right) \left(\frac{y_{\text{max}} - y_{\text{min}}}{N_y}\right)
$$

$$
y'_{\text{max}} = \hat{y}_{\text{ml}} + \left(\frac{1}{2} + f\right) \left(\frac{y_{\text{max}} - y_{\text{min}}}{N_y}\right).
$$

This modified version of the multi-resolution algorithm is provided in Algorithm 4.

By expanding the search area, the proposed algorithm includes areas close to the border of the selected grid point from the previous iteration. Doing so lets the search ranges of the subsequent layers to move outside the range of the initially selected grid region, effectively reducing the effect of the grid selection error. Grid expansion allows the multi-resolution grid search to span a much larger range of the detector field than a normal multi-resolution localization. An example of using grid expansion to fix an incorrect initial grid selection made in the first layer of resolution is provided in Figure 4.1. A similar multi-resolution method that contains built-in error tolerance is used for
image processing in [34].

One drawback of using grid expansion is that grid resolution is lost due to expanding the search region every layer. Hence, more layers must be performed to achieve the same resolution as a standard multi-resolution grid search. Note that two extra layers were required to achieve a similar final grid resolution in Figure 4.1. The number of required additional layers varies depending on the size of the field and the specified grid dimensions.

### 4.3 Comparison of MLE Algorithms

#### 4.3.1 Description of Experiment

The goal of this experiment was to investigate the trade-offs between single-resolution MLE, multi-resolution MLE, and multi-resolution MLE with grid expansion. These algorithms are functionally described by Algorithms 2, 3, and 4, respectively. Specifically, we wanted to determine whether the use of grid expansion allows the multi-resolution MLE to achieve accuracy similar to the single-resolution MLE.

We tested the three algorithms over all runs of the seven IRSS datasets listed in Table 2.1. For each run, we perform the corresponding MLE localization algorithm over a sliding 10-second time window ($w = 10$). For each window, we recorded the localization error of the final grid-search layer (as distance from the MLE coordinates to the ground truth coordinates) and the computation time of the algorithm. For the D-01 dataset, which contains a moving source, we used the average source location over the time window for the ground truth coordinates.
4.3.2 Selection of Algorithm Parameters

Table 4.1 provides a compiled list of all necessary parameters used for the specified algorithms in this experiment. The remainder of this section explains our choice of the values within the table.

We define the initial spatial parameter ranges \([x_{\text{min}}, x_{\text{max}}]\) and \([x_{\text{min}}, x_{\text{max}}]\) by the smallest bounding rectangle that includes each detector. Thus, the ranges are \([\min(x), \max(x)]\) and \([\min(y), \max(y)]\), respectively. For the single-resolution MLE, we consider \(32 \times 32\) grid, and for the standard multi-resolution MLE, we consider a \(2 \times 2\) grid with 5 iterations. For the multi-resolution MLE with grid expansion, we consider a \(2 \times 2\) grid with 7 iterations and an expansion factor of 1/8th. We perform 7 iterations to keep the localization resolution similar to that of the standard MLE localizations. For a complete comparison, Table 4.2 provides the exact dimensions of each individual grid cell at the specified layer of the respective algorithm. Note that dimensions of the final selected grid region for the specified single-res MLE, multi-res MLE, and multi-res MLE with grid expansion, are 24.6 cm by 24.2 cm, 24.6 cm by 24.2 cm, and 23.5 cm by 23.1 cm, respectively. These dimensions were the closest we could get the multi-res with grid expansion algorithm to the others with the provided parameters.

The source strength and source type vary between datasets, so no single hard-coded source range is efficient for all scenarios. As such, we developed an automatic source-ranging method that computes a sufficiently large source strength using the highest detector count in the current window. Our metric for the highest possible source strength in the field is given as

\[
A_{\text{max}} \leq \max(c) \times r_{\text{max}}^2,
\]  

(4.2)

where \(A_{\text{max}}\) is the upper bound on our source range, \(\max(c)\) is the highest detector count value within the current time window among all detectors, and \(r_{\text{max}}\) is the length of the hypotenuse of the entire search field. We found that \(A_{\text{max}}\) is typically much larger than needed to bound our source range and it can be further attenuated on a case by case basis, however further investigation needs to be done to provide a more reliable heuristic. With our implementation, we search over 100 evenly spaced source strengths from 1 to the value of the metric in (4.2) divided by 10. Note that the number or source values searched will drastically affect the computation time of the localization. We explore improving this source range selection heuristic in Chapter 5.
The background estimates, $\hat{B}$, were determined by taking the mean value of all detectors over all runs of an IRSS Background dataset, which is the same field set up as those shown in 2.3, but without any radiation source present. The background value differs between Cs-137 and Co-57, whose signatures correspond to bins 12 and 3 in Figure 2.4. The background values (in counts/second) used for each isotope are provided in Table 4.1.
(a) Multi-Resolution MLE performed with a 2x2 grid for 5 layers of resolution ($N_x = N_y = 32, M = 5$)

(b) Multi-Resolution MLE performed with a 2x2 grid for 7 layers of resolution and 1/8 grid expansion per layer ($N_x = N_y = 32, M = 7, f = 1/8$)

