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Characterizing Unmixed Trees with Respect to Total Domination and PMU Covers

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Characterizing unmixed trees with respect to total domination and PMU covers

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Abstract

One of the fundamental connections between commutative algebra and graph theory is the relationship between edge ideals and minimal vertex covers of a graph. Let $G = (V, E)$ be a simple graph with $V = \{x_1, \ldots, x_d\}$. The edge ideal of a graph is the ideal $I_G = (x_i x_j) \subsetneq k[x_1, \ldots, x_d]$ generated by the edges $x_i x_j \in E$. A vertex cover graph G is a set of vertices $V' \subset V$ such that every edge in G is incident to a member of V'. In 1990, Villarreal showed the edge ideal of a tree Cohen-Macaulay if and only if the tree is unmixed with respect to vertex covers, i.e. all of its minimal vertex covers have the same size.

In this dissertation, we consider two related graph domination problems and study their associated ideals in commutative algebra. The first problem is the PMU placement problem which has its roots in electrical engineering. We prove an analogous result in this setting, namely that the power edge ideal of a tree is Cohen-Macaulay if and only if the tree is unmixed with respect to PMU covers. The second problem is total domination in graph theory for which we define the open neighborhood ideal of a graph. We prove that the open neighborhood ideal of a tree is Cohen-Macaulay if and only if the tree is unmixed with respect to total domination. In each setting, we give a precise characterization of the unmixed trees.

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Chapter 1

Introduction

The study of monomial ideals in polynomial rings has proven useful in understanding the connections between commutative algebra and combinatorics. In particular, monomial ideals have been used to study properties of finite simple graphs. For a graph $G = (V, E)$ with vertex set $V = \{x_1, \ldots, x_d\}$, one may consider an associated polynomial ring $R = k[x_1, \ldots, x_d]$ with d variables over a field k. From the edges of the graph, one may define a variety of corresponding monomial ideals, including the edge ideal, the power edge ideal, and the open neighborhood ideal, each of which can tell us something about the graph G. While edges ideals have been well studied, the power edge ideal and the open neighborhood ideal have not; they are the main objects of study in this dissertation.

1.1 Edge Ideals

In 1990, Villarreal [20] defined the edge ideal I_G of a graph G to be the ideal generated by the edges of G within its associated polynomial ring $R = k[x_1, \ldots, x_d]$. Villarreal discovered a number of connections between the structure of a graph and the algebraic properties of its edge ideal. These connections to graph theory are centered around the minimal vertex covers of the graph. Given graph $G = (V, E)$, a *vertex cover* of G is a set $V' \subset V$ such that every edge in E is incident to at least one member of V'. A vertex cover of G is *minimal* if it does not properly contain another vertex cover of G. We say that G is *unmixed with respect to vertex covers* (also known as *well covered* [18]) if every minimal vertex cover of G has the same size.

A fundamental connection between the edge ideal I_G and its corresponding graph G can be found by taking the m-irreducible decomposition of an edge ideal $I_G = J_1 \cap \cdots \cap J_m$ and noticing that the generators of each J_i form a minimal vertex cover of G. From this fact, Villarreal showed that if the ring R/I_G is Cohen-Macaulay, then G is unmixed with respect to vertex covers. Villarreal went on to characterize the unmixed trees with respect to vertex covers and prove that their corresponding edge ideals are Cohen-Macaulay.

Theorem 1.1.1 ([20, Theorem 2.4 and Corollary 2.5])**.** *Let $G = (V, E)$ be a tree. The following are equivalent:*

1. G is unmixed with respect to vertex covers.

2. G is a suspension, i.e. every $v \in V$ with $\deg(v) > 1$ is adjacent to exactly one leaf.

3. I_G is Cohen-Macaulay.

These ideas form significant motivation for the new results in this dissertation, so we devote Chapter 2 to them.

1.2 Power Edge Ideals

Chapter 3 of this dissertation is devoted to a more recent algebraic construction called the power edge ideal. This notion was motivated by a desire to use ideals like those from Section 1.1 to understand the Phasor Measurement Unit (PMU) problem in electrical engineering (see [2], [3], [9], [11], [12] and [17] for more about PMU placements). This problem asks for the optimal placements of sensors, called PMUs, in an electrical power system to monitor the system for outages. If we consider a simple graph $G = (V, E)$ to be the representation of an electrical power system where the edges represent power lines and the vertices represent electrical buses, we can define a *PMU cover* of G to be a set $P \subset V$ such that voltage and current of ever power line and bus is monitored by PMU placed on the buses in P.

In 2015, Moore, Rogers, and Sather-Wagstaff [15] defined the *power edge ideal* $I_G^P \subsetneq R = k[x_1, \ldots, x_d]$ of a graph G to be the intersection of the ideals generated by the minimal PMU covers of G. This definition is analogous to the edge ideal being the intersections of ideals generated by the minimal vertex covers of G.

Our main result in Chapter 3 is the characterization of the set of unmixed trees with respect to PMU covers together with the proof that the power edge ideal of an unmixed tree is Cohen-Macaulay. This result is similar to Theorem 1.1.1 for edge ideals and vertex covers.

Theorem 1.2.1 (Theorem 3.1.1)**.** *Let $G = (V, E)$ be a tree. The following are equivalent:*

1. *G is unmixed with respect to PMU covers.*

2. *Every $v \in V$ with $\deg(v) > 2$ is adjacent to exactly two vertices of degree ≤ 2.*

3. *I_G^P is Cohen-Macaulay.*

4. *I_G^P is a complete intersection.*

The material in Chapter 3 is joint work with Michael Cowen, Alan Hahn, Frank Moore, and Keri Sather-Wagstaff.

1.3 Open Neighborhood Ideals

This section introduces the material for Chapter 4 in which we define the open neighborhood ideal of a simple graph. Our goal is to generate Cohen-Macaulay rings from unmixed graphs with respect to total domination (also known as *total well dominated graphs* [19]). Total domination is another well-studied graph domination problem (see [1], [10] and [14]). Given a graph $G = (V, E)$, a *total dominating set* is a set $D \subset V$ such that every vertex in G is adjacent to at least one member of D. A total dominating set of G is *minimal* if it does not properly contain another total dominating set of G.

We define the *open neighborhood ideal* $I_G^N \subsetneq R = k[x_1, \ldots, x_d]$ of a graph G to be the ideal generated by the product of neighbors of each vertex in G. Equivalently, I_G^N can be defined as the intersection of ideals generated by the minimal total dominating sets of G. This second definition is analogous to the edge ideal being the intersections of ideals generated by the minimal vertex covers of G and the power edge ideal being the intersection of ideals generated by the minimal PMU covers of G.

Our main results in Chapter 4 are the characterization the set of unmixed trees with respect to total domination together with the proof that the open neighborhood ideal of an unmixed tree is Cohen-Macaulay. Fundamental to our proof is the idea of giving every tree a red/blue coloring.

Theorem 1.3.1 (Theorem 4.1.1). *Let $G = (V, E)$ be a tree, let G_R be the red-interior subgraph of G (see Definition 4.2.18) and let G_B be the blue interior graph of G. The following are equivalent:*

1. *G is unmixed with respect to total domination.*

2. *I_G^N is Cohen-Macaulay.*

3. *The following three conditions hold for the interior graphs G_R and G_B:*

   - *The height of G_R and G_B is ≤ 3.*
   - *No two leaves in G_R and G_B are distance 4 apart.*
   - *Each support vertex in G_R and G_B is adjacent to no more than one height 2 vertex.*

The material in Chapter 4 is joint work with Devin Adams, Caroline Daw, Aayahna Herbert, Jounglag Lim, Vi Anh Nguyan, Yifan Qian, Keri Sather-Wagstaff, Matthew Schaller, Susan Tarabulsi, Zoe Zhou, and Yuyang Zhuo.

Chapter 2

Background

In this chapter, we will explore the connections between edge ideals and vertex covers of simple graphs in order to motivate Chapters 3 and 4. We include technical definitions and theorems, with examples, that are needed for Theorem 1.1.1. In Section 1.1, we will introduce the edge ideal of a graph. In Section 2.2, we will decribe how monomial ideals can be decomposed and written as the intersection of m-irreducible monomial ideals. In Section 2.3, we explore the connection between the vertex covers of a simple graph and the m-irreducible decomposition of its edge ideal. Finally, in Section 2.4, we give background to understand Cohen-Macaulay rings and we state the characterization of the trees whose edge ideals are Cohen-Macaulay.

Throughout this chapter, let k be a field.

2.1 Monomial Ideals

In this chapter, our central objects of study are edge ideals of simple graphs. Edge ideals are examples of monomial ideals and we will see that every monomial ideal can be decomposed into m-irreducible monomial ideals. We begin by giving the definition of a monomial ideal.

Definition 2.1.1 ([15, Definition 1.1.1])**.** For $R = k[x_1, \ldots, x_d]$, a *monomial ideal* in R is an ideal of R that can be generated by monomials in $x_1, \ldots, x_d$.

Example 2.1.2. $(x_1x_2, x_2x_3) \subsetneq k[x_1, x_2, x_3]$ is a monomial ideal.

Example 2.1.3. $(x_1x_2, x_2x_3, x_3x_4) \subsetneq k[x_1, x_2, x_3, x_4]$ is a monomial ideal.

We will now introduce the edge ideal of a simple graph.

Definition 2.1.4 ([15, Theorem 4.2.2])**.** Let $G = (V, E)$ with $V = \{x_1, \ldots, x_d\}$ be a simple graph. The *edge ideal* of G is the monomial ideal $I_G = (x_i x_j | x_i x_j \in E) \subsetneq k[x_1, \ldots, x_d]$.

Example 2.1.5. The edge ideal of the path $P_2 = (x_1 \text{——} x_2 \text{——} x_3)$ is the monomial ideal $I_{P_2} = (x_1x_2, x_2x_3) \subsetneq k[x_1, x_2, x_3]$ from Example 2.1.2.

Example 2.1.6. The edge ideal of the path $P_3 = (x_1 \text{——} x_2 \text{——} x_3 \text{——} x_4)$ is the monomial ideal $I_{P_3} = (x_1x_2, x_2x_3, x_3x_4) \subsetneq k[x_1, x_2, x_3, x_4]$ from Example 2.1.3.

We will see in Section 2.3 that the algebraic properties of the ring $k[x_1, \ldots, x_d]/I_G$ are related to the structure of the graph G.

2.2 Decompositions and Unmixedness

In this section, we will explore the concept of decomposing monomial ideals into intersections of irreducible monomial ideals. We begin by giving two definitions which are equivalent for monomial ideals in polynomial rings.

Definition 2.2.1 ([15, Theorem 3.2.1])**.** For $R = k[x_1, \ldots, x_d]$, an ideal $J \subsetneq R$ is *reducible* if there are ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. An ideal $J \subsetneq R$ is *irreducible* if it is not reducible.

Definition 2.2.2 ([15, Theorem 3.1.1])**.** For $R = k[x_1, \ldots, x_d]$, a monomial ideal $J \subsetneq R$ is *m-reducible* if there are monomial ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. A monomial ideal $J \subsetneq R$ is *m-irreducible* if it is not m-reducible.

Example 2.2.3. Let $R = k[x_1, x_2, x_3]$. We see that $I_{P_2} = (x_1x_2, x_2x_3) \subsetneq R$ is m-reducible (and hence reducible) because $(x_1x_2, x_2x_3) = (x_1, x_3) \cap (x_2)$.

Example 2.2.4. Let $R = k[x_1, x_2, x_3, x_4]$. We see that $I_{P_3} = (x_1x_2, x_2x_3, x_3x_4) \subsetneq R$ is m-reducible (and hence reducible) becuase $(x_1x_2, x_2x_3, x_3x_4) = (x_2, x_3x_4) \cap (x_1, x_3)$. Moreover, (x_2, x_3x_4) is m-reducible ideal (and hence reducible) because $(x_2, x_3x_4) = (x_2, x_3) \cap (x_2, x_4)$.

Clearly every m-reducible ideal is also reducible. The converse holds for monomial ideals in polynomial rings.

Theorem 2.2.5 ([15, Theorem 3.2.4])**.** *Let $R = k[x_1, \ldots, x_d]$. A monomial ideal $J \subsetneq R$ is irreducible if and only if it is m-irreducible.*

We now give a condition for determining if a given monomial ideal in a polynomial ring is reducible (equivalently m-irreducible):

Theorem 2.2.6 ([15, Theorem 3.1.3])**.** *Let $R = k[x_1, \ldots, x_d]$, and let J be a non-zero monomial ideal of R. Then J is m-irreducible if and only if it is generated by pure powers, i.e. $J = (x_{i_1}^{e_1}, \ldots, x_{i_n}^{e_n})R$ for some positive integers $i_1, \ldots, i_n, e_1, \ldots, e_n$ with $1 \leq i_1 < \cdots < i_n \leq d$.*

Example 2.2.7. Let $R = k[x_1, x_2, x_3]$. The ideals (x_1, x_3) and (x_2) are both m-irreducible (hence irreducible by Theorem 2.2.5) in R because they are generated by pure powers.

Example 2.2.8. Let $R = k[x_1, x_2, x_3, x_4]$. The ideals (x_1, x_3), (x_2, x_3), and (x_2, x_4) are all m-irreducible (hence irreducible by Theorem 2.2.5) in R because they are generated by pure powers.

Definition 2.2.9 ([15, Definition 3.3.4])**.** Let $J \subsetneq R$ be a monomial ideal. An *m-irreducible decomposition of* J is an intersection of monomial ideals $J = \bigcap_{i=1}^{m} J_i$. An *irredundant m-irreducible decomposition of* J is an m-irreducible decomposition $J = \bigcap_{i=1}^{m} J_i$ such that $J_i \not\subseteq J_j$ for all $i \neq j$.

Example 2.2.10. Let $R = k[x_1, x_2, x_3]$. The m-irreducible decomposition of the edge ideal $I_{P_2} = (x_1x_2, x_2x_3)$ in R is $I_{P_2} = (x_1, x_3) \cap (x_2)$.

Example 2.2.11. Let $R = k[x_1, x_2, x_3, x_4]$. The m-irreducible decomposition of the edge ideal $I_{P_3} = (x_1x_2, x_2x_3, x_3x_4)$ in R is $I_{P_3} = (x_1, x_3) \cap (x_2, x_3) \cap (x_2, x_4)$.

It turns out that every monomial ideal has an m-irreducible decomposition and that irredundant m-irreducible decompositions are unique up to reordering.

Theorem 2.2.12 ([15, Theorem 3.3.3])**.** *Let $R = k[x_1, \ldots, x_d]$ and $J \subsetneq R$ a monomial ideal. Then there exist m-irreducible ideals $J_1, \ldots, J_n$ of R such that $J = \bigcap_{i=1}^{n} J_i$.*

Theorem 2.2.13 ([15, Theorem 3.3.8])**.** *Let $R = k[x_1, \ldots, x_d]$. Let $J \subsetneq R$ be a monomial ideal with irredundent m-irreducible decompositions $J = \bigcap_{i=1}^{n} J_i$ and $J = \bigcap_{h=1}^{m} I_h$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $J_t = I_{\sigma(t)}$ for $t = 1, \ldots, n$.*

We conclude this section by defining an important notion for monomial ideals based on the m-irreducible ideals in these m-irreducible decompositions.

Definition 2.2.14 ([15, Definition 5.2.4])**.** Let $R = k[x_1, \ldots, x_d]$ and $J \subsetneq R$ be a monomial ideal with an irredundant m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. We say that J is *m-unmixed* if every m-irreducible ideal J_i has the same number of generators. We say that J is *m-mixed* if it is not m-unmixed.

Example 2.2.15. The monomial ideal $I_{P_2} = (x_1x_2, x_2x_3) = (x_1, x_3) \cap (x_2) \subsetneq k[x_1, x_2, x_3]$ is m-mixed.

Example 2.2.16. The monomial ideal $I_{P_3} = (x_1x_2, x_2x_3, x_3x_4) = (x_1, x_3) \cap (x_2, x_3) \cap (x_2, x_4) \subsetneq k[x_1, x_2, x_3, x_4]$ is m-unmixed.

2.3 Vertex Covers

In this section, we will explore the connections between the vertex covers of a simple graph and the m-irreducible decomposition of the edge ideal of the graph.

Definition 2.3.1 ([15, Theorem 4.3.4])**.** Given a graph $G = (V, E)$, a *vertex cover* of G is a set $V' \subset V$ such that every edge in E is incident to at least one member of V'. We say that a vertex cover of G is *minimal* if it does not properly contain another vertex cover of G.

Example 2.3.2. Consider the path $P_2 = (x_1 \text{———} x_2 \text{———} x_3)$. The minimal vertex covers of P_2 are $\{x_1, x_3\}$, and $\{x_2\}$.

Example 2.3.3. Consider the path $P_3 = (x_1 \text{———} x_2 \text{———} x_3 \text{———} x_4)$. The minimal vertex covers of P_3 are $\{x_1, x_3\}$, $\{x_2, x_3\}$, and $\{x_2, x_4\}$.

Definition 2.3.4 ([15, Theorem 3.2.4])**.** We say that a simple graph G is *unmixed with respect to vertex covers* if every minimal vertex cover has the same cardinality. We say that G is *mixed with respect to vertex covers* if it is not unmixed with respect to vertex covers.

Example 2.3.5. The path P_2 is mixed with respect to vertex covers because its vertex covers have different cardinality, that is, $|\{x_1, x_3\}| \neq |\{x_2\}|$.

Example 2.3.6. The path P_3 is unmixed with respect to vertex covers because its vertex covers have the same cardinality, that is, $|\{x_1, x_3\}| = |\{x_2, x_3\}| = |\{x_2, x_4\}|$.

We now give the fundamental connection between edge ideals and vertex covers.

Theorem 2.3.7 ([15, Theorem 4.3.8]). *Let $G = (V, E)$ be a simple graph with $V = \{x_1, \ldots, x_d\}$ and let $V_1', \ldots, V_n'$ be the minimal vertex covers of G. Then the edge ideal $I_G \subsetneq k[x_1, \ldots, x_d]$ has the following irredundant m-irreducible decomposition: $I_G = \bigcap_{i=1}^n (V_i')$.*

We verify the conclusion of this result for our running examples.

Example 2.3.8. From Example 2.3.2, the minimal vertex covers of P_2 are $\{x_1, x_3\}$, and $\{x_2\}$. From Example 2.2.10, the edge ideal of P_2 has irredundant m-irreducible decomposition $I_{P_2} = (x_1, x_3) \cap (x_2)$. So the minimal vertex covers of P_2 correspond with the m-irreducible ideals in the irredundant m-irreducible decomposition of the edge ideal of P_2.

Example 2.3.9. From Example 2.3.3, the minimal vertex covers of P_2 are $\{x_1, x_3\}$, $\{x_2, x_3\}$, and $\{x_2, x_4\}$. From Example 2.2.10, the edge ideal of P_2 has irredundant m-irreducible decomposition $I_{P_2} = (x_1, x_3) \cap (x_2, x_3) \cap (x_2, x_4)$. So the minimal vertex covers of P_3 correspond with the m-irreducible ideals in the irredundant m-irreducible decomposition of the edge ideal of P_2.

Corollary 2.3.10. *Let $G = (V, E)$ be a simple graph with $V = \{x_1, \ldots, x_d\}$. Then G is unmixed with respect to vertex covers if and only if $I_G \subsetneq k[x_1, \ldots, x_d]$ is m-unmixed.*

Example 2.3.11. From Example 2.2.15, Example 2.2.16, we see that I_{P_2} is m-mixed and P_2 is mixed with respect to vertex covers. From Example 2.3.5 and Example 2.3.6, we see that I_{P_2} is m-unmixed and P_2 is unmixed with respect to vertex covers.

2.4 Cohen-Macaulay Rings

For commutative rings, the property of Cohen-Macaulayness is stronger than unmixedness (see Theorem 2.4.17) and is important in commutative algebra, algebraic geometry and topology. We will introduce Cohen-Macaulay rings without using the usual homological definitions (see [4] for the standard treatment).

Throughout this section, unless otherwise specified, we let $R = k[x_1, \ldots, x_d]/J$ where J is generated by non-constant homogeneous polynomials.

Definition 2.4.1 ([15, Definition 5.1.4]). An ideal $I \subsetneq R$ is *prime* if $I \neq R$ and $R \setminus I$ is closed under multiplication.

Fact 2.4.2. *An ideal $I \subsetneq R$ is prime if and only if R/I is an integral domain.*

Example 2.4.3. For $R = k[x_1, x_2, x_3]$, the ideals (x_1), (x_1, x_2), and (x_1, x_2, x_3) are all prime ideals since $R/(x_1) \cong k[x_2, x_3]$, $R/(x_1, x_2) \cong k[x_3]$, and $R/(x_1, x_2, x_3) \cong k$ are all integral domains. However, the ideal $I_{P_2} = (x_1x_2, x_2x_3)$ is not a prime ideal since the polynomials x_2 and $x_1 + x_3$ are not in I_{P_2} but their product $x_2(x_1 + x_3)$ is in I_{P_2}.

Definition 2.4.4 ([15, Definition 5.1.4])**.** The *Krull dimension* of R, denoted $\dim(R)$, is the supremum of the lengths of chains of prime ideals in R:

$$\dim(R) = \sup\{n \geq 0 | \text{there is a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } R\}.$$

Example 2.4.5 ([4, Corollary A.13])**.** For a field k, the only prime ideal is the zero ideal. Therefore, $\dim(k) = 0$. For $R = k[x_1, \ldots, x_d]$, $\dim(R) = d$.

Theorem 2.4.6 ([15, Theorem 5.1.5])**.** *Let G be a graph with vertex set $V = \{x_1, \ldots, x_d\}$. If n is the size of the smallest vertex cover of G, then* $\dim(R/I_G) = d - n$.

Example 2.4.7. Let $R = k[x_1, x_2, x_3]/I_{P_2}$. Since the smallest vertex cover of P_2 is $\{x_2\}$, by Theorem 2.4.6, $\dim(R) = 3 - |\{x_2\}| = 2$.

Example 2.4.8. Let $R = k[x_1, x_2, x_3, x_4]/I_{P_3}$. Since every minimal vertex cover of P_3 has size 2, by Theorem 2.4.6, $\dim(R) = 4 - 2 = 2$.

We now work to define the depth of a polynomial ring modulo a ideal generated by non-constant homogeneous polynomials.

Definition 2.4.9 ([15, Definition 5.2.5])**.** A non-constant homogeneous element $g \in R$ is *R-regular* if the map $R \to R$ given by $p \to gp$ is 1-1. A sequence of non-constant homogeneous elements $g_1, \ldots, g_m \in R$ is *R-regular* if g_1 is R-regular and for $i = 2, \ldots, m$ the element g_i is regular for $R/(g_1, \ldots, g_{i-1})R$.

A theorem of Rees shows that the following notion is well defined.

Definition 2.4.10. The *depth* of our graded ring R, denoted $\operatorname{depth}(R)$, is the size of the longest R-regular sequence in R.

Example 2.4.11. Let $R = k[x_1, x_2, x_3]/I_{P_2} = k[x_1, x_2, x_3]/(x_1x_2, x_2x_3)$. We see that $x_1 - x_2 \in R/I_{P_2}$ is R-regular, and it is straightforward to show that $x_1 - x_2$ is a maximal R-regular sequence. Therefore $\operatorname{depth}(R) = 1$.

Example 2.4.12. Let $R = k[x_1, x_2, x_3, x_4]/I_{P_3} = k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4)$. We see that $x_1 - x_2, x_3 - x_4 \in R/I_{P_2}$ is R-regular, and it is straightforward to show that $x_1 - x_2, x_3 - x_4$ is a maximal R-regular sequence. Therefore $\operatorname{depth}(R) = 2$.

Theorem 2.4.13 ([15, Lemma 5.2.11])**.** *Let R be a commutative ring. Then* $\operatorname{depth}(R) \leq \dim(R)$.

Definition 2.4.14 ([15, Definition 5.2.12])**.** Let R be a commutative ring. We say that R is *Cohen-Macaulay* if $\dim(R) = \operatorname{depth}(R)$.

Example 2.4.15. The ring $k[x_1, x_2, x_3]/I_{P_2}$ is not Cohen-Macaulay since $\operatorname{depth}(R) = 1 < 2 = \dim(R)$ by Examples 2.4.7 and 2.4.11

Example 2.4.16. The ring $k[x_1, x_2, x_3, x_4]/I_{P_3}$ is Cohen-Macaulay since $\operatorname{depth}(R) = 2 = \dim(R)$ by Examples 2.4.8 and 2.4.12

The Cohen-Macaulay property is stronger than the property of being unmixed.

Theorem 2.4.17 ([15, Theorem 5.2.15])**.** *Let $R = k[x_1, \ldots, x_d]$ and $J \subsetneq R$ be a monomial ideal in R. If R/J is Cohen-Macaulay, then J is m-unmixed.*

Example 2.4.18. $(x_1x_2, x_2x_3) = (x_1, x_3) \cap (x_2) \subsetneq k[x_1, x_2, x_3]$ is mixed, and as we have seen, the ring $k[x_1, x_2, x_3]/(x_1x_2, x_2x_3)$ is not Cohen-Macaulay.

Example 2.4.19. $(x_1x_2, x_2x_3, x_3x_4) = (x_1, x_3) \cap (x_2, x_3) \cap (x_2, x_4) \subsetneq k[x_1, x_2, x_3, x_4]$ is unmixed, and as we have seen, the ring $k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4)$ is Cohen-Macaulay.

Example 2.4.20. $(x_1x_2, x_2x_3, x_3x_4, x_1x_4) = (x_1, x_3) \cap (x_2, x_4) \subsetneq k[x_1, x_2, x_3, x_4]$ is unmixed. However, the ring $k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4, x_1x_4)$ is not Cohen-Macaulay; it is straightforward to show that $k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4, x_1x_4)$ has depth 1 and dimension 2.

We continue this section by giving Villarreal's characterization of unmixed trees with respect to vertex covers and stating that the converse of Theorem 2.4.17 when J is an edge ideal of a tree.

