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Exact Controllability and Inverse Problem for the Mindlin-Timoshenko System

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Abstract

In this dissertation we primarily focus on the Mindlin–Timoshenko (MT) plate system, which is a strongly coupled two dimensional system consisting of a wave equation and a system of isotropic elasticity, that arises in modeling plate vibrations especially at high frequencies and thicker plates. We prove two results regarding the MT system. Namely, the exact controllability of the system and an inverse problem result. We demonstrate the exact controllability of the system via an indirect control technique that proves a two-level indirect inverse observability estimate for the diagonalized system. For the inverse problem, we prove the global uniqueness of recovering the plate density from a single boundary measurement under appropriate geometrical assumptions. Both results incorporate the use of several different Carleman-type estimates derived for hyperbolic equations that we apply to a diagonalized version of the MT system. These diagonalizations consist of coupled systems of wave equations where coupling is maintained only in the lower order.

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Chapter 1

Introduction

The primary focus of this dissertation is on results pertaining to the Mindlin–Timoshenko (MT) plate system, which is a strongly coupled two dimensional system consisting of a wave equation and a system of isotropic elasticity, that arises in modeling plate vibrations especially at high frequencies and thicker plates. We prove two results regarding the MT system. Namely, the exact controllability of the system and an inverse problem result. We demonstrate the exact controllability of the system via an indirect control technique that proves a two-level indirect inverse observability estimate for the diagonalized system. For the inverse problem, we prove the global uniqueness of recovering the plate density from a single boundary measurement under appropriate geometrical assumptions. Both results incorporate the use of several different Carleman-type estimates derived for hyperbolic equations that we apply to a diagonalized version of the MT system. These diagonalizations consist of coupled systems of wave equations where coupling is maintained only in the lower order. We begin with providing the necessary background for understanding Carleman estimates as they pertain to the control and inverse problem for systems of PDEs.

1.1 Carleman Estimates

The origination of the use of exponential weights can be traced to a mathematician named Carleman [8] in 1939. Carleman's intent was to apply these estimates to prove uniqueness in what is known as the Cauchy Problem in two variables. It was the mathematician Hörmander who realized the implications of this notion of Carleman's, which would lead to becoming a mainstay for all related work in the field [14, p.61]. Hörmander continued to popularize Carleman's approach and perfected the concept to a more broad class of differential operators. The general representation for this Carleman estimate is given by

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u e^{2\tau\varphi}|^2 dx \leq C \int |P(x,D) u e^{2\tau\varphi}|^2 dx, \quad u \in C_0^\infty$$

for some weight function φ and large parameter τ. Hörmander subsequently used this estimate to prove what is known as the *Unique Continuation Property*. Simply stated, given u as a solution to the PDE $P(x,D)u = 0$ on an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ and $u = 0$ for some $\varphi(x) > 0$, where the function $\varphi : \Omega \to \mathbb{R}$ defines a smooth hypersurface in Ω, then under certain conditions this implies $u = 0$ on a neighborhood of $\varphi = 0$.

These early results were only applicable, however, when involving solutions which assumed to be compactly supported. Thus, these Carleman estimates did not contain boundary terms which play a vital role in boundary control and some inverse problems. To emphasize this deficiency in the estimates lacking boundary terms, homogenizing the Cauchy data (a known simple process) produces a term in the right-hand side of the estimate involving norms of boundary traces that are a half derivative higher than the norm of u on the left-hand side of the estimate. This highlights the need for the addition of boundary terms to the classical Carleman estimate since they are deficient in providing decent results when applied to boundary value problems. This issue was addressed by two different approaches that were developed independently.

The development of improved Carleman type inequalities, which provided good results for solutions of boundary value problems can be attributed to two originating sources that addressed the issue rather differently. The first source is the mathematician D. Tataru [38] at the University of Virginia while the second is Lavrentiev–Romanov–Shishatski [26] of the Novosibirski school. These papers established two camps of thought for how to produce boundary terms in the estimates. The idea behind Tataru's work was motivated by extending the main Carleman estimate to general psuedo–differential operators. This results in certain structures that need to satisfy geometrical properties, including a surface which must be psuedo–convex. Tataru's work was developed from the work of Lasiecka-Triggiani [24] which developed a sharp Carleman type estimate specifically for second-order hyperbolic equations such as the wave equation. These estimates were obtained using a type of differential multiplier, which differed depending on the exact PDE to which it was applied. In contrast, Lavrentiev–Romanov–Shishatski [26] approached the problem of producing boundary terms in the estimate via a format which was much more computationally focused. Their method was to establish an initial pointwise Carleman estimate with the resulting integral form of this estimate. This was the inspiration behind the subsequent work of Lasiecka-Triggiani-Zhang [25], a primary source for several of the estimates used in this paper. More precisely, they worked via the method produced in the Lavrentiev camp by establishing a fundamental initial pointwise inequality for the general second order hyperbolic equation that was used to produce a one parameter family of pointwise Carleman estimates.

Here we introduce the estimates used in both of the main results presented in this paper. First, we present the standard required geometric conditions that are common assumptions for the Carleman estimates, followed by the precise Carleman estimates and a brief description of their unique attributes and derivation. Henceforth we will assume we are operating within a domain contained in $\mathbb{R}^2$ for the purposes of direct application to the MT system, but results may be generalized to n-dimensional space.

Let Ω be open and bounded in $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma$ of class C^1. The boundary, as pertaining to the control problem, is considered as the closure of the union of two disjoint parts $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Note that Γ_0 is considered the uncontrolled (unobserved) part of the boundary and Γ_1 the controlled (observed) part. Then for $c > 0$ held as constant, let $w(t, x)$ where $x = (x_1, x_2)$ satisfy the following wave equation:

$$\begin{cases} \mathcal{P}w = F(w) + f & \text{in } Q := [0, T] \times \Omega \\ w(0, \cdot) = w^0,\ w_t(0, \cdot) = w^1 & \text{in } \Omega \\ \mathcal{B}(w) \equiv 0 & \text{in } \Sigma := [0, T] \times \Gamma. \end{cases} \tag{1.1}$$

where $\mathcal{P}$ is the wave operator $\partial_t^2 - c^2\Delta$ with $\mathcal{B}$ either Dirichlet boundary conditions $(w|_\Sigma)$ or Neumann boundary conditions $\left(\frac{\partial w}{\partial \nu}\big|_\Sigma\right)$ where $\nu = (\nu_1, \nu_2)$ (recall $\Omega \subset \mathbb{R}^2$) is the unit outward normal vector on Γ. We further assume $f \in L^2(Q)$, and the first order operator F satisfies

$$|F(w)|^2 \leq C_T \left[w_t^2 + |\nabla w|^2 + w^2\right] \tag{1.2}$$

for some constant, C_T, depending on the final time.

1.1.1 Geometrical Assumptions

This section presents the main assumptions necessary to establish the continuous observability inequality where $w(t,x)$ is the solution to (1.1). These assumptions are well-known from [25, 24, 23] and are common for each of the Carleman estimates presented within. To accommodate the existence of a constant wave speed in (1.1), the assumptions must be altered and we will highlight differences. Given the triplet $\{\Omega, \Gamma_0, \Gamma_1\}$, we assume the existence of a strictly convex function $d : \bar{\Omega} \to \mathbb{R}$, of class $C^3(\bar{\Omega})$ and a vector field $h(x) = [h_1(x), h_2(x)]$, $x \in \mathbb{R}^2$ (recall $n = 2$ due to the potential application of the MT model), such that $h(x) \equiv \nabla d$ for every $x \in \Omega$ and the following properties hold:

(A.1) (i) $\nabla d \cdot \nu = h \cdot \nu \leq 0$ on Γ_0 in the Dirichlet b.c. ($h \cdot \nu = 0$ in the case of Neumann b.c.);

(ii) the Hessian matrix of $d(x)$, equivalent to the Jacobian matrix of $h(x)$ is strictly positive definite in that there exists positive constant ρ_0 where for every $x \in \bar{\Omega}$ we have

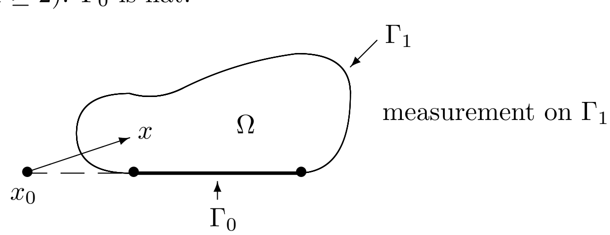
$$\mathcal{H}_d(x) = \begin{bmatrix} d_{x_1x_1} & d_{x_1x_2} \\ d_{x_2x_1} & d_{x_2x_2} \end{bmatrix} \geq \rho_0 I; \tag{1.3}$$

(A.2) $d(x)$ has no critical points within the region $\bar{\Omega}$.

Remark 1.1.1. *Note that ρ_0 depends on the coefficient, c^2, thus when applying the Carleman estimate to several different equations in a coupled system it is important to maintain a common choice for the convex function d. This enables estimates to be easily combined. By scaling and shifting d can be chosen to satisfy (A.1) for each of the equations and the various coefficient values for c.*

Here we present some examples in connection to the main geometrical assumptions (A.1), (A.2) that are meant to demonstrate the reasonableness of making such assumptions. These examples even satisfy the more stringent case of Neumann B.C. We refer to [25] for more details.

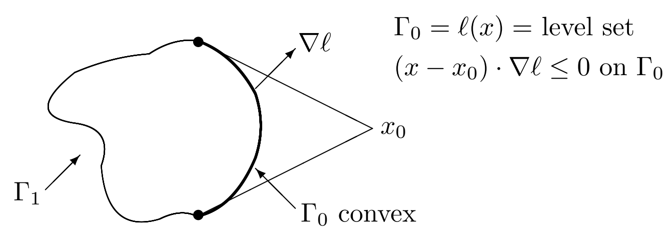
Ex. #1 (Any dimension ≥ 2): Γ_0 is flat.



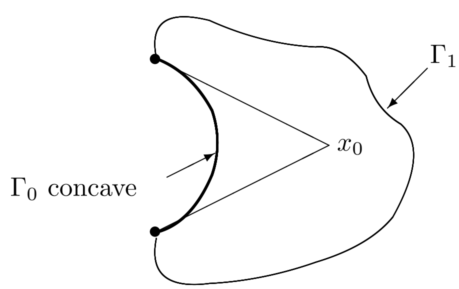
Let $x_0 \in$ hyperplane containing Γ_0, then.

$$d(x) = \|x - x_0\|^2; \quad h(x) = \nabla d(x) = 2(x - x_0).$$

Ex. #2 (A domain Ω of any dimension ≥ 2 with unobserved portion Γ_0 convex, subtended by a common point x_0): $d(x)$ in [25, Theorem. A.4.1, p. 301].



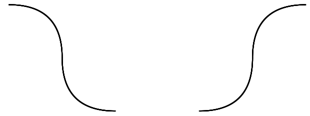
Ex. #3 (A domain Ω of any dimension ≥ 2 with unobserved portion Γ_0 concave, subtended by a common point x_0): $d(x)$ in [25, Theorem. A.4.1, p. 301].



Ex. #4 (dim = 2): Γ_0 neither convex or concave. Γ_0 is described by graph

$$y = \begin{cases} f_1(x), x_0 \leq x \leq x_1, \quad y \geq 0; \\ f_2(x), x_2 \leq x \leq x_1, \quad y < 0, \end{cases}$$

f_1, f_2 *logarithmic concave* on $x_0 < x < x_1$, e.g., $\sin x + 1,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$; $\cos x + 1,\ 0 < x < \pi$



Function $d(x)$ is given in [25, Eqn. (A.2.7), p. 289].

1.1.2 Pseudo-Convex Function

Given the existence of the strongly convex function, $d(x)$, which is merely a scaled version of the $d(x)$ in [25], we have a psuedo-convex function, $\varphi : \mathbb{R} \times \Omega \to \mathbb{R}$ of class C^3 defined as:

$$\varphi(t,x) = d(x) - k\left(t - \frac{T}{2}\right)^2; \;\; t \in [0,T],\, x \in \Omega, \tag{1.4}$$

with $T > 0$ and $k \in (0,1)$ selected following the process in 1.1.8b-d of [25]. The threshhold time $T_0 > 0$ is defined by setting

$$T_0^2 \equiv 4 \max_{x \in \bar{\Omega}} d(x). \tag{1.5}$$

The slight difference here is due to the altered assumption in (A.1) from the case in [25] where it is assumed $c = 1$. Since (1.3) inversely depends on c, the threshold time, T_0, is also inversely impacted as a result of the scaling of $d(x)$. Let $T > T_0$ be given, depending on c: a smaller (larger) threshold time permits a smaller (forces a larger) final time T. Here we also assume, without loss of generality, $d(x) > 0$ for $x \in \Omega$, since otherwise we can translate $d(x)$ to satisfy positivity over the domain and our assumptions still hold. Thus, we have the existence of $\delta > 0$ from (1.5), fixed and satisfying

$$kT^2 > 4(\max_{x \in \bar{\Omega}} d(x) + \delta) \tag{1.6}$$

so that

$$\varphi(0,x) \equiv \varphi(T,x) \leq -\delta \tag{1.7}$$

holds uniformly for $x \in \Omega$. Additionally, we have t_0, t_1 chosen symmetrically about $\frac{T}{2}$ where $0 < t_0 < \frac{T}{2} < t_1 < T$, such that the property

$$\min_{x \in \bar{\Omega}, t \in [t_0, t_1]} \varphi(t,x) \geq \sigma, \;\; 0 < \sigma < \min_{x \in \bar{\Omega}} d(x) \tag{1.8}$$

holds, recalling $d(x) > 0$ for $x \in \bar{\Omega}$. This generates the region

$$Q(\sigma) \equiv \{(t,x) : t \in [0,T],\, x \in \Omega,\, \varphi(t,x) \geq \sigma > 0\}, \tag{1.9}$$

whose relevance is shown only in the proof of Theorem 1.1.3 contained in Section A of the Appendix. Another pertinent region necessary for the proof of Theorem 1.1.3 is

$$Q^*(\sigma^*) \equiv \{(t,x) : t \in [0,T],\, x \in \Omega,\, \varphi^*(t,x) \geq \sigma^* > 0\} \tag{1.10}$$

for $0 < \sigma^* < \sigma$ where we have

$$\varphi^*(t,x) \equiv d(x) - k^2 \left(t - \frac{T}{2}\right)^2, \quad t \in [0,T],\, x \in \Omega. \tag{1.11}$$

Thus, for clarity, the ordering of containment for relevant regions is as follows:

$$[t_0, t_1] \times \Omega \subset Q(\sigma) \subset Q^*(\sigma^*) \subset [0,T] \times \Omega. \tag{1.12}$$

Remark 1.1.2. *The property* (1.8) *is only required for the estimate in* (1.13)*, but the remaining Carleman estimates maintain the additional assumption* $w(0,x) = w(T,x) = 0$*, which circumvents the need for the region* $Q(\sigma)$ *in the proof of Theorem 1.1.3.*

1.1.3 Carleman Estimates

The first Carleman estimates introduced below is similar to the estimate derived by Lasiecka–Triggiani–Zhang. It has become a canonical result and is the foundation for many other similar estimates that have followed and while it is not used explicitly in this thesis, the estimate in (1.14) and (4.26) are proved following the same process. Specifically the goal is to establish an initial pointwise estimate for a wave equation (or Riemannian wave equation in the case of (4.26)). This estimate then, via careful selection of an appropriate pseudo–convex function, as defined in (1.4), and other specifications, will ultimately yield pointwise Carleman estimates followed by the corresponding integral inequalities. The final estimate is expressed in terms of these pointwise integral inequalities. Thus we have the first of several Carleman estimates below.