Figure 4.1: An example of grid expansion lowering the effects of early errors made by a multi-resolution grid search
Table 4.1: Parameters Used for MLE Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_A^a$</th>
<th>$M$</th>
<th>$f$</th>
<th>$x_{min}, cm$</th>
<th>$x_{max}, cm$</th>
<th>$y_{min}, cm$</th>
<th>$y_{max}, cm$</th>
<th>$A_{min}, cps$</th>
<th>$A_{max}, cps$</th>
<th>$B, cps^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-Res</td>
<td>32</td>
<td>32</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>min(x)</td>
<td>max(x)</td>
<td>min(y)</td>
<td>max(y)</td>
<td>1</td>
<td>$\frac{\max(c)}{10} \times r_{max}^2$</td>
<td>2.45 or 35.71</td>
</tr>
<tr>
<td>Multi-Res</td>
<td>2</td>
<td>2</td>
<td>100</td>
<td>5</td>
<td>0</td>
<td>min(x)</td>
<td>max(x)</td>
<td>min(y)</td>
<td>max(y)</td>
<td>1</td>
<td>$\frac{\max(c)}{10} \times r_{max}^2$</td>
<td>2.45 or 35.71</td>
</tr>
<tr>
<td>Multi-Res w/ GE</td>
<td>2</td>
<td>2</td>
<td>100</td>
<td>7</td>
<td>1/8</td>
<td>min(x)</td>
<td>max(x)</td>
<td>min(y)</td>
<td>max(y)</td>
<td>1</td>
<td>$\frac{\max(c)}{10} \times r_{max}^2$</td>
<td>2.45 or 35.71</td>
</tr>
</tbody>
</table>

$^a$ For these algorithms, the autostop method stops the search at the given coordinates if a maximum is reached (see Section 3.3.1 for details). Thus, the number $N_A$ defines the resolution of the search and not the number of search points that will ultimately be used for a given time window.

$^b$ The background value used depends on the isotope of the source. 2.45 was used for Cs-137 datasets and 35.71 was used for Co-57 datasets.

Table 4.2: Dimensions of Individual Grid Cells at Specified Layer of Resolution for MLE Algorithms

<table>
<thead>
<tr>
<th>Layer #</th>
<th>Single-Res</th>
<th>Multi-Res</th>
<th>Multi-Res w/ GE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Width, cm</td>
<td>Height, cm</td>
<td>Width, cm</td>
</tr>
<tr>
<td>1</td>
<td>24.59</td>
<td>24.19</td>
<td>393.50</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>196.75</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>98.38</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>49.19</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>24.59</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Blue numbers indicate the dimensions of the grid region for the selected MLE value in final layer of resolution.
4.3.3 Results

We implemented the algorithms in Matlab on a machine running Linux Mint 17 with an Intel Core i5-3470 3.20 GHz processor and 8Gb of RAM. Table 4.3 provides the mean localization error of each MLE algorithm by dataset, along with 95% confidence of the mean. Table 4.4 provides the mean computation time of each MLE algorithm by dataset. We left out the confidence intervals for computation times because variance between runs was negligible and dependent on computer hardware, not the algorithm. Table 4.5 provides the exact number of 10-second time windows within all runs of each IRSS dataset. The total averages reported at the bottom row of Tables 4.3 and 4.4 are determined by averaging the results of each window of all datasets, not by taking the average of the means presented in the rows above.

Table 4.3: Mean Localization Error of Different MLE Algorithms (10s Windows)

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, cm</th>
<th>Multi-Res, cm</th>
<th>Multi-Res w/ GE, cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>24.27 ± 0.14</td>
<td>38.38 ± 2.91</td>
<td>22.48 ± 1.96</td>
</tr>
<tr>
<td>A-07</td>
<td>28.49 ± 1.24</td>
<td>28.95 ± 1.32</td>
<td>41.73 ± 0.81</td>
</tr>
<tr>
<td>C-01</td>
<td>42.75 ± 1.92</td>
<td>67.37 ± 6.44</td>
<td>51.73 ± 2.88</td>
</tr>
<tr>
<td>C-02</td>
<td>17.25 ± 0.47</td>
<td>15.09 ± 0.39</td>
<td>11.79 ± 1.11</td>
</tr>
<tr>
<td>C-03</td>
<td>18.38 ± 0.76</td>
<td>20.28 ± 0.90</td>
<td>16.84 ± 1.01</td>
</tr>
<tr>
<td>C-04</td>
<td>107.01 ± 2.52</td>
<td>145.84 ± 1.22</td>
<td>80.66 ± 2.68</td>
</tr>
<tr>
<td>D-01</td>
<td>30.17 ± 2.22</td>
<td>34.90 ± 2.66</td>
<td>38.39 ± 2.82</td>
</tr>
<tr>
<td>Total Avg</td>
<td>37.56 ± 0.85</td>
<td>49.33 ± 1.32</td>
<td>36.68 ± 0.76</td>
</tr>
</tbody>
</table>

Values reported with ± 95% Confidence Interval

Table 4.4: Mean Window Computation Times for Different MLE Algorithms (10s Windows)

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, s</th>
<th>Multi-Res, s</th>
<th>Multi-Res w/ GE, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>1.2865</td>
<td>0.0871</td>
<td>0.0937</td>
</tr>
<tr>
<td>A-07</td>
<td>1.2434</td>
<td>0.0866</td>
<td>0.0929</td>
</tr>
<tr>
<td>C-01</td>
<td>1.2359</td>
<td>0.0883</td>
<td>0.0959</td>
</tr>
<tr>
<td>C-02</td>
<td>1.2386</td>
<td>0.0879</td>
<td>0.0952</td>
</tr>
<tr>
<td>C-03</td>
<td>1.2383</td>
<td>0.0875</td>
<td>0.0951</td>
</tr>
<tr>
<td>C-04</td>
<td>1.2325</td>
<td>0.0871</td>
<td>0.0947</td>
</tr>
<tr>
<td>D-01</td>
<td>1.2374</td>
<td>0.0884</td>
<td>0.0951</td>
</tr>
<tr>
<td>Total Avg</td>
<td>1.2457</td>
<td>0.0874</td>
<td>0.0945</td>
</tr>
</tbody>
</table>

Figure 4.2 provides a visual representation of the total average error of each MLE algorithm by the area of the individual grid region in each layer. Compare the area of the grid cell in the Figure with the values in Table 4.2 to determine the corresponding layer of resolution. For both multi-res algorithms, the progression of layers starts in the upper-right of the plot and moves towards the bottom-left to error of the final layer.
Table 4.5: Number of 10-Second Time Windows Evaluated in Each IRSS Dataset