Theorem 2.4.21 ([20, Theorem 2.4])**.** *Let $G = (V, E)$ be a tree. Then G is unmixed with respect to vertex covers if and only if G is a suspension (also known as a corona), i.e., every $v \in V$ with* $\deg(v) > 1$ *is adjacent to exactly one leaf.*

Theorem 2.4.22 ([20, Corollary 2.5])**.** *Let $G = (V, E)$ be a tree with vertex set $V = \{x_1, \ldots, x_d\}$ and let $R = k[x_1, \ldots, x_d]$. If I_G is m-unmixed then R/I_G is Cohen-Macaulay.*

Example 2.4.23. The simple graphs P_2 and P_3 are both trees. However, P_3 is a suspension whereas P_2 is not. This is consistant with Theorem 2.4.21 since we have seen that $k[x_1, x_2, x_3, x_4]/I_{P_3}$ is Cohen-Macaulay and I_{P_2} is mixed.

We conclude this section with notion that is stronger than Cohen-Macaulayness.

Definition 2.4.24. Let $R = k[x_1, \ldots, x_d]$ and let $J = (p_1, \ldots, p_n) \subsetneq R$ generated by polynomials $p_1, \ldots, p_n$. We say that R/J is a *complete intersection* if $p_1, \ldots, p_n$ is a homogeneous R-regular sequence.

Example 2.4.25. Neither $k[x_1, x_2, x_3]/I_{P_2}$ nor $k[x_1, x_2, x_3, x_4]/I_{P_3}$ are complete intersections because I_{P_2} and I_{P_3} are not generated by regular sequences.

Example 2.4.26. Let $R = k[x_1, \ldots, x_d]/(x_1 \cdots x_{d_1}, x_{d_1+1} \cdots x_{d_2}, \ldots, x_{d_{n-1}+1} \cdots x_{d_n})$ where $1 \leq d_1 < \cdots < d_n \leq d$. Then R is a complete intersection.

Theorem 2.4.27 ([4, Sections 2.1–2.2])**.** *If a ring R is a complete intersection, then R is Cohen-Macaulay.*

The main results of Chapters 3–4 below include conditions under which the converse of Theorem 2.4.27 holds. This converse fails in general, as $k[x_1, x_2, x_3, x_4]/I_{P_3}$ is Cohen-Macaulay but not a complete intersection by Examples 2.4.16 and 2.4.25.

Chapter 3

Unmixed Trees with respect to PMU Covers

3.1 Introduction

The work in this chapter is motivated by the PMU Placement Problem in electrical engineering. This asks for the optimal placements of sensors, called PMUs, in an electrical power system to monitor the system for outages. (Definitions are in Section 3.2 below.) This problem asks how to place PMUs so that the entire system is monitored, but, because of the cost, to do so optimally. Haynes, Hedetniemi, Hedetniemi, and Henning [9] show that this problem (which they call the Power Dominating Set (PDS) Problem) is NP-complete. See the papers of Baldwin, Mili, Boisen, and Adapa [2], Brueni and Heath [3], Kavasseri and Nag [11], Kavasseri and Srinivasan [12], and Phadke [17] for more about PMU placements.

We approach this problem using tools and ideas from combinatorial commutative algebra. Specifically, if G models a power system, then Moore, Rogers, and Sather-Wagstaff [15] introduce the power edge ideal I_G^{P} of G, a monomial ideal in a polynomial ring which decomposes in terms of the minimal PMU covers of the graph. A standard problem in combinatorial commutative algebra is to determine when such a monomial ideal is Cohen-Macaulay. Since Cohen-Macaulay ideals are unmixed, this suggests that one should identify the power systems for which all minimal PMU covers have the same size. From an engineering perspective, this is reasonable: if a system is built so that all minimal PMU covers have the same size, then finding the smallest PMU covers will be easier.

The main result of this chapter solves the problem of identifying the trees for which all minimal PMU covers have the same size. It is Theorem 1.2.1 from the introduction. We prove this result over the course of Section 3.4; see Theorems 3.4.4, 3.4.10, and 3.4.11.

Theorem 3.1.1. *The following conditions on a tree T are equivalent:*

(i) I_T^P *is unmixed, i.e., all minimal PMU covers of T have the same size;*

(ii) I_T^P *is Cohen-Macaulay;*

(iii) I_T^P *is a complete intersection;*

(iv) T *is an edge linked tree (see Definition 3.4.2);*

(v) *every vertex of T with degree at least 3 is adjacent to exactly two vertices of degree at most 2.*

See Theorems 3.4.9 and 3.4.10 for computations of the minimal PMU covers and power edge ideals in general for edge linked trees. At this time, we do not know how to describe the generators for the power edge ideal of an arbitrary graph.

For power edge ideals of trees, Theorem 3.1.1 shows that the complete intersection, Gorenstein, Cohen-Macaulay, and unmixed properties are equivalent. In Section 3.2, we show that this fails for non-trees by exhibiting graphs whose power edge ideals distinguish between these properties.

3.2 Definitions, Macaulay2 Code, and Examples

In this section, we begin with relevant definitions, then we provide Macaulay2 code for computing the minimal PMU covers and the power edge ideal of a given graph. It uses Francisco, Hoefel, and Van Tuyl's `EdgeIdeals` package [6]. Then we exhibit examples of power edge ideals that distinguish between the complete intersection, Gorenstein, Cohen-Macaulay, and unmixed properties. In particular, these examples show that the tree assumption in Theorem 3.1.1 is necessary.

Definitions and Initial Examples

In an electrical power system, a *bus* is a substation where *(transmission) lines* meet. Each line connects two buses. Throughout this chapter, we model electrical power systems as graphs where vertices and edges in a graph correspond to buses and lines in a power system. For the rest of the chapter, we use the terms “graph”, “vertex”, and “edge” in place of “power system”, “bus”, and “line”, respectively.

A *phasor measurement unit (PMU)* is a device placed at a vertex of G to monitor the voltage at the vertex and the current in all edges incident to the vertex. (The name refers to the fact that PMUs measure voltage phasors and current phasors.) A *PMU placement* is a set of vertices where PMUs are placed, i.e., a PMU placement is a subset of $V(G)$. The following laws determine whether the voltage at a vertex or the current in an edge in a graph is *observed* by a PMU placement.

Incidence Law: Every vertex containing a PMU is observable, and every edge incident to a vertex containing a PMU is observable.

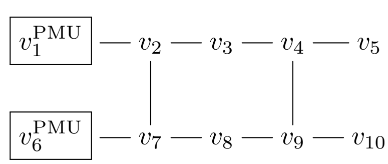
Ohm’s Law: Any edge incident to two observable vertices is observable, and every vertex incident to an observable edge is observable.

Kirchhoff’s Current Law: If a vertex v_i is incident to $k > 1$ edges, $k - 1$ of which are observable, than all k of these edges are observable.

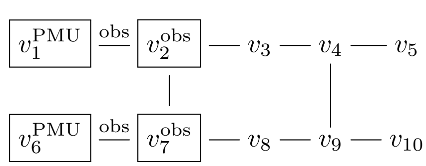
Note that the name Incidence Law is non-standard.

A *PMU cover* is a PMU placement which observes the entire graph, i.e., every edge and every vertex. A PMU cover is *minimal* if it does not properly contain another PMU cover.

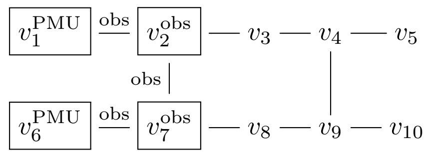
Example 3.2.1. In the following graph we place PMUs as indicated.



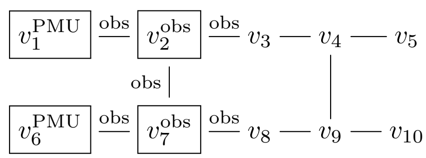
The Incidence Law guarantees the observability of the following edges and vertices.



Ohm's Law shows that another edge is observable



and Kirchhoff's Current Law applies to make two more edges observable.



Continuing in this way, one checks that this PMU placement observes the entire graph, i.e., it is a PMU cover. Moreover, if either vertex is removed from this PMU cover, then the resulting set is not a PMU cover, so the set $\{v_1, v_6\}$ is a minimal PMU cover of this graph. It is straightforward (though time consuming) to show that the complete list of minimal PMU covers of the above graph is

$$\{v_1, v_6\}, \{v_1, v_7\}, \{v_1, v_8\}, \{v_1, v_9\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_2, v_9\},$$
$$\{v_2, v_{10}\}, \{v_3, v_6\}, \{v_3, v_7\}, \{v_3, v_8\}, \{v_3, v_9\}, \{v_3, v_{10}\}, \{v_4, v_6\}, \{v_4, v_7\},$$
$$\{v_4, v_8\}, \{v_4, v_9\}, \{v_4, v_{10}\}, \{v_5, v_7\}, \{v_5, v_8\}, \{v_5, v_9\}, \{v_5, v_{10}\}.$$

Such computations are simplified using our Macaulay2 [8] code which is described below in this section; see Example 3.2.8. Note that the sets $\{v_1, v_{10}\}$ and $\{v_5, v_6\}$ are not PMU covers.

Here is an algorithm of Haynes, et al. [9, p. 520] containing notation for use throughout the sequel.

Algorithm 3.2.2. Let G be a graph and P a PMU placement on G. The paper [9] gives an algorithm to determine the sets of observable vertices $C_P(G)$ and edges $F_P(G)$. We will state that algorithm with slightly different notation:

Set $C_P^0(G) = P$ and set $F_P^0(G)$ to be the set of all edges incident to a vertex in P.

For each postive integer i starting at $i = 1$, define $C_P^i(G)$ to be the set of all vertices in G incident to an edge in $F_P^{i-1}(G)$ and F_P^i to be the set of all edges $x - y$ in G such that either

1. $x, y \in C_P^i(G)$ or

2. $x \in C_P^i(G)$ has degree greater than 1 and all other edges incident to x are in $F_P^{i-1}(G)$ or

3. $y \in C_P^i(G)$ has degree greater than 1 and all other edges incident to y are in $F_P^{i-1}(G)$

Finally, note that each $C_P^i(G) \subset C_P^{i+1}(G)$ and $F_P^i(G) \subset F_P^{i+1}(G)$ for all $i \in \{0, 1, \ldots\}$ Denote $C_P(G) = \bigcup_{i=1}^{\infty} C_P^i(G)$ and $F_P(G) = \bigcup_{i=1}^{\infty} F_P^i(G)$. Note that $(C_P(G), F_P(G))$ is the set of vertices and edges of G observable by P.

Now we are ready for our algebraic notions.

Definition 3.2.3. Let the vertex set of G be $V = \{v_1, \ldots, v_d\}$, and set $R = k[X_1, \ldots, X_d]$ where k is a field. For each subset $V' \subseteq V$, consider the ideal $P_{V'} = \langle X_i \mid v_i \in V' \rangle$ of R. The *power edge ideal* of G is

$$I_G^{\mathrm{P}} = \bigcap_{V'} P_{V'}$$

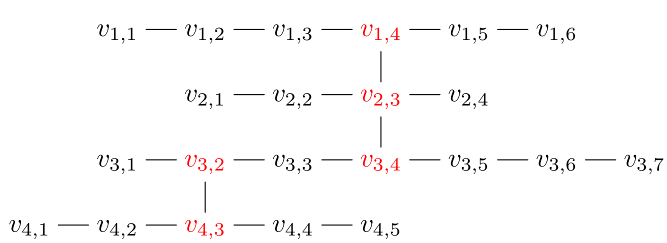
where the intersection is taken over all PMU covers V' of G, equivalently, over all minimal PMU covers V' of G.

Example 3.2.4. For the graph of Example 3.2.1, one can use the list of minimal PMU covers found there to show by definition that the power edge ideal is $I_G^{\mathrm{P}} =$

$$\langle X_6X_7X_8X_9X_{10}, X_1X_2X_3X_4X_5, X_1X_2X_3X_4X_7X_8X_9X_{10}, X_2X_3X_4X_5X_6X_7X_8X_9 \rangle.$$

As with minimal PMU covers, our Macaulay2 code below computes power edge ideals; see Example 3.2.8.

Example 3.2.5. Here is a tree satisfying the equivalent conditions of Theorem 3.1.1 (condition (v) may be the easiest to check), where the vertices of degree at least 3 are red.



One checks readily from the definitions that the minimal PMU covers of this tree are exactly the sets of the form $\{v_{1,a}, v_{2,b}, v_{3,c}, v_{4,d}\}$ and the power edge ideal of this tree is the ideal

$$\begin{aligned} I_G^{\mathrm{P}} &= \bigcap_{a=1}^{6} \bigcap_{b=1}^{4} \bigcap_{c=1}^{7} \bigcap_{d=1}^{5} \langle X_{1,a}, X_{2,b}, X_{3,c}, X_{4,d} \rangle \\ &= \langle X_{1,1} \cdots X_{1,6}, X_{2,1} \cdots X_{2,4}, X_{3,1} \cdots X_{3,7}, X_{4,1} \cdots X_{4,5} \rangle. \end{aligned}$$

In words, the minimal PMU covers are obtained by choosing one vertex from each horizontal path, and the generators of I_G^{P} are the products of the variables from the horizontal paths.

Macaulay2 Code and Further Examples

The following Macaulay2 code is based on the algorithm in Definition 3.2.2. See also Remark 3.2.7 below.

Code 3.2.6. For Example 3.2.8 below, the following code is stored in the file `PMU.m2`.

```
loadPackage "EdgeIdeals"

ohmClosure = method()
ohmClosure (Graph, List, List) := (G, C, F) -> (
   newC := unique (C | flatten F);
   newF := unique(F | select(subsets(newC, 2), p -> member(p, edges G)));
   (reverse sort newC, reverse sort newF)
)

kirchhoffClosure = method()
kirchhoffClosure (Graph, List, List) := (G, C, F) -> (
   newF := F;
   for v in C do (
      incidentToV := select(edges G, e -> member(v,e));
      if #incidentToV > 1 and #select(F, e -> member(v,e)) ==
```

```
      (#incidentToV - 1) then newF = unique (newF | incidentToV);
   );
   (reverse sort C, reverse sort newF)
)

observedVerticesEdges = method()
observedVerticesEdges (Graph, List) := (G,C) -> (
   oldC := C;
   oldF := {};
   newC := C;
   newF := select(edges G, e -> any(C, v -> member(v,e)));
   while oldC != newC or oldF != newF do (
      oldC = newC;
      oldF = newF;
      (newC,newF) = ohmClosure(G,newC,newF);
      (newC,newF) = kirchhoffClosure(G,newC,newF);
   );
   (newC,newF)
)

pmuCoversHelper = method()
pmuCoversHelper (Graph, List) := (G,C) -> (
   (obsVert,obsEdge) := observedVerticesEdges(G,C);
   if obsVert == vertices G then return {C};
   newVerts := select(vertices G, v' -> not member(v',C) and
   (any(select(edges G, e -> member(v',e)), f -> not member(f,obsEdge))
   or not member(v', obsVert)));
   flatten for v in newVerts list (
      newC := reverse sort (C | {v});
      unique pmuCoversHelper(G,newC)
   )
```

```
)

pmuCovers = method()
pmuCovers Graph := G -> (
   rawPMUCovers := unique pmuCoversHelper(G,{});
   -- now need to select those that are minimal wrt inclusion
   select(rawPMUCovers, pmu -> #select(rawPMUCovers, pmu' ->
   isSubset(pmu',pmu)) == 1)
)

powerEdgeIdeal = method()
powerEdgeIdeal Graph := G -> (
   pmuCovs := pmuCovers G;
   intersect apply(pmuCovs, cov -> ideal cov)
)
```

Here is a discussion of some aspects of the above code.

Remark 3.2.7. The `ohmClosure` method takes as input a graph, a list of observable vertices, and a list of observable edges; it then adds the new vertices and edges that are observable by Ohm's Law. The `kirchhoffClosure` method works similarly using Kirchhoff's Current Law. The `observedVerticesEdges` method takes as input a graph and a PMU placement, and it outputs the lists of observable vertices and edges obtained by an application of the Incidence Law followed by repeated application of `ohmClosure` and `kirchhoffClosure`.

The `pmuCoversHelper` method takes as input a graph `G` and a list `C` of vertices. This method uses a divide-and-conquer algorithm to find a list of PMU covers that contain `C`. In practice, it is applied with `C={}` the empty list; in this case, the method returns a list of PMU covers of `G` that contains all the minimal ones as follows:

Step 1. For each vertex `v` not in `C`, create a PMU cover candidate `newC` by adding `v` to `C`.

Step 2. If `newC` is a PMU cover of `G`, return the set `newC`; else, recursively apply Step 1 to `newC`.

This description is not entirely faithful to our code. In Step 1, we do not create a new PMU cover candidate for every `v` not in `C`: we do not use `v` to create a new PMU cover candidate if `v` is observed by `C` and all edges incident to `v` are also observed by `C`. We do this because placing a PMU at `v` does not change the observable edges or vertices. This tweak seems to improve run time by a factor of 10-20.

Example 3.2.8. Here we show how the code above can verify the conclusions of Examples 3.2.1 and 3.2.4, and we show that the power edge ideal in that example is not Cohen-Macaulay over $\mathbb{Q}$. In particular, it provides a power edge ideal that is unmixed but not Cohen-Macaulay.

```
i1 : load "PMU.m2"

i2 : R = QQ[x_1..x_10];

i3 : G = graph(R, {x_1*x_2,x_2*x_3,x_3*x_4,x_4*x_5,x_6*x_7,x_7*x_8,x_8*x_9,
    x_9*x_10,x_2*x_7,x_4*x_9});

i4 : pmuCovers G

o4 = {{x , x }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x },
        1   6     1   7     1   8     1   9     2   6     2   7     2   8
     ------------------------------------------------------------------------
     {x , x }, {x , x  }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x  },
       2   9     2   10     3   6     3   7     3   8     3   9     3   10
     ------------------------------------------------------------------------
     {x , x }, {x , x }, {x , x }, {x , x }, {x , x  }, {x , x }, {x , x },
       4   6     4   7     4   8     4   9     4   10     5   7     5   8
     ------------------------------------------------------------------------
     {x , x }, {x , x  }}
       5   9     5   10
```

```
o4 : List

i5 : IPG = powerEdgeIdeal G

o5 = ideal (x x x x x  , x x x x x , x x x x x x x x  , x x x x x x x x )
             6 7 8 9 10   1 2 3 4 5   1 2 3 4 7 8 9 10   2 3 4 5 6 7 8 9

o5 : Ideal of R

i6 : isCM(hyperGraph IPG)

o6 = false
```

The properties of the remaining examples of this section are verified as in Example 3.2.8.

Example 3.2.9. The following graph

$$\begin{array}{ccccccc} v_1 & — & v_2 & — & v_3 & — & v_4 \\ & & | & & | & & \\ v_5 & — & v_6 & — & v_7 & — & v_8 \end{array}$$

has the following 22 minimal PMU covers

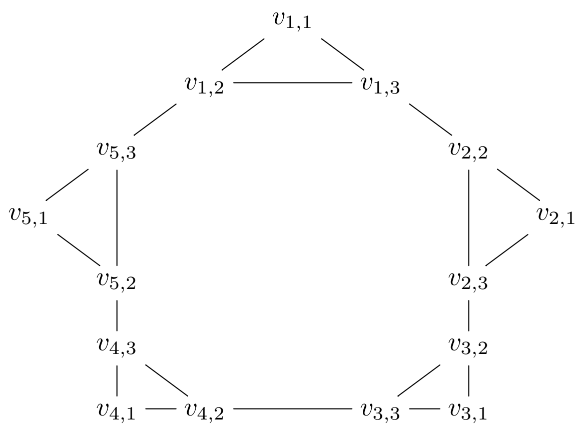
$$\{v_1,v_3\},\{v_1,v_4\},\{v_1,v_5\},\{v_1,v_6\},\{v_1,v_7\},\{v_2,v_3\},\{v_2,v_4\},\{v_2,v_5\},$$
$$\{v_2,v_6\},\{v_2,v_7\},\{v_2,v_8\},\{v_3,v_5\},\{v_3,v_6\},\{v_3,v_7\},\{v_3,v_8\},$$
$$\{v_4,v_6\},\{v_4,v_7\},\{v_4,v_8\},\{v_5,v_7\},\{v_5,v_8\},\{v_6,v_7\},\{v_6,v_8\}$$

and the following power edge ideal

$$\langle X_3X_4X_5X_6X_7X_8, X_1X_2X_5X_6X_7X_8, X_1X_2X_3X_6X_7X_8,$$
$$X_1X_2X_3X_4X_7X_8, X_2X_3X_4X_5X_6X_7, X_1X_2X_3X_4X_5X_6\rangle$$

which is Cohen-Macaulay and is not Gorenstein over $\mathbb{Q}$.

Example 3.2.10. The following graph has a power edge ideal which is Gorenstein and is not a complete intersection.



This graph has 135 minimal PMU covers, namely, all the sets of the following form:

$$\{v_{1,a}, v_{2,b}, v_{4,c}\}, \{v_{2,a}, v_{3,b}, v_{5,c}\}, \{v_{3,a}, v_{4,b}, v_{1,c}\}, \{v_{4,a}, v_{5,b}, v_{2,c}\}, \{v_{5,a}, v_{1,b}, v_{3,c}\}.$$

Its power edge ideal is

$$\langle X_{1,1}X_{1,2}X_{1,3}X_{2,1}X_{2,2}X_{2,3}, \\ X_{2,1}X_{2,2}X_{2,3}X_{3,1}X_{3,2}X_{3,3}, X_{3,1}X_{3,2}X_{3,3}X_{4,1}X_{4,2}X_{4,3}, \\ X_{4,1}X_{4,2}X_{4,3}X_{5,1}X_{5,2}X_{5,3}, X_{5,1}X_{5,2}X_{5,3}X_{1,1}X_{1,2}X_{1,3}\rangle$$

which is Gorenstein and is not a complete intersection over $\mathbb{Q}$.

3.3 PMU Covers and Associated Sets

This section consists of combinatorial results about PMU covers, for use in our algebraic results in Section 3.4. We begin with the following.

Definition 3.3.1. Let G be a simple graph, P a PMU cover of G and $v \in P$. We say v is a *minimal vertex* of P if $P - \{v\}$ is not a PMU cover of G. We define $P_{min} \subset P$ to be all minimal vertices of P.

Proposition 3.3.2. *Let G be a simple graph, P a PMU cover of G and $v \in P$. The following are true.*

1. *P is minimal if and only if $P = P_{min}$*

2. *If $v \in P_{min}$ and $\deg(v) = 1$ then incident its edge is not in $F_{P-\{v\}}(G)$.*

3. *If $v \in P_{min}$ and $\deg(v) \geq 2$. Then there exist at least two edges incident to v not in $F_{P-\{v\}}(G)$.*

Proof. Proof of (1)

($\implies$) By definition

($\impliedby$) Proving the contrapositive, suppose P not minimal, meaning there exists a proper subset $P' \subset P$ that is a PMU cover, and let $p \in P - P'$. Since P' is a PMU cover and $P' \subset P - \{p\}$, $P - \{p\}$ is a PMU cover and so $p \notin P_{min}$.

Proof of (2)

By way of contradiction, suppose va is in $F^i_{P-\{v\}}(G)$ for some $i \geq 0$. Then $F^0_P(G) = F^0_{P-\{v\}}(G) \cup \{va\} \subset F^i_{P-\{v\}}(G)$ and since $v \in C^{i+1}_{P-\{v\}}(G)$, $C^0_P(G) \subset C^{i+1}_{P-\{v\}}(G)$ which means $F^j_P(G) \subset F^{i+j}_{P-\{v\}}(G)$ and $C^j_P(G) \subset C^{i+j+1}_{P-\{v\}}(G)$ for all $j \geq 0$. Therefore $F_P(G) \subset F_{P-\{v\}}(G)$ and $C_P(G) \subset C_{P-\{v\}}(G)$ and since P is a PMU cover of G, $P - \{v\}$ is a PMU cover of G which contradicts the minimality of v in P.

Proof of (3)

By way of contradiction, suppose that at most one edge incident to v is not in $F_{P-\{v\}}(G)$. Then $v \in C_{P-\{v\}}(G)$ and by Kirkoff's Law, every edge incident to v is in $F^i_{P-\{v\}}(G)$ for some i. So just as in the proof of (2), $F^0_P(G) \subset F^i_{P-\{v\}}(G)$ and $C^0_P(G) \subset C^{i+1}_{P-\{v\}}(G)$ which means $F^j_P(G) \subset F^{i+j}_{P-\{v\}}(G)$ and $C^j_P(G) \subset C^{i+j+1}_{P-\{v\}}(G)$ for all $j \geq 0$. Therefore $F_P(G) \subset F_{P-\{v\}}(G)$ and $C_P(G) \subset C_{P-\{v\}}(G)$ and since P is a PMU cover of G, $P - \{v\}$ is a PMU cover of G which contradicts the minimality of v in P. □

Definition 3.3.3. Let G be a simple graph and P a PMU placement on G. We say an edge ab is *directed away* from a *towards* b if for some integer i, $ab \in F^i_P(G)$ but $b \notin C^i_P(G)$.

Not every edge will be directed. Furthermore, note that by Algorithm 3.2.2, it is impossible for ab to be directed towards b and away from b.

Proposition 3.3.4. *Let G be a simple graph, P a PMU placement of G and v a vertex not in P but in $C_P(G)$. The following are true.*

1. *There is at least one edge directed towards v.*

2. *There is at most one edge directed away from v.*

Proof. Proof of (1)

Let i be the smallest integer for which $v \in C_P^{i+1}(G)$. Since $v \notin P$, by Algorithm 3.2.2, there is some edge $va \in F_P^i(G)$ and thus va is directed towards v.

