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Reliability in a Dynamic Stochastic Environment

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Reliability in a Dynamic Stochastic Environment

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Abstract

Consider a device or structure whose lifetime under perfect conditions follows a distribution determined by a hazard function. In a perfect world, the distribution of the device is already known, but suppose environmental conditions are imposed. We consider a model where the device is subjected to a series of shocks that are determined by a point process, but allow for the shocks to leave a residual effect as time goes on. Furthermore, we would also allow for the hazard function to have an effect on the intensity function of the point process. We are interested in the effects of this on the original lifetime distribution, as well as performing estimation and inference using this model.

Chapter 1

Overview

1.1 Introduction

Whether it be modeling how long people live or how long a new piece of technology will last, even under perfect circumstances, these things are subject to failure, and the failure time is subject to randomness. The motivation behind modeling lifetime is clear, since the implications behind the failure of say a system that provides power to a city can be quite severe. In reliability, some general assumptions that are made is that the lifetime distribution is continuous and differentiable, which is not completely unreasonable, given that we are operating in continuous time. Thus, it is common to identify a lifetime distribution by its hazard function. Simply put, a hazard function tells us the likelihood of the device failing at a particular time t, given that it has not failed up to time t. This interpretation yields that a hazard function h can be written in the following way:

$$h(t) = \frac{f(t)}{1 - F(t)},$$

where $f(t)$ is the density and $F(t)$ is the distribution function of the lifetime. Assuming $F(0) = 0$, and taking integrals on both sides of this equation from 0 to x, we find that

$$\int_0^x h(t)dt = \int_0^x \frac{f(t)}{1 - F(t)}dt = -\log(1 - F(x)),$$

which gives that $F(x) = 1 - \exp\left(-\int_0^x h(t)dt\right)$. This is a relation that will be continually used throughout the paper.

In many cases, the hazard function can be solved directly from the distribution. For example, consider the Weibull distribution whose distribution we specify by $F(x) = 1 - \exp(-\mu x^k)$, for $k, \mu > 0$. So, with the characterization of the distribution in terms of the hazard function, $\int_0^x h(t)dt = \mu x^k$, and therefore $h(t) = \mu k x^{k-1}$. Generally speaking, for $k < 1$, the hazard function is decreasing, for $k > 1$, the hazard function is increasing, and if $k = 1$ the hazard function is constant, and in particular represents the exponential distribution.

The gamma distribution is another common distribution in reliability, given by

$$F(x) = \int_0^x \frac{\mu(\mu t)^{k-1}}{\Gamma(k)} e^{-\mu t}dt,$$

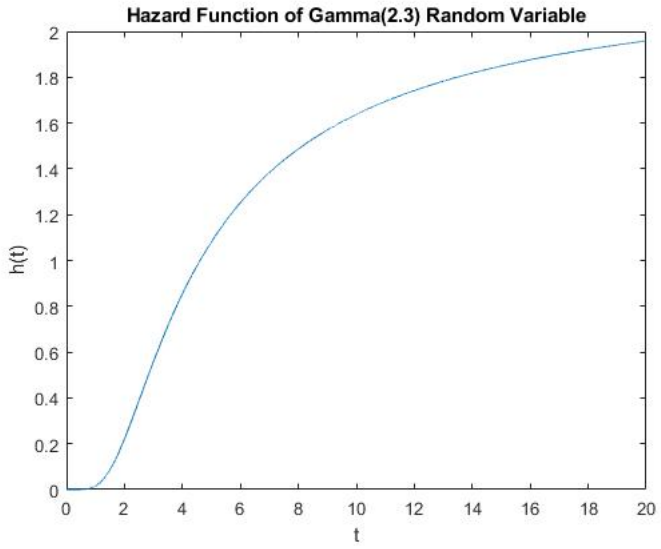
for $\mu, k > 0$. If $k \in \mathbb{N}$, then

$$F(x) = 1 - \sum_{n=0}^{k-1} \frac{(\mu x)^n}{n!} e^{-\mu x}s,$$

so that

$$h(t) = \frac{\mu(\mu t)^{k-1}}{(k-1)! \sum_{n=0}^{k-1} \frac{(\mu t)^n}{n!}}.$$

However, if k is not an integer, then $F(x)$ cannot be computed in closed form and so $h(t)$ does not have an explicit form. But, the hazard function can be computed numerically. Given below is an example of a Gamma hazard function:



Another common distribution in reliability that is used is the lognormal distribution, which, like the gamma, does not have a closed form expression for its distribution, but the hazard function can also be computed numerically.