<table>
<thead>
<tr>
<th>Dataset</th>
<th># of Windows</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>990</td>
</tr>
<tr>
<td>A-07</td>
<td>981</td>
</tr>
<tr>
<td>C-01</td>
<td>770</td>
</tr>
<tr>
<td>C-02</td>
<td>770</td>
</tr>
<tr>
<td>C-03</td>
<td>880</td>
</tr>
<tr>
<td>C-04</td>
<td>763</td>
</tr>
<tr>
<td>D-01</td>
<td>330</td>
</tr>
<tr>
<td>Total</td>
<td>5484</td>
</tr>
</tbody>
</table>

Figure 4.2: Overall error of MLE algorithms by area of individual grid region

4.3.4 Discussion

Table 4.3 indicates that, on average, the multi-res w/ GE algorithm had significantly lower localization error than the standard multi-res algorithm and slightly lower localization error than the single-res algorithms. This result extends somewhat to the individual datasets, where the GE algorithm had less localization error than the multi-res algorithm for all datasets except A-07 and
D-01, and had lower localization error than the single-res algorithm for all datasets except A-07, C-01, and D-01. It is interesting that the multi-res w/ GE algorithm performed the worst on the D-01 dataset, which contained a moving source. All MLE algorithms seemed to have the most difficulty, by far, with the C-04 dataset, where the source was the farthest from the center of the field (see 2.3). In this case the multi-res w/ GE algorithm performed significantly better than both the multi-res and single-res algorithms. The improvement of multi-res with GE over standard multi-res is most prominent in Figure 4.2. The average localization error of the multi-res with GE was lower than the error of the standard multi-res algorithm over all comparable grid resolutions with the given search parameters.

Computationally, the results were as expected from the discussion midway through Section 3.4. Table 4.4 indicates that the single-res MLE took significantly longer than both the multi-res and multi-res w/ GE algorithms to compute per 10-second window. The two extra layers required by the resolution reduction of grid expansion cause the multi-res w/ GE algorithm to take slightly longer on average than the standard multi-res algorithm. That said, it is still a significant computational speedup in comparison to the single-res algorithm.

Of course, the results presented only show that grid expansion improves the standard multi-res MLE algorithm for the exact parameters listed in Table 4.1, in the specific IRSS datasets. A more thorough examination of possible algorithm parameters and analysis of results is necessary to make broader conclusions on the effectiveness of grid expansion for improving the multi-resolution MLE algorithm.

4.4 Summary

In this chapter we presented the drawbacks of the multi-resolution MLE algorithm, and provided a possible solution in the form of grid expansion. Specifically, the use of grid expansion was meant to mitigate the negative effects that initial grid selection errors have on the standard multi-resolution MLE grid search. We compared the performance of specific implementations of the single-resolution, multi-resolution, and multi-resolution with grid expansion MLE algorithms using the seven indoor IRSS datasets. We found that, on average, the use of grid expansion provided a moderate decrease in localization error over the standard multi-resolution algorithm, while only incurring a relatively small penalty on computational time.
Chapter 5

Regression Estimates for MLE Localization with Unknown Background

5.1 Overview

Previously, we assumed that the background intensity level was known beforehand. This assumption required us to compute background values for the IRSS datasets before performing the MLE localization. In a real scenario, the assumption of a known prior background may not always be true or effective, given that background levels can change over time.

In this chapter we develop a mid-point MLE localization that treats the background as an unknown parameter. To determine intensity parameter search ranges of this new algorithm, we present a linear regression (LR) model to estimate source the and background intensities. We then re-do the experiment from Chapter 4 using the new algorithm, and compare the results. The work in this chapter is based on ideas that were presented in [36] and briefly described in [1].
5.2 MLE with Search over Background Intensity

Recall the radiation and network model from Chapter 3. In this model, the knowns are the background intensity, $B$, detector number, $N$, detector x-coordinates, $x = [x_1, x_2, \ldots, x_N]^T$, detector y-coordinates, $y = [y_1, y_2, \ldots, y_N]^T$, and counts/second values of the $i$th detector, $c_i = [c_{i1}, c_{i2}, \ldots, c_{iw}]^T$, over time window, $w$, for all $i \in \{1, 2, \ldots, N\}$. The unknowns are the ground truth parameters, $x_g$, $y_g$, and $A_g$. Henceforth, we consider a model in which the background value is also unknown. In this scenario, the ground truth parameter vector becomes $\theta_g = [x_g, y_g, A_g, B_g]^T$, and the ML estimate of the parameters is denoted $\hat{\theta}_{ml} = [\hat{x}_{ml}, \hat{y}_{ml}, \hat{A}_{ml}, \hat{B}_{ml}]^T$.

With this addition we are now required to search over background values in addition to x-coordinate, y-coordinate, and source intensity. The new set of parameters for a mid-point grid search are given by

\[
\begin{align*}
x_m &= x_{\text{min}} + \left(m - \frac{1}{2}\right) \frac{x_{\text{max}} - x_{\text{min}}}{N_x} \\
y_n &= y_{\text{min}} + \left(n - \frac{1}{2}\right) \frac{y_{\text{max}} - y_{\text{min}}}{N_y} \\
A_p &= A_{\text{min}} + \left(p - \frac{1}{2}\right) \frac{A_{\text{max}} - A_{\text{min}}}{N_A} \\
B_q &= B_{\text{min}} + \left(q - \frac{1}{2}\right) \frac{B_{\text{max}} - B_{\text{min}}}{N_B},
\end{align*}
\]

where the search over the background intensity covers range $[B_{\text{min}}, B_{\text{max}}]$ over $N_B$ evenly spaced values. The rest of the parameters in (5.1) are the same as those described in (3.9). The new mid-point MLE with background search is provided in Algorithm 5. Observe that we also extended the autostop technique developed in 3.3.1 to the background search.

