Comparison of Unit Cell Geometry for Bloch Wave Analysis in Two Dimensional Periodic Beam Structures

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ABSTRACT

The wave propagation behavior for one-dimensional rods, beams, and two-dimensional periodic lattice structures are studied using Bloch wave finite element analysis. Dispersion relations relating wave vector components and frequency are obtained by enforcing periodic conditions on a unit cell and solving an eigenvalue problem.

The one-dimensional Bloch wave finite element analysis is performed for continuous rod and beam structures treated as periodic structures with repeating unit cells in order to validate the frequency-wavenumber dispersion relationships obtained with exact solutions. In the case of the rod structure, the frequency-wavenumber relation is linear with a constant wave speed, whereas for the beam structure, the frequency-wavenumber relation is nonlinear and manifests dispersive behavior. For the beam structure, both classical Bernoulli-Euler beam theory and Timoshenko beam theory which includes transverse shear deformation and mass rotary inertia effects are compared. Results from the Bloch wave finite element analysis are shown to converge to the exact solutions with mesh refinement.

For two-dimensional Bloch wave analysis, both periodic rectangular grid lattices and hexagonal honeycomb structures are considered for both in-plane and out-of-plane bending free-wave propagation. For the rectangular grid lattice, there is only one unique choice of unit cell and basis vectors for Bloch wave analysis. Results for this case display
expected anisotropic dispersion behavior with wave direction verified with results in the literature.

For hexagonal honeycomb structures, the periodic unit cell used for Bloch wave analysis is not unique. In the literature, truncated rectangular unit cells with rectangular basis, and different rhombic unit cells in skew coordinates with wave analysis in contra-variant basis directions have been used to study frequency response from Bloch wave analysis. Rhombic unit cell with contra variant basis is scaled and transformed to a rectangular basis. The frequency-wavenumber relationship for truncated hexagonal unit cell is compared to the frequency-wavenumber relationship of rhombic unit cell in contra variant basis. Both in-plane and out of plane wave propagation analysis is performed.
DEDICATION

I would like to dedicate this thesis to my Family and Friends, who are my pillar of strength and constant motivation in the journey of graduate studies.
ACKNOWLEDGEMENT

My sincere thank you to my Advisor and Committee Chair Dr. Lonny Thompson, for his constant help and advice in my Research work. I would like to thank my committee members Dr. Gang Li & Dr. Huijuan Zhao for being part of my research committee and also helping me in this process to graduate here at Clemson University. I would also like to thank my colleagues and friends here at Clemson for their support and advice.
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CHAPTER 1: INTRODUCTION

Periodic lattices, crystals or structures are comprised of a number of very small identical elements or cells that are attached or joined together in a repeating manner. These elements or cells can be repetitive along ‘n’ number of directions and the periodic systems can be classified as n-dimensional periodic systems. The studies on the behavior of these periodic systems in various conditions have been an utmost interest because of their applications in various optical, acoustic, electrical and other engineering fields. The study of wave motion in periodic systems started more than 300 years ago when Newton used the simple one-dimensional lattice of a point mass system to formulate velocity of sound (L. Brillouin, 1946). He derived the formula by assuming the propagation of sound in air is similar to propagation of elastic waves in the lattice. Likewise many engineers and physicists have investigated the properties of crystals, optics etc. The expansion of wave motion studies for engineering periodic structures has increased in recent years because of their applications for lightweight sandwich beams, trusses, beam grillage, acoustic resonators, etc.

The analysis of wave propagation in periodic structures has been performed by considering the whole structure as crystal lattice and applying the technique of Bloch wave analysis. The smallest element or unit cell is considered for Bloch wave analysis for the study of wave propagation in the structure. Heckl (Heckl, 1964) has established propagation constants as a measure of attenuation and change of phase of
wave motion between the unit cells. The frequency and wavenumber relationships have been derived from the Bloch wave analysis to study the characteristics of the wave behavior in the structure.

1.1 Literature Review:

Mead (Mead D. J., 1970) has considered a very long beam, which is considered to be infinite with supports to study the analysis of free wave propagation. The beam is considered as infinite to eliminate the need for analyzing the wave motion with different modes. Beam elements located adjacently undergo identical modes of vibration with different phase are dependent on the direction and magnitude of the velocity caused by the wave motion in the system. The wave propagating in the beam is considered as a sinusoidal wave and the propagation constant is assumed to be a complex number, which describes the rate of decay and phase change of the wave motion per unit length. In this work, it was observed that for an assigned frequency to each propagation constant, different wavelengths and wave velocities exists. These wavelengths and velocities are mostly attributed to the imaginary part of the propagation constant while considering no attenuation in the wave propagation. The characteristic of a non-constant wave velocity is defined as a dispersive medium. Different types of beam supports and its effect on the behavior of the propagation constants were explained. It is observed that the energy flow in the system is restricted to certain frequencies.

The method of calculating the magnitudes of wavelength and velocities was not explained in Mead (Mead D. J., 1970). The method used to analyze the free and forced
wave propagations has been explained in Mead (Mead D. J., 1971). The differential equation of flexural wave motion of beam element is solved by imposing the boundary conditions and phase relationship between two ends of the beam. An exact solution is obtained by solving the equations. An eigenvalue problem is obtained when the external pressure is considered to be zero which leads to free wave propagation in the system. Mead explains that setting up the equations by the response of the conditions and solving for the unknown coefficients of the fourth order differential equation is a simple way to calculate the behavior of various parameters.

Mead (Mead D. J., 1973) has also studied wave propagation analysis in one-dimensional periodic systems with multiple coupling. A more generalized theory of wave propagation analysis has been introduced which is not restricted to just one kind of wave motion or uniform elements. The adjacent elements in the periodic structure can be linked with any number of coupling coordinates which implies multiple degrees of freedom at the ends of each individual periodic element. The number of waves associated with propagation constants in the system is equal to number of coupled coordinates between two adjacent elements. The wave motion at any given point in an element is equal to an exponential factor times the wave motion of the corresponding point in the adjacent element. Propagation constants corresponding to wave motion has been discussed in detail along with kinetic and potential energies of the wave propagation. A Rayleigh-Ritz method has been applied to free wave propagation and it is recommended that this method can be used to obtain the relationship between frequency and propagation constants. A special case of damping is also considered and studied in this paper. In
addition, analysis has been done on the two-dimensional periodic structures where the wave is propagated in the system at an angle.

The main focus of the literature presented in (Bardell, 1997) is to have a theoretical and experimental study of beam grillage when it is exposed to out-of-plane point harmonic loading. As beam grillage is a two-dimensional periodic frame structure, undergoing bending and torsion. The beams in the considered single period unit cell are subjected to out of plane wave propagation in torsion and out of plane bending. In addition, a hierarchical finite element method is employed to model the unit cell. Bloch theorem is applied to the finite element model to obtain the phase constant surfaces which is used for the purpose of forced analysis of the structure. In order to verify the analysis experimentally, beam grillage is treated with a very high level of damping to make the finite system non-reverberant and behaves like an infinite system. It is observed from both theoretical and experimental analysis that wave beaming exists in the structure. The beaming effect can be used as passive isolation mechanism.

Ruzzene (Massimo Ruzzene, 2003) has considered a honeycomb grid like structure for analysis of out-of plane wave propagation. A non-standard rectangular unit cell has been cut from the entire hexagonal honeycomb grid to perform the analysis. Each beam element has out of plane displacement by which bending and torsion occurs at each node. These nodal degrees of freedom are used to model the entire unit cell. Finite element method and Bloch theorem (theory of periodic structures) techniques are applied to obtain a eigenvalue problem whose solution when assigned the values of propagation constants yields corresponding frequencies. Ruzzene has considered the propagation
constant as purely imaginary (propagating with no amplitude decay) and their values ranging from \(-\pi, +\pi\). The phase constant surfaces obtained by incorporating the values of propagation constants can be used to predict the direction of wave propagation in the structure. Further, the analysis of phase constant surfaces helped to identify the frequency ranges (stopping bands) where there is no wave propagation and also indicated the dependency of the behavior of wave propagation on the unit cell geometry. Various geometries of grid structures are considered and analyzed. Their results are compared to regular hexagonal honeycomb grid structure with more emphasis on re-entrant grids which have negative Poisson’s ratio. By means of these studies and observations made it is concluded that the phase constant surfaces can be used as effective tools to achieve optimal design of cellular structures with desired acoustic performance having vibration isolation capability within certain frequency ranges.

Phani (A. Srikantha Phani, 2006) in his paper has explored the analysis of wave propagation in two-dimensional periodic lattices by considering four different types of structures; hexagonal honeycomb, square, triangular and Kagomé lattices. The geometrical section properties of the beams in all four lattices are considered to be same for the analysis. From each one of the lattices, a basic rectangular unit cell is taken and is divided into a network of Timoshenko beams. The beams undergo in-plane wave propagation having three nodal degrees of freedom at each node, two translations in x, y plane and a rotation about the z axis. The finite element method and Bloch theorem techniques are used to obtain the phase constant surfaces describing the frequencies of wave propagation. It is explained in the paper that dependence of frequency over wave
number is helpful in revealing the stopping band structure. The existence of bandgaps with blocked waves for different lattices are demonstrated and the wave directionality plots at high frequencies reveal that the hexagonal, Kagomé and triangular lattices exhibit isotropic behavior whereas square lattice exhibits strongly anisotropic behavior over the full frequency range.

As discussed earlier, Ruzzene (Massimo Ruzzene, 2003) worked on out of plane wave propagation in honeycomb structures. In (Stefano Gonella, 2007) the focus is on analyzing the in-plane wave propagation in both hexagonal and re-entrant lattices. The rhombic unit cell considered for the hexagonal lattice consists of three elements, where the periodicity is defined by set of lattice vectors in skew coordinates. The elements of the unit cell are discretized into a number of beams which undergoes two translations and a rotation comprising of three nodal degrees of freedom at each node. The same technique employed in (Ruzzene 2003) is used here but now with skew coordinates to obtain an eigenvalue problem which can be solved to obtain the frequencies. The phase constant surfaces or dispersion surfaces obtained for varying values of propagation constants describes the wavenumber and frequency relationship for each mode of wave propagation and can be used to obtain phase and group velocities. These dispersion surface and velocity plots are used to study the effect of unit cell geometry and frequency of the propagating waves on the directional behavior of the lattices. The discussion of existence of bandgaps are also mentioned in this paper. Various hexagonal lattice geometries with varying internal angles and especially the re-entrant configurations have been analyzed and shown to have great difference in the characteristics of wave
propagation. While citing the earlier paper by (Phani, 2006) in the introduction, no comparison or validation of the results in skew coordinates was given for the wave propagation analysis for hexagonal honeycombs already performed by Phani in rectangular coordinates.

Tie (B. Tie, 2013) has also conducted a wave propagation analysis on hexagonal and rectangular beam lattices. Both in-plane and out-of-plane waves are considered for the analysis. Reference unit cell from respective hexagonal and rectangular lattices are taken on which the beam elements are modeled and an eigenvalue problem is obtained using the Bloch reduction technique. The unit cell used for hexagonal honeycomb lattice is rhombic with skew coordinates, the same as used by Gonella, 2008. The phase constant surfaces obtained by solving the eigenvalue problem in the first Brillouin zone and detailed contour plots obtained in the irreducible Brillouin zone are used to define the bandgap characteristics of wave propagation. Similar to previous works it shown that for the in-plane wave propagation in hexagonal lattices the existence of stop bands are observed at low frequency ranges whereas for out-of-plane wave propagation these bands exists at much higher frequency ranges. For the rectangular lattices these bands are not observed in the considered frequency range. Significant work has been performed to show the band gaps dependency on the elastic and geometric properties of the unit cell. With the increasing frequency ranges hexagonal lattices exhibit more anisotropic behavior and this is observed with help of phase and group velocities. Comparisons have also been made between the hexagonal beam lattice and a homogenized orthotropic plate model. The paper also shows the existence of retro-propagating waves having negative
group velocities. This Property can be used a tool to design lattices with desired energy transmission properties. No comparison or validation of wavenumber-frequency results for hexagonal or rectangular lattices with earlier works are given.