Since the hazard function has a nice interpretation, completely determines the lifetime distribution, and has guaranteed existence under minimal assumptions, it is commonly used for modeling lifetime. However, in this paper we would like to generalize this notion by letting $\{h(t)\}_{t>0}$ itself be a stochastic process.

For example, suppose you want to model the lifetime of a transformer, that ensures power is provided to a neighborhood that is a mile away from the beach. During hurricane season, the likelihood of the transformer failing is greater than that during the winter months. This seasonal variation could be taken into account via the hazard function, but we also note that the severity and quantity of hurricanes can vary greatly from hurricane season to hurricane season. Conversely, it is reasonable to expect that as the transformer ages, it would become more sensitive to less severe weather events than when it was originally installed. In this particular example, simply modeling the lifetime via a fixed hazard function may not be very accurate,

because we have a system in which the likelihood of failure is dependent upon the severity of the hurricane season, as well as the number of past severe weather events that have impacted the transformer.

In this case, it may give a better understanding of the lifetime to lift the assumption that the hazard function is fixed. So, a reasonable next step is to make the hazard function a stochastic process which is dependent upon some underlying environmental process. In this model, we will consider a stochastic environmental process governed by a point process, where there are interactions between the intensity of the point process and the stochastic hazard function. When considered separately, these two ideas have been extensively studied, but in this paper we take a look at how these interact with one another.

1.2 Literature Review

Introducing a random environment process to a lifetime model has been well studied in reliabilility. Kiessler and Klutke [3] introduce an enivronment process where the rate of degradation of the system occurs at a rate which is governed by a continuous time Markov chain (CTMC) with a finite state space. Thus, we can ascertain from these assumptions that the hazard function is of the form

$$h_a(t) = h\left(\int_0^t \beta_{X(s)} ds\right),$$

where $\{X(t); t \geq 0\}$ is a CTMC with state space $\{0, ..., n\}$, and h is the hazard function of the original lifetime distribution. In other words, the true hazard function is speeding up the deterioration of the device, and the rate of deterioration at any given point is determined by the state of the CTMC. Since this process has condition-

ally stationary and independent increments and satisfies the strong Markov property, many results from the theory of Markov processes can be applied to further study this process. For example, one can use renewal theory to find limits using the stationary distribution of the CTMC.

Riascos-Ochoa and Sánchez-Silva [4] introduce a much more general environmental process where the deterioration of the lifetime is governed by a nonhomogeneous increasing Lévy process; that is, a process which has independent increments and is continuous in probability. This is a much different model for an environmental process, in that an increasing Lévy process $\{X(t); t \geq 0\}$ can be decomposed as $X(t) = X_1(t) + X_2(t)$, where $X_1(t)$ is deterministic, and $X_2(t)$ is a discontinuous jump process. In this particular context, the lifetime distribution is given by $P(V_0 - D(t) > k^*)$, where V_0 is the initial lifetime of the system, $D(t)$ is the Lévy process governing the deterioration, and k^* is some fixed safety threshold. Given the nature of $D(t)$, we cannot guarantee that the distribution is absolutely continuous, and hence the hazard function may not exist. Although this model allows for much flexibility, the independent increments assumption is still a strong assumption to put on a stochastic process.