5.3 Regression Model for Source and Background Estimates

In this section we develop a linear regression model for estimating the source intensity, $A$, and background intensity, $B$, using a given set of detector measurements. These estimates are intended for use to bootstrap the intensity search ranges $[A_{\text{min}}, A_{\text{max}}]$ and $[B_{\text{min}}, B_{\text{max}}]$ over the MLE grid search described in Algorithm 5. The final result of this section is provided in Algorithm 6.
Algorithm 5 MLE Localization using Mid-Point Grid Search with Search over Background and with Autostop

1: $L_{\text{max}} = \text{arbitrarily low value}$
2: for $m = 1$ to $N_x$ do
3:  for $n = 1$ to $N_y$ do
4:  $L_A = \text{arbitrarily low value}$
5:  for $p = 1$ to $N_A$ do
6:  $L_B = \text{arbitrarily low value}$
7:  for $q = 1$ to $N_B$ do
8:    $x_m = x_{\text{min}} + (m - \frac{1}{2}) \frac{x_{\text{max}} - x_{\text{min}}}{N_x}$
9:    $y_n = y_{\text{min}} + (n - \frac{1}{2}) \frac{y_{\text{max}} - y_{\text{min}}}{N_y}$
10:   $A_p = A_{\text{min}} + (p - \frac{1}{2}) \frac{A_{\text{max}} - A_{\text{min}}}{N_A}$
11:   $B_q = B_{\text{min}} + (q - \frac{1}{2}) \frac{B_{\text{max}} - B_{\text{min}}}{N_B}$
12:   for each $i$th detector do
13:     $\lambda_i = \frac{A_p}{(x_i - x_m)^2 + (y_i - y_n)^2} + B_q$
14:   end for
15:   $L = \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \ln \lambda_i - \lambda_i$
16:   if $(L > L_B)$ then
17:     $L_B \leftarrow L$
18:     $B_B \leftarrow B_q$
19:   else
20:     break
21:   end if
22: end for
23: if $L_B > L_A$ then
24:   $L_A \leftarrow L_B$
25:   $B_A \leftarrow B_B$
26: else
27:   break
28: end if
29: end for
30: if $L_A > L_{\text{max}}$ then
31:   $L_{\text{max}} \leftarrow L_A$
32:   $\hat{x}_{\text{ml}} \leftarrow x_m$
33:   $\hat{y}_{\text{ml}} \leftarrow y_n$
34:   $\hat{A}_{\text{ml}} \leftarrow A_p$
35:   $\hat{B}_{\text{ml}} \leftarrow B_A$
36: end if
37: end for
38: end for
39: return $\hat{\theta}_{\text{ml}} = [\hat{x}_{\text{ml}}, \hat{y}_{\text{ml}}, \hat{A}_{\text{ml}}, \hat{B}_{\text{ml}}]^T$

Define sample mean of measurements of detector $m_i$,

\[
\bar{c}_i = \frac{1}{w} \sum_{j=1}^{w} c_{ij} \tag{5.2}
\]
and distance from detector \( m_i \) to ground truth,

\[
d_i = \sqrt{(x_i - x_g)^2 + (y_i - y_g)^2}.
\] (5.3)

since the ground truth coordinates \((x_g, y_g)\) are not known, all \( d_i \) are unknown.

The sample mean of a Poisson process is the Maximum Likelihood estimate of the Poisson parameter [19]. Since each measurement of detector \( m_i, c_{ij} \), is a sample of the same Poisson process with parameter \( \lambda_i \), \( \bar{\tau}_i \) is a Maximum Likelihood estimate of \( \lambda_i \). Furthermore, the ML estimator is consistent, which means that the estimated value approaches the true value as the sample size increases,

\[
\lim_{w \to \infty} \bar{\tau}_i = \lambda_i. \tag{5.4}
\]

The consistency relationship in Equation (5.4) can be expressed as

\[
\lim_{w \to \infty} \bar{\tau}_i = \frac{1}{d_i^2} A_g + B_g, \tag{5.5}
\]

where \( \lambda_i \) is replaced with the right side of Equation (3.1) and previously known \( B \) is replaced with unknown \( B_g \). Equation (5.5) provides the basis for the regression estimates we derive in this section.

When background radiation is constant over the time window, \( w \), Equation (5.5) is linear with respect to \( \frac{1}{d_i^2} \). If the window size \( w \) is adequate, then we can estimate the source and background intensities using a least-squares regression model with \( \bar{\tau}_i \) and \( \frac{1}{d_i^2} \) as dependent variable and regressor [37]:

\[
[\hat{B}_g, \hat{A}_g]^T = (D^T D)^{-1} D^T \bar{\tau}, \tag{5.6}
\]

where \( \bar{\tau} = [\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_N]^T \), and the matrix \( D \) is given by

\[
D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\
\frac{1}{d_1^2} & \frac{1}{d_2^2} & \cdots & \frac{1}{d_N^2} \end{bmatrix}^T.
\]
5.3.1 Counts-Weighted Average Location Estimate

The regression model (5.6) requires knowledge of either the source location or distances \(d_i\), which are not known \textit{a priori}. Since regression is designed to infer the best estimate from sets of noise corrupted data, it suffices to find an initial source location estimate that can be used to calculate distance estimates \(\hat{d}_i\) for use in (5.6).

Define the counts-weighted average for the initial source location estimate

\[
\begin{bmatrix}
\hat{x}_g \\
\hat{y}_g
\end{bmatrix} = \left(v^T \ell\right)^{-1} X \ell,
\]

where \(v = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^P\), and \(X\) is the matrix of detector coordinates

\[
X = \begin{bmatrix}
x_1 & x_2 & \cdots & x_P \\
y_1 & y_2 & \cdots & y_P
\end{bmatrix},
\]

Note that the components of Equation (5.7) are individually represented by

\[
\hat{x}_g = \frac{\sum_{i=1}^{P} \ell_i x_i}{\sum_{i=1}^{P} \ell_i},
\]

and

\[
\hat{y}_g = \frac{\sum_{i=1}^{P} \ell_i y_i}{\sum_{i=1}^{P} \ell_i}.
\]