Proof of (2)

By way of contradiction, suppose there are two edges va and vb directed away from v. In other words, there exist integers i_a and i_b such that $va \in F_P^{i_a}(G)$ and $vb \in F_P^{i_b}(G)$ but $a \notin C_P^{i_a}(G)$ and $b \notin C_P^{i_b}(G)$. This implies that $a, b \notin P$ and since $v \notin P$, we know that $va, vb \notin F_P^0(G)$. Set $i = \min\{i_a, i_b\}$ and without loss of generality, assume $i_a \leq i_b$. Since $va \in F_P^{i_a}(G)$ and $a \notin C_P^{i_a}(G)$, by Algorithm 3.2.2, $v \in C_P^{i_a}(G)$ and all other edges incident to v are in $F_P^{i_a-1}(G)$. So $vb \in F_P^{i_a-1}(G) \implies b \in C_P^{i_a} \implies b \in C_P^{i_b}$. This is a contradiction. □

Recall that there exists a unique path between any two vertices in a tree.

Definition 3.3.5. Let G be a tree, and ab an edge in G. We define $\text{branch}_a(b) \subset G$ to be the smallest connected subgraph containing a and all vertices x such that the unique path from a to x contains ab.

Lemma 3.3.6. *Let G be a tree and P a PMU placement observing* $\text{branch}_a(b)$. *If ab is directed towards a, then* $F^i_{P \cap \text{branch}_a(b)}(G) \cap \text{branch}_a(b) = F_P^i(G) \cap \text{branch}_a(b)$ *for all i. In particular,* $P \cap \text{branch}_a(b)$ *observes all of* $\text{branch}_a(b)$ *since P observes all of* $\text{branch}_a(b)$.

Proof. Let $P' = P \cap \text{branch}_a(b)$ and note that $F_{P'}^i(G) \cap \text{branch}_a(b) \subset F_P^i(G) \cap \text{branch}_a(b)$ for all $i \geq 0$. We induct on i:

Base Case: $i = 0$: Both $F_P^0(G) \cap \text{branch}_a(b)$ and $F_{P'}^0(G) \cap \text{branch}_a(b)$ consist precisely of the edges in $\text{branch}_a(b)$ adjacent to a PMU in P'. Therefore, $F_{P'}^0(G) \cap \text{branch}_a(b) = F_P^0(G) \cap \text{branch}_a(b)$.

Inductive Step: $i \geq 1$ Suppose $F_{P'}^{i-1}(G) \cap \text{branch}_a(b) = F_P^{i-1}(G) \cap \text{branch}_a(b)$. By Algorithm 3.2.2, for every edge xy in $F_P^i(G) \cap \text{branch}_a(b)$ either $x, y \in C_P^i(G)$ or $x \in C_P^i(G)$ has degree greater than 1 and all other edges incident to x are in $F_P^{i-1}(G)$ or $y \in C_P^i(G)$ has degree greater than 1 and all other edges incident to y are in $F_P^{i-1}(G)$. We show that $xy \in F_{P'}^i(G)$ by addressing each case as well as the case that $xy = ab$:

$xy = ab$: Since ab is directed towards a and $ab \in F^i_P(G)$, $a \notin C^i_P(G)$ and so by Algorithm 3.2.2, $b \in C^i_P(G)$ has degree greater than 1 and all other edges incident to b are in $F^{i-1}_P(G)$. Since we assumed $F^{i-1}_{P'}(G) \cap \text{branch}_a(b) = F^{i-1}_P(G) \cap \text{branch}_a(b)$, and all edges incident to b are in $\text{branch}_a(b)$, we conclude that $b \in C^i_{P'}(G)$ and all edges other that ab incident to b are in $F^{i-1}_{P'}(G)$ and so by Algorithm 3.2.2, $xy = ab \in F^i_{P'}(G)$.

$xy \neq ab$ and $x, y \in C^i_P(G)$: By Algorithm 3.2.2, there exist edges wx and yz in $F^{i-1}_P(G)$. Since we assumed $F^{i-1}_{P'}(G) \cap \text{branch}_a(b) = F^{i-1}_P(G) \cap \text{branch}_a(b)$ and since wx and yz are in $\text{branch}_a(b)$ (because $xy \neq ab$), we conclude that $x, y \in C^i_{P'}(G)$ and so by Algorithm 3.2.2, $xy \in F^i_{P'}(G)$.

$xy \neq ab$ and one of x or y is in $C^i_P(G)$ and has degree greater than 1 with all other edges incident to it in $F^{i-1}_P(G)$: Without loss of generality, suppose $x \in C^i_P(G)$ has degree greater than 1 and all other edges incident to x are in $F^{i-1}_P(G)$. Since we assumed $F^{i-1}_{P'}(G) \cap \text{branch}_a(b) = F^{i-1}_P(G) \cap \text{branch}_a(b)$, and all edges incident to x are in $\text{branch}_a(b)$ (because $xy \neq ab$), we conclude that $x \in C^i_{P'}(G)$ and all edges other than xy incident to x are in $F^{i-1}_{P'}(G)$ and so by Algorithm 3.2.2, $xy \in F^i_{P'}(G)$.

So we have shown that $F^i_P(G) \cap \text{branch}_a(b) \subset F^i_{P'}(G) \cap \text{branch}_a(b)$ and therefore $F^i_P(G) \cap \text{branch}_a(b) \subset F^i_{P'}(G) \cap \text{branch}_a(b)$ for all i. □

Lemma 3.3.7. *Let G be a tree and P a PMU cover of G. If $P' = P \cap \text{branch}_a(b)$ observes all of $\text{branch}_a(b)$ and consists exclusively of degree ≤ 2 vertices, then there exists a PMU cover $P_a = (P - \{p\}) \cup \{a\}$ for some $p \in P'$.*

Proof. First of all, if $a \in P$, then we just let $P_a = (P - \{a\}) \cup \{a\} = P$ and we are done. Suppose $a \notin P$. Throughout the proof, we will talk about edges being directed with respect to the PMU placement $P' = P \cap \text{branch}_a(b)$. Since P' observes ab there exists a smallest integer i such that $ab \in F^i_{P'}(G)$. We induct on i and we denote all vertices adjacent to b (other than a) as $c_1, c_2, \ldots, c_n$.

Base case, $i = 0$: If $ab \in F^0_{P'}(G)$ then $b \in C^0_P(G) = P$ since $a \notin P \implies a \notin P'$. Let $P_a = (P - \{b\}) \cup \{a\}$. We show P_a is a PMU cover on G. We start with the fact that $(P \cup \{a\})$ is a PMU cover on G. Since $b \in \text{branch}_a(b)$, by our assumption $\deg(b) \leq 2$. The PMU placement $\{a\}$ observes b and both edges incident to b, and so by Proposition 3.3.2, $b \notin (P \cup \{a\})_{min}$. Thus $P_a = (P - \{b\}) \cup \{a\}$ is also a PMU cover on G.

Inductive step, $i \geq 1$: Suppose $i \geq 1$ the above statement is true for $i-1$. Since $ab \notin F^0_{P'}(G)$, we know that $a, b \notin P'$. Also, since P' observes all of $\text{branch}_a(b)$ every vertex in $\text{branch}_a(b)$ not in P' has at least one edge directed towards it by Proposition 3.3.4. Note that $a, b, c_1, c_2, \ldots, c_n$ are all in $\text{branch}_a(b)$ and $a, b \notin P'$. We will show exactly which edges are directed towards a and b with respect to P'.

ab is directed towards a: By Proposition 3.3.4, some edge xa is directed towards a. Suppose $x \neq b$. This would imply by Lemma 3.3.6 that xa is observable by $P' \cap \text{branch}_a(x) = \emptyset$ which is impossible. Therefore ab must be directed towards a with respect to P'.

bc_k is directed towards b for some $k \in \{1, \ldots, n\}$: Since ab is directed away from b, one of the bc_k must be directed towards b.

Every c_j is observable on $\text{branch}_b(c_j)$: If $c_j \in P'$, c_j is observable on $\text{branch}_b(c_j)$. Suppose $c_j \notin P'$ and recall that we have already shown that ab is directed away from b. By Proposition 3.3.4, at most one edge can be directed away from b. Therefore, bc_j cannot be directed toward c_j and so there must be some other vertex adjacent to c_j, call it d_j, such that c_jd_j is directed towards c_j. By Lemma 3.3.6, c_j is observable by $P' \cap \text{branch}_{c_j}(d_j) \subset P' \cap \text{branch}_b(c_j)$.

We now go back to our assumption that $ab \in F^i_{P'}(G)$ and since ab is directed towards a, then $a \notin C^i_{P'}(G)$. Therefore by Algorithm 3.2.2, $b \in C^i_{P'}(G)$ and every edge $bc_j \in F^{i-1}_{P'}(G)$. Since bc_k is directed towards b, by Lemma 3.3.6 implies that $P' \cap \text{branch}_b(c_j)$ observes $\text{branch}_b(c_j)$ and $bc_k \in F^{i-1}_{P' \cap \text{branch}_b(c_k)}(G)$ since $bc_k \in F^{i-1}_{P'}(G)$. Furthermore, $P \cap \text{branch}_b(c_k)$ consists exclusively of vertices of degree ≤ 2 since $P \cap \text{branch}_b(c_k) \subset P'$ and P' consists exclusively of degree ≤ 2. Therefore, by our inductive hypothesis, there exists a PMU cover $P_b = (P - \{p\}) \cup \{b\}$ for some $p \in P \cap \text{branch}_b(c_k)$. We conclude by showing that $P_a = (P_b - \{b\}) \cup \{a\}$ is a PMU cover on G. Note that $(P_b \cup \{a\})$ is a PMU cover on G. We claim that $b \notin (P_b \cup \{a\})_{min}$. From above, for every $j \neq k$, c_j is observable by $P' \cap \text{branch}_b(c_j) \subset P_a$. Also $\{a\}$ observes ab and b and so by Ohms Law, P_a observes every bc_j for $j \neq k$. Since P_a observes all but one edge incident to b, by Proposition 3.3.2, $b \notin (P_a \cup \{b\})_{min}$ and so $P_a = (P_b - \{b\}) \cup \{a\} = (P - \{p\}) \cup \{a\}$ is also a PMU cover on G. □

Lemma 3.3.8. *Let G be a tree and P is the set of leaves of G. Then P is a PMU Cover of G.*

Proof. We begin by showing the for every edge ab in G, $P \cap \text{branch}_a(b)$ observes $\text{branch}_a(b)$. We induct on $V = |V(\text{branch}_a(b))|$.

Base case, $V = 2$: Suppose $V(\text{branch}_a(b)) = \{a, b\}$ and $E(\text{branch}_a(b)) = \{ab\}$. Then b is a leaf in G, and so $P \cap \text{branch}_a(b) = \{b\}$ which observes $\text{branch}_a(b)$.

Inductive step, $V \geq 2$: If b is not a leaf of G, then b is adjacent to vertices $c_1, \ldots, c_n$ in addition to a. Since $|V(\text{branch}_b(c_i))| < V$ for each i, by the induction hypothesis, $P \cap \text{branch}_b(c_i)$ observes $\text{branch}_b(c_i)$. Therefore $P \cap \text{branch}_a(b) = \bigcup_{i=1}^{n} P \cap \text{branch}_b(c_i)$ observes $\bigcup_{i=1}^{n} \text{branch}_b(c_i)$. Since the vertex b and all the edges bc_i are observed by $P \cap \text{branch}_a(b)$, ab is also observed by $P \cap \text{branch}_a(b)$ by Kirkoff's Law, and a is observed by Ohm's Law. Therefore, $P \cap \text{branch}_a(b)$ observes $\text{branch}_a(b)$.

We conclude by observing that when a is a leaf in G, $\text{branch}_a(b) = G$ and $P \cap \text{branch}_a(b) = P$. Therefore, P is a PMU Cover of G. □

Corollary 3.3.9. *Let G be a tree. There exists a minimal PMU cover of G consisting only of leaves.*

Lemma 3.3.10. *Let G be a tree and P a PMU cover of G consisting only of leaves. Suppose $a_1b_1, a_2b_2, \ldots, a_nb_n$ are edges in G with each a_ib_i directed towards a_i with respect to P. Additionally, suppose $\text{branch}_{a_i}(b_i)$ and $\text{branch}_{a_j}(b_j)$ are disjoint for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then for any $m \in \{0, \ldots, n\}$, there exists a PMU cover $P_{a_1,\ldots,a_m}$ of G with the following properties:*

1. *$a_1, \ldots, a_m \in P_{a_1,\ldots,a_m}$,*
2. *For any $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$, $q \in P_{a_1,\ldots,a_m}$,*
3. *$|P_{a_1,\ldots,a_m}| = |P|$, and*
4. *For each $i \in [m+1, n]$, $P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$.*

Proof. We induct on m.

$m = 0$: We verify that P itself satisfies all three conditions of Lemma 3.3.10. The first condition is satisfied vacuously, the second and third conditions are trivial, and the fourth condition is satisfied because for each $i \in [1, n]$, $P \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$ by Lemma 3.3.6, since a_ib_i is directed towards a_i with repect to P.

$m \geq 1$: Assume that there exists a PMU cover $P_{a_1,\ldots,a_{m-1}}$ satisfying all the conditions of Lemma 3.3.10. Then $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_m}(b_m)$ observes all of $\text{branch}_{a_m}(b_m)$. Therefore, by

Lemma 3.3.7, there exists a PMU cover $P' = (P_{a_1,\ldots,a_{m-1}} - \{p\}) \cup \{a_m\}$ for some $p \in \text{branch}_{a_m}(b_m)$. We set $P_{a_1,\ldots,a_m} = P'$ and verify that $P_{a_1,\ldots,a_m}$ satisfies all three conditions: The first and third conditions are trivial. The second condition holds because, for any $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$, $q \in P_{a_1,\ldots,a_{m-1}} = (P_{a_1,\ldots,a_{m-1}} - \{p\}) \cup \{a_m\}$ and $q \neq p$ since $p \in \text{branch}_{a_m}(b_m)$. To verify the fourth condition, we note that for any $i \in [m+1, n]$, $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$. Furthermore, $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i) = P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i)$ since $\text{branch}_{a_i}(b_i)$ and $\bigcup_{j=1}^{m-1} \text{branch}_{a_j}(b_j)$ are disjoint. Therefore, $P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i) = P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$. □

Corollary 3.3.11. *Let G be a tree and P a PMU cover of G consisting only of leaves. Suppose $a_1b_1, a_2b_2, \ldots, a_nb_n$ are edges in G with each a_ib_i directed towards a_i. Additionally, suppose $\text{branch}_{a_i}(b_i)$ and $\text{branch}_{a_j}(b_j)$ are disjoint for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then there exists a PMU cover $P_{a_1,\ldots,a_n}$ of G with $|P_{a_1,\ldots,a_n}| = |P|$ containing $a_1, \ldots, a_m$ and all $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$*

Proof. This is Lemma 3.3.10 with $m = n$. □

3.4 Power Unmixed Trees

In this section, we introduce the class of edge linked trees and show that all minimal PMU covers of an edge linked tree have the same size. Then we prove (i) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) from Theorem 3.1.1. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) from this result are standard. Here are the relevant definitions.

Definition 3.4.1. A *pointed path* is a path P equipped with a *connecting vertex set*, i.e., a subset $W \subset V(P)$ such that

(P1) the set W does not contain either endpoint of P, and

(P2) the set W is independent in P, i.e., if $v_1 \in W$ and v_2 adjacent to v_1, then $v_2 \notin W$.

Definition 3.4.2. An *edge linked tree* is a tree T containing pointed path subgraphs $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively, such that

(T1) one has $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$, and $V(T) = V(P_1) \cup \cdots \cup V(P_n)$;

(T2) each $e \in E(T) \setminus E(P_1) \cup \cdots \cup E(P_n)$ is of the form $e = w_i w_j$ for some $w_i \in W_i$ and $w_j \in W_j$ with $i \neq j$; and

(T3) each $w \in W_1 \cup \cdots \cup W_n$ has $\deg(w) \geq 3$.

In (T3) above, the set $W_1 \cup \cdots \cup W_n$ is called the *connecting vertex set* of T, and in (T2), the set $E(T) \setminus E(P_1) \cup \cdots \cup E(P_n)$ is called the *connecting edge set* of T.

Example 3.4.3. The tree from Example 3.2.5 is edge linked. Indeed, the connecting vertices are red, the horizontal sub-paths are the pointed paths, and the vertical edges are the connecting edges.

The next result contains the implications (iv) $\iff$ (v) from Theorem 3.1.1.

Theorem 3.4.4. *A tree T is edge linked if and only if every vertex of T of degree at least 3 is adjacent to exactly two vertices of T of degree at most 2.*

Proof. ($\implies$) Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ and corresponding connecting vertex sets $W_1, \ldots, W_n$. Note that Definition 3.4.2 implies that the connecting vertex set of T is precisely the set of vertices of degree at least 3. Let $w_i \in W_i \subseteq V(P_i)$ be an arbitrary connecting vertex. Suppose $v \in V(T)$ is adjacent to w_i. Then Definition 3.4.2(T2) implies either $v \in W_j$ for some $j \neq i$, or $v \in V(P_i)$. For $v \in W_j$ for some $1 \leq j \leq n$, $j \neq i$, Definition 3.4.2(T3) implies $\deg(v) \geq 3$. If $v \in V(P_i)$, then Definition 3.4.1(P2) implies $v \notin W_1 \cup \cdots \cup W_n$. Thus $\deg(v) \leq 2$. Now, Definition 3.4.1(P1) implies that w_i is adjacent to two vertices in $V(P_i)$. Thus, w_i is adjacent to precisely two vertices of degree at most 2.

($\impliedby$) Suppose every vertex of T of degree at least 3 is adjacent to precisely two vertices of T of degree at most 2. Note that if there are no vertices of T of degree at least 3, then T is a path. Thus, set $P_1 =$ T, $W_1 = \emptyset$ so that T is an edge linked tree with pointed path P_1. Now, suppose there is a vertex of degree at least 3. Set the connecting vertex set of T to be equal to the set of vertices of T of degree at least 3. Thus Definition 3.4.2(T3) is satisfied. Note that by assumption, every vertex of T of degree at least 3 is adjacent to at least one other vertex of degree at least 3. Set the connecting edge set of T to be the set of all edges which connect vertices of degree at least 3. Pick an arbitrary w_1 of the connecting vertex set of T. Let P_1 be the the induced subgraph on the vertices which are contained in a path which contains w_1 and which does not contain a connecting edge of T, considered as a subgraph of T.

The claim is that P_1 is a pointed path. For the sake of contradiction, suppose P_1 is not a path. Then there exists $v \in V(P_1)$ such that the degree of v in P_1 is at least 3. As $P_1 \subseteq T$, the degree of v in T is at least 3. Let $\overline{v_1}, \overline{v_2}, \overline{v_3} \in V(P_1)$ and $v\overline{v_1}, v\overline{v_2}, v\overline{v_3} \in E(P_1)$. As P_1 does not contain connecting edges of T, $\overline{v_1}, \overline{v_2}, \overline{v_3}$ have degree at most 2 in T. This contradicts the assumption that every vertex of T of degree at least 3 is adjacent to precisely two vertices of T of degree at most 2. Thus, P_1 is a path.

It remains to show that P_1 is a pointed path.

Let $W_1 = \{v \in V(P_1) : \text{the degree of } v \text{ in } T \text{ is at least } 3\}$. By construction of P_1, Definition 3.4.1(P2) is satisfied. Let $\widetilde{w_1} \in W_1$ be a connecting vertex of P_1. As $\deg(\widetilde{w_1}) \geq 3$, $\widetilde{w_1}$ is adjacent to exactly two vertices of degree at most 2, v_1, v_2. Note that $\widetilde{w_1}v_1$, $\widetilde{w_1}v_2$ are not in the connecting edge set of T. Thus $\widetilde{w_1}v_1$, $\widetilde{w_1}v_2 \in E(P_1)$, and v_1, $v_2 \in V(P_1)$. Thus, $\widetilde{w_1}$ is not a leaf of P_1. Thus Definition 3.4.1(P1) is satisfied. Thus P_1 is a pointed path with connecting vertex set W_1.

Now, choose an arbitrary vertex w_2 of the connecting vertex set of T such that $w_2 \notin V(P_1)$. Let P_2 be defined as above, i.e., let P_2 be the induced subgraph on the vertices which are contained in a path which contains w_2 and which does not contain a connecting edge of T, considered as a subgraph of T, and let $W_2 = \{v \in V(P_2) : \text{the degree of } v \text{ in } T \text{ is at least } 3\}$. Note that $V(P_1) \cap V(P_2) = \emptyset$. Continuing, choose an arbitrary w_3 of the connecting vertex set of T such that $w_3 \notin V(P_1) \cup V(P_2)$, and define P_3, W_3 as above. Then $P_1 \cap P_3, P_2 \cap P_3 = \emptyset$, and P_3 is a pointed path with connecting vertex set W_3. Continuing in this way, pointed paths $P_1, \ldots, P_n$ are obtained with connecting vertex sets $W_1, \ldots, W_n$, respectively, such that $W_1 \cup \cdots \cup W_n$ is the connecting vertex set of T, and $V(P_i) \cap V(P_j) = \emptyset$ for $i \neq j$. Thus, Definition 3.4.2(T1) is satisfied. [(T1) now has 2 parts]

It remains to show that Definition 3.4.2(T2) is satisfied. To this end, observe that for each $1 \leq i \leq n$, $E(P_i)$ does not contain any connecting edges of T. Let $e = \widetilde{v_1}\widetilde{v_2} \in E(T)$ be arbitrary. If $\deg(\widetilde{v_1}), \deg(\widetilde{v_2}) \geq 3$, then e is in the connecting edge set of T. By construction, $\widetilde{v_1} \in W_i \subseteq V(P_i)$ for some $1 \leq i \leq n$ and $\widetilde{v_2} \in W_j \subseteq V(P_j)$ for some $1 \leq j \leq n$ so that in this case it remains to show $i \neq j$. For the sake of contradiction, suppose $i = j$. Then either $e \in E(P_i)$ or T contains a cycle. Note that $e \in E(P_i)$ contradicts the observation that $E(P_i)$ does not contain any connecting edges of T, and T containing a cycle contradicts the assumption that T is a tree. Thus $i \neq j$. Without loss of generality, if $\deg(\widetilde{v_1}) \geq 3$, $\deg(\widetilde{v_2}) \leq 2$, then $e \in E(P_i) = E(P_j) \subseteq E(P_1 \cup \cdots \cup P_n)$. If $\deg(\widetilde{v_1}), \deg(\widetilde{v_2}) \leq 2$, then there exists $\widetilde{v_3} \in W_k$ for some $1 \leq k \leq n$ such that there is a path which contains e and v_3 which does not contain a connecting edge of T. Thus, $e \in E(P_k) = E(P_i) = E(P_j) \subseteq E(P_1 \cup \cdots \cup P_n)$. Therefore Definition 3.4.2(T2) is satisfied. Thus, T is an edge linked tree with pointed paths $P_1, \ldots, P_n$. $\square$

Lemma 3.4.5. *Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively. Then $\exists\ 1 \leq i \leq n$ such that $|W_i| \leq 1$ and deg(w_i) $= 3$ for $w_i \in W_i$. In particular, if $n \geq 2$, $\exists\ 1 \leq i \leq n$ such that $|W_i| = 1$ and deg(w_i) $= 3$ for $w_i \in W_i$.*

Proof. Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively. If $n = 1$, then $T = P_1$ and $W_1 = \emptyset$ so that $|W_1| = 0$.

Let $n \geq 2$. Choose an arbitrary P_i, $1 \leq i \leq n$. As T is connected, $|W_i| \geq 1$ or $|W_i| = 1$ and $\deg(w) \geq 3$ for $w \in W_i$; if $|W_i| = 1$ and $\deg(w) = 3$ for $w \in W_i$, stop and the lemma holds. If $|W_i| > 1$ or $|W_i| = 1$ and $\deg(w) > 3$ for $w \in W_i$, then go to one of the neighboring P_j, i.e., one of the P_js for which $\exists w_j \in W_j$ such that w_j is adjacent to some $v \in W_i$. Again, if $|W_j| = 1$ and $\deg(w) = 3$ for $w \in W_j$, stop and the lemma holds. If $|W_j| > 1$ or $|W_j| = 1$ and $\deg(w) > 3$ for $w \in W_j$, choose a new neighboring P_k and do not choose the previous path, in this case P_i. Continue this process. Note that at each stage when a new neighboring P_i is chosen, the P_i chosen has not been chosen previously as T does not contain cycles. As n is finite this process terminates at some P_t with $|W_t| = 1$ and $\deg(w) = 3$ for $w \in W_t$ as if $|\{i, j, \ldots, t\}| < n$, if $|W_t| > 1$ or $|W_t| = 1$ and $\deg(w) > 3$ for $w \in W_t$, then the process could be continued, and if $|\{i, j, \ldots, t\}| = n$ with $|W_t| > 1$ or $|W_t| = 1$ and $\deg(w) > 3$ for $w \in W_t$, then T would contain a cycle. $\square$

Remark 3.4.6. Note that the pointed paths P for which $|W| \leq 1$ and $\deg(w) = 3$ for $w \in W$ are analogous to leaves of trees and as such the above proof is similar to proving a tree contains a leaf.

Definition 3.4.7. If a vertex v is observable and every line incident to v is observable, then v is called strongly observable.

Remark 3.4.8. Note that v being strongly observable is equivalent to having a PMU placed at v.

Theorem 3.4.9. *Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$. the minimal PMU covers of T are exactly sets of the form $\{v_1, \ldots, v_n\}$, where $v_i \in P_i$.*

Proof. The claim that a PMU cover must contain a vertex from each of the pointed paths is first proven.

Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ and connecting vertex sets $W_1, \ldots, W_n$, respectively. For some i, consider placing a PMU on all vertices in $\{v \in T : v \notin V(P_i)\}$.