Hawkes [1] introduces a specific type of point process for which much theory has since been developed called self-exciting point processes. Self exciting point processes have been used extensively in modeling environmental processes, specifically in modeling the occurrences of earthquakes and the aftershocks that follow. The nature of this process yields that each point that occurs generates a subsequent point process independent of the original process. Specifically, we can write the stochastic intensity of the process as follows:

$$\lambda(t) = \lambda_0(t) + \int_0^t g(t-s)\eta(ds),$$

where λ_0 is a deterministic intensity function, and g is a function that determines the intensity of each subsequent point process generated, and is generally assumed to decay over time. Though this is a popular model, when considered coupled with a lifetime model, it does not allow interactions between the deterioration of the system and the environmental process itself. However, this notion has served as inspiration for the model presented in this paper.

Chapter 2

The Model

2.1 Some Results for Point Processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let η be a Poisson process on $(\mathcal{B}(\mathbb{R}_+), \mathbb{R}_+)$ with intensity rate $\lambda(t)$. Viewing η as a counting process with Poisson arrivals, we can characterize the η in the following way:

- $P(\eta(t+\delta) - \eta(t) = 1) = \delta\lambda(t) + o(\delta)$,
- $P(\eta(t+\delta) - \eta(t) \geq 2) = o(\delta)$,
- $P(\eta(t+\delta) - \eta(t) = 0) = 1 - \delta\lambda(t) - o(\delta)$,

for small δ. However, it can be more convenient to consider η as a random counting measure on $\mathcal{B}(\mathbb{R}_+)$, written so that

$$\eta(A) = \sum_{n=1}^{\infty} 1(T_n \in A), \qquad A \in \mathcal{B}(\mathbb{R}_+),$$

where $\{T_n\}$ is a collection of points on $\mathbb{R}_+$, and

$$\eta(A) \sim \text{Poisson}\left(\int_A \lambda(t)dt\right).$$

We call $\gamma(\cdot) := \int 1(t \in \cdot)\lambda(t)dt$, the intensity measure of η, which indeed is a deterministic measure in it's own right. This is a preferable characterization of η, since it allows us to define integrals over η.

Since η is a counting measure, it may be more intuitive to think of an integral over η as a random sum. For example, define the random process $\{\xi_t; t \geq 0\}$ by

$$\xi_t = \int_0^t \alpha e^{-\beta(t-s)}\eta(ds) = \sum_{n=1}^{\eta([0,t])} \alpha e^{-\beta(t-T_n)}.$$

If we are modeling the effects of shocks to a system, this might be a reasonable model, since the effect is at its greatest when a shock occurs, and naturally decays over time. ξ is an example of a Compound Poisson process, and specifically a Poisson driven shot noise. Finding quantities such as the expectation, when treating this as a sum, may be difficult. However, treating this as an integral, we have by Campbell's theorem [2] that

$$E[\xi_t] = \alpha \int_0^t e^{-\beta(t-s)}\gamma(ds) = \alpha \int_0^t e^{-\beta(t-s)}\lambda(s)ds.$$

Thus, we have an explicit expression for the expectation and we also gather that $\xi_t < \infty$ a.s. if $\int_0^t e^{-\beta(t-s)}\gamma(ds) < \infty$. Thus, a sufficient condition for ξ_t to be finite is for λ to be integrable on any bounded interval. If we make the assumption that γ is absolutely continuous with respect to Lebesgue measure, and bounded on finite intervals, we get that $\xi_t < \infty$ a.s. and we also can calculate $E[\xi_t]$ for each $t \geq 0$. However, we note that this holds specifically when η is a Poisson process. Under the model, the intensity itself will be a stochastic process, so η will follow into a more

general class of point processes; namely, simple point processes.

A simple point process is defined as a point process η such that $\eta(\{t\}, \omega) \leq 1, \forall t > 0, \omega \in A$, where $P(A) = 1$ [2]. A sufficient condition for this is that the intensity measure of η is absolutely continuous with respect to the Lebesgue measure. Noting that we make the assumption that λ is the stochastic intensity of η, and so η is simple by the definition of the intensity measure. So, define $\mathcal{F}_t := \sigma(\{\lambda(s); s \leq t\})$. Then, since η is simple, it follows that for $t > 0$,

