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Minimal Differential Graded Resolutions of Fiber Products

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Minimal Differential Graded Resolutions of Fiber Products

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Abstract

This dissertation is the culmination of my work in [16–18]. These papers add to a body of work focused on rings known as fiber products. A ring F is said to be a fiber product if there exist ring homomorphisms $S \xrightarrow{\pi_S} W \xleftarrow{\pi_T} T$ and we have $F \cong S \times_W T := \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}$. In this set-up, we say that F is the fiber product of S and T over W.

For our purpose we only consider rings that are commutative, noetherian rings with identity. We further consider the case where R is a regular local (or standard graded polynomial) ring and study fiber products that can be realized as homomorphic images of R. To this end, we study quotients $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle$ where $\mathcal{I}'$, $\mathcal{I}$, $\mathcal{J}'$, and $\mathcal{J}$ are ideals in R. In Chapter 3 we impose conditions as these ideals that allow us to realize these quotients as fiber products (see Proposition 3.2.1). We then explicitly construct a minimal resolution of the quotient over R. From this, we recover formulas for the Betti numbers of the quotient as well as the Poincaré series.

We further establish sufficient conditions on the four ideals above to impose a differential graded structure on our minimal resolution. In Chapter 4, we address the case $\mathcal{I}' = 0 = \mathcal{J}'$ and obtain a minimal differential graded algebra resolution. We then use the techniques of [2, 27] to establish further homological properties of the fiber product. In particular, we show that these fiber products are Golod and are Tor-friendly.

In Chapter 5, we allow for $\mathcal{I}' \neq 0$ but maintain $\mathcal{J}' = 0$. We obtain minimal differential graded modules and then establish sufficient conditions on $\mathcal{I}'$ to extend the module structure to that of an algeba. We again apply techniques from [2, 27] to obtain the Golod and Tor-friendly properties for this larger class of fiber products.

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Chapter 1

Introduction

Throughout this dissertation all rings are commutative and noetherian with identity. In particular, we let R be a regular, local (or standard graded polynomial) ring. Many of the results were initially realized for the power series and polynomial rings over any field k. The main focus of this dissertation is the study of rings known as fiber products which we realize as quotients of R.

There has been extensive study of fiber products over the past several decades. Much of the interest arose due to the use of fiber products in the work of Grothendieck (see [20]) as well as Avramov, Foxby, and Herzog (see [3]). In [20], Grothendieck considers complete intersection (CI) rings which can be presented as the quotient of a regular local ring by a regular sequence. He uses the fiber product of two presentations of a CI ring to show that the definition of a CI ring is independent of the presentation considered. This technique then inspired the technique used in [3, Theorem 1.2] to show that any two Cohen factorizations of a local homomorphism have a common deformation.

Due to the utility of fiber products, the interest in their homological properties has grown (see, e.g., [9, 10, 15, 31, 34–37, 39–41]). For this dissertation, we are particularly interested in [36, Theorem 1.1] which says that fiber products are "Tor-friendly" (see Definition 2.2.8). To prove this theorem, Nasseh and Sather-Wagstaff use the explicit construction of minimal free resolutions of modules over the fiber product exhibited by Moore (see [35]). We are interested in this result since the Tor-friendly property can be established for other rings using differential graded (DG) algebra methods [4, 5].

Chapters 3–5 of this dissertation, consisting of my work in [16–18], aim to establish the necessary criteria to apply the work of [4, 5] and re-establish [36, Theorem 1.1] using DG-methods.

To that end, we establish partial results but also add to the study overall study of fiber products. We do this through explicit constructions of related commutative and homological objects. The basis of this is the construction of minimal free resolutions of fiber products in Chapter 3.

Free resolutions of R-modules encode information about the module which we can extract using a variety of techniques. This includes results the projective dimension, Betti numbers, and Poincaré series of the module amongst other invariants of the module. This information is more readily accessible when the free resolution is minimal (see Chapter 2 for definitions and background results).

In Chapter 3, our focus is the work from [16]. We give an explicit construction for a free resolution of the fiber product over a regular local (or standard graded polynomial) ring (see Constructions 3.3.1 and 3.4.5 along with Theorem 3.4.8). Moreover, Theorem 3.5.3 gives sufficient conditions for this resolution to be minimal. From there we recover homological data on fiber products in the form of Betti numbers and Poincaré series in Corollaries 3.5.4 and 3.5.5, respectively.

In Chapter 4, our focus is the work from [17]. We introduce sufficient conditions for imposing a multiplicative structure on the star product originally introduced in Chapter 3 (see Construction 3.3.1), allowing us to realize the construction as a DG algebra. We conclude the chapter by using this construction to show a certain family of fiber products are Tor-friendly.

In Chapter 5, our focus is the work from [18]. We combine the work of the previous two chapters to establish further differential graded structures on the minimal resolutions of certain fiber products. The general result establishes a DG-module structure but we give further conditions to obtain a DG-algebra. As a consequence, we show another family of fiber products are Tor-friendly.

In Chapter 6, we outline future directions of this work. We identify questions that would be useful in determining if and how to extend the results in this dissertation. For each of these questions, we give brief descriptions of how we plan to initially address these questions.

Chapter 2

Background

2.1 Complexes

In this section we introduce the necessary tools from homological and commutative algebra for understanding the material in the later chapters. In this and later sections, $(R, \mathfrak{M}_R)$ is a commutative, local (or standard graded) ring with maximal ideal $\mathfrak{M}_R$. In the polynomial case, this is the maximal ideal generated by the variables of R. Our primary means for working with R-modules and R-algebras comes from the following definitions.

Definition 2.1.1. A **chain complex** over R (or R**-complex** for short) is a family of R-modules $\mathcal{X} = \{\mathcal{X}_i\}_{i\in\mathbb{Z}}$ with a family of R-modules homomorphisms $\partial^{\mathcal{X}} = \{\partial_i^{\mathcal{X}} : \mathcal{X}_i \to \mathcal{X}_{i-1}\}_{i\in\mathbb{Z}}$ such that $\partial_{i+1}^{\mathcal{X}} \circ \partial_i^{\mathcal{X}} = 0$ for all $i \in \mathbb{Z}$. We call $\partial^{\mathcal{X}}$ the **differential** of $\mathcal{X}$. It is common (and useful) to express R-complexes with the following diagram.

$$\mathcal{X} = \qquad \cdots \longrightarrow \mathcal{X}_{i+1} \xrightarrow{\partial_{i+1}^{\mathcal{X}}} \mathcal{X}_i \xrightarrow{\partial_i^{\mathcal{X}}} \mathcal{X}_{i-1} \longrightarrow \cdots$$

Moreover, R-complexes have a natural, underlying graded module $\mathcal{X}^\natural = \bigoplus_{i\in\mathbb{Z}} \mathcal{X}_i$. Given an element $m \in \mathcal{X}_i$, we say the **homological degree of** m (**degree** for short) is $|m| = i$.

The **homology of** $\mathcal{X}$ is the chain complex $\mathrm{H}(\mathcal{X})$ where $\mathrm{H}_i(\mathcal{X}) = \ker \partial_i^{\mathcal{X}} / \operatorname{Im} \partial_{i+1}^{\mathcal{X}}$ and the differential is the zero map $\partial^{\mathrm{H}(\mathcal{X})} = 0$. For $m \in \ker \partial_i^{\mathcal{X}}$, we write $\overline{m} := m + \operatorname{Im} \partial_{i+1}^{\mathcal{X}} \in \mathrm{H}_i(\mathcal{X})$.

Fact 2.1.2. *If $\partial^{\mathcal{X}}$ is the zero map, then the homology of $\mathcal{X}$ is $\mathcal{X}$ itself, i.e., $\mathrm{H}(\mathcal{X}) = \mathcal{X}$.*

Example 2.1.3. Any R-module M can be viewed as an R-complex. Traditionally this is done by setting the module in degree 0 to be M and all other modules to be 0

$$M = \quad \cdots \to 0 \to M \to 0.$$

The differential for this complex is the zero map which yields $\mathrm{H}(M) = M$. When viewing an R-module as an R-complex in this way, we say it a complex **concentrated in degree zero**.

Definition 2.1.4. Let $\mathcal{X}$ be an R-complex $\mathcal{X}$.

1. $\mathcal{X}$ is **graded free** if for all $i \in \mathbb{Z}$, the module $\mathcal{X}_i$ is free.

2. $\mathcal{X}$ is **graded projective** if for all $i \in \mathbb{Z}$, the module $\mathcal{X}_i$ is projective.

3. $\mathcal{X}$ is **bounded below** if there exists some $n \in \mathbb{Z}$ such that $\mathcal{X}_i = 0$ for all $i < n$.

4. $\mathcal{X}$ is **bounded above** if there exists some $m \in \mathbb{Z}$ such that $\mathcal{X}_i = 0$ for all $i > m$.

5. $\mathcal{X}$ is said to be **bounded** if it is bounded both above and below.

6. $\mathcal{X}$ is **positively graded** if $\mathcal{X}_i = 0$ for all $i < 0$.

7. $\mathcal{X}$ is **exact** if $\mathrm{H}_i(\mathcal{X}) = 0$ for all $i \in \mathbb{Z}$.

8. When $\mathcal{X}$ is positively graded, we say that $\mathcal{X}$ is **acyclic** if $\mathrm{H}_i(\mathcal{X}) = 0$ for all $i \neq 0$.

In all the future sections, we work with positively graded, graded free R-complexes. Due to the context in which we use them, these complexes will often be bounded above but this is not necessary for many of the results in the coming sections.

Definition 2.1.5. The **ℓth suspension (or shift)** of the R-complex $\mathcal{X}$ is the complex $\Sigma^\ell \mathcal{X}$ where $\left(\Sigma^\ell \mathcal{X}\right)_n := \mathcal{X}_{n-\ell}$ and $\partial_n^{\Sigma^\ell \mathcal{X}} := (-1)^\ell \partial_{n-\ell}^{\mathcal{X}}$ for all $n \in \mathbb{Z}$. In the case of $\ell = 1$, we simply write $\Sigma\mathcal{X}$ and $\partial^{\Sigma\mathcal{X}}$.

Definition 2.1.6. The **tensor product complex** of two R-complexes $\mathcal{X}$ and $\mathcal{Y}$, denoted $\mathcal{X} \otimes_R \mathcal{Y}$, is defined as

$$(\mathcal{X} \otimes_R \mathcal{Y})_\ell := \bigoplus_{p \in \mathbb{Z}} \mathcal{X}_p \otimes_R \mathcal{Y}_{\ell-p}$$

and, for $\alpha \in \mathcal{X}_p$ and $\beta \in \mathcal{Y}_{\ell-p}$, the differential satisfies $\partial_\ell^{\mathcal{X}\otimes\mathcal{Y}}(\alpha\otimes\beta) := \partial_p^{\mathcal{X}}(\alpha)\otimes\beta+(-1)^p\alpha\otimes\partial_{\ell-p}^{\mathcal{Y}}(\beta)$.

Definition 2.1.7. The **Hom complex** of two R-complexes $\mathcal{X}$ and $\mathcal{Y}$, denoted $\operatorname{Hom}_R(\mathcal{X}, \mathcal{Y})$, is defined as

$$\operatorname{Hom}_R(\mathcal{X}, \mathcal{Y})_\ell := \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(\mathcal{X}_p, \mathcal{Y}_{p+\ell})$$

where the differential satisfies $\partial_\ell^{\operatorname{Hom}(\mathcal{X},\mathcal{Y})}(\{\phi_p\}) := \{\partial_{p+\ell}^{\mathcal{Y}} \circ \phi_p - (-1)^\ell \phi_{p-1} \circ \partial_p^{\mathcal{X}}\}$.

Writing ϕ in place of $\{\phi_p\}_{p \in \mathbb{Z}}$, we say it is a **chain map** if $\phi \in \ker \partial_0^{\operatorname{Hom}(\mathcal{X},\mathcal{Y})}$. A chain map $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be an **isomorphism** if it has a two-sided inverse and we write $\mathcal{X} \cong \mathcal{Y}$.

By definition of a chain map $\phi \in \ker \partial_0^{\operatorname{Hom}(\mathcal{X},\mathcal{Y})}$, we must have $0 = \partial_p^{\mathcal{Y}} \circ \phi_p - \phi_{p-1} \circ \partial_p^{\mathcal{X}}$ for all $p \in \mathbb{Z}$. In computations, we will often rewrite this as $\partial_p^{\mathcal{Y}} \circ \phi_p = \phi_{p-1} \circ \partial_p^{\mathcal{X}}$. When speaking more generally, we drop the subscripts and simply write $\partial^{\mathcal{Y}} \circ \phi = \phi \circ \partial^{\mathcal{X}}$. For the more visually inclined, we say $\phi : \mathcal{X} \to \mathcal{Y}$ is a chain map if it makes the following diagram commute.

$$\begin{array}{cccccccccc}
\mathcal{X} = & \cdots & \longrightarrow & \mathcal{X}_{p+1} & \xrightarrow{\partial_{p+1}^{\mathcal{X}}} & \mathcal{X}_p & \xrightarrow{\partial_p^{\mathcal{X}}} & \mathcal{X}_{p-1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \phi} & & & \downarrow{\scriptstyle \phi_{p+1}} & & \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \phi_{p-1}} & & \\
\mathcal{Y} = & \cdots & \longrightarrow & \mathcal{Y}_{p+1} & \xrightarrow[\partial_{p+1}^{\mathcal{Y}}]{} & \mathcal{Y}_p & \xrightarrow[\partial_p^{\mathcal{Y}}]{} & \mathcal{Y}_{p-1} & \longrightarrow & \cdots
\end{array}$$

Fact 2.1.8. *Given a chain map $\phi : \mathcal{X} \to \mathcal{Y}$ there is an induced map*

$$\operatorname{H}(\phi) = \{\operatorname{H}_p(\phi) : \operatorname{H}_p(\mathcal{X}) \to \operatorname{H}_p(\mathcal{Y})\}$$

where $\operatorname{H}_p(\phi)$ is the well-defined R-module homomorphism given by $\operatorname{H}_p(\phi)(\overline{\alpha}) := \overline{\phi_p(\alpha)}$ for all $p \in \mathbb{Z}$.

Definition 2.1.9. If $\epsilon : \mathcal{X} \to \mathcal{Y}$ and $\tau : \mathcal{Y} \to \mathcal{S}$ are chain maps such that $\ker \tau = \operatorname{Im} \epsilon$ then we say the following sequence of R-complexes is **short exact**

$$0 \longrightarrow \mathcal{X} \xrightarrow{\epsilon} \mathcal{Y} \xrightarrow{\tau} \mathcal{S} \longrightarrow 0$$

and yields the commutative diagram below with exact rows.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}_i & \xrightarrow{\epsilon_i} & \mathcal{Y}_i & \xrightarrow{\tau_i} & \mathcal{S}_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i^{\mathcal{X}}} & & \downarrow{\scriptstyle \partial_i^{\mathcal{Y}}} & & \downarrow{\scriptstyle \partial_i^{\mathcal{S}}} & & \\
0 & \longrightarrow & \mathcal{X}_{i-1} & \xrightarrow[\epsilon_{i-1}]{} & \mathcal{Y}_{i-1} & \xrightarrow[\tau_{i-1}]{} & \mathcal{S}_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Fact 2.1.10. *Given the short exact sequence in the previous definition, there exists a connecting homomorphism* $\eth_i : \mathrm{H}_i(\mathcal{S}) \to \mathrm{H}_{i-1}(\mathcal{X})$ *for all* i. *This yields a corresponding* **long exact sequence** *in homology, written below.*

$$\cdots \longrightarrow \mathrm{H}_i(\mathcal{X}) \xrightarrow{\mathrm{H}_i(\epsilon)} \mathrm{H}_i(\mathcal{Y}) \xrightarrow{\mathrm{H}_i(\tau)} \mathrm{H}_i(\mathcal{S}) \xrightarrow{\eth_i} \mathrm{H}_{i-1}(\mathcal{X}) \xrightarrow{\mathrm{H}_{i-1}(\epsilon)} \mathrm{H}_{i-1}(\mathcal{Y}) \longrightarrow \cdots$$

In the proof of this fact, one first constructs $\eth_i$ and then proves it is a well-defined R-module homomorphism for all i with the desired exactness property. We only apply this fact in the context of mapping cones (which we defined later in this section) and in that case we are able to state explicitly the map $\eth_i$.

Definition 2.1.11. Given a chain map $\phi : \mathcal{X} \to \mathcal{Y}$, if the induced map on homology is an isomorphism then we say ϕ is a **quasiisomorphism** and write $\mathcal{X} \xrightarrow{\simeq} \mathcal{Y}$.

Fact 2.1.12. *Every isomorphism is a quasiisomorphism but the converse does not hold in general.*

Definition 2.1.13. Given a chain map $\phi : \mathcal{X} \to \mathcal{Y}$ the **mapping cone** of ϕ is the R-complex $\mathrm{Cone}(\phi)$ where $\mathrm{Cone}(\phi)_\ell = \mathcal{Y}_\ell \oplus \mathcal{X}_{\ell-1}$ which we write as $\begin{smallmatrix}\mathcal{Y}_\ell \\ \oplus \\ \mathcal{X}_{\ell-1}\end{smallmatrix}$ to align with the following differential.

$$\partial_\ell^{\mathrm{Cone}(\phi)} = \begin{pmatrix} \partial_\ell^{\mathcal{Y}} & \phi_{\ell-1} \\ 0 & -\partial_{\ell-1}^{\mathcal{X}} \end{pmatrix}$$

Fact 2.1.14. *Suppose* $\epsilon : \mathcal{Y} \to \mathrm{Cone}(\phi)$ *and* $\tau : \mathrm{Cone}(\phi) \to \Sigma\mathcal{X}$ *are, respectively, the natural* *embedding and natural surjection that makes the following a short exact sequence for all ℓ.*

$$0 \longrightarrow \mathcal{Y}_\ell \xrightarrow{\binom{1}{0}} \begin{matrix}\mathcal{Y}_\ell \\ \oplus \\ \mathcal{X}_{\ell-1}\end{matrix} \xrightarrow{(0\ 1)} \mathcal{X}_{\ell-1} \longrightarrow 0$$

Then ϵ and τ are chain maps that make the following diagram a short exact sequence of complexes.

$$0 \longrightarrow \mathcal{Y} \xrightarrow{\epsilon} \operatorname{Cone}(\phi) \xrightarrow{\tau} \Sigma\mathcal{X} \longrightarrow 0$$

Moreover, the connecting map for the long exact sequence in homology is given by $\eth_\ell = \mathrm{H}_{\ell-1}(\phi)$.

The first reason for considering mapping cones is their ability to detect quasiisomorphisms as we see in the following fact. Other applications are given in subsequent chapters.

Fact 2.1.15. *The chain map $\phi : \mathcal{X} \to \mathcal{Y}$ is a quasiisomorphism if and only if* $\operatorname{Cone}(\phi)$ *is exact.*

2.2 Free Resolutions

Definition 2.2.1. Given an R-module M and a positively graded, graded free R-complex $\mathcal{X}$, we say $\mathcal{X}$ **resolves** M **over** R if there exists a quasiisomorphism $\phi : \mathcal{X} \to M$.

Equivalently, we say $\mathcal{X}$ is a **free resolution** of M over R if $\mathcal{X}$ is acyclic with $\mathrm{H}_0(\mathcal{X}) \cong M$. The associated **augmented free resolution** is the exact R-complex

$$\mathcal{X}^+ = \cdots \longrightarrow \mathcal{X}_2 \xrightarrow{\partial_2^{\mathcal{X}}} \mathcal{X}_1 \xrightarrow{\partial_1^{\mathcal{X}}} \mathcal{X}_0 \longrightarrow M \xrightarrow{\phi} 0$$

If for some integer $n < \infty$ we have $\mathcal{X}_i = 0$ for all $i > n$ and $\mathcal{X}_n \neq 0$, we say the **length** of $\mathcal{X}$ is n. When this happens we say that $\mathcal{X}$ is a **finite** resolution.

Example 2.2.2. With the notation of Definition 2.2.1, since $\phi : \mathcal{X} \to M$ is a quasiisomorphism, we have $\operatorname{Cone}(\phi)$ is exact. In fact, one can show that $\operatorname{Cone}(\phi) \cong \Sigma\mathcal{X}^+$.

We will often replace R-modules with a free resolution of the module over R. The free resolution provides more homological tools to study the module. Since we often consider maps between R-modules, we need a way to translate those R-module homomorphisms to maps between R-complexes.

Fact 2.2.3. *Let M and N be R-modules with respective free resolutions $\mathcal{X}$ and $\mathcal{Y}$. If $\varphi : M \to N$ is an R-module homomorphism, then there exists a chain map $\Phi : \mathcal{X} \to \mathcal{Y}$ such that $\mathrm{H}_0(\Phi) = \varphi$ (and $\mathrm{H}_i(\Phi) = 0$ for $i \neq 0$).*

While we omit the proof of this fact, we point out that the proof utilizes the exactness of $\mathcal{X}^+$ and $\mathcal{Y}^+$ to build a chain map $\Phi^+ : \mathcal{X}^+ \to \mathcal{Y}^+$ with $\Phi^+_{-1} = \varphi$. One then takes $\Phi_i = \Phi^+_i$ for $i \geq 0$ and $\Phi_j = 0$ for $j < 0$ to obtain a chain map with the desired properties.

Similar to the free resolution of M, one can define a projective resolution; the difference is that $\mathcal{X}$ is a graded projective complex. Free resolutions are also projective resolutions but the converse fails in many cases.

Definition 2.2.4. If the R-module M has a finite projective resolution then the **projective dimension** of M over R is the minimum length among all projective resolutions of M and is written $\mathrm{pd}_R(M)$. If M has no finite projective resolution, the convention is to say $\mathrm{pd}_R(M) = \infty$.

There are several results that relate the projective dimension of a module to the following definition; see, e.g., Corollaries 4.6.12 and 5.7.3.

Definition 2.2.5. Given an R-module M with projective resolution $\mathcal{X}$, if N is also an R-module, then the **ℓth Tor-module** of M and N over R is the module $\mathrm{Tor}^R_\ell(M,N) := \mathrm{H}_\ell(\mathcal{X} \otimes_R N)$. As complexes, we have $\mathrm{Tor}^R(M,N) = \mathrm{H}(\mathcal{X} \otimes_R N)$.

Fact 2.2.6. *If $\mathcal{X}$ is a projective resolution of M and $\mathcal{Y}$ a projective resolution of N, both over R, then we have quasiisomorphisms $\mathcal{X} \otimes_R N \xleftarrow{\simeq} \mathcal{X} \otimes_R \mathcal{Y} \xrightarrow{\simeq} M \otimes_R \mathcal{Y}$. Consequently, we have*

$$\mathrm{Tor}^R(M,N) = \mathrm{H}(\mathcal{X} \otimes_R N) \cong \mathrm{H}(\mathcal{X} \otimes_R \mathcal{Y}) \cong \mathrm{H}(M \otimes_R \mathcal{Y}).$$

Moreover, the Tor-modules are independent of the choice of projective resolutions $\mathcal{X}$ and $\mathcal{Y}$.

Definition 2.2.7. Two R-modules M and N are said to be **Tor-independent** if $\mathrm{Tor}^R_i(M,N) = 0$ for all $i \geq 1$. If $\mathcal{I}, \mathcal{J} \subseteq R$ are ideals such that $R/\mathcal{I}$ and $R/\mathcal{J}$ are Tor-independent, then we say $\mathcal{I}$ and $\mathcal{J}$ are **Tor-independent**.

Definition 2.2.8. We say R is **Tor-friendly** if for all finitely generated R-modules M and N we have the property that bounded $\mathrm{Tor}^R(M,N)$ implies $\mathrm{pd}_R(M) < \infty$ or $\mathrm{pd}_R(N) < \infty$.

While there are several known examples (see [4, 5, 15, 36] for a partial list), it is important to note that not all rings are Tor-friendly. The following example of this comes from [4].

Example 2.2.9. Let k be an arbitrary field and set $R = k[x,y]/(x^2,y^2)$. We will find that neither $M = R/(x)$ nor $N = R/(y)$ have finite projective dimension even though we have $\operatorname{Tor}_i^R(M,N) = 0$ for all $i > 0$. Since R is a non-negatively graded, noetherian algebra over a field we know that every finitely generated graded projective R-module is free over R. Thus every graded projective resolution over R of a graded R-module is a graded free resolution. Since M is a graded R-module, we have the following minimal graded projective resolution over R.

$$\mathcal{X} = \cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow 0$$

The minimality shows us that $\operatorname{pd}_R(M) = \infty$. We can build a similar resolution for N where the differentials are given by multiplication by y instead of x. Doing this shows that $\operatorname{pd}_R(N) = \infty$. To see that R is not Tor-friendly, one computes

$$\operatorname{Tor}_i^R(M,N) = \operatorname{H}_i(\mathcal{X} \otimes_R N) = \begin{cases} k & i = 0 \\ 0 & i > 0 \end{cases}.$$

Since we have a bounded $\operatorname{Tor}^R(M,N)$ for two modules with infinite projective dimension, we have shown that R is not Tor-friendly.

For much of this research area it is especially helpful when we have a minimal free resolution. There are several equivalent definitions of a minimal free resolution but we will use the following definition.

Definition 2.2.10. A free resolution $\mathcal{X}$ of M over R is said to be **minimal** if $\operatorname{Im}\partial^{\mathcal{X}} \subseteq \mathfrak{M}_R\mathcal{X}$; that is, if $\operatorname{Im}\partial_{i+1}^{\mathcal{X}} \subseteq \mathfrak{M}_R\mathcal{X}_i$ for all i.

Fact 2.2.11. *Let M be a finitely generated module over a noetherian ring R. If $\mathcal{X}$ is a minimal free resolution of M over R, then the length of $\mathcal{X}$ is precisely $\operatorname{pd}_R(M)$.*

Fact 2.2.12. *Set $k = R/\mathfrak{M}_R$. If $\mathcal{X}$ is a minimal resolution of the R-module M, then* $\operatorname{Tor}^R(M,k) = \operatorname{H}(\mathcal{X} \otimes_R k) = \mathcal{X} \otimes_R k$.

Chapter 3

Minimal Free Resolutions of Fiber Products

3.1 Introduction

Throughout the chapter, let $(S, \mathfrak{M}_S, k)$ and $(T, \mathfrak{M}_T, k)$ both be commutative, local (or standard graded), Noetherian rings with $S \cong A/I'$ and $T \cong B/J'$ where A and B are regular local rings (or polynomial rings over k). Both S and T come equipped with a natural surjection $S \xrightarrow{\pi_S} k \xleftarrow{\pi_T} T$ which are used to build the fiber product of S and T over k given by $S \times_k T := \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}$. Much research has been conducted comparing and contrasting the homological properties of S and T with those of $S \times_k T$, e.g., the Cohen-Macaulay, Gorenstein, Golod, finite representation type, and Arf properties (see [9, 10, 15, 31, 34–37, 39–41]).

We continue this line of inquiry by explicitly constructing a minimal free resolution of the fiber product $F = S \times_k T$ over an appropriate regular local ring or polynomial ring; see Theorem 3.5.3 and Corollary 3.5.6. In particular, this construction yields the following formulas for Poincaré series; see Corollary 3.5.5.

Theorem 3.1.1. *Set $\underline{x} = x_1, \ldots, x_m$ and $\underline{y} = y_1, \ldots, y_n$. Suppose $\mathcal{I}' \subseteq \langle \underline{x} \rangle^2$ and $\mathcal{J}' \subseteq \langle \underline{y} \rangle^2$. Consider either of the two following cases with k a field;*

1. *$A = k[\underline{x}]$ and $B = k[\underline{y}]$ with $R = k[\underline{x}, \underline{y}]$;*

2. $A = k[[\underline{x}]]$ *and* $B = k[[\underline{y}]]$ *with* $R = k[[\underline{x}, \underline{y}]]$.

Set $S = A/I'$ *and* $T = B/J'$ *and consider the fiber product* $F := S \times_k T$. *Then one has the following formulas for Poincaré series:*

$$\frac{P^R_F(t) - P^R_{R/(\mathcal{I}'+\mathfrak{M}_B)}(t) - P^R_{R/(\mathfrak{M}_A+\mathcal{J}')}(t) + P^R_{R/\mathfrak{M}_R}(t)}{P^R_{\langle \underline{x}\underline{y}\rangle}(t)} = t(1+t)$$

as well as

$$\frac{P^R_F(t) - (1+t)^n P^A_S(t) - (1+t)^m P^B_T(t) + (1+t)^{m+n}}{((1+t)^m - 1)\,((1+t)^n - 1)} = \frac{t+1}{t}.$$

Section 2 of this chapter documents background material for use in Sections 3 and 4. In addition, this section contains a few results about indicator functions (inspired by the use of measure theory within probability and stochastics) which we use to reduce significantly the number of cases needed for our results and proofs.

Section 3 is devoted to Construction 3.3.1 that builds a minimal resolution over a polynomial ring for a quotient of that polynomial ring by certain products of ideals. One can then recover the results of [43] on edge ideals of complete bipartite graphs by specializing Theorem 3.3.4 using $\mathcal{I} = \langle \underline{x} \rangle$ and $\mathcal{J} = \langle \underline{y} \rangle$.

Section 4 utilizes (hard) truncations, tensor products, and shifts in order to manipulate the minimal resolutions of S and T over distinct polynomial rings into resolutions over our desired ring. Constuction 3.4.5 shows how to build specific chain maps from these new resolutions to the one constructed in Theorem 3.4.8 so that the associated mapping cones provide resolutions of the desired quotients.

Section 5 is dedicated to the criteria for and consequences of the construction in Theorem 3.4.8 being minimal. Corollary 3.5.6 specifically identifies the application to fiber products. The section concludes by giving formulas for the Betti numbers (Corollary 3.5.4) and Poincaré series (Corollary 3.5.5) for the minimal construction.

3.2 Notation and Background

Throughout the chapter, $(R, \mathfrak{M}_R)$ is a commutative, regular local (or standard graded) ring. If $\mathcal{S}$ is a chain complex of finite-rank free R-modules such that $\mathcal{S}_i = 0$ for all $i < 0$, then its

generating function is

$$\mathbb{P}_{\mathcal{S}}^{R}(t) = \sum_{n \geq 0} \operatorname{rank}_R \mathcal{S}_n t^n.$$

If $\mathcal{S}$ is acyclic and minimal, then the **Poincaré series** for $\mathrm{H}_0(\mathcal{S})$ is $P_{\mathrm{H}_0(\mathcal{S})}(t) = \mathbb{P}_{\mathcal{S}}(t)$.

We denote the ℓth **suspension** (or **shift**) of $\mathcal{S}$ as $\Sigma^\ell \mathcal{S}$. In the case $\ell = 1$, we set $\Sigma\mathcal{S} := \Sigma^1\mathcal{S}$. We write $\mathcal{S}_{\geq p}$ for the (hard) truncation of $\mathcal{S}$ in degrees greater than or equal to p. This complex is given by

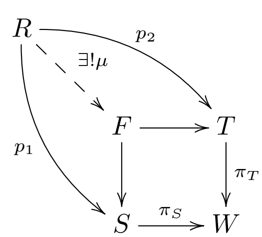
$$(\mathcal{S}_{\geq p})_q = \begin{cases} \mathcal{S}_q & q \geq p \\ 0 & q < p \end{cases} \qquad \text{and} \qquad \partial_q^{\mathcal{S}_{\geq p}} = \begin{cases} \partial_q^{\mathcal{S}} & q > p \\ 0 & q \leq p \end{cases}$$

Two R-modules M and N are said to be **Tor-independent** if $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. If $\mathcal{I}, \mathcal{J} \subseteq R$ are ideals such that $R/\mathcal{I}$ and $R/\mathcal{J}$ are Tor-independent, then we say $\mathcal{I}$ and $\mathcal{J}$ are Tor-independent.

Proposition 3.2.1. *Consider the ideals $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$, all in R, such that $\mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}$, e.g., such that $\mathcal{I}$ and $\mathcal{J}$ are Tor-independent. Set $W = R/(\mathcal{I} + \mathcal{J})$. We then have*

$$\frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle} \cong \frac{R}{\mathcal{I}' + \mathcal{J}} \times_W \frac{R}{\mathcal{I} + \mathcal{J}'}.$$

Proof. Set $F = \frac{R}{\mathcal{I}' + \mathcal{J}} \times_W \frac{R}{\mathcal{I} + \mathcal{J}'} = S \times_W T$ and note that $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$ gives us natural surjections $\pi_S : S \to W \leftarrow T : \pi_T$. Moreover, fiber products come with a universal mapping property making the following diagram commute



where p_1 and p_2 are the natural surjections from R to S and T, respectively. It is straightforward to check that μ is surjective and yields the following isomorphism.

$$F \cong \frac{R}{\ker \mu} = \frac{R}{\ker p_1 \cap \ker p_2} = \frac{R}{(\mathcal{I}' + \mathcal{J}) \cap (\mathcal{I} + \mathcal{J}')}.$$

From here we observe that

$$\mathcal{I}' + \mathcal{I}\mathcal{J} + \mathcal{J}' \subseteq (\mathcal{I}' + \mathcal{J}) \cap (\mathcal{I} + \mathcal{J}').$$

To obtain the other containment, we consider $\alpha \in (\mathcal{I}' + \mathcal{J}) \cap (\mathcal{I} + \mathcal{J}')$. We can then write $\alpha = \alpha_{\mathcal{I}'} + \alpha_{\mathcal{J}} = \alpha_{\mathcal{I}} + \alpha_{\mathcal{J}'}$ where the subscript denotes which ideal each element comes from, i.e., $\alpha_{\mathcal{I}'} \in \mathcal{I}'$. Set $\beta = \alpha_{\mathcal{I}'} - \alpha_{\mathcal{I}}$ and note that $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$ means we have

$$\beta = \alpha_{\mathcal{I}'} - \alpha_{\mathcal{I}} = \alpha_{\mathcal{J}'} - \alpha_{\mathcal{J}} \in \mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}.$$

From here we can conclude that

$$\alpha = \alpha_{\mathcal{I}} + \alpha_{\mathcal{J}'} = \alpha_{\mathcal{I}'} - \beta + \alpha_{\mathcal{J}'} \in \mathcal{I}' + \mathcal{I}\mathcal{J} + \mathcal{J}',$$

which then gives the desired containment and isomorphism of rings. □

Example 3.2.2. Consider rings $S \cong k[[\underline{x}]]/I'$ and $T \cong k[[\underline{y}]]/J'$. Set $R = k[[\underline{x}, \underline{y}]]$ and consider the ideals of $\mathcal{I}' = I'R$ and $\mathcal{J}' = J'R$. We then have $S \cong R/(\mathcal{I}' + \langle \underline{y} \rangle)$ and $T \cong R/(\langle \underline{x} \rangle + \mathcal{J}')$. Combining these isomorphisms with Proposition 3.2.1 yields the following.

$$S \times_k T \cong \frac{R}{\mathcal{I}' + \langle \underline{y} \rangle} \times_{\frac{R}{\langle \underline{x} \rangle + \langle \underline{y} \rangle}} \frac{R}{\langle \underline{x} \rangle + \mathcal{J}'} \cong \frac{R}{\langle \mathcal{I}', \underline{x}\underline{y}, \mathcal{J}' \rangle}.$$

We note that this isomorphism holds if the power series rings are replaced with polynomial rings.

Throughout this chapter, we utilize indicator functions $\mathbf{1}_{[*]}$. These functions return 1 if the input makes the statement $*$ true and 0 if false. In particular, we make use of the following properties.

Lemma 3.2.3. *Let W_1 and W_2 be statements that can be evaluated as true or false, and let W_1^c represent "not W_1". Then the following hold.*

(a) $1 = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_1^c]}$

(b) $\mathbf{1}_{[W_1 \cap W_2]} = \mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]}$

(c) $\mathbf{1}_{[W_1 \cup W_2]} = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_2]} - \mathbf{1}_{[W_1 \cap W_2]}$

(d) *If* W_1 *implies* W_2*, then* $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = \mathbf{1}_{[W_1]}$*, and we say that* $\mathbf{1}_{[W_2]}$ *is redundant.*

(e) *If* W_1 *implies* W_2^c*, then* $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = 0$.

(f) *If* W_1 *is a logical statement such that* W_1 *implies* $x = x_0$*, then for any function* f *with* x_0 *in the domain of* f*, we have* $\mathbf{1}_{[W_1]}f(x) = \mathbf{1}_{[W_1]}f(x_0)$.

Proof. To prove part (a) we use cases. When W_1 is true, we have W_1^c is false. Thus, we have $\mathbf{1}_{[W_1]} + \mathbf{1}_{[W_1^c]} = 1 + 0 = 1$. The case where W_1 is false follows similarly.

The proofs of (b) through (e) can be proven similarly. To prove (f) in a similar fashion, we first set $g(x) = \mathbf{1}_{[W_1]}f(x) - \mathbf{1}_{[W_1]}f(x_0) = \mathbf{1}_{[W_1]}(f(x) - f(x_0))$. One can then use cases to show $g(x) = 0$ for all x, which returns the desired result. □

3.3 Free Resolutions for Products of Certain Ideals

We next consider a construction that takes the minimal resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ over R and outputs the minimal resolution of $R/\mathcal{I}\mathcal{J}$. We then specialize the construction in a corollary to reproduce the minimal, cellular resolution constructed in [43] for fiber products of the form $k[\underline{x}] \times_k k[\underline{y}] \cong k[\underline{x}, \underline{y}]/\langle \underline{x}\underline{y}\rangle$.