The claim is that this is not a PMU cover. From the placement of the PMUs, the Incidence Law implies $T \setminus P_i$ is observable. Note that for $w \in W_i$, Definition 3.4.2(T2) and Definition 3.4.2(T3) imply that w is adjacent to a vertex $\tilde{w}$ such that $\tilde{w} \notin V(P_i)$. Thus by assumption, $\tilde{w}$ has a PMU so that edge $w\tilde{w}$ is observable by the Incidence Law and thus w is observable by Ohm's Law. Thus W_i is observable. Note that Definition 3.4.1(P1) implies that for $w \in W_i$, w is adjacent to two vertices in $V(P_i)$, v_1 and v_2, with v_1, $v_2 \notin W_i$ by Definition 3.4.1(P2). As $w \in V(P_i)$, w does not have a PMU so that the Incidence Law does not apply for edge wv_1 to be observable. Also, v_1 is not observable so that Ohm's Law does not apply. None of the lines in $E(P_i)$ are observable so that neither Ohm's Law nor Kirchhoff's Current Law applies and thus T remains unobservable and the PMU placement is not a PMU cover. Thus a vertex is needed from each pointed path P_i for a PMU cover.

That sets of the form $\{v_1, \ldots, v_n\}$ with $v_i \in V(P_i)$ are vertex covers is proven next.

Base Case, $n = 1$: $T = P_1$ is a path and the result is clear.

Assume the statement is true for $l \leq k$, and that $n > 1$: Suppose T is a tree with pointed paths $P_1, \ldots, P_{k+1}$ and with connecting vertex sets $W_1, \ldots, W_{k+1}$, respectively. Consider the set $\{v_1, \ldots, v_k, v_{k+1}\}$, where each $v_i \in V(P_i)$. The lemma above says that there is an index i for which $|W_i| = 1$ and $\deg(w) = 3$ for $w \in W_i$. Without loss of generality, let $i = k+1$ and let $w_{k+1} \in W_{k+1}$. Definition 3.4.2(T2) and Definition 3.4.2(T3) imply that w_{k+1} is adjacent to a vertex $w_j \in W_j$ for some index $j \neq k+1$. Consider $\widetilde{T} = T \setminus (P_{k+1} \cup w_j w_{k+1})$, i.e., the induced subgraph on the vertices not in P_{k+1}. Note that $\widetilde{T}$ is an edge linked tree with pointed paths $P_1, \ldots, P_k$ and by the inductive hypothesis, sets of the form $\{v_1, \ldots, v_k\}$ with $v_i \in V(P_i)$ are PMU covers for $\widetilde{T}$. This implies that in T, w_j is observable and one edge in $E(P_j)$ incident to w_j is observable. By the Incidence Law v_{k+1} is observable and edges incident to v_{k+1} are observable. If $v_{k+1} = w_{k+1}$ then as all remaining vertices in P_{k+1} are of degree at most 2, Ohm's Law and Kirchhoff's Current Law apply so that P_{k+1} is observable.

If $v_{k+1} \neq w_{k+1}$, again note that $\deg(v) \leq 2$ for all $v \in V(P_{k+1}) \setminus w_{k+1}$. The Incidence Law applies so that the vertices adjacent to v_{k+1} are observable and Ohm's Law and Kirchhoff's Current Law applied $\mathrm{d}(w_{k+1}, v_{k+1}) - 1$ times shows that w_{k+1} is observable. By Ohm's Law, $w_j w_{k+1}$ is observable, and by Ohm's Law and Kirchhoff's Current Law the remaining vertices in P_{k+1} are observable in T. It remains to show that $\widetilde{T}$ is observable as a subgraph of T.

Case 1, the degree of w_j in T is 3: Kirchhoff's Current Law may be applied so that the remaining edge incident to w_j is observable and thus w_j is strongly observable. This is equivalent to having a PMU placed at w_j. This implies that $\widetilde{T}$ is observable in T and thus T is observable.

Case 2, the degree of w_j in T is greater than 3: Suppose w_j is adjacent to $w_{j_1}, \ldots, w_{j_m}, w_{k+1}$, where $w_{j_i} \in W_{j_i}$ and $m \geq 1$ as the degree of w_j in T is greater than 3. For each $1 \leq r \leq m$, remove $w_{j_r} w_j$ and consider the connected subgraph which contains w_{j_r}, $\widehat{T}$. $\widehat{T}$ is an edge linked tree whose number of pointed paths s is less than k. Denote by $\widehat{P_1}, \ldots, \widehat{P_s}$ such paths. Then by the inductive hypothesis, sets of the form $\{v_1, \ldots, v_s\}$ with $v_i \in V(\widehat{P_i})$ are PMU covers of $\widehat{T}$. This implies that in T, w_{j_r} is observable. By Ohm's Law, $w_j w_{j_r}$ is observable, and by Kirchhoff's Current Law, the remaining edge in P_j is observable. Thus, w_j is strongly observable in T, which is equivalent to having a PMU placed at w_j. This implies $\widetilde{T}$ is observable in T and thus T is observable. □

Our next result follows directly from Theorem 3.4.9. It contains the implication (iv) $\Longrightarrow$ (iii) from Theorem 3.1.1.

Theorem 3.4.10. *Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$. Then*

$$I_T^P = \left\langle \prod_{x_j \in P_i} x_j \mid i = 1, \ldots, n \right\rangle.$$

In particular, I_T^P is a complete intersection.

We conclude by proving that prove (i) $\Longrightarrow$ (v) from Theorem 3.1.1.

Theorem 3.4.11. *Let G be a unmixed tree. Then every vertex of degree ≥ 3 is adjacent to exactly 2 vertices of degree ≤ 2.*

Proof. Suppose G is an unmixed tree and P is a minimal PMU cover of G containing only leaves. We will show each of the following:

1. If an edge ab is directed towards b with respect to P, then either $\deg(a) \leq 2$ or $\deg(b) \leq 2$.

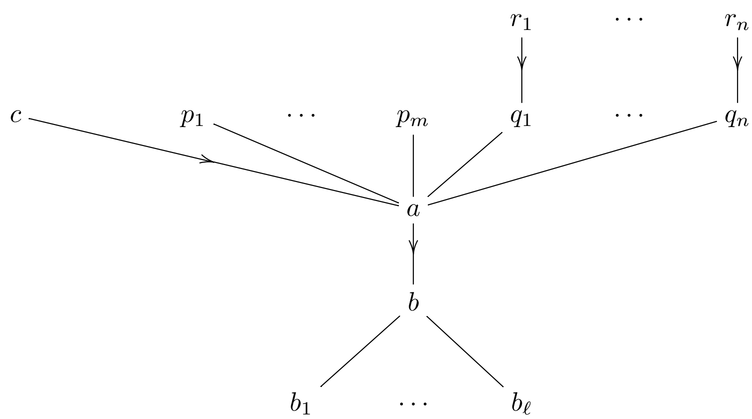
2. If an edge ab is undirected with respect to P, then both $\deg(a) \geq 3$ and $\deg(b) \geq 3$.

3. For any vertex a with $\deg(a) \geq 3$, there is *exactly* one edge directed towards a.

4. For any vertex a with $\deg(a) \geq 3$, there is *exactly* one edge directed away from a.

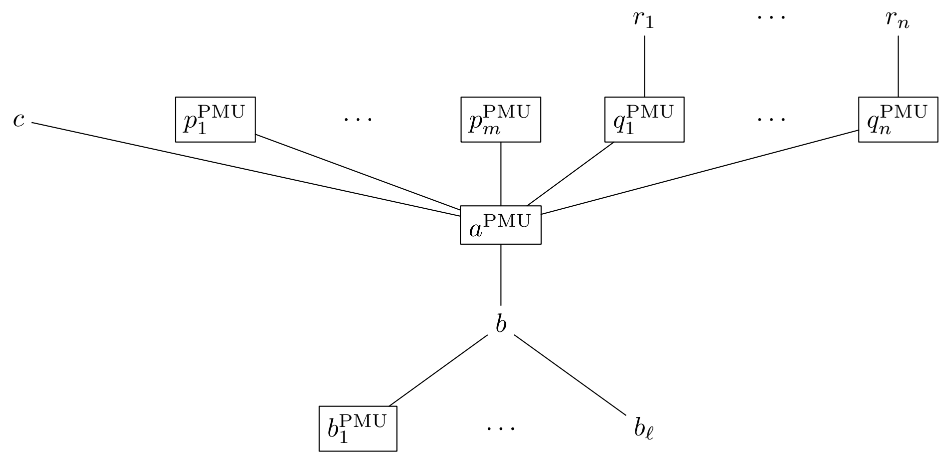
Note that the following imply that if $\deg(a) \geq 3$ then a must be adjacent to exactly 2 vertices of degree ≤ 2.

Proof of (1): Suppose, by way of contradiction, that ab is directed towards b with respect to P, and both $\deg(a) \geq 3$ and $\deg(b) \geq 3$. Then $a \notin P$, and by Proposition 3.3.4, there is at least one edge directed towards a, call it ca. We denote the other neighbors of a as $p_1, \ldots, p_m \in P$ and $q_1, \ldots, q_n \notin P$ with $m + n \geq 1$. We denote the other neighbors of b as $b_1, \ldots, b_l$ with $l \geq 2$.

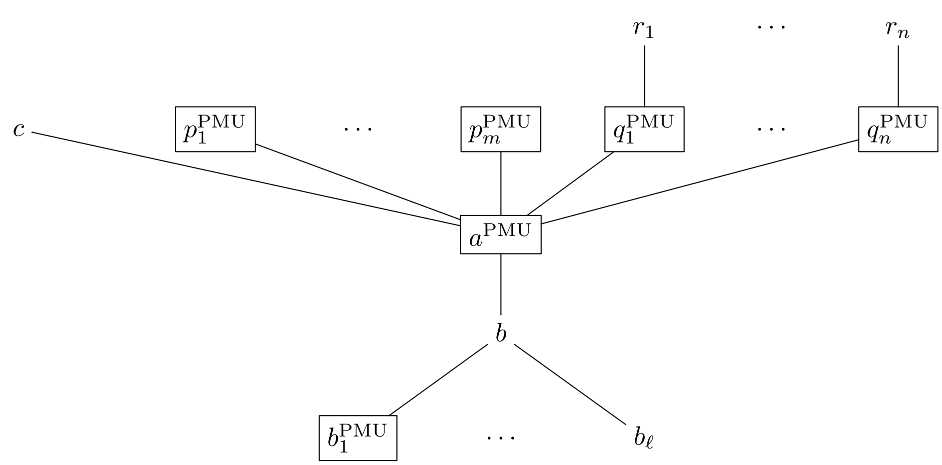


By Proposition 3.3.4, ab is the only edge directed away from a. Therefore, each aq_i is not directed towards q_i. By Proposition 3.3.4, there exists at least 1 edge directed towards each q_i call them r_iq_i. By Proposition 3.3.4, at most one of the bb_i are directed away from b. Since $l \geq 2$, there exists a bb_i that is not directed towards b_i. Without loss of generality, assume that bb_1 is not directed towards b_1. We divide the proof of (1) into 2 subcases and show that G is mixed:

Case 1*:* $b_1 \in P$*:* Since ca is directed towards a, each r_iq_i is directed towards q_i, $b_1, p_1, \ldots, p_m \notin \text{branch}_a(c) \cup \text{branch}_{q_1}(r_1) \cup \cdots \cup \text{branch}_{q_n}(r_n)$, and $\text{branch}_a(c), \text{branch}_{q_1}(r_1), \ldots, \text{branch}_{q_n}(r_n)$ are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,q_1,\ldots,q_n}$ containing $a, q_1, \ldots, q_n$, $p_1, \ldots, p_m, b_1$ with $|P_{a,q_1,\ldots,q_n}| = |P|$. However, $\{q_1, \ldots, q_n, p_1, \ldots, p_m, b_1\} \subseteq P_{a,q_1,\ldots,q_n} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,q_1,\ldots,q_n} \setminus \{a\}$ is a PMU cover, and $|P_{a,q_1,\ldots,q_n} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.



Case 2: $b_1 \notin P$: By Proposition 3.3.4, there is at least one edge directed towards b_1, call it $b_1 d$. Since ca is directed towards a, since each $r_i q_i$ is directed towards q_i, since $b_1 d$ is directed towards b_1, $p_1, \ldots, p_m \notin \text{branch}_a(c) \cup \text{branch}_{q_1}(r_1) \cup \cdots \cup \text{branch}_{q_n}(r_n) \cup \text{branch}_{b_1}(d)$, and since $\text{branch}_a(c), \text{branch}_{q_1}(r_1), \ldots, \text{branch}_{q_n}(r_n), \text{branch}_{b_1}(d)$ are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,q_1,\ldots,q_n,b_1}$ containing $a, q_1, \ldots, q_n, p_1, \ldots, p_m, b_1$ with $|P_{a,q_1,\ldots,q_n,b_1}| = |P|$. However, $\{q_1, \ldots, q_n, p_1, \ldots, p_m, b_1\} \subseteq P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}$ is a PMU cover, and $|P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.



Proof of (2): Suppose, by way of contradiction, that ab is undirected with respect to P, and $\deg(a) \leq 2$. We divide the proof into four cases and show that there exists a PMU cover containing both a and b, leading to a contradiction:

Case 1*: $a, b \in P$:* We note that $\{b\} \subseteq P \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P \setminus \{a\}$ is a PMU cover, and $|P \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 2*: $a \in P, b \notin P$:* By Proposition 3.3.4, there is at least one edge directed towards b, call it bd. Since bd is directed towards b, and $a \notin \text{branch}_b(d)$, by Corollary 3.3.11, there exists a PMU Cover P_b containing a and b with $|P_b| = |P|$. However, $\{b\} \subseteq P_b \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_b \setminus \{a\}$ is a PMU cover, and $|P_b \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 3*: $a \notin P, b \in P$:* By Proposition 3.3.4, there is at least one edge directed towards a, call it ac. Since ac is directed towards a, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU Cover P_a containing a and b with $|P_a| = |P|$. However, $\{b\} \subseteq P_a \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_a \setminus \{a\}$ is a PMU cover, and $|P_a \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 4*: $a \notin P, b \notin P$:* By Proposition 3.3.4, there is at least one edge directed towards a, call it ac, and one edge directed towards b, call is bd.

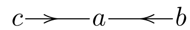
$$c \longrightarrow a \text{———} b \longleftarrow d$$

Since ac is directed towards a, bd is directed towards b, and $\text{branch}_a(c)$ and $\text{branch}_b(d)$ are disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,b}$ containing a and b with $|P_{a,b}| = |P|$.

$$c \text{———} \boxed{a^{\text{PMU}}} \text{———} \boxed{b^{\text{PMU}}} \text{———} d$$

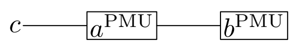
However, $\{b\} \subseteq P_{a,b} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,b} \setminus \{a\}$ is a PMU cover, and $|P_{a,b} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Proof of (3): Suppose $\deg(a) \geq 3$ and there are two edges ba and ca directed towards a.



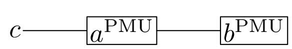
By (1), $\deg(b) \leq 2$ and $\deg(c) \leq 2$. We divide the proof into two cases and show that there exists a PMU cover containing both b and c, leading to a contradiction:

Case 1: $b, c \in P$: Since ac is directed towards a, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU Cover P_a containing a and b with $|P_a| = |P|$.



However, $\{a\} \subseteq P_a \setminus \{b\}$ observes all edges incident to b. Therefore, by Proposition 3.3.2, $P_a \setminus \{b\}$ is a PMU cover, and $|P_a \setminus \{b\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

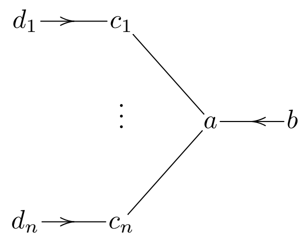
Case 2: At least one of $b, c \notin P$: Without loss of generality, assume $b \notin P$. By Proposition 3.3.4, there is at least one edge directed towards b, call it bd. Since ac is directed towards a, bd is directed towards b, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU Cover P_a containing a and b with $|P_a| = |P|$.



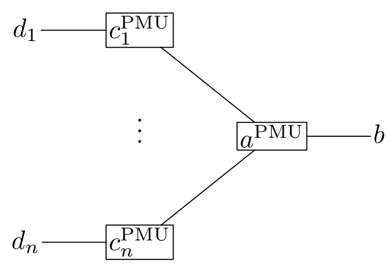
However, $\{a\} \subseteq P_a \setminus \{b\}$ observes all edges incident to b. Therefore, by Proposition 3.3.2, $P_a \setminus \{b\}$ is a PMU cover, and $|P_a \setminus \{b\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Proof of (4): Suppose $\deg(a) \geq 3$ and there are no edges directed away from a. By Proposition 3.3.4, there is at least one edge directed towards a, call it ab and let $ac_1, ac_2, \ldots, ac_n$ be $n \geq 2$ additional undirected edges. By (2), $c_1, \ldots, c_n$ all have degree ≥ 3 and thus $c_1, \ldots, c_n \notin P$. So by

Proposition 3.3.4, there is at least one edge directed towards each c_i, call them $c_i d_i$.



Since ab is directed towards a, each $c_i d_i$ is directed towards c_i, and $\text{branch}_a(b)$, $\text{branch}_{c_1}(d_1)$, $\ldots, \text{branch}_{c_n}(d_n)$ are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,c_1,\ldots,c_n}$ containing $a, c_1, \ldots, c_n$ with $|P_{a,c_1,\ldots,c_n}| = |P|$.



However, $\{c_1, \ldots, c_n\} \subseteq P_{a,c_1,\ldots,c_n} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,c_1,\ldots,c_n} \setminus \{a\}$ is a PMU cover, and $|P_{a,c_1,\ldots,c_n} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Conclusion: Therefore, every vertex $a \in G$ with $\deg(a) \geq 3$ must be adjacent to exactly 2 vertices of degree ≤ 2.

$\square$

Chapter 4

Unmixed Trees with respect to Total Domination

4.1 Introduction

Commutative algebra and combinatorics have a rich history of intersections. In this chapter, we focus on the relation between commutative algebra and graph theory. The root of this research began with the construction of an ideal from a graph by Villarreal. He came up with the notion of edge ideal $I(G)$ of a graph G which is an ideal 'generated by edges of the graph G'. Many research showed the relation between the algebraic properties of $I(G)$ and the combinatorial properties of G.

Mathematicians began to make variations to the construction of edge ideals and showed tied properties between graphs and their new ideals. Conca and De Negri [5] introduced the r-path ideal $I_r(G)$ of a graph G, when G is a tree. This construction covers Villarreal's construction as a subcase which is when $r = 1$, hence $I_1(G) = I(G)$. Paulsen and Sather-Wagstaff [16] introduced the edge ideal of a weighted graph G_w where $w : E(G) \to \mathbb{N}$ is a weight function which assigns some natural value (weight) to every edge of G. In case when w is a constant function $w = 1$ where $1 : E(G) \to \{1\}$, we have that $I(G_1) = I(G)$. Later, Kubik and Sather-Wagstaff [13] introduce the r-path ideals on weighted graph G_w which is the generalization of weighted edge ideals to the r-path ideals.

In this chapter, we introduce a new graph ideal called the open neighbor ideal I_G^N of a graph

G and characterize when I_G^N is Cohen-Macaulay when G is a tree. The main result of the chapter is the following; it is Theorem 1.3.1 from the introduction.

Theorem 4.1.1. *Let T be a tree. Then the following are equivalent:*

1. *T is unmixed.*

2. *T_B and T_R has height less than or equals to 3, has no two leaves of distance 4 apart, and no support vertex is adjacent to more than one height 2 vertex.*

3. *I_G^N is Cohen-Macaulay.*

In Section 4.2, we focus on the characterization of unmixed trees (*1* if and only if *2*). In Section 4.3, we construct a simplicial complex Δ_T (called ruled complex) out of a tree T whose Stanley-Reisner ideal is I_T^N and prove that Δ_T is shellable which implies that I_T^N is Cohen-Macaulay [7]. Finally, the Stanley-Reisner ideal has to be m-unmixed [15, Theorem 5.3.16.(a)], so Δ_T is pure, thus T is unmixed, thus shows *3* implies *1*. In the remaining part of Section 4.1, we introduce the relation between open neighborhood ideal and total domination problem on graphs.

4.1.1 Open Neighborhood Ideals and Total Dominating Sets

Definition 4.1.2. Let G be a finite simple graph (we assume that all graphs considered further are finite and simple). For $v \in V(G)$, the *open neighborhood of v in G* is the set of vertices $N(v) = \{u \in V(G) : uv \in E(G)\}$; for any subset $A \subseteq V(G)$, we define $N(A) = \bigcup_{v \in A} N(v)$. A subset $D \subseteq V(G)$ is a *total dominating (TD) set of G* if $N(D) = V(G)$. We say D is a *minimal TD-set* if no proper subset of D is a TD-set. We say G is *unmixed* if every minimal TD-set of G has the same size.

The minimal TD-set problem of graph itself is a well-studied subject in graph theory [1],[10],[19] and finding the size of the smallest minimal TD-set of a graph turns out to be NP-complete problem [14]. However, we focus on the mixedness property to establish our main goal.

Fact 4.1.3. *Let G be a graph. Then*

(i) *For any $V', V'' \subseteq V$, if V' is a TD-set and $V' \subseteq V''$, then V'' is a TD-set.*

(ii) *Every TD-set that is not minimal contains a minimal TD-set.*

Let G be a graph with the vertex set $V = \{v_1, v_2, \ldots, v_d\}$, and let $R = A[X_1, X_2, \ldots, X_d]$ be a polynomial ring over some commutative ring A. *The open neighborhood ideal of* G is the ideal "generated by the open neighborhood of each vertex in G":

$$I_G^N = (\underline{N}(v_1), \ldots, \underline{N}(v_d))R \quad \text{where} \quad \underline{N}(v_i) = \prod_{v_j \in N(v_i)} X_j.$$

Also, given a subset $V' \subseteq V$, we define $P_{V'}$ to be the ideal
"generated by the elements in V'":

$$P_{V'} = (\{X_i \mid v_i \in V'\})R.$$

By definition open neighborhood ideals are monomial ideals, hence has a irredundant m-irreducible decomposition. So, we find such decomposition for any open neighborhood ideal of a given graph.

Lemma 4.1.4. *Let G be a graph with the vertex set $V = \{v_1, v_2, \ldots, v_d\}$, and let $R = A[X_1, X_2, \ldots, X_d]$ be a polynomial ring over some commutative ring A. Let $V' \subseteq V(G)$. Then V' is a TD-set if and only if $I_G^N \subseteq P_{V'}$.*

Proof. First, suppose that V' is a TD-set. Let $f \in \{\underline{N}(v_1), \ldots, \underline{N}(v_d)\}$. Then $f = \underline{N}(v_i)$ for some i. Since v_i is a vertex in G and V' is a TD-set, $\exists v_j \in V'$ such that $v_i \in N(v_j)$. In particular, we have $X_j \mid \underline{N}(v_i)$. So, we get $\underline{N}(v_i) \in (X_j)R \subseteq (\{X_k : v_k \in V'\})R = P_{V'}$.

Now, suppose that $I_G^N \subseteq P_{V'}$. Let $u \in V$. Then $\underline{N}(u) \in I_G^N \subseteq P_{V'}$. Hence there exists $X_j \in \{X_k : v_k \in V'\}$ such that $X_j \mid \underline{N}(u)$. By definition of $\underline{N}(u)$, we have $v_j \in N(u)$, hence $u \in N(v_j) \subseteq N(V')$. Since u is an arbitrary vertex in G, we get $V \subseteq N(V')$, therefore V' is a TD-set. □

Theorem 4.1.5. *Let G be a graph with the vertex set $V = \{v_1, v_2, \ldots, v_d\}$, and let $R = A[X_1, X_2, \ldots, X_d]$ be a polynomial ring over some commutative ring A. The open neighborhood ideal has the following m-irreducible decomposition*

$$I_G^N = \bigcap_{V'} P_{V'} = \bigcap_{V' \ min.} P_{V'}$$

where the first intersection is taken over all TD-set in G, and the second intersection is taken over all minimal TD-set in G. In particular, the second decomposition is irredundant.

Proof. Since for any $A, B \subseteq V$ we have $P_A \subseteq P_B$ iff $A \subseteq B$, the second intersection is irredundant. Let $V'' \subseteq V$ be a TD-set which is not minimal. Then V'' contains a minimal TD-set. So, we have $\bigcap_{V'} P_{V'} = \bigcap_{V' \neq V''} P_{V'}$. Since V is finite, by repeating the same argument finitely many times, we conclude that $\bigcap_{V'} P_{V'} = \bigcap_{V'\ min.} P_{V'}$.

By Lemma 4.1.4, we have $I_G^N \subseteq \bigcap_{V'} P_{V'}$. For the containment $I_G^N \supseteq \bigcap_{V'} P_{V'}$, notice that I_G^N is a square-free monomial ideal, thus there are subsets $V_1, \ldots, V_k \subseteq V$ such that $I_G^N = \bigcap_{i=1}^k P_{V_i}$. For each index j, we have $I_G^N \subseteq P_{V_j}$ which implies that V_j is a TD-set by Lemma 4.1.4. Thus we get $I_G^N = \bigcap_{i=1}^k P_{V_i} \supseteq \bigcap_{V'} P_{V'}$. □

Example 4.1.6. Consider a path with 5 edges, P_5, with $V(P_5) = \{v_1, v_2, \ldots, v_6\}$ (Figure 4.1).