Theorem 1.1.3. *Consider* $w \in \mathcal{M}^2(\mathbb{R}_t \times \mathbb{R}_x^2)$ *as the solution to the wave equation in* (1.1) *without the boundary conditions, and let* $F(w)$ *satisfy* (1.2) *and* $f \in L^2(Q)$*. Then for pseudo-convex function,* $\varphi(t,x)$*, as defined in* (1.4)*, and the same assumptions as in Corollaries A.2.3.1, and A.2.3.2 of the appendix section A, we have the one parameter family of estimates given* $\tau > 0$ *sufficiently large and*

$\epsilon > 0$ small:

$$
\begin{aligned}
BT|_\Sigma + 2\int_0^T\int_\Omega e^{2\tau\varphi} f^2\,dxdt + C_T e^{2\tau\sigma}\int_0^T\int_\Omega w^2\,dxdt \\
\geq (\tau\epsilon\rho - 2C_T)\int_0^T\int_\Omega e^{2\tau\varphi}[w_t^2 + c|\nabla w|^2]\,dxdt \\
+ [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]\int_{Q(\sigma)} e^{2\tau\varphi} w^2\,dxdt - c_T\tau^3 e^{-2\tau\delta}[E(0)+E(T)],
\end{aligned}
\tag{1.13}
$$

for $\delta > 0$, and $\sigma > 0$ as specified in (1.7) *and* (1.8) *and the where the region $Q(\sigma)$ is as defined in* (1.9)*. Furthermore, notice $BT|_\Sigma$ represents the boundary terms, which are impacted by the constant c and whose derivation is shown below:*

$$
\begin{aligned}
BT|_\Sigma \equiv & 2\tau c^2\int_0^T\int_{\Gamma_1} e^{2\tau\varphi}(w_t^2 - c^2|\nabla w|^2)h\cdot\nu\,d\Gamma dt \\
& + 8k\tau c^2\int_0^T\int_\Gamma e^{2\tau\varphi}c^2\left(t-\frac{T}{2}\right)w_t\frac{\partial w}{\partial\nu}\,d\Gamma dt \\
& + 4\tau c^2\int_0^T\int_\Gamma c^2 e^{2\tau\varphi}(h\cdot\nabla w)\frac{\partial w}{\partial\nu}\,d\Gamma dt \\
& + 4\tau^2 c^2\int_0^T\int_\Gamma e^{2\tau\varphi}\left(c^2|h|^2 - 4k^2\left(t-\frac{T}{2}\right)^2 + \frac{\alpha}{2\tau}\right)w\frac{\partial w}{\partial\nu}\,d\Gamma dt \\
& + 2\tau c^2\int_0^T\int_{\Gamma_1} e^{2\tau\varphi}\left[2\tau^2\left(c^2|h|^2 - 4k^2\left(t-\frac{T}{2}\right)^2\right) + \tau(\alpha - c^2\Delta d - 2k)\right]w^2 h\cdot\nu\,d\Gamma dt.
\end{aligned}
$$

Also, the energy term $E(t)$ for $t\in[0,T]$ is defined to be

$$
E(t) = \int_\Omega [w_t^2(t,x) + c^2|\nabla w(t,x)|^2 + w^2(t,x)]\,dxdt.
$$

For the convenience of the reader, and since (1.13) differs slightly from that of [25, Theorem 5.1], the proof of Theorem (1.1.3) is included in Appendix A.

The following two estimates are used in the proof of the controllability result for the MT system and are introduced by Imanuvilov in [15] and Fu–Yong–Zhang in [13]. We provide a brief description and reference for the proof of each.

Theorem 1.1.4. *Under assumptions (A.1) and (A.2), let w satisfy the equation* (1.1)*, and let $\omega\subset\Omega$. Then for pseudo-convex function, φ, as defined in* (1.4)*, there exists a $\lambda_0 > 1$ such that for* *all* $\lambda \geq \lambda_0$ *and all* $w \in H_0^1(Q)$ *with* $\mathcal{P}w \in L^2(Q)$ *the following estimate holds*

$$\lambda \int_Q (\lambda^2 w^2 + w_t^2 + |\nabla w|^2) e^{2\lambda\varphi}\, dxdt \leq C \left[\left\| e^{\lambda\varphi} \mathcal{P}w \right\|_{L^2(Q)}^2 + \lambda^2 \int_0^T \int_\omega (\lambda^2 w^2 + w_t^2) e^{2\lambda\varphi}\, dxdt \right] \quad (1.14)$$

The proof of Theorem 1.1.4 can be found in [13, Theorem 5.1], but follows the proof of Theorem 1.1.3 closely (as originally proved in [25, Theorem 5.1]). The reason for the additional assumption $w(0,\cdot) = w(T,\cdot) = 0$ is that it enables integration over the entire cylinder Q instead of the case in Theorem 1.1.3 where integration is over a subdomain bounded by a level surface of φ. Thus there is no longer a need for φ to satisfy (1.8).

Theorem 1.1.5. *Under assumptions (A.1) and (A.2), let* w *satisfy the equation* (1.1) *under the regularity assumption* $w \in C([0,T]; L^2(\Omega))$ *and satisfies* $w(0,x) = w(T,x) = 0$ *for* $x \in \Omega$*. Then for the same choice of* φ *as in Theorem 1.1.4 and any* $\lambda \geq \lambda_0 \geq 1$*, if* $\mathcal{P}w \in H^{-1}(Q)$*, and*

$$(w, \mathcal{P}\eta)L^2(Q) = \langle \mathcal{P}w, \eta \rangle_{H^{-1}(Q), H_0^1(Q)} \quad \forall \eta \in H_0^1(Q) \text{ with } \mathcal{P}\eta \in L^2(Q), \quad (1.15)$$

it holds that

$$\lambda \int_Q w^2 e^{2\lambda\varphi}\, dxdt \leq C \left(\left\| e^{\lambda\varphi} \mathcal{P}w \right\|_{H^{-1}(Q)}^2 + \lambda^2 \int_0^T \int_\omega w^2 e^{2\lambda\varphi}\, dxdt \right), \quad (1.16)$$

where ω *is an arbitrary subset of the domain.*

The estimate in (1.16) was first proved by Imanuvilov in [15], and then Fu–Yong–Zhang in [13]. The basic idea of the proof relies on taking advantage of the required property in (1.15) and apply it to a particular η where $\mathcal{P}\eta = r_{1t} + r_2 + \lambda w e^{2\lambda\varphi}$, where $r_1, r_2 \in [H^1([0,T]; L^2(\Omega))]^2$ satisfy certain necessary properties. This gives the desired term $\lambda \int_Q w^2 e^{2\lambda\varphi}\, dxdt$ on the smaller side and the estimate is reduced to an estimate for $\|\eta\|_{H_0^1(Q)}$ that is produced through no small effort. The remainder of the introduction is thus organized to provide a better understanding of both the exact controllability/observability of coupled systems of PDEs and inverse problems for those systems and their use of Carleman estimates.

1.2 Control Theory

For the past few decades, the area of control theory for PDEs has been an active area of research leading to discoveries that can be applied to a broad range of applications. Considering an evolution system described by a PDE (or ODE), the controllability problem can be defined as the existence of a control, and corresponding solution, that drives the system within some time, T, from an initial state to a desired final state [30]. The content of this thesis relies mostly on results pertaining to the exact boundary controllability of the single wave equation, or equations consisting of hyperbolic operators in the more general cases.

Qualitatively, by exact controllability of a wave equation or a hyperbolic system we mean the property of steering any initial condition at time $t = 0$ to 0 (and hence any state in suitable function spaces due to the time-reversibility of the system) at target time T, by means of a non-homogeneous function referred to as the *control* function (in a suitable function space). The control function can act on the entire or a portion of the boundary of an open bounded domain in which the hyperbolic system is defined, or within a region contained in the domain. The first is known as boundary control while the latter is interior control.

Much advancement has been made regarding the controllability of the single wave equation under various types of applied controls and boundary conditions. One such article that has become a standard reference in this area of research is the SIAM Review paper [28] by Lions, in which he established the exact boundary controllability for hyperbolic type and Petrowsky type systems via the method known therein as the *Hilbert Uniqueness Method* (HUM). The crucial part of the HUM is that it reduces the exact controllability problem to an observability problem for the adjoint system. Lions specifically applies these results to the wave equation in this paper under both Dirichlet and Neumann boundary control functions. Around the same time, Lasiecka and Triggiani also demonstrated the exact controllability of second order hyperbolic operators under both Dirchlet and Neumann boundary controls by exploiting the relationship with the observability of the adjoint problem [23]. Here, the focus is more on the ontoness of the "control-to-solution" operator that maps the boundary control to the final state of the solution under a preassigned target space. Moreover, Laseicka and Triggiani were able to better characterize the optimal spaces of regularity for the Neumann case.

The aforementioned observability problem for the adjoint system refers to establishing an *observability inequality* that essentially states the initial energy can be "observed" through a suitable boundary trace of the solution to the adjoint system, which is homogeneous on the boundary in the same boundary condition. In the case of interior control, the desired observed data occurs within the specified region within the domain in lieu of the boundary. Traditionally, observability inequality for the wave equation may be established by the moment method in one dimension [30], and the multiplier method for a general dimension as it was done in [28] and [23]. However, such methods may only work with the constant coefficients wave equation and are not robust enough to account for lower order terms or variable coefficients. These were eventually overcome by applying a much more powerful tool, called Carleman estimates, in establishing the observability inequalities for those systems as can be seen in [48, 25, 47], which gave rise to sharper inequalities that included boundary terms and yielded previously assumed uniqueness results for the over-determined system. We refer the reader to Section 1.1 for more literature on Carleman estimates.

With the progress of understanding the control problem for single equations, came applications which led to the desire to understand the control problem in the context of systems of PDEs under various coupling conditions. Until recently, little progress was made regarding the establishment of observability inequalities for systems of PDEs with principal level coupling.

1.3 Inverse Problems

The essential idea behind *inverse problems* is identifying a root cause out of some sort of knowledge of their effects. The field has been motivated in part out of necessity for practical applications in a variety of areas such as geophysical explorations, reflection seismology, biomedical imaging, weather predictions, remote sensing, and mine detection [46]. In regards to a PDE system satisfied over some given domain with respect to time, this can be particularized as the recovery of a coefficient, or multiple coefficients, of the system from some sort of measurement taken in a region within the domain or on the boundary. Specific applications from inverse problems involving hyperbolic systems of PDEs, as is the focus of this paper, include but are not limited to, electromagnetic, acoustic, and elastic waves.

The inverse problem of focus in the present paper is more akin to the multidimensional inverse problem for second-order hyperbolic equations where the measurement is of the single boundary observation type. This sort of problem was pioneered by Bukhgeim and Kilbanov [7], wherein the methodology was grounded in the use of Carleman estimates (discussed with greater detail in 1.1). The ensuing development in the many years since [7] has seen a rich development in the process for determining the uniqueness of coefficients establishing a somewhat routine algorithm for second-order hyperbolic or parabolic type equations.

The typical method involved, developed since its initial appearance in [7], persists in its use of appropriate Carleman type estimates for the underlying system. While variations in determining stability and uniqueness of coefficients transpired, two primary techniques are worth noting here. Imanuvilov and Yammamoto in [16] used Carleman estimates to develop a direct approach for the stable recovery (and resulting implied uniqueness) of coefficients for the wave equation. This process, however, led to increased restrictions on the damping or potential coefficient often denoted as q. In lieu of the typical requirement of $q \in L^{\infty}(\Omega)$ their approach necessitated the need for q to be in an admissible set that imposed more regularity. A second strategy originating with Isakov, as can be observed in [18], takes advantage of a post-Carleman technique to first demonstrate a coefficient as uniquely recoverable and uses controllability results for the system to demonstrate stability separately. It is the latter that is the inspiration behind the inverse problem result proved within. For more details about inverse problems with a single measurement formulation, we refer to the monographs [6, 17, 19, 29] and the many references therein. Although our approach to solve the inverse problem also relies on Carleman estimates, due to the strongly coupled nature of the MT system (introduced in Chapter 2), it is not straight forward to get an appropriate Carleman estimate for the system. To overcome that, we will perform a crucial diagonalization process for the principal part of the MT system first (see Section 2.3 below) and make it a system of wave-like equations with two more variables. Such a diagonalization method was motivated by [12] and is described in further detail in Chapters 2 and 4.

1.4 Coupled Systems

The primary focus of this paper is the application of Carleman estimates en route to establishing an observability inequality for the Mindlin-Timoshenko (MT) plate system and inverse problem for recovering the density coefficient. As the MT system is a strongly coupled system of hyperbolic equations, it is necessary to explore the control theory and inverse problems of similarly coupled hyperbolic equations with different coefficients in the principal part of the operator. A brief exposition of such results are contained within the present section of the introduction.

The exact controllability, or observability, of systems of coupled wave equations has recently gained more attention, but still lacks the robust exploration that exists for the single wave equation. Following the earlier ideas for the single wave equation, Lasiecka and Triggiani [24] considered the exact controllability for a system of coupled non-conservative wave equations for both Dirichlet and Neumann boundary control through the use of Carleman estimates. This, however, was prior to their work with Zhang in [25] that produced much sharper estimates and no longer required the uniqueness assumption for the over-determined system. Later work produced results for exact controllability of systems of wave equations with variable coefficients, but has parallel coupling and considers the Dirichlet case [45].

The case of showing exact controllability for the MT system differs from the standard set up of two weakly coupled hyperbolic equations due to the strong coupling. Since the typical approach for addressing this concern is to diagonalize the principal part of the operator, we introduce a variable substitution for an expression of first order terms within the system. These introduced components thus operate on a lower energy level and ultimately should not appear as observed values in order to imply exact controllability for the original system. This type of observability inequality, where there are components at multiple energy levels and where the lower energy level terms are not observed quantities, is referred to as *indirect observability* and is a matter of more recent research.

1.4.1 Indirect Control

The idea behind *indirect controllability* is to show exact controllability using fewer controls than components for a given system of coupled PDEs. While some results deal with systems coupled in cascade, as in [39] and [40], this is not what is needed for our result. As it is a topic that has only recently gained attention, there is still much to be known regarding using a single control for a coupled system of two hyperbolic PDEs. We present a basic overview of current results below.

Much of the existing literature showing indirect controllability in different energy levels, relies more on energy estimates and multiplier techniques instead of the use of Carleman estimates. This is the case for [2, 3, 4], wherein all show indirect boundary observability for a coupled system of hyperbolic equations. The compromise is then a restriction on the coupling parameters, as is the case in [2, 3]. The third reference shows more the geometry requirements on the coupling region in relation to the control region in order to satisfy the Geometric Control Conditions (GCC) and assumes the coupling parameters do not depend on time. The limitation in all three cases is only zero order coupling. Without Carleman estimates, the task of absorbing first order terms is more of a challenge. Similarly, [10] considers a coupled system of hyperbolic equations with fewer controls in the more abstract setting of a compact manifold. Controllability is proved through the GCCs, which are not satisfied unless the equations have the same wave speed. This case also only considers the case of a single zero order coupling term on one of the components. In [27] a system of wave equations are considered under Neumann boundary control, but the system maintains the same speed of wave propagation between all of the equations in the system and there is only zero order coupling with constant coefficients. Thus, something more is needed for the exact controllability of the MT system.

Tebou considers the interior indirect control, namely, a single interior control for a coupled system of wave equations with lower order terms in [41], and is the inspiration behind much of the indirect observability estimate established in Chapter 3. In [41], Carleman estimates, as derived in [15] and [13], with norms in negative Sobolev spaces are used to absorb lower order terms. There is also no smallness constraint on the coupling parameters, but the controllability is interior controllability from a region close to the boundary. This is the only known indirect control result, in the multi–dimensional case, with control not done in cascade that contains first order terms, and even then, only first order terms for the higher regularity component. The MT system, once diagonalized, also contains first order terms for the lower regularity component as coupling terms present in the other equations for higher regularity components. There are no known results that handle this sort of coupling in the multi–dimensional setting due to the difficulty of absorbing these terms into the estimate. Moreover, there is only one known control result for the MT system in [5], but this is for the one–dimensional case, which reduces the MT system to a beam system instead of plate. This reference uses similar techniques as in [41], but also incorporates elements within the process that can not be generalized to two–dimensions. Outcomes from these coupled systems can be applied to systems that model physical phenomena to include electromagnetism, elasticity, thermoelastic, and various plate models [10]. This, naturally, is also the goal for demonstrating the observability of the system in Chapter 3, that the results may be applied to the more general setting to accommodate many types of systems.