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} P(\eta(t+\delta) - \eta(t) > 1 | \mathcal{F}_t) = 0,$$

which can easily be verified. Therefore we have a similar infinitesimal characterization of η to that of a Poisson process:

- $P(\eta(t+\delta) - \eta(t) = 1 | \mathcal{F}_t) = \lambda(t)\delta + o(\delta)$
- $P(\eta(t+\delta) - \eta(t) \geq 2 | \mathcal{F}_t) = o(\delta)$
- $P(\eta(t+\delta) - \eta(t) = 0 | \mathcal{F}_t) = 1 - \lambda(t)\delta - o(\delta)$

This, combined with the infinite divisibility property of Poisson processes makes simulating realizations of η quite simple, since for sufficiently small δ,

$$\eta(t+\delta) - \eta(t) | \mathcal{F}_t \overset{app.}{\sim} \text{Bernoulli}(\lambda(t)\delta).$$

Shifting our focus back to the compound Poisson process ξ_t, this paper considers a version of this process where η is a simple point process with a stochastic intensity. Calculating expectation is not quite as simple here, since we cannot apply Campbell's theorem directly. However, from the tower property of conditional expectation,

$$\begin{aligned} E[\xi_t] &= E[E[\xi_t|\mathcal{F}_t]] = \int_\Omega \int_0^t \alpha e^{-\beta(t-s)} \lambda(s,\omega) ds P(d\omega) \\ &= \int_0^t \alpha e^{-\beta(t-s)} E[\lambda(s)] ds, \end{aligned}$$

where we can change the order of integration since the integrand is nonnegative. Thus, we have if $E[\lambda(s)]$ is integrable on bounded intervals, then $E[\xi_t] < \infty$ almost surely. Moreover, if $\{\lambda(s)\}_{s \leq t}$ is uniformly integrable, then $E[\xi_t] < \infty$ almost surely.

In the context of the problem, we will consider the situation where $\lambda(t)$ is actually a function of ξ_t; specifically, one that depends on the hazard function as well as ξ_t. Defining $\lambda(t) = g(t, h, \xi_t)$, we have

$$E[\xi_t] = \int_0^t \alpha e^{-\beta(t-s)} E[g(s, h, \xi_s)] ds,$$

and so we have $E[\xi_t]$ is a solution to the differential equation

$$\frac{d}{dt} y(t) + \beta y(t) = \alpha E[g(t, h, \xi_t)],$$

which can be solved directly if g is linear in ξ_t.

2.2 Interactions between Lifetime and Environment

To represent the interaction between the lifetime and the environment, we define the following stochastic process (existence to be shown later):

$$\mathbf{X} = \{(\xi_t, h_a(t), \lambda_a(t), \eta_a(t)); t \geq 0\}.$$

Each individual process is defined in the following way:

- $\xi_t = \int_0^t \alpha e^{-\beta(t-s)} \eta_a(ds), \alpha, \beta \geq 0,$
- $h_a(t) = h(t + \xi_t),$
- $\lambda_a(t) = g(t, h_a(t)),\ g \geq 0,$
- η_a is a simple point process driven by the stochastic intensity λ_a

Note that no single element of the process can be defined alone, and so the key feature that we have is that we are starting with a basic hazard function h, which we assume is nondecreasing and also satisfies $h(t) \to \infty, t \to \infty$. We also start with a basic Poisson process which we assume has intensity rate $\lambda(t) = g(t, h(t))$. So, initially the process is relatively simple, but as points occur over time, the process becomes increasingly more complex. It is also noteworthy that the points of η_a drive the stochastic component of the system, which are at most countable, so the points of η_a completely determine the process.

There is good reason to believe that this modeling of the adjusted lifetime of a structure with hazard function h_a is useful in practice, since external stimuli are bound to have an effect on lifetime. The original hazard function included in the model can be useful to account for the expected external forces that affect the lifetime of a structure, but it may not be sufficient to compensate for the innumerable potential failures of individual components or a structure not to mention the effects of sudden outside damage to the structure. It is also reasonable to assume that shocks to the system can have lingering and cumulative effects on the system, and may effect the frequency of future shocks to the system.