Construction 3.3.1. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be complexes of free* R*-modules. The* **star product** *of* $\mathcal{X}$ *and* $\mathcal{Y}$ *over* R*, denoted* $\mathcal{X} *_R \mathcal{Y}$*, is the chain complex given by*

$$(\mathcal{X} *_R \mathcal{Y})_n = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{n+1} & n \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & n = 0 \\ 0 & n < 0 \end{cases} \quad \text{and} \quad \partial_n^{\mathcal{X}*\mathcal{Y}} = \begin{cases} \partial_{n+1}^{\mathcal{X}_{\geq 1}\otimes \mathcal{Y}_{\geq 1}} & n \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & n = 1 \\ 0 & n \leq 0 \end{cases}.$$

In summary, $\mathcal{X} *_R \mathcal{Y}$ is obtained by truncating $\mathcal{X}$ and $\mathcal{Y}$, tensoring the truncations, then shifting and augmenting the tensor product. In particular, it is straightforward to show that $\mathcal{X} *_R \mathcal{Y}$ is a bounded below complex of free R-modules. We denote simple tensors of positive degree in $\mathcal{X} *_R \mathcal{Y}$ as $\alpha * \beta$ where $\alpha \in \mathcal{X}$ and $\beta \in \mathcal{Y}$.

Example 3.3.2. Set $\mathcal{X} = K^{k[\underline{x},\underline{y}]}(\underline{x})$ be the Koszul complex for $\underline{x} = x_1, \ldots, x_m$. Similarly, set $\mathcal{Y} = K^{k[\underline{x},\underline{y}]}(\underline{y})$ be the Koszul complex for $\underline{y} = y_1, \ldots, y_n$. Then we have $(\mathcal{X} *_R \mathcal{Y})_0 \cong k[\underline{x}, \underline{y}]$ for

$R = k[\underline{x}, \underline{y}]$ and, for $\ell \geq 1$,

$$(\mathcal{X} *_R \mathcal{Y})_\ell = \bigoplus_{t=1}^{\ell} k[\underline{x}, \underline{y}]^{\binom{m}{t}} \otimes_R k[\underline{x}, \underline{y}]^{\binom{n}{\ell+1-t}} \cong k[\underline{x}, \underline{y}]^{\binom{m+n}{\ell+1} - \binom{m}{\ell+1} - \binom{n}{\ell+1}}.$$

In particular, if we set $R = k[\underline{x}, \underline{y}]$ and consider $m = 2 = n$, then $\mathcal{X} *_R \mathcal{Y}$ has the following form.

$$0 \to R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x_1y_1 & x_1y_2 & x_2y_1 & x_2y_2 \end{pmatrix}} R \to 0$$

The following lemma allows to express the differential $\partial^{\mathcal{X} *_R \mathcal{Y}}$ in terms of $\partial^{\mathcal{X}}$, $\partial^{\mathcal{Y}}$, and indicator functions. In particular, it removes the need to call on the truncated complexes and will simplify later proofs.

Lemma 3.3.3. *For $n \geq 1$, the differential $\partial^{\mathcal{X} * \mathcal{Y}}$ acts on $(\mathcal{X} *_R \mathcal{Y})_n$ by*

$$\partial^{\mathcal{X} * \mathcal{Y}}(a * b) = \mathbf{1}_{[|a|>1]} \partial^{\mathcal{X}}(a) * b + \mathbf{1}_{[|b|>1]} (-1)^{|a|} a * \partial^{\mathcal{Y}}(b) + \mathbf{1}_{[|a|,|b|=1]} \partial^{\mathcal{X}}(a) * \partial^{\mathcal{Y}}(b)\,.$$

Proof. Since n is a positive integer, we have $\mathbf{1}_{[n \neq 1]} = \mathbf{1}_{[|a|+|b|-1 \geq 2]}$. Moreover, we have $n = 1$ if and only if $|a| = 1 = |b|$, thus $\mathbf{1}_{[n=1]} = \mathbf{1}_{[|a|,|b|=1]}$. Thus, Lemma 3.2.3(a) and (f) allow us to conclude

$$\partial_n^{\mathcal{X} * \mathcal{Y}}(a * b) = \mathbf{1}_{[|a|+|b| \geq 3]} \Big(\partial_{|a|}^{\mathcal{X}_{\geq 1}}(a) * b + (-1)^{|a|} a * \partial_{|b|}^{\mathcal{Y}_{\geq 1}}(b) \Big) + \mathbf{1}_{[|a|,|b|=1]} \partial^{\mathcal{X}}(a) * \partial^{\mathcal{Y}}(b)\,.$$

Lemma 3.2.3(a), (e), and (f) implies $\partial_{|a|}^{\mathcal{X}_{\geq 1}}(a) * b = \mathbf{1}_{[|a|>1]} \partial^{\mathcal{X}}(a) * b$ and $a * \partial_{|b|}^{\mathcal{Y}_{\geq 1}}(b) = \mathbf{1}_{[|b|>1]} a * \partial^{\mathcal{Y}}(b)$. Using Lemma 3.2.3(d), we note that $\mathbf{1}_{[|a|+|b| \geq 3]}$ is redundant in the presence of both $\mathbf{1}_{[|a|>1]}$ and $\mathbf{1}_{[|b|>1]}$ and can then be dropped to produce the desired formula. □

Revisiting Example 3.3.2, one can check that the complex given in the case of $m, n = 2$ is in fact a free resolution of $k[\underline{x}, \underline{y}]/\langle \underline{xy} \rangle$ over $k[\underline{x}, \underline{y}]$. This matches with the construction by Visscher in [43], which is a particular case in the following theorem. It should also be noted that the following theorem was proved independently and simultaneously by VandeBogert in [42].

Theorem 3.3.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be free resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ over R, respectively. If $\mathcal{I}, \mathcal{J} \subseteq R$ are* Tor*-independent ideals, then the star product $\mathcal{X} *_R \mathcal{Y}$ is a free resolution of $R/\mathcal{I}\mathcal{J}$ over R. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are minimal, then $\mathcal{X} *_R \mathcal{Y}$ is also minimal.*

Proof. To verify the first conclusion, we need only show that the construction is acyclic and resolves the desired ring. For exactness in degrees $n \geq 2$, we observe that

$$\begin{aligned}
\mathrm{H}_n\left(\mathcal{X} *_R \mathcal{Y}\right) &= \mathrm{H}_{n+1}\left(\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1}\right) \\
&= \mathrm{Tor}_{n-1}^R\left(\mathcal{I}, \mathcal{J}\right) \\
&\cong \mathrm{Tor}_{n+1}^R\left(\frac{R}{\mathcal{I}}, \frac{R}{\mathcal{J}}\right) \\
&= 0.
\end{aligned}$$

To treat the case of $n = 1$, we consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{X}_1 \otimes_R \mathcal{Y}_1 & \xrightarrow{\partial_1^{\mathcal{X} *_R \mathcal{Y}}} & \mathcal{I}\mathcal{J} \\
{\scriptstyle \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}} \searrow & & \nearrow {\scriptstyle \mu} \\
& \operatorname{Im} \partial_1^{\mathcal{X}} \otimes_R \operatorname{Im} \partial_1^{\mathcal{Y}} = \mathcal{I} \otimes_R \mathcal{J} &
\end{array}$$

and observe that $\ker\left(\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}\right) \subseteq \ker \partial_1^{\mathcal{X} *_R \mathcal{Y}}$ with equality if the map μ sending $\alpha \otimes \beta \in \mathcal{I} \otimes_R \mathcal{J}$ to $\alpha\beta \in \mathcal{I}\mathcal{J}$ is injective. To show that μ is injective, consider the long exact sequence in Tor obtained by applying $\mathcal{I} \otimes_R -$ to the short exact sequence

$$0 \to \mathcal{J} \to R \to R/\mathcal{J} \to 0.$$

This yields the exact sequence

$$0 \to \underbrace{\mathrm{Tor}_1^R\left(\mathcal{I}, \frac{R}{\mathcal{J}}\right)}_{\cong \mathrm{Tor}_2^R\left(\frac{R}{\mathcal{I}}, \frac{R}{\mathcal{J}}\right) = 0} \to \mathcal{I} \otimes_R \mathcal{J} \to \underbrace{\mathcal{I} \otimes_R R}_{\cong \mathcal{I}} \to \underbrace{\mathcal{I} \otimes_R R/\mathcal{J}}_{\cong \mathcal{I}/\mathcal{I}\mathcal{J}} \to 0$$

from which we deduce the exact sequence

$$0 \longrightarrow \mathcal{I} \otimes_R \mathcal{J} \xrightarrow{\ \mu\ } \mathcal{I}\mathcal{J} \longrightarrow 0.$$

Thus, μ is an isomorphism so $\ker\left(\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}\right) = \ker \partial_1^{\mathcal{X} *_R \mathcal{Y}}$.

We note that $\partial_1^{\mathcal{X}} : \mathcal{X}_1 \to \operatorname{Im} \partial_1^{\mathcal{X}}$ and $\partial_1^{\mathcal{Y}} : \mathcal{Y}_1 \to \operatorname{Im} \partial_1^{\mathcal{Y}}$ are both surjections. It follows that

$$\mathcal{X}_1 \otimes_R \mathcal{Y}_1 \xrightarrow{\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}} \operatorname{Im} \partial_1^{\mathcal{X}} \otimes_R \operatorname{Im} \partial_1^{\mathcal{Y}} \longrightarrow 0$$

is exact. Furthermore, [32, p. 267] shows us that kernel of this surjection is given by

$$\begin{aligned} \ker \partial_1^{\mathcal{X} * \mathcal{Y}} &= \ker \left(\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}\right) \\ &= \ker \partial_1^{\mathcal{X}} \otimes_R \mathcal{Y}_1 + \mathcal{X}_1 \otimes_R \ker \partial_1^{\mathcal{Y}} \\ &= \operatorname{Im} \partial_2^{\mathcal{X}} \otimes_R \mathcal{Y}_1 + \mathcal{X}_1 \otimes_R \operatorname{Im} \partial_2^{\mathcal{Y}} \\ &= \operatorname{Im} \partial_2^{\mathcal{X} * \mathcal{Y}}. \end{aligned}$$

Moreover, we note

$$H_0\left(\mathcal{X} *_R \mathcal{Y}\right) = \frac{\mathcal{X}_0 \otimes_R \mathcal{Y}_0}{\operatorname{Im}\left(\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}\right)} \cong \frac{R \otimes_R R}{\mathcal{I} \otimes_R \mathcal{J}} \cong \frac{R}{\mathcal{I}\mathcal{J}}.$$

Lastly, it is straightforward to check minimality. □

As we discuss in the introduction, setting $\mathcal{I} = \langle \underline{x} \rangle$ and $\mathcal{J} = \langle \underline{y} \rangle$ in the case of $R = k[\underline{x}, \underline{y}]$ in Theorem 3.3.4 recovers the main result of Visscher in [43]. Next, we document consequences for Betti numbers.

Corollary 3.3.5. *We have the following formulas for the Betti numbers and graded Betti Numbers of $R/\mathcal{I}\mathcal{J}$ for $\ell > 0$.*

$$\beta_\ell^R(R/\mathcal{I}\mathcal{J}) = \sum_{i=1}^{\ell} \beta_i^R(R/\mathcal{I}) \beta_{\ell+1-i}^R(R/\mathcal{J})$$
$$\beta_{\ell,k}^R(R/\mathcal{I}\mathcal{J}) = \sum_{i=1}^{\ell} \sum_{j=0}^{k} \beta_{i,j}^R(R/\mathcal{I}) \beta_{\ell+1-i,k-j}^R(R/\mathcal{J})$$

Proof. Use the fact that $\mathcal{X} *_R \mathcal{Y}$ is minimal and obtained by truncating, tensoring, shifting, and augmenting minimal resolutions to get the desired formulas. □

We conclude this section by interpreting Corollary 3.3.5 in terms of Poincaré series.

Corollary 3.3.6. *We have the following relationship of Poincaré series:*

$$P_{R/\mathcal{I}\mathcal{J}}^R(t) = 1 + \frac{1}{t}(P_{R/\mathcal{I}}^R(t) - 1)(P_{R/\mathcal{J}}^R(t) - 1).$$

3.4 Extending the Star Product

In this section, we use $\mathcal{X} *_R \mathcal{Y}$ to build a resolution of $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle$ over R where $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$. Taking $R = k[[\underline{x}, \underline{y}]]$, we will show how this specializes to fiber products of the form $k[[\underline{x}]]/I' \times_k k[[\underline{y}]]/J'$ and note the same can be done in the polynomial case.

Notation 3.4.1. Throughout this section, we fix the following notation.

1. Let $\mathcal{X}$ resolve $R/\mathcal{I}$ over R.
2. Let $\mathcal{Y}$ resolve $R/\mathcal{J}$ over R.
3. Let $\mathcal{S}$ resolve $R/\mathcal{I}'$ over R.
4. Let $\mathcal{T}$ resolve $R/\mathcal{J}'$ over R.
5. The pairs $\{\mathcal{I}, \mathcal{J}\}$, $\{\mathcal{I}', \mathcal{J}\}$, and $\{\mathcal{I}, \mathcal{J}'\}$ are Tor-independent ideals in R.

The next example shows that some ideals from Example 3.2.2 are Tor-independent for use in our proof of Theorem 3.1.1.

Example 3.4.2. Suppose that we either have one of the following cases.

1. $A = k[\underline{x}]$ and $B = k[\underline{y}]$ with $R = k[\underline{x}, \underline{y}]$.
2. $A = k[[\underline{x}]]$ and $B = k[[\underline{y}]]$ with $R = k[[\underline{x}, \underline{y}]]$.

If M is an A-module and N is a B-module, both finitely generated, then a standard prime filtration argument using induction on dimension shows that $M \otimes_A R$ and $N \otimes_B R$ are Tor-independent R-modules.

The Tor-independence in Notation 3.4.1(5) is needed to ensure the vanishing of homology in Construction 3.3.1 and in the following lemma.

Lemma 3.4.3. *Under the assumptions in Notation 3.4.1, we have $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ resolves $\mathcal{I}' \cdot R/\mathcal{I}\mathcal{J}$ over R while $\Sigma^{-1}(\mathcal{X} \otimes_R \mathcal{T}_{\geq 1})$ resolves $\mathcal{J}' \cdot R/\mathcal{I}\mathcal{J}$.*

Proof. We prove the case of $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$. Due to their Tor-independence, we have $\mathcal{I}' \cap \mathcal{J} = \mathcal{I}'\mathcal{J}$. Combining this with the fact that $\mathcal{I}' \subseteq \mathcal{I}$, we have

$$\mathcal{I}'\mathcal{J} \subseteq \mathcal{I}' \cap \mathcal{I}\mathcal{J} \subseteq \mathcal{I}' \cap \mathcal{J} = \mathcal{I}'\mathcal{J}$$

yielding $\mathcal{I}'\mathcal{J} = \mathcal{I}' \cap \mathcal{I}\mathcal{J}$. Thus, we have the following chain of isomorphisms:

$$\mathcal{I}' \cdot \frac{R}{\mathcal{I}\mathcal{J}} = \frac{\mathcal{I}' + \mathcal{I}\mathcal{J}}{\mathcal{I}\mathcal{J}} \cong \frac{\mathcal{I}'}{\mathcal{I}' \cap \mathcal{I}\mathcal{J}} = \frac{\mathcal{I}'}{\mathcal{I}'\mathcal{J}} \cong \mathcal{I}' \otimes_R \frac{R}{\mathcal{J}}.$$

It is straightforward to check that $\mathrm{H}_0\left(\Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right)\right) = \mathcal{I}' \otimes_R R/\mathcal{J}$. For $i \geq 1$, we use the Tor-independence of $\mathcal{I}'$ and $\mathcal{J}$ to get

$$\mathrm{H}_i\left(\Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right)\right) = \mathrm{Tor}_i^R\left(\mathcal{I}', \frac{R}{\mathcal{J}}\right) \cong \mathrm{Tor}_{i+1}^R\left(\frac{R}{\mathcal{I}'}, \frac{R}{\mathcal{J}}\right) = 0.$$

Thus, $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ is the desired resolution. $\square$

Note 3.4.4. *Since $\mathcal{I}' \subseteq \mathcal{I}$, we have a natural surjection $R/\mathcal{I}' \to R/\mathcal{I}$ which lifts to a chain map $\phi : \mathcal{S} \to \mathcal{X}$. We also lift the surjection $R/\mathcal{J}' \to R/\mathcal{J}$ to a chain map $\psi : \mathcal{T} \to \mathcal{Y}$. In the next construction, we lift ϕ and ψ to chain maps $\Phi : \Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right) \to \mathcal{X} *_R \mathcal{Y}$ and $\Psi : \Sigma^{-1}\left(\mathcal{X} \otimes_R \mathcal{T}_{\geq 1}\right) \to \mathcal{X} *_R \mathcal{Y}$.*

Construction 3.4.5. *For $\alpha \otimes \beta \in \Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right)$, we define*

$$\Phi(\alpha \otimes \beta) := \begin{cases} (-1)^{|\alpha|+|\beta|}\phi(\alpha) * \beta & |\beta| > 0 \\ \partial_1^{\mathcal{S}}(\alpha) * \beta & |\beta| = 0,\ |\alpha| = 1 \\ 0 & |\beta| = 0,\ |\alpha| > 1 \end{cases}$$
$$= \mathbf{1}_{[|\beta|>0]}(-1)^{|\alpha|+|\beta|}\phi(\alpha) * \beta + \mathbf{1}_{[|\beta|=0]}\mathbf{1}_{[|\alpha|=1]}\partial_1^{\mathcal{S}}(\alpha) * \beta.$$

In the case that $\alpha \otimes \beta \in \Sigma^{-1}\left(\mathcal{X} \otimes_R \mathcal{T}_{\geq 1}\right)$, we define

$$\Psi(\alpha \otimes \beta) := \begin{cases} (-1)^{|\alpha|+|\beta|-1}\alpha * \psi(\beta) & |\alpha| > 0 \\ \alpha * \partial_1^{\mathcal{T}}(\beta) & |\alpha| = 0, |\beta| = 1 \\ 0 & |\alpha| = 0, |\beta| > 1 \end{cases}$$
$$= -\mathbf{1}_{[|\alpha|>0]}(-1)^{|\alpha|+|\beta|}\alpha * \psi(\beta) + \mathbf{1}_{[|\alpha|=0]}\mathbf{1}_{[|\beta|=1]}\alpha * \partial_1^{\mathcal{T}}(\beta).$$

The second equality in both definitions follow as in the proof of Lemma 3.3.3.

Lemma 3.4.6. *The maps Φ and Ψ, as defined in Construction 3.4.5, are chain maps.*

Proof. For the Φ computations, we use the following formula

$$\begin{aligned}
\partial_i^{\Sigma^{-1}(\mathcal{S}_{\geq 1}\otimes_R \mathcal{Y})}(\alpha\otimes\beta) &= -\,\partial_{i+1}^{\mathcal{S}_{\geq 1}\otimes_R\mathcal{Y}}(\alpha\otimes\beta)\\
&= -\left(\partial^{\mathcal{S}_{\geq 1}}(\alpha)\otimes\beta + (-1)^{|\alpha|}\alpha\otimes\partial^{\mathcal{Y}}(\beta)\right) \qquad (3.4.6.1)\\
&\overset{3.2.3(\mathrm{f})}{=} -\,\mathbf{1}_{[|\alpha|>1]}\partial^{\mathcal{S}}(\alpha)\otimes\beta - (-1)^{|\alpha|}\alpha\otimes\partial^{\mathcal{Y}}(\beta)\,.
\end{aligned}$$

Note that the first two equalities follow by definition.

To see that Φ is a chain map, we use Lemma 3.2.3(d) to get $\mathbf{1}_{[|\beta|=1]} = \mathbf{1}_{[|\beta|=1]}\mathbf{1}_{[|\beta|>0]}$ and $\mathbf{1}_{[|\beta|>1]} = \mathbf{1}_{[|\beta|>1]}\mathbf{1}_{[|\beta|>0]}$. Since ϕ is a chain map with $\phi_0 = 1_{k[\underline{x}]}$, we have $\partial_1^{\mathcal{S}}(\alpha) = \partial^{\mathcal{X}}(\phi(\alpha))$. Lastly, we note that Lemma 3.2.3(f) yields

$$\mathbf{1}_{[|\beta|=0]}\mathbf{1}_{[|\alpha|=1]}\partial^{\mathcal{X}*\mathcal{Y}}\big(\partial_1^{\mathcal{S}}(\alpha)*\beta\big) = 0. \qquad (3.4.6.2)$$

Thus, setting $a = |\alpha|$ and $b = |\beta|$, we see that for all $i \in \mathbb{Z}$ we have

$$\begin{aligned}
\left(\Phi_{i-1}\circ\partial_i^{\Sigma^{-1}(\mathcal{S}_{\geq 1}\otimes_R\mathcal{Y})}\right)(\alpha\otimes\beta) &\overset{3.4.6.1}{=} -\,\mathbf{1}_{[a>1]}\Phi_{i-1}\left(\partial^{\mathcal{S}}(\alpha)\otimes\beta\right) - (-1)^a\Phi_{i-1}\left(\alpha\otimes\partial^{\mathcal{Y}}(\beta)\right)\\
&\overset{3.4.5}{=} -\,\mathbf{1}_{[a>1]}\mathbf{1}_{[b>0]}(-1)^{a-1+b}\left(\phi\circ\partial^{\mathcal{S}}\right)(\alpha)*\beta\\
&\qquad -\,\mathbf{1}_{[|\partial^{\mathcal{Y}}(\beta)|>0]}(-1)^{2a+b-1}\phi(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\qquad -\,\mathbf{1}_{[|\partial^{\mathcal{Y}}(\beta)|=0]}\mathbf{1}_{[a=1]}(-1)^a\partial^{\mathcal{S}}(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\overset{*}{=} \mathbf{1}_{[a>1]}\mathbf{1}_{[b>0]}(-1)^{a+b}\left(\partial^{\mathcal{X}}\circ\phi\right)(\alpha)*\beta\\
&\qquad +\,\mathbf{1}_{[b>1]}(-1)^{2a+b}\phi(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\qquad +\,\mathbf{1}_{[b=1]}\mathbf{1}_{[a=1]}(-1)^{a+b}\left(\partial^{\mathcal{X}}\circ\phi\right)(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\overset{3.2.3(\mathrm{d})}{=} \mathbf{1}_{[a>1]}\mathbf{1}_{[b>0]}(-1)^{a+b}\left(\partial^{\mathcal{X}}\circ\phi\right)(\alpha)*\beta\\
&\qquad +\,\mathbf{1}_{[b>1]}\mathbf{1}_{[b>0]}(-1)^{2a+b}\phi(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\qquad +\,\mathbf{1}_{[b=1]}\mathbf{1}_{[b>0]}\mathbf{1}_{[a=1]}(-1)^{a+b}\left(\partial^{\mathcal{X}}\circ\phi\right)(\alpha)*\partial^{\mathcal{Y}}(\beta)\\
&\overset{3.3.3}{=} \mathbf{1}_{[b>0]}(-1)^{a+b}\partial^{\mathcal{X}*\mathcal{Y}}(\phi(\alpha)*\beta)\\
&\overset{3.4.6.2}{=} \mathbf{1}_{[b>0]}(-1)^{a+b}\partial^{\mathcal{X}*\mathcal{Y}}(\phi(\alpha)*\beta) + \mathbf{1}_{[b=0]}\mathbf{1}_{[a=1]}\partial^{\mathcal{X}*\mathcal{Y}}\big(\partial_1^{\mathcal{S}}(\alpha)*\beta\big)\\
&= \partial^{\mathcal{X}*\mathcal{Y}}\big(\mathbf{1}_{[b>0]}(-1)^{a+b}\phi(\alpha)*\beta + \mathbf{1}_{[b=0]}\mathbf{1}_{[a=1]}\partial_1^{\mathcal{S}}(\alpha)*\beta\big)\\
&\overset{3.4.5}{=} \left(\partial_i^{\mathcal{X}*_R\mathcal{Y}}\circ\Phi_i\right)(\alpha\otimes\beta)\,.
\end{aligned}$$

The unmarked equality follows from the linearity of $\partial^{\mathcal{X} *_R \mathcal{Y}}$ while starred equality follows from the fact that ϕ is a chain map and that $|\partial^{\mathcal{Y}}(b)\,| = |b| - 1$. The proof that Ψ is a chain map is similar. □

The following result is a special case of Theorem 3.1.1.

Theorem 3.4.7. *The R-complexes* $\operatorname{Cone}(\Phi)$ *and* $\operatorname{Cone}(\Psi)$ *resolve* $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle$ *and* $R/\langle \mathcal{I}\mathcal{J}, \mathcal{J}'\rangle$*.*

Proof. Since $\Sigma\left(\Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right)\right) = \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}$, the short exact sequence of complexes associated with the mapping cone is given by

$$0 \longrightarrow \mathcal{X} *_R \mathcal{Y} \longrightarrow \operatorname{Cone}(\Phi) \longrightarrow \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \longrightarrow 0.$$

Since $\mathcal{X} *_R \mathcal{Y}$ only has nonzero homology in degree 0 and $\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}$ only has homology in degree 1, the corresponding long exact sequence in homology reduces to the top row in the following commutative diagram. Moreover, the Short-Five Lemma tells us that the diagram is an isomorphism of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_1\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right) & \longrightarrow & \mathrm{H}_0\left(\mathcal{X} *_R \mathcal{Y}\right) & \longrightarrow & \mathrm{H}_0\left(\operatorname{Cone}(\Phi)\right) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathcal{I}' \cdot \frac{R}{\mathcal{I}\mathcal{J}} & \longrightarrow & \frac{R}{\mathcal{I}\mathcal{J}} & \longrightarrow & \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle} & \longrightarrow & 0
\end{array}$$

The argument for $\operatorname{Cone}(\Psi)$ follows nearly identical arguments. □

We use a short exact sequence to intertwine the mapping cones from Theorem 3.4.7.

Theorem 3.4.8. *Let Φ and Ψ be as in construction 3.4.5, set*

$$\Omega = \begin{pmatrix} \Phi & \Psi \end{pmatrix} : \Sigma^{-1}\left(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}\right) \oplus \Sigma^{-1}\left(\mathcal{X} \otimes_R \mathcal{T}_{\geq 1}\right) \longrightarrow \mathcal{X} *_R \mathcal{Y}.$$

Then $\operatorname{Cone}(\Omega)$ *is a free resolution of* $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}'\rangle$ *over R.*

Proof. By definition of Ω, it is straightforward to show that the following sequence is exact:

$$0 \longrightarrow \mathcal{X} *_R \mathcal{Y} \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}} \begin{matrix} \operatorname{Cone}(\Phi) \\ \bigoplus \\ \operatorname{Cone}(\Psi) \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \operatorname{Cone}(\Omega) \longrightarrow 0.$$

We use the associated long exact sequence in homology to see that $\mathrm{H}_i(\mathrm{Cone}(\Omega)) = 0$ for $i > 1$, leaving the case $i = 1$.

$$\begin{aligned}
&\mathrm{H}_1(\mathrm{Cone}(\Omega)) \\
&\qquad \cong \ker\left[\mathrm{H}_0(\mathcal{X} *_R \mathcal{Y}) \to \mathrm{H}_0(\mathrm{Cone}(\Phi))\right] \cap \ker\left[\mathrm{H}_0(\mathcal{X} *_R \mathcal{Y}) \to \mathrm{H}_0(\mathrm{Cone}(\Psi))\right] \\
&\qquad \cong \ker\left[\frac{R}{\mathcal{I}\mathcal{J}} \to \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle}\right] \cap \ker\left[\frac{R}{\mathcal{I}\mathcal{J}} \to \frac{R}{\langle \mathcal{I}\mathcal{J}, \mathcal{J}'\rangle}\right] \\
&\qquad = \left(\mathcal{I}' \cdot \frac{R}{\mathcal{I}\mathcal{J}}\right) \cap \left(\mathcal{J}' \cdot \frac{R}{\mathcal{I}\mathcal{J}}\right)
\end{aligned}$$

The assumptions $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$, along with Tor-independence, imply that

$$(\mathcal{I}' + \mathcal{I}\mathcal{J}) \cap (\mathcal{I}\mathcal{J} + \mathcal{J}') \subseteq \mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}$$

so we get $\mathrm{H}_1(\mathrm{Cone}(\Omega)) = 0$.

The long exact sequence in homology now reduces to the top row of the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_0(\mathcal{X} *_R \mathcal{Y}) & \xrightarrow{\overline{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}} & \begin{matrix} \mathrm{H}_0(\mathrm{Cone}(\Phi)) \\ \bigoplus \\ \mathrm{H}_0(\mathrm{Cone}(\Psi)) \end{matrix} & \xrightarrow{\overline{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}} & \mathrm{H}_0(\mathrm{Cone}(\Omega)) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \frac{R}{\mathcal{I}\mathcal{J}} & \xrightarrow[\begin{pmatrix} 1 \\ -1 \end{pmatrix}]{} & \begin{matrix} R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle \\ \bigoplus \\ R/\langle \mathcal{I}\mathcal{J}, \mathcal{J}'\rangle \end{matrix} & \xrightarrow[\begin{pmatrix} 1 & 1 \end{pmatrix}]{} & \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}'\rangle} & \longrightarrow & 0
\end{array}$$

Thus, $\mathrm{Cone}(\Omega)$ resolves the desired quotient. □

3.5 Minimality Conditions

To make full use of the mapping cone in Theorem 3.4.8 we want to ensure it is minimal. We saw in Theorem 3.3.4 that if the resolutions $\mathcal{X}$ and $\mathcal{Y}$ are minimal, then $\mathcal{X} *_R \mathcal{Y}$ is minimal. In this section, we give conditions guaranteeing $\mathrm{Cone}(\Omega)$ from Theorem 3.4.8 is minimal.

Lemma 3.5.1. *Using the notation of 3.4.1, suppose $\mathcal{I}$ is generated by a regular sequence with minimal resolution $\mathcal{X}$ with underlying graded module $\mathcal{X}^\natural$. If $M \subseteq \mathcal{X}^\natural$ is a submodule such that*

$\partial^{\mathcal{X}}(M) \subseteq (\mathcal{I})^2 \mathcal{X}$, *then* $M \subseteq \mathcal{I}\mathcal{X}$.

Proof. Since $\mathcal{I}$ is generated by a regular sequence, we know that $\mathcal{X}$ is a Koszul complex. For any $m \in M$, we have $\partial^{\mathcal{X}}(m) \in (\mathcal{I})^2 \mathcal{X}$. By [41, Lemma 2.8(ii)], we have $m \in \mathcal{I}\mathcal{X}$ and thus $M \subseteq \mathcal{I}\mathcal{X}$. □

Proposition 3.5.2. *With the notation of 3.4.1, suppose the ideal* $\mathcal{I} \subset R$ *is generated by a regular sequence and that* $\mathcal{I}' \subseteq \mathcal{I}^2$. *If* $\mathcal{X}$ *and* $\mathcal{S}$ *are minimal, then* $\phi : \mathcal{S} \to \mathcal{X}$ *from Note 3.4.4 can be chosen such that* $\phi_0 = 1$ *and* $(\operatorname{Im}\phi)_{\geq 1} \subseteq \mathcal{I}\mathcal{X} \subseteq \mathfrak{M}_R\mathcal{X}$.

Proof. Since $\mathcal{I}' \subseteq \mathcal{I}^2$, the natural surjection $R/\mathcal{I}' \to R/\mathcal{I}$ can be factored through $R/\mathcal{I}^2$, so it suffices to check that $R/\mathcal{I}^2 \to R/\mathcal{I}$ lifts to a chain map ϕ with the claimed property. Thus, we assume without loss of generality that $\mathcal{I}' = \mathcal{I}^2$. Since $\mathcal{S}$ and $\mathcal{X}$ resolve powers of $\mathcal{I}$, [14, Corollary A2.13, Exercise A2.17(d)] tells us that $\operatorname{Im}\partial_i^{\mathcal{S}} \subseteq \mathcal{I}\mathcal{S}$ and $\operatorname{Im}\partial_i^{\mathcal{X}} \subseteq \mathcal{I}\mathcal{X}$ for all i.

Consider $\operatorname{Im}\phi_1 \subseteq \mathcal{X}$ and note that $\partial_1^{\mathcal{S}}(\mathcal{S}_1) = \mathcal{I}^2$. It follows that

$$\partial^{\mathcal{X}}(\operatorname{Im}\phi_1) = \partial^{\mathcal{X}}(\phi_1(\mathcal{S}_1)) = \phi_0\left(\partial^{\mathcal{S}}(\mathcal{S}_1)\right) = \phi_0\left(\mathcal{I}^2\right) = \mathcal{I}^2.$$

It follows from Lemma 3.5.1 that $\operatorname{Im}\phi_1 \subseteq \mathcal{I}\mathcal{X}_1$. Proceeding by induction, we observe that if $\operatorname{Im}\phi_{j-1} \subseteq \mathcal{I}\mathcal{X}_{j-1}$, then

$$\partial^{\mathcal{X}}(\operatorname{Im}\phi_j) = \partial^{\mathcal{X}}(\phi_j(\mathcal{S}_j)) = \phi_{j-1}\left(\partial^{\mathcal{S}}(\mathcal{S}_j)\right) \subseteq \mathcal{I}\phi_{j-1}(\mathcal{S}_{j-1}) \subseteq (\mathcal{I})^2 \mathcal{X}_{j-1}.$$

Applying Lemma 3.5.1, we obtain $\operatorname{Im}\phi_j \subseteq \mathcal{I}\mathcal{X}_j$. By induction, we observe that $(\operatorname{Im}\phi)_{\geq 1} \subseteq \mathcal{I}\mathcal{X}$. □

Theorem 3.5.3. *Suppose* $\mathcal{I}' \subseteq \mathcal{I}^2$ *and* $\mathcal{J}' \subseteq \mathcal{J}^2$ *with both* $\mathcal{I}$ *and* $\mathcal{J}$ *generated by regular sequences. If the resolutions* $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{S}$, *and* $\mathcal{T}$ *are all minimal and we choose* ϕ *and* ψ *as in Proposition 3.5.2, then* $\operatorname{Cone}(\Omega)$ *from Theorem 3.4.8 is minimal.*

Proof. To prove minimality we just need to show that $\operatorname{Im}\partial^{\operatorname{Cone}(\Omega)} \subseteq \mathfrak{M}_R \operatorname{Cone}(\Omega)$. To see this, we first note that

$$\partial_i^{\operatorname{Cone}(\Omega)} = \begin{pmatrix} \partial_i^{\mathcal{X} *_R \mathcal{Y}} & \Phi_{i-1} & \Psi_{i-1} \\ 0 & \partial_i^{\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}} & 0 \\ 0 & 0 & \partial_i^{\mathcal{X} \otimes_R \mathcal{T}_{\geq 1}} \end{pmatrix}.$$

By the minimality of our four resolutions, we know the entries on the main diagonal all lie in $\mathfrak{M}_R$. Thus, we need only address Φ_{i-1} and Ψ_{i-1}. We are argue the case of Φ_{i-1}.

For any $\alpha \otimes \beta$ in the domain of Φ, we have $\alpha \in \mathcal{S}_{\geq 1}$. It follows that $\phi(\alpha) \in \mathcal{I}\mathcal{X}$ and thus $\phi(\alpha) * \beta \in \mathcal{I}(\mathcal{X} *_R \mathcal{Y}) \subseteq \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$. Moreover, we know $\partial^{\mathcal{S}}(\alpha) \in \mathcal{I}\mathcal{S}$ and thus $\partial^{\mathcal{S}}(\alpha) * \beta \in \mathcal{I}(\mathcal{X} *_R \mathcal{Y}) \subseteq \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$. Combining these observation with the definition of Φ in Construction 3.4.5 we have $\Phi(\alpha \otimes \beta) \in \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$ for all $\alpha \otimes \beta \in \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ as desired. □

Corollary 3.5.4. *If $\mathcal{I}$ and $\mathcal{J}$ are respectively generated by m and n elements, then for $\ell \geq 1$ and $F := R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle$, we have the following formula for Betti numbers.*

$$\beta_\ell^R(F) = \sum_{t=1}^{\ell} \left(\beta_t^R(R/\mathcal{I}') \binom{n}{\ell - t} + \binom{m}{\ell - t} \beta_t^R(R/\mathcal{J}') \right) + \binom{m+n}{\ell+1} - \binom{m}{\ell+1} - \binom{n}{\ell+1} \tag{3.5.4.1}$$

In addition, if all the ideals are homogeneous then we get the following formulas for the graded Betti numbers: for $\ell = 0$, we have $\beta_{0,k}^R(F) = \beta_{0,k}^R\left(\frac{R}{\mathcal{I}\mathcal{J}}\right)$, while for $\ell > 0$, we have

$$\beta_{\ell,k}^R(F) = \beta_{\ell,k}^R\left(\frac{R}{\mathcal{I}\mathcal{J}}\right) + \sum_{i=1}^{\ell}\sum_{j=0}^{k} \beta_{i,j}^R\left(\frac{R}{\mathcal{I}'}\right)\beta_{\ell-i,k-j}^R\left(\frac{R}{\mathcal{J}}\right) + \sum_{i=1}^{\ell}\sum_{j=0}^{k} \beta_{i,j}^R\left(\frac{R}{\mathcal{I}}\right)\beta_{\ell-i,k-j}^R\left(\frac{R}{\mathcal{J}'}\right).$$

Proof. Since $\mathcal{I}$ and $\mathcal{J}$ are generated by regular sequences, we have $\beta_i^R\left(\frac{R}{\mathcal{I}}\right) = \binom{m}{i}$ and $\beta_i^R\left(\frac{R}{\mathcal{J}}\right) = \binom{n}{i}$.