1.2 Motivation for Present Work and Goals:

In the literature described in the previous section, the analysis of rectangular and hexagonal lattices have been performed by various authors in both rectangular and skew coordinates but have not been compared or validated with each other. In particular, for two-dimensional wave propagation, there are several choice of repeating unit cells for the lattice that can be used and the wave vector direction can be decomposed into either cartesian or skew (contravariant) components.

The goals of this thesis are to carefully examine wave propagation in beam lattices with comparisons of results for both rectangular and rhombic unit cells and for wave vectors decomposed in both cartesian and skew (contravariant) components. Transformations are used to directly show that the directionality results using the two different coordinate systems are the same. While the skew coordinates are expected to give a more efficient Bloch wave analysis for reduced zones, the transformation to cartesian coordinates helps give a clear physical interpretation of anisotropic directionality in contour plots.

In this thesis, both one-dimensional and two-dimensional periodic structures are examined for wave propagation analysis. In the present work the emphasis is laid on explaining in detail the Bloch wave technique and its application in periodic structures.
Different unit cell configurations having different basis vectors for various periodic lattices are considered for obtaining the dispersion relationships between wavenumber and frequencies. A main objective of this study is to verify the results of dispersion relationships when different unit cell geometries having the different basis vector configurations are analyzed and compared for both in-plane and out of plane wave propagation in two-dimensional periodic structures. In particular, truncated hexagonal and rhombic hexagonal unit cells are compared.

1.3 Thesis Overview:

Chapter 1 gives a brief introduction to periodic structures and to the analysis of wave propagation in them. Literature from previous work is presented to provide motivation for the goals of the present work.

Chapter 2 details the wave propagation analysis in one-dimensional periodic structures. Both rod and beam unit cells are considered for the analysis. The finite element method and Bloch technique used to formulate the eigenvalue problem to obtain the frequencies of wave propagation has been clearly explained. These results are compared to the analytical solutions of the rod and beam elements.

Chapter 3 provides a detailed explanation of wave propagation in two-dimensional periodic structures. Both rectangular and hexagonal periodic structures are used for analysis. Finite Element modeling and the application of Bloch technique on such structures are described in detail. Different types of unit cells considered for the hexagonal periodic structure are presented. These unit cell analysis having different basis
vectors have been compared to each other to understand the behavior of the wave propagation.

Chapter 4 concludes the main results obtained by the objectives considered in this work.
CHAPTER 2: WAVE PROPAGATION IN 1-D PERIODIC STRUCTURES

2.1 Introduction:

In this chapter, a wave propagation analysis is performed on one dimensional periodic rod and beam finite elements. The analysis is carried out by using a Bloch wave technique to study free wave propagation in these structures. The Bloch wave analysis provides a method for computing frequency-wavenumber relationships for waves propagating in periodic structures. The numerical solutions obtained are compared to the analytical solutions of the bending waves on rod and beam structures.

2.2 Bloch Theorem in One-Dimensional Structures:

Before going into the details of the finite element analysis of the periodic structure to determine the wave propagation characteristics it is important to discuss the Bloch wave Theorem, used in the analysis. For general periodic structures, waves travel in different directions which can be expressed in terms of basis vectors. In case of one dimensional periodic structure waves propagate along only one direction.

For the analysis of a one-dimensional periodic structure a unit cell is selected in which the end nodes of the cell are called as lattice points and the direct lattice vector i.e., the basis vector is associated with the unit cell. Let $x$ be the global coordinate system and $x_j$ be a lattice point in the reference unit cell having length $a_x$. Let $u(x_j)$ be the displacement of the point in the unit cell. When a sinusoidal wave condition is imposed
on the structure, the solution to the 1-D wave condition assuming purely propagating waves yields:

\[ u(x_t) = u_0 e^{(jk_x x_t)} \]  \hspace{1cm} (2.1)

where \( u_0 \) is amplitude, \( k_x \) is the wavenumber and \( j \) is an imaginary number. The displacement of the lattice point \( x_t + na_x \), when evaluating at the lattice points is given by:

\[ u(x_t + na_x) = u_0 e^{(jk_x (x_t + na_x))} \]  \hspace{1cm} (2.2)

where \( n \) is an integer and \( a_x \) is the unit cell length. As \( u(x_t) = u_0 e^{(jk_x x_t)} \), substituting this into the above equation (1.2) we get

\[ u(x_t + na_x) = u(x_t) e^{\mu_n} \]  \hspace{1cm} (2.3)

where \( \mu_n = jk_x a_x \) is the propagation constant in the direction of the basis vector, which in this case is in the \( x \) direction. If the wave in the structure is propagating with attenuation the propagation constant is:

\[ \mu_n = \delta_n + i\varepsilon_n \]  \hspace{1cm} (2.4)

where the propagation constant is a complex number in which the real part \( \delta_n \) is the “attenuation constant” and the imaginary part is the “phase constant”. The characteristics of the propagating waves inside a structure is determined by the propagation constants. If the propagation constant is purely imaginary then the waves are
free to propagate without any attenuation. When the real part of the propagation constant is not zero then the waves undergo attenuation. In our case we are only concerned with the free wave propagation without attenuation, therefore the propagation constant in the eq. (2.4) becomes:

\[ \mu_x = i \varepsilon_x \]  

(2.5)

As \( \mu_x = jk_x a_x \), that implies:

\[ \mu_x = jk_x a_x = i \varepsilon_x \]  

(2.6)

\[ \Rightarrow \varepsilon_x = k_x a_x \]  

(2.7)

According to the definition of the Bloch theorem the location of the unit cell in a periodic structure is not responsible for the change in amplitude across the cells due to the wave propagation that is travelling without attenuation. With the help of this definition the analysis of the whole structure can be simplified by considering the wave propagation in a single unit cell. This reduces the computational time in the analysis of wave propagation in periodic structures.

2.2.1 Matrix Reduction using Bloch Theorem:

Application of Bloch wave theorem in the finite element analysis of the unit cell helps in investigating the propagation of elastic waves in the entire structure or lattice. The generalized displacements and generalized forces are very much useful in carrying
out these studies. The displacement vector is designated as \( \{ u \} \) and the force vector as \( \{ F \} \).

When a time-harmonic condition is imposed on the structure, the equation of motion is obtained by relating the displacement and force vectors in the following way:

\[
\left( [K] - \omega^2 [M] \right) \{ u \} = \{ F \}
\]

(2.8)

Where \([ K ]\) is the global stiffness matrix, \([ M ]\) is the global mass matrix, \( \omega \) is the frequency of the wave propagation.

The displacement vector \( \{ u \} = \{ u_1, u_2, u_i \}^T \) i.e., the two end lattice points of the unit cell are labeled as \( u_1 \) and \( u_2 \) while the nodes that are in between the two end nodes are named as \( u_i \). Similarly the forces at the end lattice points and middle nodes are labeled, by making the force vector \( \{ F \} = \{ F_1, F_2, F_i \}^T \)

Bloch theorem comprises the conditions that the propagating wave has and reduces the dimensions of the equation of motion. According to Bloch Theorem the displacements at the end lattice points are related as follows:

\[
u_2 = e^{i\mu} u_1
\]

(2.9)

Similarly the forces at the end lattice points are related as:
\[ F_2 = -e^{\mu_i} F_1 \] (2.10)

The displacement and force equations after the Bloch reduction can be combined in a matrix forms as:

\[
\{ u \} = [A] \{ u_i \} 
\] (2.11)

\[
\{ F \} = [B] \{ F_i \}
\] (2.12)

Where \( \{ u_i \} = \{ u_1, u_j \}^T \), \( \{ F_i \} = \{ F_1, F_j \}^T \), \( [A] \) and \( [B] \) are the complex valued matrices that are equal to:

\[
[A] = \begin{bmatrix}
[I] & [0] \\
e^{\mu_i} \cdot [I] & [0] \\
[0]^T & [I]
\end{bmatrix} 
\] (2.13)

\[
[B] = \begin{bmatrix}
[I] & [0] \\
-e^{\mu_i} \cdot [I] & [0] \\
[0]^T & [I]
\end{bmatrix} 
\] (2.14)

Where \( [I] \) is the identity matrix and \( [0] \) is the zero matrix.

The size of these matrices are given by:

Size of \( [A] \) and size of \( [B] \) = (number of degrees of freedom at each node*number of nodes \times \) (number of interior nodes + number of left corner node)*number of degrees of freedom at each node)
Substituting the displacement and force reduction equation into the equation of motion we get:

$$\left( [K] - \omega^2 [M] \right) [A] \{ u_r \} = [B] \{ F_r \}$$  \hspace{1cm} (2.15)

Pre-multiplying the above equation with the Hermitian (Complex conjugate) transpose $[A^H]$ on both sides yields

$$[A^H] \left( [K] - \omega^2 [M] \right) [A] \{ u_r \} = [A^H] [B] \{ F_r \}$$  \hspace{1cm} (2.16)

Implies

$$\left( [A^H] [K] [A] - \omega^2 [A^H] [M] [A] \right) \{ u_r \} = \{ 0 \}$$  \hspace{1cm} (2.17)

The left hand side is equal to zero as $[A^H] [B] \{ F_r \} = \{ 0 \}$.

The resultant reduced equation of motion after the above sequence of operations performed gives

$$\left( [K, (\mu_s) - \omega^2 M, (\mu_s)] \right) \{ u_r \} = \{ 0 \}$$  \hspace{1cm} (2.18)

Where $[K,] = [A^H] [K] [A]$ is the reduce stiffness matrix and $[M,] = [A^H] [M] [A]$ is the reduced mass matrix. Both these matrices are the functions of the propagation constant $\mu_s = jk_s a_s = j\varepsilon_s$. 

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2.2.2 Eigen-Value Problem:

The reduced equation of motion obtained can be solved for obtaining the frequencies by assigning values for the propagation constant in the following equation:

$$\det \left( \left[ K_{\mu}^s - \omega^2 M_{\mu}^s \right] \right) = 0 \quad (2.19)$$

The above equation is now an eigenvalue problem. The solution obtained by solving this eigenvalue problem yields the frequencies for the given values of propagation constant. As we are concerned with only free wave propagation and not the attenuation, the real part of the propagation constant $\delta$, is set to zero. That is, the propagation constant is now purely imaginary which is defined by:

$$\mu_s = je_s = jk_s a_s \quad (2.20)$$

$e_s$ is a dimensionless quantity called the “phase constant” which is equal to wave number ($k$) multiplied by unit-cell length ($a_s$). By assigning the values of $e_s$ and substituting these values into the eigenvalue problem yields a solution that generates the frequency corresponding to the wave propagation.

2.3 One-Dimensional Rod:

A rod of length “$L$” is considered as shown in the figure 2-1 on which finite element analysis is performed by the application of Bloch theorem to find out the frequency-wavenumber relations.
2.3.1 Unit Cell Configuration:

The rod itself is taken as the unit cell because of its simple structure and replicability. It is divided into “n” number of elements. Each element of the unit cell is of equal length that is equal to “L/n”. The geometrical and material properties of the rod that are considered are given by: Length $L = 0.2 \text{m}$; Radius $r = 0.5 \text{m}$; Young’s Modulus $E = 71.9 \text{GPa}$; Poisson’s ratio $\nu = 0.3$; Density $\rho = 2700 \text{kg/m}^3$.

2.3.2 Finite Element Method

2.3.2.1 Mass and Stiffness Matrices:

Each element of the unit cell is modeled as a rod element having circular cross-section. The rod element is subjected to axial loading in the x-direction resulting in the axial displacement at each node as shown in the figure 2-2. As the rod undergoes only axial displacement, it has only one degree of freedom.
The elemental stiffness and mass matrices of a rod are calculated using the total potential energy and the stress-strain and strain-displacement relationships using standard linear interpolation functions which are given by (Tirupathi R. Chandrupatla, 2002):

\[
k_e = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]  \hspace{1cm} (2.21)

\[
m_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]  \hspace{1cm} (2.22)

where \( k_e \) and \( m_e \) are the elemental stiffness and mass matrices respectively, \( E \) is the young’s modulus, \( \rho \) is the density, \( A_e = \pi r^2 \) is the cross-sectional area and \( l_e \) is the elemental length.