In general, it will be difficult to find a closed form expression for the lifetime distribution. Based on our construction, even if the initial environmental process is a stationary Poisson process, the nice properties that it enjoys will quickly dissipate

as the interaction between the hazard function and the intensity function takes over. For example, suppose we define $g(t,s) = \lambda(t)f(s)$, where $\lambda(t)$ is the original Poisson intensity. Assuming that η is a stationary point process with rate λ_0, we still get that η_a is a point process with stochastic intensity $\lambda_a = \lambda_0 f \circ h_a$. However, later on we will present some central limit theorem results that allow us to approximate the resulting integral of the hazard function of the actual lifetime, and relate it to how we can compare the lifetime distributions based upon the information we are given.

2.3 Construction and Properties of X

Given how $\mathbf{X}$ is defined, it is not necessarily clear that this definition is valid, or that this indeed is a stochastic process. We will show that this indeed is a stochastic process, and then give some properties of the lifetime based upon this construction.

First, define $\eta_a(\cdot) = \sum_{n=1}^{\infty} 1(T_n \in \cdot)$, where $\{T_n\} \subset \mathbb{R}$ is a collection of points that are later defined.

Theorem 1 *Suppose $T_n \to \infty$ a.s. as $n \to \infty$. Then $\boldsymbol{X}$ is defined on $\mathbb{R}$.*

Proof. We can define this process in terms of the points of η_a.

Step 1: First observe that since $T_n \to \infty$ a.s., we can partition $\mathbb{R}$ so that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [T_{n-1}, T_n),$$

where we define $T_0 := 0$. Then observe, before the first point occurs, the stochastic system is purely deterministic. That is, for $t < T_1$,

- $\xi_t = 0$,

- $h_a(t) = h(t)$,
- $\lambda_a(t) := g(t, h(t))$,
- $\eta_a([0,t]) = 0$.

Now, the distribution of T_1 is given by

$$F_{T_1}(x) = 1 - \exp\left\{-\int_0^x g(s, h(s))ds\right\}$$

Thus, the point T_1 is generated by λ_a.

Step 2: Suppose $t \in [T_1, T_2)$. Then,

- $\xi_t = \alpha e^{-\beta(t-T_1)}$,
- $h_a(t) = h(t + \xi_t)$,
- $\lambda_a(t) = g(t, h_a(t))$,
- $\eta_a([0,t]) = 1$

Now, given that T_1 is already known, the system is deterministic. So, define $T_2 = T_1 + X_2$, where $X_2|T_1$ has the distribution

$$F_{X_2|T_1}(x) = 1 - \exp\left\{-\int_0^x \lambda_a(s + T_1)ds\right\}.$$

Step 3: We lastly consider the general case, where $t \in [T_n, T_{n+1})$. Then,

- $\xi_t = \sum_{k=1}^n \alpha e^{-\beta(t-T_k)} = \int_0^t \alpha e^{-\beta(t-s)}\eta_a(ds)$,
- $h_a(t) = h(t + \xi_t)$,
- $\lambda_a(t) = g(t, h_a(t))$,

- $\eta_a([0,t]) = n$.

Now, define $T_{n+1} = T_n + X_{n+1}$, where the distribution of $X_{n+1}|(T_1, ..., T_n)$ is given by

$$F_{X_{n+1}|(T_1,...T_n)}(x) = 1 - \exp\left\{-\int_0^x \lambda_a(s + T_n)ds\right\},$$

noting here that it is not enough to merely consider $X_{n+1}|T_n$, since the points $T_1, ...T_n$ determine λ_a.

Now, it is easy to see from here that η_a is a point process with stochastic intensity λ_a, which can be defined on $\mathcal{B}(\mathbb{R})$. Therefore, the process $\mathbf{X}$ exists and can be defined as stated in the introduction.

Moreover, since the process at time t is determined by $\mathcal{F}_t = \sigma(\{T_n : T_n \leq t\})$, it follows that $\mathbf{X}$ is indeed a stochastic process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

□

Observe that if the function g is bounded, our assumption that $T_n \to \infty$ a.s. is guaranteed. However, it may be of interest to observe the dynamics of the system when this restriction is removed.