We note that $\mathrm{Cone}(\Omega)^\natural \cong (\mathcal{X} *_R \mathcal{Y})^\natural \oplus (\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})^\natural \oplus (\mathcal{X} \otimes_R \mathcal{T}_{\geq 1})^\natural$ as R-modules. Combining this observation with the fact that $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ resolves $\mathcal{I}' \otimes_R \frac{R}{\mathcal{J}}$ and $\Sigma^{-1}(\mathcal{X} \otimes_R \mathcal{T}_{\geq 1})$ resolves $\frac{R}{\mathcal{I}} \otimes_R \mathcal{J}'$, we have

$$\beta_\ell^R(F) = \beta_\ell^R\left(\frac{R}{\mathcal{I}\mathcal{J}}\right) + \beta_{\ell-1}^R\left(\mathcal{I}' \otimes_R \frac{R}{\mathcal{J}}\right) + \beta_{\ell-1}^R\left(\frac{R}{\mathcal{I}} \otimes_R \mathcal{J}'\right). \tag{3.5.4.2}$$

For $\ell = 0$, the last two terms vanish and the problem reduces to Corollary 3.3.5.

For the case of $\ell \geq 1$, we address each of three terms in Equation (3.5.4.2). For the first term, we combine Vandermonde's identity

$$\binom{m+n}{r} = \sum_{t=0}^{r} \binom{m}{t}\binom{n}{r-t}. \tag{3.5.4.3}$$

with Corollary 3.3.5 to get

$$\begin{aligned}\beta_\ell^R\left(\frac{R}{\mathcal{I}\mathcal{J}}\right) &\overset{3.3.5}{=} \sum_{i=1}^{\ell} \beta_i^R\left(\frac{R}{\mathcal{I}}\right)\beta_{\ell+1-i}^R\left(\frac{R}{\mathcal{J}}\right)\\ &= \sum_{i=0}^{\ell+1}\binom{m}{i}\binom{n}{\ell+1-i} - \binom{m}{0}\binom{n}{\ell+1} - \binom{m}{\ell+1}\binom{n}{0}\\ &\overset{3.5.4.3}{=} \binom{m+n}{\ell+1} - \binom{m}{\ell+1} - \binom{n}{\ell+1}.\end{aligned}$$

For the other two terms, we use the Tor-independence. Moreover, by construction we have

$$\begin{aligned}\beta_{\ell-1}^R\left(\mathcal{I}' \otimes_R \frac{R}{\mathcal{J}}\right) &= \sum_{i=0}^{\ell-1}\beta_i^R(\mathcal{I}')\beta_{\ell-1-i}^R\left(\frac{R}{\mathcal{J}}\right)\\ &= \sum_{i=0}^{\ell-1}\beta_{i+1}^R\left(\frac{R}{\mathcal{I}'}\right)\binom{n}{\ell-(i+1)}\\ &= \sum_{i=1}^{\ell}\beta_i^R\left(\frac{R}{\mathcal{I}'}\right)\binom{n}{\ell-i}.\end{aligned}$$

The same reasoning is used to obtain the desired formula $\beta_{\ell-1}^R\left(\frac{R}{\mathcal{I}} \otimes_R \mathcal{J}'\right)$. Substituting these into equation 3.5.4.2 yields equation 3.5.4.1, which is our desired result.

For the graded Betti numbers, we note that there is an analogous expression of equation 3.5.4.2. For $\ell > 0$, the argument follows the non-graded case using the formula

$$\beta_{\ell-1,k}^R\left(\mathcal{I}' \otimes_R \frac{R}{\mathcal{J}}\right) = \sum_{i=0}^{\ell-1}\sum_{j=0}^{k}\beta_{i,j}^R(\mathcal{I}')\beta_{\ell-1-i,k-j}^R\left(\frac{R}{\mathcal{J}}\right).$$

The case of $\ell = 0$ reduces to Corollary 3.3.5. □

We now obtain the Poincaré series portion of Theorem 3.1.1. In the case of fiber products, setting $\mathcal{I} = \langle \underline{x} \rangle$ and $\mathcal{J} = \langle \underline{y} \rangle$ with $R = k[\underline{x}, \underline{y}]$ (or its completion) recovers [35, Corollary 5.1.3].

Corollary 3.5.5. *The Poincaré series of $F := R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle$ satisfies the formulas below.*

$$\frac{P_F^R(t) - P_{R/(\mathcal{I}'+\mathcal{J})}^R(t) - P_{R/(\mathcal{I}+\mathcal{J}')}^R(t) + P_{R/(\mathcal{I}+\mathcal{J})}^R(t)}{P_{\mathcal{I}\mathcal{J}}^R(t)} = t(1+t) \tag{3.5.5.1}$$

$$\frac{P_F^R(t) - (1+t)^n P_{R/\mathcal{I}'}^R(t) - (1+t)^m P_{R/\mathcal{J}'}^R(t) + (1+t)^{m+n}}{((1+t)^m - 1)((1+t)^n - 1)} = \frac{t+1}{t}. \tag{3.5.5.2}$$

Proof. The unlabeled equalities in the following display are by definition.

$$\begin{aligned}
P^R_F(t) &= \mathbb{P}^R_{\mathcal{X}*_R\mathcal{Y}}(t) + \mathbb{P}^R_{\mathcal{S}_{\geq 1}\otimes_R\mathcal{Y}}(t) + \mathbb{P}^R_{\mathcal{X}\otimes_R\mathcal{T}_{\geq 1}}(t) \\
&= P^R_{R/\mathcal{I}\mathcal{J}}(t) + \mathbb{P}^R_{\mathcal{S}_{\geq 1}}(t)\mathbb{P}^R_{\mathcal{Y}}(t) + \mathbb{P}^R_{\mathcal{X}}(t)\mathbb{P}^R_{\mathcal{T}_{\geq 1}}(t) \\
&= 1 + t\cdot P^R_{\mathcal{I}\mathcal{J}}(t) + (P^R_{R/\mathcal{I}'}(t)-1)P^R_{R/\mathcal{J}}(t) + P^R_{R/\mathcal{I}}(t)(P^R_{R/\mathcal{J}'}(t)-1).
\end{aligned}$$

Rearranging and applying Corollary 3.3.6 yields

$$P^R_F(t) - P^R_{R/\mathcal{I}'}(t)P^R_{R/\mathcal{J}}(t) - P^R_{R/\mathcal{I}}(t)P^R_{R/\mathcal{J}'}(t) = 1 + t\cdot P^R_{\mathcal{I}\mathcal{J}}(t) - P^R_{R/\mathcal{J}}(t) - P^R_{R/\mathcal{I}}(t). \tag{3.5.5.3}$$

Our Tor-independence assumptions in Notation 3.4.1 guarantee the following equations

$$\begin{aligned}
P^R_{R/\mathcal{I}'}(t)P^R_{R/\mathcal{J}}(t) &= P^R_{R/(\mathcal{I}'+\mathcal{J})}(t) \\
P^R_{R/\mathcal{I}}(t)P^R_{R/\mathcal{J}}(t) &= P^R_{R/(\mathcal{I}+\mathcal{J}')}(t) \\
P^R_{R/\mathcal{I}}(t)P^R_{R/\mathcal{J}}(t) &= P^R_{R/(\mathcal{I}+\mathcal{J})}(t) \\
t^2\cdot P^R_{\mathcal{I}\mathcal{J}}(t) &= \left(P^R_{R/\mathcal{I}}(t)-1\right)\left(P^R_{R/\mathcal{J}}(t)-1\right).
\end{aligned} \tag{3.5.5.4}$$

Combining these four equations with equation 3.5.5.3, yields

$$P^R_F(t) - P^R_{R/(\mathcal{I}'+\mathcal{J})}(t) - P^R_{R/(\mathcal{I}+\mathcal{J}')}(t) = tP^R_{\mathcal{I}\mathcal{J}}(t) + t^2P^R_{\mathcal{I}\mathcal{J}}(t) - P^R_{R/(\mathcal{I}+\mathcal{J})}(t).$$

Isolating $t(1+t)$ on the right-hand side provides equation 3.5.5.1.

To get equation 3.5.5.2, recall that $\mathcal{I}$ and $\mathcal{J}$ are generated by a regular sequence and thus $\mathcal{X}$ and $\mathcal{Y}$ are Koszul complexes. It follows that $P^R_{R/\mathcal{I}}(t) = (1+t)^m$ and $P^R_{R/\mathcal{J}}(t) = (1+t)^n$. These expessions and equation 3.5.5.4 combine with equation 3.5.5.1 to produce equation 3.5.5.2. □

Lastly, we explicitly combine Example 3.4.2 with Theorem 3.5.3 to obtain Theorem 3.1.1.

Corollary 3.5.6. *Suppose the rings R, A, and B satisfy either case of Example 3.4.2. If $I' \subseteq \mathfrak{M}_A^2$ and $J' \subseteq \mathfrak{M}_B^2$, then* $\operatorname{Cone}(\Omega)$ *is a minimal free resolution over R of $\frac{A}{I'} \times_k \frac{B}{J'} \cong \frac{R}{\langle \mathcal{I}', \underline{xy}, \mathcal{J}' \rangle}$ where $\mathcal{I}' = I' \otimes_A R$ and $\mathcal{J}' = J' \otimes_B R$.*

Chapter 4

Differential Graded Algebra Resolutions from Tensor Products of Truncations

4.1 Introduction

Throughout the chapter, let $(R, \mathfrak{M}_R)$ be a local or standard graded ring. We work with free R-complexes possessing a multiplicative structure known as a commutative differential graded algebras (DG R-algebras or DG algebra for short) over R (see Definition 4.2.1). There is particular interest in demonstrating certain families of free resolutions always possess the structure of a DG R-algebra.

The existence of the DG structure allows for the utilization of a wider variety of homological tools in research. For an exposition on recent applications of DG structures one should refer to [38]. As a particular example, the work of [4, 5] shows how, under the right conditions, one can use DG structures to show that R satisfies a property known as "Tor-friendly."

It has been shown that any R-algebra can be resolved by a DG R-algebra [2, Proposition 6.1.4]. These constructions are often far from minimal, so one might question when minimal free resolutions possess the structure of a DG algebra. The answers turns out to be sometimes, but not always. Avramov gives examples of R-algebras whose minimal resolution cannot possess the structure of a DG R-algebra (see [2, Theorem 2.3.1]).

On the other hand, for minimal free resolutions of length one or two, one can always impose a DG algebra structure. Similarly, the minimal resolution of any Gorenstein ideal I of the local ring R with $\operatorname{pd}_R(R/I) = 4$ possesses the structure of a DG R-algebra [29, 30]. As one might expect, explicitly describing the multiplicative structure adds a layer of complexity to proving the existence of a DG structure. Often, to write down rules describing the multiplication table, one must restrict families of complexes, a sample of which is listed below.

- Hilbert-Burch Resolutions (see [7, 22])
- Buchsbaum-Eisenbud Resolutions (see [7, 8])
- Though not always a resolution, the Koszul Complex (see [7])
- Taylor Resolutions (see [6])

The main result of this chapter expands this list by adding another family of resolutions based off of the star-product from [16, 42]; see Corollary 4.6.9. The corollary follows from the theorem below.

Theorem 4.1.1. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are DG R-algebras. If for some positive integer n, $\mathcal{Y}$ is supported on the simplex Δ^{n-1} with simplicial multiplication then the binary operation*

$$\begin{aligned}(\alpha * f_\Omega)(\beta * f_\Gamma) =& \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial(\beta) * f_\Omega f_\Gamma \\ &+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\ &- \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\partial(\alpha)\,\beta * f_\Omega f_\Gamma.\end{aligned}$$

makes $\mathcal{X} *_R \mathcal{Y}$ a DG R-algebra.

We then use this theorem to reprove a result of Vandebogert [42, Corollary 4.6] and Herzog and Steurich [24]. We note that our proof differs from the existing ones due to our use of differential graded algebras.

Theorem 4.1.2. *Suppose the ideals $\mathcal{I}, \mathcal{J} \subset R$ satisfying the following;*

1. *the minimal free resolution of $R/\mathcal{I}$ over R possesses the structure of a DG R-algebra;*

2. *the quotient* $R/\mathcal{J}$ *is minimally resolved over* R *by either the Koszul complex or the Taylor resolution;*

3. *the modules* $\operatorname{Tor}_i^R(R/\mathcal{I}, R/\mathcal{J}) = 0$ *for* $i > 0$.

If $W = R/(\mathcal{I} + \mathcal{J})$, *then the fiber product* $\frac{R}{\mathcal{I}} \times_W \frac{R}{\mathcal{J}} \cong R/\mathcal{I}\mathcal{J}$ *is Golod.*

Section 2 of this chapter introduces the reader to DG algebras, indicator functions, and the star product. The indicator functions are used frequently throughout every section and drastically reduce the number of cases needed to prove various results.

Section 3 establishes definitions and results about complexes supported on simplicial resolutions. The focus of this section is defining and understanding the ℓth vertex removal map (see Definition 4.3.4).

Section 4 defines the binary operation found in Theorem 4.1.1. While doing this, we also fix notation to be used in the following sections We also prove the operation is unital, distributive, and graded commutative; see Theorem 4.4.4 for the first two and Theorem 4.4.5 for the last.

The first half of section 5 is dedicated to proving Equations (4.5.7.1) and (4.5.7.2). These equations are then used to prove that the binary operation from section 4 satisfies what is known as the Leibniz Rule; see Theorem 4.5.9.

The first half of section 6 is dedicated to proving Proposition 4.6.5. The equation in this proposition then paves the way for establishing Theorem 4.6.7. As an immediate corollary (Corollary 4.6.8), we obtain Theorem 4.1.1. We then use this theorem and the work of [2] to show that these fiber products are Golod, thereby establishing Theorem 4.1.2. As a consequence, we show that these fiber products are Tor-friendly.

4.2 Notation and Background

Throughout the chapter, we predominantly work with the R-complexes $\mathcal{X}$ and $\mathcal{Y}$. At times we will want to work with (hard) truncations of these complexes. We write $\mathcal{X}_{\geq p}$ for the (hard) truncation of $\mathcal{X}$ in degrees greater than or equal to p. This complex is given by

$$(\mathcal{X}_{\geq p})_q = \begin{cases} \mathcal{X}_q & q \geq p \\ 0 & q < p \end{cases} \qquad \text{and} \qquad \partial_q^{\mathcal{X}_{\geq p}} = \begin{cases} \partial_q^{\mathcal{X}} & q > p \\ 0 & q \leq p \end{cases}$$

Definition 4.2.1 ([7, Definition 5.1])**.** A **commutative differential graded algebra over** R (DG algebra for short) is a positively graded R-complex $\mathcal{X}$ equipped with a binary operation $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ called the product and written $(\alpha, \beta) \mapsto \alpha\beta$, that is unital, distributive, associative, and satisfies the following properties:

- **graded commutative:** for all $\alpha, \beta \in \mathcal{X}$ we have $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha \in \mathcal{X}_{|\alpha|+|\beta|}$ and $\alpha^2 = 0$ if $|\alpha|$ is odd;
- **Leibniz rule:** for all $\alpha, \beta \in \mathcal{X}$, we have $\partial^{\mathcal{X}}(\alpha\beta) = \partial^{\mathcal{X}}(\alpha)\,\beta + (-1)^{|\alpha|}\alpha\partial^{\mathcal{X}}(\beta)$.

To establish the results of this chapter, we utilize indicator functions $\mathbf{1}_{[*]}$ established in [16]. These functions return 1 if the input is true and 0 if it is false. We establish several arithmetic properties of these functions in order to properly utilize them in later results. We omit the proof of the following lemma since one can quickly check each equality by considering cases.

Lemma 4.2.2 ([16, Lemma 2.3])**.** *Let W_1 and W_2 be statements that can be evaluated as true or false and let W_1^c represent "not W_1", then the are following hold.*

(a) $1 = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_1^c]}$

(b) $\mathbf{1}_{[W_1 \cap W_2]} = \mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]}$

(c) $\mathbf{1}_{[W_1 \cup W_2]} = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_2]} - \mathbf{1}_{[W_1 \cap W_2]}$

(d) *If W_1 implies W_2, then $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = \mathbf{1}_{[W_1]}$, and we say that $\mathbf{1}_{[W_2]}$ is redundant.*

(e) *If W_1 implies W_2^c, then $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = 0$.*

(f) *If W_1 is a logical statement such that W_1 implies $x = x_0$, then for any function f with x_0 in the domain of f, we have $\mathbf{1}_{[W_1]}f(x) = \mathbf{1}_{[W_1]}f(x_0)$.*

The main construction of interest is the *star product* of two complexes. This definition was established simultaneously but independently in [16] and [42].

Construction 4.2.3 ([16,42])**.** *The* **Star Product** *of $\mathcal{X}$ and $\mathcal{Y}$, denoted $\mathcal{X} * \mathcal{Y}$, is the chain complex* *given by*

$$(\mathcal{X} * \mathcal{Y})_n = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{n+1} & n \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & n = 0 \\ 0 & n < 0 \end{cases} \quad \text{and} \quad \partial_n^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{n+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & n \geq 2 \\ \partial_1^{\mathcal{X}} \otimes_R \partial_1^{\mathcal{Y}} & n = 1 \\ 0 & n \leq 0 \end{cases} .$$

For computational purposes, we will want to rewrite the definition $\partial^{\mathcal{X} *_R \mathcal{Y}}$. To do this, we utilize indicator functions to obtain the following lemma.

Lemma 4.2.4 ([16, Lemma 3.3])**.** *The differential $\partial^{\mathcal{X} * \mathcal{Y}}$ acts on $\mathcal{X} *_R \mathcal{Y}$ by*

$$\partial^{\mathcal{X} * \mathcal{Y}}(a * b) = \mathbf{1}_{[|a|>1]} \partial^{\mathcal{X}}(a) * b + \mathbf{1}_{[|b|>1]} (-1)^{|a|} a * \partial^{\mathcal{Y}}(b) + \mathbf{1}_{[|a|,|b|=1]} \partial^{\mathcal{X}}(a) * \partial^{\mathcal{Y}}(b) .$$

Moving forward, we drop the superscript on ∂ when it should be understood from the context. This greatly reduces the space occupied by the numerous equations throughout this chapter, especially in the sections dedicated to the Leibniz Rule and Associativity.

The authors of [16,42] both used Construction 3.3.1 to establish similar versions the following Theorem. We will use this Theorem along with Corollary 4.6.8 to answer a question from [42, Question 5.7].

Theorem 4.2.5 ([16,42])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be free resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$, respectively, over R. If $\operatorname{Tor}_i^R(R/\mathcal{I}, R/\mathcal{J}) = 0$ for all $i > 0$, then the star product $\mathcal{X} *_R \mathcal{Y}$ is a free resolution of $R/\mathcal{I}\mathcal{J}$ over R.*

4.3 Simplicial Resolutions

Definition 4.3.1. Given a set of vertices $V = \{v_1, \ldots, v_n\}$ consider its power set 2^V. We say the $\Delta \subseteq 2^V$ is a **simplicial complex** if

1. $\{v_i\} \in \Delta$ for all i,
2. if $\Omega \in \Delta$ and $\Omega' \subset \Omega$, then $\Omega' \in \Delta$.

Moreover, if $\Delta = 2^V$ then we say it is an $(n-1)$**-simplex** and write Δ^{n-1}.

If a free R-complex $\mathcal{Y}$ has basis $\{f_\Omega\}_{\Omega\in\Delta}$ where the homological degree of f_Ω is equal to cardinality of Ω (i.e., $|f_\Omega| = |\Omega|$), we say $\mathcal{Y}$ is **supported on the simplicial complex** Δ. When $\mathcal{Y}$ is a free resolution with this property, we simply say it is a **Simplicial Resolution**.

Remark 4.3.2. Throughout this chapter we will work with arbitrary faces $\Omega \in \Delta$. To avoid the need of double subscripts, i.e. $\Omega = \{v_{i_1}, \ldots, v_{i_\ell}\}$, we refer to the vertices via their enumeration, i.e. $V = \{1, 2, 3, \ldots, n\}$. Along with removing the need for double subscripts, this practice puts a natural ordering on the vertices of which we use frequently. Thus, when we work with the vertices of a face $\Omega \in \Delta$ we write

$$\Omega = \{\omega_1, \ldots, \omega_{|\Omega|}\} \subseteq \{1, 2, \ldots, n\}$$

with the ordering $\omega_i < \omega_j$ if $i < j$.

A natural consequence of $\mathcal{Y}$ being supported on Δ is seen in the differential:

$$\partial^{\mathcal{Y}}(f_\Omega) = \sum_{i=1}^{|\Omega|} c(\Omega, \Omega\backslash\{\omega_i\}) f_{\Omega\backslash\{\omega_i\}}$$

with $c(\Omega, \Omega\backslash\{\omega_i\}) \in R$. In order for $\partial^{\mathcal{Y}} \circ \partial^{\mathcal{Y}} = 0$, given any pair $\omega_i, \omega_j \in \Omega$, the coefficients of the differential must satisfy

$$0 = c(\Omega, \Omega\backslash\{\omega_i\})c(\Omega\backslash\{\omega_i\}, \Omega\backslash\{\omega_i, \omega_j\}) + c(\Omega, \Omega\backslash\{\omega_j\})c(\Omega\backslash\{\omega_j\}, \Omega\backslash\{\omega_i, \omega_j\}).$$

Example 4.3.3. Consider the ideal $I = (m_1, \ldots, m_n) \subseteq R$. The following complexes both have $c(\Omega, \Omega\backslash\{\omega_i\}) = (-1)^{i+1}\frac{m_\Omega}{m_{\Omega\backslash\{\omega_i\}}}$ but use different definitions for the term m_Ω given below.

1. For $\Omega \in \Delta$, the Koszul complex $K^R(I)$ uses $m_\Omega = \prod_{\omega\in\Omega} m_\omega$.
2. Supposing each m_i is a monomial, the Taylor Resolution $T^R(I)$ uses $m_\Omega = \operatorname{lcm}\{m_\omega : \omega \in \Omega\}$ (see [33, Theorem 3.4]).

We now define the ℓ the vertex removal map. The motivation is to take any face $\Omega \in \Delta$ and delete the ℓth vertex ω_ℓ of Ω for $1 \le \ell \le |\Omega|$. For example, if $\Omega = \{1, 2, 4, 7\}$ then the 4th vertex removal deletes the 7 from the set, i.e. $\Omega\backslash\{\omega_4\} = \{1, 2, 4\}$. We extended this idea to a linear map on the complex $\mathcal{Y}$ via the elements $\{f_\Omega\}_{\Omega\in\Delta}$ in such a way that it interacts well with the expression $\partial\,(f_\Omega)$.

Definition 4.3.4. Let $\mathcal{Y}$ be a complex supported on Δ with n vertices. For all $1 \leq t \leq n$ and $1 \leq \ell \leq t$ we define the **ℓth vertex removal** as the R-linear map $\mathrm{P}_\ell(\cdot) : \mathcal{Y}_t \to \mathcal{Y}_{t-1}$ whose action on the basis elements, f_Ω, is given by

$$\mathrm{P}_\ell(f_\Omega) = \mathrm{proj}_{f_{\Omega \backslash \{\omega_\ell\}}}\left(\partial^{\mathcal{Y}}(f_\Omega)\right) = c\left(\Omega, \Omega\backslash\{\omega_\ell\}\right) f_{\Omega\backslash\{\omega_\ell\}}.$$

In particular, we have

$$\partial^{\mathcal{Y}}(f_\Omega) = \sum_{\Omega' \subset \Omega} c(\Omega, \Omega') f_{\Omega'} = \sum_{\ell=1}^{|\Omega|} \mathrm{P}_\ell(f_\Omega)\,.$$

Lemma 4.3.5. *Suppose $\mathcal{Y}$ is supported on Δ and consider the face $\Omega \in \Delta$. If $1 \leq \ell < k \leq |\Omega|$, then*

$$\mathrm{P}_\ell(\mathrm{P}_k(f_\Omega)) = -\,\mathrm{P}_{k-1}(\mathrm{P}_\ell(f_\Omega))\,.$$

Proof. Since $\ell < k$, we have $\omega_\ell < \omega_k$, thus making ω_k the $(k-1)^{\text{th}}$ vertex of $\Omega\backslash\{\omega_\ell\}$, which implies

$$\mathrm{P}_{k-1}\big(f_{\Omega\backslash\{\omega_\ell\}}\big) = c(\Omega\backslash\{\omega_\ell\}, \Omega\backslash\{\omega_\ell, \omega_k\}) f_{\Omega\backslash\{\omega_\ell,\omega_k\}}.$$

Using the R-linearity of $\mathrm{P}_\ell(\cdot)$ along with the fact that

$$0 = c(\Omega, \Omega\backslash\{\omega_k\}) c(\Omega\backslash\{\omega_k\}, \Omega\backslash\{\omega_\ell, \omega_k\}) + c(\Omega, \Omega\backslash\{\omega_\ell\}) c(\Omega\backslash\{\omega_\ell\}, \Omega\backslash\{\omega_\ell, \omega_k\})$$

we obtain the following expression.

$$\begin{aligned}
\mathrm{P}_\ell(\mathrm{P}_k(f_\Omega)) &= c(\Omega, \Omega\backslash\{\omega_k\})\,\mathrm{P}_\ell\big(f_{\Omega\backslash\{\omega_k\}}\big) \\
&= c(\Omega, \Omega\backslash\{\omega_k\}) c(\Omega\backslash\{\omega_k\}, \Omega\backslash\{\omega_\ell, \omega_k\}) f_{\Omega\backslash\{\omega_\ell,\omega_k\}} \\
&= -c(\Omega, \Omega\backslash\{\omega_\ell\}) c(\Omega\backslash\{\omega_\ell\}, \Omega\backslash\{\omega_\ell, \omega_k\}) f_{\Omega\backslash\{\omega_\ell,\omega_k\}} \\
&= -c(\Omega, \Omega\backslash\{\omega_\ell\})\,\mathrm{P}_\ell\big(f_{\Omega\backslash\{\omega_k\}}\big) \\
&= -\,\mathrm{P}_{k-1}(\mathrm{P}_\ell(f_\Omega))
\end{aligned}$$

□

Definition 4.3.6. Suppose $\mathcal{Y}$ is a DG algebra and is supported on Δ. We say $\mathcal{Y}$ has **Simplicial Multiplication** if given any two faces $\Omega, \Gamma \in \Delta$ where $\Omega \cup \Gamma \in \Delta$ we have $f_\Omega f_\Gamma \in \mathrm{span}_R\left(f_{\Omega\cup\Gamma}\right)$.

Examples of complexes with simplicial multiplication include Koszul complexes and Taylor resolutions [14]. Moreover, squarefree monomial ideals that are minimally resolved by the scarf complex have a unique DG algebra structure with simplicial multiplication [28, Proposition 3.1].

Lemma 4.3.7. *Suppose $\mathcal{Y}$ is a DG algebra supported on Δ and has simplicial multiplication. If $\Omega, \Gamma \in \Delta$ such that $\Omega \cup \Gamma \in \Delta$ and $\Omega \cap \Gamma \neq \emptyset$, then $f_\Omega f_\Gamma = 0$.*

Proof. Suppose for contradiction that $f_\Omega f_\Gamma \neq 0$. Then the graded commutatihave of $\mathcal{Y}$ tells us $|f_\Omega f_\Gamma| = |f_\Omega| + |f_\Gamma|$. By simplicial multiplication and $\Omega \cup \Gamma \in \Delta$, we must have $f_\Omega f_\Gamma \in \operatorname{span}_R(f_{\Omega\cup\Gamma})$ and thus $|f_\Omega f_\Gamma| = |f_{\Omega\cup\Gamma}|$. Combining these facts, we obtain

$$|\Omega| + |\Gamma| = |f_\Omega| + |f_\Gamma| = |f_{\Omega\cup\Gamma}| = |\Omega \cup \Gamma|.$$

However, this is a contradiction since $\Omega \cap \Gamma \neq \emptyset$ implies that $|\Omega \cup \Gamma| < |\Omega| + |\Gamma|$. Thus, we must conclude $f_\Omega f_\Gamma = 0$. □

Lemma 4.3.8. *Suppose $\mathcal{Y}$ is supported on the simplex Δ with simplicial multiplication. Let $\Omega, \Gamma \in \Delta$ such that they have the same lead vertex, i.e., $\omega_1 = \gamma_1$, then f_Ω and f_Γ satisfy the following relations.*

(a) $\mathrm{P}_1(f_\Omega)\, f_\Gamma = (-1)^{|\Omega|-1} f_\Omega\, \mathrm{P}_1(f_\Gamma)$

(b) $\mathrm{P}_1(f_\Omega)\, \partial(f_\Gamma) = \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma) + (-1)^{|\Omega|} f_\Omega \partial(\mathrm{P}_1(f_\Omega))$

Proof. We start by proving (a). Since $\omega_1 = \gamma_1$, Lemma 4.3.7 tells us that $f_\Omega f_\Gamma = 0$ and $f_{\Omega\setminus\{\omega_i\}} f_\Gamma = 0$ for all $2 \leq i \leq |\Omega|$. It follows by definition of $\mathrm{P}_i(f_\Omega)$ that we then $\mathrm{P}_i(f_\Omega)\, f_\Gamma = 0$ for all $2 \leq i \leq |\Omega|$. Similar reasoning tells us that $f_\Omega\, \mathrm{P}_i(f_\Gamma) = 0$ for all $2 \leq i \leq |\Gamma|$. We now deduce the following.

$$\begin{aligned}
0 &= \partial(f_\Omega f_\Gamma) \\
&= \partial(f_\Omega)\, f_\Gamma + (-1)^{|\Omega|} f_\Omega \partial(f_\Gamma) \\
&= \sum_{i=1}^{|\Omega|} \mathrm{P}_i(f_\Omega)\, f_\Gamma + (-1)^{|\Omega|} \sum_{j=1}^{|\Gamma|} f_\Omega\, \mathrm{P}_j(f_\Gamma) \\
&= \mathrm{P}_1(f_\Omega)\, f_\Gamma + (-1)^{|\Omega|} f_\Omega\, \mathrm{P}_1(f_\Gamma)\,.
\end{aligned}$$

Rearranging yields the desired equation.

We now prove (b) using (a). For $2 \leq j \leq |\Gamma|$, we observe that

$$\mathrm{P}_1(f_\Omega)\,\mathrm{P}_j(f_\Gamma) \overset{(a)}{=} (-1)^{|\Omega|-1} f_\Omega\,\mathrm{P}_1(\mathrm{P}_j(f_\Gamma)) \overset{4.3.5}{=} (-1)^{|\Omega|} f_\Omega\,\mathrm{P}_{j-1}(\mathrm{P}_1(f_\Gamma))\,.$$

We then use this equality to obtain the following.

$$\begin{aligned}
\mathrm{P}_1(f_\Omega)\,\partial\,(f_\Gamma) &= \sum_{j=1}^{|\Gamma|} \mathrm{P}_1(f_\Omega)\,\mathrm{P}_j(f_\Gamma) \\
&= \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) + (-1)^{|\Omega|} \sum_{j=2}^{|\Gamma|} f_\Omega\,\mathrm{P}_{j-1}(\mathrm{P}_1(f_\Gamma)) \\
&= \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) + (-1)^{|\Omega|} \sum_{k=1}^{|\Gamma|-1} f_\Omega\,\mathrm{P}_k(\mathrm{P}_1(f_\Gamma)) \\
&= \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) + (-1)^{|\Omega|} f_\Omega \partial\,(\mathrm{P}_1(f_\Gamma))
\end{aligned}$$

□

Lemma 4.3.9. *Suppose $\mathcal{Y}$ is supported on the simplex Δ and has simplicial multiplication. If $\Omega, \Gamma \in \Delta$ with $\Omega \cup \Gamma \in \Delta$, then*

$$\mathrm{P}_1(f_\Omega f_\Gamma) = \mathbf{1}_{[\omega_1 < \gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\gamma_1 < \omega_1]}(-1)^{|\Omega|} f_\Omega\,\mathrm{P}_1(f_\Gamma)\,.$$

Proof. We start by noting that if $(\Omega \backslash \{\omega_1\}) \cap (\Gamma \backslash \{\gamma_1\}) \neq \emptyset$ then simplicial multiplication tells

$$0 = f_\Omega f_\Gamma = \mathrm{P}_1(f_\Omega)\,f_\Gamma = f_\Gamma\,\mathrm{P}_1(f_\Gamma)\,.$$

This causes both sides of the desired equation to vanish, so there is nothing to show.

If we now suppose $(\Omega \backslash \{\omega_1\}) \cap (\Gamma \backslash \{\gamma_1\}) = \emptyset$ but $\omega_1 \in \Gamma \backslash \{\gamma_1\}$, then these exists some $2 \leq j \leq |\Gamma|$ such that $\omega_1 = \gamma_j > \gamma_1$. Thus, we have

$$\mathbf{1}_{[\omega_1 < \gamma_1]} = 0 = f_\Omega f_\Gamma = f_\Omega\,\mathrm{P}_1(f_\Gamma)$$

which, again, causes both sides of the desired equation to vanish. Similarly, we can show that everything vanishes if $(\Omega \backslash \{\omega_1\}) \cap (\Gamma \backslash \{\gamma_1\}) = \emptyset$ and $\gamma_1 \in \Omega \backslash \{\omega_1\}$. This reasoning also applies to the case of $\omega_1 = \gamma_1$.

For our final case, we suppose $\Omega \cap \Gamma = \emptyset$. In this case, $f_\Omega f_\Gamma$ is a nonzero multiple of $f_{\Omega\cup\Gamma}$. It then follows from R-linearity that $\mathrm{P}_1(f_\Omega f_\Gamma)$ is then a nonzero multiple of $\mathrm{P}_1(f_{\Omega\cup\Gamma})$. Since ω_1 and γ_1 are first vertices of their respective faces, we have $\min\{\omega_1, \gamma_1\}$ is the first vertex of $\Omega \cup \Gamma$. When working with the indicator function $\mathbf{1}_{[\omega_1<\gamma_1]}$, we see that $\omega_1 = \min\{\omega_1, \gamma_1\}$ and thus

$$\mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_{\Omega\cup\Gamma}) \in \mathrm{span}_R\left(f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}\right).$$

From here, we note that $0 = \mathrm{proj}_{f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}}\left(f_\Omega \partial\left(f_\Gamma\right)\right)$ which allows us to compute the following.

$$\begin{aligned}
\mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) &= \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{proj}_{f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}}\left(\partial\left(f_\Omega f_\Gamma\right)\right) \\
&= \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{proj}_{f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}}\left(\partial\left(f_\Omega\right) f_\Gamma\right) \\
&\quad + \mathbf{1}_{[\omega_1<\gamma_1]}(-1)^{|\Omega|}\,\mathrm{proj}_{f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}}\left(f_\Omega \partial\left(f_\Gamma\right)\right) \\
&= \mathbf{1}_{[\omega_1<\gamma_1]} \sum_{i=1}^{|\Omega|} \mathrm{proj}_{f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}}\left(\mathrm{P}_i(f_\Omega)\, f_\Gamma\right) \\
&= \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\, f_\Gamma
\end{aligned}$$

Similar reasoning provides us with $\mathbf{1}_{[\gamma_1<\omega_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) = \mathbf{1}_{[\gamma_1<\omega_1]}(-1)^{|\Omega|} f_\Omega\,\mathrm{P}_1(f_\Gamma)$. From here we finally conclude the following.

$$\begin{aligned}
\mathrm{P}_1(f_\Omega f_\Gamma) &\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) + \mathbf{1}_{[\omega_1=\gamma_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) + \mathbf{1}_{[\gamma_1<\omega_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) \\
&\overset{3.2.3(f)}{=} \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\, f_\Gamma + \mathbf{1}_{[\gamma_1<\omega_1]}(-1)^{|\Omega|} f_\Omega\,\mathrm{P}_1(f_\Gamma)
\end{aligned}$$

□

4.4 A Binary Operation on the Star Product

Notation 4.4.1. For the rest of the chapter, we fix the following notation.

- The elements $\alpha, \beta \in \mathcal{X}$ have degrees $a := |\alpha|$ and $b := |\beta|$.
- For $\Omega, \Gamma \in \Delta$, we have $u := |\Omega| = |f_\Omega|$ and $v = |\Gamma| = |f_\Gamma|$.

Definition 4.4.2. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are DG R-algebras. Further suppose $\mathcal{Y}$ is supported on a simplicial complex Δ with a fixed ordering on its vertices. For $\alpha * f_\Omega, \beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$, we define their product with the following expression:

$$\begin{aligned}(\alpha * f_\Omega)(\beta * f_\Gamma) = &\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &- \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial\,(\beta) * f_\Omega f_\Gamma \\ &+ \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}(-1)^{(u-1)b}\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\ &- \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\partial\,(\alpha)\,\beta * f_\Omega f_\Gamma.\end{aligned}$$

In the case of $\Omega = \{\omega_1\}$, we do not have an ω_2 to use with the indicator $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}$. To accomodate this case when ω_2 does not exist, we adopt the convention $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]} = \mathbf{1}_{[\omega_1 \leq \gamma_1]}$ and $\mathbf{1}_{[\gamma_1 < \omega_2]} = 1$. In the case that $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_0$ and $\beta * f_\Gamma \in \mathcal{X} *_R \mathcal{Y}$, the product is given by

$$(\alpha * f_\Omega)(\beta * f_\Gamma) = (\beta * f_\Gamma)(\alpha * f_\Omega) = \alpha\beta * f_\Omega f_\Gamma.$$

Remark 4.4.3. While the presence of these indiciator functions may seem excessive and convoluted, they are essential for the following proof to be manageable. Without the indicator functions, the naive case-by-case approach to prove just associativity of the stated product would take 720 cases; the indicator functions allow us to reduce the number of cases down to just one case.