We refer to the estimates in Equations (5.7), (5.8), and (5.9) as the Counts-Weighted average (CW) estimate of the ground truth coordinates. For these equations, integer \(P\) is the cardinality of a subset of detectors \(S \subset M\) deemed \textit{significant}. Significant detectors are selected as those whose average counts are likely to be greater than the cumulative average counts of all detectors. This separation is done to mitigate bias that would be introduced by detectors with counts primarily due to background. Since the mean detector measurements are ML estimates of Poisson means, we perform this selection using a chi-squared test with one degree of freedom. The general chi-squared
The test statistic for comparing two Poisson means is given by
\[ \chi^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2}, \] (5.10)
where \( O_n \) and \( E_n \) are the observed and expected values of group \( n \) for \( n \in \{1, 2\} \). Define group 1 as the sum of the measurements of potentially significant detector, \( m_k \), and group 2 as the sum of the measurements of all other detectors, \( m_i \), for \( i \in \{1, 2, ..., N : i \neq k\} \). With the measurements accumulated over time window, \( w \), formulate the hypotheses for the significance of a detector as follows:
\[ H_0 : w\lambda_k = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \] and \( H_1 : w\lambda_k \neq \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \). The components of (5.10) corresponding to this hypothesis test are:
\[ O_1 = \sum_{j=1}^{w} c_{kj} \]
\[ O_2 = \sum_{i=1, i\neq k}^{N} \sum_{j=1}^{w} c_{ij} \]
\[ E_1 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \]
\[ E_2 = \frac{N - 1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij}. \]
Plug these values into (5.10) and the chi-squared test statistic can be given the form
\[ \chi^2 = \frac{N^2 (\sum_{j=1}^{w} c_{kj} - \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij})^2}{\sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij}}. \] (5.11)
To perform the test, compare \( \chi^2 \) with a critical value determined by the chi-squared distribution at a given confidence level and one degree-of-freedom. If \( \chi^2 \) is greater than the critical value, then reject \( H_0 \).

Note that this chi-squared test is a comparison of Poisson means, and detectors that have both significantly large and small values will cause the rejection of \( H_0 \). We are only interested in detectors that are significantly larger than the average. Thus, if the chi-squared test rejects \( H_0 \) for
detector \( m_k \), we then employ the decision rule,

\[
    d(\bar{c}_k) = \begin{cases} 
    1 & \text{if } \bar{c}_k > \frac{1}{N} \sum_{i=1}^{N} c_i \\
    0 & \text{if } \bar{c}_k \leq \frac{1}{N} \sum_{i=1}^{N} c_i 
    \end{cases},
\]

(5.12)

to filter out the detectors that have significantly low counts values. Thus, \( m_k \in S \) only if the chi-squared test rejects \( H_0 \) and \( d(\bar{c}_k) = 1 \).

The detectors in \( S \) are intended as those likely to be near the source since their counts are generally higher than others within the field. The selection method will only work with a constant background over the time window and one source within the field. These detectors are the only ones selected for use in (5.7) because the counts-weighted location estimates are inherently biased towards the locations of the detectors included.

The full process to compute the regression intensity estimates is then:

1. Determine set of significant detectors using (5.11) and (5.12)

2. Use the set of significant detectors in (5.7) to find the CW location estimates

3. Compute the distance from all detectors to the CW estimates and plug into (5.6) to determine source and background intensity estimates

This process is also provided in Algorithm 6. Note that the algorithm uses a confidence level of 90% and one degree-of-freedom for the chi-squared critical value. Figure 5.1 shows an example of a linear regression model built over a 10 second time window \((w = 10)\) using the algorithm.

### 5.3.2 Regression Estimates for MLE Localization with Background Search

Recall that the coordinate search ranges \([x_{min}, x_{max}]\) and \([y_{min}, y_{max}]\) are naturally defined by the bounds of the detector field. The intensity search ranges \([A_{min}, A_{max}]\) and \([B_{min}, B_{max}]\) required by Algorithm 5 have no such natural bounds. In the previous chapter, we used a large heuristic (4.2) to determine a very large search range \([A_{min}, A_{max}]\), which was less than ideal. Consider the linear regression (LR) estimates for the source and background intensity, \( A_g \) and \( B_g \), determined using Algorithm 6. These estimates give us a solid starting point to build the search ranges \([A_{min}, A_{max}]\) and \([B_{min}, B_{max}]\) for the MLE grid search described in Algorithm 5. If these LR estimates are close to the ML estimates, then they can be used to initialize the search ranges.
Algorithm 6 Linear Regression Intensity Estimation using Counts-Weighted Average
(chi-squared critical value for 90% confidence and one degree-of-freedom)

1: \( \chi^2_{\text{crit}} = 2.7055 \)
2: \( S = \emptyset \)
3: for \( k = 1 \) to \( N \) do
4: \( \chi^2 = N \left( \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} - \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{w} c_{ij} \right)^2 \)
5: if \( (\chi^2 > \chi^2_{\text{crit}}) \) and \( \left( \frac{1}{N} \sum_{i=1}^{N} \tau_i > \bar{\tau} \right) \) then
6: \( S \leftarrow S \cup \{ k \} \)
7: end if
8: end for
9: \( \hat{x}_g = \frac{\sum_{i \in S} \tau_i x_i}{\sum_{i \in S} \tau_i} \)
10: \( \hat{y}_g = \frac{\sum_{i \in S} \tau_i y_i}{\sum_{i \in S} \tau_i} \)
11: for \( i = 1 \) to \( N \) do
12: \( \hat{d}_i = \sqrt{(x_i - \hat{x}_g)^2 + (y_i - \hat{y}_g)} \)
13: end for
14: \( D = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \frac{1}{\hat{d}_1^2} & \frac{1}{\hat{d}_2^2} & \cdots & \frac{1}{\hat{d}_N^2} \end{bmatrix}^T \)
15: \( [\hat{B}_g, \hat{A}_g]^T = (D^T D)^{-1} D^T \pi \)
16: return \( \hat{B}_g, \hat{A}_g \)

over the respective parameters. For example, we can set the intensity search range to be centered at the LR estimate and then search some over some range about the estimate. In our case, we set the bounds of the source search range to

\[
A_{\text{min}} = \frac{1}{10} \hat{A}_g \\
A_{\text{max}} = 10 \hat{A}_g,
\]

(5.13)

and the bounds of the background search range

\[
B_{\text{min}} = \frac{1}{2} \hat{B}_g \\
B_{\text{max}} = \frac{3}{2} \hat{B}_g.
\]

(5.14)

We selected these bounds simply because the source intensity is typically much higher order than the background intensity (see Figure 5.1), thus the source parameter should cover a wider range of values. Still, these bounds are heuristic and their tuning is left for future work. Furthermore, tests using these bounds should still provide insight into how well the regression estimates perform in general.