The main focus of this thesis is to consider the two problems of first showing exact controllability of the MT model and second, uniquely recover one of the coefficients in the MT system.

Since the MT model is a system of hyperbolic equations with strong coupled terms we diagonalize the system into a system of coupled wave equations for both problems. The diagonalization of the MT model presented in Chapter 2 is motivated by existing diagonalizations of other elastic systems that introduce divergence and curl terms into the system and is the diagonalization used in Chapter 3. The remainder of the thesis is organized as follows: a chapter detailing the MT system, its derivation, and the derivation of the diagonalization used for the control problem, the subsequent chapter then shows the controllability of the MT system, followed by the inverse problem and future/ongoing research.

Chapter 2

Mindlin-Timoshenko Plate System

2.1 Background

In the study of thin, rigid structures possessing a certain amount of flexibility and referred to as *plates*, a few models have become the accepted models of choice for engineering applications due to their modeling accuracy. Among these is the Mindlin-Timoshenko (MT) model, which is useful in situations where one must account for transverse shear effects. Thin plates, assumed to possess a uniform thickness h that is small in comparison to the dimensions describing the lower and upper surfaces of the plate, have what is known as a middle surface. The middle surface is a plane existing halfway between the two faces of the plate and running parallel to them. The primary assumption in many classical models, such as the Kirchoff plate model [21], is that filaments that are initially straight and perpendicular to the middle surface remain so during middle surface deformation. This implies the transverse shear is negligible, at least in comparison to the thickness of the plate, contrary to the MT model. These models are limited in their description of plates or beams experiencing high-frequency vibrations [9, 33].

Prior to the development of the MT model, a few models existed which did take into account shear deformations. Such a model accounting for the transverse shear deformation occurring to the plate involving two shear angles was considered by Reissner [9, 35]. Reissner's model [35] possessed several deviations from classical plate theory, including allowing for a change in the thickness of the plate due to stresses. These were also changes from an earlier model proposed by Timoshenko, which considered the displacement of a beam taking into account a single shear angle of its filament [9, 33, 43]. Eventually a model was proposed by Mindlin, independent of Reissner, that also considered two shear angles, and has been foundational in the development of modern plate theory [9, 31, 33]. The Mindlin-Timoshenko model considered for this present paper was considered in Lagnese [21] with explorations of the systems stability and well-posedness researched by Pei et al. [33], Jorge Silva et al. [37], Grobbelaar-Van Dalsen [9], and Fernandez Sare [36]. It is the derivation of the MT model in Lagnese's *Boundary Stabilization of Thin Plates* [21] that we follow for the purposes of presenting the model here.

2.2 Summary of the Derivation

Under the assumptions that the material described by the model is perfectly elastic and isotropic, we provide a few details in the derivation for the MT model. For the development of the model, we assume the traditional notation for stress and strain tensors as $\epsilon_{ij}, \sigma_{ij}$ respectively where we use the rectangular coordinates (x_1, x_2, x_3) with $x_3 = 0$ corresponding to the middle surface. Under the assumption that the thin plates are perfectly elastic, isotropic, and maintain a uniform thickness throughout deformation the in-plane stress-strain relationships are described by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1-\mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} \tag{2.1}$$

where E is the (Young's) modulus of elasticity in tension and μ is Poisson's ratio constrained by $0 < \mu < \frac{1}{2}$ for physical applications [21, 43]. For structural steel, E, is quite small (0.001) and Poisson's ratio is 0.3, but is taken as 0.25 in most situations [43]. The normal stress, σ_{33}, is assumed to be negligible in plate theory, and the shear stress-strain throughout the thickness of the plate are

$$\sigma_{13} = k\frac{E}{1+\mu}\epsilon_{13}, \qquad \sigma_{23} = k\frac{E}{1+\mu}\epsilon_{23}$$

where k is the shear correction coefficient [21]. Then, representing displacement at time t by $S_i(x_1, x_2, x_3, t)$ for $i = 1, 2, 3$ and $s_i(x_1, x_2, t)$ for in-plane displacement we assume the linear strain displacement relation

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial S_i}{\partial x_j} + \frac{\partial S_j}{\partial x_i}\right), \qquad i,j = 1,2,3$$

as in [21]. Since the filament is not assumed to maintain perpendicularity with the middle surface we introduce the rotation angles ψ and ϕ and we define the displacements as

$$S_1 = s_1 + x_3\psi, \quad S_2 = s_2 + x_3\phi, \quad S_3 = w.$$

where, again, s_1 and s_2 represent in-plane displacement (stretching/compression). Note the strain energy $\mathbb{P}$ and kinetic energy of a plate $\mathbb{K}$ are defined as

$$\mathbb{P} = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\Omega} \epsilon_{ij}\sigma_{ij}dx_1dx_2dx_3, \quad \mathbb{K} = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\Omega} \rho\frac{\partial S_i}{\partial t}\frac{\partial S_j}{\partial t}dx_1dx_2dx_3 \tag{2.2}$$

where, as can be seen in Chapter 1 of [22], the in-plane deformations s_1 and s_2 separate from the components related to bending, namely w, ψ, and ϕ. Since the primary motivation is the stabilization of the energy due to bending in $\mathbb{P}+\mathbb{K}$, labeled as $\mathbb{P}_b+\mathbb{K}_b$, this is all that is considered in the development of the model. Other models, such as the von Karman model, do not allow for such uncoupling and must consider the total energy $\mathbb{P}+\mathbb{K}$ without separating the bending energy. For the complete heuristic derivation, see [22, Ch. 1], but essentially the equations of motion for w, ψ and ϕ are obtained by setting

$$\int_0^T \mathbb{K}_b(t) + \mathbb{W}_b(t) - \mathbb{P}_b(t)dt$$

to zero, where $\mathbb{W}_b$ is the part of the work resulting from forces acting on the plate that contributes to bending. Making the assumption of no external loading on either plate face yields the system (*for the remainder of the chapter we shall use* (x,y,z) *in place of* (x_1,x_2,x_3) *to ease notation*)

$$\begin{cases} \rho h w_{tt} - K\Delta w - K(\psi_x + \phi_y) = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\psi_{tt} - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}\right) - D\left(\frac{1+\mu}{2}\phi_{xy}\right) + K(\psi + w_x) = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\phi_{tt} - D\left(\phi_{yy} + \frac{1-\mu}{2}\phi_{xx}\right) - D\left(\frac{1+\mu}{2}\psi_{xy}\right) + K(\phi + w_y) = 0 & \text{in } [0,T]\times\Omega \end{cases} \tag{2.3}$$

stated here without boundary conditions, where $\Delta = \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplacian operator. The positive constants $D = \frac{Eh^3}{12(1-\mu^2)}$, $K = \frac{kEh}{2(1+\mu)}$, and ρ are the *modulus of flexural rigidity*, *shear modulus*, and density respectively for some $\Omega \subset \mathbb{R}^2$ [21, 36, 33, 37]. Moreover, due to the substitution $S_3 = w$ we have that w is displacement of the plate from the central plane in the normal direction to the mid-surface plane, while ϕ and ψ are the angles of shear deformation [36]. For the well-posedness of the MT system, it is necessary to include the initial conditions

$$\begin{cases} (w(0,x,y), \psi(0,x,y), \phi(0,x,y)) = (w^0, \psi^0, \psi^0) \in (H^1(\Omega))^3 \\ (w_t(0,x,y), \psi_t(0,x,y), \phi_t(0,x,y)) = (w^1, \psi^1, \psi^1) \in (L^2(\Omega))^3. \end{cases} \tag{2.4}$$

The typical boundary conditions that accompany this system are either Dirichlet boundary conditions

$$w = f_1,\ \psi = f_2,\ \phi = f_3 \quad \text{on } [0,T] \times \partial\Omega \tag{2.5}$$

or the associated Neumann type boundary conditions for the MT operator as presented in [25]

$$\begin{aligned} &K\left(\frac{\partial w}{\partial \nu} + \nu_1 \psi + \nu_2 \phi\right) = g_1 \\ &D\left[\nu_1 \frac{\partial \psi}{\partial x} + \mu\nu_1 \frac{\partial \phi}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_2\right] = g_2 \\ &D\left[\nu_2 \frac{\partial \phi}{\partial y} + \mu\nu_2 \frac{\partial \psi}{\partial x} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_1\right] = g_3 \end{aligned} \tag{2.6}$$

where the regularity of f_i, g_i for $i = 1, 2, 3$ depends on the particular application. For stability results of the MT system under certain boundary feedback, either of these conditions or a mixture thereof are what is typically presented. For the purposes of our results we assume Dirichlet boundary conditions, but the inverse problem thus necessitates measurement of the type in Section 2.6.

2.3 Diagonalization

The goal for the current section is to demonstrate a sample transform for the system shown in (2.3) into a form that more readily lends itself to applications. The diagonalizations used in Chapter 3 is the same as the derivation that follows and the diagonalization in Chapter 4 follows a similar process. To further clarify, the difficulty in applying Carleman estimates to such a system, as is needed in both the inverse problem and controllability results, arises from the coupling terms. These terms become especially problematic when the coupling is of the same order as the PDE such as the mixed second order coupling terms that are present in the MT system, which are unable to be absorbed on the lower side of observability estimates using established methods. The system is thus transformed in a way that circumvents this issue. All diagonalizations used within are inspired by the diagonalization in [12] of certain Maxwell and elastic system.

To formulate (2.3) in a way that permits the application of a certain Carleman type estimate and ensuing results contained within this paper, we introduce a substitution of variables and apply through differentiation and algebraic manipulation. Consider $\eta = \psi_y - \phi_x$. Then via differentiation we have the relationships

$$\eta_{tt} = \psi_{ytt} - \phi_{xtt}, \quad \eta_{xx} = \psi_{yxx} - \phi_{xxx}, \quad \eta_{yy} = \psi_{yyy} - \phi_{xyy}. \tag{2.7}$$

Moreover, differentiating the second and third equations in (2.3) with respect to y and x respectively, we have the relations

$$\begin{cases} \frac{\rho h^3}{12}(\psi_{tt})_y - D\left(\psi_{xxy} + \frac{1-\mu}{2}\psi_{yyy}\right) - D\left(\frac{1+\mu}{2}\phi_{xyy}\right) + K(\psi_y + w_{xy}) = 0 \\ \frac{\rho h^3}{12}(\phi_{tt})_x - D\left(\phi_{yyx} + \frac{1-\mu}{2}\phi_{xxx}\right) - D\left(\frac{1+\mu}{2}\psi_{xyx}\right) + K(\phi_x + w_{yx}) = 0. \end{cases} \tag{2.8}$$

Manipulating the equations in (2.8) and taking their difference produces

$$\begin{aligned} 0 &= \frac{\rho h^3}{12}(\psi_{ytt} - \phi_{xtt}) - D\left(\psi_{xxy} + \frac{1-\mu}{2}\psi_{yyy} - \phi_{yyx} - \frac{1-\mu}{2}\phi_{xxx}\right) \\ &\qquad - D\left(\frac{1+\mu}{2}\phi_{xyy} - \frac{1+\mu}{2}\psi_{xyx}\right) + K(\psi_y - \phi_x) \\ &= \frac{\rho h^3}{12}(\psi_{ytt} - \phi_{xtt}) - D\frac{1-\mu}{2}\left(\psi_{yxx} - \phi_{xxx} + \psi_{yyy} - \phi_{xyy}\right) + K(\psi_y - \phi_x). \end{aligned} \tag{2.9}$$

Performing the substitution shown in (2.7) yields

$$\frac{\rho h^3}{12}\eta_{tt} - D\left(\frac{1-\mu}{2}\right)\Delta\eta + K\eta = 0. \tag{2.10}$$

Motivated by this substitution, it is then possible to rewrite the second equation in (2.3) as

$$\begin{aligned}
0 =& \frac{\rho h^3}{12}\psi_{tt} - D\left(\psi_{xx} + \psi_{yy} + \frac{1-\mu}{2}\psi_{yy} - \psi_{yy}\right) - D\left(\frac{1+\mu}{2}\phi_{xy}\right) + K\left(\psi + w_x\right) \\
=& \frac{\rho h^3}{12}\psi_{tt} - D\Delta\psi + D\left(\frac{1+\mu}{2}\right)\left(\psi_{yy} - \phi_{xy}\right) + K\left(\psi + w_x\right) \\
=& \frac{\rho h^3}{12}\psi_{tt} - D\Delta\psi + D\left(\frac{1+\mu}{2}\right)\eta_y + K\left(\psi + w_x\right) \qquad (2.11)
\end{aligned}$$

Following a similar process for the third equation in (2.3) gives

$$\frac{\rho h^3}{12}\phi_{tt} - D\Delta\phi - D\left(\frac{1+\mu}{2}\right)\eta_x + K\left(\phi + w_y\right) = 0. \qquad (2.12)$$

In a similar manner, consider $\beta = \psi_x + \phi_y$. Then, differentiating the second equation in (2.3) with respect to x, and the third equation with respect to y gives

$$\begin{cases}
\frac{\rho h^3}{12}(\psi_{tt})_x - D\left(\psi_{xxx} + \frac{1-\mu}{2}\psi_{xyy}\right) - D\left(\frac{1+\mu}{2}\phi_{xxy}\right) + K(\psi_x + w_{xx}) = 0 \\
\frac{\rho h^3}{12}(\phi_{tt})_y - D\left(\phi_{yyy} + \frac{1-\mu}{2}\phi_{yxx}\right) - D\left(\frac{1+\mu}{2}\psi_{xyy}\right) + K(\phi_y + w_{yy}) = 0.
\end{cases} \qquad (2.13)$$

Taking the sum of the two equations in (2.13) after some manipulation yields,

$$\frac{\rho h^3}{12}\beta_{tt} - D\Delta\beta + K(\beta + \Delta w) = 0. \qquad (2.14)$$

Using a substitution from the first equation we arrive at the following equation

$$\frac{\rho h^3}{12}\beta_{tt} - D\Delta\beta + \rho h w_{tt} = 0. \qquad (2.15)$$

Thus, from the derivations in (2.10), (2.11), (2.15) and (2.12) we can then express the system (2.3) using the four resulting equations

$$\begin{cases} \rho h w_{tt} - K\Delta w - K(\psi_x + \phi_y) = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\beta_{tt} - D\Delta\beta + \rho h w_{tt} = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\eta_{tt} - D\left(\frac{1-\mu}{2}\right)\Delta\eta + K\eta = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\psi_{tt} - D\Delta\psi + D\left(\frac{1+\mu}{2}\right)\eta_y + K\left(\psi + w_x\right) = 0, & \text{in } [0,T]\times\Omega \\ \frac{\rho h^3}{12}\phi_{tt} - D\Delta\phi - D\left(\frac{1+\mu}{2}\right)\eta_x + K\left(\phi + w_y\right) = 0, & \text{in } [0,T]\times\Omega \end{cases} \tag{2.16}$$

which is a more ideal form since the mixed second order coupling terms are no longer present. Initial conditions are inherited from (2.4). This is the diagonalization used in Chapter 3.