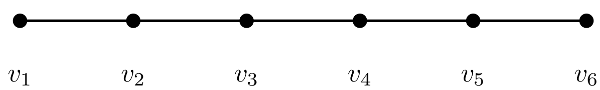


Figure 4.1: Path with 5 edges, P_5

Let $R = A[X_1, X_2, \ldots, X_6]$ be a polynomial ring over some commutative ring A. Then the open neighborhood ideal of P_5 is

$$\begin{aligned} N^o(P_5) &:= (\underline{N}(v_1), \underline{N}(v_2), \underline{N}(v_3), \underline{N}(v_4), \underline{N}(v_5), \underline{N}(v_6))\, R \\ &= (X_2,\ X_1X_3,\ X_2X_4,\ X_3X_5,\ X_4X_6,\ X_5)R. \end{aligned}$$

The irredundant m-irreducible decomposition of $N^o(P_5)$ is

$$\begin{aligned} N^o(P_5) &= (X_2,\ X_1X_3,\ X_2X_4,\ X_3X_5,\ X_4X_6,\ X_5)R \\ &= (X_2, X_1X_3, X_4X_6, X_5)R \\ &= (X_2, X_1, X_4X_6, X_5)R \cap (X_2, X_3, X_4X_6, X_5)R \\ &= (X_2, X_1, X_4, X_5)R \cap (X_2, X_1, X_6, X_5)R \\ &\quad \cap\ (X_2, X_3, X_4, X_5)R \cap (X_2, X_3, X_6, X_5)R \\ &= (X_1, X_2, X_4, X_5)R \cap (X_1, X_2, X_5, X_6)R \\ &\quad \cap\ (X_2, X_3, X_4, X_5)R \cap (X_2, X_3, X_5, X_6)R. \end{aligned}$$

Now, consider one of the m-irreducible ideal factor, $(X_1, X_2, X_4, X_5)R$. We can easily check that the set $\{v_1, v_2, v_4, v_5\}$ is a minimal TD-set in P_5.

4.2 Graph Theoretical Results

In this section, we nail down graph theoretical results related to minimal TD-set. The main goal is to characterize unmixed trees.

4.2.1 Preliminary Results

Definition 4.2.1. Let $x, y \in V(G)$. A *path from x to y* is a finite sequence of distinct vertices $\text{path}(x, y) := (v_0, v_1, \ldots, v_d)$ such that $l \in \mathbb{N}$, $v_0 = x$, $v_d = y$, and for any $0 \leq i < l$ we have $v_i v_{i+1} \in E(G)$. The *distance from x to y* is the positive integer d.

If a vertex z is one of the v_j in $\text{path}(x, y)$, we write $z \in \text{path}(x, y)$ or $z \in (v_0, v_1, \ldots, v_k)$.

Definition 4.2.2. The *degree* of a vertex $v \in V(G)$ denoted $\deg(v)$ is the number of edges incident to v. A vertex $l \in V(G)$ is a *leaf* if $\deg(l) = 1$. A vertex $s \in V(G)$ is a *support vertex* if s is adjacent to a leaf. Every vertex adjacent to a support vertex is called *supported vertex*.

Definition 4.2.3. Let G be a graph with at least one leaf. The *height* of a vertex $v \in V(G)$ is given by

$$\text{height}(v) := \min\{\text{distance}(v, x) \mid x \text{ a leaf in } G\}.$$

We denote $V_k(G) := \{v \in V(G) \mid \text{height}(v) = k\}$, and the height of G is the integer

$$\text{height}(G) := \max\{k \in \mathbb{N} : V_k(G) \neq \emptyset\}.$$

For instance, every leaf of a graph has height 0, and every non-leaf vertex adjacent to a leaf has height 1.

Fact 4.2.4. *If T is a tree, and $a, b \in V(T)$ with a adjacent to b, then $|\text{height}(a) - \text{height}(b)| \leq 1$.*

Lemma 4.2.5. *Let G be a graph with a support vertex s. Let $L = N(s) \cap V_0(G)$. Then for any minimal TD set D, we have $|D \cap L| \leq 1$.*

Proof. Assume not, then there are two vertices $x, y \in D \cap L$. Then $N(D \setminus \{x\}) = V(G)$ as $s \in N(y)$, which contradicts the fact that D is minimal. □

Let G and H be graphs. A subgraph H of G is an *induced subgraph* if two vertices of $V(H)$ are adjacent in H iff they are adjacent in G. In other words, induced graphs can be obtained from G by deleting a vertex v in G together with all edges that are incident to v; thus an induced subgraph is determined by its vertex set. We also say a subgraph of G is induced by its vertex set.

Definition 4.2.6. A *graph homomorphism* is a function $\sigma : V(G) \to V(H)$ such that for all $u, v \in V(G)$, we have $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. σ is an *embedding* if σ is injective. We say that an embedding σ *respects support* if for all $v \in V(G)$, v is a support vertex in G iff $\sigma(v)$ is a support vertex in H.

Theorem 4.2.7. ***(Mixedness is independent from number of leaves adjacent to support vertices).*** *Let G, H be a graphs. Let $\sigma : V(G) \hookrightarrow V(H)$ be an embedding which respects support, and every $v \in V(H) \setminus \sigma(V(G))$ is a leaf. Then G is unmixed iff H is unmixed.*

Proof. For proving the forward direction, we show its contrapositive. So suppose that H is mixed. Hence there are two minimal TD-sets D_1 and D_2 in H such that $|D_1| < |D_2|$. For every support vertex of $s \in V(H)$, choose a leaf in the image of σ and label it l_s. Then define a function $\tau : V(H) \to V(H)$ by

$$\tau(v) = \begin{cases} v & \text{if } \deg(v) > 1; \\ l_x & \text{if } \deg(v) = 1, \end{cases}$$

where x is the support vertex adjacent to v. Then $\tau(D_1)$ and $\tau(D_2)$ are still a minimal TD-set in H where the only possible change is the switch of leaves by definition of τ. Hence by Lemma 4.2.5, we get $|\tau(D_1)| = |D_1|$ and $|\tau(D_2)| = |D_2|$. Since $\tau(D_1)$ and $\tau(D_2)$ are in the image of σ, there exists $A_1, A_2 \subseteq V(G)$ such that $\sigma(A_1) = \tau(D_1)$ and $\sigma(A_2) = \tau(D_2)$. By definition of σ, A_1 and A_2 are TD-sets in G. To show that A_1 is a minimal TD-set in G, assume for a contradiction that A_1 is not a minimal TD-set. Hence there is a minimal TD-set $C_1 \subset A_1$ with $N(C_1) = N(A_1)$. By definition of σ, $\sigma(C_1)$ is also a minimal TD-set in H, which is a proper subset of $\tau(D_1)$, which is a contradiction. Similarly, A_2 is also a minimal TD-set. So, as $|A_1| = |\sigma(A_1)| = |\tau(D_1)| < |\tau(D_2)| = |\sigma(A_2)| = |A_2|$, G is mixed.

Now, suppose that H is unmixed. Since G embeds into H with respecting the supports, any minimal TD set in G is also a minimal TD set in H. Since every minimal TD set in H has the same cardinality, so is the minimal TD sets in G. Therefore, G is unmixed. □

Theorem 4.2.7 states that for any graph with some support vertices, its mixedness does not depend on attaching more leaves on those support vertices or removing leaves from the support vertices, but leaving at least one for each support vertex.

Lemma 4.2.8. *Let G be a graph with a TD-set D. Then D is minimal if and only if*

$$\forall v \in D,\ \exists u \in V(G) \text{ such that } N(u) \cap D = \{v\}.$$

Proof. First, suppose that D is minimal. Assume for a contradiction that there exists a vertex $c \in D$ such that for all $x \in N(c)$ we have $(N(x) \cap D) \setminus \{v\} \neq \emptyset$. Then the set $D \setminus \{c\}$ is also a TD set, a contradiction.

Now suppose that $\forall v \in D, \exists u \in V(G)$ s.t. $N(u) \cap D = \{v\}$. Assume for a contradiction that D is not minimal. Hence there is $c \in D$ such that $D \setminus \{c\}$ is a TD set. By hypothesis, there is some vertex $y \in N(c)$ such that $N(y) \cap D = \{c\}$. Hence $N(D \setminus \{c\}) \not\ni y$, a contradiction. □

Hence by Lemma 4.2.8, we can construct the following function.

Definition 4.2.9. Let D be a minimal TD set in G. For each $v \in D$, choose a vertex $x \in N(v)$ such that $N(x) \cap D = \{v\}$, and label it v_{ud} (i.e., $v_{ud} := x$). We define a function called a *domination selector of* D (in G) as $\mathcal{D}_D : D \to N(D)$ so that $\mathcal{D}_D(v) = v_{ud}$.

Domination selectors have to be injective by definition. Note that for every $v \in D$, the vertex $\mathcal{D}_D(v)$ is dominated by a *unique* vertex v. Next we extend the definition of *minimal set*(under subsets).

Definition 4.2.10. Let G be a graph with $D \subseteq V(G)$. D is a *minimal set* if there is no $C \subset D$, such that $N(C) = N(D)$.

Note that the domination selector of a minimal set is not unique. Now by extending the definition of domination selector for minimal sets, we can state the following lemma without proof.

Corollary 4.2.11. *Given a graph G and a set $D \subseteq V(G)$, D is a minimal set if and only if there is a domination selector of D. So, given that $N(D) = V(G)$, D is a minimal TD-set if and only if D has a domination selector in G.*

Example 4.2.12. Consider the following graph Y (Figure 4.2).

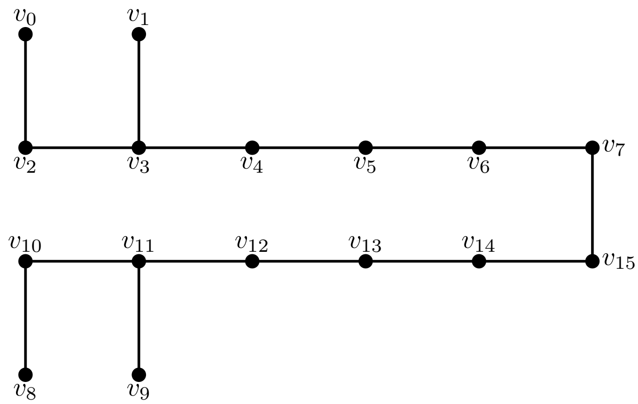


Figure 4.2: Graph Y, the Yifan graph

Consider the set $D := \{v_2, v_3, v_5, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$. We can simply check that D is a TD-set in Y. So, we show the existence of a domination selector for D to show that D is minimal. Define a function $\mathcal{D} : D \to V(Y)$ by

$$\mathcal{D}(x) = \begin{cases} v_0 & \text{if } x = v_2 \\ v_1 & \text{if } x = v_3 \\ v_6 & \text{if } x = v_5 \\ v_5 & \text{if } x = v_6 \\ v_8 & \text{if } x = v_{10} \\ v_9 & \text{if } x = v_{11} \\ v_{15} & \text{if } x = v_{14} \\ v_{14} & \text{if } x = v_{15} \end{cases}.$$

Since for all $v \in D$ we have $|N(\mathcal{D}(v)) \cap D| = 1$, $\mathcal{D}$ is a domination selector of D in Y. This means that for all $v \in D$, we have $\mathcal{D}(v) \notin N(D \setminus \{v\})$; for instance, set $v = v_{14}$, then $N(D \setminus \{v_{14}\}) = V(Y) \setminus \{v_{15}\} = V(Y) \setminus \{\mathcal{D}(v_{14})\}$. Hence no proper subset of D is a TD-set, thus D is minimal.

The domination selector is based on the fact that no proper subset of a non-TD-set is a TD-set. In other words, the existence of a domination selector assures us that given a set D of size k, every subset of D of size $k-1$ is not a TD-set, hence D is minimal. The usage of domination selectors comes in handy when we have to show the minimality of a subset without knowing its concrete (precise) structure. So, in the later sections, Lemma 4.2.11 will be our main tool to prove that certain sets are minimal (TD-sets).

4.2.2 Red Blue Colorings on Trees

From this point on, we will be looking exclusively at tree which are bipartite and whose vertices may be assigned a red/blue coloring such that no two adjacent vertices have the same color. If we denote the red vertices V_R and the blue vertices V_B, we notice that $N(V_R) = V_B$ and $N(V_B) = V_R$. Therefore, we define the following:

Definition 4.2.13. Let $T = (V, E)$ be a tree with red vertices V_R and blue vertices V_B such that no two adjacent vertices have the same color. Let D be a TD-set. We call $D \cap V_R$ a *red dominating set (RD-set)* and denote it $D_R = D \cap V_R$. We call $D \cap V_B$ a *blue dominating set (BD-set)* and denote it $D_B = D \cap V_B$.

Fact 4.2.14. *For a tree T, any two vertices which are even distance away have the same colour.*

Lemma 4.2.15. *Let T be a tree, $D' \subseteq V_R(T)$ and $D'' \subseteq V_B(T)$. Then $D = D' \cup D''$ is a minimal TD-set iff D' is a minimal RD-set and D'' is a minimal BD-set.*

Proof. Suppose that D is a minimal TD-set and set $D \cap V_R(T) = D'$ and $D \cap V_B(T) = D''$. D' is a RD-set since $N(D'') = V_R(T)$. D'' is an BD-set since and $N(D') = V_B(T)$, Also, D' and D'' are minimal because D is minimal.

Now, suppose that D' is a minimal RD-set and D'' is a minimal BD-set. Then $D = D' \cup D''$ is a TD-set because $N(D) = N(D') \cup N(D'') = V_B(T) \cup V_R(T) = V(T)$. Also, D is minimal because D' or D'' are minimal. □

Corollary 4.2.16. *Let T be a tree. Then T is unmixed iff every minimal BD-set has the same size and every RD-set has the same size.*

Example 4.2.17. Consider the following graph, Y, with vertices colored red and blue:

The set $D := \{v_2, v_3, v_5, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ is a minimal TD-set of Y (Example 4.2.12). Now, we partition the set D under coloring: so, set $D_R := \{v_2, v_6, v_{11}, v_{15}\}$ and $D_B := \{v_3, v_5, v_{10}, v_{14}\}$. Then we have

$$N(D_R) = N(v_2) \cup N(v_6) \cup N(v_{11}) \cup N(v_{15})$$

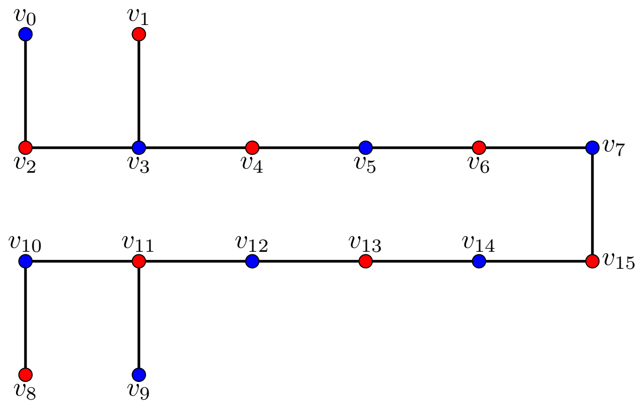


Figure 4.3: Graph Y, a two colored Yifan graph

$$
\begin{aligned}
&= \{v_0, v_3\} \cup \{v_5, v_7\} \cup \{v_9, v_{10}, v_{12}\} \cup \{v_7, v_{14}\} \\
&= \{v_0, v_3, v_5, v_7, v_9, v_{10}, v_{12}, v_{14}\} \\
&= V_B(Y),
\end{aligned}
$$

and

$$
\begin{aligned}
N(D_B) &= N(v_3) \cup N(v_5) \cup N(v_{10}) \cup N(v_{14}) \\
&= \{v_1, v_2, v_4\} \cup \{v_4, v_6\} \cup \{v_8, v_{11}\} \cup \{v_{13}, v_{15}\} \\
&= \{v_1, v_2, v_4, v_6, v_8, v_{11}, v_{13}, v_{15}\} \\
&= V_R(Y).
\end{aligned}
$$

Now, define functions $\mathcal{D}_R : D_R \to V(Y)$ and $\mathcal{D}_B : D_B \to V(Y)$ by

$$
\mathcal{D}_R(x) = \begin{cases} v_0 & \text{if } x = v_2 \\ v_5 & \text{if } x = v_6 \\ v_9 & \text{if } x = v_{11} \\ v_{14} & \text{if } x = v_{15} \end{cases} \qquad \text{and} \qquad \mathcal{D}_B(x) = \begin{cases} v_1 & \text{if } x = v_3 \\ v_6 & \text{if } x = v_5 \\ v_8 & \text{if } x = v_{10} \\ v_{15} & \text{if } x = v_{14} \end{cases}.
$$

Notice that $\mathcal{D}_R$ and $\mathcal{D}_B$ are just function $\mathcal{D}$ from Example 4.2.12 restricted to D_R and D_B. Hence the functions $\mathcal{D}_R$ and $\mathcal{D}_B$ are domination selectors for D_R and D_B respectively, hence both sets are minimal; furthermore, D_R and D_B are RD-set and BD-set, respectively.

So far, we found that checking whether a graph is unmixed can be boiled down to 2 subproblems: checking whether all minimal RD-sets have the same size and all minimal BD-sets have the same size (Corollary 4.2.16). So, given a tree T with vertices colored red and blue, we construct two subgraphs of T to break down our problem into two smaller problems.

Definition 4.2.18. Define T_B to be the *blue interior graph* obtained from T deleting all blue support vertices and supported red vertices, and all edges incident to them. Define T_R to be the *red interior graph* obtained from T by deleting all red support vertices and supported blue vertices, and all edges incident to them. Let $T_{B1}, T_{B2}, \ldots, T_{Bm}$ denote the m connected components of T_B, and let $T_{R1}, T_{R2}, \ldots, T_{Rn}$ denote the n connected components of T_R.

Example 4.2.19. The blue and red interior graph of the graph Y from Figure 4.3 and the path with 5 edges (P_5) are shown in Figure 4.4.

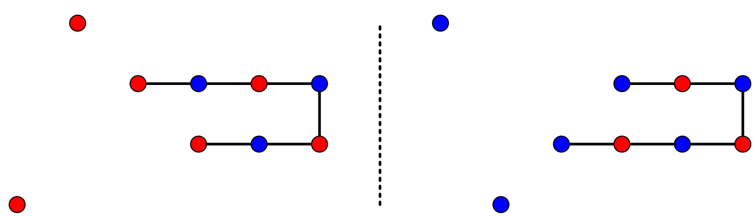


Figure 4.4: red(left) and blue(right) interior graphs of Y

In these examples, it turns out that the interiors are isomorphic graphs. However, this is not true in general.

The next lemma will demonstrate the importance of studying unmixedness in the interior graphs of trees.

Lemma 4.2.20. *Every minimal RD-set (BD-set) in T has the same size iff every minimal RD-set (BD-set) in T_R (T_B) has the same size.*

Proof. WLOG, we prove the claim just for BD-sets.

($\Rightarrow$) Suppose that every minimal BD-set in T has the same size. Assume for a contradiction that there are minimal BD-sets D_1, D_2 in T_B with $|D_1| \neq |D_2|$. Let $\mathcal{D}_1 : D_1 \to N_{T_B}(D_1)$ and $\mathcal{D}_2 : D_2 \to N_{T_B}(D_2)$ be domination selectors for D_1 and D_2 respectively. Let $A = V_1(T) \cap V_B(T)$. We claim that the sets $D := D_1 \cup A$ and $D' := D_2 \cup A$ are minimal BD-sets in T, to derive a contradiction. For each $s \in A$, choose a leaf $l_s \in N(s)$. Since $N(A)$ is the set of red vertices which are deleted while constructing T_B, the sets D and D' are BD-sets in T. Now, define $\mathcal{D} : D \to N_T(D)$ and $\mathcal{D}' : D' \to N(D')$ by

$$\mathcal{D}(x) = \begin{cases} \mathcal{D}_1(x) & \text{if } x \notin A, \\ l_x & \text{if } x \in A. \end{cases} \quad \text{and} \quad \mathcal{D}'(x) = \begin{cases} \mathcal{D}_2(x) & \text{if } x \notin A, \\ l_x & \text{if } x \in A. \end{cases}$$

Since $\mathcal{D}_1(D_1) \subseteq V(T_B)$ and $\mathcal{D}_2(D_2) \subseteq V(T_B)$, the functions $\mathcal{D}$ and $\mathcal{D}'$ are domination selectors for D and D' respectively. Thus D and D' are minimal BD-sets in T by Lemma 4.2.11, which is false as $|D| \neq |D'|$ but T is unmixed.

($\Leftarrow$) We show its contrapositive; so prove that there are two minimal BD-sets of different sizes in T_B if there are minimal BD-sets of different sizes in T. Let D_1 and D_2 be minimal BD-sets in T with $|D_1| \neq |D_2|$. Let $A := V_1(T) \cap V_B(T)$ and set $D := D_1 \setminus A$ and $D' := D_2 \setminus A$. We claim that D and D' are minimal BD-sets in T_B. Well, D and D' are minimal (else D_1 and D_2 are not minimal). Also, if there is some vertex $c \in V_R(T_B)$ such that $N(D) \not\ni c$, then D_1 is not a BD-set since $c \notin N(A)$ hence $c \notin N(D_1)$. Therefore, D and D' are minimal BD-sets in T_B, and since $|D| = |D_1| - |A|$ and $|D'| = |D_2| - |A|$, we get $|D| \neq |D'|$ as $|D_1| \neq |D_2|$ and $A \subseteq D_1 \cap D_2$. $\square$

Remark 4.2.1. Proof of Lemma 4.2.20 states that a set $D \subseteq V_B(T)$ is a minimal BD-set iff $D = D' \cup (V_B(T) \cap V_1(T))$ where D' is a minimal BD-set in T_B (and similar for RD-sets).

Corollary 4.2.21. *A tree T is unmixed iff every minimal RD-set of T_R has the same size and every BD-set of T_B has the same size.*

Proof. Combine Corollary 4.2.16 and Lemma 4.2.20. $\square$

At this point, we have reduced the problem of characterizing unmixed trees to characterizing the graphs for which every minimal BD-set of T_B has the same size and every RD-set of T_R has the same size. We will devote the remainder of this section to asking the question 'For which T_R (equivalently T_B) does every minimal BD-set (RD-set) have the same size?' This question is significantly easier than determining if a tree is unmixed due to a very special property of T_R and T_B, namely that no two adjacent vertices have the same height. Thus, in order to understand the unmixedness of T_R and T_B we notice that these interior graphs have an important property, namely that no vertices of the same height in T_R (equivalently T_B) are adjacent. We call such trees Δ-trees, and we spend the rest of this section studying their unmixedness in order to characterize all unmixed trees.

Definition 4.2.22. Let Δ be the set of all trees T (and forests) such that no two vertices of the same height in T are adjacent. For all $T \in \Delta$, we call T a Δ-tree (forest).

Proposition 4.2.23. *Let T be a tree with a given red/blue coloring. The following are equivalent:*

1. *T is a Δ-tree.*
2. *No two vertices of the same height in T are adjacent.*
3. *Any two vertices of the same height have the same color.*
4. *Every leaf has the same color.*

Proof. (1) $\Leftrightarrow$ (2) holds by definition of Δ-tree.

(4 $\Rightarrow$ 3) Suppose that every leaf of T has the same color. For any vertices $u, v \in V(T)$ with the same height h, there exist leaves $u', v' \in V_0(T)$ such that distance$(u, u') = h$ and distance$(u, u') = h$. Without loss of generality, assume u' and v' are red. If h is even, by Fact 4.2.14, u and v will also both be red. If h is odd, by Fact 4.2.14, u and v will both be blue. Therefore, u and v have the same color.

(3 $\Rightarrow$ 2) Suppose that any two vertices of the same height have the same color. Then for any vertices $u, v \in V(T)$ with the same height h, u and v cannot be adjacent since they share the same color.

(2 $\Rightarrow$ 4) Suppose that no two vertices of the same height in T are adjacent. Assume for a contradiction that there are leaves l_1 and l_2 such that l_1 is red colored and l_2 is blue colored. Since T is a tree, there is a unique path from l_1 to l_2, so set path$(l_1, l_2) = (v_0, v_1, \ldots, v_m)$ where m is a positive integer, $v_0 = l_1$ and $v_m = l_2$. By Fact 4.2.14 m must be odd. Consider the vertices in path(l_1, l_2). Since v_i and v_{i+1} are adjacent for all $i : 0 \leq i < m$, we must have height$(v_i) \neq$ height(v_{i+1}) by hypothesis. In particular, by definition of height, we must have height$(v_i) =$ height$(v_{i+1}) \pm 1$. Define two integers

$$a := |\{v_i : 0 < i \leq m, \text{height}(v_i) = \text{height}(v_{i-1}) + 1\}|$$

and

$$d := |\{v_i : 0 < i \leq m, \text{height}(v_i) = \text{height}(v_{i-1}) - 1\}|.$$

Note that $a + d = m$. Thus $0 = \text{height}(l_2) = \text{height}(v_m) = \text{height}(v_0) + a - d = \text{height}(l_1) + a - d = 0 + a - d \neq 0$ where the last inequality follows from the fact that m is odd, a contradiction. □

Example 4.2.24. Let T be a tree. Then both T_B are T_R are Δ-trees (forest) since their leaves all have the same color.

Hence by Lemma 4.2.24 and Theorem 4.2.21, we can characterize all unmixed trees by characterizing all unmixed Δ-trees.

Lemma 4.2.25. *Let $T \in \Delta$ of height d. Then $|V_{d-1}(T)| > |V_d(T)|$.*

Proof. Assume for a contradiction that we have $|V_{d-1}(T)| \leq |V_d(T)|$. Since no vertices in $V_d(T)$ is a leaf, every vertex in $V_d(T)$ is adjacent to at least 2 vertices in $V_{d-1}(T)$. Hence there are at least $2|V_d(T)|$ edges between the vertices in $V_d(T)$ and $V_{d-1}(T)$. Thus the subgraph induced by $V_d(T) \cup V_{d-1}(T)$ is not a tree; any tree H has exactly $|V(H)| - 1$ number of edges, but the subgraph induced by $V_d(T) \cup V_{d-1}(T)$ has $2|V_d(T)| \geq |V_d(T)| + |V_{d-1}(T)|$. Since a subgraph of T is not a tree, so is T, a contradiction. □

Define *the radar of x of distance d* (in G) to be the set

$$R_G(x, d) := \{a \in V(T) : \text{distance}(x, a) = d\}.$$

Let T be a tree, and choose $r \in V(T)$ to be the root of T. Let $v \in V(T)$. Define *the branch generated by r with respect to* (*the root*) r to be $\text{branch}_r(v) := \{x \in V(T) : v \in \text{path}(x, r)\}$.

Example 4.2.26. Let G be the given in Figure 4.5.