Theorem 4.4.4. *The product from Definition 4.4.2 is unital and distributive.*

Proof. It immediately follows from Definition 4.4.2 that $1_{\mathcal{X} *_R \mathcal{Y}} = 1_R * 1_R$, and thus the product is unital. The distrubitive property is almost immediate; the product is defined for elements of the form $\alpha * f_\Omega$ and extended R-linearly so we have

$$(\alpha * f_\Omega + \beta * f_\Gamma)(\rho * f_\Lambda) := (\alpha * f_\Omega)(\rho * f_\Lambda) + (\beta * f_\Gamma)(\rho * f_\Lambda).$$

Thus the question of distributivity reduces to a question on whether or not the product respects linearity in the $\mathcal{X}$ term, i.e., if the following holds.

$$\begin{aligned}(\alpha * f_\Omega + \beta * f_\Omega)(\rho * f_\Lambda) &= ((\alpha + \beta) * f_\Omega)(\rho * f_\Lambda) \\ &= (\alpha * f_\Omega)(\rho * f_\Lambda) + (\beta * f_\Omega)(\rho * f_\Lambda)\end{aligned}$$

The first equality holds by definition of the star product, see Construction 3.3.1. The second equality also follows by the definition of the star product after noting $\partial(\alpha+\beta) = \partial(\alpha) + \partial(\beta)$. $\square$

Theorem 4.4.5. *Let $\mathcal{X}$ be a DG-algebra and $\mathcal{Y}$ a Koszul complex or Taylor Resolution. The product given in Definition 4.4.2 is graded commutative.*

Proof. We start by showing that $(\alpha * f_\Omega)(\beta * f_\Gamma) \in (\mathcal{X} *_R \mathcal{Y})_{|\alpha * f_\Omega| + |\beta * f_\Gamma|}$. This amounts to showing that each term on the right-hand side of Definition 4.4.2 has degree $|\alpha * f_\Omega| + |\beta * f_\Gamma|$. We show this for the term $\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma$ and note that the same reasoning can be used to prove the other three terms in the product have the correct degree.

$$\begin{aligned}
|\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma| &= |\alpha\beta| + |\,\mathrm{P}_1(f_\Omega)\, f_\Gamma| - 1 \\
&= |\alpha| + |\beta| + |\,\mathrm{P}_1(f_\Omega)\,| + |f_\Gamma| - 1 \\
&= |\alpha| + |\beta| + |f_\Omega| - 1 + |f_\Gamma| - 1 \\
&= (|\alpha| + |f_\Omega| - 1) + (|\beta| + |f_\Gamma| - 1) \\
&= |\alpha * f_\Omega| + |\beta * f_\Gamma|
\end{aligned}$$

Since we show the other terms in Definition 4.4.2 have the same degree, we can conclude that $(\alpha * f_\Omega)(\beta * f_\Gamma) \in (\mathcal{X} *_R \mathcal{Y})_{|\alpha * f_\Omega| + |\beta * f_\Gamma|}$.

We now show this product satisfies graded commutativity. To do this, we treat two separate cases; $\omega_1 = \gamma_1$ and $\omega_1 \neq \gamma_1$. In the former case, we immediately have the following calculation where the starred equality follows from the graded commutativity of $\mathcal{X}$ and $\mathcal{Y}$.

$$\begin{aligned}
(\alpha * f_\Omega)(\beta * f_\Gamma) &\overset{4.4.2}{=} (-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
&\overset{4.3.8(a)}{=} (-1)^{(u-1)b}\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\
&\overset{*}{=} (-1)^{(u-1)b+ab+u(v-1)}\beta\alpha * \mathrm{P}_1(f_\Gamma)\, f_\Omega \\
&= (-1)^{(a+u-1)(b+v-1)+(v-1)(a-1)}\beta\alpha * \mathrm{P}_1(f_\Gamma)\, f_\Omega \\
&\overset{4.4.2}{=} (-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|}(\beta * f_\Gamma)(\alpha * f_\Omega)
\end{aligned}$$

In the case of $\omega_1 \neq \gamma_1$, Lemma 3.2.3(b) allows us replace the indicator function $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}$ with $\mathbf{1}_{[\omega_1 < \gamma_1 < \omega_2]}$ in Definition 4.4.2. This observation, along with the graded commutativity of $\mathcal{X}$ and $\mathcal{Y}$, reduces this proof to "sign chasing." Part of this sign chasing utilizes the following relations.

1. $\mathbf{1}_{[a=1]}(-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|} \overset{3.2.3(f)}{=} \mathbf{1}_{[a=1]}(-1)^{u(b+v-1)}$

2. $\mathbf{1}_{[b=1]}(-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|} \overset{3.2.3(f)}{=} \mathbf{1}_{[b=1]}(-1)^{(a+u-1)v}$

We now show the product is graded commutative in the case of $\omega_1 \neq \gamma_1$.

$$\begin{aligned}
(\alpha * f_\Omega)(\beta * f_\Gamma) &\overset{4.4.2}{=} \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
&\quad - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial\,(\beta) * f_\Omega f_\Gamma \\
&\quad + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\
&\quad - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\partial\,(\alpha)\,\beta * f_\Omega f_\Gamma \\
&= \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)+ab+(u-1)v}\beta\alpha * f_\Gamma\, \mathrm{P}_1(f_\Omega) \\
&\quad - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}(-1)^{uv}\partial\,(\beta)\,\alpha * f_\Gamma f_\Omega \\
&\quad + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b+ab+u(v-1)}\beta\alpha * \mathrm{P}_1(f_\Gamma)\, f_\Omega \\
&\quad - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)+uv}\beta\partial\,(\alpha) * f_\Gamma f_\Omega \\
&\overset{4.4.2}{=} (-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|}(\beta * f_\Gamma)(\alpha * f_\Omega)
\end{aligned}$$

Finally, we need to show that $(\alpha * f_\Omega)^2 = 0$ whenever $|\alpha * f_\Omega|$ is odd. When applying Definition 4.4.2, we find that

$$(\alpha * f_\Omega)^2 = (-1)^{(u-1)(a-1)}\alpha^2 * \mathrm{P}_1(f_\Omega)\, f_\Omega.$$

If $|\Omega| > 1$, then we have $(\Omega\backslash\{\omega_1\}) \cap \Omega \neq \emptyset$ and thus $\mathrm{P}_1(f_\Omega)\, f_\Omega = 0$. This then causes $(\alpha * f_\Omega)^2 = 0$, regardless of the degree of the entire term.

In the case that $\Omega = \{\omega_1\}$, we have $u = |\Omega| = 1$ which yields $|\alpha * f_\Omega| = a + u - 1 = a$. The assumption that $|\alpha * f_\Omega|$ is odd then means that $a = |\alpha|$ is odd and thus $\alpha^2 = 0$. Thus we conclude that $(\alpha * f_\Omega)^2 = 0$ whenever $|\alpha * f_\Omega|$ is odd. □

4.5 Proof of the Leibniz Rule

The main goal of this section is show that the product in Definition 4.4.2 and the diffential on $\mathcal{X} *_R \mathcal{Y}$ interact "nicely." For us, this means showing that the product satisfies the Leibniz rule:

$$\partial^{\mathcal{X}*\mathcal{Y}}((\alpha * f_\Omega)(\beta * f_\Gamma)) = \partial^{\mathcal{X}*\mathcal{Y}}(\alpha * f_\Omega)\,(\beta * f_\Gamma) + (-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial^{\mathcal{X}*\mathcal{Y}}(\beta * f_\Gamma)\,.$$

Since the last section concluded by showing the product is graded commutative, we make the additional observation that

$$(\alpha * f_\Omega)\partial^{\mathcal{X}*\mathcal{Y}}(\beta * f_\Gamma) = (-1)^{|\alpha * f_\Omega|\cdot(|\beta * f_\Gamma|-1)}\partial^{\mathcal{X}*\mathcal{Y}}(\beta * f_\Gamma)\,(\alpha * f_\Omega).$$

Thus, our strategy is to use Definition 4.4.2 to construct a formula for the product

$$\begin{aligned}\partial^{\mathcal{X}*\mathcal{Y}}(\alpha * f_\Omega)\,(\beta * f_\Gamma) = \mathbf{1}_{[a>1]}(\partial\,(\alpha) * f_\Omega)(\beta * f_\Gamma) + \mathbf{1}_{[u>1]}(\alpha * \partial\,(f_\Omega))(\beta * f_\Gamma)\\ + \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial\,(\alpha)\,\beta * \partial\,(f_\Omega)\,f_\Gamma\end{aligned}$$

which we then use to prove the Leibniz rule (see Theorem 4.5.9). Communicating this result in an effective manner requires several intermediate equations.

We start by using Lemma 4.3.5 to establish several equations necessary for making sense of the expression $(\alpha * \partial\,(f_\Omega))(\beta * f_\Gamma)$. This is particularly important since the product in Definition 4.4.2 is defined using the depends on the basis elements $\{f_\Omega\}_{\Omega\in\Delta}$ and then extended linearly.

Lemma 4.5.1. *Given faces $\Omega, \Gamma \in \Delta$ we have the following expressions involving the basis elements f_Ω and f_Γ.*

$$\begin{aligned}-\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\partial\,(\mathrm{P}_1(f_\Omega))\,f_\Gamma = \mathbf{1}_{[\omega_2\leq\gamma_1<\omega_3]}\,\mathrm{P}_1(\mathrm{P}_1(f_\Omega))\,f_\Gamma + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_3]}\,\mathrm{P}_1(\mathrm{P}_2(f_\Omega))\,f_\Gamma\\ + \sum_{i=3}^{u}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\,\mathrm{P}_1(\mathrm{P}_i(f_\Omega))\,f_\Gamma\end{aligned} \tag{4.5.1.1}$$

$$-\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\omega_1<\gamma_1]}\partial\,(f_\Omega)\,f_\Gamma = \mathbf{1}_{[\omega_2<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \sum_{i=2}^{u}\mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_i(f_\Omega)\,f_\Gamma \tag{4.5.1.2}$$

$$(\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]} - \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]})\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\partial\,(f_\Omega)\,f_\Gamma \tag{4.5.1.3}$$
$$= \mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \sum_{i=2}^{u}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\,\mathrm{P}_i(f_\Omega)\,f_\Gamma$$

$$\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\gamma_1<\omega_1]}\partial\,(f_\Omega)\,f_\Gamma = \mathbf{1}_{[\gamma_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \sum_{i=2}^{u}\mathbf{1}_{[\gamma_1<\omega_1]}\,\mathrm{P}_i(f_\Omega)\,f_\Gamma \tag{4.5.1.4}$$

Proof. Recall that Definition 4.3.4 tells us that

$$\partial\,(f_\Omega) = \sum_{i=1}^{u}\mathrm{P}_i(f_\Omega)\,.$$

We make use of this equality in the proofs of Equations (4.5.1.2)-(4.5.1.4) and, for (4.5.1.1), note that it extends linearly to produce

$$\partial\,(\mathrm{P}_1(f_\Omega)) = \sum_{i=1}^{u-1}\mathrm{P}_i(\mathrm{P}_1(f_\Omega))\,.$$

To prove Equation (4.5.1.1), we observe that

$$\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_3]}\,\mathrm{P}_2(\mathrm{P}_1(f_\Omega))\,f_\Gamma \overset{4.3.5}{=} -\,\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_3]}\,\mathrm{P}_1(\mathrm{P}_1(f_\Omega))\,f_\Gamma$$
$$\overset{3.2.3(a)}{=} -\,\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\,\mathrm{P}_1(\mathrm{P}_1(f_\Omega))\,f_\Gamma - \mathbf{1}_{[\omega_2\leq\gamma_1<\omega_3]}\,\mathrm{P}_1(\mathrm{P}_1(f_\Omega))\,f_\Gamma.$$

We now use this equality and another application of Lemma 4.3.5 to obtain the following.

$$-\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\partial\,(\mathrm{P}_1(f_\Omega))\,f_\Gamma = -\,\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\partial\,(\mathrm{P}_1(f_\Omega))\,f_\Gamma - \sum_{i=2}^{u-1}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\,\mathrm{P}_i(\mathrm{P}_1(f_\Omega))\,f_\Gamma$$
$$\overset{4.3.5}{=}\mathbf{1}_{[\omega_2\leq\gamma_1<\omega_3]}\,\mathrm{P}_1(\mathrm{P}_1(f_\Omega))\,f_\Gamma + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_3]}\,\mathrm{P}_2(\mathrm{P}_1(f_\Omega))\,f_\Gamma$$
$$+\sum_{i=2}^{u-1}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\,\mathrm{P}_1(\mathrm{P}_{i+1}(f_\Omega))\,f_\Gamma$$

Reindexing the summation produces the desired formula.

For Equation (4.5.1.2), since $\omega_1 < \omega_2$ by definition, we have

$$\mathbf{1}_{[\omega_1<\gamma_1]} = \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]} + \mathbf{1}_{[\omega_2\leq\gamma_1]}.$$

By further noting that $\mathbf{1}_{[\omega_2=\gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma = 0$, we observe that

$$\mathbf{1}_{[\omega_2\le\gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma = \mathbf{1}_{[\omega_2<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma.$$

From this we observe the following.

$$-\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\omega_1<\gamma_1]}\partial\,(f_\Omega)\,f_\Gamma = \mathbf{1}_{[\omega_2<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma + \mathbf{1}_{[\omega_1<\gamma_1]}(\partial\,(f_\Omega)\,f_\Gamma - \mathrm{P}_1(f_\Omega)\,f_\Gamma)$$

The expression $\partial\,(f_\Omega)-\mathrm{P}_1(f_\Omega)$ produces the desired summation and so we recover Equation (4.5.1.2).

Equations (4.5.1.3) and (4.5.1.4) are established using the same methods though the latter uses the relation

$$\mathbf{1}_{[\gamma_1<\omega_2]} = \mathbf{1}_{[\gamma_1<\omega_1]} + \mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}$$

noting that $\mathbf{1}_{[\gamma_1<\omega_1]} = \mathbf{1}_{[\gamma_1<\omega_1<\omega_2]}$. □

We now use these equations to develop the following formulas.

Lemma 4.5.2. *We have the following three equalities.*

$$\begin{aligned}\mathbf{1}_{[a>1]}(\partial\,(\alpha) * f_\Omega)(\beta * f_\Gamma) = {} & \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\partial\,(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\,f_\Gamma \\ & - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[a>1]}\mathbf{1}_{[b=1]}\partial\,(\alpha)\,\partial\,(\beta) * f_\Omega f_\Gamma \\ & + \mathbf{1}_{[a>1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\partial\,(\alpha)\,\beta * f_\Omega\,\mathrm{P}_1(f_\Gamma)\end{aligned} \tag{4.5.2.1}$$

$$\begin{aligned}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial\,(\alpha)\,\beta * \partial\,(f_\Omega)\,f_\Gamma = {} & \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial\,(\alpha)\,\beta * \partial\,(f_\Omega)\,f_\Gamma \\ & + \mathbf{1}_{[\omega_1\le\gamma_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial\,(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\,f_\Gamma\end{aligned} \tag{4.5.2.2}$$

$$\begin{aligned}\mathbf{1}_{[u>1]}(-1)^a(\alpha * \partial\,(f_\Omega))(\beta * f_\Gamma) = {} & -\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{u(b-1)+a}\alpha\beta * \partial\,(\mathrm{P}_1(f_\Omega))\,f_\Gamma \\ & - \mathbf{1}_{[u>1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[\omega_1<\gamma_1]}(-1)^a\alpha\partial\,(\beta) * \partial\,(f_\Omega)\,f_\Gamma \\ & + \mathbf{1}_{[u>1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^a\alpha\partial\,(\beta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma \\ & + \mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{ub+a}\alpha\beta * \partial\,(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ & + \mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\end{aligned} \tag{4.5.2.3}$$

$$
\begin{aligned}
&- \mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\\
&+ \mathbf{1}_{[u>1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[\gamma_1<\omega_1]}(-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma\\
&+ \mathbf{1}_{[u>1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma
\end{aligned}
$$

Proof. Equation (4.5.2.1) follows directly from Definition 4.4.2 and noting that $\partial^2 = 0$. Equation (4.5.2.2) follows from the following set of equalities.

$$
\begin{aligned}
\mathbf{1}_{[u=1]}\partial(f_\Omega)\, f_\Gamma &\overset{3.2.3(a)}{=} \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[u=1]}\partial(f_\Omega)\, f_\Gamma + \mathbf{1}_{[\omega_1\le\gamma_1]}\mathbf{1}_{[u=1]}\partial(f_\Omega)\, f_\Gamma\\
&\overset{3.2.3(f)}{=} \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[u=1]}\partial(f_\Omega)\, f_\Gamma + \mathbf{1}_{[\omega_1\le\gamma_1]}\mathbf{1}_{[u=1]}\,\mathrm{P}_1(f_\Omega)\, f_\Gamma
\end{aligned}
$$

Finally, we turn our attention Equation (4.5.2.3). For spacing, we omit $\mathbf{1}_{[u>1]}(-1)^a$ and observe that the bilinearity of the star products yields the following equality

$$
(\alpha * \partial(f_\Omega))(\beta * f_\Gamma) = \sum_{i=1}^{u}(\alpha * \mathrm{P}_i(f_\Omega))(\beta * f_\Gamma).
$$

By definition, $\mathrm{P}_i(f_\Omega)$ differs from $f_{\Omega\setminus\{\omega_i\}}$ by a non-zero constant. Thus, $(\alpha * \mathrm{P}_i(f_\Omega))(\beta * f_\Gamma)$ differs from $(\alpha * f_{\Omega\setminus\{\omega_i\}})(\beta * f_\Gamma)$ by the same non-zero constant in R. This allows us to apply Definition 4.4.2 directly to each product in the summation. For $i \ge 3$, the deleted vertex ω_i does not appear in any of the indicator functions and thus we have

$$
\begin{aligned}
(\alpha * \mathrm{P}_i(f_\Omega))(\beta * f_\Gamma) =& \mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{(u-2)(b-1)}\alpha\beta * \mathrm{P}_1(\mathrm{P}_i(f_\Omega))\, f_\Gamma\\
&- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial(\beta) * \mathrm{P}_i(f_\Omega)\, f_\Gamma\\
&+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-2)b}\alpha\beta * \mathrm{P}_i(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\\
&- \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \mathrm{P}_i(f_\Omega)\, f_\Gamma.
\end{aligned}
$$

In the case of $i = 2$, we find that the vertex ω_2 has been deleted from Ω. Thus, ω_3 is now the second vertex and we have

$$
\begin{aligned}
(\alpha * \mathrm{P}_2(f_\Omega))(\beta * f_\Gamma) =& \mathbf{1}_{[\omega_1\le\gamma_1<\omega_3]}(-1)^{(u-2)(b-1)}\alpha\beta * \mathrm{P}_1(\mathrm{P}_2(f_\Omega))\, f_\Gamma\\
&- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial(\beta) * \mathrm{P}_2(f_\Omega)\, f_\Gamma\\
&+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-2)b}\alpha\beta * \mathrm{P}_2(f_\Omega)\,\mathrm{P}_1(f_\Gamma)
\end{aligned}
$$

$$- \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u-1)(b-1)}\partial\left(\alpha\right)\beta * \mathrm{P}_2(f_\Omega)\, f_\Gamma.$$

Similar reasoning tells us that when $i = 1$, ω_2 and ω_3 are, respectively, the first and second vertices in $\Omega\backslash\{\omega_1\}$. It follows that

$$\begin{aligned}(\alpha * \mathrm{P}_1(f_\Omega))(\beta * f_\Gamma) =& \mathbf{1}_{[\omega_2\leq\gamma_1<\omega_3]}(-1)^{(u-2)(b-1)}\alpha\beta * \mathrm{P}_1(\mathrm{P}_1(f_\Omega))\, f_\Gamma \\ &- \mathbf{1}_{[\omega_2<\gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial\left(\beta\right) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &+ \mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}(-1)^{(u-2)b}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ &- \mathbf{1}_{[\gamma_1<\omega_2]}\mathbf{1}_{[a=1]}(-1)^{(u-1)(b-1)}\partial\left(\alpha\right)\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma.\end{aligned}$$

When we sum these expressions together we encounter the left-hand side of the four equations from Lemma 4.5.1. For example, summing the first line from each of the above equations yields Equation (4.5.1.1) with $(-1)^{(u-2)(b-1)}\alpha\beta * \mathrm{P}_1(\mathrm{P}_i(f_\Omega))\, f_\Gamma$ in place of $\mathrm{P}_1(\mathrm{P}_i(f_\Omega))\, f_\Gamma$. Similarly, the second line corresponds to (4.5.1.2), the third with (4.5.1.3), and the fourth with (4.5.1.4) noting that Lemma 3.2.3(f) gives us $-\mathbf{1}_{[a=1]}(-1)^a = \mathbf{1}_{[a=1]}$. Further noting that $(-1)^{u-2} = (-1)^u$ gives us the desired coefficients, thereby recovering Equation (4.5.2.3). □

While we have sufficient information to produce a formula for $\partial\left(\alpha * f_\Omega\right)(\beta * f_\Gamma)$, we postpone that result until after the following set of lemmas. These lemmas will allow to produce a version of the formula that is more easily identified with terms coming from $\partial\left((\alpha * f_\Omega)(\beta * f_\Gamma)\right)$ as defined in Definition 4.4.2.

Lemma 4.5.3. *We have the following equations related to $\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma$ where α, β, f_Ω, and f_Γ all have positive degree.*

$$\begin{aligned}&\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\partial\left(\alpha\right)\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \qquad (4.5.3.1)\\ &\quad= \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\partial\left(\alpha\right)\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &\qquad+ \mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\partial\left(\alpha\right)\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &\qquad+ \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\omega_1\leq\gamma_1]}\partial\left(\alpha\right)\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma\end{aligned}$$

$$\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)(b-1)+a+b}\alpha\beta * \partial\left(\mathrm{P}_1(f_\Omega)\right) f_\Gamma \qquad (4.5.3.2)$$

$$= -\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{u(b-1)+a}\alpha\beta * \partial\left(\mathrm{P}_1(f_\Omega)\right) f_\Gamma$$

Proof. For Equation (4.5.3.1) we observe the following.

$$\begin{aligned}
\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma &\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a>1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\quad + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a=1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a>1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\quad + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\quad + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\overset{3.2.3(f)}{=} \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a>1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\quad + \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(u-1)(b-1)}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma \\
&\quad + \mathbf{1}_{[\omega_1\leq\gamma_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma
\end{aligned}$$

The desired equation follows immediately.

For Equation (4.5.3.2), recall that if $u = 1$, then we have $\partial\left(f_\Omega\right) = \mathrm{P}_1(f_\Omega)$. Lemma 3.2.3(f) tells us that $\mathbf{1}_{[u=1]}\partial\left(\mathrm{P}_1(f_\Omega)\right) = \mathbf{1}_{[u=1]}\partial\left(\partial\left(f_\Omega\right)\right) = 0$. This then gives the unmarked equality in the following expression.

$$\begin{aligned}
\mathbf{1}_{[u+v>2]}\partial\left(\mathrm{P}_1(f_\Omega)\right) &\overset{3.2.3(a)}{=} \mathbf{1}_{[u+v>2]}\mathbf{1}_{[u>1]}\partial\left(\mathrm{P}_1(f_\Omega)\right) + \mathbf{1}_{[u+v>2]}\mathbf{1}_{[u=1]}\partial\left(\mathrm{P}_1(f_\Omega)\right) \\
&= \mathbf{1}_{[u+v>2]}\mathbf{1}_{[u>1]}\partial\left(\mathrm{P}_1(f_\Omega)\right) \\
&\overset{3.2.3(d)}{=} \mathbf{1}_{[u>1]}\partial\left(\mathrm{P}_1(f_\Omega)\right).
\end{aligned}$$

The desired equation follows immediately from this expression. □

For the sake of brevity, we forego the proofs of the next three lemmas as they are proved by the same means as Lemma 4.5.3.

Lemma 4.5.4. *We have the following equations related to* $\mathbf{1}_{[b=1]}\alpha\partial\left(\beta\right) * f_\Omega f_\Gamma$ *where* α*,* β*,* f_Ω*, and* f_Γ *all have positive degree.*

$$-\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}(-1)^a\alpha\partial\left(\beta\right) * \partial\left(f_\Omega\right) f_\Gamma = -\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u>1]}(-1)^a\alpha\partial\left(\beta\right) * \partial\left(f_\Omega\right) f_\Gamma \quad (4.5.4.1)$$

$$- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u=1]}(-1)^a \alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma$$

$$\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[b=1]}(-1)^a \alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma = \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u=1]}(-1)^a \alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma + \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u>1]}(-1)^a \alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \quad (4.5.4.2)$$

Lemma 4.5.5. *We have the following equations related to* $\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma)$ *where* α, β, f_Ω, *and* f_Γ *all have positive degree.*

$$\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) = \mathbf{1}_{[a>1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) + \mathbf{1}_{[a=1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \quad (4.5.5.1)$$

$$\begin{aligned}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)b+a+b}\alpha\beta * \partial(f_\Omega)\, \mathrm{P}_1(f_\Gamma) &= \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u>1]}(-1)^{ub+a}\alpha\beta * \partial(f_\Omega)\, \mathrm{P}_1(f_\Gamma)\\ &\quad + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u=1]}\mathbf{1}_{[v>1]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma)\end{aligned} \quad (4.5.5.2)$$

$$\begin{aligned}-\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma) &= -\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u=1]}\mathbf{1}_{[v>1]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma)\\ &\quad - \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u>1]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma)\end{aligned} \quad (4.5.5.3)$$

Lemma 4.5.6. *We have the following equations related to* $\mathbf{1}_{[a=1]}\partial(\alpha)\,\beta * f_\Omega f_\Gamma$ *where* α, β, f_Ω, *and* f_Γ *all have positive degree.*

$$\begin{aligned}-\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)+b}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma &= \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma\\ &\quad + \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma\end{aligned} \quad (4.5.6.1)$$

Proposition 4.5.7. *We have the following formula for* $\partial(\alpha * f_\Omega)(\beta * f_\Gamma)$.

$$\begin{aligned}
\partial(\alpha * f_\Omega)(\beta * f_\Gamma) = {} & \mathbf{1}_{[\omega_1 \le \gamma_1 < \omega_2]}(-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
& + \mathbf{1}_{[\omega_1 \le \gamma_1 < \omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)(b-1)+a+b}\alpha\beta * \partial(\mathrm{P}_1(f_\Omega))\, f_\Gamma \\
& - \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[a>1]}\partial(\alpha)\,\partial(\beta) * f_\Omega f_\Gamma \\
& - \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}(-1)^{a}\alpha\partial(\beta) * \partial(f_\Omega)\, f_\Gamma \\
& + \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)b+a+b}\alpha\beta * \partial(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
& - \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)+b}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma \\
& + \mathbf{1}_{[\omega_1 < \gamma_1 < \omega_2]}\mathbf{1}_{[b=1]}(-1)^{a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
& + \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}\mathbf{1}_{[a>1]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\,\mathrm{P}_1(f_\Gamma) \\
& - \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
& + \mathbf{1}_{[\gamma_1 < \omega_2 < \gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)
\end{aligned} \tag{4.5.7.1}$$

In the case that we have $\omega_1 = \gamma_1$, *this reduces to the following formula.*

$$\begin{aligned}
\partial(\alpha * f_\Omega)(\beta * f_\Gamma) = {} & (-1)^{(u-1)(b-1)}\partial(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
& + \mathbf{1}_{[u+v>2]}(-1)^{(u-1)(b-1)+a+b}\alpha\beta * \partial(\mathrm{P}_1(f_\Omega))\, f_\Gamma \\
& + \mathbf{1}_{[\omega_2 < \gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)
\end{aligned} \tag{4.5.7.2}$$

Proof. Establishing equation (4.5.7.1) is predominantly "equation chasing." Applying Lemma 3.3.3, we obtain the following expression.

$$\begin{aligned}
\partial(\alpha * f_\Omega)(\beta * f_\Gamma) = {} & \mathbf{1}_{[a>1]}(\partial(\alpha) * f_\Omega)(\beta * f_\Gamma) + \mathbf{1}_{[u>1]}(\alpha * \partial(f_\Omega))(\beta * f_\Gamma) \\
& + \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma
\end{aligned}$$

Directly substituting in Equations (4.5.2.1) - (4.5.2.3) generates a formula with 13 terms. Among these terms are those needed to substitute in Equations (4.5.3.1), (4.5.3.2), and (4.5.6.1). We have now established lines 1-3, 6, and 8 of (4.5.7.1).

We obtain lines 4 and 7 by adding together Equations (4.5.4.1) and (4.5.4.2). This shows us that the sum of these two terms are equal to the expression

$$\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u>1]}(-1)^a\alpha\partial(\beta)*\mathrm{P}_1(f_\Omega)\,f_\Gamma-\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\mathbf{1}_{[u>1]}(-1)^a\alpha\partial(\beta)*\partial(f_\Omega)\,f_\Gamma$$

which is found in Equation (4.5.2.3). Summing Equations (4.5.5.2) and (4.5.5.3) yields lines 5 and 9. To establish term 10, we need to show that $\mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}=\mathbf{1}_{[u+v>2]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}$. To do this, recall that $\mathbf{1}_{[u=1]}\mathbf{1}_{[\gamma_1<\omega_2]}=\mathbf{1}_{[u=1]}$ by the convention established in Definition 4.4.2. Now consider the following.

$$\begin{aligned}\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}&\overset{3.2.3(b)}{=}\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\gamma_1<\omega_2]}\mathbf{1}_{[\omega_2<\gamma_2]}\\&\overset{4.4.2}{=}\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\omega_2<\gamma_2]}\\&\overset{3.2.3(a)}{=}\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}-\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\gamma_2\leq\omega_2]}\\&\overset{4.4.2}{=}\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}-\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\\&=0\end{aligned}$$

Note that the presence of $\mathbf{1}_{[v>1]}$ is necessary to be able to make sense of $\mathbf{1}_{[\omega_2<\gamma_2]}$ since the established convention assumes at least one of $|\Omega|>1$ and $|\Gamma|>1$ is true. We now conclude the following.

$$\mathbf{1}_{[u+v>2]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}\overset{3.2.3(a)}{=}\mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}+\mathbf{1}_{[v>1]}\mathbf{1}_{[u=1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}=\mathbf{1}_{[u>1]}\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}$$

Thus, we have established the last term in Equation (4.5.7.1).

To obtain Equation (4.5.7.2), note that the assumption $\omega_1=\gamma_1$ causes the equalities below.

1. $\mathbf{1}_{[\gamma_1<\omega_1]}=0=\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}$
2. $f_\Omega f_\Gamma=0=\mathrm{P}_i(f_\Omega)\,f_\Gamma$ for $i>1$.
3. $\mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}=\mathbf{1}_{[\omega_2<\gamma_2]}$
4. $\mathbf{1}_{[\omega_1\leq\gamma_1]}=1=\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}$

The first two formulas above kill off terms 3-9. Thus, the only terms remaining are terms 1, 2, and 10 whose coefficients are reduced by the above observations to produce the desired formula. □

The next lemma helps relate $\partial((\alpha*f_\Omega)(\beta*f_\Gamma))$ to Equations (4.5.7.1) and (4.5.7.2). We omit the proof of the lemma as it is a direct result of applying the differential $\partial^{\mathcal{X} *_R \mathcal{Y}}$.

Lemma 4.5.8. *We have the following equations related to $\partial\left((\alpha * f_\Omega)(\beta * f_\Gamma)\right)$.*

$$\begin{aligned}\partial\left(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma\right) = {} & \partial(\alpha)\,\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma + (-1)^a \alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ & + \mathbf{1}_{[u+v>2]}(-1)^{a+b}\alpha\beta * \partial\left(\mathrm{P}_1(f_\Omega)\right) f_\Gamma \\ & + \mathbf{1}_{[u+v>2]}(-1)^{a+b+u-1}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\partial(f_\Gamma)\end{aligned} \tag{4.5.8.1}$$

$$\begin{aligned}\mathbf{1}_{[b=1]}\partial\left(\alpha\partial(\beta) * f_\Omega f_\Gamma\right) = {} & \mathbf{1}_{[a>1]}\mathbf{1}_{[b=1]}\partial(\alpha)\,\partial(\beta) * f_\Omega f_\Gamma + \mathbf{1}_{[b=1]}(-1)^a\alpha\partial(\beta) * \partial(f_\Omega)\, f_\Gamma \\ & + \mathbf{1}_{[b=1]}(-1)^{a+u}\alpha\partial(\beta) * f_\Omega\partial(f_\Gamma)\end{aligned} \tag{4.5.8.2}$$

$$\begin{aligned}\partial\left(\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma)\right) = {} & \partial(\alpha)\,\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) + (-1)^a\alpha\partial(\beta) * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\ & + \mathbf{1}_{[u+v>2]}(-1)^{a+b}\alpha\beta * \partial(f_\Omega)\, \mathrm{P}_1(f_\Gamma) + \mathbf{1}_{[u+v>2]}(-1)^{a+b+u}\alpha\beta * f_\Omega\partial\left(\mathrm{P}_1(f_\Gamma)\right)\end{aligned} \tag{4.5.8.3}$$

$$\begin{aligned}\mathbf{1}_{[a=1]}\partial\left(\partial(\alpha)\,\beta * f_\Omega f_\Gamma\right) = {} & \mathbf{1}_{[b>1]}\mathbf{1}_{[a=1]}\partial(\alpha)\,\partial(\beta) * f_\Omega f_\Gamma + \mathbf{1}_{[a=1]}(-1)^b\partial(\alpha)\,\beta * \partial(f_\Omega)\, f_\Gamma \\ & + \mathbf{1}_{[a=1]}(-1)^{b+u}\partial(\alpha)\,\beta * f_\Omega\partial(f_\Gamma)\end{aligned} \tag{4.5.8.4}$$

Theorem 4.5.9. *The product from Definition 4.4.2 satisfies the Leibniz rule.*

Proof. While we employ the indicator functions to reduce the number of cases we need to consider, we find that for the sake of clarity that it is best to treat the cases $\omega_1 = \gamma_1$ and $\omega_1 \neq \gamma_1$ seperately.

In both of these cases we argue

$$0 = \mathbf{1}_{[\omega_2=\gamma_2]}\mathbf{1}_{[u+v>2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \tag{4.5.9.1}$$

Since u and v are both positive, the only way for their sum to be greater than 2 is if $u > 1$ or $v > 1$. This means at least one ω_2 and γ_2 must exist, so the only way they can be equal is if they both exist. However, if they both exist and are equal then we have $\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) = 0$ since $(\Omega\backslash\{\omega_1\}) \cap (\Gamma\backslash\{\gamma_1\}) \supseteq \{\omega_2\}$.

Case 1: Since the case of $\omega_1 = \gamma_1$ requires extra care, we first prove this case. Under this assumption, Definition 4.4.2 reduces to $(\alpha * f_\Omega)(\beta * f_\Gamma) = (-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma$. We use Equation (4.5.7.2) and the graded commutativity of $\mathcal{X}$ and $\mathcal{Y}$ to get the starred equality below.

$$
\begin{aligned}
(-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma) &\overset{4.4.5}{=} (-1)^{|\alpha * f_\Omega| + |\alpha * f_\Omega|(|\beta * f_\Gamma|-1)}\partial(\beta * f_\Gamma)(\alpha * f_\Omega) \\
&= (-1)^{(a+u-1)(b+v-1)}\partial(\beta * f_\Gamma)(\alpha * f_\Omega) \\
&\overset{*}{=} (-1)^{ub+a+b}\alpha\partial(\beta) * f_\Omega\,\mathrm{P}_1(f_\Gamma) \\
&\quad + \mathbf{1}_{[u+v>2]}(-1)^{ub+a+u}\alpha\beta * f_\Omega\partial(\mathrm{P}_1(f_\Gamma)) \\
&\quad + \mathbf{1}_{[\gamma_2<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\overset{4.3.8(a)}{=} (-1)^{(u-1)(b-1)+a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
&\quad + \mathbf{1}_{[u+v>2]}(-1)^{ub+a+u}\alpha\beta * f_\Omega\partial(\mathrm{P}_1(f_\Gamma)) \\
&\quad + \mathbf{1}_{[\gamma_2<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\overset{4.3.8(b)}{=} (-1)^{(u-1)(b-1)+a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
&\quad + \mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\partial(f_\Gamma) \\
&\quad - \mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\quad + \mathbf{1}_{[\gamma_2<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\overset{3.2.3(a)}{=} (-1)^{(u-1)(b-1)+a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\
&\quad + \mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\partial(f_\Gamma) \\
&\quad - \mathbf{1}_{[\omega_2\le\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)
\end{aligned}
$$

Since $(-1)^{ub+a} = (-1)^{(u-1)(b-1)+a+b+(u-1)}$, adding Equation (4.5.7.2) to the above and comparing with Equation (4.5.8.1) produces the next expression.

$$
\begin{aligned}
\partial((\alpha * f_\Omega)(\beta * f_\Gamma)) &\overset{(4.5.8.1)}{=} \partial(\alpha * f_\Omega)(\beta * f_\Gamma) + (-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma) \\
&\quad - \mathbf{1}_{[\omega_2<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\quad + \mathbf{1}_{[\omega_2\le\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\overset{3.2.3(a)}{=} \partial(\alpha * f_\Omega)(\beta * f_\Gamma) + (-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma) \\
&\quad + \mathbf{1}_{[\omega_2=\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
&\overset{(4.5.9.1)}{=} \partial(\alpha * f_\Omega)(\beta * f_\Gamma) + (-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma)
\end{aligned}
$$

Case 2: We now treat the case $\omega_1 \neq \gamma_1$. When we consider Equation (4.5.7.1) under this assumption we replace every weak inequality with a strict inequality, i.e., we replace $\mathbf{1}_{[\omega_1 \leq \gamma_1]}$ with $\mathbf{1}_{[\omega_1 < \gamma_1]}$. As in the previous case, we make use of Theorem 4.4.5 to get

$$(-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma) = (-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|}\partial(\beta * f_\Gamma)(\alpha * f_\Omega).$$

We can now apply Equation (4.5.7.1) to the right-hand side of the above equaltiy. When doing this, one should note that Lemma 3.2.3(f) gives us the following.