Chapter 3

Exact controllability for MT system

3.1 Introduction

Let $\Omega \subset \mathbb{R}^2$, which represents the mid-surface of the plate, be an open bounded domain with C^1 boundary Γ and recall earlier definitions $Q = [0,T] \times \Omega$ and $\Sigma = [0,T] \times \Gamma$. Moreover, let ω be a proper nonempty subset of Ω with characteristic function, χ_ω. Given $T > 0$ we consider the following controlled MT system of equations with an internal local controller acting on ω:

$$\begin{cases} \rho h w_{tt} - K\Delta w - K(\psi_x + \phi_y) = f_1 \chi_\omega, & \text{in } Q \\ \frac{\rho h^3}{12}\psi_{tt} - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}\right) - D\left(\frac{1+\mu}{2}\phi_{xy}\right) + K(\psi + w_x) = f_2\chi_\omega, & \text{in } Q \\ \frac{\rho h^3}{12}\phi_{tt} - D\left(\phi_{yy} + \frac{1-\mu}{2}\phi_{xx}\right) - D\left(\frac{1+\mu}{2}\psi_{xy}\right) + K(\phi + w_y) = f_3\chi_\omega & \text{in } Q \end{cases} \tag{3.1}$$

with initial and boundary conditions

$$\begin{cases} (w(0,x,y), \psi(0,x,y), \phi(0,x,y)) = (w^0, \psi^0, \phi^0) & \text{in } \Omega \\ (w_t(0,x,y), \psi_t(0,x,y), \phi_t(0,x,y)) = (w^1, \psi^1, \phi^1) & \text{in } \Omega \\ w = \phi = \psi = 0 & \text{in } \Sigma. \end{cases}$$

The state of the system in (3.1) is represented by a functional vector $(w, w_t, \psi, \psi_t, \phi, \phi_t)$ where the component $w = w(t, x, y)$ corresponds to the displacement of the plate from the central plane in the normal direction to the mid-surface plane at point $(x, y) \in \Omega$ and time $t > 0$, and $\psi = \psi(t, x, y)$, $\phi = \phi(t, x, y)$ are the angles of shear deformation. The controls are represented by $f_1(t, x, y)$, $f_2(t, x, y)$ and $f_3(t, x, y)$ which act on the system in the specified subset of the domain, ω. We also place an assumption on the thickness of the plate for the purposes of controllability in that h must satisfy $h^3 < 36/E$ where E is Young's modulus of elasticity.

Demonstrating the controllability of the system in (3.1), as a strongly coupled system, follows the HUM [28], wherein controllability of the diagonalized system can be shown via an observability inequality for the homogeneous adjoint system henceforth referred to as the dual system. To show the observability estimate for the dual system, we first diagonalize the principal part of the differential operator and use an indirect observability technique that yields an observability inequality in two different energy levels, with the variables introduced for the diagonalization not appearing in the observed part. This then implies the controllability of the original system. We begin with expressing the dual system in the next section and its associated diagonalization. This is then followed by the derivation for the indirect observability result.

3.2 Diagonalization

Introducing the dual system to (3.1) we have (z_1, z_2, z_3) as the dual components to (w, ψ, ϕ), satisfy the system

$$\begin{cases} \rho h z_{1tt} - K\Delta z_1 - K(z_{2x} + z_{3y}) = 0, & \text{in } Q \\ \frac{\rho h^3}{12} z_{2tt} - D\left(z_{2xx} + \frac{1-\mu}{2} z_{2yy}\right) - D\left(\frac{1+\mu}{2} z_{3xy}\right) + K(z_2 + z_{1x}) = 0, & \text{in } Q \\ \frac{\rho h^3}{12} z_{3tt} - D\left(z_{3yy} + \frac{1-\mu}{2} z_{3xx}\right) - D\left(\frac{1+\mu}{2} z_{2xy}\right) + K(z_3 + z_{1y}) = 0 & \text{in } Q \end{cases} \tag{3.2}$$

with initial and boundary conditions

$$\begin{cases} (z_1(0, x, y), z_2(0, x, y), z_3(0, x, y)) = (z_1^0, z_2^0, z_3^0) & \text{in } \Omega \\ (z_{1t}(0, x, y), z_{2t}(0, x, y), z_{3t}(0, x, y)) = (z_1^1, z_2^1, z_3^1) & \text{in } \Omega \\ z_1 = z_2 = z_3 = 0 & \text{in } \Sigma. \end{cases}$$

For the diagonalization of (3.2) let $u_1 = z_{2x} + z_{3y}$ and $u_2 = z_{2y} - z_{3x}$. Then, following a similar diagonalization process as shown in 2.3 we can express the system (3.1) using the resulting equations

$$\begin{cases} \rho h z_{1tt} - K\Delta z_1 - K u_1 = 0, & \text{in } Q \quad (3.3a) \\ \dfrac{\rho h^3}{12} u_{1tt} - D\Delta u_1 + \rho h z_{1tt} = 0, & \text{in } Q \quad (3.3b) \\ \dfrac{\rho h^3}{12} u_{2tt} - D\left(\dfrac{1-\mu}{2}\right)\Delta u_2 + K u_2 = 0, & \text{in } Q \quad (3.3c) \\ \dfrac{\rho h^3}{12} z_{2tt} - D\Delta z_2 + D\left(\dfrac{1+\mu}{2}\right) u_{2y} + K\left(z_2 + z_{1x}\right) = 0, & \text{in } Q \quad (3.3d) \\ \dfrac{\rho h^3}{12} z_{3tt} - D\Delta z_3 - D\left(\dfrac{1+\mu}{2}\right) u_{2x} + K\left(z_3 + z_{1y}\right) = 0, & \text{in } Q \quad (3.3e) \end{cases}$$

with initial and boundary conditions

$$\begin{cases} (z_1(0,x,y), z_2(0,x,y), z_3(0,x,y)) = (z_1^0, z_2^0, z_3^0) & \text{in } \Omega \\ (z_{1t}(0,x,y), z_{2t}(0,x,y), z_{3t}(0,x,y)) = (z_1^1, z_2^1, z_3^1) & \text{in } \Omega \\ (u_1(0,x,y), u_2(0,x,y)) = (u_1^0, u_2^0) & \text{in } \Omega \\ (u_{1t}(0,x,y), u_{2t}(0,x,y)) = (u_1^1, u_2^1) & \text{in } \Omega \\ z_1 = z_2 = z_3 = 0,\ u_1 = g_1,\ u_2 = g_2. & \text{in } \Sigma. \end{cases}$$

3.3 Exact Controllability

3.3.1 Introduction and Main Results

The primary result regarding the exact controllability of the system in (3.1) is presented as an indirect observability result for the system shown in (3.3). This implies an observability result for the dual system (3.2), which, as is well known, is equivalent to the desired exact controllability result. Thus our focus is on demonstrating the indirect observability estimate. For the statement of the main result and subsequent proof we maintain the notation convention $\sum_i$ (resp. $\sum_j$) for the summation $\sum_{i=1}^3$ ($\sum_{j=1}^2$).

Theorem 3.3.1. *Let ω be a neighborhood of Γ. Then, under the strength of assumptions (A.1) and (A.2) and for sufficiently large time, $T > 0$, there is a constant C such that for all $(z_1^0, z_1^1; z_2^0, z_2^1; z_3^0, z_3^1) \in \left[H_0^1(\Omega) \times L^2(\Omega)\right]^3$, and $(u_1^0, u_1^1; u_2^0, u_2^1) \in [L^2(\Omega) \times H^{-1}(\Omega)]^2$, with the added time regularity assump-*

tion $z_2, z_3 \in C^2([0,T]; L^2(\Omega))$, *the observability estimate*

$$\mathbb{E}(0) \leq C \int_0^T \int_\omega (\sum_i |z_{it}|^2 + \sum_i |z_i|^2)\, dxdydt \tag{3.4}$$

holds for the corresponding solution 5-tuple $(z_1, z_2, z_3, u_1, u_2)$ *of* (3.3). *Here, the energy term is defined for each* $t \in [0,T]$ *as follows,*

$$\mathbb{E}(t) = \sum_i E_{z_i}(t) + \sum_j \check{E}_{u_i}(t)$$

where, for a given u in the appropriate space we have

$$E_u(t) = \int_\Omega u_t^2(t) + |\nabla u(t)|^2 dydx, \quad \check{E}_u(t) = \|u(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{H^{-1}(\Omega)}\,.$$

3.3.2 Energy Estimates

For the remainder of this chapter, consider the use of C to denote a generic positive constant depending on the various domains and coefficients and parameters introduced, but not on the initial data. The proof of the main theorem not only relies on the use of the Carleman estimates introduced in the introduction of this paper, but also invokes a well known energy estimate result. One obtains this energy estimate, appearing as follows. using the typical energy method,

Lemma 3.3.2. *Under the same hypotheses as Theorem 3.3.1 on has the following energy estimates:*

$$\tilde{E}_{u_i}(t) \leq C\tilde{E}(s), \qquad \textit{for } i = 1,2,\ \forall s,t \in [0,T]$$
$$E_{z_i}(t) \leq CE(s), \qquad \textit{for } i = 1,2,3,\ \forall s,t \in [0,T]$$

implying the estimate

$$\mathbb{E}(t) \leq C\mathbb{E}(s) \qquad \forall s,t \in [0,T]. \tag{3.5}$$

3.4 Proof of Main Result

The proof for Theorem 3.3.1 is separated into three main steps. The initial step uses a multiplier technique to estimate the terms in the Sobolov spaces of negative order and the second step applies the various Carleman estimates introduced in 1.1 to derive an initial observability estimate, which includes u_1^2 and u_2^2 as observed quantities. The final steps seeks to absorb the lower regularity components u_1 and u_2 into the estimate so only the original components of the system (3.2) appear as observed components. For Step 1, it can be assumed $\langle \cdot,\cdot\rangle \triangleq \langle \cdot,\cdot\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$, while the duality brackets will switch to imply $\langle \cdot,\cdot\rangle_{H^{-1}(Q),H_0^1(Q)}$ in the subsequent steps.

Step 1 claim:

$$\mathbb{E}(0) \le C\int_{Q_0} \sum_i {z_i}_t^2 + z_2^2 + z_3^2 + |\nabla z_1|^2 + \sum_j |u_j|^2\, dQ_0 dt \tag{3.6}$$

Proof. We will first provide an $L^2(\Omega)$ estimate for the energy terms with a negative Sobolev norm. Then, we will derive an estimate for the gradient terms on z_2 and z_3. For some $T_0 > 0$ let $f \in C^1([T_0, T_0'])$ be a nonnegative function such that $f(T_0) = f(T_0') = 0$ and $|f'|/f \in L^\infty(T_0, T_0')$. Also let $G = (-\Delta)^{-1}$. Multiplying equation (3.3c) by fGu_2 and integrating by parts over $Q_0 \triangleq [T_0, T_0'] \times \Omega$ yields

$$\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f\langle u_{2t}, Gu_{2t}\rangle\, dt = -\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f'\langle u_{2t}, Gu_2\rangle\, dt + \int_{Q_0} D\left(\frac{1-\mu}{2}\right) f u_2^2\, dQ_0 + K\int_{Q_0} f u_2 G u_2\, dQ_0. \tag{3.7}$$

Working term by term on the RHS of (3.7) we have by Cauchy-Schwarz and trivial analysis inequalities

$$\begin{aligned}
\left|\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f'\langle u_{2t}, Gu_2\rangle\, dt\right| &\le \frac{|f'|^2}{2f}\left(\frac{\rho h^3}{12}\right)^2 \int_{Q_0} u_2 G u_2\, dQ_0 + \frac{1}{2}\int_{T_0}^{T_0'} f\langle u_{2t}, Gu_{2t}\rangle\, dt \\
&\le C\int_{Q_0} u_2^2\, dQ_0 + \frac{1}{2}\int_{T_0}^{T_0'} f\langle u_{2t}, Gu_{2t}\rangle\, dt \\
\left|\int_{Q_0} D\left(\frac{1-\mu}{2}\right) f u_2^2\, dQ_0\right| &\le C\int_{Q_0} u_2^2\, dQ_0 \\
\left|K\int_{Q_0} f u_2 G u_2\, dQ_0\right| &\le C\int_{Q_0} u_2^2 dQ_0.
\end{aligned}$$

Thus, combining these estimate we are able to derive

$$\int_{T_0}^{T_0'} f\tilde{E}_{u_2}(t)\, dt \le C\int_{Q_0} u_2^2\, dQ_0. \tag{3.8}$$

Repeating the process for the second equation in (3.3) we multiply equation (3.3b) by fGu_1 and integrating by parts over $Q_0 \triangleq [T_0, T_0'] \times \Omega$ yields

$$\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f\langle u_{1t}, Gu_{1t}\rangle\,dt = -\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f'\langle u_{1t}, Gu_1\rangle\,dt + \int_{Q_0} Dfu_1^2\,dQ_0 + K\int_{Q_0} fu_1Gu_1\,dQ_0 + K\int_{Q_0} fu_1z_1\,dQ_0. \tag{3.9}$$

Note here we use the equation satisfied by z_1 in (3.3) to replace ρhz_{1tt}. Working term by term on the RHS of (3.9) we have by Cauchy-Schwarz and trivial analysis inequalities

$$\begin{aligned}
\left|\frac{\rho h^3}{12}\int_{T_0}^{T_0'} f'\langle u_{1t}, Gu_1\rangle\,dt\right| &\le \left(\frac{\rho h^3}{12}\right)^2\int_{Q_0}\frac{|f'|^2}{2f}u_1Gu_1\,dQ_0 + \frac{1}{2}\int_{T_0}^{T_0'} f\langle u_{1t}, Gu_{1t}\rangle\,dt\\
&\le C\int_{Q_0} u_1^2\,dQ_0 + \frac{1}{2}\int_{T_0}^{T_0'} f\langle u_{1t}, Gu_{1t}\rangle\,dt\\
\left|\int_{Q_0} Dfu_1^2\,dQ_0\right| &\le C\int_{Q_0} u_1^2\,dQ_0\\
\left|K\int_{Q_0} fu_1Gu_1\,dQ_0\right| &\le C\int_{Q_0} u_1^2 dQ_0\\
\left|K\int_{Q_0} fu_1z_1\,dQ_0\right| &\le C\int_{Q_0} u_1^2 + z_1^2 dQ_0.
\end{aligned}$$

Thus, combining these estimate we are able to derive

$$\int_{T_0}^{T_0'} f\tilde{E}_{u_1}(t)\,dt \le C\int_{Q_0} u_1^2 + z_1^2\,dQ_0. \tag{3.10}$$

Next, we will provide an estimate for the energy terms that removes the gradient terms on z_1 and z_2. Multiplying the equation (3.3d) by fz_2, integrating over Q_0, applying Green's theorems and integration by parts produces

$$\begin{aligned}
D\int_{Q_0} f|\nabla z_2|^2\,dQ_0 =& \frac{\rho h^3}{12}\int_{Q_0} fz_{2t}^2\,dQ_0 + \frac{\rho h^3}{12}\int_{Q_0} f'z_{2t}z_2\,dQ_0 + D\left(\frac{1+\mu}{2}\right)\int_{Q_0} fu_2z_{2y}\,dQ_0\\
&- K\int_{Q_0} f(z_2 + z_{1x})z_2\,dQ_0 + \frac{\rho h^3}{12}\int_{Q_0}(r''z_2 + 2r'z_{2t})\tilde{z}_2\,dQ_0.
\end{aligned} \tag{3.11}$$

Examining the terms on the larger side of the estimate we have by Cauchy Schwarz and other common analysis inequalities

$$\left|\frac{\rho h^3}{12}\int_{Q_0} f{z_2}_t^2\,dQ_0\right| \le C\int_{Q_0} {z_2}_t^2\,dQ_0$$
$$\left|\frac{\rho h^3}{12}\int_{Q_0} f' z_{2t} z_2\,dQ_0\right| \le C\int_{Q_0} z_{2t}^2\,dQ_0 + C\int_{Q_0} z_2^2 dQ_0$$
$$\left|D\left(\frac{1+\mu}{2}\right)\int_{Q_0} f u_2 z_{2y}\,dQ_0\right| \le C\int_{Q_0} u_2^2\,dQ_0 + D\left(\frac{1+\mu}{2}\right)\int_{Q_0} |\nabla z_2|^2\,dQ_0$$
$$\left|K\int_{Q_0} f(z_2 + z_{1x})z_2\,dQ_0\right| \le C\int_{Q_0} (z_2^2 + |\nabla z_1|^2)\,dQ_0$$
$$\left|\frac{\rho h^3}{12}\int_{Q_0} f(r'' z_2 + 2r' z_{2t})z_2\,dQ_0\right| \le C\int_{Q_0} (z_2^2 + z_{2t}^2)\,dQ_0.$$