The process to perform MLE localization using the LR estimate bounds is then:

1. perform Algorithm 6 to get the LR intensity estimates, \( \hat{A}_g \) and \( \hat{B}_g \),
2. determine source intensity bounds \([A_{\text{min}}, A_{\text{max}}]\) using 5.13 and background intensity search
bounds $[B_{min}, B_{max}]$ using 5.14,

3. perform MLE localization using Algorithm 5.

We refer to this process as an MLE localization with LR initialization. Note that the method is for a single-resolution MLE localization. To modify this method for multi-resolution (Algorithm 3) or multi-resolution with grid expansion (Algorithm 4) simply use their respective algorithms in Step 3 of the process and then replace the call to Algorithm 2 on line 2 with a call to Algorithm 5.

### 5.4 Repeat of Multi-Resolution Experiment

To test the performance of the MLE algorithm with LR initialization developed in the prior section, we performed the same experiment as that in Section 4.3 with the core algorithm replaced with the one of interest. By repeating the same test, we are able to compare results between sections and determine how LR Initialization affects the localization error and computation time of the MLE localization algorithms.

The exact parameters for the algorithms tested are given in Tables 5.1 and 5.2. See Section
4.3 for explanation of the parameters used in Table 5.1.
Table 5.1: Resolution Parameters Used for MLE Algorithms using LR Initialization

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_A^a$</th>
<th>$N_B^b$</th>
<th>$M$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-Res</td>
<td>32</td>
<td>32</td>
<td>100</td>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Multi-Res</td>
<td>2</td>
<td>2</td>
<td>100</td>
<td>10</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Multi-Res w/ GE</td>
<td>2</td>
<td>2</td>
<td>100</td>
<td>10</td>
<td>7</td>
<td>1/8</td>
</tr>
</tbody>
</table>

$^a$ For these algorithms, the autostop method stops the search at the given coordinates if a maximum is reached (see Section 3.3.1 for details). Thus, the numbers $N_A$ and $N_B$ define the resolution of the search over those parameters and not the number of search points that will ultimately be used.

Table 5.2: Search Range Parameters Used for MLE Algorithms using LR Initialization

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$x_{min}, cm$</th>
<th>$x_{max}, cm$</th>
<th>$y_{min}, cm$</th>
<th>$y_{max}, cm$</th>
<th>$A_{min}, cps$</th>
<th>$A_{max}, cps$</th>
<th>$B_{min}, cps$</th>
<th>$B_{max}, cps$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-Res</td>
<td>min($x$)</td>
<td>max($x$)</td>
<td>min($y$)</td>
<td>max($y$)</td>
<td>$\frac{1}{10}A_g^a$</td>
<td>$10A_g$</td>
<td>$\frac{1}{2}B_g$</td>
<td>$\frac{1}{2}B_g$</td>
</tr>
<tr>
<td>Multi-Res</td>
<td>min($x$)</td>
<td>max($x$)</td>
<td>min($y$)</td>
<td>max($y$)</td>
<td>$\frac{1}{10}A_g$</td>
<td>$10A_g$</td>
<td>$\frac{1}{2}B_g$</td>
<td>$\frac{1}{2}B_g$</td>
</tr>
<tr>
<td>Multi-Res w/ GE</td>
<td>min($x$)</td>
<td>max($x$)</td>
<td>min($y$)</td>
<td>max($y$)</td>
<td>$\frac{1}{10}A_g$</td>
<td>$10A_g$</td>
<td>$\frac{1}{2}B_g$</td>
<td>$\frac{1}{2}B_g$</td>
</tr>
</tbody>
</table>

$^a$ $A_g$ and $B_g$ are source and background intensity estimates computed using Algorithm 6.
5.4.1 Results

We implemented the algorithms in Matlab on a machine running Linux Mint 17 with an Intel Core i5-3470 3.20 GHz processor and 8Gb of RAM. Table 5.3 provides the mean localization error of each MLE algorithm by dataset, along with 95% confidence of the mean. Table 5.4 provides the mean computation time of each MLE algorithm by dataset. Tables 5.5 and 5.6 provide the absolute and percent change of the localization error that was incurred by using LR initialization on the MLE algorithms. These values were determined by comparing Tables 5.3 and 5.4 with Tables 4.3 and 4.4 from the results in Section 4.3.

As before, the total averages reported at the bottom row of Tables 5.3, 5.4, 5.5, 5.6, 5.7, and 5.8 were determined by averaging the results of each window of all datasets, not by taking the average of the means presented in the above rows. See Table 4.5 for the exact number of 10-second time windows within all runs of each IRSS dataset.