After computing the elemental stiffness and mass matrices for each element, they are assembled into global stiffness \( K \) and mass \( M \) matrices respectively.
2.3.2.2 Calculation of Frequencies:

The equation of motion \((K - \omega^2 M)u = F\) is obtained after assembling the global stiffness and mass matrices. Once done, the matrices are reduced by the application of bloch theorem as discussed in the section 2.2.1.

The size of the complex valued matrix \([A]\) that is responsible for the matrix reduction of the rod is given by:

Number of nodes = \((n+1)\), where \(n\) are the number of elements

Number of degrees of freedom at each node = 1

Number of corner nodes = 2

Number of Interior nodes = Number of nodes - Number of corner nodes = \((n+1)-2\)

Number of left corner nodes = 1

Size of \([A]\) = \((n+1) \times n\)

Once the matrix reduction is done, the equation of motion and the reduced equation of motion takes the form same as in equations (2.19) and (2.20) respectively.

The reduced equation obtained is an eigenvalue problem and is solved for frequencies, which is explained in detail in section 2.2.2. The values of \(\varepsilon_x\) range between \((-\pi, \pi)\), due to periodicity of the solution and is known to be the first Brillouin zone. The wave propagation characteristics can be determined in this first Brillouin zone.
2.3.3 Results of a One-Dimensional Rod:

A Matlab code is written detailing the procedure discussed above to solve for the frequencies which are in the form of eigenvalues. The epsilon values range between $\epsilon_x = (0, \pi)$. As the Brillouin zone is symmetric, a portion of its values can be considered for the analysis and when the $\epsilon_x$ values are divided by the unit cell length $L$ gives the wave number values i.e., $k_x = \frac{\epsilon_x}{L}$. The number of elements considered for the discretization of the unit cell is calculated for the smallest wavelength $\lambda_{\min}$, whereas this smallest wavelength is obtained from the maximum wavenumber i.e., $\lambda_{\min} = \frac{2\pi}{k_{\max}}$.

The frequency versus wavenumber plot is plotted by considering 20 elements per the smallest wavelength for the rod having a circular cross section and using the geometric and material properties as given in the section (2.3.1).
The frequency versus wavenumber plot in fig 1-3 depends on the value of $L$ used and the material properties. However, the solution is independent of the cross-section area $A_r = \pi r^2$. The above plot is plotted for two roots of the eigenvalues i.e., the values of the frequencies for the first root are obtained for the wavenumber range $k_x = [0, \pi]/L$ and for the second root are obtained for the wavenumber range $k_x = [\pi, 2\pi]/L$. For the first root the frequencies are ranging from 0-1273.24 Hz approximately and for the second root the frequencies are ranging from 1273.24-2546.48 Hz approximately. When
observed in the figure the second root has been mirror imaged to display the results in the considered epsilon values i.e., $\epsilon = (0, \pi)$.

2.3.4 Analytical Solution for Wave Propagation in One-Dimensional Rod:

The following governing equation represents the wave propagation in one-dimensional rod in x-direction (Graff, 1975):

\[ E \frac{\partial^2 u(x,t)}{\partial x^2} = \rho \frac{\partial^2 u(x,t)}{\partial t^2} \]  \hspace{1cm} (2.23)

Which can be written, \( \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2} \) \hspace{1cm} (2.24)

Where \( c \) is the wave speed which is given by:

\[ c = \sqrt{\frac{E}{\rho}} \] \hspace{1cm} (2.25)

\( E \) - Young’s Modulus

\( \rho \) - Density

The governing equation of the wave propagation is independent of \( A \). Solving the second order differential equation 2.25 using spatial transformation takes the form:

\[ \frac{d^2 u(x, \omega)}{d x^2} + \frac{\omega^2}{c^2} u(x, \omega) = 0 \] \hspace{1cm} (2.26)
The solution of the propagating waves positively directed to the equation 2.26 is:

\[ u(x, \omega) = A e^{ikx} \tag{2.27} \]

where \( k = \frac{\omega}{c} \) is the wavenumber.

\( \omega \) is the frequency of the wave propagation defined by:

\[ \omega = ck \tag{2.28} \]

The frequency equation has a linear relation with constant slope \( c \).

The characteristics of the wave propagation are determined by calculating the frequency, which is obtained by imposing the values of wave number and wave speed. The analytical solution has no amplitude decay. Thus, the Bloch theorem is exact for the rod.

2.3.5 Comparison of Results:

A plot comparing the finite element periodic solution for Bloch wave and analytical solution is plotted to verify whether the procedure adopted to determine the characteristics of the wave propagation is appropriate.
Figure 2-4 Plot Showing the Comparison between Bloch Wave Finite Element Solution and Analytical Solution of a Rod

The figure 2-4 is the plot showing the comparison for the two solutions for two roots of the frequency for 20 elements. As the analytical solution has no amplitude decay the Bloch wave finite element solution and the analytical solution match each other for the two roots of the frequencies.

2.3.6 Normalization of Rod:

Normalization is done so that the solutions obtained can be verified for any material properties used and for any of the reference cell length $L$. 
2.3.6.1 Normalization of Finite Element Solution:

The stiffness and mass matrices in the equation of motion takes the form

\[
\frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \omega^2 \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \{0\} \tag{2.29}
\]

Since \([K - \omega^2 M] = 0\]

Multiplying the above equation with \(\frac{l_e}{EA}\) gives

\[
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \omega^2 \left(\frac{\rho l_e^2}{6E}\right) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \{0\} \tag{2.30}
\]

\[
\omega^2 \left(\frac{\rho l_e^2}{6E}\right) = \frac{1}{6} \left(\frac{\omega l_e}{c}\right)^2 = \frac{\dot{\omega}^2}{6} \tag{2.31}
\]

\(l_e\) in terms of length \(L\) can be written as \(L/n\), substituting this in the equation 2.33 gives the following expression

\[
\omega^2 \left(\frac{\rho l_e^2}{6E}\right) = \frac{1}{6} \left(\frac{\omega l_e}{c}\right)^2 = \frac{\omega^2}{6} \tag{2.32}
\]

The normalized equation of motion becomes
where \( \hat{\omega} = \frac{\omega L}{c} \) is the normalized frequency.

The above obtained equation of motion is used to calculate the normalized frequencies for any material properties and cross sections used for the \( \varepsilon \) values.

### 2.3.6.2 Normalization of Analytical Solution:

Given frequency of wave propagation in one-dimensional rod for the exact solution is:

\[
\omega = kc \quad \text{(from eq. 2.31)}
\]

\[
\Rightarrow k = \frac{\omega}{c} \quad (2.34)
\]

Multiplying the above equation with \( L \) on both sides gives:

\[
kL = \frac{\omega L}{c} \quad (2.35)
\]

\[
\Rightarrow \hat{k} = \hat{\omega} \quad (2.36)
\]

where \( \hat{k} = \varepsilon = kL \) and the equation \( \hat{k} = \hat{\omega} \) is a linear relationship with unit slope.

Imposing the given \( \varepsilon \) values in the obtained normalized equation yields the normalized frequencies.
2.3.7 Normalized Frequency Vs Normalized Wavenumber Plots:

A figure showing a comparison between the Bloch wave finite element and analytical normalized frequency is plotted against the normalized wave number as shown below:

As observed from the figure the Bloch wave finite element solution is matching with the analytical solution and is true for any material properties and periodic cell length $L$ used.

Figure 2-5 Plot showing the comparison between the normalized Bloch wave Finite Element solution and the Analytical Solution of a rod
2.4 One-Dimensional Beam:

A one-dimensional flexural beam is considered as shown in the figure 2-6 with periodic cell length $L$ on which Bloch wave finite element analysis is performed to determine the frequencies for propagating waves at different wavenumbers and compared the results with the analytical solution as did for the one dimensional rod. Both Euler-Bernoulli Beam and Timoshenko Beam which includes deformation due to shear are considered for the analysis.

![Figure 2-6 One Dimensional Beam](image)

2.4.1 Unit Cell Configuration:

The unit cell is divided into $n$ number of beam elements. The length of each element is equal to $L/n$. For the beam the length is given by $L = 2m$, having a rectangular cross-section with thickness $t = 0.006848m$ and width $w = 1m$. The area of the cross-section of the beam $A_y = wt$ and moment of inertia $I = \frac{wt^3}{12}$. The material properties of the beam are considered to be Young’s Modulus $E = 71.9GPa$, Poisson’s ratio $\nu = 0.3$, Density $\rho = 2700kg/m^3$. 
2.4.2 Bloch Wave Analysis:

2.4.2.1 Mass and Stiffness Matrices (Euler-Bernoulli Beam):

The unit cell is subdivided into \( n \) number Euler Bernoulli beam elements. The Euler Bernoulli beam is structured in a way that one of its dimensions is much larger than the other two dimensions and the axis of the beam is usually defined on the larger dimension. In this beam the cross-section is perpendicular to the bending line. Each node in the beam element is subjected to two degrees of freedom with a vertical displacement \( v \) and a rotation \( \theta \). Figure 2-7 represents the schematic diagram of a beam element with two degrees of freedom.

![Figure 2-7 One Dimensional Beam Element with Two Degrees of Freedom at Each Node](image)

The mass and stiffness matrices for each element are calculated using Hermitian polynomials as shape functions that describe the bending caused by the two degrees of freedom. These are given by the expressions (Tirupathi R. Chandrupatla, 2002):

\[
m_e = \frac{\rho A l_e}{420} \begin{bmatrix} 156 & 22 l_e & 54 & -13 l_e \\ 22 l_e & 4 l_e^2 & 13 l_e & -3 l_e^2 \\ 54 & 13 l_e & 156 & -22 l_e \\ -13 l_e & -3 l_e^2 & -22 l_e & 4 l_e^2 \end{bmatrix}
\]

\[(2.37)\]
\[
\begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2
\end{bmatrix}
\]

(2.38)

Where \( m_e, k_e \) are the elemental mass and stiffness matrices respectively, \( E \) is the young’s modulus, \( I \) is the moment of inertia of the cross-section of the beam, \( A \) is the area of the cross-section of the beam, \( l_e \) is the beam elemental length, \( \rho \) is the density of the beam material.

Once the elemental mass and stiffness matrices are computed, they are assembled into global stiffness \( K \) and mass \( M \) matrices respectively.

2.4.2.2 Mass and Stiffness Matrices (Timoshenko Beam):

When beam is modeled as a Timoshenko beam it takes into account the shear deformation and rotational inertia effects. In this beam rotation is allowed between the cross-section and bending line. The beam has also two degrees of freedom at each nodes with a vertical displacement \( v \) and a rotation \( \theta \). The mass and stiffness matrices of an element of a Timoshenko beam with a consistent interpolation are given by (Thompson, 2013):

The total stiffness of a Timoshenko beam include both the stiffness due to bending and the stiffness due to shear i.e.,
\[ k_e = k_{\text{bending}}^e + k_{\text{shear}}^e \]  

(2.39)

where

\[ k_{\text{bending}}^e = \frac{EI}{l_e} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \]  

(2.40)

\[ k_{\text{shear}}^e = \frac{GA_\kappa}{l_e} \begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ \frac{l_e}{2} & l_e^2/4 & -l_e^2/2 & l_e^2/4 \\ -1 & \frac{l_e}{2} & 1 & -\frac{l_e}{2} \\ \frac{l_e}{2} & l_e^2/4 & -l_e^2/2 & l_e^2/4 \end{bmatrix} \]  

(2.41)

Where \( G \) is the shear modulus, \( \kappa \) is the Timoshenko shear coefficient and \( \kappa = \frac{5}{6} \) for a rectangular cross-section.

The mass matrix is given by the expression:

\[ m_e = \int x N_e^T \begin{bmatrix} \rho A_\phi & 0 \\ 0 & \rho I \end{bmatrix} N_e dx \]  

(2.42)

Where \( N_e = \begin{bmatrix} N_1 & \phi_1 & N_2 & \phi_2 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \) in which \( N_1, \phi, N_2, \phi_2 \) are the shape functions for consistent approximation.
After the computation of elemental mass and stiffness matrices, they are assembled into Global mass and stiffness matrices.