Notice for the gradient term of z_2 on the RHS of the third estimate, we have by assumption on the parameter μ for physical applications that $\frac{1}{2} < \frac{1+\mu}{2} < \frac{3}{4}$ implying $D - D\left(\frac{1+\mu}{2}\right) > 0$ permitting this term to be absorbed into the smaller side. Reporting the remaining estimates into (3.11) gives the estimate

$$\int_{T_0}^{T_0'} f E_{z_2}(t)\,dt \le C\int_{Q_0} (z_2^2 + {z_2}_t^2 + |\nabla z_1| + u_2^2)\,dQ_0 \tag{3.12}$$

In a similar manner, multiplying equation (3.16e) by $\tilde{z}_3$ integrating over Q and following the same process results in

$$\int_{T_0}^{T_0'} f E_{z_3}(t)\,dt \le C\int_{Q_0} (z_3^2 + {z_3}_t^2 + |\nabla z_1| + u_2^2)\,dQ_0 \tag{3.13}$$

Thus, combining (3.8), (3.10), (3.12), and (3.13) yields

$$\int_{T_0}^{T_0'} f\mathbb{E}(t)\,dt \le C\int_{Q_0} \sum_i {z_i}_t^2 + z_2^2 + z_3^2 + |\nabla z_1|^2 + \sum_j |u_j|^2\,dQ_0 dt$$

and referencing the energy estimate (3.5) gives the desired inequality. □

Step 2 claim:

We claim the following holds

$$\lambda \int_Q r^2(\lambda^2 z_1^2 + z_{1t}^2 + |\nabla z_1|^2)e^{2\lambda\varphi}\, dQ + \lambda \int_Q r^2(z_2^2 + z_3^2 + u_1^2 + \lambda^2 u_2^2)e^{2\lambda\varphi}\, dQ \leq$$
$$Ce^{-c_1\lambda}\mathbb{E}(0) + C\lambda^2 \int_0^T r^2 \int_\omega e^{2\lambda\varphi}(\lambda^2 z_1^2 + z_{1t}^2 + z_2^2 + z_3^2)\, d\omega dt + C\lambda^2 \int_0^T r^2 \int_{\omega_0} e^{2\lambda\varphi}(u_1^2 + \lambda^2 u_2^2)\, d\omega_0 dt. \tag{3.14}$$

for some positive constant c_1 and large parameter $\lambda > 0$ where we define the cut-off function $r \in C_0^\infty((0,T))$ so that $r \equiv 1$ in some time interval, $[T_1, T_1']$ so that $r < 1$ on $\tilde{Q} \triangleq ([0,T_1) \cup (T_1', T] \times \Omega)$ and $[T_0, T_0'] \subset [T_1, T_1']$ (in other words $Q_0 \subset Q/\tilde{Q}$).

Proof. This step provides an intermediary estimate that will be combined with an inequality in the subsequent step in order to provide an estimate for the larger side of (3.6). Let r be as defined above and also let ω_0 represent a neighborhood of Γ such that $\omega_0 \subset\subset \omega$. Since (1.7) holds for our choice of φ, we can select $T_1 > 0$ and $T_1' > 0$ such that we have $\varphi(t,x,y) \leq -\gamma$ for some $\gamma > 0$ for all (t,x,y) on $\tilde{Q}$. Then the functions

$$\tilde{z_1}(t,x,y) = r(t)z_1(t,x,y),\ \tilde{z_2}(t,x,y) = r(t)z_2(t,x,y),\ \tilde{z_3}(t,x,y) = r(t)z_3(t,x,y)$$
$$\tilde{u_2}(t,x,y) = r(t)u_2(t,x,y),\ \tilde{u_1}(t,x,y) = r(t)u_1(t,x,y), \tag{3.15}$$

satisfy the system

$$\begin{cases} \rho h\tilde{z}_{1tt} - K\Delta\tilde{z}_1 - K\tilde{u_1} = \rho h(r''z_1 + 2r'z_{1t}), & \text{in } Q \quad (3.16a)\\ \frac{\rho h^3}{12}\tilde{u}_{2tt} - D\left(\frac{1-\mu}{2}\right)\Delta\tilde{u_2} + K\tilde{u_2} = \frac{\rho h^3}{12}(r''u_2 + 2r'u_{2t}), & \text{in } Q \quad (3.16b)\\ \frac{\rho h^3}{12}\tilde{u}_{1tt} - D\left(\frac{1-\mu}{2}\right)\Delta\tilde{u_1} + \rho h r z_{1tt} = \frac{\rho h^3}{12}(r''u_1 + 2r'u_{1t}) + \rho h(r''z_1 + 2r'z_{1t}), & \text{in } Q \quad (3.16c)\\ \frac{\rho h^3}{12}\tilde{z}_{2tt} - D\Delta\tilde{z}_2 + D\left(\frac{1+\mu}{2}\right)\tilde{u}_{2y} + K\left(\tilde{z}_2 + \tilde{z}_{1x}\right) = \frac{\rho h^3}{12}(r''z_2 + 2r'z_{2t}), & \text{in } Q \quad (3.16d)\\ \frac{\rho h^3}{12}\tilde{z}_{3tt} - D\Delta\tilde{z}_3 - D\left(\frac{1+\mu}{2}\right)\tilde{u}_{2x} + K\left(\tilde{z}_3 + \tilde{z}_{1y}\right) = \frac{\rho h^3}{12}(r''z_3 + 2r'z_{3t}), & \text{in } Q\ . \quad (3.16e) \end{cases}$$

In this step we will apply the Carleman estimates (1.14) and (1.16) to the appropriate equations in 3.16. This will produce the observability terms on the left and enable us to absorb most of the unwanted terms when combining the estimates for λ sufficiently large.

i) Applying the Carleman estimate in (1.14) to equation (3.16a) gives

$$\lambda \int_Q (\lambda^2 \tilde{z_1}^2 + \tilde{z_1}_t^2 + |\nabla \tilde{z_1}|^2) e^{2\lambda\varphi}\, dxdt \le C \left[\left\| e^{\lambda\varphi} \mathcal{P} \tilde{z_1} \right\|^2_{L^2(Q)} + \lambda^2 \int_0^T \int_\omega (\lambda^2 \tilde{z_1}^2 + \tilde{z_1}_t^2) e^{2\lambda\varphi}\, dxdt \right] \tag{3.17}$$

where

$$\left\| e^{\lambda\varphi} \mathcal{P} \tilde{z_1} \right\|^2_{L^2(Q)} = \int_Q e^{2\lambda\varphi} \left\{ \rho h (r'' z_1 + 2r' z_{1t}) + Kru_1 \right\}^2 dQ.$$

Notice, using $r' = r'' = 0$ on Q and $\varphi < -\gamma$ on $\tilde{Q}$ and by Lemma 3.3.2 we have

$$\int_Q e^{2\lambda\varphi} \rho h (r'' z_1 + 2r' z_{1t})\, dQ = \int_{\tilde{Q}} \rho (r'' z_1 + 2r' z_{1t})\, d\tilde{Q} \le C e^{-\gamma\lambda} E_{z_1}(0). \tag{3.18}$$

Hence, from (3.17) and (3.18) we derive the estimate

$$\lambda \int_Q (\lambda^2 \tilde{z_1}^2 + \tilde{z_1}_t^2 + |\nabla \tilde{z_1}|^2) e^{2\lambda\varphi}\, dxdt$$
$$\le C \left[e^{-\gamma\lambda} E_{z_1}(0) + \int_Q e^{2\lambda\varphi} (u_1^2) r^2\, dQ + \lambda^2 \int_0^T \int_\omega e^{2\lambda\varphi} (\lambda^2 z_1^2 + z_{1t}^2)\, dxdt \right]. \tag{3.19}$$

Focusing on the LHS of (3.19) we can derive the following lower estimate for the $\tilde{z_1}_t$ term:

$$\int_Q e^{2\lambda\varphi} |\tilde{z_1}_t|^2\, dQ = \int_Q e^{2\lambda\varphi} |r' z_1 + r z_{1t}|^2\, dQ$$
$$\ge \frac{1}{2} \int_Q e^{2\lambda\varphi} r^2 |z_{1t}|^2\, dQ - \int_{\tilde{Q}} e^{2\lambda\varphi} |r'|^2 |z_1|^2\, d\tilde{Q}. \tag{3.20}$$

Thus, applying (3.20) to (3.19) we can further derive

$$\lambda \int_Q r^2 (\lambda^2 z_1^2 + z_{1t}^2 + |\nabla z_1|^2) e^{2\lambda\varphi}\, dQ$$
$$\le C \left[e^{-\gamma\lambda} E_{z_1}(0) + \int_Q e^{2\lambda\varphi} u_1^2\, dQ + \lambda^2 \int_0^T \int_\omega e^{2\lambda\varphi} (\lambda^2 z_1^2 + z_{1t}^2)\, d\omega \right]. \tag{3.21}$$

ii) Applying the Carleman estimate in (1.16) to equation (3.16b) we have

$$\lambda \left\| e^{\lambda\varphi} \tilde{u_2} \right\|^2_{L^2(Q)} \le C \left(\left\| e^{\lambda\varphi} (\mathcal{P} \tilde{u_2}) \right\|^2_{H^{-1}(Q)} + \lambda^2 \left\| e^{\lambda\varphi} \tilde{u_2} \right\|^2_{L^2(0,T;L^2(\omega_0))} \right) \tag{3.22}$$

Notice

$$\begin{aligned}
\left\| e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' u_2 + 2r' u_{2t}) \right\} \right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1} \left\langle e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' u_2 + 2r' u_{2t}) \right\}, f \right\rangle \\
&\le C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{ \int_{\tilde{Q}} e^{\lambda\varphi} u_2 (-2r' f_t - r'' f - 2\lambda \varphi_t r' f) \, d\tilde{Q} \right\} \\
&\le C e^{-\gamma\lambda} \lambda \|u_2\|_{L^2(Q)} \\
&\le C e^{-\gamma\lambda} \lambda \tilde{E}_{u_2}(0).
\end{aligned}$$

Thus, (3.22) becomes

$$\lambda \left\| e^{\lambda\varphi} \tilde{u_2} \right\|^2_{L^2(Q)} \le C e^{-\gamma\lambda} \lambda \tilde{E}_{u_2}(0) + C \int_Q r^2 e^{\lambda\varphi} u_2^2 \, dQ + C\lambda^2 \left\| e^{\lambda\varphi} u_2 \right\|^2_{L^2(0,T;L^2(\omega_0))}.$$

Choosing λ large enough to absorb the unwanted term on the larger side results in

$$\lambda \int_Q r^2 e^{\lambda\varphi} u_2^2 \, dQ \le C e^{-\gamma\lambda} \lambda \tilde{E}_{u_2}(0) + C\lambda^2 \left\| e^{\lambda\varphi} u_2 \right\|^2_{L^2(0,T;L^2(\omega_0))}. \tag{3.23}$$

iii) Applying the Carleman estimate (1.16) to (3.16d) we have

$$\begin{aligned}
\lambda \left\| e^{\lambda} \tilde{z_2} \right\|^2_{L^2(Q)} &\le C \left(\left\| e^{\lambda\varphi} (\mathcal{P} \tilde{z_2}) \right\|^2_{H^{-1}(Q)} + \lambda^2 \left\| e^{\lambda\varphi} \tilde{z_2} \right\|^2_{L^2(0,T;L^2(\omega))} \right) \\
&\le C \left\| e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' z_2 + 2r' z_{2t}) - D \left(\frac{1+\mu}{2} \right) \tilde{u_{2y}} - K \left(\tilde{z_2} + \tilde{z_{1x}} \right) \right\} \right\|^2_{H^{-1}(Q)} \\
&+ C\lambda^2 \left\| e^{\lambda\varphi} z_2 \right\|^2_{L^2(0,T;L^2(\omega))}.
\end{aligned} \tag{3.24}$$

Examining the larger side yields

$$\begin{aligned}
\left\| e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' z_2 + 2r' z_{2t}) \right\} \right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1} \left\langle e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' z_2 + 2r' z_{2t}) \right\}, f \right\rangle \\
&\le C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{ \int_Q e^{\lambda\varphi} r'' z_2 f \, dQ + \int_Q e^{\lambda\varphi} 2r' z_{2t} f \, dQ \right\} \\
&= C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{ \int_{\tilde{Q}} e^{\lambda\varphi} r'' z_2 f \, d\tilde{Q} + \int_{\tilde{Q}} e^{\lambda\varphi} 2 z_2 (-r' f_t - r'' f - \lambda \varphi_t r' f) \, d\tilde{Q} \right\} \\
&\le C e^{-\gamma\lambda} (1+\lambda) \|z_2\|_{L^2(Q)} \\
&\le C e^{-\gamma\lambda} (1+\lambda) E_{z_2}(0)
\end{aligned}$$

and by Lemma 3.3.2. For the next term withing the $H^{-1}(Q)$ norm we have

$$\begin{aligned}\left\| e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\tilde{u_{2y}}\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1}\left\langle e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\tilde{u_{2y}}\right\}, f\right\rangle \\ &\leq C \sup_{\|f\|_{H_0^1(Q)}=1}\left\{\int_Q e^{\lambda\varphi} r u_2 (f_y + \lambda\varphi_y f)\, dQ\right\} \\ &\leq C(1+\lambda)\int_Q r^2 e^{2\lambda\varphi} {u_2}^2\, dQ.\end{aligned}$$

For the final term we have

$$\begin{aligned}\left\| e^{\lambda\varphi}\left\{-K(\tilde{z_2}+\tilde{z_{1x}})\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1}\left\langle e^{\lambda\varphi}\left\{-K(\tilde{z_2}+\tilde{z_{1x}})\right\}, f\right\rangle \\ &\leq C\int_Q r^2 e^{\lambda\varphi}(z_2^2 + |\nabla z_1|^2) dQ.\end{aligned}$$

Substituting these estimates for the larger side of (3.24) yields

$$\begin{aligned}\lambda \left\| e^{\lambda\varphi}\tilde{z_2}\right\|^2_{L^2(Q)} \leq & C e^{-\gamma\lambda}(1+\lambda)E_{z_2}(0) + C(1+\lambda)\int_Q r^2 e^{2\lambda\varphi}{u_2}^2\, dQ \\ & + C\int_Q r^2 e^{\lambda\varphi}(z_2^2+|\nabla z_1|^2)dQ + C\lambda^2\left\| e^{\lambda\varphi} z_2\right\|^2_{L^2(0,T;L^2(\omega))}. \qquad (3.25)\end{aligned}$$

iv) Applying the Carleman estimate (1.16) to (3.16e) we have

$$\begin{aligned}\lambda \left\| e^{\lambda\varphi}\tilde{z_3}\right\|^2_{L^2(Q)} &\leq C\left(\left\| e^{\lambda\varphi}(\mathcal{P}\tilde{z_3})\right\|^2_{H^{-1}(Q)} + \lambda^2\left\| e^{\lambda\varphi}\tilde{z_3}\right\|^2_{L^2(0,T;L^2(\omega))}\right) \\ &\leq C\left\| e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}(r'' z_3 + 2r' z_{3t}) - D\left(\frac{1+\mu}{2}\right)\tilde{u_{2x}} - K\left(\tilde{z_3}+\tilde{z_{1y}}\right)\right\}\right\|^2_{H^{-1}(Q)} \\ &\quad + C\lambda^2\left\| e^{\lambda\varphi} z_3\right\|^2_{L^2(0,T;L^2(\omega))}. \qquad (3.26)\end{aligned}$$