Proposition 2 *Suppose that* $\lim_{n\to\infty} T_n(\omega) = t_\infty$. *Then,* $Y(\omega) \leq t_\infty$.

Proof. If $t_\infty = \infty$, then the result is obvious. So, suppose that $t_\infty < \infty$. Then, for $\delta > 0, \exists N \in \mathbb{N}$ s.t. $\{T_n(\omega)\}_{n\geq N} \subset (t_\infty - \delta, t_\infty)$. Thus,

$$\xi_{t_\infty}(\omega) = \sum_{n=1}^{\infty} \alpha e^{-\beta(t_\infty - T_n(\omega))} \geq \sum_{n\geq N} \alpha e^{-\beta(t_\infty - T_n(\omega))} \geq \sum_{n\geq N} \alpha e^{-\beta\delta} = \infty$$

Thus, since $h(t) \to \infty, t \to \infty, h(t_\infty + \xi_{t_\infty}(\omega)) = \infty$. Since h is non-decreasing, $h_a(t) = \infty$ for $t \geq t_\infty$. Thus, for $\epsilon > 0$, $\int_{t_\infty}^{t_\infty+\epsilon} h_a(t)(\omega)dt = \infty$, so $Y(\omega) \leq t_\infty + \epsilon$. Therefore, $Y(\omega) \leq t_\infty$. □

A consequence of this result is that the construction of the stochastic process given is guaranteed to be well-defined on the support set of Y. We can also note that h_a will actually be discontinuous in general, but $\int_0^t h_a(s)ds$ will still be continuous, since the discontinuities only appear at countably many values.

The construction of $\mathbf{X}$ provides that the points of η_a completely determine the process as a whole, which gives us a good interpretation of the nature of the process, and it also makes simulation relatively straightforward.

Chapter 3

Applications

3.1 Simulation

This paper presents two different approaches of generating realizations of the lifetime. The first approach is to exploit the infinitesimal characterization of the point process; that is, by breaking an interval into small enough increments, we can treat the point process in that given time increment as a Bernoulli trial. Then, ξ, h_a, and λ_a can be updated at each increment accordingly. Since we can write $\mathbb{P}(Y \geq y) = e^{-\int_0^y h_a(t)dt}$, the tail distribution can be approximated by summation. Finally, generate $U \sim \text{Uniform}(0,1)$, and the realization $y_0 = \inf\{y > 0 | P(Y \geq y) < U\}$ can be approximated as stated above along with the fact that $P(Y \geq y)$ is decreasing in y.

A fallback of this method is potential convergence issues. To see this, note we are constructing Bernoulli trials where at the nth trial, the probability of a point occurring from $n\delta$ to $(n+1)\delta$ is $\lambda_a(n\delta)\delta$. Thus, if λ_a is growing very quickly, δ will need to be adjusted so that $\lambda_a(n\delta)\delta < 1$, which would slow down the convergence as well as the stability of the algorithm. Next, we provide another algorithm that is

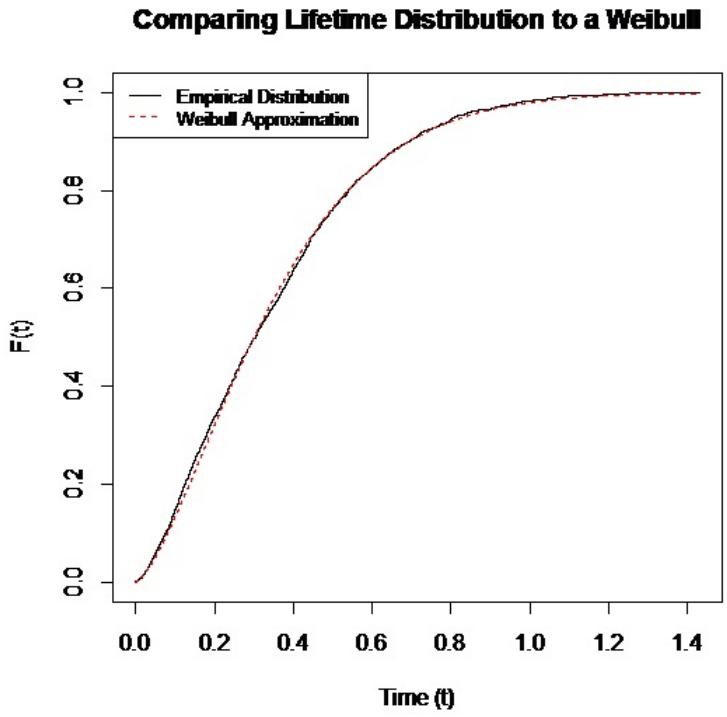
more numerically stable.