2.4.2.3 Calculation of Frequencies:

The calculation of frequencies of the beam is done by following the same procedure as did for the rod discussed in the section 2.3.2.2.

The size of the Hermitian matrix \([A]\) that is responsible for the matrix reduction of the beam is:

Number of nodes = \((n+1)\), where \(n\) are the number of elements

Number of degrees of freedom at each node = 2

Number of corner nodes = 2

Number of Interior nodes = Number of nodes-Number of corner nodes = \((n+1)-2\)

Number of left corner nodes = 1

Size of \([A]\) = \(((2n+2)\times2n)\)

After the matrix reduction is done the reduced equation can be written as follows:

\[
\begin{vmatrix}
K_r (\mu_x) - \omega^2 M_r (\mu_x)
\end{vmatrix}
= 0
\]

(from eq 2.20)

where \(\mu_x = j\varepsilon_x\)
The above equation is an eigenvalue problem which is solved for the eigenvalues that represent the frequencies by imposing the $\varepsilon_x$ values as discussed in detail in section 2.3.2.2.

2.4.3 Results of a One-Dimensional Beam:

The frequencies of both the Euler-Bernoulli beam and the Timoshenko beam are obtained by writing a MATLAB code following the procedures discussed in the above sections. For the imposed normalized wavenumbers $\varepsilon_x$, the frequencies obtained are clearly demonstrated in the following plots.

2.4.3.1 Euler-Bernoulli Beam:

Figure 1-8 displays the frequency versus wavenumber for an Euler-Bernoulli beam:
Figure 2-8 Frequency versus Wavenumber Plot of a Euler-Bernoulli Beam Element with $L=2m$

The frequencies for the wave propagation are calculated in the $\epsilon_x$ range $(0, \pi)$ for two roots. The number of elements used for the discretization of the unit cell are determined by calculating the minimum wavelength. In this case the number of elements considered are 5 for the smallest wavelength. The geometric and material properties used for the calculations are presented in the section 2.4.1. The frequency versus wavenumber plot obtained is dependent on the length $L$ and the material properties used but independent of the moment of inertia $I$ and area $A_x$. The frequencies are exponentially increasing for the given wavenumbers which is observed in the figure 2-8. For the first
root the frequencies are ranging from 0-3.98 Hz approximately and for the second root the frequencies are ranging from 3.98-15.91 Hz approximately. When observed the second root of the frequency has been mirror imaged to be plotted in the given range.

2.4.3.2 Timoshenko Beam:

A plot of frequency versus wavenumber is shown below for a Timoshenko Beam. The material and geometric properties used are same as used for the Euler-Bernoulli Beam. The epsilon values that are used to calculate the wavenumbers are ranging from $0-\pi$.

*Figure 2-9 Frequency versus wavenumber plot of a Timoshenko beam element with L=2m*
In this case the number of elements used are 30 for the smallest wavelength and for two roots of frequencies. For the first root the frequencies are ranging from 0-4.01 Hz approximately and for the second root the frequencies are ranging from 4.01-16.07 Hz. These frequency ranges obtained are close to the frequencies obtained in case of Euler-Bernoulli beam.

2.4.4 Analytical Solution for Wave Propagation in a Beam:

The analytical solutions for both the Euler-Bernoulli beam and a Timoshenko beam are determined to make the comparison between the Bloch wave finite element solution and the analytical solution.

2.4.4.1 Analytical Solution for Euler-Bernoulli Beam:

The governing equation of motion for an Euler Bernoulli beam takes the form (Graff, 1975)

\[
EI \frac{d^4 y}{dx^4} + \rho A \frac{d^2 y}{dt^2} = q(x,t)
\]  

(2.43)

Where \(q(x,t)\) is the distributed load acting on the beam. If the distributed load is neglected the equation (2.48) reduces to the form

\[
EI \frac{d^4 y}{dx^4} + \rho A \frac{d^2 y}{dt^2} = 0
\]  

(2.44)


\[ \Rightarrow \frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI} \frac{\partial^2 y}{\partial t^2} = 0 \]  
(2.45)

\[ \Rightarrow \frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} = 0 \]  
(2.46)

Where \( a^2 = \frac{EI}{\rho A} \)

The condition for the propagation of waves with no amplitude decay in the beam can be assumed by considering the following equation

\[ y = Ae^{i(kx - \omega \epsilon)} \]  
(2.47)

By substituting the above equation in the equation of motion (2.52) and solved results in a relation between wavenumber and frequency which is given by

\[ k^4 - \frac{\omega^2}{a^2} = 0 \]  
(2.48)

\[ \Rightarrow k^4 = \frac{\omega^2}{a^2} \]  
(2.49)

\[ \Rightarrow k^4 = \omega^2 \frac{\rho A}{EI} \]  
(2.50)
\[ \Rightarrow \omega = k^2 \sqrt{\frac{EI}{\rho A}} \quad (2.51) \]

From the above equation we can infer that the relationship between the frequency \( \omega \) and wavenumber \( k \) is not linear which shows bending waves in a beam are dispersive.

### 2.4.4.2 Analytical Solution for Timoshenko Beam:

The solution for the wave equation in the Timoshenko beam yields a polynomial equation as given below (Bendigiri, 2014):

\[
a_1c^4 + a_2c^2 + a_3 = 0 \quad (2.52)
\]

As \( c = \frac{\omega}{k} \), substituting this in the above equation gives:

\[
a_1 \left( \frac{\omega}{k} \right)^4 + a_2 \left( \frac{\omega}{k} \right)^2 + a_3 = 0 \quad (2.53)
\]

The coefficients in the above quadratic expression are given by:

\[
a_i = \left( \frac{A}{I} - \frac{\omega^2}{\left(k'c_g^2 \right)^2} \right) \quad (2.54)
\]
\[ a_2 = \left( \frac{c_e^2}{(k'c_g)^2} + 1 \right) \omega^2 \] (2.55)

\[ a_3 = -c_e^2 \omega^2 \] (2.56)

where \( c_e = \sqrt{\frac{E}{\rho}} \), \( c_g = \sqrt{\frac{G}{\rho}} \)

By substituting the above terms into the quadratic expression (2.52) results in the following expression:

\[
\left( \frac{-1}{(k'c_g)^2} \right) \omega^4 + \left( \frac{A_x}{I} + \left( \frac{c_e^2}{(k'c_g)^2} + 1 \right) k^2 \right) \omega^2 - c_e^2 k^4 = 0
\]

(2.57)

The above quadratic equation in \( \omega^2 \) when solved for the roots gives the frequencies for the given wavenumbers \( k \).

2.4.5 Comparison of Results:

The solutions obtained using the Bloch wave finite element method and analytical method for both the beam models are compared as did for the rod by plotting frequencies over the range of wavenumbers.

The following figure 2-10 shows the plot for an Euler-Bernoulli beam model:
Figure 2-10 Plot Showing the Comparison between Bloch Wave Finite Element Solution and Analytical Solution of a Euler-Bernoulli Beam

The above plot is plotted for 5 elements. As observed, the finite element solution is matching the analytical solution for two roots of frequencies.

The following figure 2-11 shows the plot for a Timoshenko beam model:
The plot shown above is obtained by taking 30 elements in the unit cell. Two roots of frequencies are shown in the plot. As observed the Bloch wave finite element solution matches with the analytical solution for the two roots.

2.4.6 Normalization of Beam:

The normalization of beam is also done as did for the rod to verify whether the solutions obtained by both the finite element method and analytical solution hold good for any material and cross section chosen.
2.4.6.1 Normalization of Finite Element Solution:

The mass and stiffness matrices in the equation of motion take the form:

\[
\begin{bmatrix}
\frac{EI}{l_e^3} & \frac{E l_e^2}{l_e^3} & -\frac{E l_e^2}{l_e^3} & \frac{E l_e^2}{l_e^3} \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2 \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\omega^2}{420} & \frac{\rho A l_e^4}{EI} \\
\frac{22l_e}{54} & 13l_e & 156 & -22l_e \\
-13l_e & -3l_e^2 & -22l_e & 4l_e^2 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\end{bmatrix} = 0 \quad (2.58)
\]

Dividing the above equation by \( \frac{l_e^3}{EI} \) we get:

\[
\begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\omega^2}{420} & \frac{\rho A l_e^4}{EI} \\
\frac{22l_e}{54} & 13l_e & 156 & -22l_e \\
-13l_e & -3l_e^2 & -22l_e & 4l_e^2 \\
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\end{bmatrix} = 0 \quad (2.59)
\]

The element lengths \( l_e \) inside the matrices can be manipulated in the following way so that there are only constant values inside them.

\[
\begin{bmatrix}
12 & 6 & -12 & 6 \\
6 & 4 & -6 & 2 \\
-12 & -6 & 12 & -6 \\
6 & 2 & -6 & 4 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\omega^2}{420} & \frac{\rho A l_e^4}{EI} \\
\frac{22l_e}{54} & 13l_e & 156 & -22l_e \\
-13l_e & -3l_e^2 & -22l_e & 4l_e^2 \\
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\end{bmatrix} = 0 \quad (2.60)
\]

\[
\omega^2 \frac{\rho A l_e^4}{EI} = \omega^2 \frac{\rho A (L/n)^4}{EI} = \left( \frac{\omega}{c_o} \right)^2 \left( \frac{L}{r} \right)^2 \frac{1}{n^4 420} = \frac{\omega^2}{n^4 420} \quad (2.61)
\]
In the above equation $l_e$ is substituted by $\frac{L}{n}$ and

$$\hat{\omega} = \omega L, \quad c_o = \frac{E}{\sqrt{\rho}}, \quad r = \sqrt{\frac{I}{A}}$$

$$\bar{\omega}^2 = \left(\frac{\hat{\omega}}{c_o}\right)^2 \left(\frac{L}{r}\right)^2$$  \hspace{1cm} (2.62)

Hence $\bar{\omega}$ is a normalized frequency and the equation (2.66) becomes

$$\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \hat{\theta}_1 \\ v_2 \\ \hat{\theta}_2 \end{bmatrix} - \frac{\bar{\omega}^2}{n^4 420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \hat{\theta}_1 \\ v_2 \\ \hat{\theta}_2 \end{bmatrix} = 0$$  \hspace{1cm} (2.63)

Where $\hat{\theta}_1 = l\theta_1$ and $\hat{\theta}_2 = l\theta_2$

The above equation can be solved for the normalized frequencies for any material and cross-section.

2.4.6.2 Normalization of Analytical Solution

The analytical solution of the beam wave equation is characterized by the relation between frequency and wavenumber given by:

$$\omega^2 = k^4 \frac{EI}{\rho A}$$  \hspace{1cm} (2.64)
Multiplying the above equation with the unit cell length $L$ in terms of $4^{th}$ power i.e., $L^4$ we get:

$$\omega^2 L^4 = (kL)^4 \frac{EI}{\rho A}$$  \hspace{1cm} (2.65)

$$\Rightarrow (\omega L)^2 L^2 = (kL)^4 \frac{EI}{\rho A}$$  \hspace{1cm} (2.66)

$$\Rightarrow (\omega L)^2 L^2 \frac{\rho A}{EI} = (kL)^4$$  \hspace{1cm} (2.67)

$$\Rightarrow \left( \frac{\omega}{c_0} \right)^2 \left( \frac{L}{r} \right)^2 = \hat{k}^4$$  \hspace{1cm} (2.68)

Substituting the above equation with equation (2.68) we get:

$$\bar{\omega}^2 = \hat{k}^4$$  \hspace{1cm} (2.69)

$$\Rightarrow \bar{\omega} = \hat{k}^2$$  \hspace{1cm} (2.70)

where $\hat{k} = \epsilon_{x} = kL$. Therefore substituting $\epsilon_{x}$ values in the above equation gives normalized frequencies $\bar{\omega}$.