Then the radar of a of distance 2, denoted $R_G(a, 2)$, is the set of red and orange vertices. The branch generated by d with respect to c, denoted $\text{branch}_c(d)$, is the set of all blue vertices. Let H be the subgraph induced by the vertex set $\text{branch}_b(a)$. Then the radar of a of distance 2 in H which is $R_H(a, 2)$ is the set of orange vertices.

4.2.3 Characterizing the unmixed Δ-trees

In this section, we characterize the set of unmixed Δ-trees. Theorem 4.2.28, Theorem 4.2.31, and Theorem 4.2.33 will each give a condition which forces a Δ-tree to be mixed. For the Δ-trees

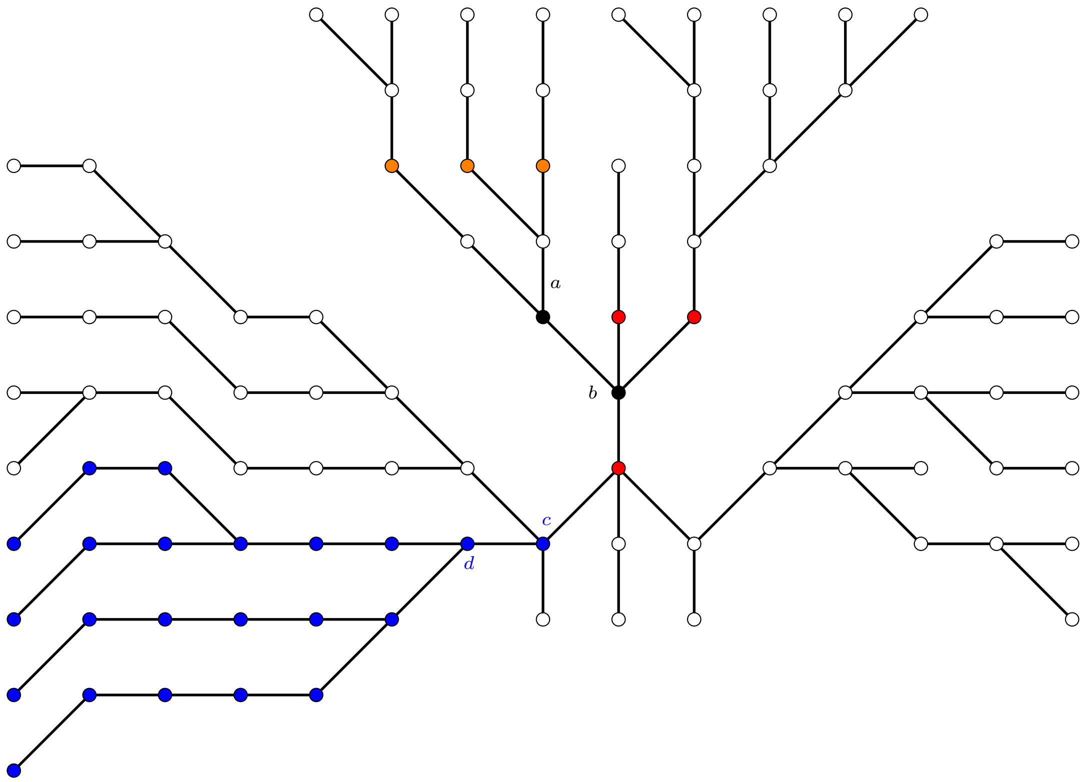


Figure 4.5: branch and radar example on G

throughout this section, we assign a red/blue coloring so that the even height vertices are blue and odd height vertices are red. (this is a valid coloring by Proposition 4.2.23).

Note 4.2.27. *Theorem 4.2.28, 4.2.31, and 4.2.33 are proved using the following technique: Given a* Δ*-tree* $T = (V, E)$*, choose a subgraph* $T' = (V', E')$ *of* T*. Then construct two minimal BD-sets in* T'*, say* M_1 *and* M_2*, so that* $N(M_1) = N(M_2) = V'_R$ *and* $|M_1| \neq |M_2|$*; hence there are domination selectors* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *of* M_1 *and* M_2 *such that* $\mathcal{D}_1(M_1), \mathcal{D}_2(M_2) \subseteq V'_R$*. Then construct a minimal set* $D \subseteq V \setminus V'$ *such that* $N(D) = V_R \setminus V'_R$*; so, there is a domination selector* $\mathcal{D}$ *of* D *such that* $\mathcal{D}(D) \subseteq V_R \setminus V'_R$*. Later, we will show that* $D' := D \cup M_1$ *and* $D'' := D \cup M_2$ *are minimal BD-set in* T*. But the problem is, we cannot guarantee that* D' *and* D'' *are minimal from the fact that* M_1, M_2*, and* D *are minimal. In order to show that* D' *and* D'' *are minimal, we construct domination selectors for each sets* D' *and* D''*, namely* $\mathcal{D}'$ *and* $\mathcal{D}''$*. The most natural way of constructing such functions* *is*

$$\mathcal{D}'(v) = \begin{cases} \mathcal{D}_1(v) & v \in M_1 \\ \mathcal{D}(v) & v \in D \end{cases} \quad \textit{and} \quad \mathcal{D}''(v) = \begin{cases} \mathcal{D}_2(v) & v \in M_2 \\ \mathcal{D}(v) & v \in D \end{cases}.$$

The issue with the above selectors is that there might be vertices $m_1 \in M_1$, $m_2 \in M_2$, and $d_1, d_2 \in D$ such that $\mathcal{D}_1(m_1) = \mathcal{D}(d_1)$ and $\mathcal{D}_2(m_2) = \mathcal{D}(d_2)$. WLOG, consider the case when $\mathcal{D}_1(m_1) = \mathcal{D}(d_1)$. In this case, we have $N(m_1) \cap N(d_1) \neq \emptyset$. In other words, d_1 is a vertex in D that is distance 2 apart from the set M_1. So, to ensure that the $\mathcal{D}(d_1) \notin V'$, we construct D from the set $V_B \setminus (V'_B \cap N(N(d_1)))$ instead of $V_B \setminus V'_B$; If $N(N(d_1)) = distance(d_1, 2)$ is not in D, then we can always force the value of $\mathcal{D}(d_1)$ to be in $V_R \setminus V'_R$ (by choosing a vertex in $N(distance(d_1, 2))$). So, set $C := distance(distance(M_1 \cup M_2, 2) \setminus V_B, 2)$, and we construct $D \subseteq C$ so that D is minimal and $N(D) = V_R \setminus V'_R$. We cannot tell if such set D exists all the time, but the existence of such sets becomes clear in each proof of the theorem by what we choose for the subgraph T'.

Theorem 4.2.28. *Let T be a Δ-tree. If T has two leaves l_1 and l_2 such that $distance(l_1, l_2) = 4$, then T has two minimal BD-sets of different sizes, hence T is mixed.*

Proof. Let s_1 and s_2 be the support vertices adjacent to l_1 and l_2 respectively, and let $v \in V_2(T)$ be the vertex which is adjacent to both s_1 and s_2. By Corollary 4.2.16, we show that there are two minimal BD-sets of different sizes. By Theorem 4.2.7, we assume that there is a unique leaf for each support vertex in T. For some non-negative integer k, let $u_1, u_2, \dots, u_k$ be vertices which are adjacent to v which are not the support vertices. For some integer $m \geq 2$, let $s_1, s_2, \dots, s_m$ be the support vertices adjacent to v, and finally, set l_i be the leaf adjacent to s_i for each possible i. Note that $u_1, \dots, u_k \in V_R(T)$ as $v \in V_B(T)$ as T is a Δ-tree (see Figure 4.6).

For each $i : 1 \leq i \leq k$, define G_{u_i} to be the subgraph of G induced by $\text{branch}_v(u_i) =: B_{u_i}$. For each G_{u_i}, consider the set $R_{u_i} := R_{G_{u_i}}(u_i, 1) \cup \left(\bigcup_{j=1}^{\infty} R_{G_{u_i}}(u_i, 3 + 2j)\right)$. Since u_i is red color, the set R_{u_i} is colored blue by Fact 4.2.14. Also, one can verify that R_{u_i} is an BD-set in G_{u_i}; i.e., every red vertex in G_{u_i} is dominated by some vertex in R_{u_i}. Since R_{u_i} is an BD-set in G_{u_i}, there is some minimal BD-set $D_{u_i} \subseteq R_{u_i}$. Set $D_u := \cup_{i=1}^{k} D_{u_i}$.

Next, for each $j : 1 \leq j \leq m$, define G_{s_j} be the subgraph of G induced by the set $\text{branch}_v(s_j) \setminus \{l_j\} =: B_{s_j}$. For each G_{s_j}, define the set $R_{s_j} = \bigcup_{i=0}^{\infty} R_{G_{s_j}}(s_i, 3 + 2i)$ (see Figure 4.7). Then $R_{s_j} \subseteq V_B(T)$, and R_{s_j} dominates all red vertices in G_{s_j} but s_j. Let $D_{s_j} \subseteq R_{s_j}$ be a

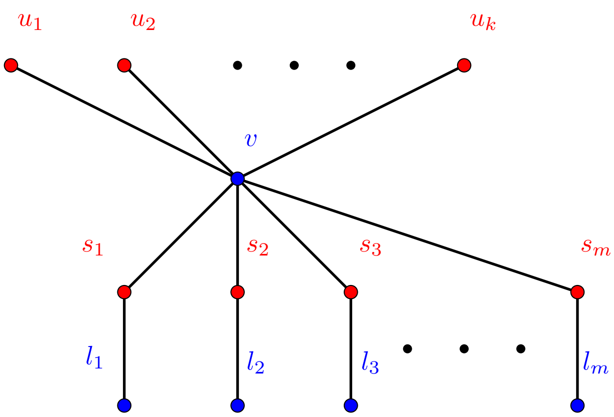


Figure 4.6: Tree T with vertices specified and their colorings

minimal set of R_{s_j}. Set $D_s = \bigcup_{j=1}^{m} D_{s_j}$.

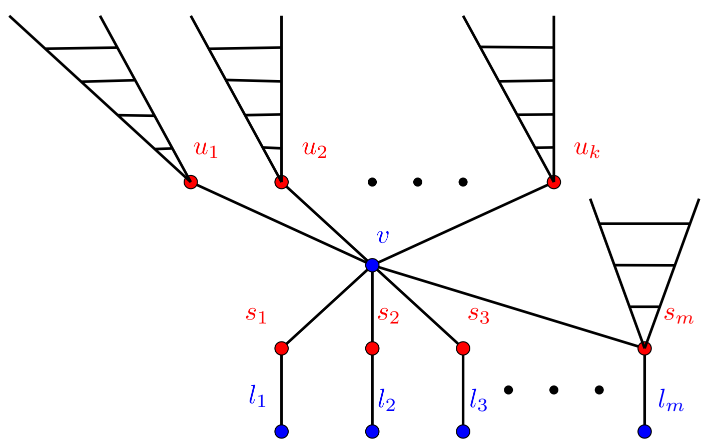


Figure 4.7: Tree T with branches (there are branches growing from all s_i's like s_m)
(View the horizontal lines as the set of vertices)

Let $D = D_u \cup D_s \cup \{v\}$ and let $D' = D_u \cup D_s \cup \{l_1, l_2, \dots, l_m\}$. We claim that D and D' are minimal BD-sets in G. Since $N(D_u \cup D_s) = V_R(T) \setminus \{s_1, s_2, \dots, s_k\}$, we have $N(D) = N(D') = V_R(T)$. So, we check the minimality of the two sets.

Assume for a contradiction that D and D' are not minimal. Then there are minimal RD-sets $C \subset D$ and $C' \subset D'$. Set $A := \left(\cup_{i=1}^{k} D_{u_i}\right) \cup \left(\cup_{j=1}^{m} D_{s_j}\right)$ and let $\mathcal{D}$ be the domination selector of A. Since A is constructed in the graphs induced by each B_{u_i} and B_{s_j} , we may have $\mathcal{D}(A) \subseteq \left(\bigcup_{i=1}^{k} B_{u_i}\right) \cup \left(\bigcup_{j=1}^{m} B_{s_j}\right)$; i.e., since $D_{u_i} \subseteq A$ is constructed so that it is minimal in G_{u_i}, for each i and for each $x \in D_{u_i}$, we can choose $\mathcal{D}(x)$ to be in B_{u_i}, and same for the D_{s_j}'s. Hence we must have

$A \subseteq C$ and $A \subseteq C'$ by Lemma 4.2.8. Since $A \subseteq C \subset D$ and $A \cup \{v\} = D$, we must have $A = C$, which is false since $N(A) \not\ni s_1$ (thus D is minimal). Also, since $A \subseteq C' \subset D'$ and $D' = A \cup \{l_1, l_2, \ldots, l_m\}$, there is some $j : 1 \leq j \leq m$ such that $l_j \notin C'$. Then $s_j \notin N(C')$ as $l_j \notin C'$, a contradiction (so D' is minimal). □

Example 4.2.29 (distance 4 leaves implies mixed)**.** Consider the tree in Figure 4.8.

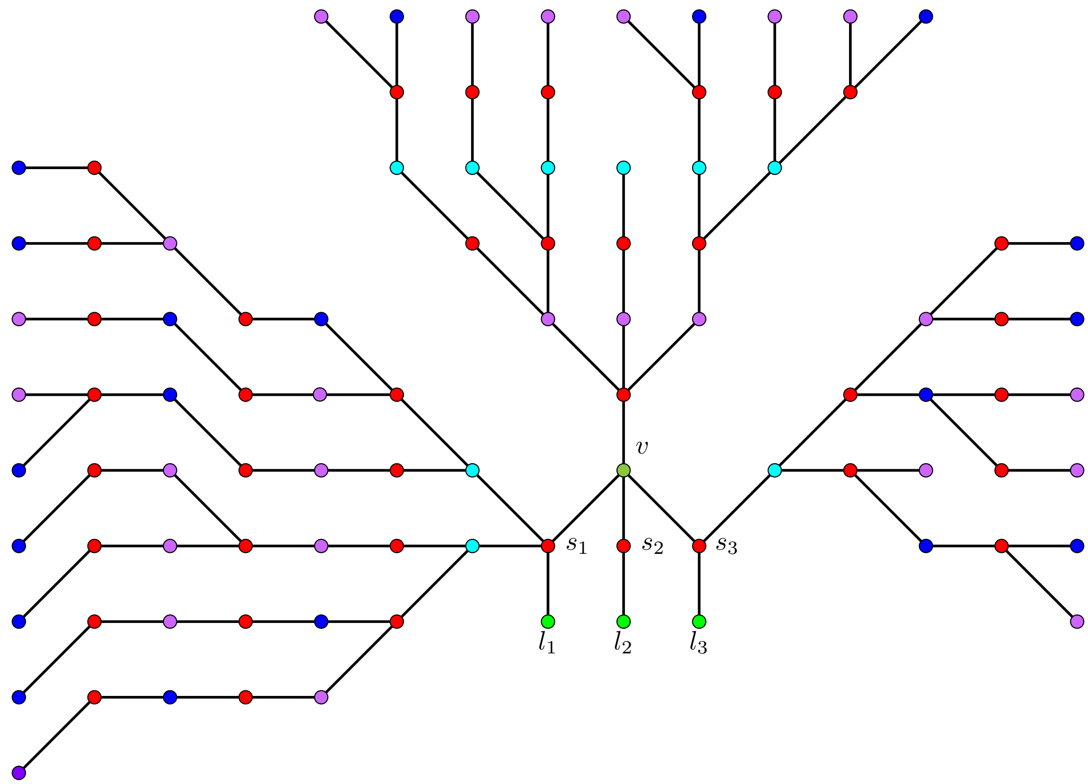


Figure 4.8: Δ-tree with distance 4 apart leaves

In this example, we will construct two minimal BD-sets of different sizes. We use the notations used in the proof of Theorem 4.2.28 (and Note 4.2.27). Let $M_1 := \{v\}$ and $M_2 := \{l_1, l_2, l_3\}$. The cyan color vertices represent the even height vertices (blue vertices) that were excluded when we constructed R_{u_i} and R_{s_i}. Set P be the set of all purple vertices (P corresponds to the set $D_u \cup D_s$). Notice that P is minimal, and that $N(P) = V_R \setminus \{s_1, s_2, s_3\}$. One can compute that the sets $D := P \cup M_1$ and $D' := P \cup M_2$ are minimal BD-sets.

Also, one can choose a different purple set (minimal and dominates all red vertices but the s_i's) which does not include any cyan color vertices and see that the union of the new purple set with either M_1 or M_2 form a minimal BD-set; minimality of the union is preserved by not picking the cyan vertices.

Corollary 4.2.30. *Let T be a Δ-tree with height$(T) = 2$. Then T is mixed.*

Proof. Since T is a Δ-tree with height$(T) = 2$, T has two leaves of distance 4 apart, then T is mixed. $\square$

Theorem 4.2.31. *Let T be a Δ-tree with height$(T) \geq 4$; in particular, there are two minimal BD-sets of different sizes. Then T is mixed.*

Proof. It is sufficient to show that there are two minimal BD-sets of different sizes. Since height$(T) \geq 4$, there is some vertex v of height 4. Hence there are vertices w_0, x, y, z where z is a leaf such that path$(v, z) = (v, w_0, x, y, z)$. By Theorem 4.2.28, we may assume that there are no two leaves of distance 4 apart; so, for any $a \in V_2(T)$, we have $|N(a) \cap V_1(T)| = 1$. For some non-negative integer k, let $w_1, w_2, \ldots, w_k$ be vertices which are adjacent to v where none of them is w_0. Choose v to be the root of T. For each $i : 1 \leq i \leq k$, let G_i to be the subgraph induced by $\text{branch}_v(w_i) =: B_i$, and let G_0 to be the subgraph induced by $B_0 := \text{branch}_v(w_0) \setminus \{x, y, z\}$. For each $i : 1 \leq i \leq k$, set $R_i = R_{G_i}(w_i, 1) \cup \left(\bigcup_{j=1}^{\infty} R_{G_i}(w_i, 3 + 2j)\right)$ and $R_0 = R_{G_0}(w_0, 3) \cup \left(\bigcup_{j=1}^{\infty} R_{G_0}(w_0, 5 + 2j)\right)$ (note that R_0 is not empty as $R_{G_0}(w_0, 3)$ is not empty since height$(w_0) = 3$).

First, notice that the R_i's for all $i \neq 0$ is an BD-set of G_i's, and R_0 dominates all red vertex in G_0 *except* w_0. Since R_i's are BD-sets in G_i for $i > 0$, there exists a minimal BD-set $D_i \subseteq G_i$ for each $i > 0$. Also, there is a minimal set $D_0 \subseteq R_0$ which dominates all red vertex in G_0 but w_0. Set $D_w = \cup_{i=0}^{k} D_i$ and $W = \{w_0, w_1, \ldots, w_k\}$.

Next, let G_x and G_y be subgraphs of G induced by the sets $\text{branch}_v(x) \setminus \text{branch}_v(y) =: B_x$ and $\text{branch}_v(y) \setminus \{z\}$, respectively. Let $R_x := R_{G_x}(x, 2) \cup (\bigcup_{i=0}^{\infty} R_{G_x}(v, 6 + 2i))$ and $R_y = \bigcup_{i=0}^{\infty} R_{G_y}(y, 3 + 2i)$. Then R_x dominates all red vertices in G_x, and R_y dominates all red vertices in G_y but y (also, as we assumed that there are no two leaves of distance 4 apart, we should realize that $R_{G_x}(x, 1) \subseteq V_3(T)$ which implies that $R_{G_x}(x, 4) \neq \emptyset$, and that $R_{G_y}(y, 1) \subseteq V_2(T)$ by Theorem 4.2.7 which implies that $R_{G_y}(y, 3) \neq \emptyset$). Let $D_x \subseteq R_x$ and $D_y \subseteq R_y$ be minimal sets of R_x and R_y.

We claim that the sets $D := D_w \cup D_x \cup D_y \cup \{v, z\}$ and $D' := D_w \cup D_x \cup D_y \cup \{x\}$ are minimal BD-sets in T. Since $N(D_w \cup D_x \cup D_y) = V_R(T) \setminus \{w_0, y\}$ and $N(\{v, z\}) \supseteq N(\{x\}) \supseteq \{w_0, y\}$, the sets D and D' do dominate all red vertices in T. Hence we are left to show their minimality.

For all $i : 1 \leq i \leq k$, let $\mathcal{D}_i$ be a domination selector of D_i in G_i. For each i, since we have $R_{G_i}(w_i, 3) \cap R_i = \emptyset$, we may assume that $w_i \notin \mathcal{D}_i(D_i)$. Overall for each $i > 0$, we have $\mathcal{D}_i(D_i) \subseteq B_i \setminus \{w_i\}$.

Similarly, by letting $\mathcal{D}_0 : D_0 \to B_0$ be a domination selector of D_0 in G_0, and we may assume that $\mathcal{D}_0(D_0) \subseteq B_0 \setminus \{w_0\}$ as $R_0 \cap R_{G_0}(w_0, 1) = \emptyset$. Lastly, let $\mathcal{D}_x : D_x \to B_x$ and $\mathcal{D}_y : D_y \to B_y$ be domination selectors of D_x and D_y, respectively. Consider the set $R_{G_x}(x, 2)$. Since $R_{G_x}(x, 4) \cap D_x = \emptyset$ and for any two vertices $v_1, v_2 \in R_{G_x}(x, 2)$ we have $N(v_1) \cap N(v_2) \cap R_{G_x}(v, 3) = \emptyset$ (else G_x contains a loop), we may assume that $\mathcal{D}_x(D_x) \subseteq B_x \setminus R_{G_x}(v, 1)$. Also, as $R_{G_y}(y, 1) \cap D_y = \emptyset$, we may assume that $\mathcal{D}_y(D_y) \subseteq B_y \setminus \{y\}$.

Now, define $\mathcal{D} : D \to N(D)$ and $\mathcal{D}' : D' \to N(D')$ by

$$\mathcal{D}(e) = \begin{cases} \mathcal{D}_i(e) & \text{if } e \in D_i \text{ for } 0 \leq i \leq k, \\ \mathcal{D}_x & \text{if } e \in D_x, \\ \mathcal{D}_y & \text{if } e \in D_y, \\ w_0 & \text{if } e = v, \\ y & \text{if } e = z. \end{cases} \quad \text{and} \quad \mathcal{D}'(e) = \begin{cases} \mathcal{D}_i(e) & \text{if } e \in D_i \text{ for } 0 \leq i \leq k, \\ \mathcal{D}_x & \text{if } e \in D_x, \\ \mathcal{D}_y & \text{if } e \in D_y, \\ w_0 & \text{if } e = x. \end{cases}$$

By the constructions above, we see that both $\mathcal{D}$ and $\mathcal{D}'$ are (well-defined) domination selectors for D and D' respectively. Therefore, D and D' are minimal by Lemma 4.2.11. □

Example 4.2.32. Consider the tree, T, in Figure 4.9.

We show that there are two minimal BD-sets of different sizes in T. We use the notation used in Theorem 4.2.31 and Note 4.2.27. Set $M_1 := \{v, z\}$ and $M_2 := \{x\}$, and let P be the set of all purple vertices (the set P corresponds to the set $D_w \cup D_x \cup D_y$ in the proof). The set of cyan vertices represent the blue vertices (even height vertices) excluded when we construct R_x, R_y, and R_i for $i : 1 \leq i \leq 3$ in this example. Just like we did with Example 4.8, the sets $D := P \cup M_1$ and $D' := P \cup M_2$ form a minimal BD-sets. One can choose a different purple vertices which does not

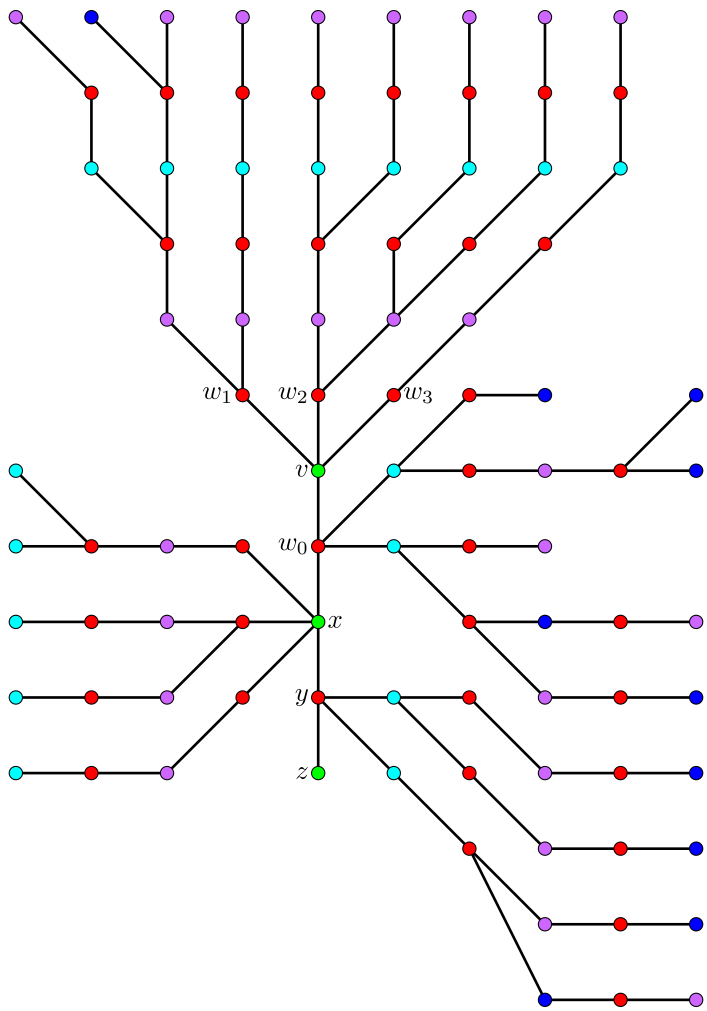


Figure 4.9: a Δ-tree T with no distance 4 apart leaves

choose any cyan color vertex and construct a different BD-set; not allowing to choose a cyan vertex for the new set of purple vertices reserves the minimality when we take the union.