The construction of $\mathbf{X}$ can be exploited in that rather than constructing many Bernoulli trials, the points can be generated by using the fact that $T_{n+1} = T_n + X_n$, and the form of the conditional distribution of X_n given $(T_1, ..., T_n)$ has been previously defined.

The distribution of Y can be estimated via the empirical distribution function, which is an asymptotically consistent estimator of the true distribution of Y, and also has a central limit theorem. The R code for the simulation is given in the appendix for the specific case given in the next section.

3.2 An Application to the Weibull Distribution

Suppose that g is of the form $g(t, h_a(t)) = \lambda(t + h_a(t))$ for some function λ (which we may assume is the intensity of some point process). Empirical evidence has suggested that if the original hazard function h is of the form $h(t) = \mu t^p$, with $\mu, p > 0$, then the actual lifetime distribution can be well approximated by a Weibull distribution. An illustration of this is provided below for the conditions $h(t) = 5t^2$, $\lambda(t) = 2 + \sin(t)$, $\alpha = 3, \beta = 2$:



As one can see, the approximation is a very accurate one, which may lead us to ask why this is occurring. It is however noteworthy that this relationship is less consistent as p increases. So, it may also be worth considering, for the more general case, looking at mixtures of Weibull random variables to approximate the distribution.

3.3 Estimation

A primary goal of the model is to be able to provide some meaningful estimation or learning methods given a data set. So we will assume that $Y_1, ...Y_n$ are iid random variables with common distribution F, where F is the true lifetime distribution taking into account the environmental process. Since the integral $\int_0^t h_a(s)ds$ is absolutely continuous for any $t > 0$, it is clear that F is a continuous distribution,

and so we can write F in the following way:

$$F(t) = 1 - \exp\left\{-\int_0^t G_a(s)ds\right\},$$

where $G_a(s)$ represents the true hazard function.

First, some asymptotic properties of the empirical distribution function will be discussed.

Define $\hat{F}_n(t)$ to be the empirical distribution function of Y. Then, some properties of $\hat{F}_n$ are that $n\hat{F}_n(t) \sim \text{Binomial}(n, F(t))$, and the central limit theorem provides

$$\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{d} \mathcal{N}\big(0, F(t)(1 - F(t))\big), \forall t > 0.$$

Further, Donsker's Theorem [5] provides that the collection of stochastic processes $\{\sqrt{n}(\hat{F}_n(t) - F(t))\}_{t>0}$ converges to a Brownian bridge as $n \to \infty$.

An application of the central limit theorem result is to find an asymptotic distribution for $\int_0^t G_a(s)ds$. With this in mind, consider the relation of the distribution to the hazard function

$$-\log\big(1 - F(t)\big) = \int_0^t G_a(s)ds.$$

Recalling that Brownian motions are not differentiable, an asymptotic distribution for G_a itself may not actually exist, so we will focus on $\int_0^t G_a(s)ds$. Hence, we can apply the delta method to $\hat{F}_n$, so that

$$\sqrt{n}\left(\log(1 - \hat{F}_n(t)) - \int_0^t G_a(s)ds\right) \to \mathcal{N}\left(0, \frac{F(t)}{1 - F(t)}.\right)$$