2.4.7 Normalized Frequency Versus Normalized Wavenumber Plot:

A normalized frequency versus normalized wavenumber plot is presented below in figure 2-12 which shows the comparison between the finite element solution and the analytical solution.
Figure 2-12 Plot Showing the Comparison between Normalized Bloch Wave Finite Element Solution and Analytical Solution of a Euler-Bernoulli Beam

The above plot is obtained by taking 5 elements in the unit cell and the epsilon values are ranging from $0 - \pi$. The normalized frequencies are calculated for two roots. As observed the normalized Bloch wave finite element solution matches the analytical solution. Hence we can conclude that the solutions are true for any material and cross-section.
CHAPTER 3: WAVE PROPAGATION IN 2-D PERIODIC STRUCTURES

3.1 Introduction:

This chapter details the wave propagation analysis in two dimensional periodic rectangular and regular hexagonal structures. The application of Bloch wave theorem in these structures is explained. After performing the analysis by the application of Bloch wave theorem a frequency-wavenumber relationship for propagating waves in two-dimensions are demonstrated. In addition, a comparison is made to show that the frequency results for a truncated rectangular unit cell with the rectangular basis and a rhombic unit cell geometry with contra-variant basis are similar.

3.2 Bloch Wave Theorem in Rectangular 2-D Structures:

The Bloch wave theorem for one-dimensional structures was discussed earlier in Section 2.2. In this section the discussion is extended to two-dimensional structures. In two-dimensional structures waves can propagate in a direction defined by phase magnitude and orientation angle. The magnitude and angle are expressed in x and y components in two-dimensions.

Consider a simple rectangular periodic structure as shown in figure 3.1(a)
To perform the wave propagation analysis a generic reference unit cell having a horizontal length $a_x$ and a vertical length $a_y$ is selected as displayed in the figure 3-1(b) and the unit cell is associated with the direct lattice vectors, also known as basis vectors The corner nodes of the unit cell are known as lattice points. Let $x, y$ be the global coordinate
system and \(x_l, y_l\) be the local coordinate system. Consider a point \(P\) in the reference unit cell, the position vector \(\mathbf{r}\) of the point \(P\) in the local coordinate system is given by:

\[
\mathbf{r}_l = x_l \hat{i} + y_l \hat{j}
\]  

(3.1)

where \(\hat{i}, \hat{j}\) are unit vectors in the \(x\) and \(y\) directions, respectively. Let \(\mathbf{u}_l\) be the displacement of the point \(P\) in the unit cell. When a sinusoidal wave condition is imposed on the structure the displacement is given by the solution for the wave condition assuming purely propagating waves, which is:

\[
\mathbf{u}_l = u_0 e^{i k \cdot r}
\]  

(3.2)

where \(u_0\) is the amplitude and \(k\) is the wave vector given by

\[
k = k_x \hat{i} + k_y \hat{j}
\]  

(3.3)

where \(k_x, k_y\) are the components of the wave vector in the \(x, y\) directions respectively. The position vector of the point \(P\) from the global coordinate system is denoted as \(\mathbf{r}\). The periodic shift of reference unit cell is defined by

\[
\mathbf{r}_{n,m} = na \hat{i} + ma \hat{j}
\]  

(3.4)

where \((n,m)\) are integers describing the location of the unit cell in the periodic structure within a periodic unit cell. The local position vector \(\mathbf{r}_l\) is related to the global position vector by:
\[ \mathbf{r}_i = \mathbf{r} - \left( n_a \hat{i} + m_a \hat{j} \right) \] 
(3.5)

or rearranging

\[ \mathbf{r} = \mathbf{r}_i + \left( n_a \hat{i} + m_a \hat{j} \right) \] 
(3.6)

The displacement of the point \( P \) from the global coordinate system is given by

\[ \mathbf{u} = u_0 e^{jkr} \] 
(3.7)

Substituting eq. (3.3) and eq. (3.6) in the equation (3.7) gives

\[ \mathbf{u} = u_0 e^{jkr} e^{j(\{k,a,n\}+\{k,a,m\})} \] 
(3.8)

As \( \mathbf{u}_i = u_0 e^{jkr} \), the equation (3.8) can be written as

\[ \mathbf{u} = u_0 e^{j(\{k,a,n\}+\{k,a,m\})} \] 
(3.9)

\[ \Rightarrow \mathbf{u} = u_j e^{(\mu_{a+n},\mu_{a+m})} \] 
(3.10)

\[ \Rightarrow \mathbf{u} = u_j e^{\mu_{a+n}} \] 
(3.11)

where

\[ \mu = \mu_{a} \hat{i} + \mu_{b} \hat{j} \] 
(3.12)

\[ \mathbf{n} = n \hat{i} + m \hat{j} \] 
(3.13)
The terms $\mu_x = jk_x a_x$ and $\mu_y = jk_y a_y$ in eq. (3.12) are the propagation constants in the x and y directions respectively. If the wave is propagating with attenuation the propagation constants are given by:

\[
\mu_x = \delta_x + j\varepsilon_x \quad (3.14)
\]
\[
\mu_y = \delta_y + j\varepsilon_y \quad (3.15)
\]

As we are concerned only with free wave propagation without attenuation, the attenuation terms in the equations (3.14) and (3.15) are equal to zero i.e., $\delta_x = 0$ and $\delta_y = 0$. Therefore the above equations becomes

\[
\mu_x = jk_x a_x = j\varepsilon_x \quad (3.16)
\]
\[
\mu_y = jk_y a_y = j\varepsilon_y \quad (3.17)
\]

where $\varepsilon_x$ and $\varepsilon_y$ are the dimensionless quantities, known as “phase constants”. From the equations (3.16) and (3.17), the phase constants are given by the expression:

\[
\varepsilon_x = k_x a_x \quad (3.18)
\]
\[
\varepsilon_y = k_y a_y \quad (3.19)
\]

3.2.1 Matrix Reduction Using Bloch Wave theorem:

Similar to the one-dimensional case discussed earlier, the application of Bloch wave theorem with finite element analysis helps to investigate the propagation of elastic
waves in the two-dimensional periodic structure. The reference unit cell geometry is modeled using beam finite elements. The displacements acting at the nodes of the elements are designated by the displacement vector \( \{u\} \) and the forces acting on the nodes are designated by the force vector \( \{f\} \). The generalized forces and generalized displacements for beam elements include both translational and rotational degrees of freedom, and are used in the analysis of the unit cell to determine the wave characteristics. For time-harmonic waves the generalized displacements and generalized forces are related in the following way, similar to the discussion for the case of one-dimensional structures:

\[
([K] - \omega^2 [M])\{u\} = \{f\}
\]

(3.20)

where \( [K] \) and \( [M] \) are the assembled global stiffness and global mass matrices respectively for the beam elements modeling the reference unit cell and \( \omega \) is the frequency.

Figure 3-2 Rectangular Unit cell with displacements and forces labeled
The displacement and the force vectors are arranged by:

\[ \{u\} = \{u_L, u_R, u_B, u_T, u_{LB}, u_{LT}, u_{RB}, u_{RT}, u_i\}^T \]  \hspace{1cm} (3.21)

\[ \{f\} = \{f_L, f_R, f_B, f_T, f_{LB}, f_{LT}, f_{RB}, f_{RT}, f_i\}^T \]  \hspace{1cm} (3.22)

in which the displacements \( u_L, u_R, u_B, u_T, u_{LB}, u_{LT}, u_{RB}, u_{RT}, u_i \) and the forces \( f_L, f_R, f_B, f_T, f_{LB}, f_{LT}, f_{RB}, f_{RT}, f_i \) represent the displacements and forces acting on the left, right, top, bottom, left bottom corner, right bottom corner, right top corner, right bottom corner and internal nodes respectively that is clearly indicated in the figure 3-2.

The left, right, top, bottom, left bottom, right bottom, right top, right bottom nodes are the lattice points on the perimeter of the reference unit cell which are the subset of the nodes for the finite element analysis. It is assumed that there are no external forces acting on the internal nodes i.e., \( f_i = 0 \)

According to the Bloch theorem and periodicity as discussed in Section 3.2 the displacements acting on the lattice points are related by:

\[ u_R = e^{\mu_i} u_L \] \hspace{1cm} (3.23)

\[ u_T = e^{\mu_i} u_B \] \hspace{1cm} (3.24)

\[ u_{LT} = e^{\mu_i} u_{LB} \] \hspace{1cm} (3.25)

\[ u_{RB} = e^{\mu_i} u_{LB} \] \hspace{1cm} (3.26)

\[ u_{RT} = e^{(\mu_i + \mu_r)} u_{LB} \] \hspace{1cm} (3.27)
From equilibrium the forces acting on the lattice points are related as follows:

\[
\begin{align*}
    f_R &= -e^{\mu_i} f_L \\
    f_T &= -e^{\mu_J} f_B \\
    f_{LT} &= -e^{\mu_i} f_{LB} \\
    f_{RB} &= -e^{\mu_i} f_{LB} \\
    f_{RT} &= -e^{(\mu_i + \mu_J)} f_{LB}
\end{align*}
\]  

After the application of Bloch theorem the displacement and the force vectors are given as follows:

\[
\begin{align*}
    \{u\} &= [A]\{u_i\} \\
    \{f\} &= [B]\{f_i\}
\end{align*}
\]  

where \(\{u_i\} = \{u_L, u_B, u_{LB}, u_J\}^T\) and \(\{f_i\} = \{f_L, f_B, f_{LB}, f_J\}^T\) are the reduced displacement and force vectors respectively, \([A]\) and \([B]\) are the complex valued matrices given by:
\[
[A] = \begin{bmatrix}
[I] & [0] & [0] & [0] \\
[0] & [I] & [0] & [0] \\
[0] & [0] & [I] & [0] \\
[0] & [0] & [0] & [I] \\
\end{bmatrix}
\]

(3.35)

\[
[B] = \begin{bmatrix}
[I] & [0] & [0] & [0] \\
[0] & [I] & [0] & [0] \\
[0] & [0] & [I] & [0] \\
[0] & [0] & [0] & [I] \\
\end{bmatrix}
\]

(3.36)

where \([I]\) is the identity matrix and \([0]\) is the zero matrix.

The size of these matrices are given by:

Size of \([A]\) and size of \([B]\) = (number of degrees of freedom at each node * number of nodes \times (number of interior nodes + number of left end lattice points + number of bottom end lattice point) * number of degrees of freedom at each node)
Substituting the equations of reduced displacement and force equations (3.33; 3.34) into (3.20) gives:

\[
\left( [K] - \omega^2 [M] \right) [A] \{ u_r \} = [B] \{ f_r \} \tag{3.37}
\]

Now multiplying the above equation with the Hermitian transpose $[A^H]$ on both sides, the equation takes the form:

\[
\left( [A^H] [K] [A] - \omega^2 [A^H] [M] [A] \right) \{ u_r \} = \{ 0 \} \tag{3.38}
\]

The left hand side of the equation is zero as $[A^H] [B] \{ f_r \} = [0]$

Thus the reduced equation of motion obtained is given by:

\[
\left( [K_r (\mu_x, \mu_y)] - \omega^2 [M_r (\mu_x, \mu_y)] \right) \{ u_r \} = \{ 0 \} \tag{3.39}
\]

where $K_r$ and $M_r$ are the reduced stiffness and mass matrices respectively, in which $[K_r] = [A^H] [K] [A]$ and $[M_r] = [A^H] [M] [A]$ are functions of the propagation constants $\mu_x = jk_x a_x = j\epsilon_x$ and $\mu_y = jk_y a_y = j\epsilon_y$.

3.2.2 Eigen-Value Problem:

The reduced equation of motion is used to solve for the frequencies of the free wave propagation in the considered periodic structure. The eigenvalues are solved from the characteristic equation:
\[
\begin{align*}
\det \left( \left[ K, \left( \mu_x, \mu_y \right) - \omega^2 M, \left( \mu_x, \mu_y \right) \right] \right) = 0
\end{align*}
\]

(3.40)

The equation (3.40) is an eigenvalue problem, which when solved for the given values of propagation constants yields eigenvalues that are the frequencies of the free wave propagation in the structure. The values of propagation constants implies imposing the values of \( \epsilon_x \) and \( \epsilon_y \) in the equation (3.40) as \( \mu_x = jk_x a_x = j \epsilon_x \) and \( \mu_y = jk_y a_y = j \epsilon_y \).