(a) $\mathbf{1}_{[a=1]}(-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|} = \mathbf{1}_{[a=1]}(-1)^{u(b+v-1)}$

(b) $\mathbf{1}_{[b=1]}(-1)^{|\alpha * f_\Omega|\cdot|\beta * f_\Gamma|} = \mathbf{1}_{[b=1]}(-1)^{(a+u-1)v}$

These equations combined with Equation (4.5.7.1) and the graded commutativity of $\mathcal{X}$ and $\mathcal{Y}$ produce the next equation.

$$\begin{aligned}
(-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial(\beta * f_\Gamma) = {} & \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)(b-1)+a+b+u-1}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\partial(f_\Gamma) \\
& - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}(-1)^{a+u}\alpha\partial(\beta) * f_\Omega\partial(f_\Gamma) \qquad (4.5.9.2) \\
& + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b+a}\alpha\partial(\beta) * f_\Omega\,\mathrm{P}_1(f_\Gamma) \\
& + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{(u-1)b+a+b+u}\alpha\beta * f_\Omega\partial(\mathrm{P}_1(f_\Gamma)) \\
& - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[b>1]}(-1)^{u(b-1)}\partial(\alpha)\,\partial(\beta) * f_\Omega f_\Gamma \\
& - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\partial(f_\Gamma) \\
& + \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[b>1]}(-1)^{(u-1)(b-1)+a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma \\
& + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[a=1]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\,\mathrm{P}_1(f_\Gamma) \\
& - \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\
& + \mathbf{1}_{[\omega_1<\gamma_2<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)
\end{aligned}$$

To add Equations (4.5.7.1) and (4.5.9.2), one uses Lemma 3.2.3(a) to combine the 7th term of each equation to produce the term $\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)+a}\alpha\partial(\beta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma$. Similarly, one combines the 8th terms of both equations to produce $\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\partial(\alpha)\,\beta * f_\Omega\,\mathrm{P}_1(f_\Gamma)$. From here, one uses Equations (4.5.8.1) - (4.5.8.4) and Definition 4.4.2 to obtain the following equality.

$$\begin{aligned}\partial\left((\alpha * f_\Omega)(\beta * f_\Gamma)\right) = {} & \partial\left(\alpha * f_\Omega\right)\left(\beta * f_\Gamma\right) + (-1)^{|\alpha * f_\Omega|}(\alpha * f_\Omega)\partial\left(\beta * f_\Gamma\right) \\ & + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ & - \mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ & + \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ & - \mathbf{1}_{[\omega_1<\gamma_2<\omega_2]}\mathbf{1}_{[u+v>2]}(-1)^{ub+a}\alpha\beta * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\end{aligned}$$

We now prove the extra four terms cancel with each other. Since we are working in the case where $\omega_1 \neq \gamma_1$, we are able to replace $\omega_1 \leq \gamma_1$ with $\omega_1 < \gamma_1$ in the following equalities.

$$\begin{aligned}&(\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]} - \mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]})\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ &\qquad = \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2\leq\omega_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) - \mathbf{1}_{[\omega\leq\gamma_1<\omega_2<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ &\qquad = \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) - \mathbf{1}_{[\omega_1<\gamma_1<\omega_2<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\end{aligned}$$

Similarly, $\gamma_2 \leq \omega_2$ is replaced with $\gamma_2 < \omega_2$ since $\mathbf{1}_{[\omega_2=\gamma_2]}\mathbf{1}_{[u+v>2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) = 0$. This same logic produces the relation

$$\begin{aligned}&(\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]} - \mathbf{1}_{[\omega_1<\gamma_2<\omega_2]})\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) \\ &\qquad = \mathbf{1}_{[\omega_1<\gamma_1<\omega_2<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma) - \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\end{aligned}$$

from which we conclude that

$$0 = (\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]} - \mathbf{1}_{[\gamma_1<\omega_2<\gamma_2]} + \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]} - \mathbf{1}_{[\omega_1<\gamma_2<\omega_2]})\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,.$$

Thus, we can conclude the additional terms vanish and our product satisfies the Leibniz rule. □

4.6 Proof of Associativity

To prove the product defined in Definition 4.4.2 is associative, we use a similar strategy to that used in the previous section. That is, given $\alpha, \beta, \rho \in \mathcal{X}$ and $\Omega, \Gamma, \Lambda \in \Delta$, we develop a formula for the expression $[(\alpha * f_\Omega)(\beta * f_\Gamma)](\rho * f_\Lambda)$ (see Lemma 4.6.5). We can then apply this formula with the graded commutativity of $\mathcal{X} *_R \mathcal{Y}$ to obtain a formula for

$$(\alpha * f_\Omega)[(\beta * f_\Gamma)(\rho * f_\Lambda)] = (-1)^{(a+u-1)(b+v+p+\ell)}[(\beta * f_\Gamma)(\rho * f_\Lambda)](\alpha * f_\Omega),$$

where we set $\ell := |f_\Lambda|$ and $p := |\rho|$ in addition to Notation 4.4.1.

To create the desired formula, we first note that a direct application of Definition 4.4.2 yields the following expression.

$$\begin{aligned}[(\alpha * f_\Omega)(\beta * f_\Gamma)](\rho * f_\Lambda) =& \mathbf{1}_{[\omega_1 \le \gamma_1 < \omega_2]}(-1)^{(u-1)(b-1)}(\alpha\beta * \mathrm{P}_1(f_\Omega) f_\Gamma)(\rho * f_\Lambda) \\ &- \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}(\alpha\partial(\beta) * f_\Omega f_\Gamma)(\rho * f_\Lambda) \\ &+ \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}(-1)^{(u-1)b}(\alpha\beta * f_\Omega \mathrm{P}_1(f_\Gamma))(\rho * f_\Lambda) \\ &- \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\left(\partial^{\mathcal{X}}(\alpha)\beta * f_\Omega f_\Gamma\right)(\rho * f_\Lambda).\end{aligned}$$

To apply Definition 4.4.2 to the four products produced on the right-hand side of the equation, we need to express $f_\Omega f_\Gamma$, $\mathrm{P}_1(f_\Omega) f_\Gamma$, and $f_\Omega \mathrm{P}_1(f_\Gamma)$ as linear combinations of the basis elements of $\mathcal{Y}$. If $\mathcal{Y}$ is supported on Δ^{n-1} for some n with simplicial multiplication, then this is straightforward:

(a) $f_\Omega f_\Gamma \in \operatorname{span}_R(f_{\Omega\cup\Gamma})$,

(b) $\mathrm{P}_1(f_\Gamma) f_\Omega \in \operatorname{span}_R\left(f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}\right)$,

(c) $f_\Omega \mathrm{P}_1(f_\Gamma) \in \operatorname{span}_R\left(f_{\Omega\cup(\Gamma\setminus\{\gamma_1\})}\right)$.

These observations along with Lemma 4.3.9 yield the following necessary results.

Lemma 4.6.1. *If $\mathcal{Y}$ is supported on Δ^{n-1} with simplicial multiplication, then we have the following.*

(a) $\mathbf{1}_{[\omega_1<\gamma_1]}\mathrm{P}_1(f_\Omega f_\Gamma) = \mathbf{1}_{[\omega_1<\gamma_1]}\mathrm{P}_1(f_\Omega) f_\Gamma$

(b) $\mathbf{1}_{[\gamma_1<\omega_1]}\mathrm{P}_1(f_\Omega f_\Gamma) = \mathbf{1}_{[\gamma_1<\omega_1]}(-1)^u f_\Omega \mathrm{P}_1(f_\Gamma)$

(c) $\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathrm{P}_1(f_\Omega \mathrm{P}_1(f_\Gamma)) = \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathrm{P}_1(f_\Omega)\mathrm{P}_1(f_\Gamma)$

(d) $\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}\mathrm{P}_1(\mathrm{P}_1(f_\Omega) f_\Gamma) = \mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{u-1}\mathrm{P}_1(f_\Omega)\mathrm{P}_1(f_\Gamma)$

Proof. The proof of Equations (a) and (b) are essentially the same, so we only show the proof of the former. For this proof, we first observe that every indicator function satisfies the equation

$x^2 - x = 0$. We next observe that $\omega_1 < \gamma_1$ implies that $\gamma_1 < \omega_1$ is false, thus Lemma 3.2.3(e) yields $\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\gamma_1<\omega_1]} = 0$. These observations give us the final equality in the following expression.

$$\begin{aligned}\mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega f_\Gamma) &\overset{4.3.9}{=} \left(\mathbf{1}_{[\omega_1<\gamma_1]}\right)^2 \mathrm{P}_1(f_\Omega)\, f_\Gamma + \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\gamma_1<\omega_1]}(-1)^u f_\Omega\,\mathrm{P}_1(f_\Gamma)\\ &= \mathbf{1}_{[\omega_1<\gamma_1]}\,\mathrm{P}_1(f_\Omega)\, f_\Gamma\end{aligned}$$

We now use Equation (a) to obtain Equation (c).

$$\begin{aligned}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\,\mathrm{P}_1(f_\Omega\,\mathrm{P}_1(f_\Gamma)) &\overset{3.2.3(b)}{=} \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\omega_1<\gamma_2]}\,\mathrm{P}_1(f_\Omega\,\mathrm{P}_1(f_\Gamma))\\ &\overset{4.6.1(a)}{=} \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\omega_1<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\\ &\overset{3.2.3(b)}{=} \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\end{aligned}$$

Similarly, one uses Equation (b) to prove Equation (d). □

Lemma 4.6.2. *Tthe following formulas hold for all* $\alpha, \beta, \rho \in \mathcal{X}_{\geq 1}$ *and all non-empty* $\Omega, \Gamma, \Lambda \in \Delta^{n-1}$.

$$\begin{aligned}&\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\rho * f_\Lambda) \qquad (4.6.2.1)\\ &\quad= \mathbf{1}_{[\omega_1\leq\gamma_1\leq\lambda_1<\min\{\omega_2,\gamma_2\}]}(-1)^{(u+v)(p-1)+(u-1)b}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\, f_\Lambda\\ &\qquad- \mathbf{1}_{[\omega_1=\gamma_1<\lambda_1]}\mathbf{1}_{[p=1]}(-1)^{(u-1)(b-1)}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda\\ &\qquad- \mathbf{1}_{[\omega_1<\gamma_1<\min\{\omega_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)(b-1)}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda\\ &\qquad+ \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}(-1)^{(u+v)p+(u-1)(b-1)}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma\,\mathrm{P}_1(f_\Lambda)\end{aligned}$$

$$\begin{aligned}&\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}(\alpha\beta * f_\Omega\,\mathrm{P}_1(f_\Gamma))(\rho * f_\Lambda) \qquad (4.6.2.2)\\ &\quad= \mathbf{1}_{[\gamma_1<\omega_1\leq\lambda_1<\min\{\omega_2,\gamma_2\}]}(-1)^{(u+v)(p-1)+(u-1)b}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\, f_\Lambda\\ &\qquad- \mathbf{1}_{[\gamma_1<\omega_1<\min\{\gamma_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)b}\alpha\beta\partial\,(\rho) * f_\Omega\,\mathrm{P}_1(f_\Gamma)\, f_\Lambda\\ &\qquad+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{(u+v)p+(u-1)b}\alpha\beta\rho * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\end{aligned}$$

Proof. We establish Equation (4.6.2.1) and note that Equation (4.6.2.2) can be established using the same means and observations. We consider $\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\rho * f_\Lambda)$.

Recall that $\mathrm{P}_1(f_\Omega)\, f_\Gamma \in \mathrm{span}_R\left(f_{(\Omega\setminus\{\omega_1\})\cup\Gamma}\right)$, so we need to know the first and second vertices of $(\Omega\backslash\{\omega_1\}) \cup \Gamma$ in order to apply Definition 4.4.2. Since ω_2 and γ_1 are the first vertices of $\Omega\backslash\{\omega_1\}$ and Γ, respectively, we know the first vertex of $(\Omega\backslash\{\omega_1\})\cup\Gamma$ is $\min\{\omega_2, \gamma_1\}$. However, the presence of the indicator function of $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}$ tells us that $\gamma_1 = \min\{\omega_2, \gamma_1\}$. From here, the same reasoning tells us that the second vertex must then be $\min\{\omega_2, \gamma_2\}$. Before we use this information to apply Definition 4.4.2, we note the following equalities.

1. $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\gamma_1 \leq \lambda_1 < \min\{\omega_2,\gamma_2\}]} = \mathbf{1}_{[\omega_1 \leq \gamma_1 \leq \lambda_1 < \min\{\omega_2,\gamma_2\}]}$

2. $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\gamma_1 < \lambda_1]} = \mathbf{1}_{[\omega_1 = \gamma_1 < \lambda_1]} + \mathbf{1}_{[\omega_1 < \gamma_1 < \min\{\omega_2,\lambda_1\}]}$

Moreover, since $a, b \geq 1$, we have $|\alpha\beta| = a + b \geq 2$ and so $\mathbf{1}_{[|\alpha\beta|=1]} = 0$. This observation with the above equalities yields the following expression.

$$
\begin{aligned}
&\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\rho * f_\Lambda) \\
&\quad \overset{4.4.2}{=} \mathbf{1}_{[\omega_1 \leq \gamma_1 \leq \lambda_1 < \min\{\omega_2,\gamma_2\}]}(-1)^{(u+v-2)(p-1)}\alpha\beta\rho * \mathrm{P}_1(\mathrm{P}_1(f_\Omega)\, f_\Gamma)\, f_\Lambda \\
&\qquad - \mathbf{1}_{[\omega_1 = \gamma_1 < \lambda_1]}\mathbf{1}_{[p=1]}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\qquad - \mathbf{1}_{[\omega_1 < \gamma_1 < \min\{\omega_2,\lambda_1\}]}\mathbf{1}_{[p=1]}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\qquad + \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\lambda_1 < \gamma_1 < \lambda_2]}(-1)^{(u+v-2)p}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma\, \mathrm{P}_1(f_\Lambda) \\
&\quad \overset{4.6.1}{=} \mathbf{1}_{[\omega_1 \leq \gamma_1 \leq \lambda_1 < \min\{\omega_2,\gamma_2\}]}(-1)^{(u+v)(p-1)+u-1}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&\qquad - \mathbf{1}_{[\omega_1 = \gamma_1 < \lambda_1]}\mathbf{1}_{[p=1]}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\qquad - \mathbf{1}_{[\omega_1 < \gamma_1 < \min\{\omega_2,\lambda_1\}]}\mathbf{1}_{[p=1]}\alpha\beta\partial\,(\rho) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\qquad + \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\lambda_1 < \gamma_1 < \lambda_2]}(-1)^{(u+v)p}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma\, \mathrm{P}_1(f_\Lambda)
\end{aligned}
$$

Multiplying the expression by $(-1)^{(u-1)(b-1)}$ produces the desired equation. □

Along with Equations (4.6.2.1) and (4.6.2.2), we need two more equations to obtain a meaningful formula for the product $[(\alpha * f_\Omega)(\beta * f_\Gamma)](\rho * f_\Lambda)$.

Lemma 4.6.3. *The following formulas hold for all $\alpha, \beta, \rho \in \mathcal{X}_{\geq 1}$ and all non-empty $\Omega, \Gamma, \Lambda \in \Delta^{n-1}$.*

$$
\begin{aligned}
&-\mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}(\alpha\partial\,(\beta) * f_\Omega f_\Gamma)(\rho * f_\Lambda) \qquad (4.6.3.1)\\
&\quad = -\mathbf{1}_{[\omega_1 < \lambda_1 < \min\{\omega_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial\,(\beta)\,\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\qquad - \mathbf{1}_{[\omega_1 = \lambda_1 < \gamma_1]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial\,(\beta)\,\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda
\end{aligned}
$$

$$+\mathbf{1}_{[\omega_1<\min\{\gamma_1,\lambda_1\}]}\mathbf{1}_{[b=1=p]}\alpha\partial(\beta)\partial(\rho)*f_\Omega f_\Gamma f_\Lambda$$
$$-\mathbf{1}_{[\lambda_1<\omega_1<\min\{\lambda_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)p}\alpha\partial(\beta)\rho*f_\Omega f_\Gamma \mathrm{P}_1(f_\Lambda)$$
$$+\mathbf{1}_{[\lambda_1<\omega_1<\gamma_1]}\mathbf{1}_{[a=1=b]}(-1)^{(u+v)(p-1)}\partial(\alpha)\partial(\beta)\rho*f_\Omega f_\Gamma f_\Lambda$$

$$\begin{aligned}
-\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}(\partial(\alpha)\beta*f_\Omega f_\Gamma)(\rho*f_\Lambda) \qquad (4.6.3.2)\\
=-\mathbf{1}_{[\gamma_1<\lambda_1<\min\{\omega_1,\gamma_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial(\alpha)\beta\rho*f_\Omega \mathrm{P}_1(f_\Gamma)f_\Lambda\\
-\mathbf{1}_{[\gamma_1=\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial(\alpha)\beta\rho*f_\Omega \mathrm{P}_1(f_\Gamma)f_\Lambda\\
+\mathbf{1}_{[\gamma_1<\min\{\omega_1,\lambda_1\}]}\mathbf{1}_{[a=1=p]}(-1)^{u(b-1)}\partial(\alpha)\beta\partial(\rho)*f_\Omega f_\Gamma f_\Lambda\\
-\mathbf{1}_{[\lambda_1<\gamma_1<\min\{\omega_1,\lambda_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)p+u(b-1)}\partial(\alpha)\beta\rho*f_\Omega f_\Gamma \mathrm{P}_1(f_\Lambda)\\
+\mathbf{1}_{[\lambda_1<\gamma_1<\omega_1]}\mathbf{1}_{[a=1=b]}(-1)^{(u+v)(p-1)}\partial(\alpha)\partial(\beta)\rho*f_\Omega f_\Gamma f_\Lambda
\end{aligned}$$

Proof. Establishing Equations (4.6.3.1) and (4.6.3.2) uses nearly identical reasoning. For brevity, we show only show the former but note that the latter requires the additional observation that Lemma 3.2.3(f) gives $\mathbf{1}_{[b=1]}(-1)^{u(b-1)}=\mathbf{1}_{[b=1]}$.

We establish (4.6.3.1) by considering the expression $\mathbf{1}_{[\omega_1<\gamma_1]}(\alpha\partial(\beta)*f_\Omega f_\Gamma)(\rho*f_\Lambda)$, since $\mathbf{1}_{[b=1]}\partial(\beta)\in\mathcal{X}_0=R$. To apply Definition 4.4.2 to this product, we need to determine the first and second vertices of $\Omega\cup\Gamma$ since $f_\Omega f_\Gamma\in \mathrm{span}_R(f_{\Omega\cup\Gamma})$. Similar to our approach in Lemma 4.6.2, we use $\mathbf{1}_{[\omega_1<\gamma_1]}$ to note that the first vertex is ω_1 and the second vertex is $\min\{\omega_2,\gamma_1\}$. We also note the following equalities.

$$\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\omega_1\le\lambda_1<\min\{\omega_2,\gamma_1\}]}\overset{3.2.3(\mathrm{d})}{=}\mathbf{1}_{[\omega_1<\lambda_1<\min\{\omega_2,\gamma_1\}]}+\mathbf{1}_{[\omega_1=\lambda_1<\gamma_1]}$$
$$\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\omega_1<\lambda_1]}\overset{3.2.3(\mathrm{b})}{=}\mathbf{1}_{[\omega_1<\min\{\gamma_1,\lambda_1\}]}$$
$$\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}\overset{3.2.3(\mathrm{b})}{=}\mathbf{1}_{[\lambda_1<\omega_1<\min\{\gamma_1,\lambda_1\}]}$$
$$\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\lambda_1<\omega_1]}\overset{3.2.3(\mathrm{b})}{=}\mathbf{1}_{[\lambda_1<\omega_1<\gamma_1]}$$

The above equalities with Definition 4.4.2 and Lemma 4.6.1(a) yields the following.

$$\begin{aligned}
\mathbf{1}_{[\omega_1<\gamma_1]}(\alpha*f_\Omega f_\Gamma)(\rho*f_\Lambda)=&\mathbf{1}_{[\omega_1<\lambda_1<\min\{\omega_2,\gamma_1\}]}(-1)^{(u+v-1)(p-1)}\alpha\rho*\mathrm{P}_1(f_\Omega)f_\Gamma f_\Lambda\\
&+\mathbf{1}_{[\omega_1=\lambda_1<\gamma_1]}(-1)^{(u+v-1)(p-1)}\alpha\rho*\mathrm{P}_1(f_\Omega)f_\Gamma f_\Lambda
\end{aligned}$$

$$- \mathbf{1}_{[\omega_1 < \min\{\gamma_1, \lambda_1\}]} \mathbf{1}_{[p=1]} \alpha \partial(\rho) * f_\Omega f_\Gamma f_\Lambda$$
$$+ \mathbf{1}_{[\lambda_1 < \omega_1 < \min\{\lambda_2, \gamma_1\}]} (-1)^{(u+v-1)p} \alpha\rho * f_\Omega f_\Gamma \operatorname{P}_1(f_\Lambda)$$
$$- \mathbf{1}_{[\lambda_1 < \omega_1 < \gamma_1]} \mathbf{1}_{[a=1]} (-1)^{(u+v)(p-1)} \partial(\alpha) \rho * f_\Omega f_\Gamma f_\Lambda$$

Multiplying the entire expression by $-\mathbf{1}_{[b=1]}\partial(\beta)$ produces Equation (4.6.3.1). □

We want to add Equations (4.6.2.1), (4.6.2.2), (4.6.3.1), and (4.6.3.2). To do this, we will use the following relations to combine the common terms in these equations.

Lemma 4.6.4. *If $\Omega, \Gamma, \Lambda \in \Delta^{n-1}$ are non-empty, then we have the following relation.*

$$\mathbf{1}_{[\omega_1 \le \lambda_1 < \omega_2]} \mathbf{1}_{[\gamma_1 \le \lambda_1 < \gamma_2]} = \mathbf{1}_{[\omega_1 \le \gamma_1 \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} + \mathbf{1}_{[\gamma_1 < \omega_1 \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} \tag{4.6.4.1}$$

$$\mathbf{1}_{[\lambda_1 < \min\{\omega_1, \gamma_1\}]} f_\Omega f_\Gamma f_\Lambda = \mathbf{1}_{[\lambda_1 < \omega_1 < \gamma_1]} f_\Omega f_\Gamma f_\Lambda + \mathbf{1}_{[\lambda_1 < \gamma_1 < \omega_1]} f_\Omega f_\Gamma f_\Lambda \tag{4.6.4.2}$$

Proof. We start with Equation (4.6.4.1). In the following, the first equality follows from the fact that $\omega_1, \gamma_1 \le \lambda_1$ if and only if $\max\{\omega_1, \gamma_1\} \le \lambda_1$ and that $\lambda_1 \le \omega_2, \gamma_2$ if and only if $\lambda_1 \le \min\{\omega_2, \gamma_2\}$.

$$\begin{aligned}
\mathbf{1}_{[\omega_1 \le \lambda_1 < \omega_2]} \mathbf{1}_{[\gamma_1 \le \lambda_1 < \gamma_2]} &= \mathbf{1}_{[\max\{\omega_1, \gamma_1\} \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} \\
&\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1 \le \gamma_1]} \mathbf{1}_{[\max\{\omega_1, \gamma_1\} \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} + \mathbf{1}_{[\gamma_1 < \omega_1]} \mathbf{1}_{[\max\{\omega_1, \gamma_1\} \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} \\
&\overset{3.2.3(b)}{=} \mathbf{1}_{[\omega_1 \le \gamma_1 \le \lambda_1 < \min\{\omega_2, \gamma_2\}]} + \mathbf{1}_{[\gamma_1 < \omega_1 \le \lambda_1 < \min\{\omega_2, \gamma_2\}]}
\end{aligned}$$

Equation (4.6.4.2) is a consequence of Lemma 3.2.3(a),(f). □

Proposition 4.6.5. *If $\Omega, \Gamma, \Lambda \in \Delta^{n-1} \backslash \{\emptyset\}$, then we have the following formula.*

$$[(\alpha * f_\Omega)(\beta * f_\Gamma)](\rho * f_\Lambda) \tag{4.6.5.1}$$
$$\begin{aligned}
&= \mathbf{1}_{[\omega_1 \le \lambda_1 < \omega_2]} \mathbf{1}_{[\gamma_1 \le \lambda_1 < \gamma_2]} (-1)^{(u+v)(p-1)+(u-1)b} \alpha\beta\rho * \operatorname{P}_1(f_\Omega) \operatorname{P}_1(f_\Gamma) f_\Lambda \\
&\quad + \mathbf{1}_{[\omega_1 \le \gamma_1 < \omega_2]} \mathbf{1}_{[\lambda_1 < \gamma_1 < \lambda_2]} (-1)^{(u+v)p+(u-1)(b-1)} \alpha\beta\rho * \operatorname{P}_1(f_\Omega) f_\Gamma \operatorname{P}_1(f_\Lambda) \\
&\quad + \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]} \mathbf{1}_{[\lambda_1 < \omega_1 < \lambda_2]} (-1)^{(u+v)p+(u-1)b} \alpha\beta\rho * f_\Omega \operatorname{P}_1(f_\Gamma) \operatorname{P}_1(f_\Lambda) \\
&\quad - \mathbf{1}_{[\omega_1 = \gamma_1 < \lambda_1]} \mathbf{1}_{[p=1]} (-1)^{(u-1)(b-1)} \alpha\beta\partial(\rho) * \operatorname{P}_1(f_\Omega) f_\Gamma f_\Lambda
\end{aligned}$$

$$
\begin{aligned}
&- \mathbf{1}_{[\omega_1=\lambda_1<\gamma_1]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial\left(\beta\right)\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\gamma_1=\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial\left(\alpha\right)\beta\rho * f_\Omega\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&- \mathbf{1}_{[\omega_1<\gamma_1<\min\{\omega_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)(b-1)}\alpha\beta\partial\left(\rho\right) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\gamma_1<\omega_1<\min\{\gamma_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)b}\alpha\beta\partial\left(\rho\right) * f_\Omega\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&- \mathbf{1}_{[\omega_1<\lambda_1<\min\{\omega_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial\left(\beta\right)\rho * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\lambda_1<\omega_1<\min\{\lambda_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)p}\alpha\partial\left(\beta\right)\rho * f_\Omega f_\Gamma\, \mathrm{P}_1(f_\Lambda) \\
&- \mathbf{1}_{[\gamma_1<\lambda_1<\min\{\omega_1,\gamma_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial\left(\alpha\right)\beta\rho * f_\Omega\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&- \mathbf{1}_{[\lambda_1<\gamma_1<\min\{\omega_1,\lambda_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)p+u(b-1)}\partial\left(\alpha\right)\beta\rho * f_\Omega f_\Gamma\, \mathrm{P}_1(f_\Lambda) \\
&+ \mathbf{1}_{[\lambda_1<\min\{\omega_1,\gamma_1\}]}\mathbf{1}_{[a=1=b]}(-1)^{(u+v)(p-1)}\partial\left(\alpha\right)\partial\left(\beta\right)\rho * f_\Omega f_\Gamma f_\Lambda \\
&+ \mathbf{1}_{[\gamma_1<\min\{\omega_1,\lambda_1\}]}\mathbf{1}_{[a=1=p]}(-1)^{u(b-1)}\partial\left(\alpha\right)\beta\partial\left(\rho\right) * f_\Omega f_\Gamma f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\min\{\gamma_1,\lambda_1\}]}\mathbf{1}_{[b=1=p]}\alpha\partial\left(\beta\right)\partial\left(\rho\right) * f_\Omega f_\Gamma f_\Lambda
\end{aligned}
$$

Proof. We stated at the beginning of this section that

$$
\begin{aligned}
\left[\left(\alpha * f_\Omega\right)\left(\beta * f_\Gamma\right)\right]\left(\rho * f_\Lambda\right) =& \mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{(u-1)(b-1)}\left(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma\right)\left(\rho * f_\Lambda\right) \\
&- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}\left(\alpha\partial\left(\beta\right) * f_\Omega f_\Gamma\right)\left(\rho * f_\Lambda\right) \\
&+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(u-1)b}\left(\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma)\right)\left(\rho * f_\Lambda\right) \\
&- \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\left(\partial^{\mathcal{X}}(\alpha)\,\beta * f_\Omega f_\Gamma\right)\left(\rho * f_\Lambda\right).
\end{aligned}
$$

From here, we substitute in Equations (4.6.2.1), (4.6.2.2), (4.6.3.1), and (4.6.3.2). When doing this we use Equation (4.6.4.1) to combine the $\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\, f_\Lambda$ terms from (4.6.2.1) and (4.6.2.2). Similarly, to combine the $\partial\left(\alpha\right)\partial\left(\beta\right)\rho * f_\Omega f_\Gamma f_\Lambda$ terms of (4.6.3.1) and (4.6.3.2), we use Equation (4.6.4.2). □

Before we prove the product is associative, we need one last lemma.

Lemma 4.6.6. *If $\mathcal{Y}$ has simplicial multiplication, then the following holds.*

$$
\begin{aligned}
0 =& (\mathbf{1}_{[\omega_1\le\lambda_1<\omega_2]}\mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]} - \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]})(-1)^{u+v}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&+ (\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}\mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]} - \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]})(-1)^{u-1}\,\mathrm{P}_1(f_\Omega)\, f_\Gamma\, \mathrm{P}_1(f_\Lambda)
\end{aligned}
$$

$$+\,(\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]} - \mathbf{1}_{[\gamma_1\le\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1\le\omega_1<\lambda_2]})f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)$$

Proof. To prove that this expression vanishes, we first simplify each line on the right-hand side. For the first line, we observe the following.

$$\begin{aligned}
&(\mathbf{1}_{[\omega_1\le\lambda_1<\omega_2]}\mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]} - \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]})(-1)^{u+v}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda\\
&\qquad = \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}(\mathbf{1}_{[\omega_1\le\lambda_1<\omega_2]} - \mathbf{1}_{[\omega_1<\lambda_1<\omega_2]})(-1)^{u+v}\,\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda\\
&\qquad \overset{3.2.3(a)}{=} \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1=\lambda_1]}(-1)^{u+v+(v-1)\ell}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda\,\mathrm{P}_1(f_\Gamma)\\
&\qquad \overset{4.3.8(a)}{=} \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1=\lambda_1]}(-1)^{(v-1)(\ell-1)}f_\Omega\,\mathrm{P}_1(f_\Lambda)\,\mathrm{P}_1(f_\Gamma)\\
&\qquad \overset{3.2.3(b)}{=} \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1=\lambda_1]}f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{3.2.3(a)}{=} \mathbf{1}_{[\gamma_1<\lambda_1=\omega_1<\gamma_2]}f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda) + \mathbf{1}_{[\gamma_1=\lambda_1=\omega_1]}f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)
\end{aligned}$$

We next simplify the second line of the original expression.

$$\begin{aligned}
&(\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}\mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]} - \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]})(-1)^{u-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad = \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}(\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]} - \mathbf{1}_{[\omega_1<\gamma_1<\omega_2]})(-1)^{u-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{3.2.3(a)}{=} \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}\mathbf{1}_{[\omega_1=\gamma_1]}(-1)^{u-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{3.2.3(b)}{=} \mathbf{1}_{[\lambda_1<\omega_1=\gamma_1<\lambda_2]}(-1)^{u-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{4.3.8(a)}{=} \mathbf{1}_{[\lambda_1<\omega_1=\gamma_1<\lambda_2]}f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)
\end{aligned}$$

Finally, we check the last line of the original expression.

$$\begin{aligned}
&(\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]} - \mathbf{1}_{[\gamma_1\le\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1\le\omega_1<\lambda_2]})f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{3.2.3(a)}{=} -(\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1=\omega_1]} + \mathbf{1}_{[\gamma_1=\omega_1]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]})f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\qquad\qquad - \mathbf{1}_{[\lambda_1=\omega_1]}\mathbf{1}_{[\gamma_1=\omega_1]})f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\qquad \overset{3.2.3(b)}{=} -(\mathbf{1}_{[\gamma_1<\lambda_1=\omega_1<\gamma_2]} + \mathbf{1}_{[\lambda_1<\omega_1=\gamma_1<\lambda_2]} + \mathbf{1}_{[\gamma_1=\lambda_1=\omega_1]})f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)
\end{aligned}$$

From here, one can see the sum of the first two lines cancels with the third, thus proving the entire expression vanishes as desired. □

Theorem 4.6.7. *If for some n we have $\mathcal{Y}$ is supported on Δ^{n-1} with simplicial multiplication, then* *the product in Definition 4.4.2 is associative.*

Proof. To compare $[(\alpha * f_\Omega)(\beta * f_\Gamma)](\rho * f_\Lambda)$ with $(\alpha * f_\Omega)[(\beta * f_\Gamma)(\rho * f_\Lambda)]$, we need a formula for the latter. To obtain such a formula, recall that graded commutativity gives us

$$(\alpha * f_\Omega)[(\beta * f_\Gamma)(\rho * f_\Lambda)] = (-1)^{(a+u-1)(b+v+p+\ell)}[(\beta * f_\Gamma)(\rho * f_\Lambda)](\alpha * f_\Omega)$$

and note that we can apply Equation (4.6.5.1) to the right-hand side of the expression. From there, one uses the graded commutativity of $\mathcal{X}$ and $\mathcal{Y}$ to rearrange the resulting expression to match the ordering of $(\alpha * f_\Omega)[(\beta * f_\Gamma)(\rho * f_\Lambda)]$; i.e.,

$$\mathbf{1}_{[p=1]}\beta\partial(\rho)\,\alpha * f_\Gamma f_\Lambda\,\mathrm{P}_1(f_\Omega) \overset{3.2.3(\mathrm{f})}{=} \mathbf{1}_{[p=1]}(-1)^{ab+(v+\ell)(u-1)}\alpha\beta\partial(\rho) * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda.$$

In the process of applying the graded commutativity, which we omit purely for spacing reasons, one will need the following two expressions.

$$\begin{aligned}
\mathbf{1}_{[\gamma_1=\omega_1<\lambda_1]}\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda f_\Omega &= \mathbf{1}_{[\gamma_1=\omega_1<\lambda_1]}(-1)^{(v+\ell-1)u}f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&\overset{4.3.8(\mathrm{a})}{=} \mathbf{1}_{[\gamma_1=\omega_1<\lambda_1]}(-1)^{(v+\ell-1)u+u-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
\mathbf{1}_{[\lambda_1=\omega_1<\gamma_1]}f_\Gamma\,\mathrm{P}_1(f_\Lambda)\,f_\Omega &\overset{4.3.8(\mathrm{a})}{=} \mathbf{1}_{[\lambda_1=\omega_1<\gamma_1]}(-1)^{\ell-1}f_\Gamma f_\Lambda\,\mathrm{P}_1(f_\Omega) \\
&= \mathbf{1}_{[\lambda_1=\omega_1<\gamma_1]}(-1)^{(v+\ell)(u-1)+\ell-1}\,\mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda
\end{aligned}$$

These expressions with graded commutativity of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X} *_R \mathcal{Y}$ yields the following formula.

$$\begin{aligned}
(\alpha * f_\Omega)[(\beta * f_\Gamma)(\rho * f_\Lambda)] = {}& \mathbf{1}_{[\gamma_1\leq\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1\leq\omega_1<\lambda_2]}(-1)^{(u+v)p+(u-1)b}\alpha\beta\rho * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda) \qquad (4.6.7.1)\\
&+ \mathbf{1}_{[\gamma_1\leq\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}(-1)^{(u+v)(p-1)+(u-1)b}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&+ \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^{(u+v)p+(u-1)(b-1)}\alpha\beta\rho * \mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda) \\
&- \mathbf{1}_{[\gamma_1=\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial(\alpha)\,\beta\rho * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&- \mathbf{1}_{[\omega_1=\gamma_1<\lambda_1]}\mathbf{1}_{[p=1]}(-1)^{(u-1)(b-1)}\alpha\beta\partial(\rho) * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\omega_1=\lambda_1<\gamma_1]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial(\beta)\,\rho * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\gamma_1<\lambda_1<\min\{\omega_1,\gamma_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)(p-1)+ub}\partial(\alpha)\,\beta\rho * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&- \mathbf{1}_{[\lambda_1<\gamma_1<\min\{\omega_1,\lambda_2\}]}\mathbf{1}_{[a=1]}(-1)^{(u+v-1)p+u(b-1)}\partial(\alpha)\,\beta\rho * f_\Omega f_\Gamma\,\mathrm{P}_1(f_\Lambda)
\end{aligned}$$

$$
\begin{aligned}
&- \mathbf{1}_{[\gamma_1<\omega_1<\min\{\gamma_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)b}\alpha\beta\partial(\rho)*f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda\\
&- \mathbf{1}_{[\omega_1<\gamma_1<\min\{\omega_2,\lambda_1\}]}\mathbf{1}_{[p=1]}(-1)^{(u-1)(b-1)}\alpha\beta\partial(\rho)*\mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda\\
&- \mathbf{1}_{[\lambda_1<\omega_1<\min\{\lambda_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)p}\alpha\partial(\beta)\,\rho*f_\Omega f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&- \mathbf{1}_{[\omega_1<\lambda_1<\min\{\omega_2,\gamma_1\}]}\mathbf{1}_{[b=1]}(-1)^{(u+v-1)(p-1)}\alpha\partial(\beta)\,\rho*\mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda\\
&+ \mathbf{1}_{[\omega_1<\min\{\gamma_1,\lambda_1\}]}\mathbf{1}_{[b=1=p]}\alpha\partial(\beta)\,\partial(\rho)*f_\Omega f_\Gamma f_\Lambda\\
&+ \mathbf{1}_{[\lambda_1<\min\{\omega_1,\gamma_1\}]}\mathbf{1}_{[a=1=b]}(-1)^{(u+v)(p-1)}\partial(\alpha)\,\partial(\beta)\,\rho*f_\Omega f_\Gamma f_\Lambda\\
&+ \mathbf{1}_{[\gamma_1<\min\{\omega_1,\lambda_1\}]}\mathbf{1}_{[a=1=p]}(-1)^{u(b-1)}\partial(\alpha)\,\beta\partial(\rho)*f_\Omega f_\Gamma f_\Lambda
\end{aligned}
$$

Except for the first three lines of Equation (4.6.7.1), every line has a perfect match with a line from Equation (4.6.5.1). Thus, taking the difference of these two equations provides the following relation.

$$
\begin{aligned}
&[(\alpha*f_\Omega)(\beta*f_\Gamma)](\rho*f_\Lambda)-(\alpha*f_\Omega)[(\beta*f_\Gamma)(\rho*f_\Lambda)]\\
&\quad= \mathbf{1}_{[\omega_1\le\lambda_1<\omega_2]}\mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}(-1)^{(u+v)(p-1)+(u-1)b}\alpha\beta\rho*\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda\\
&\qquad- \mathbf{1}_{[\gamma_1\le\lambda_1<\gamma_2]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}(-1)^{(u+v)(p-1)+(u-1)b}\alpha\beta\rho*\mathrm{P}_1(f_\Omega)\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda\\
&\qquad+ \mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}\mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}(-1)^{(u+v)p+(u-1)(b-1)}\alpha\beta\rho*\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad- \mathbf{1}_{[\lambda_1<\gamma_1<\lambda_2]}\mathbf{1}_{[\omega_1<\gamma_1<\omega_2]}(-1)^{(u+v)p+(u-1)(b-1)}\alpha\beta\rho*\mathrm{P}_1(f_\Omega)\,f_\Gamma\,\mathrm{P}_1(f_\Lambda)\\
&\qquad+ \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{(u+v)p+(u-1)b}\alpha\beta\rho*f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\qquad- \mathbf{1}_{[\gamma_1\le\omega_1<\gamma_2]}\mathbf{1}_{[\lambda_1\le\omega_1<\lambda_2]}(-1)^{(u+v)p+(u-1)b}\alpha\beta\rho*f_\Omega\,\mathrm{P}_1(f_\Gamma)\,\mathrm{P}_1(f_\Lambda)\\
&\quad\overset{4.6.6}{=} 0.
\end{aligned}
$$

Thus, we have shown the product satisfies the associative property. ☐

Corollary 4.6.8. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are DG R-algebras. If $\mathcal{Y}$ is supported on Δ^{n-1} with simplicial multiplication, then Definition 4.4.2 gives $\mathcal{X} *_R \mathcal{Y}$ the structure of a DG R-algebra.*

Proof. By construction, $\mathcal{X} *_R \mathcal{Y}$ is positively graded. Theorem 4.4.4 shows the product is unital and distributive. Theorem 4.4.5 shows the product is graded commutative. Theorem 4.5.9 shows the product satisfies the Leibniz rule. Theorem 4.6.7 shows the product is associative. ☐

The next corollary gives a partial answer to a question by Vandebogert [42, Question 5.7].