Theorem 4.2.33. *Let T be a Δ-tree. If there is a support vertex $s \in V(T)$ such that $|N(s) \cap V_2(T)| \geq 2$, then there are two minimal BD-sets of different sizes, hence T is mixed.*

Proof. By Theorem 4.2.31 and Corollary 4.2.30, we may assume that $\text{height}(T) = 3$ as $\text{height}(T) \geq 2$ as $V_2(T) \neq \emptyset$. Now, let $v_1, v_2 \in N(s) \cap V_2(T)$. Let $u_1, u_2 \in V_3(T)$ where $v_1u_1, v_2u_2 \in E(T)$. Since $\deg(u_1), \deg(u_2) \geq 2$, there are vertices $v_1', v_2' \in V_2(T)$ where $v_1'u_1, v_2'u_2 \in E(T)$. Since $\text{height}(v_1') = \text{height}(v_2') = 2$, there are leaves $l_1, l_2 \in V(T)$ and support vertices $s_1, s_2 \in V(T)$ such that $\text{path}(v_1', l_1) = (v_1', s_1, l_1)$ and $\text{path}(v_2', l_2) = (v_2', s_2, l_2)$ (see Figure 4.10).

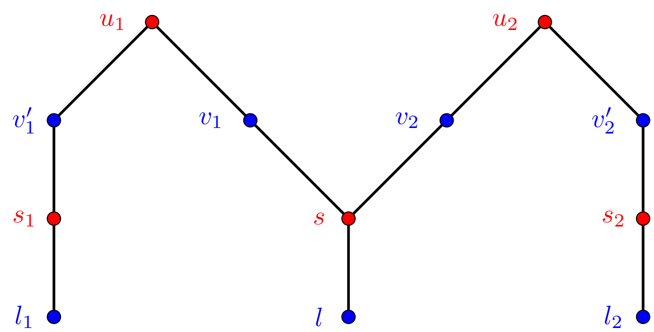


Figure 4.10: Tree T with specified vertices and coloring

Set $S_1 = V_2(T) \setminus N(u_1, u_2, s, s_1, s_2)$ and $S_2 = V_0(T) \setminus \{l, l_1, l_2\}$. We show that $N(S_1 \cup S_2) = (V_1(T) \cup V_3(T)) \setminus \{u_1, u_2, s_1, s_2, s\}$. The containment $N(S_1 \cup S_2) \subseteq (V_1(T) \cup V_3(T)) \setminus \{u_1, u_2, s_1, s_2, s\}$ is clear by construction of S_1 and S_2. To show that $N(S_1 \cup S_2) \supseteq (V_1(T) \cup V_3(T)) \setminus \{u_1, u_2, s_1, s_2, s\}$, assume for a contradiction that $N(S_1 \cup S_2) \not\supseteq (V_1(T) \cup V_3(T)) \setminus \{u_1, u_2, s_1, s_2, s\}$. Hence there is a vertex $y \in (V_1(T) \cup V_3(T)) \setminus \{u_1, u_2, s_1, s_2, s\}$ which is not in $N(S_1 \cup S_2)$.

Case 1: height$(y) = 1$. Then there is some leaf l' which is adjacent to y. Since $y \notin \{s_1, s_2, s\}$, we have $l' \notin \{l_1, l_2, l\}$. Thus $y \in N(l') \subseteq N(S_2) \subseteq N(S_1 \cup S_2)$, a contradiction.

Case 2: height$(y) = 3$. Since y is not a leaf, $\exists x_1, x_2 \in N(y)$ s.t. height$(x_1) =$ height$(x_2) = 2$ with $x_1 \neq x_2$. Now, since $y \notin N(S_1 \cup S_2)$, this implies that $\forall x \in N(y), x \notin S_1 \cup S_2$. Hence for all $x \in N(y)$, we have $x \in N(u_1, u_2, s_1, s_2, s) \cup \{l, l_1, l_2\}$. Since height$(x) = 2$, we must have $x \in N(u_1, u_2, s_1, s_2, s)$. Thus we get $N(y) \subseteq N(u_1, u_2, s_1, s_2, s)$. In particular, the vertices x_1, x_2 are adjacent to some vertices $c_1, c_2 \in \{u_1, u_2, s_1, s_2, s\}$, respectively. But, as $y \notin$ path$(s_1, s_2) = (s_1, v_1', u_1, v_1, s, v_2, u_2, v_2', s_2)$ and path$(x_1, x_2) = (x_1, y, x_2)$, the fact that x_1, x_2 are adjacent to vertices c_1, c_2 respectively creates a loop in the tree, which is a contradiction.

Therefore, we conclude that $N(S_1 \cup S_2) = V_1 \cup V_3 \setminus \{u_1, u_2, s_1, s_2, s\}$. Since $N(S_1 \cup S_2) = V_1 \cup V_3 \setminus \{u_1, u_2, s_1, s_2, s\}$, there is some minimal set $D_m \subseteq N(S_1 \cup S_2)$ such that $N(D_m) = V_1 \cup V_3 \setminus \{u_1, u_2, s_1, s_2, s\}$. Set $D_1 = \{l_1, l_2, v_1, v_2\}$ and $D_2 = \{v_1', v_2', l\}$. We claim that the sets $D' := D_m \cup D_1$ and $D'' := D_m \cup D_2$ are minimal BD-sets.

Since $N(D_1) \supseteq \{u_1, u_2, s_1, s_2, s\}$ and $N(D_2) \supseteq \{u_1, u_2, s_1, s_2, s\}$, it is clear that D' and D'' are BD-sets; so it remains to show that D' and D'' are minimal. Let $\mathcal{D}_m : D_m \to V_1(T) \cup V_3(T)$ be a domination selector for the minimal set D_m, and define the domination selectors $\mathcal{D}_1 : D_1 \to V_1(T) \cup V_3(T)$ and $\mathcal{D}_2 : D_2 \to V_1(T) \cup V_3(T)$ to be

$$\mathcal{D}_1(z) = \begin{cases} s_1 & \text{if } z = l_1 \\ s_2 & \text{if } z = l_2 \\ u_1 & \text{if } z = v_1 \\ u_2 & \text{if } z = v_2 \end{cases} \qquad \text{and} \qquad \mathcal{D}_2(z) = \begin{cases} s_1 & \text{if } z = v_1' \\ s_2 & \text{if } z = v_2' \\ l & \text{if } z = s. \end{cases}$$

Since $S_1 = V_2(T) \setminus N(u_1, u_2, s_1, s_2, s)$ and $S_2 = V_0(T) \setminus \{l, l_1, l_2\}$, for all $z \in D_m$ we have $\mathcal{D}_m(z) \notin \{u_1, u_2, s_1, s_2, s, l, l_1, l_2\}$. Therefore, the functions $\mathcal{D}' : D' \to V_1(T) \cup V_3(T)$ and $\mathcal{D}'' : D'' \to V_1(T) \cup V_3(T)$ defined by

$$\mathcal{D}'(z) = \begin{cases} \mathcal{D}_m(z) & \text{if } z \notin D_1 \\ \mathcal{D}_1 & \text{if } z \in D_1 \end{cases} \qquad \text{and} \qquad \mathcal{D}''(z) = \begin{cases} \mathcal{D}_m(z) & \text{if } z \notin D_2 \\ \mathcal{D}_2 & \text{if } z \in D_2 \end{cases}$$

are injective, hence are domination selectors for the sets D' and D''. Thus by Lemma 4.2.11, the sets D' and D'' are minimal, and as $|D'| \neq |D''|$, this proves the claim. □

Corollary 4.2.34. *Let T be a Δ-tree. If there is index i with $0 \leq i < height(T)$ such that $|V_i(T)| < |V_{i+1}(T)|$, then T is mixed.*

Proof. Suppose that there is some level l such that $|V_l(T)| < |V_{l+1}(T)|$. By Theorem 4.2.31, we may assume that height$(T) \leq 3$. Since $|V_0(T)| \geq |V_1(T)|$ for all tree, we must have height$(T) > 1$. If height$(T) = 2$, then by Lemma 4.2.25 we have $|V_1(T)| > |V_2(T)|$, thus there is no index that satisfies our hypothesis. Thus we must assume that height$(T) = 3$ and that $i = 1$; so $|V_1(T)| < |V_2(T)|$. Since every height 2 vertex is connected to some height 1 vertex, there are at least $|V_2(T)|$ edges between the vertices $V_1(T)$ and $V_2(T)$. Now, as $|V_1(T)| < |V_2(T)|$, by Pigeon Hole Principle, there is some vertex $s \in V_1(T)$ which is adjacent to more than 1 height 2 vertices. Thus by Theorem 4.2.33, tree T is mixed. □

In other words, Corollary 4.2.34 states that in order for a Δ-tree to be unmixed, the number of vertices in each level must be non-increasing as we go deeper in height.

Theorem 4.2.35. *Let T be a Δ-tree of height 3. If T is mixed, then either there are two leaves l_1, l_2 with $d(l_1, l_2) = 4$, or there is $s \in V_1(T)$ such that $|N(s) \cap V_2(T)| > 1$.*

Proof. We prove the contrapositive: If there are no two leaves l_1, l_2 with $d(l_1, l_2) = 4$ and there is no vertex $s \in V_1(T)$ such that $|N(s) \cap V_2(T)| > 1$, then T is unmixed. WLOG, we assume that $|V_0(T)| = |V_1(T)|$. To show that $|V_2(T)| = |V_1(T)|$, assume for a contradiction that $|V_2(T)| \neq |V_1(T)|$. Then by hypothesis that there is no vertex $s \in V_1(T)$ such that $|N(s) \cap V_2(T)| > 1$, we must have $|V_2(T)| < |V_1(T)|$. Since T is connected, every vertex in $V_1(T)$ must be connected to some vertex in $V_2(T)$. Hence there are at least $|V_1(T)|$ number of edges incident to both vertex sets $V_1(T)$ and $V_2(T)$. As $|V_2(T)| < |V_1(T)|$, there is a vertex $c \in V_2(T)$ which is adjacent to more than 1 vertex in $V_1(T)$, which contradicts the fact that there is no leaf of distance 4 apart. So, we must have $|V_2(T)| = |V_1(T)|$, which implies $|V_2(T)| = |V_0(T)|$. Let D be a minimal BD-set. Then $|D| \geq |V_1(T)|$ as we need at least $|V_1(T)|$ number of vertices to dominate vertices in $V_1(T)$. Now, we claim that $|D| = |V_1(T)|$. So, assume for a contradiction that $|D| > |V_1(T)|$. For every $v \in V_1(T)$, we have $|N(v)| = 2$ and the two vertices in $N(v)$ is a *toggled vertex* of the other. Hence there must be a vertex $x \in V_1(T)$ such that $N(x) \subseteq D$ as $D \subseteq N(V_1(T))$, which contradicts that D is minimal.

Finally, let D_B be a minimal BD-set. Since all support vertex must be in any minimal TD-set, the vertices $V_1(T) \subseteq D_B$ as $V_1(T) \subseteq V_R(T)$. Since $V_B(T) = V_0(T) \cup V_2(T)$ and $N(V_1(T)) = V_0(T) \cup V_2(T)$, we must have $D_B \subseteq V_1(T)$. Thus we get $D_B = V_1(T)$, and we conclude that minimal BD-set of T is unique. So, by Corollary 4.2.16, T is unmixed. □

Theorem 4.2.36 (Characterization of Unmixed Δ-trees)**.** *Let T be a Δ-tree. Then T is unmixed if and only if*

(i) $height(T) \leq 3$;

(ii) $\forall l_1, l_2 \in V_0(T)$, $distance(l_1, l_2) \neq 4$;

(iii) $\forall s \in V_1(T)$, $|N(s) \cap V_2(T)| = 1$.

Proof. (⇒) Suppose that T is unmixed. Then Theorem 4.2.31, Theorem 4.2.28, and Theorem 4.2.33 implies what we desire.
(⇐) Suppose that $(i), (ii), (iii)$ holds. By (i) and Corollary 4.2.30, we have height$(T) \in \{0, 1, 3\}$. If height$(T) \in \{0, 1\}$, then T is unmixed (as all tree of height 0,1 are unmixed). If height$(T) = 3$, then by Theorem 4.2.35, T is unmixed. □

Finally, we can characterize all unmixed trees.

Theorem 4.2.37. *[Characterization of Unmixed Trees] A tree T is unmixed iff every connected component of T_B and T_R has height less than or equal to 3, has no two leaves of distance 4 apart, and has no support vertex adjacent to more than one height 2 vertex.*

Proof. Combine Theorem 4.2.21, Lemma 4.2.24, and Theorem 4.2.36. □

4.3 Shellability of Unmixed Trees

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a *ruled set* if

(**DS1**) $\forall v \in D,\ \exists u \in V \setminus D$ s.t. $vu \in E$ (D is dominated by its complement);

(**DS2**) $\nexists v \in V \setminus D$ s.t. $N(v) \subseteq D$ (D never isolates a vertex in its complement).

We say that D is *maximal* if no subset $C \subseteq V$ with $D \subset C$ is a ruled set.

Lemma 4.3.1. *Let $G = (V, E)$ be a graph and let $D, T \subseteq V$ such that $T = V \setminus D$. Then*

1. *D is a ruled set iff T is a TD-set;*

2. *D is a maximal ruled set iff T is a minimal TD-set.*

Proof. Suppose that D is a ruled set. Since for all $v \in D$, there is $u \in T$ such that $vu \in E$, we have $D \subseteq N(T)$. Let $u \in T$. Since $N(u) \not\subseteq D$, there is some vertex $v \in N(u)$ such that $v \notin D$, hence $v \in T$. Thus $u \in N(T)$ and therefore $T \subseteq N(T)$, so $D \cup T = V \subseteq N(T)$. Now, suppose that T is a TD-set. Since $V \subseteq N(T)$, we have $D \subseteq N(T)$ which satisfies DS1. To show (DS2), assume for a contradiction that there is $v \in T$ such that $N(v) \subseteq V \setminus T = D$. Then $v \notin N(T)$ hence T is not a TD-set.

Suppose that D is a maximal ruled set. Assume for a contradiction that T is not minimal. Then there is $T' \subset T$ which is a TD-set. Hence $V \setminus T' = (V \setminus T) \cup (T \setminus T') = D \cup (T \setminus T')$ is a ruled set, that contains D, which is a contradiction. Finally, suppose that T is a minimal TD-set, and assume for a contradiction that D is not maximal. Then there is $D' \supset D$ which is a ruled set. Hence the set $V \setminus D' = (V \setminus D) \setminus (D' \setminus D) = T \setminus (D' \setminus D)$ is a TD-set, contained in T, which is a contradiction. $\square$

Let $\mathcal{D}$ be the set of all minimal TD-set of G. Let $\mathcal{F}_G := \{V \setminus A : A \in \mathcal{D}\}$ and let Δ_G be the set closed under subsets with the base set $\mathcal{F}_G$ (we also denote this set by $\langle \mathcal{F}_G \rangle$, the simplicial complex generated by $\mathcal{F}_G$). The set Δ_G is called the *ruled complex of* G; in particular, by Lemma 4.3.1, we know that the facets of Δ_G represent the set of all maximal ruled set in G, hence we alternatively call Δ_G by the *ruled complex of* G. Let Δ be a simplicial complex. Δ is *pure* if all facets have the same size, and in case Δ is pure, the *dimension of* Δ is $\dim(\Delta) := |F| - 1$ where F is a facet in Δ. Suppose that Δ is pure. Let $\prec$ be a total order on the set of all facets in Δ. By indexing the facets of Δ under $\prec$, obtain $F_1, F_2, \ldots, F_n$. We say $\prec$ is a *shelling* if for all $1 \leq i \leq n-1$, the simplicial complex generated by the first i facets intersecting with the simplicial complex generated by F_{i+1} ($\langle F_1, F_2, \ldots, F_i \rangle \cap \langle F_{i+1} \rangle$) forms a simplicial complex which is pure of dimension $\dim(\Delta) - 1$. If Δ has a shelling, then we say that Δ is *shellable*. Also, in case when the ruled complex of a given graph is shellable, we say that the graph is *shellable*.

An alternative definition for shelling is: for any i, if there is $j < i$ such that $|F_j \cap F_i| < \dim(\Delta)$, then there exists $k < i$ such that $F_k \cap F_i \supset F_j \cap F_i$; in particular, this implies that there is some facet $F_{k'}$ with $k' < i$ such that $F_{k'} \cap F_i \supset F_j \cap F_i$ and $|F_{k'} \cap F_i| = \dim(\Delta)$. We will use this alternative definition of shelling in the following theorems. Before further theorems, we state some facts that are derived from the characterization of unmixed Δ-trees:

Fact 4.3.2. *Let T be an unmixed Δ-tree of height 3 with support vertices $s_1, s_2, \ldots, s_k$ for some $k \in \mathbb{Z}^+$, and let D be a minimal TD-set of T. Then*

(i) $|D| = 2k$;

(ii) for all $i, j : 1 \leq i < j \leq k$, *we have* $N(s_i) \cap N(s_j) = \emptyset$.

(iii) $D = \{s_1, \ldots, s_k\} \cup \{u_1, \ldots, u_k\}$ *where for each* $i : 1 \leq i \leq k$, *we have* $u_i \in N(s_i)$.

Let T be an unmixed Δ-tree with support vertices $s_1, \ldots, s_k$. Let $N(s_i) := \{u_{i,1}, \ldots, u_{i,m_i}\}$ where $m_i = |N(s_i)|$. The *maximal (ruled) complex of* T (denoted by Δ_T^M) is the simplicial complex generated by the set

$$\mathcal{F}_T^M := \{V(T) \setminus (\{s_1, \ldots, s_k\} \cup \{u_{1,i_1}, \ldots, u_{k,i_k}\}) : 1 \leq i_j \leq m_j\}.$$

Note that we have $\mathcal{F}_T \subseteq \mathcal{F}_T^M$, hence $\Delta_T \subseteq \Delta_T^M$. Also, we have $\dim(\Delta_T) = \dim(\Delta_T^M)$. It is obvious that all height 0 or 1 unmixed Δ-trees have shellable ruled complex. Hence we assume that all unmixed Δ-tree we consider are of height 3. Starting this point, let T to be an unmixed Δ-tree with support vertices $s_1, \ldots, s_k$ with $N(s_i) = \{u_{i,1}, \ldots, u_{i,m_i}\}$ where $m_i = |N(s_i)|$ and $\text{height}(u_{i,1}) = 2$ (see Figure 4.11).

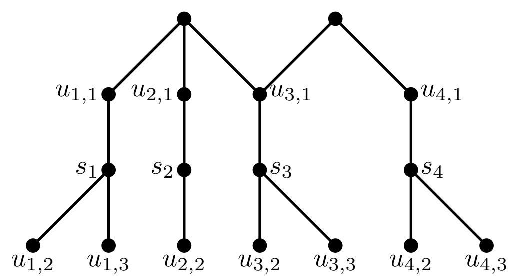


Figure 4.11: a Δ-tree with promised configuration

Lemma 4.3.3. *Let* $P := \{(n_1, \ldots, n_k) : 1 \leq n_i \leq m_i \text{ for all } i\}$. *Then there is a bijection between* $\mathcal{F}_T^M$ *and* P.

Proof. Consider the map ν which sends $F := V(T) \setminus (\{s_1, \ldots, s_k\} \cup \{u_{1,i_1}, \ldots, u_{k,i_k}\})$ to $(i_1, \ldots, i_k)$. We claim that ν is a bijection from $\mathcal{F}_T^M$ to P. Let $F_1, F_2 \in \mathcal{F}_T^M$ with $\nu(F_1) = \nu(F_2)$. Set $\nu(F_1) = (a_1, \ldots, a_k)$ and $\nu(F_2) = (b_1, \ldots, b_k)$. Then by definition of ν, we have

$$F_1 = V(T) \setminus (\{s_1, \ldots, s_k\} \cup \{u_{1,a_1}, \ldots, u_{k,a_k}\})$$

and

$$F_2 = V(T) \setminus (\{s_1, \ldots, s_k\} \cup \{u_{1,b_1}, \ldots, u_{k,b_k}\}).$$

Since $a_i = b_i$ for all i, we have $F_1 = F_2$, so ν is injective. Now, let $a \in P$. Then we can write $a = (a_1, \ldots, a_k)$. Consider $F = V(T) \setminus (\{s_1, \ldots, s_k\} \cup \{u_{1,a_1}, \ldots, u_{k,a_k}\})$. Then we have $\nu(F) = a$, hence ν is surjective. $\square$

Until the end of this section, ν will refer to the bijection between the facets and k vectors in the proof of Lemma 4.3.3.

Example 4.3.4. Consider the yellow vertices in Figure 4.12:

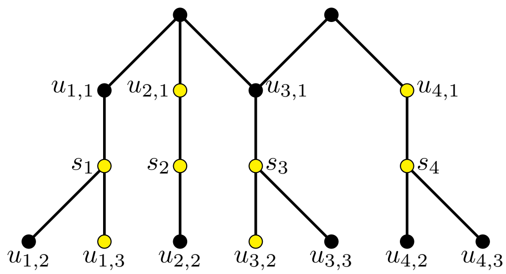


Figure 4.12: a Δ-tree with promised configuration

In this example, the set of yellow vertices, Y, form a minimal TD-set. The set of vectors is $P = \{(a_1, a_2, a_3, a_4) : 1 \leq a_1, a_3, a_4 \leq 3,\ 1 \leq a_2 \leq 2\}$. The corresponding vector for $V \setminus Y$ (the facet corresponding to Y) is $\nu(V \setminus Y) = (3, 1, 2, 1)$.

Lemma 4.3.5. *Let $F_1, F_2 \in \mathcal{F}_T^M$ with $\nu(F_1) = (a_1, \ldots, a_k)$ and $\nu(F_2) = (b_1, \ldots, b_k)$. Then*

$$|F_1 \cap F_2| = (\dim(\Delta_M^T) + 1) - |\{i : a_i \neq b_i\}|.$$

Proof. Set $A = \{i : a_i = b_i\}$ and $B = \{i : a_i \neq b_i\}$. Since we have $F_1 = V_3(T) \cup \left(\bigcup_{i=1}^k N(s_i) \setminus \{u_{i,a_i}\}\right)$ and $F_2 = V_3(T) \cup \left(\bigcup_{i=1}^k N(s_i) \setminus \{u_{i,b_i}\}\right)$, we have

$$\begin{aligned} F_1 \cap F_2 &= V_3(T) \cup \left(\bigcup_{i \in A} N(s_i) \setminus \{u_{i,a_i}\}\right) \cup \left(\bigcup_{i \in B} N(s_i) \setminus \{u_{i,a_i}, u_{i,b_i}\}\right) \\ &= V_3(T) \cup \left(\bigcup_{i=1}^{k} N(s_i) \setminus \{u_{i,a_i}\}\right) \setminus \bigcup_{i \in B} \{u_{i,b_i}\}. \end{aligned}$$

Thus we get

$$\begin{aligned} |F_1 \cap F_2| &= \left| V_3(T) \cup \left(\bigcup_{i=1}^{k} N(s_i) \setminus \{u_{i,a_i}\}\right) \setminus \bigcup_{i \in B} \{u_{i,b_i}\} \right| \\ &= \left| V_3(T) \cup \left(\bigcup_{i=1}^{k} N(s_i) \setminus \{u_{i,a_i}\}\right) \right| - \left| \bigcup_{i \in B} \{u_{i,b_i}\} \right| \end{aligned}$$

$$= |F_1| - |B|$$

$$= (\dim(\Delta_T^M) + 1) - |\{i : a_i \neq b_i\}|. \qquad \text{(since } F_1 \text{ is a facet)}$$

□

Example 4.3.6. For instance, consider the two colored sets in Figure 4.13:

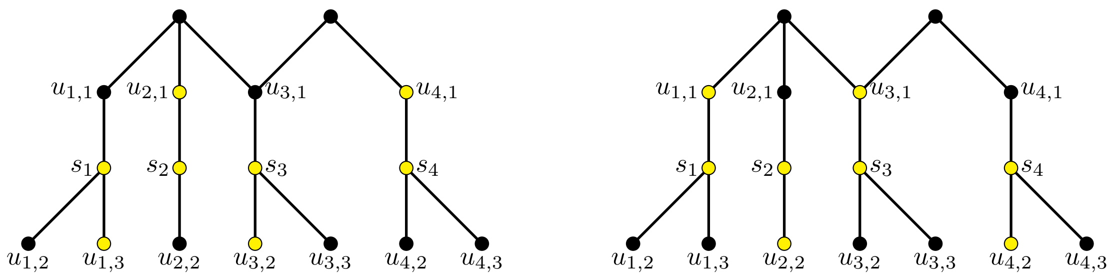


Figure 4.13: two minimal TD-sets D_1 (left) and D_2 (right)

Call this tree T. Denote the left set by D_1 and right set by D_2. Let $F_1 = V(T) \setminus D_1$ and $F_2 = V(T) \setminus D_2$. Then we have $|F_1 \cap F_2| = 5$. Now, T is unmixed by Theorem 4.2.36. Hence the ruled complex of T is pure of dimension $|F_1| - 1 = 8$. Also, we have $\nu(F_1) = (3, 1, 2, 1)$ and $\nu(F_2) = (1, 2, 1, 2)$. Hence by Lemma 4.3.5, we have $|F_1 \cap F_2| = (8+1) - 4 = 5$ where 4 is the number of digits where the two vectors $\nu(F_1)$ and $\nu(F_2)$ differ (which is all four digits in this example).