### 3.3 Case Studies:

#### 3.3.1 Rectangular Periodic Structure:

A 2-D periodic rectangular grid lattice structure is considered to perform the Bloch wave analysis for finding the frequencies of the wave propagation as shown in the figure. For the analysis of the structure both in-plane and out-of plane bending waves are considered.

#### 3.3.1.2 Unit Cell Configuration:

The lattice is composed of rigidly joined elastic beams. The unit cell considered for the analysis consists of a horizontal and vertical beam, which is presented in the figure 3-3.
As seen in the figure 3-3 the unit cell is composed of a horizontal beam and a vertical beam, perpendicular to each other. The geometric and material properties considered for the structure are taken from the literature (B. Tie, 2013). The geometrical dimensions of both the horizontal and vertical beams are equal. The length, width and thickness of both the beams are $L = a_x = a_y, w$ and $t$ respectively where $L = 3\text{mm}$, $w = 0.12\text{mm}$ and $t = 0.2\text{mm}$. The material properties considered are Young’s modulus: $E = 70\text{GPa}$; Poisson ratio: $\nu = 0.33$; Density: $\rho = 2700\text{kg/m}^3$.

3.3.2 Bloch Wave Analysis:

3.3.2.1 Mass and Stiffness Matrices:

The whole structure is considered in the global coordinate system $x, y, z$. Each member of the unit cell is modeled as an Euler-Bernoulli Beam having a rectangular
cross-section. The unit cell can be divided into \( n \) number of finite elements. Each element of the unit cell is parameterized in its local coordinate system \( l, m, z \).

**In-Plane Wave Propagation:**

In case of in-plane wave propagation, each node of the element has assumed to have three degrees of freedom, two translations in the local \( l, m \) directions and one rotation about the \( z \) direction. The two translations include an axial displacement in the \( l \) direction and a vertical displacement in the \( m \) direction. Due to the vertical displacement the beam undergoes bending that causes a rotation about the \( z \) direction. The stiffness and mass matrices for the beam are formulated according to the Euler-Bernoulli beam theory as follows:

Elemental stiffness matrix \( k_i \) and mass matrix \( m_i \) for in-plane wave propagation in local system is formed due to axial and bending contributions respectively:

**Stiffness matrix due to axial contribution:**

\[
\frac{E A}{l_c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

(3.41)

**Mass matrix due to axial contribution:**

\[
\frac{\rho A l_c}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

(3.42)

Stiffness matrix due to bending contribution
Mass matrix due to bending contribution:

\[
\begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2
\end{bmatrix} \begin{bmatrix}
v_1 \\
\theta_1 \\
v_2 \\
\theta_2
\end{bmatrix}
\]  
(3.43)

where \(u_1, u_2\) are the axial displacements, \(v_1, v_2\) are the vertical displacements and \(\theta_1, \theta_2\) are the rotations at the elemental nodes respectively, \(l_e\) is the elemental length, \(A_t = tw\) is the cross-sectional area, \(I = \frac{wt^3}{12}\) is the moment of inertia in case of in-plane bending.

Out-of Plane Wave Propagation:

In case of out-of plane wave propagation, each node of the element is also assumed to have three degrees of freedom: one translation about the \(z\) direction and two rotations about the \(l, m\) directions respectively. The out-of plane displacement is about the \(z\)-direction, due to which the beam has bending moment that causes rotation in the \(l\) direction and twisting in the \(m\) direction. The stiffness and mass matrices for the beam is obtained by considering the standard interpolation functions for torsion and the Hermitian polynomials as shape functions for bending.
Elemental stiffness matrix $k_i$ and mass matrix $m_i$ for out of plane wave propagation in local system is formed due to bending and twisting contributions respectively:

Stiffness matrix due to bending contribution:

$$
\begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2 \\
\end{bmatrix}
\begin{pmatrix}
v_1 \\
\theta_1 \\
v_2 \\
\theta_2 \\
\end{pmatrix}
\begin{pmatrix}
\frac{EI}{l_e^3} \\
\frac{\rho A_l}{420} \\
GJ \\
\rho J l_e \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
2 & 1 \\
1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_1 \\
\phi_2 \\
\end{pmatrix}
$$

(3.45)

Mass matrix due to bending contribution:

$$
\begin{bmatrix}
156 & 22l_e & 54 & -13l_e \\
22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\
54 & 13l_e & 156 & -22l_e \\
-13l_e & -3l_e^2 & -22l_e & 4l_e^2 \\
\end{bmatrix}
\begin{pmatrix}
v_1 \\
\theta_1 \\
v_2 \\
\theta_2 \\
\end{pmatrix}
\begin{pmatrix}
\frac{\rho A_l}{420} \\
\frac{\rho J l_e}{6} \\
GJ \\
\rho J l_e \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
2 & 1 \\
1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_1 \\
\phi_2 \\
\end{pmatrix}
$$

(3.46)

Elemental Stiffness matrix due to twisting contribution:

$$
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\end{bmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\end{pmatrix}
\begin{pmatrix}
\frac{GJ}{l_e} \\
\frac{\rho J l_e}{6} \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
2 & 1 \\
1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_1 \\
\phi_2 \\
\end{pmatrix}
$$

(3.47)

Mass matrix due to twisting contribution:

$$
\begin{bmatrix}
2 & 1 \\
1 & 2 \\
\end{bmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\end{pmatrix}
\begin{pmatrix}
\frac{GJ}{l_e} \\
\frac{\rho J l_e}{6} \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
2 & 1 \\
1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_1 \\
\phi_2 \\
\end{pmatrix}
$$

(3.48)

where $v_1, v_2$ are the vertical displacements and $\theta_1, \theta_2$ are the rotations due to bending, $\phi_1, \phi_2$ are the angles of twist at the elemental nodes respectively, $G = \frac{E}{2(1+\nu)}$ is the...
shear modulus of elasticity, \( J = I_y + I_z \) is the polar moment of inertia, where \( I_y = \frac{tw^3}{12} \), \( I_z = \frac{wt^3}{12} \), \( I = I_y \) is the moment of inertia for out of plane wave propagation.

Once the computation of elemental stiffness and mass matrices is done in the local coordinate system, they are transformed into the global coordinate system using a transformation matrix as given below:

\[
\begin{bmatrix}
    k_x \\
    m_x
\end{bmatrix} = [T] \begin{bmatrix}
    k_i \\
    m_i
\end{bmatrix} [T]^T
\]

\[
\begin{bmatrix}
    k_y \\
    m_y
\end{bmatrix} = [T] \begin{bmatrix}
    k_i \\
    m_i
\end{bmatrix} [T]^T
\]

(3.49)

(3.50)

where \([T]\) is the transformation matrix given by:

\[
[T] = \begin{bmatrix}
    [R] & [0] \\
    [0] & [R]
\end{bmatrix}
\]

(3.51)

in which \([R]\) is the rotation matrix that is different for both in-plane and out of plane wave propagation defined as:

Rotation matrix for in-plane condition:

\[
[R] = \begin{bmatrix}
    \cos \alpha & \sin \alpha & 0 \\
    -\sin \alpha & \cos \alpha & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]

(3.52)

Rotation matrix for out of plane condition:
\[ [R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \tag{3.53} \]

where \( \alpha \) is the angle between the local coordinate system and the global coordinate system.

After the transformation of local elemental matrices into global elemental matrices, these matrices are assembled into the global mass \([M]\) and global stiffness \([K]\) matrices respectively.

3.3.2.2 Matrix Reduction:

The equation of motion \( ([K] - \omega^2 [M]) \{u\} = \{f\} \) which relates the generalized displacements and forces is obtained when the structure undergoes harmonic motion. The equation of motion is reduced to perform the analysis by applying the Bloch theorem as discussed in the section 3.2. For this particular rectangular grid structure and unit cell, the nodes in consideration are not present on the corners of the rectangle, therefore the displacement vector and the force vector are given by

\[ \{u\} = \{u_L, u_R, u_B, u_T, u_i\}^T \tag{3.54} \]

\[ \{f\} = \{f_L, f_R, f_B, f_T, f_i\}^T \tag{3.55} \]

According to the Bloch theorem, the boundary displacements and forces are related as follows:
\[ u_R = e^{ij\mu}u_L \]  
\[ u_T = e^{ij\mu}u_B \]  
\[ f_R = -e^{ij\mu}f_L \]  
\[ f_T = -e^{ij\mu}f_B \]

with reduced displacement vector \( \{u_i\} = \{u_L, u_B, u_i\}^T \) and reduced force vector \( \{f_i\} = \{f_L, f_B, f_i\}^T \). The complex valued matrix \([A]\) that is used in the matrix reduction is given by:

\[
[A] = \begin{bmatrix}
[I] & [0] & [0] \\
[0] & [I] & [0] \\
[0] & [0] & [I] \\
\end{bmatrix}
\]

The size of the complex valued matrix \([A]\) is calculated as below:

Number of nodes = \( (n+1) \), where \( n \) are the number of elements

Number of degrees of freedom at each node = 3

Number of corner nodes = 4

Number of Interior nodes = Number of nodes - Number of corner nodes = \( (n+1)-4 \)

Number of left end lattice points of the unit cell = 1
Number of bottom end lattice points of the unit cell = 1

Size of $[A] = (3n+3 \times 3n-3)$

The matrices are reduced similarly as did in section 3.2.1 for a generic rectangular unit cell. The reduced equation of motion obtained after the matrix reduction is treated as an eigenvalue problem i.e., the equation is given by:

$$\det \left( K_r \left( \mu_x, \mu_y \right) - \omega^2 M_r \left( \mu_x, \mu_y \right) \right) = 0$$

(3.61)

The above equation, when imposed the values of the propagation constants i.e.,

$$\mu_x = jk_x a_x = j \epsilon_x \quad \text{and} \quad \mu_y = jk_y a_y = j \epsilon_y$$

, where for this structure $a_x = a_y = L$ is solved for the eigenvalues. These eigenvalues are the frequencies of the wave propagation in the structure. The phase constants $\epsilon_x$ and $\epsilon_y$ values are considered in the first Brillouin zone, i.e., the values of each phase constant range between $(-\pi, \pi)$.
3.3.3 Results for the Wave Propagation of the Rectangular Unit Cell:

The unit cell consisting of a horizontal and vertical beam perpendicularly bisecting each other is meshed into finer number of elements to perform the Bloch wave finite element analysis to study its response to both in-plane and out-of plane wave propagation. The results obtained are verified with the results presented in the literature (B. Tie, 2013) to verify the analysis procedures and to interpret the results.

The figure 3-4 displayed below is the mesh refinement of the unit cell on which the analysis is performed for both in-plane and out-of plane wave propagation.

![Figure 3-4 Mesh generation of the rectangular unit cell with two beams](image)

As discussed earlier the equation of motion in the form of eigenvalue problem is solved for the frequencies of the wave propagation by imposing the values of phase constants in the first Brillouin zone i.e., $\epsilon_x, \epsilon_y$ ranges between $(-\pi, \pi)$. 
Figures 3-5(a), (b); 3-6(a), (b) are the surface and contour plots displaying frequencies obtained for the assigned phase constants $\varepsilon_x, \varepsilon_y$ ranging between $(-\pi, \pi)$. 

Figure 3-5 (a), (b) Surface and Contour Plot for Frequency versus Phase Constants (Normalized Wavenumbers) $(-\pi, \pi)$ for First Shear Wave Mode (In Plane Wave Propagation)

Figure 3-6 (a), (b) Surface and Contour Plot for Frequency versus Phase Constants (Normalized Wavenumbers) $(-\pi, \pi)$ for Second Pressure Wave Mode (In Plane Wave Propagation)
also known as normalized frequencies for the 1\textsuperscript{st} and 2\textsuperscript{nd} wave modes at In-plane wave propagation. The two wave modes displayed are the shear and pressure wave modes respectively, which are the only two physical wave modes as these two modes have zero initial frequencies which is clearly demonstrated in k-space diagram in figure 3-7.