Corollary 4.6.9. *Let $(R, \mathfrak{M}, k)$ be a regular local ring with ideals $\mathcal{I}, \mathcal{J} \subseteq R$ such that $\mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}$. Suppose $\mathcal{J} = \langle m_1 \dots m_n \rangle$ satisfies one of the following two properties.*

(1) $\mathcal{J}$ is generated by an R-regular sequence.

(2) $\mathcal{J}$ is a monomial ideal where for any $\Omega, \Gamma \subseteq \{1, 2, \dots, 2\}$, we have $\mathrm{lcm}_{i \in \Omega}\{m_i\} = \mathrm{lcm}_{j \in \Gamma}\{m_j\}$ if and only if $\Omega = \Gamma$.

If the minimal free resolution of $R/\mathcal{I}$ over R has the structure of a DG algebra, then so does the minimal free resolution of $R/\mathcal{I}\mathcal{J}$ over R.

Proof. Let $\mathcal{X}$ by the minimal DG algebra resolution of R/I over R and let $\mathcal{Y}$ be the minimal resolution of R/J over R. If J satisfies Property (1), then we can take $\mathcal{Y}$ to be the Koszul complex $K^R(J)$ which is supported on Δ^{n-1} and has simplicial multiplication. If J satisfies Property (2), then we can take $\mathcal{Y}$ to be the Taylor Resolution which is also supported on Δ^{n-1} and has simplicial multiplication. Theorem 3.3.4 tells us that $\mathcal{X} *_R \mathcal{Y}$ is then a minimal resolution of R/IJ over R while Corollary 4.6.8 tells us that it has the structure of a DG algebra. □

When one minimally resolves an R-algebra S over R with a DG algebra, a natural question to ask is whether or not S is a Golod ring (see [2, 19] for details). This question arises due to the work of [2, Section 5] in which Arvamov gives several criterion which can be used to classify a Golod ring. In [42], Vandebogert explicitly defines a trivial Massey operation on the Koszul homology of $R/\mathcal{I}\mathcal{J}$ and then uses [2, Theorem 5.2.2] to prove the quotient is Golod.

Using DG methods, we bypass the need to explictly exhibit a trivial Massey operation. To do this, we impose the restrictions from Corollary 4.6.9 on the ideal $\mathcal{J} \subseteq R$ to obtain a minimal DG R-algebra resolution of the ring

$$F := \frac{R}{\mathcal{I}} \times_W \frac{R}{\mathcal{J}} \cong \frac{R}{\mathcal{I}\mathcal{J}}$$

where $W = R/(\mathcal{I} + \mathcal{J})$. The first step of this approach is exhibited in the following proposition.

Proposition 4.6.10. *Let the ideals $\mathcal{I}, \mathcal{J} \subseteq R$ and the resolutions $\mathcal{X}$ and $\mathcal{Y}$ be as in Corollary 4.6.9, then we have $(\mathcal{X} *_R \mathcal{Y})_{\geq 1} \cdot (\mathcal{X} *_R \mathcal{Y})_{\geq 1} \subseteq \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$.*

Proof. It suffices to show for $\alpha * f_\Omega, \beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ that we have $(\alpha * f_\Omega)(\beta * f_\Gamma) \in \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$. This amounts to showing that

$$(\partial(\alpha)\, \beta * f_\Omega f_\Gamma), (\alpha \partial(\beta) * f_\Omega f_\Gamma), (\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma), (\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma)) \in (\mathfrak{M}_R \mathcal{X} *_R \mathcal{Y}).$$

Since $\mathcal{X}$ is minimal, we have $\partial(\alpha), \partial(\beta) \in \mathfrak{M}_R\mathcal{X}$ and thus $\partial(\alpha)\beta * f_\Omega f_\Gamma \in \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$ with the same holding for $\alpha\partial(\beta) * f_\Omega f_\Gamma$.

Now suppose there exists some $\Omega \in \Delta$ such that $\mathrm{P}_1(f_\Omega) \notin \mathfrak{M}_R\mathcal{Y}$. In this case, we have $\mathrm{P}_1(f_\Omega) = \tau f_{\Omega\setminus\{\omega_1\}}$ for some $\tau \in R^\times$. However, this would then imply that

$$\partial(f_\Omega) = \sum_{i=1}^{u} \mathrm{P}_i(f_\Omega) \notin \mathfrak{M}_R\mathcal{Y}$$

contradicting the minimality of $\mathcal{Y}$. Thus, we have $\mathrm{P}_1(f_\Omega) \in \mathfrak{M}_R\mathcal{Y}$ for all $\Omega \in \Delta$. This tells us that $\alpha\beta * \mathrm{P}_1(f_\Omega) f_\Gamma \in \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$ with the same holding for $\alpha\beta * f_\Omega \mathrm{P}_1(f_\Gamma)$. □

We now obtain our second main theorem (Theorem 4.1.2) in the following result.

Theorem 4.6.11. *Under the conditions of Corollary 4.6.9, the fiber product F is Golod.*

Proof. We let K^F denote the Koszul complex on the minimal generators of the maximal ideal of F. Similarly, we let K^R denote the Koszul complex on the minimal generators of $\mathfrak{M}_R$ and note that $K^R \simeq R/\mathfrak{M}_R \cong k$. Since $\mathcal{X} *_R \mathcal{Y}$ resolves F over R, we have $\mathcal{X} *_R \mathcal{Y} \simeq F$ and thus the following quasimorphisms:

$$F \otimes_R K^R \xleftarrow{\simeq} (\mathcal{X} *_R \mathcal{Y}) \otimes_R K^R \xrightarrow{\simeq} (\mathcal{X} *_R \mathcal{Y}) \otimes_R k.$$

On the left-hand side we have $K^F \cong F \otimes_R K^R$ and we thus find $K^R \simeq (\mathcal{X} *_R \mathcal{Y}) \otimes_R k$. Moreover, since $\mathcal{X} *_R \mathcal{Y}$ is a DG algebra, all of these morphisms are as DG algebras.

We combine this with the fact that $\mathcal{X} *_R \mathcal{Y}$ is minimal to obtain

$$H\left(K^F\right) \cong H((\mathcal{X} *_R \mathcal{Y}) \otimes_R k) = (\mathcal{X} *_R \mathcal{Y}) \otimes_R k.$$

Since Proposition 4.6.10 tells us that all products in $(\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ land in $\mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y})$, we have all products in $((\mathcal{X} *_R \mathcal{Y}) \otimes_R k)_{\geq 1}$ vanish. The same must then hold $H\left(K^F\right)_{\geq 1}$. We thus apply [2, Proposition 5.2.4(1)] to conclude that F is Golod. □

Corollary 4.6.12. *Under the conditions of Corollary 4.6.9 the fiber product F is Tor-friendly. That is, for any finite F-modules M, N such that $\mathrm{Tor}^F(M, N)$ is bounded, we have $\mathrm{pd}_F(M) < \infty$ or $\mathrm{pd}_F(N) < \infty$.*

Proof. Since F is Golod, [27, Theorem 3.1] tells us that F is Tor-friendly. □

Chapter 5

Differential Graded Resolutions and Golod Fiber Products

5.1 Introduction

Throughout the chapter we let $(R, \mathfrak{M}_R)$ be a commutative, regular local (or standard graded) ring. We are interested in identifying commutative, differential graded structures on minimal free resolutions of R-algebras over R. The existence of the multiplicative structure on these resolutions provides us with additional means to identify homological properties of R-algebras. For this chapter, we use the multiplicative structure to identify a family of R-algebras that are Golod rings.

An R-algebra S is a Golod ring if the Poincaré series of the residue field k of S satisfies certain maximal growth conditions [2, Eqn. (5.0.1)]. The Golod property makes it easier to study the homological properties of S-modules [2, Section 5] and thus has been the focus of several studies (see [1, 10–13, 21, 23, 24, 26]). This chapter contributes to the study of Golod rings by using DG algebras along with the results of [2, 27]; see Theorem 5.7.2. This is a consequence of the of the main results of this chapter.

Theorem 5.1.1. *Consider ideals $\mathcal{I}, \mathcal{J} \subseteq R$ generated by regular sequences with $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}}, \frac{R}{\mathcal{J}}\right) = 0$ for all $i > 0$ and set $W = R/(\mathcal{I} + \mathcal{J})$. Given any ideal $\mathcal{I}' \subseteq R$ satisfying $\mathcal{I}' \subseteq \mathcal{I}^2$ and $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}'}, \frac{R}{\mathcal{J}}\right) = 0$*

for all $i > 0$, we have the minimal free resolution of

$$F = \frac{R}{\mathcal{I}' + \mathcal{J}} \times_W \frac{R}{\mathcal{I}} \cong \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle}$$

over R can be realized as DG module over the minimal resolution of $\frac{R}{\mathcal{I}} \times_W \frac{R}{\mathcal{J}}$.

This theorem is realized in Corollary 5.5.5. This result uses the minimal free resolutions of the R-algebras $R/\mathcal{I}'$, $R/\mathcal{I}$, and $R/\mathcal{J}$ all over R; see [16, Theorem 4.7] for the construction of the minimal free resolution of F over R. While this result depends on the minimal resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ both being DG R-algebras, no such restriction is required for the minimal resolution of $R/\mathcal{I}'$. However, when it does have the DG structure, we obtain our next main theorem.

Theorem 5.1.2. *Suppose the ideals $\mathcal{I}', \mathcal{I}, \mathcal{J} \subseteq R$ satisfy the assumptions of Theorem 5.1.1. In addition, if the minimal resolution of $R/\mathcal{I}'$ is a DG R-algebra with a DG morphism to the minimal resolution of $R/\mathcal{I}$, then the minimal resolution of $F \cong R/\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle$ is a DG R-algebra.*

This result is attained in Corollary 5.6.9 and lays the foundation for our final main theorem which is realized in Theorem 5.7.2.

Theorem 5.1.3. *Let $\mathcal{S}$ and $\mathcal{Y}$ denote the minimal free resolutions of $R/\mathcal{I}'$ and $R/\mathcal{J}$, respectively. If $\mathcal{S}_{\geq 1} \cdot \mathcal{S}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{S}$ or $\mathcal{Y}_{\geq 1} \cdot \mathcal{Y}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{Y}$, then F is Golod.*

In section 2 of this chapter, we document results from [16, 17] needed to construct the minimal resolution of F over R. We also give the DG algebra structure of the minimal resolution of $\frac{R}{\mathcal{I}} \times_W \frac{R}{\mathcal{J}}$ over R. Lastly, Notation 5.2.9 establishes the notation we use throughout the rest of the chapter.

Section 3 introduces us to the binary operation we use to realize the minimal resolution of F as a DG module. From there, we prove that this binary operation is unital, distributive, and graded (see Proposition 5.3.5).

Section 4 is entirely dedicated to showing the binary operation satisfies the Leibniz rule for DG-modules. The first part of this section establishes intermediate results that greatly simplify the proof of the Leibniz rule in Theorem 5.4.9.

Section 5 is dedicated to proving the associative property for our binary operation, see Theorem 5.5.4. As in section 4, the first part of the section introduces intermediate results that simplifies the proof of associativity. The section ends by recovering Theorem 5.1.1 as a corollary.

Section 6 is dedicated to realizing the minimal resolution of F over R as a DG R-algebra. The section starts by expanding Notation 5.2.9 to Notation 5.6.1. The additional notation and assumptions gives us Theorem 5.1.2 in Corollary 5.6.9.

Section 7 is dedicated to establishing conditions that make F a Golod ring. To do this, we show that products from the minimal DG algebra resolution of F vanish when tensored with $R/\mathfrak{M}_R$. This then gives us Theorem 5.7.2 which is a rephrasing of Theorem 5.1.3. As a corollary, we establish that F is a Tor-friendly ring; see Corollary 5.7.3.

5.2 Notation and Background

Throughout the chapter, we predominantly work with the R-complexes $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{S}$. At times we will want to work with (hard) truncations of these complexes. We write $\mathcal{X}_{\geq p}$ for the (hard) truncation of $\mathcal{X}$ in degrees greater than or equal to p. This complex is given by

$$\left(\mathcal{X}_{\geq p}\right)_q = \begin{cases} \mathcal{X}_q & q \geq p \\ 0 & q < p \end{cases} \qquad \text{and} \qquad \partial_q^{\mathcal{X}_{\geq p}} = \begin{cases} \partial_q^{\mathcal{X}} & q > p \\ 0 & q \leq p \end{cases}$$

We also utilize the ℓth suspension (or shift) of $\mathcal{X}$, denoted $\Sigma^\ell \mathcal{X}$. In the case $\ell = 1$, we set $\Sigma\mathcal{X} := \Sigma^1\mathcal{X}$.

Definition 5.2.1 ([7, Definition 5.1])**.** A **commutative differential graded algebra over** R (DG algebra for short) is a positively graded R-complex $\mathcal{X}$ equipped with a binary operation $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ called the product and written $(\alpha, \beta) \mapsto \alpha\beta$, that is unital, distributive, associative, and satisfies the following properties:

- **graded commutative:** for all $\alpha, \beta \in \mathcal{X}$ we have $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha \in \mathcal{X}_{|\alpha|+|\beta|}$ and $\alpha^2 = 0$ if $|\alpha|$ is odd;
- **Leibniz rule:** for all $\alpha, \beta \in \mathcal{X}$, we have $\partial^{\mathcal{X}}(\alpha\beta) = \partial^{\mathcal{X}}(\alpha)\,\beta + (-1)^{|\alpha|}\alpha\partial^{\mathcal{X}}(\beta)$.

Definition 5.2.2 ([7, Definition 5.5])**.** A **DG morphism** is a chain map $\phi : \mathcal{S} \to \mathcal{X}$ between two DG R-algebras satisfying:

- $\phi(\rho\delta) = \phi(\rho)\phi(\delta)$ for all $\rho, \delta \in \mathcal{S}$;
- $\phi(1_{\mathcal{S}}) = 1_{\mathcal{X}}$.

If ϕ is a quasiisomorphism, we say it is a **quasiisomorphism of DG algebras**.

Definition 5.2.3 ([7, Definition 5.10])**.** Let $\mathcal{X}$ be a DG R-algebra. A **differential graded module over** $\mathcal{X}$ (DG $\mathcal{X}$-module for short) is an R-complex M equipped with a binary operator $\mathcal{X} \times M \to M$ called the **scalar multiplication** and written $(\alpha, m) \mapsto \alpha m$ that is unital, distributive, associative, graded, and satisfies the following version of the Leibniz Rule:

$$\partial^M(\alpha m) = \partial^{\mathcal{X}}(\alpha)\, m + (-1)^{|\alpha|}\alpha\partial^M(m).$$

To establish the results of this chapter, we utilize indicator functions $\mathbf{1}_{[*]}$ established in [16]. These functions return 1 if the input is true and 0 if it is false. We establish several arithmetic properties of these functions in order to properly utilize them in later results. We omit the proof of the following lemma since one can quickly check each equality by considering cases.

Lemma 5.2.4 ([16, Lemma 2.3])**.** *Let W_1 and W_2 be statements that can be evaluated as true or false and let W_1^c represent "not W_1", then the are following hold.*

(a) $1 = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_1^c]}$

(b) $\mathbf{1}_{[W_1 \cap W_2]} = \mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]}$

(c) $\mathbf{1}_{[W_1 \cup W_2]} = \mathbf{1}_{[W_1]} + \mathbf{1}_{[W_2]} - \mathbf{1}_{[W_1 \cap W_2]}$

(d) *If W_1 implies W_2, then $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = \mathbf{1}_{[W_1]}$, and we say that $\mathbf{1}_{[W_2]}$ is redundant.*

(e) *If W_1 implies W_2^c, then $\mathbf{1}_{[W_1]}\mathbf{1}_{[W_2]} = 0$.*

(f) *If W_1 is a logical statement such that W_1 implies $x = x_0$, then for any function f with x_0 in the domain of f, we have $\mathbf{1}_{[W_1]}f(x) = \mathbf{1}_{[W_1]}f(x_0)$.*

The main construction of interest is the *star product* of two complexes. This definition was established simultaneously but independently in [16] and [42].

Construction 5.2.5 ([16, 42])**.** *The* **Star Product** *of $\mathcal{X}$ and $\mathcal{Y}$, denoted $\mathcal{X} * \mathcal{Y}$, is the chain complex* *given by*

$$(\mathcal{X} * \mathcal{Y})_n = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{n+1} & n \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & n = 0 \\ 0 & n < 0 \end{cases} \quad \textit{and} \quad \partial_n^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{n+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & n \geq 2 \\ \partial_1^{\mathcal{X}} \otimes_R \partial_1^{\mathcal{Y}} & n = 1 \\ 0 & n \leq 0 \end{cases} .$$

In order to impose a DG algebra structure on this construction in [17], $\mathcal{Y}$ is restricted to DG R-algebras supported on a simplicial complex (see Definition 5.2.6). Moreover, the multiplicative structure needs to respect unions of faces from Δ, a structure referred to as simplicial multiplication (see Definition 5.2.7).

Definition 5.2.6 ([17, Definition 3.1])**.** Given a set of vertices $V = \{1, \ldots, n\}$ consider its power set 2^V. We say the $\Delta \subseteq 2^V$ is a **simplicial complex** if

1. $\{v_i\} \in \Delta$ for all i,
2. if $\Omega \in \Delta$ and $\Omega' \subset \Omega$, then $\Omega' \in \Delta$.

Moreover, if $\Delta = 2^V$ then we say it is an $(n-1)$**-simplex** and write Δ^{n-1}.

If a free R-complex $\mathcal{Y}$ has basis $\{f_\Omega\}_{\Omega \in \Delta}$ where the homological degree of f_Ω is equal to cardinality of Ω (i.e., $|f_\Omega| = |\Omega|$), we say $\mathcal{Y}$ is **supported on the simplicial complex** Δ. When $\mathcal{Y}$ is a free resolution with this property, we simply say it is a **Simplicial Resolution**.

Definition 5.2.7. Suppose $\mathcal{Y}$ is a DG algebra and is supported on Δ. We say $\mathcal{Y}$ has **Simplicial Multiplication** if given any two faces $\Omega, \Gamma \in \Delta$ where $\Omega \cup \Gamma \in \Delta$ we have $f_\Omega f_\Gamma \in \operatorname{span}_R(f_{\Omega \cup \Gamma})$.

When we refer to a face $\Omega \in \Delta$ of degree $1 \leq u \leq n$, we write

$$\Omega = \{\omega_1, \ldots, \omega_u\} \subseteq \{1, \ldots, n\}.$$

where we enumerate the vertices in ascending order, i.e., $\omega_i < \omega_j$ if $i < j$. This imposes a natural ordering on the vertices which allows one to define the ℓth vertex removal of a face, $\Omega \mapsto \Omega \backslash \{\omega_\ell\}$. This is the motivating idea for the following definition.

Definition 5.2.8 ([17, Definition 3.4])**.** Let $\mathcal{Y}$ be a complex supported on Δ with n vertices. For all $1 \leq t \leq n$ and $1 \leq \ell \leq t$ we define the ℓ**th vertex removal** as the R-linear map $\mathrm{P}_\ell(\cdot) : \mathcal{Y}_t \to \mathcal{Y}_{t-1}$

whose action on the basis elements, f_Ω, is given by

$$\mathrm{P}_\ell(f_\Omega) = \mathrm{proj}_{f_{\Omega\setminus\{\omega_\ell\}}}\left(\partial^{\mathcal{Y}}(f_\Omega)\right).$$

Extending this notion to $\mathrm{P}_1(\cdot) : \mathcal{Y}_0 \to \mathcal{Y}_{-1} = 0$, i.e., $\mathrm{P}_1(1) = 0$, is useful for later applications.

A natural consequence of this definition is that we have

$$\partial^{\mathcal{Y}}(f_\Omega) = \sum_{\ell=1}^{|\Omega|} \mathrm{P}_\ell(f_\Omega).$$

More consequences are exhibited in [17] but we use them sparringly, so we cite them as they arise.

Notation 5.2.9. Throughout the chapter we will work with $\alpha * f_\Omega, \beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ and fix the following notation.

- The elements $\alpha, \beta \in \mathcal{X}$ have degrees $a := |\alpha|$ and $b := |\beta|$.
- For $\Omega, \Gamma \in \Delta$, we have $u := |\Omega| = |f_\Omega|$ and $v = |\Gamma| = |f_\Gamma|$.
- In $\mathcal{X} *_R \mathcal{Y}$, we have $|\alpha * f_\Omega| = a + u - 1$ and $|\beta * f_\Gamma| = b + v - 1$.

We conclude this section by stating the theorem that gives us the DG algebra we want to build our modules over.

Theorem 5.2.10 ([17, Corollary 6.9])**.** *Consider ideals $\mathcal{I}, \mathcal{J} \subseteq R$ and suppose that;*

1. *we have $\mathrm{Tor}_i^R(R/\mathcal{I}, R/\mathcal{J}) = 0$ for all $i > 0$;*
2. *the minimal resolution $\mathcal{Y}$ of $R/\mathcal{J}$ is either a Koszul complex or Taylor resolution.*

If $\mathcal{X}$ is a minimal DG R-algebra resolution of $R/\mathcal{I}$, then $\mathcal{X} *_R \mathcal{Y}$ is a minimal DG R-algebra where the product of two elements $\alpha * f_\Omega, \beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ is given by the following formula.

$$\begin{aligned}(\alpha * f_\Omega)(\beta * f_\Gamma) = &\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(-1)^{(u-1)(b-1)}\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma \\ &- \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}\alpha\partial(\beta) * f_\Omega f_\Gamma \\ &+ \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}(-1)^{(u-1)b}\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma) \\ &- \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}(-1)^{u(b-1)}\partial(\alpha)\,\beta * f_\Omega f_\Gamma.\end{aligned}$$

5.3 DG Resolutions of Certain Fiber Products

Building off the work of [16], we turn our attention to fiber products of the form

$$F := \frac{R}{\mathcal{I}' + \mathcal{J}} \times_W \frac{R}{\mathcal{I}} \cong \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle}$$

where we have the following restrictions.

1. We have $W = R/(\mathcal{I} + \mathcal{J})$.

2. The minimal resolution $\mathcal{Y}$ of $R/\mathcal{J}$ is either a Koszul complex or Taylor resolution.

3. The ideal $\mathcal{I}$ is generated by a regular sequence and satisfies $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}}, \frac{R}{\mathcal{J}}\right) = 0$ for all $i > 0$.

4. The ideals $\mathcal{I}'$ and $\mathcal{J}$ satisfy $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}'}, \frac{R}{\mathcal{J}}\right) = 0$ for all $i > 0$.

5. We have the containment $\mathcal{I}' \subseteq \mathcal{I}^2$.

To minimally resolve F over R, the author makes use of $\mathcal{X} *_R \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are Koszul complexes resolving $R/\mathcal{I}$ and $R/\mathcal{J}$, respectively, over R. The work of [17, Corollary 6.9] tells us that $\mathcal{X} *_R \mathcal{Y}$ has the structure of a DG R-algebra, so naturally we investigate whether this structure extends to the minimal resolution in [16, Theorem 4.7].

We start by briefly summarizing how one obtains the minimal free resolution of F over R. We let $\mathcal{S}$ be a minimal free resolution of $R/\mathcal{I}'$ and note that the natural surjection $R/\mathcal{I}' \to R/\mathcal{I}$ lifts to a chain map $\phi : \mathcal{S} \to \mathcal{X}$ where ϕ_0 is the identity map on R. This is then used to build a chain map [16, Lemma 4.6] $\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \to \mathcal{X} *_R \mathcal{Y}$ given by

$$\Phi(\delta \otimes f_\Lambda) = \mathbf{1}_{[\ell > 0]}(-1)^{d+\ell}\phi(\delta) * f_\Lambda + \mathbf{1}_{[\ell = 0]}\mathbf{1}_{[d=1]}\partial^{\mathcal{S}}(\delta) * f_\Lambda,$$

where $d := |\delta|$ and $\ell = |\Lambda| = |f_\Lambda|$ (as in the previous section).

Theorem 5.3.1 ([16, Theorem 4.7]). *The R-complex* $\operatorname{Cone}(\Phi)$ *minimally resolves F.*

Remark 5.3.2. Since we are using a mapping cone to resolve F over R, one might reasonably hope to use the work of [25] to establish $\operatorname{Cone}(\Phi)$ as a DG-algebra. This is something we considered but eventually ruled out. In order to apply [25, Lemma 2.1], three conditions need to be met:

1. $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ is a two sided DG $(\mathcal{X} *_R \mathcal{Y})$-module;

2. Φ is a DG module morphism, i.e., it is $(\mathcal{X} *_R \mathcal{Y})$-linear;

3. $\Phi(m)n = m\Phi(n)$ for all $m, n \in \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$.

While we cannot officially rule out the first two criteria, we have an example of why the choice of Φ from [16] cannot possibly satisfy the third criterion. Consider $\rho \otimes 1, \delta \otimes 1 \in \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ such that $|\rho| = 2$ and $|\delta| = 1$. On one hand, we see that $\Phi(\rho \otimes 1) = 0$ and thus

$$\Phi(\rho \otimes 1)(\delta \otimes 1) = 0 \cdot (\delta \otimes 1) = 0.$$

On the other hand, we see that $\Phi(\delta \otimes 1) = \partial_1^{\mathcal{S}}(\delta) * 1 = \partial_1^{\mathcal{S}}(\delta) \in (\mathcal{X} *_R \mathcal{Y})_0 = R$ and thus

$$(\rho \otimes 1)\Phi(\delta \otimes 1) = (\rho \otimes \delta)(\partial_1^{\mathcal{S}}(\delta) * 1) = \rho\partial_1^{\mathcal{S}}(\delta) \otimes 1 \neq 0.$$

Unfortunately, the later results of [25] fail to apply. To apply those results we first need $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ to be a DG algebra resolution of $R/(\mathcal{I}\mathcal{J} : f)$ for some $f \in R$. However, by construction $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$ resolves $\mathcal{I}' \otimes_R \frac{R}{\mathcal{J}} \cong \mathcal{I}' \cdot \frac{R}{\mathcal{I}\mathcal{J}}$ (see [16, Lemma 4.3] and stuff cannot be recognized as the desired quotient ring.

To put a DG structure on $\mathrm{Cone}(\Phi)$, we make use of the underlying graded modules structure:

$$\begin{aligned}
\mathrm{Cone}(\Phi)^\natural &:= \bigoplus_{i \geq 0} \mathrm{Cone}(\Phi)_i \\
&= \left(\bigoplus_{i \geq 1} (\mathcal{X} *_R \mathcal{Y})_i\right) \oplus \left(\bigoplus_{i \geq 1} (\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})_i\right) \\
&= (\mathcal{X} *_R \mathcal{Y})^\natural \oplus (\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})^\natural.
\end{aligned}$$

Using this decomposition, we can write elements of $\mathrm{Cone}(\Phi)$ as linear combinations of elements of the form $\beta * f_\Gamma$ and $\rho \otimes f_\Lambda$. Thus, to define an action of $\mathcal{X} *_R \mathcal{Y}$ on $\mathrm{Cone}(\Phi)$, we simply need to define actions $(\mathcal{X} *_R \mathcal{Y}) \times (\mathcal{X} *_R \mathcal{Y}) \to \mathrm{Cone}(\Phi)$ and $(\mathcal{X} *_R \mathcal{Y}) \times (\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \to \mathrm{Cone}(\Phi)$.

Definition 5.3.3. The DG algebra $\mathcal{X} *_R \mathcal{Y}$ acts on $\mathrm{Cone}(\Phi)$ by the following relations.

1. For $\alpha * f_\Omega \in \mathcal{X} *_R \mathcal{Y}$ and $\beta * f_\Gamma \in \mathcal{X} *_R \mathcal{Y} \subset \mathrm{Cone}(\Phi)$, we have $(\alpha * f_\Omega)(\beta * f_\Gamma)$ is given by multiplication on $\mathcal{X} *_R \mathcal{Y}$ (see Theorem 5.2.10).

2. For $\alpha * f_\Omega \in \mathcal{X} *_R \mathcal{Y}$ and $\rho \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \mathcal{X} *_R \mathcal{Y}$ we have the following cases.

   (a) If $|\alpha * f_\Omega| = 0$, then $(\alpha * f_\Omega)(\rho \otimes f_\Lambda) = \alpha\rho \otimes f_\Omega f_\Lambda$.

   (b) If $|\alpha * f_\Omega| > 0$, then

$$(\alpha * f_\Omega)(\delta \otimes f_\Lambda) = -\mathbf{1}_{[a=1]}(-1)^{(d-1)u}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega f_\Gamma + \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\alpha\phi(\delta) * f_\Omega f_\Gamma.$$

Remark 5.3.4. For elements $\delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}$, it is possible for $f_\Lambda \in \mathcal{Y}_0$. In this case, $\Lambda = \emptyset$ and $f_\Lambda = 1_R$. Moreover, we adopt the convention that $\mathbf{1}_{[\omega_1<\lambda_1]} = 1$ when $\Lambda = \emptyset$; this is done to match the convention establish in Theorem 5.2.10.

We claim that Definition 5.3.3 makes Cone(Φ) into a DG $\mathcal{X} *_R \mathcal{Y}$-module, defined below.

Proposition 5.3.5. *The action of $\mathcal{X} *_R \mathcal{Y}$ on* Cone(Φ) *as given in Definition 5.3.3 is unital, distributive, and graded.*

Proof. Definition 5.3.3(1) satisfies these properties since they are inherited from the DG algebra structure on $\mathcal{X} *_R \mathcal{Y}$. It is straightforward to check that $1_R * 1_R$ make the action defined in 5.3.3(2) unital. Since the scalar multiplication is defined using the basis elements $\{f_\Omega\}_{\Omega\in\Delta}$, it is naturally extended linearly to all elements of $\mathcal{Y}$. Consequently, the distributive property follows since $\partial^{\mathcal{X}}$ and ϕ are R-linear.

To check that 5.3.3(2) is graded, the case of $|\alpha * f_\Omega| = 0$ follows from the fact that we must then have $a = 0 = u$. For the case of $a+u-1 = |\alpha * f_\Omega| > 0$, we observe that $|\delta \otimes f_\Lambda| = |\delta| + |f_\Lambda| = d + \ell$ and $|\phi(\delta)| = |\delta| = d$. This leads to equalities below.

$$\begin{aligned}
|\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega f_\Gamma| &= |\partial^{\mathcal{X}}(\alpha)\,\delta| + |f_\Omega f_\Lambda| \\
&= a - 1 + d + u + \ell \\
&= |\alpha * f_\Omega| + |\delta \otimes f_\Lambda| \\
|\alpha\phi(\delta) * f_\Omega f_\Gamma| &= |\alpha\phi(\delta)| + |f_\Omega f_\Gamma| - 1 \\
&= a + d + u + \ell - 1 \\
&= |\alpha * f_\Omega| + |\delta \otimes f_\Lambda|
\end{aligned}$$

It follows from this that $|(\alpha * f_\Omega)(\delta \otimes f_\Lambda)| = |\alpha * f_\Omega| + |\delta \otimes f_\Lambda|$. □

5.4 Leibniz Rule for the Mapping Cone

We would like to claim that Definition 5.3.3 satisfies the Leibniz rule for DG-modules. To do this, we introduce some intermediate formulas and results to make our proof more tractable. Throughout this section we will see expressions such as $\mathbf{1}_{[a=1]}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{S}}(\phi)\otimes f_\Omega f_\Lambda$. As such, we will appropriately mark which differentials ∂ come from $\mathcal{X}$ and which come from $\mathcal{S}$. However, the differential from $\mathcal{Y}$ will continue to be clear from context, so we continue to write ∂ in place of $\partial^{\mathcal{Y}}$.

The first of our intermediate results gives us explicit formulas for $\partial^{\mathrm{Cone}(\Phi)}$. These will be particularly important for making sense of the expression

$$\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega)\,(\delta\otimes f_\Lambda) + (-1)^{a+u-1}(\alpha * f_\Omega)\partial^{\mathrm{Cone}(\Phi)}(\delta\otimes f_\Lambda)\,.$$

Lemma 5.4.1. *We have the following expressions for* $\partial^{\mathrm{Cone}(\Phi)}$.

$$\begin{aligned}\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega) = {} & \mathbf{1}_{[a>1]}\partial^{\mathcal{X}}(\alpha) * f_\Omega + \mathbf{1}_{[u>1]}(-1)^a\alpha * \partial^{\mathcal{Y}}(f_\Omega) \\ & + \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial^{\mathcal{X}}(\alpha) * \partial^{\mathcal{Y}}(f_\Omega)\,.\end{aligned} \tag{5.4.1.1}$$

$$\begin{aligned}\partial^{\mathrm{Cone}(\Phi)}(\delta\otimes f_\Lambda) = {} & \mathbf{1}_{[\ell>0]}(-1)^{d+\ell}\partial^{\mathcal{S}}(\delta) * f_\Lambda + \mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}\partial^{\mathcal{S}}(\delta)\otimes f_\Lambda \\ & + \mathbf{1}_{[d>1]}\partial^{\mathcal{S}}(\delta)\otimes f_\Lambda + (-1)^d\delta\otimes\partial\,(f_\Lambda)\end{aligned} \tag{5.4.1.2}$$

Proof. By definition of $\mathrm{Cone}\,(\Phi)$, we have $\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega) = \partial^{\mathcal{X}*\mathcal{Y}}(\alpha * f_\Omega)$. The first equation then results from [16, Lemma 3.3]. For the second equation, we have

$$\partial^{\mathrm{Cone}(\Phi)}(\delta\otimes f_\Lambda) = \Phi(\delta\otimes f_\Lambda) + \partial^{\mathcal{S}_{\geq 1}\otimes\mathcal{Y}}(\delta\otimes f_\Lambda).$$

The full expression then follows by definition of the respective maps. □

These expressions lay the foundation to develop formulas for $\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega)\,(\delta\otimes f_\Lambda)$ and $(\alpha * f_\Omega)\partial^{\mathrm{Cone}(\Phi)}(\delta\otimes f_\Lambda)$. In order to get workable formulas, we need the following lemma to simplify expressions so that we can properly combine similar terms.

Lemma 5.4.2. *If $u \geq 1$ and $\ell \geq 0$, then we have the following equalities.*

$$\begin{aligned}&\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}\partial\,(f_\Omega)\,f_\Lambda - \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda \\ &\quad= \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1]}\partial\,(f_\Omega)\,f_\Lambda - \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda\end{aligned} \tag{5.4.2.1}$$

$$\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda = \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda \tag{5.4.2.2}$$

$$\mathbf{1}_{[\ell>0]}\,\mathrm{P}_1(f_\Lambda) = \mathrm{P}_1(f_\Lambda) \tag{5.4.2.3}$$

$$\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\lambda_1<\omega_1]} = \mathbf{1}_{[\lambda_1<\omega_1]} \tag{5.4.2.4}$$

Proof. To start, we observe that we have the following decompositions of $\mathbf{1}_{[u+\ell>1]}$.

$$\mathbf{1}_{[u+\ell>1]} = \mathbf{1}_{[u>1]} + \mathbf{1}_{[u=1]}\mathbf{1}_{[\ell>0]} = \mathbf{1}_{[\ell>0]} + \mathbf{1}_{[\ell=0]}\mathbf{1}_{[u>1]}$$

We use the first expression for (5.4.2.1) while we use the second for (5.4.2.2). Using the first line, we have the following.

$$\begin{aligned}&\mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda - \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda \\ &\qquad= \mathbf{1}_{[u=1]}\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda \\ &\qquad\overset{3.2.3(\mathrm{f})}{=} \mathbf{1}_{[u=1]}\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\omega_1<\lambda_1]}\partial\,(f_\Omega)\,f_\Lambda \\ &\qquad= \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1]}\partial\,(f_\Omega)\,f_\Lambda - \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}\partial\,(f_\Omega)\,f_\Lambda\end{aligned}$$

Rearranging this expression provides Equation (5.4.2.1).