Examining the larger side yields

$$\begin{aligned}
\left\| e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}(r'' z_3 + 2r' z_{3t})\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1} \left\langle e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}(r'' z_3 + 2r' z_{3t})\right\}, f\right\rangle \\
&\leq C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{\int_Q e^{\lambda\varphi} r'' z_3 f \, dQ + \int_Q e^{\lambda\varphi} 2r' z_{3t} f \, dQ\right\} \\
&= C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{\int_{\tilde{Q}} e^{\lambda\varphi} r'' z_3 f \, d\tilde{Q} + \int_{\tilde{Q}} e^{\lambda\varphi} 2 z_3(-r' f_t - r'' f - \lambda\varphi_t r' f) \, d\tilde{Q}\right\} \\
&\leq C e^{-\gamma\lambda}(1+\lambda) \|z_3\|_{L^2(Q)} \\
&\leq C e^{-\gamma\lambda}(1+\lambda) E_{z_3}(0)
\end{aligned}$$

by Lemma 3.3.2. For the next term within the $H^{-1}(Q)$ norm we have

$$\begin{aligned}
\left\| e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\tilde{u_{2x}}\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1} \left\langle e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\tilde{u_{2x}}\right\}, f\right\rangle \\
&\leq C \sup_{\|f\|_{H_0^1(Q)}=1} \left\{\int_Q e^{\lambda\varphi} r u_2 (f_x + \lambda\varphi_x f) \, dQ\right\} \\
&\leq C(1+\lambda) \int_Q r^2 e^{2\lambda\varphi} {u_2}^2 \, dQ
\end{aligned}$$

Again, for the final term we have the estimate

$$\begin{aligned}
\left\| e^{\lambda\varphi}\left\{-K(\tilde{z_3} + \tilde{z_{1y}})\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1} \left\langle e^{\lambda\varphi}\left\{-K(\tilde{z_3} + \tilde{z_{1y}})\right\}, f\right\rangle \\
&\leq C \int_Q r^2 e^{2\lambda\varphi}(z_3^2 + |\nabla z_1|^2) dQ.
\end{aligned}$$

Substituting these estimates for the larger side of (3.24) yields

$$\begin{aligned}
\lambda \left\| e^{\lambda\varphi}\tilde{z_3}\right\|^2_{L^2(Q)} \leq & C e^{-\gamma\lambda}(1+\lambda) E_{z_3}(0) + C(1+\lambda)\int_Q r^2 e^{2\lambda\varphi} u_2^2 \, dQ \\
& + C\int_Q r^2 e^{2\lambda\varphi}(z_3^2 + |\nabla z_1|^2) dQ + C\lambda^2 \left\| e^{\lambda\varphi} z_3\right\|^2_{L^2(0,T;L^2(\omega))}.
\end{aligned} \tag{3.27}$$

v) Applying the Carleman estimate in (1.16) to equation (3.16c) we have

$$\lambda \left\| e^{\lambda\varphi} \tilde{u_1} \right\|^2_{L^2(Q)} \leq C \left(\left\| e^{\lambda\varphi} (\mathcal{P} \tilde{u_1}) \right\|^2_{H^{-1}(Q)} + \lambda^2 \left\| e^{\lambda\varphi} \tilde{u_1} \right\|^2_{L^2(0,T;L^2(\omega_0))} \right) \tag{3.28}$$

Notice

$$\begin{aligned}
\left\| e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' u_1 + 2r' u_{1t}) \right\} \right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H^1_0(Q)}=1} \left\langle e^{\lambda\varphi} \left\{ \frac{\rho h^3}{12} (r'' u_1 + 2r' u_{1t}) \right\}, f \right\rangle \\
&\leq C \sup_{\|f\|_{H^1_0(Q)}=1} \left\{ \int_{\tilde{Q}} e^{\lambda\varphi} u_1 (-2r' f_t - r'' f - 2\lambda \varphi_t r' f) \, d\tilde{Q} \right\} \\
&\leq C e^{-\gamma\lambda} \lambda \left\| u_1 \right\|_{L^2(Q)} \\
&\leq C e^{-\gamma\lambda} \lambda \tilde{E}_{u_1}(0).
\end{aligned}$$

and

$$\begin{aligned}
\left\| e^{\lambda\varphi} \{ \rho h r z_{1tt} \} \right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H^1_0(Q)}=1} \left\langle e^{\lambda\varphi} \rho h r z_{1tt}, f \right\rangle \\
&\leq C \sup_{\|f\|_{H^1_0(Q)}=1} \left\{ \int_Q e^{\lambda\varphi} z_{1t} (-r' f - r f_t) \, dQ + \int_Q e^{\lambda\varphi} z_{1t} (-\lambda \varphi_t r f) \, dQ \right\} \\
&\leq C \sup_{\|f\|_{H^1_0(Q)}=1} \left\{ \left\| e^{\lambda\varphi} z_{1t} \right\|_{L^2(Q)} + \int_{\tilde{Q}} e^{\lambda\varphi} z_1 (\lambda^2 \varphi_{tt} r f + \lambda \varphi_t r' f + \lambda_t r f_t) \, d\tilde{Q} \right\} \\
&\leq C \left[\left\| e^{\lambda\varphi} z_{1t} \right\|_{L^2(Q)} + (\lambda^2 + \lambda) \left\| e^{\lambda\varphi} z_1 \right\|_{L^2(Q)} \right]
\end{aligned}$$

Thus, (3.28) becomes

$$\begin{aligned}
\lambda \left\| e^{\lambda\varphi} \tilde{u_1} \right\|^2_{L^2(Q)} &\leq C e^{-\gamma\lambda} \lambda \tilde{E}_{u_1}(0) + C \left\| e^{\lambda\varphi} z_{1t} \right\|_{L^2(Q)} \\
&\quad + C(\lambda^2 + \lambda) \left\| e^{\lambda\varphi} z_1 \right\|_{L^2(Q)} + C\lambda^2 \left\| e^{\lambda\varphi} u_1 \right\|^2_{L^2(0,T;L^2(\omega_0))}.
\end{aligned} \tag{3.29}$$

Hence, combining (3.21), (3.23), (3.25), (3.27), and (3.29) (where we multiply (3.23) by λ^2), and for

λ sufficiently large to absorb unwanted terms we have

$$\lambda \int_Q r^2(\lambda^2 z_1^2 + {z_1}_t^2 + |\nabla z_1|^2) e^{2\lambda\varphi}\, dQ + \lambda \int_Q r^2(z_2^2 + z_3^2 + u_1^2 + \lambda^2 u_2^2) e^{2\lambda\varphi}\, dQ$$
$$\leq Ce^{-\gamma\lambda} E_{z_1}(0) + Ce^{-\gamma\lambda}\lambda[\tilde{E}_{u_1}(0) + \tilde{E}_{u_2}(0)] + Ce^{-\gamma\lambda}(1+\lambda)[E_{z_2}(0) + E_{z_3}(0)]$$
$$+ C\lambda^2 \int_0^T \int_\omega e^{2\lambda\varphi}(\lambda^2 z_1^2 + {z_1}_t^2 + z_2^2 + z_3^2)\, d\omega dt + C\lambda^2 \int_0^T r^2 \int_{\omega_0} e^{2\lambda\varphi}(u_1^2 + \lambda^2 u_2^2)\, d\omega_0 dt.$$

Finally, using the energy estimate in (3.5) produces the inequality shown in (3.14).

Step 3 claim:

We will show

$$\int_{Q_0} \sum_i {z_i}_t^2 + z_2^2 + z_3^2 + |\nabla z_1|^2 + \sum_j |u_j|^2\, dQ_0 dt$$
$$\leq Ce^{-c_1\lambda}\mathbb{E}(0) + Ce^{C\lambda} \int_0^T r^2 \int_{\omega_0} (|\nabla z_2|^2 + |\nabla z_3|^2 + u_2^2 + u_1^2)\, d\omega_0 dt$$
$$+ Ce^{C\lambda} \int_0^T \int_\omega (\sum_i {z_i^2}_t + \sum_i z_i^2)\, d\omega dt. \tag{3.30}$$

where c_1 is a positive constant.

Proof. For this step it suffices to show an estimate for $\int_{Q_0}(z_{2t}^2 + z_{3t}^2)\, dQ_0$. Consider the function r from the previous step and let us introduce the function ξ satisfying

$$\xi \in C_0^\infty(\bar{\Omega}), \qquad 0 \leq \xi \leq 1, \qquad \xi = 1 \text{ in } \Omega\backslash\omega.$$

Here let us define the region $Q_\omega \triangleq [0,T] \times \omega$. If we differentiate the equations (3.3d) and (3.3e) in time, then we have that $\hat{z}_{2t} = rz_{2t}\xi$ and $\hat{z}_{3t} = rz_{3t}\xi$ satisfy the following equations

$$\begin{cases} \dfrac{\rho h^3}{12}\hat{z}_{2ttt} - D\Delta\hat{z}_{2t} + D\left(\dfrac{1+\mu}{2}\right)\hat{u}_{2ty} + K\left(\hat{z}_{2t} + \hat{z}_{1tx}\right) = \\ \qquad \dfrac{\rho h^3}{12}\xi(r'' z_{2t} + 2r' z_{2tt}) - Kr(\Delta\xi z_2 + 2\nabla\xi\cdot\nabla z_2) + D\left(\dfrac{1+\mu}{2}\right) r\xi_y u_2 + Kr\xi_x z_1, \quad \text{in } Q \quad (3.31a) \\ \dfrac{\rho h^3}{12}\hat{z}_{3ttt} - D\Delta\hat{z}_{3t} - D\left(\dfrac{1+\mu}{2}\right)\hat{u}_{2tx} + K\left(\hat{z}_{3t} + \hat{z}_{1ty}\right) = \\ \qquad \dfrac{\rho h^3}{12}\xi(r'' z_{3t} + 2r' z_{3tt}) - Kr(\Delta\xi z_3 + 2\nabla\xi\cdot\nabla z_3) + D\left(\dfrac{1+\mu}{2}\right) r\xi_x u_2 + Kr\xi_y z_1, \quad \text{in } Q\ . \quad (3.31b) \end{cases}$$

Applying the Carleman estimate (1.16) to (3.31b) we have

$$
\begin{aligned}
\lambda \left\|e^{\lambda\varphi}\hat{z}_{2t}\right\|^2_{L^2(Q)} &\le C\left(\left\|e^{\lambda\varphi}(\mathcal{P}\hat{z}_{2t})\right\|^2_{H^{-1}(Q)} + \lambda^2 \left\|e^{\lambda\varphi}\hat{z}_{2t}\right\|^2_{L^2(0,T;L^2(\omega))}\right) \\
&\le C\left\|e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}\xi(r''z_{2t}+2r'z_{2tt}) - D\left(\frac{1+\mu}{2}\right)\hat{u}_{2ty} - K\left(\hat{z}_{2t}+\hat{z}_{1tx}\right)\right\}\right\|^2_{H^{-1}(Q)} \\
&\quad + C\left\|-Kr(\Delta\xi z_2 + 2\nabla\xi\cdot\nabla z_2) + D\left(\frac{1+\mu}{2}\right)r\xi_y u_2 + Kr\xi_x z_1\right\|^2_{H^{-1}(Q)} \\
&\quad + C\lambda^2\left\|e^{\lambda\varphi}z_{2t}\right\|^2_{L^2(0,T;L^2(\omega_0))}. \qquad (3.32)
\end{aligned}
$$

Providing an estimate for each term in the negative Sobolev norm we have

$$
\begin{aligned}
\left\|e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}\xi(r''z_{2t}+2r'z_{2tt})\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H^1_0(Q)}=1}\left\langle e^{\lambda\varphi}\left\{\frac{\rho h^3}{12}\xi(r''z_{2t}+2r'z_{2tt})\right\}, f\right\rangle \\
&\le C\sup_{\|f\|_{H^1_0(Q)}=1}\left\{\int_Q e^{\lambda\varphi}r''z_{2t}f\,dQ + \int_Q e^{\lambda\varphi}2r'z_{2tt}f\,dQ\right\} \\
&= C\sup_{\|f\|_{H^1_0(Q)}=1}\left\{\int_{\hat{Q}} e^{\lambda\varphi}r''z_{2t}f\,d\hat{Q} + \int_{\hat{Q}} e^{\lambda\varphi}2z_{2t}(-r'f_t - r''f - \lambda\varphi_t r'f)\,d\hat{Q}\right\} \\
&\le Ce^{-\gamma\lambda}(1+\lambda)\left\|z_{2t}\right\|_{L^2(Q)} \\
&\le Ce^{-\gamma\lambda}(1+\lambda)E_{z_2}(0)
\end{aligned}
$$

by Lemma 3.3.2. The next term can be bounded as follows,

$$
\begin{aligned}
\left\|e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\hat{u}_{2ty}\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H^1_0(Q)}=1}\left\langle e^{\lambda\varphi}\left\{-D\left(\frac{1+\mu}{2}\right)\hat{u}_{2ty}\right\}, f\right\rangle \\
&\le C\sup_{\|f\|_{H^1_0(Q)}=1}\left\{\int_Q e^{\lambda\varphi}ru_{2t}(f_y\xi + \lambda\varphi_y f\xi + f\xi_y)\,dQ\right\} \\
&\le C\sup_{\|f\|_{H^1_0(Q)}=1}\left\{\int_Q e^{\lambda\varphi}r'u_2(-f_y\xi - \lambda\varphi_y f\xi - f\xi_y)\,dQ\right. \\
&\quad + C\int_Q e^{\lambda\varphi}\lambda\varphi_t ru_2(-f_y\xi - \lambda\varphi_y f\xi - f\xi_y)\,dQ \\
&\quad \left. + C\int_Q e^{\lambda\varphi}ru_2(-\lambda\varphi_y f_t\xi - f_t\xi_y)\,dQ + \left\langle -e^{\lambda\varphi}ru_2 f_{yt}, r\xi\right\rangle\right\} \\
&\le Ce^{-\gamma\lambda}\left\|u_2\right\|_{L^2(Q)} + C(1+\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}u_2^2\,dQ.
\end{aligned}
$$

Again, for the next term (where ξ is absorbed into the constant immediately) we have

$$\begin{aligned}\left\|e^{\lambda\varphi}\left\{-K(\hat{z_{2t}}+\hat{z_{1tx}})\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1}\left\langle e^{\lambda\varphi}\left\{-K(\hat{z_{2t}}+\hat{z_{1tx}})\right\}, f\right\rangle \\ &\le C\sup_{\|f\|_{H_0^1(Q)}=1}\left\{\int_Q e^{\lambda\varphi}(rz_{2t}f + z_{1x}(-rf_t - r'f - \lambda\varphi_t rf))\,dQ\right\} \\ &\le C\int_Q r^2 e^{2\lambda\varphi}(z_{2t}^2+|\nabla z_1|^2)dQ + C\int_Q e^{\lambda\varphi}z_1(\lambda\varphi_t rf_x + \lambda^2\varphi_t\varphi_x rf)\,dQ. \\ &\le C\int_Q r^2e^{2\lambda\varphi}(z_{2t}^2+|\nabla z_1|^2)\,dQ + C(\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}z_1^2\,dQ.\end{aligned}$$

The final two terms are thus bounded by

$$\begin{aligned}\left\|e^{\lambda\varphi}\left\{-Kr(\Delta\xi z_2 + 2\nabla\xi\cdot\nabla z_2)\right\}\right\|^2_{H^{-1}(Q)} &= \sup_{\|f\|_{H_0^1(Q)}=1}\left\langle e^{\lambda\varphi}\left\{-Kr(\Delta\xi z_2+2\nabla\xi\cdot\nabla z_2)\right\}, f\right\rangle \\ &\le C\int_Q e^{2\lambda\varphi}z_2^2\,dQ + C\int_0^T\int_{\omega_0} e^{2\lambda\varphi}r^2|\nabla z_2|^2\,d\omega_0 dt\end{aligned}$$

and

$$\begin{aligned}\left\|e^{\lambda\varphi}\left\{D\left(\frac{1+\mu}{2}\right)r\xi_y u_2 + Kr\xi_x z_1\right\}\right\|^2_{H^{-1}(Q)} &\le C\sup_{\|f\|_{H_0^1(Q)}=1}\left\langle e^{\lambda\varphi}\left\{r\xi_y u_2 + Kr\xi_x z_1\right\}, f\right\rangle \\ &\le C\int_Q r^2 e^{2\lambda\varphi}(u_2^2+z_1^2)dQ.\end{aligned}$$