As the rectangular structure is symmetric about the x and y axis, and with a rectangular unit cell defined with rectangular basis, the propagation of the waves through the structure is expected to be symmetric with respect to the x and y axis, which is evident by observing the contour plots for both wave modes. Figure 3-5 (b) shows that the wave directionality is ani-isotropic.
Fig 3-7(a) Contour Plot Displaying the Irreducible Brillouin Zone, (b) k-space Diagram for In-Plane Wave Propagation

As the propagation of waves is symmetric through the Brillouin zone, a small portion of it is sufficient to study the wave propagation. This small portion is known to be irreducible zone. Figure 3-7 (b) is the k-space diagram plotted in the irreducible zone as shown in figure 3-7(a) for In-plane wave propagation. The k-space diagram is representing the frequencies for the irreducible zone (k-space) for eight eigenvalues. The k-space curve with the lowest frequency corresponds to the first eigenvalue. When observed, two curves have zero frequency that has been explained earlier having two physical modes shear and pressure modes. The curves represent wave propagation along directions that envelope the full range. The plots obtained for the results are matching
with the results presented in the literature (B. Tie, 2013). The results shown in the k-space diagrams are consistent with the data from the contour and surface plots shown earlier. From point 0 to point Z1 corresponds to waves propagating in the positive x-direction only with $\epsilon_y = 0$. From Z1 to Z2 corresponds to wave propagating with fixed $\epsilon_x$, and variable $\epsilon_y$. From Z2 to Z3 is the symmetry of Z1 to Z2, while Z3 to 0 is the symmetry of 0 to Z1.

Figures 3-8 (a), (b) are the surface and contour plots displaying frequencies obtained for the assigned phase constants $\epsilon_x, \epsilon_y$ ranging between $(-\pi, \pi)$ also known as normalized frequencies for out of plane wave propagation. One physical wave mode, which is the shear wave made as only one frequency mode has zero frequency which can be evident in figure 3-9. The wave propagation through the structure is symmetric about

![Figure 3-8 (a), (b) Surface and Contour Plot for Frequency versus Phase Constants (Normalized Wavenumbers) (-\pi, \pi) for First Shear Wave Mode (Out of Plane Wave Propagation)]
x and y axis as discussed earlier for In-plane wave propagation. It is observed from figure 3-8 (b), there is isotropy in the wave directionality for lower frequencies and anisotropy while approaching higher frequencies.

[Figure 3-9: k-space Diagram for Out of Plane Wave Propagation]

Figure 3-9 is the k-space diagram for out of plane wave propagation, displayed for 8 eigenvalues. As discussed earlier for out of plane wave propagation only one physical mode is present which is evident from k-space diagram, where only one curve has zero frequency. Even the out of plane wave propagation plots are matching with the results presented in the literature (B. Tie, 2013). This indicates that the wave propagation analysis of the structure using Bloch wave theorem is done properly. From Z2 to Z3 is the symmetry of Z1 to Z2, while Z3 to 0 is the symmetry of 0 to Z.
3.3.4 Regular Hexagonal Periodic Structure:

A two-dimensional regular hexagonal periodic structure is considered. Both in-plane and out of plane wave propagation analysis is performed on the structure. The choice of unit cell in this case is not unique. Truncated hexagonal and a rhombic unit cell are considered on which the Bloch wave analysis is performed. Also, a comparison is done to show that the frequency contours obtained for the wave vector components are similar.

3.3.4.1 Unit-Cell Configuration:

Truncated Hexagonal Unit Cells:
A truncated hexagonal unit cell is considered as shown in the figure 3-10(b). The unit cell is divided into $n$ number of beam elements. The geometry of both the unit cells are defined using their cell lengths $a_x$ and $a_y$. The unit cell has an internal angle $\theta = 30^\circ$, for a regular hexagonal. Each element of the unit cell is modeled as an Euler-Bernoulli beam element having a square cross section. The material and geometric properties, considered for the study are taken from the literature (Massimo Ruzzene, 2003), which are given by Young’s modulus $E = 5$ GPa; Poisson’s ratio $\nu = 0.3$; Density $\rho = 1300$ kg/m$^3$; the cell length $a_x = L = 21$ cm; the square cross section has a side dimension $t = 3$ mm.

For the truncated hexagonal $\frac{a_x}{a_y} = \sqrt{3}$. This ratio is calculated based on the considered unit cell geometry and whose internal angle $\theta = 30^\circ$. 

---

*Fig 3-10(a) Honeycomb Structure Highlighting the Unit Cell, (b) Truncated Hexagonal Unit Cell*
3.3.5 Bloch Wave Analysis:

3.3.5.1 Mass and Stiffness Matrices:

As discussed earlier each element of the unit cell is modeled as an Euler-Bernoulli beam element. The elemental mass and stiffness matrices are the same for these elements of the unit cell as discussed in the section (3.3.2.1). As the cross section is now a square in this case, area of cross section \( A_r = l^2 \); moment of inertia \( I = \frac{l^4}{12} \); polar moment of inertia \( J = 2I \). Once the elemental stiffness and mass matrices are calculated as described in the section (3.3.2.1), they are assembled into the global stiffness and mass matrices.

3.3.5.2 Calculation of Frequencies:

The equation of motion \(( [K] - \omega^2 [M]) \{u\} = \{f\} \) is reduced by applying the Bloch theorem as described in section (3.2). The size of the complex valued matrix \([A]\) for truncated hexagonal unit cell is given by:

**Truncated Hexagonal Unit Cell:**

Number of nodes = \( n+1 \), where \( n \) are the number of elements

Number of degrees of freedom at each node = 3

Number of corner nodes = 6

Number of Interior nodes = Number of nodes-Number of corner nodes = \((n+1)-6\)

Number of left end lattice points of the unit cell = 1

Number of bottom end lattice points of the unit cell = 2

Size of \([A]\) = \((3n+3 \times 3n-3)\)
The matrices are reduced similarly as did in section (3.2.1) for a generic rectangular unit cell. The reduced equation of motion obtained after the matrix reduction is treated as an eigenvalue problem i.e., the equation is given by:

\[
\det\left[ K_r \left( \mu_x, \mu_y \right) - \omega^2 M_r \left( \mu_x, \mu_y \right) \right] = 0 \tag{3.62}
\]

The equation (3.62), when imposed the values of the propagation constants i.e., \( \mu_x = jk_x a_x = j\epsilon_x \) and \( \mu_y = jk_y a_y = j\epsilon_y \), where for the full hexagonal unit cell \( a_x = L \) and \( a_y = \frac{\sqrt{3}}{2} L \); for the truncated hexagonal unit cell \( a_x = L \) and \( a_y = \sqrt{3}L \) is solved for the eigenvalues. These eigenvalues are the frequencies of the wave propagation in the structure. The phase constants \( \epsilon_x \) and \( \epsilon_y \) values are considered in the first Brillouin zone, i.e., the values of each phase constant range between \((-\pi, \pi)\).

### 3.3.6 Results for Truncated Hexagonal Unit Cell:

The other unit cell that is being considered for the wave propagation analysis is the truncated hexagonal unit cell. In this case a 24 element truncated hexagonal unit cell is presented after verifying the results that are converging for different mesh (9 and 18 elements).

#### 3.3.6.1 Results for Truncated Hexagonal Unit Cell (24-Elements):

Presented in the figure 3-11 is the truncated hexagonal unit cell with 24 elements. The node numbering is done according to the proper alignment as discussed in the theory section.
Fig 3-11 Truncated Hexagonal Unit Cell with 24 Elements

Fig 3-12 (a), (b) Surface and Contour Plot for Frequency versus Phase Constants (Normalized Wavenumbers) (-\pi, \pi) for First Shear Wave Mode (In Plane Wave Propagation)
Figure 3-12 (a), (b); 3-13 (a), (b) are the surface and contour plots showing the dispersion relationships i.e., the frequencies obtained for the assigned pair of phase constants $\varepsilon_x, \varepsilon_y$ ranging between $(-\pi, \pi)$ which is said to be the first Brillouin zone for the In-plane wave propagation. Similar to rectangular unit cell, the truncated hexagonal unit cells have two physical modes. Figures 3-12 (a), (b) represent the surface and contour plot for $1^{st}$ shear wave mode whereas $2^{nd}$ pressure wave mode is presented in Figures 3-13 (a),(b). When observed from figure 3-10 (b), the truncated unit cell is symmetric in the vertical direction. The shapes of the contour plots have significant variations in frequency especially when the normalized wavenumber reaches the Brillian zone limit of $(-\pi, \pi)$, which shows the anisotropic directionality for the honeycomb periodic lattice structure. For the in-plane case, as the normalized epsilon_x and epsilon_y approach zero the directionality approaches nearly isotropic behavior as can be
seen by the nearly circular shape to the contours. This isotropic behavior for honeycomb as the frequency approaches zero is in contrast to the rectangular lattice structure which was strongly anisotropic even in the zero limit.

![Contour Plot Displaying the Irreducible Brillouin Zone](image1)

**Fig 3-14 (a) Contour Plot Displaying the Irreducible Brillouin Zone, (b) k-space Diagram for In Plane Wave Propagation**
Figure 3-14 (b) is the k-space diagram plotted in the irreducible zone as shown in figure 3-10(a) for In-plane wave propagation. The k-space diagram is representing the frequencies for the irreducible zone (k-space) for eight eigenvalues. When observed, two curves have zero frequency that has been explained earlier having two physical modes shear and pressure modes. The curves represent wave propagation along directions that envelope the full range. The results shown in the k-space diagrams are consistent with the data from the contour and surface plots shown earlier. There are no bandgaps present for the range of frequency modes considered as observed from k-space diagram.

Figures 3-8 (a), (b) are the surface and contour plots displaying frequencies obtained for the assigned phase constants \( \epsilon_x, \epsilon_y \) ranging between \((-\pi, \pi)\) also known as normalized frequencies for out of plane wave propagation. In this case there exists only one physical wave mode (zero frequency) as opposed to two physical wave modes in case of in-plane wave propagation. It is observed from figure 3-15 (b), the contours are...
elliptic in shape, with different frequency values in x and y directions. This indicates anisotropic behavior of the wave directionality for out of plane wave propagation.

3.3.7 Rhombic Unit Cell:

A periodic hexagonal lattice is considered in which the smallest possible unit cell consisting of three beams as shown in the figure 3-16 (b) is taken.

Fig 3-16 (a) Honeycomb Structure Highlighting the Unit Cell, (b) Rhombic Unit Cell
This is different when compared to the other two unit cells that are considered, as the basis is not rectangular. This kind of unit cell can be induced into a generic unit cell taken from a periodic lattice as shown in the figure 3-17(a), (b).

Fig 3-17 (a) Rhombic Periodic Structure, (b) Generic Unit Cell
A Bloch wave analysis is performed on this unit cell to know the wave characteristics. The unit cell is associated with the lattice vectors $d_1, d_2$. Let $P$ be any point in the unit cell. Let $x, y$ be the global coordinate system and $x_l, y_l$ be the local coordinate system. The position vector $r_i$ of the point $P$ is given by:

$$ r_i = x_i \hat{i} + y_i \hat{j} \quad (3.63) $$

The reference unit cell is mapped to a cell as shown in the figure, which has $\eta_1, \eta_2$ as its coordinate system.

The local coordinates $x_l, y_l$ are related to the coordinates $\eta_1, \eta_2$ through a lattice matrix $D = \begin{bmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{bmatrix}$ in the following way:

$$ \begin{pmatrix} x_l \\ y_l \end{pmatrix} = \begin{bmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad (3.64) $$

After mapping the position vector $r_i$ can be written in terms of the lattice vectors $d_1, d_2$ as follows:

$$ r_i = x_i \hat{i} + y_i \hat{j} = \eta_1 d_1 + \eta_2 d_2 \quad (3.65) $$
Let $\mathbf{u}_l$ is the displacement of the point $P$ in the unit cell. When a sinusoidal wave condition is imposed on the structure the displacement equation is given by the solution for the wave condition assuming purely propagating waves, which is:

$$\mathbf{u}_l = u_0 e^{i \mathbf{k} \cdot \mathbf{r}} \quad (3.66)$$

where $u_0$ is the amplitude and $\mathbf{k} = k_x \hat{i} + k_y \hat{j}$ is the wave vector.