For (5.4.2.2), we recall that if ω_2 exists while λ_1 does not, then we must have $\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]} = 0$ by our established convention. We now observe the following.

$$\begin{aligned}\mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda &= \mathbf{1}_{[\ell>0]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda + \mathbf{1}_{[\ell=0]}\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda \\ &= \mathbf{1}_{[\ell>0]}\mathbf{1}_{[\omega_1\leq\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\,f_\Lambda\end{aligned}$$

For (5.4.2.3) we recall that $\mathrm{P}_1(1) = 0$ by definition and thus $\mathbf{1}_{[\ell=0]}\,\mathrm{P}_1(f_\Lambda) = 0$. From here, we observe that

$$\mathrm{P}_1(f_\Gamma) \overset{3.2.3(\mathrm{a})}{=} \mathbf{1}_{[\ell>0]}\,\mathrm{P}_1(f_\Gamma) + \mathbf{1}_{[\ell=0]}\,\mathrm{P}_1(f_\Gamma) = \mathbf{1}_{[\ell>0]}\,\mathrm{P}_1(f_\Gamma)\,.$$

Equation (5.4.2.4) is proved using the same means. □

Lemma 5.4.3. *We have the following formulas related to how $\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega)$ acts on $\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}$.*

$$\mathbf{1}_{[a>1]}(\partial^{\mathcal{X}}(\alpha) * f_\Omega)(\delta \otimes f_\Lambda) = \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \tag{5.4.3.1}$$

$$\mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}(\partial^{\mathcal{X}}(\alpha) * \partial\,(f_\Omega))(\delta \otimes f_\Lambda) = \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\,(f_\Omega)\,f_\Lambda \tag{5.4.3.2}$$

$$\begin{aligned}\mathbf{1}_{[u>1]}(-1)^a(\alpha * \partial\,(f_\Omega))(\delta \otimes f_\Lambda) = {}& \mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\,(f_\Omega)\,f_\Lambda \\ &+ \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \partial\,(f_\Omega)\,f_\Lambda \\ &- \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Lambda\end{aligned} \tag{5.4.3.3}$$

Proof. Equations (5.4.3.1) and (5.4.3.2) are direct applications of Definition 5.3.3. For Equation (5.4.3.3), note that $\mathrm{P}_i(f_\Omega) \in \mathrm{span}_R\left(f_{\Omega\setminus\{\omega_i\}}\right)$ and thus the first vertex of $\mathrm{P}_1(f_\Omega)$ is ω_2 whereas for $i > 1$ the first vertex of $\mathrm{P}_i(f_\Omega)$ is ω_1. Tthis observation yields the starred equality below.

$$\begin{aligned}\mathbf{1}_{[u>1]}(-1)^a(\alpha * \partial\,(f_\Omega))(\delta \otimes f_\Lambda) &= \mathbf{1}_{[u>1]}(-1)^a\sum_{i=1}^{u}(\alpha * \mathrm{P}_i(f_\Omega))(\delta \otimes f_\Lambda) \\ &\overset{*}{=} -\mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(d-1)(u-1)+a}\sum_{i=1}^{u}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \mathrm{P}_i(f_\Omega)\,f_\Lambda \\ &\quad + \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\sum_{i=2}^{u}\alpha\phi(\delta) * \mathrm{P}_i(f_\Omega)\,f_\Lambda \\ &\quad + \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_2<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Lambda \\ &\overset{3.2.3(\mathrm{f})}{=} \mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\,(f_\Omega)\,f_\Lambda \\ &\quad + \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\sum_{i=1}^{u}\alpha\phi(\delta) * \mathrm{P}_i(f_\Omega)\,f_\Lambda \\ &\quad - \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Lambda\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_2<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda \\
\overset{3.2.3(c)}{=}\ & \mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(d-1)(u-1)+a}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\,(f_\Omega)\, f_\Lambda \\
&+ \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \partial\,(f_\Omega)\, f_\Lambda \\
&- \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1\le\omega_2]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda
\end{aligned}
$$

To conclude the proof, we recall that $0 = \mathbf{1}_{[u>1]}\mathbf{1}_{[\lambda_1=\omega_2]}\,\mathrm{P}_1(f_\Omega)\, f_\Lambda$ since either $\ell = 0$ meaning λ_1 does not exist and $\mathbf{1}_{[\lambda_1=\omega_2]} = 0$ or $\ell > 0$ in which case $\lambda_1 = \omega_2$ means $0 = \mathrm{P}_1(f_\Omega)\, f_\Lambda$. We thus have

$$
\mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1\le\omega_2]}\,\mathrm{P}_1(f_\Omega)\, f_\Lambda \overset{3.2.3(c)}{=} \mathbf{1}_{[u>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\,\mathrm{P}_1(f_\Omega)\, f_\Lambda.
$$

From here we apply Equation (5.4.2.1) to recover Equation (5.4.3.3). ☐

Proposition 5.4.4. *For $\alpha * f_\Omega \in \mathcal{X} *_R \mathcal{Y}$ and $\delta \otimes f_\Lambda \in \mathcal{S}_{\ge 1} \otimes_R \mathcal{Y}$, we have*

$$
\begin{aligned}
\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega)\,(\delta \otimes f_\Lambda) = &\ \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\
&+ \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\,(f_\Omega)\, f_\Lambda \\
&+ \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \partial\,(f_\Omega)\, f_\Lambda \\
&- \mathbf{1}_{[u+\ell>1]}\mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda.
\end{aligned} \tag{5.4.4.1}
$$

Proof. To obtain this expression, we first note that $\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega) = \partial^{\mathcal{X}*\mathcal{Y}}(\alpha * f_\Omega)$ by definition of the mapping cone. Thus, we obtain this expression by summing Equations (5.4.3.1) - (5.4.3.3). When doing this, note that Equation (5.4.3.2) combines with the tensor term of (5.4.3.3) since

$$
\mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)} \overset{3.2.3(a)(f)}{=} \mathbf{1}_{[a=1]}\mathbf{1}_{[u>1]}(-1)^{(d-1)(u-1)} + \mathbf{1}_{[a=1]}\mathbf{1}_{[u=1]}.
$$

After combining the tensor terms, the resulting equation is precisely (5.4.4.1). ☐

To obtain a formula for $(\alpha * f_\Omega)\partial^{\mathrm{Cone}(\Phi)}(\delta \otimes f_\Lambda)$ we consider the expressions $(\alpha * f_\Omega)\Phi(\delta \otimes f_\Lambda)$ and $(\alpha * f_\Omega)\partial^{\mathcal{S}_{\ge 1}\otimes\mathcal{Y}}(\delta \otimes f_\Lambda)$.

Lemma 5.4.5. *We have the following formulas related to $(\alpha * f_\Omega)\Phi(\delta \otimes f_\Lambda)$.*

$$
\mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}(\alpha * f_\Omega)(\partial^{\mathcal{S}}(\delta) \otimes f_\Lambda) = \mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \tag{5.4.5.1}
$$

$$\mathbf{1}_{[\ell>0]}(-1)^{d+\ell}(\alpha * f_\Omega)(\phi(\delta) * f_\Lambda) = -\mathbf{1}_{[\omega_1=\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+u+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda \tag{5.4.5.2}$$
$$- \mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+u+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda$$
$$+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d=1]}\mathbf{1}_{[\ell>0]}(-1)^{\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda$$
$$+ \mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$
$$+ \mathbf{1}_{[\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\, \phi(\delta) * f_\Omega f_\Lambda$$

$$(-1)^{a+u-1}(\alpha * f_\Omega)\Phi(\delta \otimes f_\Lambda) = \mathbf{1}_{[\omega_1=\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda \tag{5.4.5.3}$$
$$+ \mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda$$
$$- \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d=1]}(-1)^{a+u+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda$$
$$- \mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{du+a+u+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$
$$- \mathbf{1}_{[\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\, \phi(\delta) * f_\Omega f_\Lambda$$

Proof. We first observe that by the construction of ϕ, we have $(\partial^{\mathcal{X}}_1 \circ \phi_1)(\delta) = (\phi_0 \circ \partial^{\mathcal{S}}_1)(\delta) = \partial^{\mathcal{S}}(\delta)$ and thus $\mathbf{1}_{[d=1]}\partial^{\mathcal{X}}(\phi(\delta)) = \mathbf{1}_{[d=1]}\partial^{\mathcal{S}}(\delta)$. This allows us to observe the following for Equation (5.4.5.1).

$$\mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \overset{3.2.3(\mathrm{f})}{=} \mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}\mathbf{1}_{[\omega_1<\lambda_1]}\alpha\partial^{\mathcal{S}}(\delta) * f_\Omega f_\Lambda$$
$$= \mathbf{1}_{[\ell=0]}\mathbf{1}_{[d=1]}(\alpha * f_\Omega)(\partial^{\mathcal{S}}(\delta) * f_\Lambda)$$

For Equation (5.4.5.2), we use Theorem 5.2.10 to expand $\mathbf{1}_{[\ell>0]}(\alpha * f_\Omega)(\phi(\delta) * f_\Lambda)$. From there, we apply Lemma 5.4.2 to deal with the indicator function $\mathbf{1}_{[\ell>0]}$.

We obtain $(\alpha * f_\Omega)\Phi(\delta \otimes f_\Lambda)$ by summing Equations (5.4.5.1) and (5.4.5.2). When doing this, we combine the $\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda$ terms by noting that $1 = \mathbf{1}_{[\ell=0]} + \mathbf{1}_{[\ell>0]}$. We then obtain Equation (5.4.5.3) by multiplying by $(-1)^{a+u-1}$ and use Lemma 3.2.3(f) to get $\mathbf{1}_{[a=1]}(-1)^a = -\mathbf{1}_{[a=1]}$ for the final line of the equation. □

Lemma 5.4.6. *We have the following formulas related to* $(\alpha * f_\Omega)\partial^{\mathcal{S}_{\geq 1} \otimes \mathcal{Y}}(\delta \otimes f_\Lambda)$.

$$\mathbf{1}_{[d>1]}(\alpha * f_\Omega)(\partial^{\mathcal{S}}(\delta) \otimes f_\Lambda) = -\mathbf{1}_{[a=1]}\mathbf{1}_{[d>1]}(-1)^{du}\partial^{\mathcal{X}}(\alpha)\, \partial^{\mathcal{S}}(\delta) \otimes f_\Omega f_\Gamma \tag{5.4.6.1}$$
$$+ \mathbf{1}_{[d>1]}\mathbf{1}_{[\omega_1<\gamma_1]}(-1)^{(d-1)(u-1)+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Gamma$$

$$(-1)^d(\alpha * f_\Omega)(\delta \otimes \partial(f_\Omega)) = \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial(f_\Gamma) \tag{5.4.6.2}$$
$$- \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega \partial(f_\Lambda)$$
$$+ \mathbf{1}_{[\lambda_1=\omega_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+u+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda$$
$$- \mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$

$$(-1)^{a+u-1}(\alpha * f_\Omega)\partial^{\mathcal{S}_{\geq 1}\otimes\mathcal{Y}}(\delta \otimes f_\Lambda) = -\mathbf{1}_{[a=1]}\mathbf{1}_{[d>1]}(-1)^{du+u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{S}}(\delta) \otimes f_\Omega f_\Lambda \tag{5.4.6.3}$$
$$+ \mathbf{1}_{[d>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda$$
$$- \mathbf{1}_{[a=1]}(-1)^{d(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial(f_\Lambda)$$
$$+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{(d-1)u+a+\ell}\alpha\phi(\delta) * f_\Omega \partial(f_\Lambda)$$
$$- \mathbf{1}_{[\lambda_1=\omega_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda$$
$$+ \mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{(d-1)u+a+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$

Proof. Equation (5.4.6.1) follows from Definition 5.3.3 along with the observation that $\partial_1^{\mathcal{S}} = \partial_1^{\mathcal{X}} \circ \phi_1$. To establish Equation (5.4.6.2), we observe that the first vertex of $\Lambda\backslash\{\lambda_1\}$ is λ_2. This observation is used along with Definition 5.3.3 in the starred equality of the following expression.

$$(-1)^d(\alpha * f_\Omega)(\delta \otimes \partial(f_\Lambda)) = (-1)^d \sum_{i=1}^{\ell}(\alpha * f_\Omega)(\delta \otimes \mathrm{P}_i(f_\Lambda))$$
$$\overset{*}{=} - \mathbf{1}_{[a=1]}(-1)^{(d-1)u+d}\sum_{i=1}^{\ell}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega\, \mathrm{P}_i(f_\Lambda)$$
$$- \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+\ell}\sum_{i=2}^{\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_i(f_\Lambda)$$
$$- \mathbf{1}_{[\omega_1<\lambda_2]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$
$$\overset{3.2.3(c)}{=} \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial(f_\Lambda)$$
$$- \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+\ell}\sum_{i=1}^{\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_i(f_\Lambda)$$
$$- \mathbf{1}_{[\lambda_1\leq\omega_1<\lambda_2]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Lambda)$$
$$\overset{3.2.3(c)}{=} \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial(f_\Lambda)$$
$$- \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega \partial(f_\Lambda)$$

$$
\begin{aligned}
&- \mathbf{1}_{[\lambda_1=\omega_1]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega \, \mathrm{P}_1(f_\Lambda) \\
&- \mathbf{1}_{[\lambda_1<\omega_1<\lambda_2]}(-1)^{du+\ell}\alpha\phi(\delta) * f_\Omega \, \mathrm{P}_1(f_\Lambda)
\end{aligned}
$$

For future application, we wish to rewrite the line term $\alpha\phi(\delta) * f_\Omega \partial\,(f_\Lambda)$. To do this we observe the following chain of equalities.

$$
\mathbf{1}_{[\omega_1<\lambda_1]} f_\Omega \partial\,(f_\Omega) \overset{5.4.2}{=} \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[\ell>0]} f_\Omega \partial\,(f_\Lambda) \overset{3.2.3(c)}{=} \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]} f_\Omega \partial\,(f_\Lambda)
$$

To obtain the Equation (5.4.6.2), we apply [17, Lemma 3.8] to obtain the first equality below.

$$
\mathbf{1}_{[\lambda_1=\omega_1]} f_\Omega \, \mathrm{P}_1(f_\Lambda) = \mathbf{1}_{[\lambda_1=\omega_1]}(-1)^u \, \mathrm{P}_1(f_\Omega)\, f_\Lambda \overset{3.2.3(d)}{=} \mathbf{1}_{[\lambda_1=\omega_1]}\mathbf{1}_{[u+\ell>1]}(-1)^u \, \mathrm{P}_1(f_\Omega)\, f_\Lambda
$$

Finally, to obtain Equation (5.4.6.3), we sum Equations (5.4.6.1) and (5.4.6.2) and then multiply by $(-1)^{a+u-1}$. When doing this, recall that $\mathbf{1}_{[a=1]}(-1)^a = -\mathbf{1}_{[a=1]}$. ☐

Proposition 5.4.7. *For all* $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ *and all* $\delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \mathrm{Cone}\,(\Phi)$ *we have the following formula.*

$$
\begin{aligned}
(-1)^{a+u-1}(\alpha * f_\Omega)\partial^{\mathrm{Cone}(\Phi)}(\delta \otimes f_\Lambda) = {} & \mathbf{1}_{[\omega_1<\lambda_1<\omega_2]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Lambda \qquad (5.4.7.1) \\
& + \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \\
& - \mathbf{1}_{[\lambda_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\
& - \mathbf{1}_{[a=1]}\mathbf{1}_{[d>1]}(-1)^{du+u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{S}}(\delta) \otimes f_\Omega f_\Lambda \\
& - \mathbf{1}_{[a=1]}(-1)^{d(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial\,(f_\Lambda) \\
& + \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{(d-1)u+a+\ell}\alpha\phi(\delta) * f_\Omega \partial\,(f_\Lambda)
\end{aligned}
$$

Proof. Since $\partial^{\mathrm{Cone}(\Phi)}(\delta \otimes f_\Lambda) = \Phi(\delta \otimes f_\Lambda) + \partial^{\mathcal{S}_{\geq 1}\otimes\mathcal{Y}}(\delta \otimes f_\Lambda)$, we sum Equations (5.4.5.3) and (5.4.6.3). When summing these equations, we use the following relation.

$$
\begin{aligned}
\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda &\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d>1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \\
&\quad + \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d=1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \\
&\overset{3.2.3(f)}{=} \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d>1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda
\end{aligned}
$$

$$- \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[d=1]}(-1)^{u+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda$$

Using this relation establishes the second line of Equation (5.4.7.1) and thus the entire equation. □

Before we prove that Definition 5.3.3 satisfies the Leibniz rule, we need one more relation. In particular, we want to combine the first line of Equation (5.4.4.1) and the third line of (5.4.7.1). The main tool for this is the following lemma.

Lemma 5.4.8. *Let a be a positive integer and let $\Omega, \Lambda \in \Delta$, then we have*

$$\mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda - \mathbf{1}_{[a=1]}\mathbf{1}_{[\lambda_1<\omega_1]}f_\Omega f_\Lambda = \mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda - \mathbf{1}_{[a=1]}f_\Omega f_\Lambda \tag{5.4.8.1}$$

Proof. To start, we have $\mathbf{1}_{[\omega_1=\lambda_1]}f_\Omega f_\Lambda = 0$. We use this to obtain the starred equality below.

$$\begin{aligned} f_\Omega f_\Lambda &= \mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda + \mathbf{1}_{[\omega_1=\lambda_1]}f_\Omega f_\Lambda + \mathbf{1}_{[\lambda_1<\omega_1]}f_\Omega f_\Lambda \\ &\overset{*}{=} \mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda + \mathbf{1}_{[\lambda_1<\omega_1]}f_\Omega f_\Lambda \end{aligned}$$

Rearranging the above gives the first equality below.

$$\begin{aligned} \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda - \mathbf{1}_{[a=1]}\mathbf{1}_{[\lambda_1<\omega_1]}f_\Omega f_\Lambda &= \mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda + \mathbf{1}_{[a=1]}\mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda - \mathbf{1}_{[a=1]}f_\Omega f_\Lambda \\ &\overset{3.2.3(a)}{=} \mathbf{1}_{[\omega_1<\lambda_1]}f_\Omega f_\Lambda - \mathbf{1}_{[a=1]}f_\Omega f_\Lambda \end{aligned}$$

□

Theorem 5.4.9. *The action of $\mathcal{X} *_R \mathcal{Y}$ on* Cone(Φ) *from Definition 5.3.3 satisfies the Leibniz rule.*

Proof. To start, consider the case where $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ acts on $\delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \text{Cone}(\Phi)$. In the equalities below, the starred equality follows by summing Equations (5.4.4.1) and (5.4.7.1).

$$\begin{aligned} \partial^{\text{Cone}(\Phi)}((\alpha * f_\Omega)(\delta \otimes f_\Lambda)) &\overset{5.3.3}{=} - \mathbf{1}_{[a=1]}(-1)^{(d-1)u}\Phi(\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega f_\Lambda) \\ &\quad - \mathbf{1}_{[a=1]}(-1)^{(d-1)u}\partial^{\mathcal{S}_{\geq 1}\otimes\mathcal{Y}}(\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega f_\Lambda) \\ &\quad + \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}*\mathcal{Y}}(\alpha\phi(\delta) * f_\Omega f_\Lambda) \\ &= - \mathbf{1}_{[a=1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\ &\quad - \mathbf{1}_{[a=1]}\mathbf{1}_{[d>1]}(-1)^{(d-1)u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{S}}(\delta) \otimes f_\Omega f_\Lambda \end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\left(f_\Omega\right) f_\Lambda \\
&- \mathbf{1}_{[a=1]}(-1)^{d(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial\left(f_\Lambda\right) \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \partial\left(f_\Omega\right) f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{(d-1)u+a+\ell}\alpha\phi(\delta) * f_\Omega \partial\left(f_\Lambda\right) \\
\overset{(5.4.8.1)}{=} &\mathbf{1}_{[a>1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\
&+ \mathbf{1}_{[a=1]}(-1)^{(d-1)(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes \partial\left(f_\Omega\right) f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{du+a+\ell}\alpha\phi(\delta) * \partial\left(f_\Omega\right) f_\Lambda \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{d(u-1)+a+\ell}\alpha\partial^{\mathcal{X}}(\phi(\delta)) * f_\Omega f_\Lambda \\
&- \mathbf{1}_{[a=1]}\mathbf{1}_{[\lambda_1<\omega_1]}(-1)^{d(u-1)+\ell}\partial^{\mathcal{X}}(\alpha)\,\phi(\delta) * f_\Omega f_\Lambda \\
&- \mathbf{1}_{[a=1]}\mathbf{1}_{[d>1]}(-1)^{(d-1)u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{S}}(\phi) \otimes f_\Omega f_\Lambda \\
&- \mathbf{1}_{[a=1]}(-1)^{d(u-1)}\partial^{\mathcal{X}}(\alpha)\,\delta \otimes f_\Omega \partial\left(f_\Lambda\right) \\
&+ \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[u+\ell>1]}(-1)^{(d-1)u+a+\ell}\alpha\phi(\delta) * f_\Omega \partial\left(f_\Lambda\right) \\
\overset{*}{=} &\,\partial^{\mathrm{Cone}(\Phi)}(\alpha * f_\Omega)\,(\delta \otimes f_\Lambda) \\
&+ (-1)^{a+u-1}(\alpha * f_\Omega)\partial^{\mathrm{Cone}(\Phi)}(\delta \otimes f_\Lambda)
\end{aligned}
$$

Next, we consider the case of $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ and $\beta * f_\Gamma \in \mathcal{X} *_R \mathcal{Y} \subset \mathrm{Cone}\,(\Phi)$. Since $\partial^{\mathrm{Cone}(\Phi)} = \partial^{\mathcal{X} *_R \mathcal{Y}}$ when we restrict to the subcomplex $\mathcal{X} *_R \mathcal{Y}$, which is a DG algebra, we inherit the Leibniz rule from [17, Theorem 5.9].

The final case is when we have $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_0 \cong R$. In this case, we automatically have Leibniz rule due to the R-linearity of $\partial^{\mathrm{Cone}(\Phi)}$. □

5.5 Associativity for the Mapping Cone

This next section is dedicated to showing that Definition 5.3.3 satisfies the associative property. Similar to the Leibniz rule, we see that [17] gives us this property when we restrict to $\mathcal{X} *_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ since $\mathcal{X} *_R \mathcal{Y}$ is a DG algebra. Thus, we turn our attention to proving

$$(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)] = [(\alpha * f_\Omega)(\beta * f_\Gamma)]\,(\delta \otimes f_\Lambda).$$

While this will be less computationally intensive than proving the Leibniz rule, we will still benefit from introducing some intermediate results.

Lemma 5.5.1. *Given $\Omega, \Gamma, \Lambda \in \Delta$, we have the following relations of indicator functions.*

$$\begin{aligned}
\mathbf{1}_{[\omega_1<\min\{\lambda_1,\gamma_1\}]} &= \mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[\omega_1<\gamma_1]}\\
\mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]} &= \mathbf{1}_{[\omega_1\le\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}\\
\mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\min\{\gamma_2,\lambda_1\}]} &= \mathbf{1}_{[\gamma_1<\omega_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}
\end{aligned}$$

Proof. These equalities can be checked by cases or shown using Lemma 3.2.3. The first equation follows immediately from Lemma 3.2.3(b). The second and third equalities use similar reasoning, so we omit their proofs. □

Proposition 5.5.2. *For $\alpha * f_\Omega, \beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\ge 1}$ and $\delta \otimes f_\Lambda \in \mathcal{S}_{\ge 1} \otimes_R \mathcal{Y}$, we have the following expression for $(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)]$.*

$$\begin{aligned}
(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)] = {}& \mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(d-1)(u+v)}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[b=1]}\mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[\omega_1<\gamma_1]}(-1)^{d(u+v)+d+\ell}\alpha\partial^{\mathcal{X}}(\beta)\,\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&- \mathbf{1}_{[\omega_1\le\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1\le\gamma_1<\omega_2]}(-1)^{bu+du+dv+b+u+\ell}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
&+ \mathbf{1}_{[\gamma_1<\omega_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{bu+du+dv+b+\ell}\alpha\beta\phi(\delta) * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&- \mathbf{1}_{[a=1]}\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{bu+du+dv+d+u+\ell}\partial^{\mathcal{X}}(\alpha)\,\beta\phi(\delta) * f_\Omega f_\Gamma f_\Lambda
\end{aligned} \tag{5.5.2.1}$$

Proof. To start, we apply Definition 5.3.3 to make the observation below.

$$\begin{aligned}
(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)] = {}& -\mathbf{1}_{[b=1]}(-1)^{(d-1)v}(\alpha * f_\Omega)(\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Gamma f_\Lambda)\\
&+ \mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{d(v-1)+\ell}(\alpha * f_\Omega)(\beta\phi(\delta) * f_\Gamma f_\Lambda)
\end{aligned}$$

To address $(\alpha * f_\Omega)(\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Gamma f_\Lambda)$, we first note that $\mathcal{Y}$ is a Koszul complex on $\mathcal{J} = \langle m_1, \ldots, m_n \rangle$, so it is supported on the $(n-1)$-simplex Δ^{n-1}. Thus, $\Gamma\cup\Lambda \in \Delta^{n-1}$ and the simplicial multiplication of $\mathcal{Y}$ tells us that $f_\Gamma f_\Lambda \in \operatorname{span}_R(f_{\Lambda\cup\Gamma})$. Thus, when we get the following expression where apply Lemma 5.5.1 in the same equality marked with Lemma 3.2.3(f).

$$\begin{aligned}
\mathbf{1}_{[b=1]}(\alpha * f_\Omega)(\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Gamma f_\Lambda) &\overset{5.3.3}{=} -\mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(b+d-2)u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[b=1]}\mathbf{1}_{[\omega_1<\min\{\lambda_1,\gamma_1\}]}(-1)^{(b+d-1)(u-1)+v+\ell}\alpha\partial^{\mathcal{X}}(\beta)\,\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\overset{3.2.3(f)}{=} -\mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(d-1)u}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[b=1]}\mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[\omega_1<\gamma_1]}(-1)^{d(u-1)+v+\ell}\alpha\partial^{\mathcal{X}}(\beta)\,\phi(\delta) * f_\Omega f_\Gamma f_\Lambda
\end{aligned}$$

Multiplying by $(-1)^{(d-1)v+1}$ gives the first two lines of Equation (5.5.2.1). To obtain the remaining three lines, we again call upon a consequence of $\mathcal{Y}$ being supported on Δ^{n-1}:

$$\mathbf{1}_{[\gamma_1<\lambda_1]}\,\mathrm{P}_1(f_\Gamma f_\Lambda) = \mathbf{1}_{[\gamma_1<\lambda_1]}\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda.$$

This equality comes from [17, Lemma 6.1] and is used in the product $(\alpha * f_\Omega)(\beta\phi(\delta)\otimes f_\Gamma f_\Lambda)$. We also note that by construction $\delta \in \mathcal{S}_{\geq 1}$ and by initial condition $\beta * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$. This, in turn, means $b+d \geq 2$ and therefore $\mathbf{1}_{[b+d=1]} = 0$ which we use in the following.

$$\begin{aligned}
\mathbf{1}_{[\gamma_1<\lambda_1]}(\alpha * f_\Omega)(\beta\phi(\delta) * f_\Gamma f_\Lambda) &\overset{5.3.3}{=} \mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(b+d-1)(u-1)}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
&\quad - \mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b+d=1]}\alpha\partial^{\mathcal{X}}(\beta\phi(\delta)) * f_\Omega f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\min\{\gamma_2,\lambda_1\}]}(-1)^{(b+d)(u-1)}\alpha\beta\phi(\delta) * f_\Omega\,\mathrm{P}_1(f_\Gamma f_\Lambda) \\
&\quad - \mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(b+d-1)u}\partial^{\mathcal{X}}(\alpha)\,\beta\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\overset{5.5.1}{=} \mathbf{1}_{[\omega_1\leq\gamma_1<\lambda_1]}\mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(b+d-1)(u-1)}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\,f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[\gamma_1<\omega_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{(b+d)(u-1)}\alpha\beta\phi(\delta) * f_\Omega\,\mathrm{P}_1(f_\Gamma)\,f_\Lambda \\
&\quad - \mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(b+d-1)u}\partial^{\mathcal{X}}(\alpha)\,\beta\phi(\delta) * f_\Omega f_\Gamma f_\Lambda
\end{aligned}$$

Multiplying by $(-1)^{d(v-1)+\ell}$ produces the final three lines of Equation (5.5.2.1). □

Now that we have a formula for $(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta\otimes f_\Lambda)]$, we consider the case where we first expand $(\alpha * f_\Omega)(\beta * f_\Gamma)$. In order to make sense of this expansion, we use the following lemma.

Lemma 5.5.3. *Given $\alpha,\beta \in \mathcal{X}_{\geq 1}$ and $f_\Omega, f_\Gamma \in \mathcal{Y}_{\geq 1}$, we have the following formulas for products* *with* $\delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}$.

$$\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(-1)^{(b-1)(u-1)}(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\delta \otimes f_\Lambda) \tag{5.5.3.1}$$
$$= -\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\omega_1 \leq \gamma_1 < \lambda_1]}(-1)^{bu+du+dv+b+u+\ell}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda$$

$$-\mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[b=1]}(\alpha\partial^{\mathcal{X}}(\beta) * f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \tag{5.5.3.2}$$
$$= \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(d-1)(u+v)}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda$$
$$- \mathbf{1}_{[\omega_1 < \gamma_1]}\mathbf{1}_{[\omega_1 < \lambda_1]}\mathbf{1}_{[b=1]}(-1)^{d(u+v)+d+\ell}\alpha\partial^{\mathcal{X}}(\beta)\,\phi(\delta) * f_\Omega f_\Gamma f_\Lambda$$

$$\mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}(-1)^{b(u-1)}(\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma))(\delta \otimes f_\Lambda) \tag{5.5.3.3}$$
$$= \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]}\mathbf{1}_{[\gamma_1 < \omega_1 < \lambda_1]}(-1)^{bu+du+dv+b+\ell}\alpha\beta\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda$$

$$-\mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}(-1)^{(b-1)u}(\partial^{\mathcal{X}}(\alpha)\,\beta * f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \tag{5.5.3.4}$$
$$= \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(d-1)(u+v)}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda$$
$$- \mathbf{1}_{[\gamma_1 < \omega_1]}\mathbf{1}_{[\gamma_1 < \lambda_1]}\mathbf{1}_{[a=1]}(-1)^{bu+du+dv+d+u+\ell}\partial^{\mathcal{X}}(\alpha)\,\beta\phi(\delta) * f_\Omega f_\Gamma f_\Lambda$$

Proof. The proofs of Equations (5.5.3.1) and (5.5.3.3) are nearly identical, so we only show the proof of the former. Similarly, we will show the proof of Equation (5.5.3.4) and omit the proof of (5.5.3.2).

For Equation (5.5.3.1), we have $\alpha, \beta \in \mathcal{X}_{\geq 1}$, so $a+b \geq 2$ and thus $\mathbf{1}_{[a+b=1]} = 0$. Moreover, the first vertex removal along with simplicial multiplication tells us that $\mathrm{P}_1(f_\Omega)\, f_\Gamma \in \mathrm{span}_R\left(f_{(\Omega \setminus \{\omega_1\}) \cup \Gamma}\right)$. It follows that the first vertex of $\mathrm{P}_1(f_\Omega)\, f_\Gamma$ is either ω_2 or γ_1. Thus, we find Definition 5.3.3 reduces to the following expression:

$$(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\delta \otimes f_\Lambda) = \mathbf{1}_{[\min\{\omega_2,\gamma_1\} < \lambda_1]}(-1)^{d(u+v)+\ell}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda.$$

From here, we multiply by $\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}(-1)^{(b-1)(u-1)}$ and use the following equalities to obtain the desired equation.

$$\mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\min\{\omega_2,\gamma_1\} < \lambda_1]} \overset{3.2.3(\mathrm{f})}{=} \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\gamma_1 < \lambda_1]} = \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]}\mathbf{1}_{[\omega_1 \leq \gamma_1 < \lambda_1]}.$$

Equation (5.5.3.4) is established similarly but we point out some subtle differences. In this case, we note that $f_\Omega f_\Gamma \in \operatorname{span}_R(f_{\Omega\cup\Gamma})$, so the first vertex is $\min\{\omega_1, \gamma_1\}$. Thus, we find

$$\mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\min\{\omega_1,\gamma_1\}<\lambda_1]} \overset{3.2.3(f)}{=} \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\gamma_1<\lambda_1]}.$$

The other subtlety is using Lemma 3.2.3(f) to get $\mathbf{1}_{[b=1]}(-1)^{(b-1)u} = \mathbf{1}_{[b=1]}$. The rest follows via Definition 5.3.3. $\square$

Theorem 5.5.4. *The scalar multiplication from Definition 5.3.3 is associative.*

Proof. As stated at the beginning of this section, we know scalar multiplication is associative when we restrict to the subcomplex $\mathcal{X} *_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ since $\mathcal{X} *_R \mathcal{Y}$ is a DG algebra and thus a DG-module over itself. We turn our attention to showing the scalar multiplication on $\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ is associative.

To show this, we will sum Equations (5.5.3.1)-(5.5.3.4). When we do this, we use the relation to $\mathbf{1}_{[\omega_1=\gamma_1]} f_\Omega f_\Gamma$ to see that $f_\Omega f_\Gamma = \mathbf{1}_{[\omega_1<\gamma_1]} f_\Omega f_\Gamma + \mathbf{1}_{[\gamma_1<\omega_1]} f_\Omega f_\Gamma$. We use this relation with Lemma 5.5.3 below.

$$\begin{aligned}
[(\alpha * f_\Omega)(\beta * f_\Gamma)](\delta \otimes f_\Lambda) &\overset{5.3.3}{=} \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}(-1)^{(b-1)(u-1)}(\alpha\beta * \mathrm{P}_1(f_\Omega)\, f_\Gamma)(\delta \otimes f_\Lambda) \\
&\quad - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[b=1]}(\alpha\partial^{\mathcal{X}}(\beta) * f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \\
&\quad + \mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{b(u-1)}(\alpha\beta * f_\Omega\, \mathrm{P}_1(f_\Gamma))(\delta \otimes f_\Lambda) \\
&\quad - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[a=1]}(-1)^{(b-1)u}(\partial^{\mathcal{X}}(\alpha)\,\beta * f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \\
&\overset{5.5.3}{=} \mathbf{1}_{[a=1]}\mathbf{1}_{[b=1]}(-1)^{(d-1)(u+v)}\partial^{\mathcal{X}}(\alpha)\,\partial^{\mathcal{X}}(\beta)\,\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&\quad - \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\omega_1<\lambda_1]}\mathbf{1}_{[b=1]}(-1)^{d(u+v)+d+\ell}\alpha\partial^{\mathcal{X}}(\beta)\,\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\quad - \mathbf{1}_{[\omega_1\leq\gamma_1<\omega_2]}\mathbf{1}_{[\omega_1\leq\gamma_1<\lambda_1]}(-1)^{bu+du+dv+b+u+\ell}\alpha\beta\phi(\delta) * \mathrm{P}_1(f_\Omega)\, f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[\gamma_1<\omega_1<\lambda_1]}\mathbf{1}_{[\gamma_1<\omega_1<\gamma_2]}(-1)^{bu+du+dv+b+\ell}\alpha\beta\phi(\delta) * f_\Omega\, \mathrm{P}_1(f_\Gamma)\, f_\Lambda \\
&\quad - \mathbf{1}_{[\gamma_1<\omega_1]}\mathbf{1}_{[\gamma_1<\lambda_1]}\mathbf{1}_{[a=1]}(-1)^{bu+du+dv+d+u+\ell}\partial^{\mathcal{X}}(\alpha)\,\beta\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\overset{5.5.2}{=} (\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)]
\end{aligned}$$

Since $\operatorname{Cone}(\Phi)^\natural \cong (\mathcal{X} *_R \mathcal{Y})^\natural \oplus (\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})^\natural$, this is sufficient for showing that the scalar multiplication of $\mathcal{X} *_R \mathcal{Y}$ on $\operatorname{Cone}(\Phi)$ is associative. $\square$

We immediately are able to conclude the following corollary and the first of our two main theorems.

Corollary 5.5.5. *Definition 5.3.3 makes* Cone(Φ) *into a DG* $(\mathcal{X} *_R \mathcal{Y})$*-module.*

5.6 Further Structures for Fiber Products

The significance of Corollary 5.5.5 is that no DG assumptions have been put on the minimal resolution $\mathcal{S}$ of the ideal $\mathcal{I}'$ in the fiber product $F = R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle$. This is especially important since we know that there are minimal resolutions that cannot be realized as DG algebras [2, Theorem 2.3.1]. Corollary 5.5.5 tells us that even when using ideals whose minimal resolution cannot be a DG algebra, we can still make the minimal resolution of F a DG $(\mathcal{X} *_R \mathcal{Y})$-module. Naturally, one might: "ask what if $\mathcal{S}$ can be realized as a DG algebra?"