Table 5.3: Mean Localization Error of Different MLE Algorithms using LR Initialization (10s Windows)

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, cm</th>
<th>Multi-Res, cm</th>
<th>Multi-Res w/ GE, cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>25.32 ± 1.14</td>
<td>25.33 ± 1.37</td>
<td>19.57 ± 1.51</td>
</tr>
<tr>
<td>A-07</td>
<td>38.94 ± 1.46</td>
<td>34.13 ± 1.20</td>
<td>36.60 ± 1.66</td>
</tr>
<tr>
<td>C-01</td>
<td>57.94 ± 2.78</td>
<td>60.12 ± 4.37</td>
<td>61.22 ± 2.98</td>
</tr>
<tr>
<td>C-02</td>
<td>17.50 ± 0.49</td>
<td>15.85 ± 0.63</td>
<td>11.96 ± 1.06</td>
</tr>
<tr>
<td>C-03</td>
<td>19.85 ± 1.11</td>
<td>21.44 ± 1.27</td>
<td>18.69 ± 1.32</td>
</tr>
<tr>
<td>C-04</td>
<td>99.72 ± 2.62</td>
<td>124.35 ± 2.29</td>
<td>99.40 ± 2.46</td>
</tr>
<tr>
<td>D-01</td>
<td>33.57 ± 2.28</td>
<td>39.64 ± 3.21</td>
<td>36.88 ± 2.39</td>
</tr>
<tr>
<td>Total Avg</td>
<td>41.21 ± 0.83</td>
<td>44.47 ± 1.07</td>
<td>39.40 ± 0.89</td>
</tr>
</tbody>
</table>

Values reported with ± 95% Confidence Interval

Table 5.4: Mean Computation Time of Different MLE Algorithms using LR Initialization (10s Windows)

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, s</th>
<th>Multi-Res, s</th>
<th>Multi-Res w/ GE, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>1.4471</td>
<td>0.0979</td>
<td>0.1107</td>
</tr>
<tr>
<td>A-07</td>
<td>1.2849</td>
<td>0.0940</td>
<td>0.1020</td>
</tr>
<tr>
<td>C-01</td>
<td>1.2780</td>
<td>0.0951</td>
<td>0.1033</td>
</tr>
<tr>
<td>C-02</td>
<td>1.3707</td>
<td>0.0966</td>
<td>0.1045</td>
</tr>
<tr>
<td>C-03</td>
<td>1.2958</td>
<td>0.0956</td>
<td>0.1032</td>
</tr>
<tr>
<td>C-04</td>
<td>1.2706</td>
<td>0.0968</td>
<td>0.1023</td>
</tr>
<tr>
<td>D-01</td>
<td>1.3676</td>
<td>0.1023</td>
<td>0.1062</td>
</tr>
<tr>
<td>Total Avg</td>
<td>1.3300</td>
<td>0.0964</td>
<td>0.1046</td>
</tr>
</tbody>
</table>
Table 5.5: Absolute Change in Localization Error Incurred by using LR Initialization

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, cm</th>
<th>Multi-Res, cm</th>
<th>Multi-Res w/ GE, cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>1.0467</td>
<td>-13.0510</td>
<td>-2.9045</td>
</tr>
<tr>
<td>A-07</td>
<td>10.4409</td>
<td>5.1778</td>
<td>-5.1274</td>
</tr>
<tr>
<td>C-01</td>
<td>15.1982</td>
<td>-7.2570</td>
<td>9.4878</td>
</tr>
<tr>
<td>C-02</td>
<td>0.2594</td>
<td>0.7615</td>
<td>0.1727</td>
</tr>
<tr>
<td>C-03</td>
<td>1.4700</td>
<td>1.1608</td>
<td>1.8425</td>
</tr>
<tr>
<td>C-04</td>
<td>-7.2942</td>
<td>-21.4894</td>
<td>18.7317</td>
</tr>
<tr>
<td>D-01</td>
<td>3.4025</td>
<td>4.7344</td>
<td>-1.5047</td>
</tr>
<tr>
<td>Total Avg</td>
<td>3.65</td>
<td>-4.86</td>
<td>2.72</td>
</tr>
</tbody>
</table>

Beneficial changes are colored blue and detrimental changes are colored red

Table 5.6: Percent Change in Localization Error Incurred by using LR Initialization

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, %</th>
<th>Multi-Res, %</th>
<th>Multi-Res w/ GE, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>4.31</td>
<td>-34.00</td>
<td>-12.92</td>
</tr>
<tr>
<td>A-07</td>
<td>36.64</td>
<td>17.89</td>
<td>-12.29</td>
</tr>
<tr>
<td>C-01</td>
<td>35.56</td>
<td>-10.77</td>
<td>18.34</td>
</tr>
<tr>
<td>C-02</td>
<td>1.50</td>
<td>5.05</td>
<td>1.46</td>
</tr>
<tr>
<td>C-03</td>
<td>8.00</td>
<td>5.72</td>
<td>10.94</td>
</tr>
<tr>
<td>C-04</td>
<td>-6.82</td>
<td>-14.73</td>
<td>23.22</td>
</tr>
<tr>
<td>D-01</td>
<td>11.28</td>
<td>13.56</td>
<td>-3.92</td>
</tr>
<tr>
<td>Total Avg</td>
<td>9.72</td>
<td>-9.85</td>
<td>7.42</td>
</tr>
</tbody>
</table>

Beneficial changes are colored blue and detrimental changes are colored red

5.4.2 Discussion

As evident in Tables 5.5 and 5.6, the use of LR initialization incurred a generally moderate increase of localization error for the single-resolution and multi-resolution with grid expansion algorithms, and a moderate decrease in localization error for the standard multi-resolution algorithm. This result is not ideal, considering the multi-resolution with grid expansion algorithm is the most useful for a real-time scenario. That said, the multi-res w/ GE algorithm with LR initialization still achieves a 39.40 cm localization error on average, which is fairly good given that it does not require a known background.