Let $n, m$ are the integers that describe the cell location in the periodic structure. The position vector of the point $P$ when described from the global coordinate system is given by the equation:

$$r = x_l \hat{i} + y_l \hat{j} + n \mathbf{d}_1 + m \mathbf{d}_2 \quad (3.67)$$

From the global coordinate system, let $\mathbf{u}$ be the displacement of the point, then according to the Bloch theorem the displacement is given by:

$$\mathbf{u} = u_0 e^{i \mathbf{k} \cdot \mathbf{r}} \quad (3.68)$$

Substituting the equation (3.67) in the equation (3.68) we get:

$$\mathbf{u} = u_0 e^{i \mathbf{k} \cdot (x_l \hat{i} + y_l \hat{j} + n \mathbf{d}_1 + m \mathbf{d}_2)} \quad (3.69)$$

As $\mathbf{u}_l = u_0 e^{i \mathbf{k} \cdot \mathbf{r}}$, imposing this equation in the equation (3.69) and modifying it gives:

$$\mathbf{u} = \mathbf{u}_l e^{i (\mathbf{k} \cdot \mathbf{d}_1 + m \mathbf{k} \cdot \mathbf{d}_2)} \quad (3.70)$$
Consider a reciprocal lattice that is convenient to use in the wave vector space or \( k \) space. The reciprocal lattice is defined by the lattice vectors \( \mathbf{b}_1, \mathbf{b}_2 \). The basis vectors and the reciprocal lattice vectors are related in such a way that they satisfy the following equation:

\[
\mathbf{b}_i \cdot \mathbf{d}_j = \delta_{ij} \tag{3.71}
\]

Where \( \delta_{ij} \) is known as kronecker delta, whose value is 0 if \( i \neq j \) and 1 if \( i = j \). For a two dimensional structure the values of the subscripts \( i \) and \( j \) take the values 1 and 2.

The wave vector \( \mathbf{k} \) in the reciprocal lattice is expressed as:

\[
\mathbf{k} = \mu_1 \mathbf{b}_1 + \mu_2 \mathbf{b}_2 \tag{3.72}
\]

Where \( \mu_1 = j \varepsilon_1, \mu_2 = j \varepsilon_2 \) are the propagation constants in which only free wave propagation is considered without attenuation. \( \varepsilon_1, \varepsilon_2 \) are the phase constants.

Substituting the wave vector equation (3.71) into the equation (3.69) yields:

\[
\mathbf{u} = \mathbf{u}_i e^{j(\sigma_1 \mathbf{b}_1 + \sigma_2 \mathbf{b}_2)} \tag{3.73}
\]

\[
\mathbf{u} = \mathbf{u}_i e^{\mu \cdot \mathbf{u}} \tag{3.74}
\]

As

\[
\mathbf{k} \cdot \mathbf{d}_1 = \mu_1 \tag{3.75}
\]

\[
\mathbf{k} \cdot \mathbf{d}_2 = \mu_2 \tag{3.76}
\]
As mentioned earlier, a smallest unit cell consisting of three beams having a rectangular cross section. The geometry of the unit cell is defined by its internal angle \( \theta = 30^\circ \), relative cell wall length \( \alpha = \frac{H}{L} = 1 \) and wall’s slenderness ratio \( \beta = \frac{t}{L} = \frac{1}{15} \),

where \( H \) is the vertical height of the beam, \( L \) is the slant height of the beam and \( t \) is the thickness of the beam. The material and geometric properties, considered for the unit cell are taken from the literature (Massimo Ruzzene, 2003), which are given by Young’s modulus \( E = 5 \) GPa; Poisson’s ratio \( \nu = 0.3 \); Density \( \rho = 1300 \) kg/\( m^3 \); the slant length \( L = 21 \) cm = \( H \) as \( \alpha = 1 \); Thickness \( t = \frac{L}{15} \) and width \( W = 1 \) for the rectangular cross section.

3.3.8 Bloch Wave Analysis:

3.3.8.1 Mass and Stiffness Matrices:

Each beam in the unit cell is divided into \( n \) number of elements. Each element is modeled as an Euler Bernoulli Beam element, where in each node of the beam has assumed to have three degrees of freedom. The elemental mass and stiffness matrices of the beam elements are similar to those matrices discussed in section 3.3.2. As the beam elements have rectangular cross section in this case, area of cross section \( A_y = T \times W \); moment of inertia \( I = \frac{t^3 w}{12} \); polar moment of inertial \( J = I_y + I_z \), where \( I_y = \frac{t w^3}{12} \) and \( I = \frac{w t^3}{12} \). Once the elemental stiffness and mass matrices are calculated as described in the section 3.3.2, they are assembled into the global stiffness and mass matrices.
3.3.8.2 Matrix Reduction:

The equation of motion \((K - \omega^2 M)\{u\} = \{f\}\) is obtained after performing the standard finite element procedures and when the structure is exposed to the harmonic motion. The application of Bloch theorem for this particular unit cell structure and unit cell, the corner nodes present are different when compared to the rectangular unit cells, therefore the displacement vector and the force vector are given by

\[
\{u\} = \{u_1, u_2, u_3, u_4\}^T
\]

\[
\{f\} = \{f_1, f_2, f_3, f_4\}^T
\]

According to the Bloch theorem, the boundary displacements and forces are related as follows:

\[
u_2 = e^{i\mu} u_1
\]

\[
u_3 = e^{i\mu} u_1
\]

\[f_2 = -e^{i\mu} f_1
\]

\[f_3 = -e^{i\mu} f_1
\]

From the above equation now the reduced displacement vector \(\{u_r\} = \{u_1, u_1\}^T\) and reduced force vector \(\{f_r\} = \{f_1, f_1\}^T\). The displacement vector and force vector can be rewritten in the matrix form as:
The complex valued matrix $[A]$ that is used in the matrix reduction is given by:

$$[A] = \begin{bmatrix} I & 0 \\ e^{i\alpha} I & 0 \\ e^{i\beta} I & 0 \\ 0 & I \end{bmatrix}$$  \hspace{1cm} (3.85)$$

The size of the complex valued matrix $[A]$ is calculated as below:

Number of nodes = $(n+1)$, where $n$ are the number of elements

Number of degrees of freedom at each node = 3

Number of corner nodes = 3

Number of Interior nodes = Number of nodes-Number of corner nodes = $(n+1)-3$

Number of top end lattice points of the unit cell = 1

Size of $[A] = (3n+3 \times 3n-3)$

By substituting the equations (3.82) and (3.83) into the equation of motion and performing the pre-multiplication as did in section (name the section), the reduced equation of motion is

$$\left[[K_1, (\mu_1, \mu_2) - \omega^2 M_2 (\mu_1, \mu_2)]\right]\{u_1\} = \{0\}.$$

The reduced equation of motion obtained after the matrix reduction is treated as an eigenvalue problem i.e., the equation is given by:
The above equation, when imposed the values of the propagation constants i.e., \( \mu_x = j \epsilon_1 \) and \( \mu_2 = j \epsilon_2 \) is solved for the eigenvalues. These eigenvalues are the frequencies of the wave propagation in the structure. The phase constants \( \epsilon_1 \) and \( \epsilon_2 \) values are considered in the first Brillouin zone, i.e., the values of each phase constant range between \((-\pi, \pi)\). This particular range is considered as the solution is periodic only in this zone of the reciprocal lattice.

3.3.9 Results for the Rhombic Unit Cell:

Figure 3-28 shows the rhombic unit cell with 30 elements. A wave propagation analysis is performed on this unit cell for both in-plane and out-of plane wave propagation. The geometric and material properties used for the analysis is presented in the section (3.3.8).

![Fig 3-18 Mesh Generation of Rhombic Unit Cell](image-url)
Figure 3-19 (a), (b) is the surface and contour plot showing the dispersion relations in which the frequencies are obtained for the assigned pair of phase constants $\varepsilon_1$, $\varepsilon_2$ in the range $(-\pi, \pi)$ for the in-plane wave propagation respectively. As the basis considered for this unit cell is the contra variant basis, the contour plots obtained for both in-plane and wave propagation represents the direction and orientation of the wave propagation in the entire lattice which is observed to be in the direction of the basis aligned to the unit cell. Also the contour plots are symmetric about the contra variant axis. There are frequency variations all over the region which is observed from the contour plots. When the frequency is approaching zero the directionality of the waves exhibit nearly isotropic behavior that can be observed from the nearly circular contours present at lower frequencies.
Figure 3-20 is the contour plot of rhombic unit cell that has been scaled and transformed to rectangular basis by considering the lengths of the truncated unit cell. It can be observed that the shape of the contour plot is same to the contour plot before scaling and transformation for in-plane wave propagation.
Figure 3-20 (a), (b) is the surface and contour plot showing the dispersion relations in which the frequencies are obtained for the assigned pair of phase constants $\varepsilon_1, \varepsilon_2$ in the range $(-\pi, \pi)$ for the out of plane wave propagation respectively. The contour plots are symmetric about the contra variant axis. It is observed from the contour plot that the contours have oval like shape at lower frequencies and deviate further for high frequencies. It can be inferred that anisotropic behavior exists for the directionality of waves.

3.3.10 Comparison of Truncated Hexagonal and Rhombic Unit Cells:

A frequency-wavenumber comparison is done for the truncated hexagonal and rhombic unit cell structures by comparing the contour plots of both the unit cells for in-plane and out of plane wave propagation. The area of the truncated unit cell in rectangular basis is twice to the area of rhombic unit cell having contra variant basis.
Figure 3-22 (a) Contour Plot for Truncated Hexagonal Unit Cell, (b) Contour Plot for Rhombic Unit Cell (In-Plane Wave Propagation)

Figure 3-22(a), (b) represents the contour plots for truncated unit cell having rectangular basis and rhombic unit cell having contra variant basis at in-plane wave propagation condition. The shapes of the two contour plots are similar to each other. From the previous observations being made, both these contours approach isotropy for wave directionality at lower frequencies as the contour approaches nearly circular shape as observed from both the contour plots. At increasing frequencies the directionality of waves have anisotropy.
Figure 3-23(a), (b) represents the contour plots for truncated unit cell having rectangular basis and rhombic unit cell having contra variant basis at out of wave propagation condition. The frequency contours for both truncated and rhombic unit cells are comparable as they have similar wave propagation direction even though their basis directions are different. The wave directionality exhibits anisotropic behavior for both truncated and rhombic unit cells which is observed from the contour plots.
CHAPTER 4: CONCLUSIONS

Wave propagation analysis in one-dimensional and two-dimensional periodic structures are performed. In the case of one-dimensional periodic rod and beam structures, a general theory of Bloch wave technique has been explained in detail. The results obtained by numerical analysis i.e., with the help of finite element analysis and Bloch theorem are compared to the exact analytical solution to that of a rod and beam. In the given frequency modes it is observed that the dispersion curves for both numerical and analytical solutions are converging i.e., the technique used for periodic structure is proved to be accurate. Also, normalization technique has been executed to verify that the solutions obtained are independent for any material properties used and length of the unit cell.

In case of two-dimensional periodic structure analysis, the basic theory of Bloch wave technique for considered rectangular, truncated and rhombic unit cells are discussed. The results of wave propagation analysis in rectangular structure performed by considering both in-plane and out of plane wave propagation are compared to the results in literature (B. Tie, 2013). This verifies the accuracy of application of Bloch theorem in these kinds of structures. Dispersion curves obtained by the analysis of two different types of unit cell geometries of a hexagonal periodic lattice are presented for both in-plane and out-of plane wave propagation.

It is observed that the dispersion curves obtained for truncated unit cell in the rectangular basis when compared to rhombic unit cell in the contra variant basis are similar. The comparison is performed by discretizing the beam elements of the respective
unit cells until a more converging dispersion curves are obtained. It is also observed from the curves that the hexagonal structure exhibits nearly isotropic behavior at lower frequencies and anisotropic behavior at increasing frequency for in-plane wave propagation. In case of out of plane wave propagation, anisotropic behavior is observed in the hexagonal structure. The frequency contours for the wave vector components obtained for truncated hexagonal and rhombic unit cells are similar.
WORKS CITED