Substituting these estimates for the larger side of (3.32) yields

$$\begin{aligned}\lambda\left\|e^{\lambda\varphi}\hat{z_{2t}}\right\|^2_{L^2(Q)} \le & Ce^{-\gamma\lambda}(1+\lambda)E_{z_2}(0) + Ce^{-\gamma\lambda}\|u_2\|_{L^2(Q)} + C(1+\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}u_2^2 dQ \\ &+ C\int_Q r^2e^{2\lambda\varphi}(z_2^2+z_{2t}^2+|\nabla z_1|^2)dQ + C(\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}z_1^2\,dQ \\ &+ C\int_0^T\int_{\omega_0} e^{2\lambda\varphi}r^2|\nabla z_2|^2\,d\omega_0 dt + C\lambda^2\left\|e^{2\lambda\varphi}z_{2t}\right\|^2_{L^2(0,T;L^2(\omega))}.\end{aligned}$$

Notice further for the lower side we have

$$\begin{aligned}\int_Q e^{2\lambda\varphi}|\hat{z_{2t}}|^2\,dQ &\ge \int_{Q\setminus Q_\omega} e^{2\lambda\varphi}|r'z_2 + rz_{2t}|^2\,dQ \\ &\ge \frac{1}{2}\int_{Q\setminus Q_\omega} e^{2\lambda\varphi}r^2|z_{2t}|^2\,dQ - \int_{(Q\setminus Q_\omega)\cap\tilde{Q}} e^{2\lambda\varphi}|r'|^2|z_2|^2\,d\tilde{Q}.\end{aligned}$$

This gives

$$\begin{aligned}\lambda \left\|e^{\lambda\varphi} r z_{2t}\right\|^2_{L^2(Q\setminus Q_\omega)} \leq & Ce^{-\gamma\lambda}(1+\lambda)E_{z_2}(0) + Ce^{-\gamma\lambda}\left\|u_2\right\|_{L^2(Q)} + C(1+\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}u_2^2 dQ \\ & + C\int_Q r^2 e^{2\lambda\varphi}(z_2^2 + z_{2\,t}^2 + |\nabla z_1|^2)dQ + C(\lambda+\lambda^2)\int_Q e^{2\lambda\varphi} z_1^2\, dQ \\ & + C\int_0^T \int_{\omega_0} e^{2\lambda\varphi} r^2 |\nabla z_2|^2\, d\omega_0 dt + C\lambda^2 \left\|e^{2\lambda\varphi} z_{2t}\right\|^2_{L^2(0,T;L^2(\omega))}. \end{aligned} \tag{3.33}$$

In a similar way, from applying (1.16) to (3.31a), we derive

$$\begin{aligned}\lambda \left\|e^{\lambda\varphi} r z_{3t}\right\|^2_{L^2(Q\setminus Q_\omega)} \leq & Ce^{-\gamma\lambda}(1+\lambda)E_{z_3}(0) + Ce^{-\gamma\lambda}\left\|u_2\right\|_{L^2(Q)} + C(1+\lambda+\lambda^2)\int_Q e^{2\lambda\varphi}u_2^2 dQ \\ & + C\int_Q r^2 e^{2\lambda\varphi}(z_3^2 + z_{3\,t}^2 + |\nabla z_1|^2)dQ + C(\lambda+\lambda^2)\int_Q e^{2\lambda\varphi} z_1^2\, dQ \\ & + C\int_0^T \int_{\omega_0} e^{2\lambda\varphi} r^2 |\nabla z_3|^2\, d\omega_0 dt + C\lambda^2 \left\|e^{2\lambda\varphi} z_{3t}\right\|^2_{L^2(0,T;L^2(\omega))}. \end{aligned} \tag{3.34}$$

Using the fact that $Q = (Q\setminus Q_\omega) \cup Q_\omega$, we can combine the estimates (3.33) and (3.34) with (3.14), and, choosing λ large enough, can absorb the remaining terms. It is only left to handle the exponential weight. For the larger side this follows immediately by definition of the pseudo-convex function. For the smaller side, recall T_0, T_0' were arbitrarily defined. Here we set them to be symmetrically around $\frac{T}{2}$ such that (1.8) holds. Thus, Q_0 can be defined to be the region $Q(\sigma)$ in (1.9) and we have a lower bound on the exponential weight which finally yields (3.30). □

Step 4:

In the final step, we shall eliminate the terms involving u_1 and u_2 on the right hand side of (3.14) as well as absorb the first order spatial terms for z_2 and z_3. By construction we have

$$u_1^2 + u_2^2 \leq C(|\nabla z_2|^2 + |\nabla z_3|^2). \tag{3.35}$$

As such we can provide an estimate for the two gradient terms for z_2 and z_2, which will in turn necessitate an estimate for the gradient term of z_1.

First, consider the same region ω_0 where we now define this region as another neighborhood of Γ such that $\omega_0 \subset\subset \omega$. Then let us introduce the function ζ satisfying

$$\zeta \in C^\infty(\bar{\Omega}), \quad 0 \leq \zeta \leq 1, \quad \zeta = 1 \text{ in } \omega_0, \quad \zeta = 0 \text{ in } \Omega\setminus\omega. \tag{3.36}$$

Define the following functions

$$\check{z_1} = r\zeta z_1, \qquad \check{z_2} = r\zeta z_2, \qquad \check{z_2} = r\zeta z_2, \qquad \check{z_3} = r\zeta z_3, \qquad \check{u_1} = r\zeta u_1, \qquad \check{u_2} = r\zeta u_2.$$

The functions $\check{z_1}, \check{z_2}, \check{z_3}$ satisfy homogeneous boundary, initial and final conditions together with the three equations

$$\begin{cases} \rho h \check{z}_{1tt} - K\Delta \check{z_1} - K\check{u_1} = \rho h(2r' z_{1t} + r'' z_1)\zeta - Kr(\Delta\zeta z_1 + 2\nabla\zeta\cdot\nabla z_1), & \text{in } Q \quad (3.37a) \\ \dfrac{\rho h^3}{12}\check{z}_{2tt} - D\Delta\check{z_2} + D\left(\dfrac{1+\mu}{2}\right)\check{u}_{2y} + K\left(\check{z_2} + \check{z}_{1x}\right) = & \\ \quad \dfrac{\rho h^3}{12}(2r' z_{2t} + r'' z_2)\zeta - Kr(\Delta\zeta z_2 + 2\nabla\zeta\cdot\nabla z_2) + D\left(\dfrac{1+\mu}{2}\right) r\zeta_y u_2 + Kr\zeta_x z_1, & \text{in } Q \quad (3.37b) \\ \dfrac{\rho h^3}{12}\check{z}_{3tt} - D\Delta\check{z_3} - D\left(\dfrac{1+\mu}{2}\right)\check{u}_{2x} + K\left(\check{z_3} + \check{z}_{1y}\right) = & \\ \quad \dfrac{\rho h^3}{12}(2r' z_{3t} + r'' z_3)\zeta - Kr(\Delta\zeta z_3 + 2\nabla\zeta\cdot\nabla z_3) + D\left(\dfrac{1+\mu}{2}\right) r\zeta_x u_2 + Kr\zeta_y z_1 & \text{in } Q\ . \quad (3.37c) \end{cases}$$

To absorb the gradient term of z_2 we multiply (3.37b) by $\check{z_2}$ and integrate over Q where, using Green's Theorem, integration by parts we arrive at the equation

$$\int_Q D|\nabla\check{z_2}|^2\, dQ = \int_Q \frac{\rho h^3}{12}\check{z}_{2t}^2 + D\left(\frac{1-\mu}{2}\right)\check{u_2}\check{z}_{2y} - K(\check{z_2} + \check{z}_{1x})\check{z_2}\, dQ + \frac{\rho h^3}{12}\int_Q (2r' z_{2t} + r'' z_2)\zeta\check{z_2}\, dQ$$
$$- Kr\int_Q (\Delta\zeta z_2 + 2\nabla\zeta\cdot\nabla z_2)\check{z_2}\, dQ + \int_Q \left[D\left(\frac{1+\mu}{2}\right) r\zeta_y u_2 + Kr\zeta_x z_1\right]\check{z_2}\, dQ \qquad (3.38)$$

We then provide an upper estimate for each term on the RHS of (3.38) using Hölder and Cauchy Schwarz inequalities as follows

$$\left|\int_Q \rho h\check{z}_{2t}^2 - K(\check{z_2} + \check{z}_{1x})\check{z_2}\, dQ\right| \le C\int_0^T\int_\omega z_2^2 + z_{2t}^2 + r^2|\nabla z_1|\, d\omega dt$$

$$\left|D\left(\frac{1-\mu}{2}\right)\check{u_2}\check{z}_{2y}\right| \le D\left(\frac{1-\mu}{4}\right)\int_Q u_2^2\, dQ + D\left(\frac{1-\mu}{4}\right)\int_Q |\nabla z_2|^2\, dQ$$

$$\left|\frac{\rho h^3}{12}\int_Q (2r' z_{2t} + r'' z_2)\zeta\check{z_2}\, dQ\right| \le C\int_0^T\int_\omega (z_2^2 + z_{2t}^2)\, d\omega dt$$

$$\left|Kr\int_Q (\Delta\zeta z_2 + 2\nabla\zeta\cdot\nabla z_2)\check{z_2}\, dQ\right| \le C\left\|\nabla z_2\right\|_{L^2(Q)}\left(\int_0^T\int_\omega z_2^2\, d\omega dt\right)^{\frac{1}{2}} + C\int_0^T\int_\omega z_2^2\, d\omega dt$$

$$\left|\int_Q \left[D\left(\frac{1+\mu}{2}\right) r\zeta_y u_2 + Kr\zeta_x z_1\right]\check{z_2}\, dQ\right| \le C\left\|u_2\right\|_{L^2(Q)}\left(\int_0^T\int_\omega z_2^2\, d\omega dt\right)^{\frac{1}{2}} + C\int_0^T\int_\omega z_1^2 + z_2^2\, d\omega dt.$$

Notice, by assumptions on μ for physical applications we have $D - D(1-\mu)/2 > D/2 > 0$. We can thus absorb the gradient term on the larger side of the second estimate into the smaller side of (3.38). Reporting these estimates to the RHS of (3.38) and recalling the energy estimate in (3.5) gives

$$\int_\omega |\nabla \check{z}_2|^2 \, d\omega \leq C \int_0^T \int_\omega z_1^2 + z_2^2 + z_{2\,t}^2 + r^2 |\nabla z_1| \, d\omega dt + C\mathbb{E}(0)^{\frac{1}{2}} \left(\int_0^T \int_\omega z_2^2 \, d\omega dt \right) + \frac{c_2}{2} \|u_2\|_{L^2(Q)} \tag{3.39}$$

where we define $c_2 = D(1-\mu)/2$ and thus $0 < c_2 < 1$ due to physical assumptions on the MT system's coefficients. Notice for all $\delta_1 > 0$ we have by the Cauchy inequality

$$C\mathbb{E}(0)^{\frac{1}{2}} \left(\int_0^T \int_\omega z_2^2 \, d\omega dt \right)^{\frac{1}{2}} \leq C_{\delta_1} \int_0^T \int_\omega z_2^2 \, d\omega dt + \delta_1 \mathbb{E}(0)$$

Rewriting (3.39) with the above estimates and recalling properties of ζ we then arrive at

$$\int_0^T \int_{\omega_0} r^2 |\nabla z_2|^2 \, d\omega dt \leq C_{\delta_1} \int_0^T \int_\omega (z_1^2 + z_2^2 + z_{2\,t}^2 + r^2 |\nabla z_1|^2) \, d\omega dt + \delta_1 \mathbb{E}(0) + \frac{c_2}{2} \|u_2\|_{L^2(Q)} \tag{3.40}$$

for any $\delta_1 > 0$. In a similar manner we arrive at the following estimate for the gradient term of z_3

$$\int_0^T \int_{\omega_0} r^2 |\nabla z_3|^2 \, d\omega dt \leq C_{\delta_2} \int_0^T \int_\omega (z_1^2 + z_3^2 + z_{3\,t}^2 + r^2 |\nabla z_1|^2) \, d\omega dt + \delta_2 \mathbb{E}(0) + \frac{c_2}{2} \|u_2\|_{L^2(Q)} \tag{3.41}$$

for any $\delta_2 > 0$.

It remains to absorb the term with the gradient of z_1. Following the same basic process, multiply (3.37a) by $\check{z}_1$ and integrate over Q, which yields

$$\begin{aligned} \int_Q K |\nabla \check{z}_1|^2 \, dQ = & \int_Q \rho h \check{z}_{1\,t}^{\,2} - K \check{u}_1 \check{z}_1 \, dQ + \rho h \int_Q (2r' z_{1\,t} + r'' z_1) \zeta \check{z}_1 \, dQ \\ & - Kr \int_Q (\Delta \zeta z_1 + 2 \nabla \zeta \cdot \nabla z_1) \check{z}_1 \, dQ \end{aligned} \tag{3.42}$$

Thus, going term by term on the RHS of (3.42) and applying Hölder's inequality we have

$$\left|\int_Q \rho h \check{z}_{1t}^{\,2} - K\check{u}_1\check{z}_1\, dQ\right| \leq C\,\|u_1\|_{L^2(Q)}\left(\int_0^T\int_\omega z_1^2\, d\omega dt\right)^{\frac{1}{2}} + C\int_0^T\int_\omega z_{1\,t}^2\, d\omega dt$$
$$\left|\rho h\int_Q (2r' z_{1\,t} + r'' z_1)\zeta\check{z}_1\, dQ\right| \leq C\int_0^T\int_\omega (z_1^2 + z_{1\,t}^2)\, d\omega dt$$
$$\left|Kr\int_Q (\Delta\zeta z_1 + 2\nabla\zeta\cdot\nabla z_1)\check{z}_1\, dQ\right| \leq C\,\|\nabla z_1\|_{L^2(Q)}\left(\int_0^T\int_\omega z_1^2\, d\omega dt\right)^{\frac{1}{2}} + C\int_0^T\int_\omega z_1^2\, d\omega dt.$$

Combining all of these estimates and applying them to the RHS of (3.42) while recalling the energy estimate (3.5), we have

$$\int_0^T\int_\omega Kr^2|\nabla(\zeta z_1)|^2\, d\omega dt \leq C\int_0^T\int_\omega (z_1^2 + z_{1\,t}^2)\, d\omega dt + C\mathbb{E}(0)^{\frac{1}{2}}\left(\int_0^T\int_\omega z_1^2\, d\omega dt\right)^{\frac{1}{2}}.$$

Then from Young's inequality, and recalling $K > 0$ and also (3.36) we have

$$\int_0^T\int_{\omega_0} r^2|\nabla z_1|^2\, d\omega dt \leq C_\epsilon\int_0^T\int_\omega (z_1^2 + z_{1\,t}^2)\, d\omega dt + \epsilon\mathbb{E}(0) \tag{3.43}$$

holds for all $\epsilon > 0$. Finally, combining (3.6), (3.14), (3.40), (3.41), and (3.43) we have

$$\mathbb{E}(0) \leq C_{\epsilon,\delta_1,\delta_3}\int_0^T\int_\omega \sum_i z_i^2 + \sum_i z_{i\,t}^2\, d\omega dt + (Ce^{-\gamma\lambda} + C_{\delta_1}\epsilon + C_{\delta_2}\epsilon + \delta_1 + \delta_2 + c_2)\mathbb{E}(0) \tag{3.44}$$

where for λ large enough and δ_1, δ_2 fixed small enough, we then can choose $\epsilon > 0$ small enough to absorb the initial energy term into the smaller side to yield

$$\mathbb{E}(0) \leq C\int_0^T\int_\omega \sum_i z_i^2 + \sum_i z_{i\,t}^2\, d\omega dt \tag{3.45}$$

which is the desired result for Theorem 3.3.1. □

Chapter 4

An Inverse Problem for the MT System: Recovering Density

4.1 Introduction and Problem Formulation

This chapter considers an inverse problem of recovering the plate density for the Mindlin–Timoshenko (MT) system (2.3) and includes much of what can be seen in [20]. The MT model assumed for this chapter is as presented in Chapter 2. Such a system can be seen as a two dimensional extension of the Timoshenko beam [1]. It refines the classical Kirchhoff–Love model by taking into account shear deformations and thus relaxing the assumption that the filaments of the plate must remain perpendicular to its mid-plane. Such description is substantially more accurate at high frequencies and when describing thicker plates, and therefore has attracted a lot of research attention. However, to the best of our knowledge, there has not been any work concerning inverse problems for the MT system. In this thesis we provide such an attempt and prove that under appropriate assumptions one can recover the plate density of the MT system from a single boundary measurement of the solution.