We know that the complex $\mathcal{S} \otimes_R \mathcal{Y}$ is a DG algebra if both $\mathcal{S}$ and $\mathcal{Y}$ are DG algebras [7]. In particular, the product for $\mathcal{S} \otimes_R \mathcal{Y}$ is given by

$$(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) = (-1)^{dv} \rho\delta \otimes f_\Gamma f_\Lambda.$$

Using this, we extend Definition 5.3.3 under the following assumptions.

Notation 5.6.1. Consider the ideals $\mathcal{I}', \mathcal{I}, \mathcal{J} \subseteq R$ are minimally resolved by DG algebras $\mathcal{S}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively.

1. The minimal resolution $\mathcal{Y}$ is either a Koszul complex or Taylor resolution.

2. The ideal $\mathcal{I}$ is generated by a regular sequence.

3. We have the containment $\mathcal{I}' \subseteq \mathcal{I}^2$.

4. The chain map $\phi : \mathcal{S} \to \mathcal{X}$ is a DG-morphism.

5. The ideals satisfy $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}'}, \frac{R}{\mathcal{J}}\right) = 0 = \operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}}, \frac{R}{\mathcal{J}}\right)$ for all $i > 0$.

6. The elements $\alpha, \beta \in \mathcal{X}$ have respective degrees a and b.

7. The elements $\rho, \delta \in \mathcal{S}_{\geq 1}$ have respective degrees p and d.

8. The basis elements $f_\Omega, f_\Gamma, f_\Lambda \in \mathcal{Y}$ have respective degrees u, v, and ℓ.

Definition 5.6.2. For $\alpha * f_\Omega \in \mathcal{X} *_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ and $\rho \otimes f_\Gamma, \delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ we define the product on $\operatorname{Cone}(\Phi)$ via the following relations.

1. The element $\alpha * f_\Omega$ acts on $\operatorname{Cone}(\Phi)$ via Definition 5.3.3.
2. The element $\rho \otimes f_\Gamma$ acts on $\operatorname{Cone}(\Phi)$ via the following:
   (a) the action of $\rho \otimes f_\Gamma$ on $\mathcal{X} *_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ is given via
   $$(\rho \otimes f_\Gamma)(\alpha * f_\Omega) := (-1)^{(p+v)(a+u-1)}(\alpha * f_\Omega)(\rho \otimes f_\Gamma);$$
   (b) the action of $\rho \otimes f_\Gamma$ on $\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \operatorname{Cone}(\Phi)$ is given via
   $$(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) = (-1)^{dv} \rho\delta \otimes f_\Gamma f_\Lambda.$$

Proposition 5.6.3. *The product from Definition 5.6.2 is unital, distributive, and graded commutative.*

Proof. For unital, we see that $1_R * 1_R$ is the identity element; this follows from Proposition 5.3.5. Distributivity follows by the same arguments used in Proposition 5.3.5. For graded commutativity we have the following arguments:

- The product $(\alpha * f_\Omega)(\beta * f_\Gamma)$ is graded commutative since $\mathcal{X} *_R \mathcal{Y}$ is a DG algebra.
- The product $(\rho \otimes f_\Gamma)(\alpha * f_\Omega)$ is defined via graded commuativity and thus automatically satisfies the property.
- The product $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)$ inherits graded commutativity by identifying it in the DG algebra $\mathcal{S} \otimes_R \mathcal{Y}$.

For the last product, one can also check graded commutativity directly. □

While we were able to identify $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)$ in $\mathcal{S} \otimes_R \mathcal{Y}$ to obtain graded commutativity, the same does not work for the Leibniz rule since the differential in $\mathcal{S} \otimes_R \mathcal{Y}$ is different than that of $\operatorname{Cone}(\Phi)$. To obtain the Leibniz rule, we use the following lemmas.

Lemma 5.6.4. *For $\rho \otimes f_\Gamma, \delta \otimes f_\Lambda \in \mathcal{S}_{\geq 1} \otimes_R \mathcal{Y} \subset \mathrm{Cone}(\Phi)$, we have the following formulas related to the Leibniz rule.*

$$\begin{aligned}\partial^{\mathrm{Cone}(\Phi)}(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) = {} & (-1)^{dv}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda + (-1)^{dv+p+d}\rho\delta \otimes \partial(f_\Gamma)\, f_\Lambda \\ & + \mathbf{1}_{[v+\ell>0]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \end{aligned} \tag{5.6.4.1}$$

$$\begin{aligned}(\rho \otimes f_\Gamma)\partial^{\mathrm{Cone}(\Phi)}(\delta \otimes f_\Lambda) = {} & (-1)^{dv+p}\rho\partial^{\mathcal{S}}(\delta) \otimes f_\Gamma f_\Lambda + (-1)^{dv+p+d+v}\rho\delta \otimes f_\Gamma\partial(f_\Lambda) \\ & + \mathbf{1}_{[v+\ell>0]}\mathbf{1}_{[\lambda_1<\gamma_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \end{aligned} \tag{5.6.4.2}$$

Proof. To establish Equation 5.6.4.1, we consider $\Phi(\rho\otimes f_\Gamma)(\delta\otimes f_\Lambda)$ using the fact that $\partial_1^{\mathcal{X}} \circ \phi_1 = \partial_1^{\mathcal{S}}$.

$$\begin{aligned}\Phi(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) &= \mathbf{1}_{[v>0]}(-1)^{p+v}(\phi(\rho) * f_\Gamma)(\delta \otimes f_\Lambda) \\ &\quad + \mathbf{1}_{[v=0]}\mathbf{1}_{[p=1]}(\partial^{\mathcal{S}}(\rho) * f_\Gamma)(\delta \otimes f_\Lambda) \\ &\overset{5.6.2}{=} -\mathbf{1}_{[v>0]}\mathbf{1}_{[p=1]}(-1)^{dv+p}\partial^{\mathcal{X}}(\phi(\rho))\,\delta \otimes f_\Gamma f_\Lambda \\ &\quad + \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \\ &\quad + \mathbf{1}_{[v=0]}\mathbf{1}_{[p=1]}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \\ &\overset{3.2.3(a)(f)}{=} \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \\ &\quad + \mathbf{1}_{[p=1]}(-1)^{dv}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \end{aligned}$$

This allows us to establish the expression below.

$$\begin{aligned}\partial^{\mathrm{Cone}(\Phi)}(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) &= \Phi(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) + \mathbf{1}_{[p>1]}(\partial^{\mathcal{S}}(\rho) \otimes f_\Gamma)(\delta \otimes f_\Lambda) \\ &\quad + (-1)^p(\rho \otimes \partial(f_\Gamma))(\delta \otimes f_\Lambda) \\ &\overset{5.6.2}{=} \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \\ &\quad + \mathbf{1}_{[p=1]}(-1)^{dv}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \\ &\quad + \mathbf{1}_{[p>1]}(-1)^{dv}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \\ &\quad + (-1)^{d(v-1)+p}\rho\delta \otimes \partial(f_\Gamma)\, f_\Lambda \\ &\overset{3.2.3(a)}{=} \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \\ &\quad + (-1)^{dv}\partial^{\mathcal{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \end{aligned}$$

$$+ (-1)^{d(v-1)+p} \rho\delta \otimes \partial\left(f_\Gamma\right) f_\Lambda$$

To replace the indicator $\mathbf{1}_{[v>0]}$ with $\mathbf{1}_{[v+\ell>0]}$, we recall that $\mathbf{1}_{[v=0]}\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\gamma_1<\lambda_1]} = 0$ since the smaller vertex γ_1 is not defined. This gives the starred equality below.

$$\begin{aligned}
\mathbf{1}_{[v+\ell>0]}\mathbf{1}_{[\gamma_1<\lambda_1]} f_\Gamma f_\Lambda &\overset{3.2.3(c)}{=} \mathbf{1}_{[v=0]}\mathbf{1}_{[\ell>0]}\mathbf{1}_{[\gamma_1<\lambda_1]} f_\Gamma f_\Lambda + \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]} f_\Gamma f_\Lambda \\
&\overset{*}{=} \mathbf{1}_{[v>0]}\mathbf{1}_{[\gamma_1<\lambda_1]} f_\Gamma f_\Lambda
\end{aligned}$$

Thus we have established Equation (5.6.4.1). We omit the proof of (5.6.4.2) since it follows the same reasoning. □

Theorem 5.6.5. *The product in Definition 5.6.2 satisfies the Leibniz rule.*

Proof. We start with the case $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)$. The starred equalities below use the fact that ϕ is a DG-morphism. Moreover, the equality marked with Lemma 5.6.4 also uses the equality $f_\Gamma f_\Lambda = \mathbf{1}_{[\gamma_1<\lambda_1]} f_\Gamma f_\Lambda + \mathbf{1}_{[\lambda_1<\gamma_1]} f_\Gamma f_\Lambda$.

$$\begin{aligned}
\partial^{\mathrm{Cone}(\Phi)}((\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)) &\overset{5.6.2}{=} (-1)^{dv}\Phi(\rho\delta \otimes f_\Gamma f_\Lambda) + (-1)^{dv}\partial^{\mathbb{S}}(\rho\delta) \otimes f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv+p+d}\rho\delta \otimes \partial\left(f_\Gamma f_\Lambda\right) \\
&= \mathbf{1}_{[v+\ell>0]}(-1)^{dv+p+d+v+\ell}\phi(\rho\delta) * f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv}\partial^{\mathbb{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv+p}\rho\partial^{\mathbb{S}}(\delta) \otimes f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv+p+d}\rho\delta \otimes \partial\left(f_\Gamma\right) f_\Lambda \\
&\quad + (-1)^{dv+p+d+v}\rho\delta \otimes f_\Gamma \partial\left(f_\Lambda\right) \\
&\overset{*}{=} \mathbf{1}_{[v+\ell>0]}(-1)^{dv+p+d+v+\ell}\phi(\rho)\phi(\delta) * f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv}\partial^{\mathbb{S}}(\rho)\,\delta \otimes f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv+p}\rho\partial^{\mathbb{S}}(\delta) \otimes f_\Gamma f_\Lambda \\
&\quad + (-1)^{dv+p+d}\rho\delta \otimes \partial\left(f_\Gamma\right) f_\Lambda \\
&\quad + (-1)^{dv+p+d+v}\rho\delta \otimes f_\Gamma \partial\left(f_\Lambda\right) \\
&\overset{5.6.4}{=} \partial^{\mathrm{Cone}(\Phi)}(\rho \otimes f_\Gamma)\left(\delta \otimes f_\Lambda\right)
\end{aligned}$$

$$+ (-1)^{|\rho\otimes f_\Gamma|}(\rho\otimes f_\Gamma)\partial^{\mathrm{Cone}(\Phi)}(\delta\otimes f_\Lambda)$$

We now turn our attention to the two remaining cases. We already know that scalar multiplication of Cone (Φ) by $\alpha * f_\Omega$ satisfies the Leibniz. Thus, the remaining case is $(\rho\otimes f_\Gamma)(\alpha * f_\Omega)$. However, one uses the graded commutativity of the product to transform this case into the case of scalar multiplication by $\alpha * f_\Omega$. □

The remaining property to check is whether or not Definition 5.6.2 is associative. At first consideration, there would appear to be 8 cases where each of the three terms in product comes from either $\mathcal{X} *_R \mathcal{Y}$ or $\mathcal{S}_{\geq 1}\otimes_R \mathcal{Y}$. The next two lemmas reduces the the number of cases to four.

Lemma 5.6.6. *If any of the equalities below are true, then they must all be true.*

(a) $(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta\otimes f_\Lambda)] = [(\alpha * f_\Omega)(\beta * f_\Gamma)]\,(\delta\otimes f_\Lambda)$

(b) $(\alpha * f_\Omega)\,[(\delta\otimes f_\Lambda)(\beta * f_\Gamma)] = [(\alpha * f_\Omega)(\delta\otimes f_\Lambda)]\,(\beta * f_\Gamma)$

(c) $(\delta\otimes f_\Lambda)\,[(\alpha * f_\Omega)(\beta * f_\Gamma)] = [(\delta\otimes f_\Lambda)(\alpha * f_\Omega)]\,(\beta * f_\Gamma)$

Proof. For the proof, we will assume (a) is true and then prove (b) must also be true. The following set of equalities results from graded commutativity unless otherwise indicated.

$$\begin{aligned}(\alpha * f_\Omega)\,[(\delta\otimes f_\Lambda)(\beta * f_\Gamma)] &= (-1)^{(d+\ell)(b+v-1)}(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta\otimes f_\Lambda)]\\ &\overset{\text{(a)}}{=} (-1)^{(d+\ell)(b+v-1)}\,[(\alpha * f_\Omega)(\beta * f_\Gamma)]\,(\delta\otimes f_\Lambda)\\ &= (-1)^{(d+\ell+a+u-1)(b+v-1)}\,[(\beta * f_\Gamma)(\alpha * f_\Omega)]\,(\delta\otimes f_\Lambda)\\ &\overset{\text{(a)}}{=} (-1)^{(d+\ell+a+u-1)(b+v-1)}(\beta * f_\Gamma)\,[(\alpha * f_\Omega)(\delta\otimes f_\Lambda)]\\ &= [(\alpha * f_\Omega)(\delta\otimes f_\Lambda)]\,(\beta * f_\Gamma)\end{aligned}$$

The proof that any one of these equations implies another follows almost exactly as above, so we omit the proof of all the other implications. □

The following lemma uses similar reasoning as Lemma 5.6.6, so we state it without proof.

Lemma 5.6.7. *If any of the equalities below are true, then they must all be true.*

(a) $(\alpha * f_\Omega)\,[(\rho\otimes f_\Gamma)(\delta\otimes f_\Lambda)] = [(\alpha * f_\Omega)(\rho\otimes f_\Gamma)]\,(\delta\otimes f_\Lambda)$

(b) $(\rho \otimes f_\Gamma)\,[(\alpha * f_\Omega)(\delta \otimes f_\Lambda)] = [(\rho \otimes f_\Gamma)(\alpha * f_\Omega)]\,(\delta \otimes f_\Lambda)$

(c) $(\rho \otimes f_\Gamma)\,[(\delta \otimes f_\Lambda)(\alpha * f_\Omega)] = [(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)]\,(\alpha * f_\Omega)$

Theorem 5.6.8. *The product in Definition 5.6.2 is associative.*

Proof. Lemmas 5.6.6 and 5.6.7 reduce the problem of associativity to checking the following four cases where $\tau \in \mathcal{X}$ and $\eta \in \mathcal{S}$.

1. $(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\tau * f_\Lambda)] = [(\alpha * f_\Omega)(\beta * f_\Gamma)]\,(\tau * f_\Lambda)$
2. $(\alpha * f_\Omega)\,[(\beta * f_\Gamma)(\delta \otimes f_\Lambda)] = [(\alpha * f_\Omega)(\beta * f_\Gamma)]\,(\delta \otimes f_\Lambda)$
3. $(\alpha * f_\Omega)\,[(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)] = [(\alpha * f_\Omega)(\rho \otimes f_\Gamma)]\,(\delta \otimes f_\Lambda)$
4. $(\eta \otimes f_\Omega)\,[(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)] = [(\eta \otimes f_\Omega)(\rho \otimes f_\Gamma)]\,(\delta \otimes f_\Lambda)$

The first and second case are a result of Theorem 5.5.4. To obtain the third case, we first note that this is trivial if $\alpha * f_\Omega \in (\mathcal{X} *_R \mathcal{Y})_0$, so we suppose otherwise. As a consequence of this, we have $a + p \geq 2$ and thus $\mathbf{1}_{[a+p=1]} = 0$. This allows us to see the following.

$$\begin{aligned}
[(\alpha * f_\Omega)(\rho \otimes f_\Gamma)]\,(\delta \otimes f_\Lambda) &\overset{5.6.2}{=} -\mathbf{1}_{[a=1]}(-1)^{(p-1)u}(\partial^{\mathcal{X}}(\alpha)\,\rho \otimes f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \\
&\quad + \mathbf{1}_{[\omega_1<\gamma_1]}(-1)^{p(u-1)+v}(\alpha\phi(\rho) * f_\Omega f_\Gamma)(\delta \otimes f_\Lambda) \\
&\overset{5.6.2}{=} -\mathbf{1}_{[a=1]}(-1)^{(p-1)u+d(u+v)}\partial^{\mathcal{X}}(\alpha)\,\rho\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[\omega_1<\gamma_1]}\mathbf{1}_{[\omega_1<\lambda_1]}(-1)^{p(u-1)+d(u+v-1)+v+\ell}\alpha\phi(\rho)\phi(\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\overset{3.2.3(b)}{=} -\mathbf{1}_{[a=1]}(-1)^{dv+(d+p-1)u}\partial^{\mathcal{X}}(\alpha)\,\rho\delta \otimes f_\Omega f_\Gamma f_\Lambda \\
&\quad + \mathbf{1}_{[\omega_1<\min\{\gamma_1,\lambda_1\}]}(-1)^{dv+(p+d)(u-1)+v+\ell}\alpha\phi(\rho\delta) * f_\Omega f_\Gamma f_\Lambda \\
&\overset{5.6.2}{=} (-1)^{dv}(\alpha * f_\Omega)(\rho\delta \otimes f_\Gamma f_\Lambda) \\
&\overset{5.6.2}{=} (\alpha * f_\Omega)\,[(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)]
\end{aligned}$$

Finally, we obtain the fourth case by viewing it in the DG algebra $\mathcal{S} \otimes_R \mathcal{Y}$. □

Corollary 5.6.9. *With the assumptions set forth in Notation 5.6.1,* Cone$\,(\Phi)$ *is a minimal DG algebra resolution of the fiber product* $F \cong R/\langle \mathcal{I}', \mathcal{I}\mathcal{J}\rangle$ *over* R.

Proof. The minimality of $\operatorname{Cone}(\Phi)$ as the resolution of F over R comes from [16, Theorem 4.7]. The DG algebra structure comes from Definition 5.6.2 and is shown to have the necessary properties in Proposition 5.6.3, Theorem 5.6.5, and Theorem 5.6.8. □

5.7 Golod Fiber Products

In this section, we only consider fiber products that satisfy the conditions set forth in Notation 5.6.1. As established in Corollary 5.6.9, these conditions make $\operatorname{Cone}(\Phi)$ a minimal DG algebra resolution of

$$F := \frac{R}{\mathcal{I}' + \mathcal{J}} \times_W \frac{R}{\mathcal{I}} \cong \frac{R}{\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle}.$$

We now wish to use $\operatorname{Cone}(\Phi)$ to prove that some of these fiber products are Golod and Tor-friendly.

Proposition 5.7.1. *If* $\mathcal{S}_{\geq 1} \cdot \mathcal{S}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{S}$ *or* $\mathcal{Y}_{\geq 1} \cdot \mathcal{Y}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{Y}$*, then the same holds for* $\operatorname{Cone}(\Phi)$*, i.e., we have* $\operatorname{Cone}(\Phi)_{\geq 1} \cdot \operatorname{Cone}(\Phi)_{\geq 1} \subseteq \mathfrak{M}_R \operatorname{Cone}(\Phi)$*.*

Proof. Using distributivity and graded commutativity, all products in $\operatorname{Cone}(\Phi)$ can reduced to finite sums of the products below.

1. $(\alpha * f_\Omega)(\beta * f_\Gamma)$

2. $(\alpha * f_\Omega)(\delta \otimes f_\Lambda)$

3. $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda)$

Using [17, Proposition 7.1], we have $(\alpha * f_\Omega)(\beta * f_\Gamma) \in \mathfrak{M}_R(\mathcal{X} *_R \mathcal{Y}) \subset \mathfrak{M}_R \operatorname{Cone}(\Phi)$. Next, we note that $\mathbf{1}_{[a=1]}\partial^{\mathcal{X}}(\alpha) \in \mathcal{I} \subseteq \mathfrak{M}_R$ and thus $-\mathbf{1}_{[a=1]}(-1)^{(d-1)u}\partial^{\mathcal{X}}(\alpha)\, \delta \otimes f_\Omega f_\Lambda \in \mathfrak{M}_R \operatorname{Cone}(\Phi)$. Moreover, [16, Proposition 5.2] says we have $\phi(\delta) \in \mathfrak{M}_R \mathcal{X}$ and thus

$$\mathbf{1}_{[\omega_1 < \gamma_1]}(-1)^{d(u-1)+v} \alpha \phi(\delta) * f_\Omega f_\Gamma \in \mathfrak{M}_R \operatorname{Cone}(\Phi)\,.$$

It immediately follows that $(\alpha * f_\Omega)(\delta \otimes f_\Lambda) \in \mathfrak{M}_R \operatorname{Cone}(\Phi)$.

For the last product, recall that $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) = (-1)^{dv} \rho\delta \otimes f_\Gamma f_\Lambda$. For this product to have a coefficient from $\mathfrak{M}_R$, we need $\rho\delta \in \mathfrak{M}_R \mathcal{S}$ or $f_\Gamma f_\Lambda \in \mathfrak{M}_R \mathcal{Y}$. Hence, if we have $\mathcal{S}_{\geq 1} \cdot \mathcal{S}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{S}$ or $\mathcal{Y}_{\geq 1} \cdot \mathcal{Y}_{\geq 1} \subseteq \mathfrak{M}_R \mathcal{Y}$, then $(\rho \otimes f_\Gamma)(\delta \otimes f_\Lambda) \in \mathfrak{M}_R \operatorname{Cone}(\Phi)$. Since the desired holds for the three types of products listed, it must hold for all products in $\operatorname{Cone}(\Phi)_{\geq 1}$. □

Theorem 5.7.2. *The fiber product F is Golod.*

Proof. We let K^F denote the Koszul complex on the minimal generators of the maximal ideal of F. Similarly, K^R denotes the Koszul complex on the minimal generators of $\mathfrak{M}_R$, which resolves $k = R/\mathfrak{M}_R$. Since Corollary 5.6.9 tells us that $\mathrm{Cone}\,(\Phi)$ is a minimal DG algebra resolution of F over R, we obtain the following chain DG quasiisomorphisms.

$$K^F \cong F \otimes_R K^R \xleftarrow{\simeq} \mathrm{Cone}\,(\Phi) \otimes_R K^R \xrightarrow{\simeq} \mathrm{Cone}\,(\Phi) \otimes_R k$$

Since $\mathrm{Cone}\,(\Phi)$ is minimal, we also have

$$\mathrm{Cone}\,(\Phi) \otimes_R k = \mathrm{H}(\mathrm{Cone}\,(\Phi) \otimes_R k) \cong \mathrm{H}(K^F).$$

Thus, given any $Z \in H(K^F)_{\geq 1}$ we can identify it in $\mathrm{Cone}\,(\Phi) \otimes_R k$. By Proposition 5.7.1, we must have $Z^2 = 0 \in \mathrm{H}(\mathrm{Cone}\,(\Phi) \otimes_R k) \cong \mathrm{H}(K^F)$. By applying [2, Proposition 5.2.4(1)], this allows to conclude that F is Golod. □

Corollary 5.7.3. *The fiber product F is* Tor*-friendly; that is, given any finite F-modules M and N such that* $\mathrm{Tor}^F(M,N)$ *is bounded, we have* $\mathrm{pd}_F\,(M) < \infty$ *or* $\mathrm{pd}_F\,(N) < \infty$.

Proof. Since Theorem 5.7.2 tells us that F is Golod, we immediately have that it is Tor-friendly by [27, Theorem 3.1]. □

Chapter 6

Future Work

This dissertation was inspired by the desire to use DG methods to reprove the fact that fiber products are Tor-friendly. The first step in this process was to explicitly minimally resolve the fiber product $F = R/\langle \mathcal{I}', \mathcal{J}\mathcal{J}, \mathcal{J}' \rangle$ over R. We accomplished this in Chapter 3. The second step is to impose a DG-algebra structure on the minimal resolution. We partially accomplished this in Chapters 4–5. Naturally, this leaves open questions about how we might extend our result to cover more, if not all cases. In this chapter we pose some of those questions, highlight how they arose, and provide some insight on how we plan to address them.

6.1 DG-morphisms

Recall that Corollary 5.6.9 required the minimal resolutions of $R/\mathcal{I}'$ and $R/\mathcal{I}$, respectively denoted $\mathcal{S}$ and $\mathcal{X}$, both be DG algebras. Moreover, since $\mathcal{I}' \subseteq \mathcal{I}^2$ we were able to lift the natural surjection $R/\mathcal{I}' \to R/\mathcal{I}$ to a chain map $\phi : \mathcal{S} \to \mathcal{X}$. In the corollary, we required ϕ to be a DG-morphism in order to show $\operatorname{Cone}(\Phi)$ is a DG-algebra but gave no indication as to whether or not it is reasonable to expect such a DG-morphism to exist. We know of examples where one certainly can construct a DG-morphism but we still have the following question.

Question 6.1.1. *Can we always construct $\phi : \mathcal{S} \to \mathcal{X}$ to be a DG-morphism?*

This question by itself is rather broad, so we break it into smaller and easier to address questions. One might ask if it is reasonable to expect all choices of ϕ to be a DG-morphism, but the answer turns out to be no as it is not difficult to construct counterexamples. Naturally, this raises the question of whether or not ϕ is homotopy equivalent to a DG-morphism; that is, does there exist a DG-morphism ϕ' such that $\phi - \phi' \in \operatorname{Im} \partial_1^{\operatorname{Hom}_R(\mathcal{S},\mathcal{X})}$?

While we do not know the answer to this question, the following makes it seem reasonable to hope for an affirmative answer when ϕ_0 is the identity map on R. Since ϕ is a chain map, it is R-linear. Thus, if $\alpha, \beta \in \mathcal{S}_1$, then the R-linearity gives us the starred equality below.

$$\begin{aligned}
\partial_2^{\mathcal{X}}\left(\phi_2(\alpha\beta)\right) &= \phi_1(\partial_2^{\mathcal{S}}(\alpha\beta)) \\
&= \phi_1\left(\partial_1^{\mathcal{S}}(\alpha)\beta - \alpha\partial_1^{\mathcal{S}}(\beta)\right) \\
&\overset{*}{=} \partial_1^{\mathcal{S}}(\alpha)\phi_1(\beta) - \phi_1(\alpha)\partial_1^{\mathcal{S}}(\beta) \\
&= \partial_1^{\mathcal{X}}\left(\phi_1(\alpha)\right)\phi_1(\beta) - \phi_1(\alpha)\partial_1^{\mathcal{X}}\left(\phi_1(\beta)\right) \\
&= \partial_2^{\mathcal{X}}\left(\phi_1(\alpha)\phi_1(\beta)\right)
\end{aligned}$$

From this we see $\phi_2(\alpha\beta) - \phi_1(\alpha)\phi_1(\beta) \in \ker \partial_2^{\mathcal{X}} = \operatorname{Im} \partial_3^{\mathcal{X}}$ and, consequently, there exists some $\theta(\alpha,\beta) \in \mathcal{X}_3$ such that $\partial_3^{\mathcal{X}}(\theta(\alpha,\beta)) = \phi_2(\alpha\beta) - \phi_1(\alpha)\phi_1(\beta)$. Ideally, we could use this to construct a map $\tau_2 : \mathcal{S}_2 \to \mathcal{X}_3$ such that $\tau_2(\alpha\beta) = \theta(\alpha,\beta)$ thereby making the diagram below commute.

$$\begin{array}{ccc}
\mathcal{S}_1 \times \mathcal{S}_1 & \longrightarrow & \mathcal{S}_2 \\
 & \searrow & \Big\downarrow{\scriptstyle \tau_2} \\
 & & \mathcal{X}_3
\end{array}$$

We could then define $\tau = \{\ldots, 0, \tau_2, 0, \ldots\} \in \operatorname{Hom}_R(\mathcal{S},\mathcal{X})_1$ and set $\phi' = \phi - \partial_1^{\operatorname{Hom}_R(\mathcal{S},\mathcal{X})}(\tau)$. As a consequence, we would then have $\phi_2'(\alpha\beta) = \phi_1'(\alpha)\phi_1'(\beta)$. In theory, one could iterate this process for all $m, n \geq 0$ as long as we can create an analogous commutative diagram.

$$\begin{array}{ccc}
\mathcal{S}_m \times \mathcal{S}_n & \longrightarrow & \mathcal{S}_{m+n} \\
 & \searrow & \Big\downarrow \\
 & & \mathcal{X}_{m+n+1}
\end{array}$$

It is important to note that for our purposes, this process would terminate in a finite number of steps. This is due to the fact that we are working over a regular local (or standard graded polynomial) ring, and thus the minimal free resolution is bounded.

We consider the following questions as specific cases of our original question.

Question 6.1.2. *1. If $\mathcal{Y}$ and $\mathcal{Y}'$ are DG-algebra resolutions of an R-algebra M, then are they quasiisomorphic as DG-algebras?*

2. If $\mathcal{X}$ is an acyclic Koszul complex and if $\mathcal{S}$ is a DG-algebra resolution, can we find a DG-morphism $\phi : \mathcal{S} \to \mathcal{X}$?

If we are able to answer Question 6.1.2.1 in the affirmative, then we will have shown that any minimal DG-algebra resolution is unique up to DG-isomorphism. It is important to recall that not every minimal resolution has a DG-structure (see [2]), so this only applies to rings where some minimal resolution can be realized as a DG-algebra. However, we have identified a potential counterexample to Question 6.1.2.1 using the star product. We summarize the potential counterexample and the notation needed to understand it below.

Example 6.1.3. Let $R = k[[\underline{x}, y_1, \dots, y_n]]$ and consider the Koszul complex $\mathcal{Y} = K^R(y_1, \dots, y_n)$. Given any permutation $\sigma \in S_n$, we define the complex $\sigma(\mathcal{Y}) := K^R(y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(n)})$ and observe that both $\sigma(\mathcal{Y})$ and $\mathcal{Y}$ resolve $k[[\underline{x}]]$. Moreover, we can show that $\sigma(\mathcal{Y}) \cong \mathcal{Y}$ as free resolutions over R by lifting $\sigma \in S_n$ to a chain map $\sigma' : \mathcal{Y} \to \sigma(\mathcal{Y})$ defined by $\sigma'(f_\Omega) = \operatorname{sign}(\sigma, \Omega)\, f_{\sigma(\Omega)}$ where $\operatorname{sign}(\sigma, \Omega)$ tracks the number of permutations needed to put the order set $\{\sigma(\omega_1), \dots, \sigma(\omega_u)\}$ in ascending order. Another way to think of this is to write $\{\omega'_1, \dots \omega'_u\} = \sigma(\{\omega_1, \dots, \omega_u\})$ and then use the exterior algebra definition of the Koszul complex $\mathcal{Y}$.

$$\begin{aligned}\sigma(f_{\omega_1} \wedge \cdots \wedge f_{\omega_u}) &= f_{\sigma(\omega_1)} \wedge \cdots \wedge f_{\sigma(\omega_u)} \\ &= \operatorname{sign}(\sigma, \Omega)\, (f_{\omega'_1} \wedge \cdots \wedge f_{\omega'_u})\end{aligned}$$

We expect to show that this map is a DG-isomorphism.

While we expect to show that $\mathcal{Y} \cong \sigma(\mathcal{Y})$ as DG-algebras, we question whether $\mathcal{X} *_R \mathcal{Y}$ and $\mathcal{X} *_R \sigma(\mathcal{Y})$ will be isomorphic as DG-algebras. If this turns out to be a counterexample, it might help us identify restrictions necessary to guarantee a DG-morphism. If we are able to show these two resolutions are isomorphic as DG-algebras, it may help us better understand the iterative process of using homotopy equivalences to transform a chain map into a DG-morphism.

Regarding the DG-algebra structure for $\operatorname{Cone}(\Phi)$, answering Question 6.1.2.2 in the affirmative is sufficient since we are most interested in the case $\mathcal{I} = \langle \underline{x} \rangle$. Since we explicitly know the multiplicative structure of the Koszul complex, given $\alpha, \beta \in \mathcal{S}$ and formulas for $\phi(\alpha)$ and $\phi(\beta)$, we can write down a formula for the product $\phi(\alpha)\phi(\beta)$. Moreover, when we know ϕ is not a DG-morphism, the structure of the Koszul complex makes it easier to study the term $\phi(\alpha\beta) - \phi(\alpha)\phi(\beta)$ when it vanishes under the differential $\partial^{\mathcal{X}}$.

6.2 Additional Mapping Cones

The main focus of Chapter 5 was imposing various DG-structures on $\operatorname{Cone}(\Phi)$ from Chapter 3. To that end, in Corollary 5.5.5 we found that the conditions for $\operatorname{Cone}(\Phi)$ to minimally resolve $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle$ were sufficient to establish $\operatorname{Cone}(\Phi)$ as a DG $(\mathcal{X} *_R \mathcal{Y})$-module. We then established further conditions to realize $\operatorname{Cone}(\Phi)$ as a DG-algebra resolution in Corollary 5.6.9. However, we make no mention of the mapping cones $\operatorname{Cone}(\Psi)$ and $\operatorname{Cone}(\Omega) = \operatorname{Cone}(\,\Phi \;\; \Psi\,)$ from Chapter 3. Due to the way these mapping cones are constructed, it is logical to first tackle the following questions.

Question 6.2.1. *Consider* Ψ *from Construction 3.4.5.*

1. *When, if ever, is* $\operatorname{Cone}(\Psi)$ *a DG* $(\mathcal{X} *_R \mathcal{Y})$*-module?*

2. *When, if ever, is* $\operatorname{Cone}(\Psi)$ *a DG-algebra?*

Similar to the approach we used in Chapter 5, we want to first establish the DG-module structure and then use it to realize $\operatorname{Cone}(\Psi)$ as a DG-algebra resolution. For that reason, we will only address the question about the DG-module structure in this section. Before we do this, it is important to note the following observation.

The conditions needed for $\operatorname{Cone}(\Phi)$ to be a minimal resolution of $R/\langle \mathcal{I}', \mathcal{I}\mathcal{J} \rangle$ are analogous to those needed to make $\operatorname{Cone}(\Psi)$ a minimal resolution of $R/\langle \mathcal{I}\mathcal{J}, \mathcal{J}' \rangle$. Specifically, the relationship between $\mathcal{I}'$ and the ideals $\mathcal{I}$ and $\mathcal{J}$ are identical to those for $\mathcal{J}'$. Thus, one can swap the roles of $\mathcal{X}$ and $\mathcal{Y}$ along with Corollary 5.5.5 to realize $\operatorname{Cone}(\Psi)$ as a DG $(\mathcal{Y} *_R \mathcal{X})$-module. However, realizing $\operatorname{Cone}(\Psi)$ as a DG $(\mathcal{X} *_R \mathcal{Y})$-module is not so simple, especially since it is unclear whether or not $\mathcal{X} *_R \mathcal{Y}$ and $\mathcal{Y} *_R \mathcal{X}$ are isomorphic as DG-algebras.

To obtain the formulas in Definitions 4.4.2 and 5.3.3, we inductively constructed products that satisfied the Leibniz rule. The induction occurs on the degree of the product of two elements, however each of these elements consist of two smaller terms, e.g., the degree of $\alpha * f_\Gamma$ is governed by the degrees $|\alpha|$ and $|f_\Gamma|$. Thus, inducting on the degree of the product means controlling four different terms which yields many nested applications of induction.

To do this for Cone(Ψ), we take $\alpha * f_\Gamma \in (\mathcal{X} *_R \mathcal{Y})_{\geq 1}$ and $\beta \otimes \rho \in \mathcal{X} \otimes_R \mathcal{T}_{\geq 1}$ to consider

$$\partial^{\mathrm{Cone}(\Psi)}(\alpha * f_\Gamma)\,(\beta \otimes \rho) + (-1)^{|\alpha * f_\Gamma|}(\alpha * f_\Gamma)\partial^{\mathrm{Cone}(\Psi)}(\beta \otimes \rho)\,.$$

Within this expression, one must make sense of the product $(\alpha * f_\Gamma)\partial^{\mathrm{Cone}(\Psi)}(\beta \otimes \rho)$ when $|\beta| \geq 1$. In this case, we need to consider

$$(\alpha * f_\Gamma)\Psi(\beta \otimes \rho) = (-1)^{|\beta|+|\rho|-1}(\alpha * f_\Gamma)(\beta * \psi(\rho)).$$

Since this product effectively occurs in $\mathcal{X} *_R \mathcal{Y}$, we can theoretically use Definition 4.4.2 to expand this expression. Doing this means that for every $\Lambda \in \Delta$, we need to define $\psi_\Lambda(\rho) := \mathrm{proj}_{f_\Lambda}(\psi(\rho))$ and note that if $|\Lambda| \neq |\rho|$, then we have $\psi_\Lambda(\rho) = 0$. Setting $\Delta_{|\rho|} = \{\Lambda \in \Delta : |\Lambda| = |\rho|\}$ gives the formula $\psi(\rho) = \sum_{\Lambda \in \Delta_{|\rho|}} \psi_\Lambda(\rho)$ which we then use to obtain

$$(\alpha * f_\Gamma)\Psi(\beta \otimes \rho) = \sum_{\Lambda \in \Delta_{|\rho|}} (-1)^{|\beta|+|\rho|-1}(\alpha * f_\Gamma)(\beta * \psi_\Lambda(\rho)).$$

The complexity of this expression arises for $\Lambda \in \Delta_{|\rho|}$ such that $\lambda_1 < \gamma_1 < \lambda_2$. In that case, we must consider the term $\mathrm{P}_1(\psi_\Lambda(\rho)) \in \mathrm{span}_R\left(f_{\Lambda \setminus \{\lambda_1\}}\right)$ which we inevitably will want to compare with

$$\psi_{\Lambda \setminus \{\lambda_1\}}\left(\partial^{\mathcal{T}}(\rho)\right) = \sum_{\Theta \setminus \{\vartheta_i\} = \Lambda \setminus \{\lambda_1\}} \mathrm{P}_i(\psi_\Theta(\rho))$$

since $\psi : \mathcal{T} \to \mathcal{Y}$ is a chain map. Unfortunately, this seems to devolve into a mess of notation chasing and motivates the desire for a less complex alternative.

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