Lemma 4.3.7. *Let $F_1, F_2 \in \mathcal{F}_T^M$ with $\nu(F_1) = (a_1, \ldots, a_k)$ and $\nu(F_2) = (b_1, \ldots, b_k)$, and suppose that $a_i \neq b_i$ for some index i. Let F_3 be a facet such that $\nu(F_3) = (a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k)$. Then we have $F_3 \cap F_1 \supseteq F_2 \cap F_1$ and $|F_3 \cap F_1| = \dim(\Delta_T^M)$.*

Proof. By Lemma 4.3.5, we have $|F_3 \cap F_1| = \dim(\Delta_T^M)$. Now, we show that $F_3 \cap F_1 \supseteq F_2 \cap F_1$. Let $v \in F_2 \cap F_1$. If $v \in T_3$, then $v \in F_3 \cap F_1$ as every height 3 vertex of T is not in any minimal TD-set of T (by Fact 0.1 *iii*). So suppose that $\text{height}(v) \in \{0, 2\}$ (cannot be 1 as all support vertices are in any minimal TD-set). Then there is some index j such that $v \in N(s_j)$. Hence we can write $v = u_{j,m}$ for some integer $1 \leq m \leq m_j$. Since $u_{j,m}$ is in the intersection of F_1 and F_2, we must have $m \neq a_j$ and $m \neq b_j$ as u_{j,a_j} and u_{j,b_j} are in the complement of F_1 and F_2. Thus we have $v \in F_3$, and we get $v \in F_3 \cap F_1$. □

Given a vector set P induced from T, for all $n \geq 0$, define

$$P_n = \{(a_1, \ldots, a_k) \in P : |\{i : a_i \neq 1\}| = n\}$$

(i.e., P_n is the set of vectors in P which has $k - n$ ones). Now, let $<_{lex}$ be the lexicographic order on P; precisely, $(a_1, \ldots, a_k) <_{lex} (b_1, \ldots, b_k)$ iff $\exists i \in \{1, 2, \ldots, k\}$ s.t. $a_i < b_i$ and $a_j = b_j$ for all $j < i$. Define an order $<_s$ on P as follows:

For $p \in P_m$ and $q \in P_n$, we have $p <_s q$ iff

(i) $m < n$; or

(ii) $m = n$ and $p <_{lex} q$.

One can easily check that $<_s$ is a total order on P. By using the bijection from Lemma 4.3.3, we can define the order $<_s$ on $\mathcal{F}_T^M$ as well; for $F', F'' \in \mathcal{F}_T^M$, $F' <_s F''$ iff $\nu(F') <_s \nu(F'')$.

Example 4.3.8. Let P be the vector set of the graph in Figure 4.11. Then P arranged under $<_s$ with P_0 being the top row and P_4 being the bottom row is shown in Figure 4.14 (denote (a, b, c, d) by $abcd$, for instance $(1, 2, 3, 3)$ is denoted 1233):

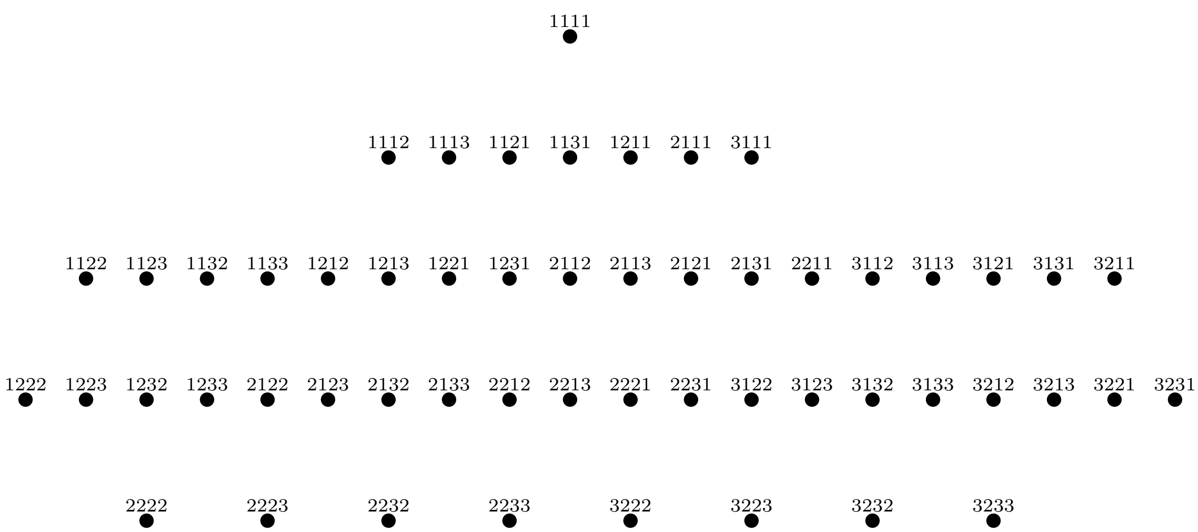


Figure 4.14: the vector set P

Hence vectors of the same row is ordered under lexicographical order (from left to right).

Theorem 4.3.9. *The order $<_s$ on $\mathcal{F}_T^M$ is a shelling. Thus Δ_T^M is shellable.*

Proof. First, we number the facets under $<_s$ and get $F_1, F_2, \ldots, F_{|\mathcal{F}_T^M|}$, and set $d := \dim(\Delta_T^M)$. Let i be an integer with $1 < i \leq |F_T^M|$. Suppose that there is $j < i$ such that $|F_j \cap F_i| < d$. Set $\nu(F_i) \in P_{l_1}$ and $\nu(F_j) \in P_{l_2}$ and write $\nu(F_i) = (a_1, \ldots, a_k)$ and $\nu(F_j) = (b_1, \ldots, b_k)$.

Case 1: $l_1 > l_2$. Then there are more ones in the vector of $\nu(F_j)$ than in $\nu(F_i)$, so, there is some index $m : 1 \leq m \leq k$ such that $a_m \neq b_m$ and $b_m = 1$. Consider some facet F such that $\nu(F) := (a_1, \ldots, a_{m-1}, 1, a_{m+1}, \ldots, a_k)$. By Lemma 4.3.7, we have $F \cap F_i \supseteq F_j \cap F_i$ and $|F \cap F_i| = d$.

Case 2: $l_1 = l_2$. Then we have $\nu(F_j) <_{lex} \nu(F_i)$. So, there is some index j such that $b_m < a_m$ and $a_r = b_r$ for all $r < m$. Set F be a facet such that $\nu(F) = (a_1, \ldots, a_{m-1}, b_m, a_{m+1}, \ldots, a_k)$; as $b_m < a_m$, we have $\nu(F) <_{lex} \nu(F_i)$, so $F <_s F_i$. Then $|F \cap F_i| = d - 1$ and $F \cap F_i \supseteq F_j \cap F_i$ by Lemma 4.3.7. $\square$

Given a vector $v \in P$, denote the i th coordinate of v by v_i. Let $p \in P_l$ and $S \subseteq P$. The *lower shadow of* p is the set $\triangle(p) := \{q \in P_{l+1} : \exists! i \text{ s.t. } p_i \neq q_i\}$ and $\triangle(S) = \bigcup_{p \in S} \triangle(p)$. For $n \geq 1$, define $\triangle_0(S) = S$ and $\triangle_n(S) = \triangle(\triangle_{n-1}(S))$. The *downset of* S is the set

$$D(S) := \bigcup_{n=0}^{\infty} \triangle_n(S).$$

Similarly, the *upper shadow of* p is the set $\bigtriangledown(p) := \{q \in P_{l-1} : \exists! i \text{ s.t. } p_i \neq q_i\}$ and $\bigtriangledown(S) = \bigcup_{p \in S} \bigtriangledown(p)$.

Example 4.3.10. Consider the vector 2113 in the vector set P from Figure 4.14. We connect two vectors in a different level if they differ by a single digits (see Figure 4.15).

The lower shadow of 2113 is $\bigtriangledown(2113) = \{2123, 2133, 2213\}$. The vertices 3122 and 3132 are not in the lower shadow of v since 3122 and 2113 differ by 3 digits, and 3132 and 2113 differ by 2 digits. The upper shadow of 2113 is the set $\triangle(2113) = \{1113, 2111\}$. The vertices 1112 and 3111 are not in $\triangle(v)$ since they both differ by 2 digit compared to 2113. After connecting all the vectors in P using the above method, the down set of 2113 is the blue vertices in Figure 4.16 (where the red vertex is 2113).

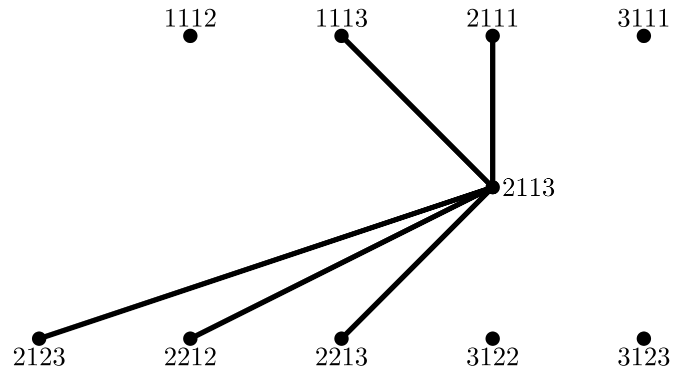


Figure 4.15: lower/upper shadow of 2113 in the vector set P

Lemma 4.3.11. *Let $p \in P$ such that $V(T) \setminus \nu^{-1}(p)$ is not a TD-set in T. Then for all $q \in \triangle(p)$, the set $V(T) \setminus \nu^{-1}(q)$ is not a TD-set in T.*

Proof. Let $q \in \triangle(p)$, and set $p = (p_1, \dots, p_k)$ and $q = (q_1, \dots, q_k)$. Set $D_p = V(T) \setminus \nu^{-1}(p)$ and $D_q = V(T) \setminus \nu^{-1}(q)$. Since $q \in \triangle(p)$, there is an index m such that $p_i = q_i$ for all $i \neq m$ and $p_m = 1 \neq q_m$. Since we have $D_p = \{s_1, \dots, s_k\} \cup \{u_{1,p_1}, \dots, u_{k,p_k}\}$ and $D_q = \{s_1, \dots, s_k\} \cup \{u_{1,q_1}, \dots, u_{k,q_k}\}$, and as we assumed that $u_{i,1}$ is of height 2, we have

$$D_q = (D_p \setminus \{u_{m,1}\}) \cup \{u_{m,q_m}\}$$

(take a non-leaf out of D_p, and add a leaf to it to obtain D_q). Thus we have $N(D_q) \subseteq N(D_p)$ as $\{s_1, \dots, s_k\} \subseteq D_p$. Since $N(D_p) \neq V(T)$ as D_p is not a TD-set, we get $N(D_q) \neq V(T)$. Therefore, D_q is not a TD-set. □

Lemma 4.3.11 really says that if you are given a minimal set which is not a TD-set, then a set which has the same size but less height 2 vertices is obviously not a TD-set.

Lemma 4.3.12. *Let $F \in \mathcal{F}_T^M$ with $\nu(F) = (a_1, \dots, a_k)$ where $V(T) \setminus F$ is a minimal TD-set in T. Let $F' \in \mathcal{F}_T^M$ with $\nu(F') = (a'_1, \dots, a'_k)$ such that $a'_i = a_i$ if $a_i = 1$. Then $V(T) \setminus F'$ is also a minimal TD-set in T.*

Proof. By the construction of the set $\mathcal{F}_T^M$, for any $A \in \mathcal{F}_T^M$, we have $N(V(T) \setminus A) \supseteq V(T) \setminus V_3(T)$. Hence all we have to show is that $V_3(T) \subseteq V(T) \setminus F'$. Since $V(T) \setminus F$ is a minimal TD-set, we have $V_3(T) \subseteq N(V(T) \setminus F)$. Also, the only vertices in $V(T) \setminus F$ that are neighboring to height 3 vertices

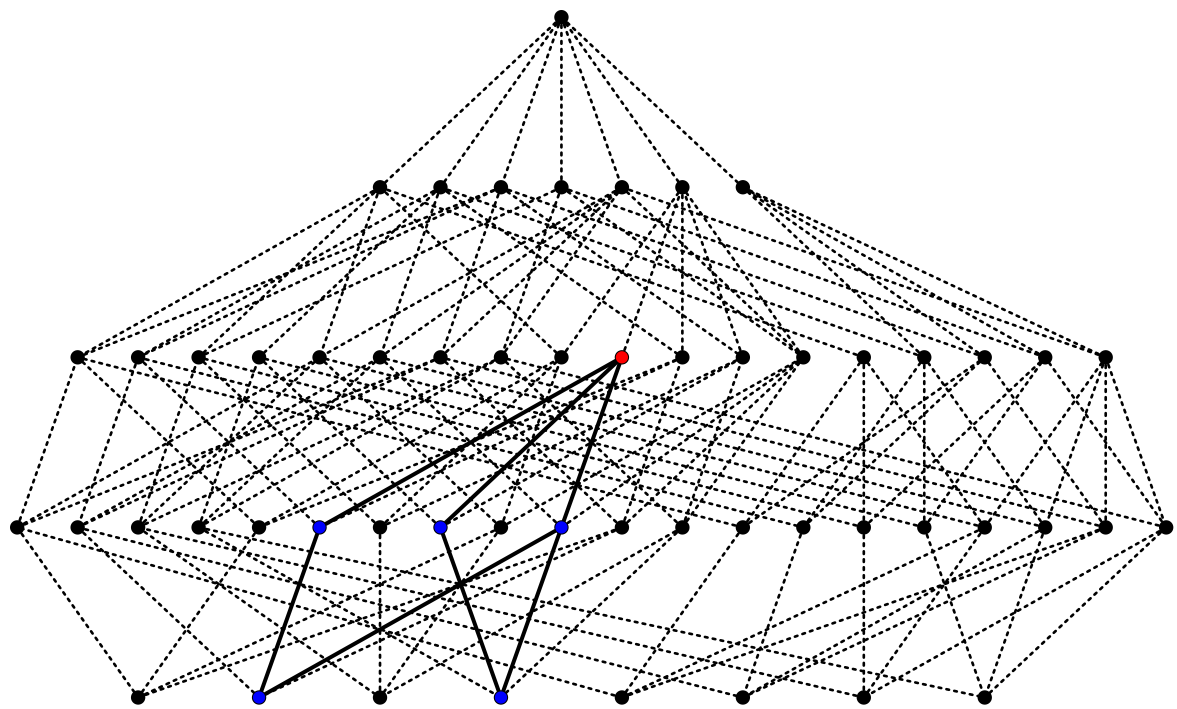


Figure 4.16: $D(\{2113\})$ in the vector set P

are height 2 vertices. Since height 2 vertices of $V(T) \setminus F'$ contains the height 2 vertices of $V(T) \setminus F$, we must have $V_3(T) \subseteq V(T) \setminus F'$. Thus $V(T) \setminus F'$ is a minimal TD-set (minimal by Fact 3.1 i). $\square$

Lemma 4.3.12 says that given a minimal TD-set D, let l be a leaf in D. Let l' be a leaf such that distance$(l, l') = 2$. Then the set $D \setminus \{l\} \cup \{l'\}$ is also a minimal TD-set; i.e., changing a leaf in D with another leaf which has the same adjacent support vertex does not affect the TD-ness.

Theorem 4.3.13 (**Ruled complex of Δ-trees are shellable**). *Let $S \subseteq P$ be the set of vectors such that for all $p \in S$, the set $V(T) \setminus \nu^{-1}(p)$ is not a TD-set in T. Let Δ be the simplicial complex generated by $\nu^{-1}(P \setminus D(S))$. Then Δ is shellable under $<_s$. Moreover, we have $\Delta = \Delta_T$ hence the ruled complex of an unmixed Δ-tree is shellable.*

Proof. Note that we have $\mathcal{F}_T = \nu^{-1}(P \setminus D(S))$ by definition of S with Lemma 4.3.11. First, we number the facets under $<_s$ and get $F_1, F_2, \ldots, F_{|\mathcal{F}_T|}$, and set $d := \dim(\Delta_T)$. Let i be an integer with $1 < i \leq |F_T|$. Suppose that there is $j < i$ such that $|F_j \cap F_i| < d$. Set $\nu(F_i) \in P_{l_1}$ and $\nu(F_j) \in P_{l_2}$ and write $\nu(F_i) = (a_1, \ldots, a_k)$ and $\nu(F_j) = (b_1, \ldots, b_k)$.

Case 1: $l_2 < l_1$. Then $\nu(F_j)$ contains more ones than $\nu(F_i)$. So, there is some index $r : 1 \leq r \leq k$ such that $a_r \neq b_r$ and $b_r = 1$. Consider the facet $F \in \mathcal{F}_T^M$ such that $\nu(F) = (a_1, \dots, a_{r-1}, 1, a_{r+1}, \dots, a_k)$. By Lemma 4.3.7, we have $F \cap F_i \supseteq F_j \cap F_i$ and $|F \cap F_i| = d$. Also, as $V(T) \setminus F_i$ is a minimal TD-set, so is the set $V(T) \setminus F$ as $F \in \bigtriangledown(F_i)$ (by Lemma 4.3.11). Thus we have $F \in \mathcal{F}_T$.

Case 2: $l_2 = l_1$. Since $\nu(F_j) <_s \nu(F_i)$, we have $\nu(F_j) <_{lex} \nu(F_i)$. So, there is some index $r : 1 \leq r \leq k$ such that $b_m = a_m$ for all $m < r$ and $b_r < a_r$. Set $\nu(F) = (a_1, \dots, a_{r-1}, b_r, a_{r+1}, \dots, a_k)$. Then by Lemma 4.3.7, we have $F \cap F_i \supseteq F_j \cap F_i$ and $|F \cap F_i| = d$, and as $b_r < a_r$ we have $F <_{lex} F_i$, hence $F <_s F_i$. All we have to show is that $F \in \mathcal{F}_T$. If $b_r = 1$, then $F \in \bigtriangledown(F_i)$, hence $F \in \mathcal{F}_T$ by Lemma 4.3.11. If $b_r \neq 1$, then by Lemma 4.3.12, the set $V(T) \setminus F$ is a minimal TD-set in T, hence $\nu(F) \notin S$. Thus $F \in \mathcal{F}_T$. □

Let Δ_1 and Δ_2 be simplicial complexes with disjoint universal sets. The *join* of Δ_1 and Δ_2 is the simplicial complex

$$\Delta_1 * \Delta_2 := \{F_1 \cup F_2 : F_1 \in \Delta_1, F_2 \in \Delta_2\}$$

define in the union of the two universal sets of Δ_1 and Δ_2. It is well known that Δ is shellable if and only if both Δ_1 and Δ_2 are shellable (see [?joinShellable]).

Let $T = (V, E)$ be a Δ-tree. Define two sets $V_{even} := \{v \in V : \text{height}(v) \text{ is even}\}$ and $V_{odd} := \{v \in V : \text{height}(v) \text{ is odd}\}$. Then the ruled complex of T is the join of two simplicial complexes:

$$\Delta_T = \Delta_T \sqcap V_{odd} * \Delta_T \sqcap V_{even}$$

where we have $\Delta_T \sqcap S = \{F \cap S : F \in \Delta_T\}$ for some $S \subseteq V$; and above equality follows directly from Lemma 4.2.15, where we set $V_B = V_{even}$ and $V_R = V_{odd}$. By Theorem 4.3.13, we know that $\Delta_T \sqcap V_{odd}$ and $\Delta_T \sqcap V_{even}$ are shellable as well. Now we are ready to prove our main theorem.

Theorem 4.3.14. *Let T be an unmixed tree. Then the ruled complex of T is shellable.*

Proof. Suppose that we two color the vertices of T, and denote $V_B(T)$ and $V_R(T)$ be the blue and red colored vertices, respectively. Set $\Delta_{even,B} := \Delta_T \sqcap V_{even}(T_B)$ and $\Delta_{even,R} := \Delta_T \sqcap V_{even}(T_R)$.

Note that $\Delta_{even,B}$ and $\Delta_{even,R}$ are shellable. By construction of T_B and T_R, we have

$$V_B(T_B) = V_B(T) \setminus V_1(T) \tag{E1}$$

$$V_R(T_R) = V_R(T) \setminus V_1(T) \tag{E2}$$

Let F be a facet of Δ_T. Then $D := V(G) \setminus F$ is a minimal TD-set in G. Hence there are $D' \subseteq V_B(G)$, $D'' \subseteq V_R(G)$ such that $D = D' \cup D''$ where D' is a minimal BD-set and D'' is a minimal RD-set (by Lemma 4.2.15). Now, we get

$$\begin{aligned} F &= V \setminus (D' \cup D'') \\ &= (V_B(T) \setminus D') \cup (V_R(T) \setminus D'') \\ &= (V_B(T_B) \setminus D') \cup (V_R(T_R) \setminus D'') && \text{(since } V_1(G) \subseteq D \text{ with E1 and E2)} \\ &= (V_{even}(T_B) \setminus D') \cup (V_{even}(T_R) \setminus D''). \end{aligned}$$

Since $V_{even}(T_B) \setminus D' \in \Delta_{even,B}$ and $V_{even}(T_R) \setminus D'' \in \Delta_{even,R}$, hence F is a facet of $\Delta_{even,B} * \Delta_{even,R}$.

Now, suppose that F' and F'' are facets in $\Delta_{even,B}$ and $\Delta_{even,R}$, respectively. Then there are $D' \subseteq V_{even,B}$ and $D'' \subseteq V_{even,R}$ such that $F' = V_{even,B} \setminus D'$ and $F' = V_{even,R} \setminus D''$. Note that D' and D'' are minimal BD-set and RD-set in T_B and T_R. Then we get

$$\begin{aligned} F' \cup F'' &= (V_{even,B} \setminus D') \cup (V_{even,R} \setminus D'') \\ &= (V_{even,B} \cup V_{even,R}) \setminus (D' \cup D'') && \text{(since } V_{even,B} \cap V_{even,R} = \emptyset) \\ &= (V_{even,B} \cup V_{even,R} \cup V_1(T)) \setminus (D' \cup D'' \cup V_1(T)) \\ &= (V_B(T_B) \cup V_R(T_R)) \setminus (D' \cup D'' \cup V_1(T)) && \text{(by E1 and E2)} \\ &= V(T) \setminus (D' \cup D'' \cup V_1(T)). \end{aligned}$$

Since $D' \cup D'' \cup V_1(T)$ is a minimal TD-set in T, we get $F' \cup F'' \in \Delta_T$. So, $F' \cup F''$ is a facet in Δ_T. Therefore, we conclude that

$$\Delta_T = \Delta_{even,B} \ * \ \Delta_{even,R}$$

and thus Δ_T is shellable. □

Example 4.3.15. Consider the graph Y from Figure 4.3. Choose a minimal TD-set in Y, which is the set of yellow vertices (see Figure 4.17). The complement of the yellow vertices form a facet of Δ_Y, the ruled complex of Y.

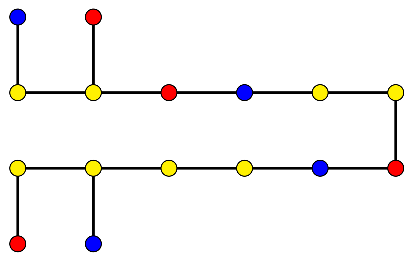


Figure 4.17: a minimal TD-set in Y

Now, partition the facet with respect to the coloring, and embed each of the pieces to Y_B and Y_R which are the big red and big blue vertices (Figure 4.18). Let Δ_B and Δ_R be the ruled complex of Y_B and Y_R, respectively. Set $\Delta_{even,B} := \Delta_B \sqcap V(Y)_{even}$ and $\Delta_{even,R} := \Delta_R \sqcap V(Y)_{even}$.

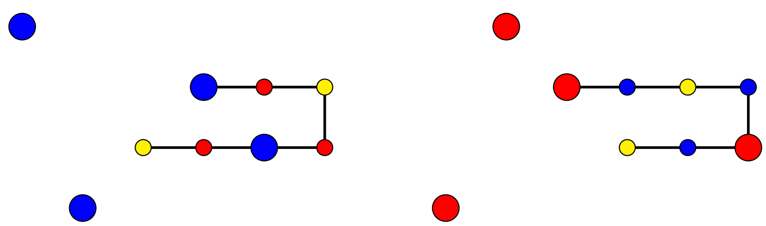


Figure 4.18: facets in the interior graphs Y_B (left) and Y_R (right)

The complement of the big blue and big red vertices restricted to the even height is the yellow vertices in each Y_B and Y_R. Since the yellow vertices in Y_B form a minimal BD-set and the yellow vertices in Y_R form a minimal RD-set, the big blue vertices in Y_B and the big red vertices in Y_R are facets of $\Delta_{even,B}$ and $\Delta_{even,R}$. One can find that the inverse implication (which is, begin with facets in $\Delta_{even,B}$ and $\Delta_{even,R}$ and construct a facet in Δ_Y) works as well; more precisely, embed the two facets in the interior graphs to the original graph, and take complement in the original graph which will give you a minimal TD-set of the whole graph.

Chapter 5

Future Work

I will conclude the dissertation by exploring the connection between our characterization of unmixed trees with respect to PMU covers and Villarreal's characterization of unmixed trees with respect to vertex covers. Recall the following two results which were stated in Chapter 1:

Theorem 5.0.1 (Theorems 1.1.1 and 1.2.1)**.** *Let T be a tree. The following hold:*

1. *T is unmixed with respect to vertex covers if and only if every vertex of degree ≥ 1 in T is adjacent to exactly 1 vertex of degree 1.*

2. *T is unmixed with respect to PMU covers if and only if every vertex of degree ≥ 2 in T is adjacent to exactly 2 vertices of degree ≤ 2.*

The similarity between these results is simultaneously clear and confounding. If all three 1's in Villarreal's result are replaced with 2's, we obtain our result for PMU covers. Is this just a coincidence?

We can think of PMU covers as vertex covers with Ohm's Law and Kirchhoff's Law. My working conjecture is that Ohm's Law reduces the algebraic complexity of the unmixed trees, making their power edge ideals complete intersections. Whereas Kirchoff's Law increases the combinatorial complexity of the unmixed trees, making all of the numbers in the characterization increase by 1. This raises the following questions:

Question 5.0.2. *1. What are the unmixed trees when we consider Kirchhoff's Law and omit Ohm's Law?*

2. *By modifying Kirchhoff's Law, can we create a graph domination problem such that the unmixed trees are the trees such that every vertex of degree* ≥ 2 *in* T *is adjacent to exactly* 2 *vertices of degree* ≤ 2.

3. *In either of the above settings, do the unmixed trees correspond to Cohen-Macaulay ideals?*

I hope to explore these problems in the near future.

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