We also note that $P(Y > t|\mathcal{F}_t) = \exp(-\int_0^t h_a(s)ds)$, which implies

$$\exp\left(-\int_0^t G_a(s)\right) = E\left[\exp\left(-\int_0^t h_a(s)ds\right)\right].$$

An extension of this notion is to compute the Laplace transform

$$\Phi_t(u) := E\left[\exp\left(-u\int_0^t h_a(s)ds\right)\right].$$

The result above provides $\Phi_t(1), t > 0$, but if $\Phi_t(u)$ can be computed on a dense subset of $(0, \infty)$, it can be inverted numerically to recover the distribution of $\int_0^t h_a(s)ds$.

3.4 Inference

It is of interest to know whether the environmental conditions are actually having an effect on the lifetime of the device. If we assume an underlying distribution for Y, we can develop a hypothesis test to determine whether or not the environment is having a statistically significant impact on the lifetime of the device.

In particular, the Kolmogorov-Smirnov hypothesis test is an application of the convergence of the scaled empirical distribution function to a Brownian bridge. It is assumed that Y has distribution F_0, and we are testing the validity of this claim. Thus, the null hypothesis is that the true distribution F of Y satisfies $||F - F_0||_\infty = 0$. Given data, the test statistic is

$$||\hat{F}_n - F_0||_\infty,$$

whose distribution can be well approximated by the distribution of

$$W = \sup_{x\in[0,1]} |B(x)|,$$

where $\{B(x)\}_{x\in[0,1]}$ is a Brownian bridge. The tail distribution of W is

$$P(W > x) = 2\sum_{j=1}^{\infty}(-1)^{j+1}e^{-2j^2x^2} \text{ [5]},$$

which can be approximated to arbitrary precision via truncation.

So, if we let h_0 be the hazard function of the lifetime under ideal conditions, and define F_0 to be the distribution with hazard function h_0, the Kolmogorov-Smirnov test can be applied to determine whether the environment is causing a statistically significant effect on the lifetime distribution.

3.5 Appendix

The code for each method is given below:

Method 1: Uses infinitesimal characterization of Poisson process

```
lifetime_d = function (delta,h,lambda,alpha,beta,n) {
y=rep(0,n);
for (i in 1:n) {
u=log(runif(1,0,1))
lambda_update=c(lambda(0)); h_actual=c(h(0))
r<-rbinom(1,1,lambda(0)*delta)
jumps=r*alpha
h_actual=c(h_actual,h(delta+jumps))
lambda_update=c(lambda_update,lambda(delta+h_actual[2]))
t_current<-delta
l_F_tail<- (-1)*mean(c(h_actual[1],h_actual[2]))*delta
n=2
while (l_F_tail>u) {
t_current=t_current+delta
r<-rbinom(1,1,lambda_update[n]*delta)
jumps=exp(-beta*delta)*jumps+alpha*r
h_actual=c(h_actual,h(t_current+jumps))
lambda_update=c(lambda_update,lambda(t_current+h_actual[n]))
l_F_tail<-l_F_tail-mean(c(h_actual[n],h_actual[n+1]))*delta
n=n+1
```

```
if (delta*lambda_update[n]>1) {
y[i]=t_current+delta
break

}
}
y[i]=t_current
}
return(y)
}
```

Method 2: Uses conditional distribution of inner arrival times

```
rand_lifetime=function(h,l,a,b,tol,n) {
y=rep(0,n);
for (i in 1:n) {
U=log(runif(1,0,1));
exph=0;
t=0;
xi=0;
while (exph>U) {
    r=log(runif(1,0,1));
    explam=0;
    while (explam>r) {
        h_t=h(t+xi);
        xi=xi*exp(-b*tol);
        h_tol=h(t+tol+xi);
        explam=explam-tol/2*(l(t+h_t)+l(t+tol+h_tol));
        exph=exph-tol/2*(h_t+h_tol);
        t=t+tol;
        if(exph<=U) {
           y[i]=t;
           break
        }
    }
    xi=xi+a;
```

```
    y[i]=t;
}
}
return(y)
}
```

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