The computation time differences presented in Tables 5.7 and 5.8, show that the use of LR initialization incurred a penalty of six to ten percent computation time on average. The increase in computation time is expected, given that the MLE with LR initialization requires a search over the background parameter. This extra parameter search is not required by the algorithms used in Chapter 4.
### Table 5.7: Absolute Change in Computation Time Incurred by using LR Initialization

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, s</th>
<th>Multi-Res, s</th>
<th>Multi-Res w/ GE, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>0.1667</td>
<td>0.0109</td>
<td>0.0169</td>
</tr>
<tr>
<td>A-07</td>
<td>0.0415</td>
<td>0.0074</td>
<td>0.0091</td>
</tr>
<tr>
<td>C-01</td>
<td>0.0421</td>
<td>0.0068</td>
<td>0.0073</td>
</tr>
<tr>
<td>C-02</td>
<td>0.1321</td>
<td>0.0087</td>
<td>0.0093</td>
</tr>
<tr>
<td>C-03</td>
<td>0.0575</td>
<td>0.0081</td>
<td>0.0080</td>
</tr>
<tr>
<td>C-04</td>
<td>0.0381</td>
<td>0.0097</td>
<td>0.0075</td>
</tr>
<tr>
<td>D-01</td>
<td>0.1303</td>
<td>0.0139</td>
<td>0.0111</td>
</tr>
<tr>
<td>Total Avg</td>
<td>0.0843</td>
<td>0.0090</td>
<td>0.0101</td>
</tr>
</tbody>
</table>

Beneficial changes are colored blue and detrimental changes are colored red

### Table 5.8: Percent Change in Computation Time Incurred by using LR Initialization

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Single-Res, %</th>
<th>Multi-Res, %</th>
<th>Multi-Res w/ GE, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-04</td>
<td>13.02</td>
<td>12.48</td>
<td>18.06</td>
</tr>
<tr>
<td>A-07</td>
<td>3.34</td>
<td>8.54</td>
<td>9.84</td>
</tr>
<tr>
<td>C-01</td>
<td>3.41</td>
<td>7.70</td>
<td>7.62</td>
</tr>
<tr>
<td>C-02</td>
<td>10.67</td>
<td>9.89</td>
<td>9.80</td>
</tr>
<tr>
<td>C-03</td>
<td>4.64</td>
<td>9.29</td>
<td>8.46</td>
</tr>
<tr>
<td>C-04</td>
<td>3.09</td>
<td>11.14</td>
<td>7.94</td>
</tr>
<tr>
<td>D-01</td>
<td>10.53</td>
<td>15.75</td>
<td>11.63</td>
</tr>
<tr>
<td>Total Avg</td>
<td>6.77</td>
<td>10.30</td>
<td>10.69</td>
</tr>
</tbody>
</table>

Beneficial changes are colored blue and detrimental changes are colored red

### 5.5 Summary

In this chapter we presented an MLE localization algorithm that no longer assumes that the background is known. This new algorithm requires a search over the background intensity parameter, in addition to the searches over the coordinate and source intensity parameters. We presented a linear regression (LR) model to estimate the source and background intensities, and use them to initialize their respective parameter searches for the localization.

We tested the performance of the new MLE localization algorithm with LR initialization by re-doing the experiment from Section 4.3. Comparing results, we found that, generally, the use of the LR estimates and an additional search over the background parameter caused a marginal increase in both localization error and computation time when compared to similar MLE algorithms that did have these modifications.

Of course, our results are nowhere near comprehensive, given that we only tested one specific implementation of the LR initialization. For future work, we should examine the effects of different intensity parameter search bounds, as well as the use of standard hill climbing algorithms to optimize
over source and background.
Chapter 6

Conclusions

6.1 Answering the Research Questions

In this work we presented improvements to MLE localization algorithms for radiation sources. Specifically, we set out to answer the following two questions:

1. Can we mitigate the drawbacks of a multi-resolution MLE localization while keeping the computational benefits?

2. Can we develop an MLE localization algorithm that does not require prior knowledge of background intensity?

6.1.1 The First Question

One of the major drawbacks to a multi-resolution MLE localization is that an incorrect grid selection on the first search layer will cause following iterations to search within an area that does not contain the source, leading to a potentially large localization error. In Chapter 4, we presented a modification to the standard multi-resolution MLE algorithm that attempted to mitigate this issue. Our modification expands the search range of each layer in a multi-resolution MLE by a factor, \( f \), before conducting the grid search. By expanding the search range each layer, this technique allows for correction of incorrect grid selections made in earlier iterations.

We tested the use of grid expansion by performing single-resolution, multi-resolution, and multi-resolution with grid expansion MLE localizations using 10-second windows over several dif-
different IRSS datasets. We found that, in general, the use of grid expansion allowed the localization error to improve over a standard multi-resolution MLE localization to similar final resolution, while only incurring a small penalty on computation time.

6.1.2 The Second Question

The MLE algorithms used in Chapter 4 required a known background estimate prior to the localization, as well as a broad search over the source parameter. In Chapter 5 we presented a Linear Regression (LR) model that can be used to estimate the source and background intensities prior to the MLE localization. We were able to use these estimates to initialize the parameter search ranges over the source and background intensity parameters. With this technique, background is not required to be known beforehand.

We performed the same single-resolution, multi-resolution, and multi-resolution with grid expansion MLE algorithms from the experiment in Chapter 4, with the addition of assuming unknown background and using LR estimates to bootstrap the parameter search. We found that by assuming unknown background and using LR initialization, we incurred a moderate increase in both localization accuracy and computation time in general over the results from Chapter 4.

6.2 Recommendations for Further Research

The conclusions of the experiments in Chapter 4 and 5 are not comprehensive. They are results for a specific set of algorithm parameters for a specific set of data. As such, more algorithm parameters should be tested, including grid sizes other than 2x2, grid expansion factors other than 1/8, and different bounds for the source and background intensity search ranges than $[\frac{1}{10} \hat{A}_g, 10 \hat{A}_g]$ and $[\frac{1}{2} \hat{B}_g, \frac{3}{2} \hat{B}_g]$. Furthermore, we would like to explore the use of optimization algorithms such as Newton’s Method and the Nedler-Mead Simplex Method to optimize over the source and background parameters. Lastly, the algorithms presented in this work need to be compared to existing methods under similar conditions to truly determine their effectiveness.

For more involved endeavors, we would like to explore limits to the coordinate bounds of the search field for the MLE search, and would like to investigate the effect of removing detectors that only measure background radiation from the localization.
Bibliography