Despite less focus on inverse problems for the MT system, there has been ample work on many different aspects of the model. For examples, [21] establishes the mathematical model and the stability theory, [32] studies the well-posedness and regularity, [33, 34] consider the semilinear MT system focusing on the interaction of nonlinear sources and damping terms, and [42] achieves the indirect stabilization of the MT system.

4.1.1 Inverse source problem. Diagonalization

Let us now formulate the inverse problem considered in this thesis. Without loss of generality, we will normalize the constant (known) parameters in (2.3) and set $h = K = D = 1$. In addition, for computational ease we use the substitution $a = \frac{1-\mu}{2}$. Thus we arrive at the following system

$$
\begin{cases}
\rho w_{tt} - \Delta w - (\psi_x + \phi_y) = 0 & \Omega \times [0,T] \quad (4.1a)\\
\dfrac{\rho}{12}\psi_{tt} - (\psi_{xx} + a\psi_{yy}) - (1-a)\phi_{xy} + (\psi + w_x) = 0 & \Omega \times [0,T] \quad (4.1b)\\
\dfrac{\rho}{12}\phi_{tt} - (a\phi_{xx} + \phi_{yy}) - (1-a)\psi_{xy} + (\phi + w_y) = 0 & \Omega \times [0,T] \quad (4.1c)
\end{cases}
$$

with the initial and boundary conditions

$$
\begin{cases}
(w(0,x,y), \psi(0,x,y), \phi(0,x,y)) = (w^0, \psi^0, \psi^0)\\
(w_t(0,x,y), \psi_t(0,x,y), \phi_t(0,x,y)) = (w^1, \psi^1, \psi^1)\\
w = f_1,\ \psi = f_2,\ \phi = f_3 \quad \text{on } [0,T] \times \Gamma.
\end{cases}
$$

Our goal is to recover the space dependent plate density $\rho = \rho(x,y)$, which is assumed to satisfy the following

$$\rho \in H^1(\Omega) \text{ and } \rho \geq \rho_0 \text{ for some constant } \rho_0 > 0 \tag{4.2}$$

from one single boundary measurement $(g_1, g_2, g_3)\,|_{[0,T]\times\Gamma_1}$ where

$$
\begin{cases}
K\left(\frac{\partial w}{\partial \nu} + \nu_1\psi + \nu_2\phi\right) = g_1 & [0,T]\times\Gamma_1\\
D\left[\nu_1\frac{\partial \psi}{\partial x} + \mu\nu_1\frac{\partial \phi}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_2\right] = g_2 & [0,T]\times\Gamma_1\\
D\left[\nu_2\frac{\partial \phi}{\partial y} + \mu\nu_2\frac{\partial \psi}{\partial x} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_1\right] = g_3 & [0,T]\times\Gamma_1
\end{cases}
$$

and where Γ_1 is an appropriate portion of the boundary Γ.

As usual the first step to solve the inverse problem is to linearize it and convert it into a corresponding inverse source problem. More precisely, let $(\bar{w}, \bar{\psi}, \bar{\phi})(\rho)$ and $(\tilde{w}, \tilde{\psi}, \tilde{\phi})(\tilde{\rho})$ be solutions of the MT system (4.1) corresponding with two different density functions ρ and $\tilde{\rho}$. Then by setting

$$w = \bar{w} - \tilde{w},\ \psi = \bar{\psi} - \tilde{\psi},\ \phi = \bar{\phi} - \tilde{\phi} \tag{4.3}$$

and for simplicity

$$f(x,y) = \tilde{\rho}(x,y) - \rho(x,y),\ \tilde{w}_{tt} = R_1(x,y,t),\ \tilde{\psi}_{tt} = R_2(x,y,t),\ \tilde{\phi}_{tt} = R_3(x,y,t) \tag{4.4}$$

we have $\{w, \psi, \phi\}$ solves the following system

$$\begin{cases} \rho w_{tt} - \Delta w - (\psi_x + \phi_y) = fR_1 & \Omega \times [0,T] \qquad (4.5a) \\ \dfrac{\rho}{12}\psi_{tt} - (\psi_{xx} + a\psi_{yy}) - (1-a)\phi_{xy} + (\psi + w_x) = \dfrac{1}{12} fR_2 & \Omega \times [0,T] \qquad (4.5b) \\ \dfrac{\rho}{12}\phi_{tt} - (a\phi_{xx} + \phi_{yy}) - (1-a)\psi_{xy} + (\phi + w_y) = \dfrac{1}{12} fR_3 & \Omega \times [0,T] \qquad (4.5c) \end{cases}$$

with the *homogeneous* initial and boundary conditions

$$(w,\psi,\phi)|_{t=0} = (w_t, \psi_t, \phi_t)|_{t=0} = (0,0,0), \qquad (w,\psi,\phi)|_{\Gamma\times[0,T]} = (0,0,0). \tag{4.6}$$

Then the corresponding inverse source problem for the system (4.5) is to show that the unknown function $f = f(x,y)$ equals to zero from the *homogeneous* boundary measurement

$$\begin{cases} K\left(\frac{\partial w}{\partial \nu} + \nu_1 \psi + \nu_2 \phi\right) = 0 & [0,T]\times\Gamma_1 \\ D\left[\nu_1 \frac{\partial \psi}{\partial x} + \mu\nu_1 \frac{\partial \phi}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_2\right] = 0 & [0,T]\times\Gamma_1 \\ D\left[\nu_2 \frac{\partial \phi}{\partial y} + \mu\nu_2 \frac{\partial \psi}{\partial x} + \frac{1-\mu}{2}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\nu_1\right] = 0 & [0,T]\times\Gamma_1, \end{cases} \tag{4.7}$$

assuming the other parameters ρ, $R_i(x,y,t)$, $i = 1,2,3$ are given suitable functions.

To solve the inverse source problem above, we need to have an appropriate Carleman estimate for the system (4.5). As the system is strongly coupled at the principle level, we will first perform a diagonalization process to diagonalize the principle part of the system and make it possible to achieve a desired estimate through the standard Carleman estimate for a single second-order hyperbolic equation. To begin, consider functions $\alpha(x,y,t)$ and $\beta(x,y,t)$ defined as

$$\alpha = \psi_x + \phi_y - \frac{12}{11}w \qquad\qquad \beta = \phi_x - \psi_y. \tag{4.8}$$

In particular notice this then gives

$$\Delta\psi = \alpha_x - \beta_y + \frac{12}{11}w_x, \qquad \Delta\phi = \alpha_y + \beta_x + \frac{12}{11}w_y. \tag{4.9}$$

We can then rewrite (4.5b) and (4.5c) in the following manner:

$$\frac{\rho}{12}\psi_{tt} - (\psi_{xx} + \psi_{yy}) + (1-a)\psi_{yy} - (1-a)\phi_{xy} + (\psi + w_x) = \frac{1}{12}fR_2$$
$$\frac{\rho}{12}\phi_{tt} - (\phi_{xx} + \phi_{yy}) + (1-a)\phi_{xx} - (1-a)\psi_{xy} + (\phi + w_y) = \frac{1}{12}fR_3.$$

Applying the substitutions from (4.8) to these equations yields

$$\frac{\rho}{12}\psi_{tt} - \Delta\psi - (1-a)\beta_y + (\psi + w_x) = \frac{1}{12}fR_2 \tag{4.10}$$
$$\frac{\rho}{12}\phi_{tt} - \Delta\phi + (1-a)\beta_x + (\phi + w_y) = \frac{1}{12}fR_3. \tag{4.11}$$

To complete the diagonalized model we now focus on deriving equations for the functions α and β. For this we first use the following implications from (4.10) and (4.11), as well as (4.9) to get

$$\begin{aligned}\psi_{tt} &= \frac{12}{\rho}\left(\Delta\psi - (1-a)\beta_y - \psi - w_x + \frac{1}{12}fR_2\right)\\ &= \frac{12}{\rho}\left(\alpha_x - a\beta_y + \frac{1}{11}w_x - \psi + \frac{f}{12}R_2\right)\end{aligned} \tag{4.12}$$
$$\begin{aligned}\phi_{tt} &= \frac{12}{\rho}\left(\Delta\phi - (1-a)\beta_x - \phi - w_y + \frac{1}{12}fR_3\right)\\ &= \frac{12}{\rho}\left(\alpha_y + a\beta_x + \frac{1}{11}w_y - \phi + \frac{f}{12}R_3\right).\end{aligned} \tag{4.13}$$

Then to find an equation for the function α we differentiate (4.5b) with respect to x and (4.5c) with respect to y. Taking their sum, and applying the substitution in (4.8), produces

$$\frac{\rho}{12}\alpha_{tt} - \Delta\alpha + \frac{\rho}{11}w_{tt} - \frac{1}{11}\Delta w + (\psi_x + \phi_y) + \frac{\rho_x}{12}\psi_{tt} + \frac{\rho_y}{12}\phi_{tt} = \frac{f_x}{12}R_2 + \frac{f}{12}R_{2x} + \frac{f_y}{12}R_3 + \frac{f}{12}R_{3y}.$$

Recalling (4.5a) we have the equivalence $\frac{\rho}{11}w_{tt} - \frac{1}{11}\Delta w = \frac{1}{11}(\psi_x + \phi_y) + \frac{1}{11}fR_1$, which we can use along with (4.12) and (4.13) to remove the second order terms resulting in

$$\frac{\rho}{12}\alpha_{tt} - \Delta\alpha + \frac{\rho_x}{\rho}\alpha_x + \frac{\rho_y}{\rho}\alpha_y + \frac{12}{11}(\psi_x + \phi_y) + \frac{\rho_x}{\rho}\left(\frac{1}{11}w_x - a\beta_y - \psi\right) + \frac{\rho_y}{\rho}\left(\frac{1}{11}w_y + a\beta_x - \phi\right)$$
$$= f\left[\frac{1}{12}\left(R_{2x} - \frac{\rho_x}{\rho}R_2\right) + \frac{1}{12}\left(R_{3y} - \frac{\rho_y}{\rho}R_3\right) - \frac{1}{11}R_1\right] + \frac{f_x}{12}R_2 + \frac{f_y}{12}R_3. \tag{4.14}$$

In a similar fashion, we derive an equation for β by first recalling (4.8), and taking the difference of (4.5c) differentiated with respect to x with (4.5b) differentiated with respect to y. This gives

$$\frac{\rho}{12}\beta_{tt} - a\Delta\beta + \frac{\rho_x}{12}\phi_{tt} - \frac{\rho_y}{12}\psi_{tt} + (\phi_x - \psi_y) = \frac{f_x}{12}R_3 + \frac{f}{12}R_{3x} - \frac{f_y}{12}R_2 - \frac{f}{12}R_{2y}.$$

Again, replacing ϕ_{tt} and ψ_{tt} with (4.12) and (4.13) respectively, permits the above to simplify as follows

$$\frac{\rho}{12}\beta_{tt} - a\Delta\beta + \frac{a\rho_x}{\rho}\beta_x + \frac{a\rho_y}{\rho}\beta_y + (\phi_x - \psi_y) + \frac{\rho_x}{\rho}\left(\frac{1}{11}w_y + \alpha_y - \phi\right) - \frac{\rho_y}{\rho}\left(\frac{1}{11}w_x + \alpha_x - \psi\right)$$
$$= \frac{f}{12}\left(\frac{\rho_y}{\rho}R_2 - \frac{\rho_x}{\rho}R_3 + R_{3x} - R_{2y}\right) - \frac{f_y}{12}R_2 + \frac{f_x}{12}R_3. \tag{4.15}$$

Thus, via the definitions in (4.8) we can combine the five equations (4.5a), (4.10), (4.11), (4.14), and (4.15) and therefore producing the diagonalized system

$$\begin{cases} \rho w_{tt} - \Delta w - (\psi_x + \phi_y) = fR_1 & \Omega \times [0,T] \quad (4.16\text{a}) \\ \frac{\rho}{12}\psi_{tt} - \Delta\psi - (1-a)\beta_y + (\psi + w_x) = \frac{1}{12}fR_2 & \Omega \times [0,T] \quad (4.16\text{b}) \\ \frac{\rho}{12}\phi_{tt} - \Delta\phi + (1-a)\beta_x + (\phi + w_y) = \frac{1}{12}fR_3 & \Omega \times [0,T] \quad (4.16\text{c}) \\ \frac{\rho}{12}\alpha_{tt} - \Delta\alpha + \frac{\rho_x}{\rho}\alpha_x + \frac{\rho_y}{\rho}\alpha_y + F_1(\beta, w, \psi, \phi) & \\ \quad = f\left[\frac{1}{12}\left(R_{2x} - \frac{\rho_x}{\rho}R_2\right) + \frac{1}{12}\left(R_{3y} - \frac{\rho_y}{\rho}R_3\right) - \frac{1}{11}R_1\right] + \frac{f_x}{12}R_2 + \frac{f_y}{12}R_3 & \Omega \times [0,T] \quad (4.16\text{d}) \\ \frac{\rho}{12}\beta_{tt} - a\Delta\beta + \frac{a\rho_x}{\rho}\beta_x + \frac{a\rho_y}{\rho}\beta_y + F_2(\alpha, w, \psi, \phi) & \\ \quad = \frac{f}{12}\left(\frac{\rho_y}{\rho}R_2 - \frac{\rho_x}{\rho}R_3 + R_{3x} - R_{2y}\right) - \frac{f_y}{12}R_2 + \frac{f_x}{12}R_3 & \Omega \times [0,T] \quad (4.16\text{e}) \end{cases}$$

where

$$F_1(\beta, w, \psi, \phi) = \frac{\rho_x}{\rho}\left(\frac{1}{11}w_x - a\beta_y - \psi\right) + \frac{\rho_y}{\rho}\left(\frac{1}{11}w_y + a\beta_x - \phi\right)$$

$$F_2(\alpha, w, \psi, \phi) = (\phi_x - \psi_y) + \frac{\rho_x}{\rho}\left(\frac{1}{11}w_y + \alpha_y - \phi\right) - \frac{\rho_y}{\rho}\left(\frac{1}{11}w_x + \alpha_x - \psi\right)$$

with the *homogeneous* initial and boundary conditions

$$(w, \psi, \phi, \alpha, \beta)|_{t=0} = (w_t, \psi_t, \phi_t, \alpha_t, \beta_t)|_{t=0} = (0,0,0,0,0), \qquad (w, \psi, \phi, \alpha, \beta)|_{\Gamma\times[0,T]} = (0,0,0,0,0). \tag{4.17}$$

Moreover, from the zero boundary measurement (4.7), as well as the homogeneous boundary condition $(w, \psi, \phi)|_{\Gamma_1\times[0,T]} = (0,0,0)$ and the definition of α and β (4.8), we also have the following proposition.

Proposition 4.1.1. *Under the assumptions of homogeneous Dirichlet boundary conditions and homogeneous boundary conditions in* Γ_1 *as shown in* (4.7) *for system* (4.16) *the following homogeneous Neumann boundary conditions hold*

$$\left(\frac{\partial w}{\partial \nu}, \frac{\partial \psi}{\partial \nu}, \frac{\partial \phi}{\partial \nu}, \frac{\partial \alpha}{\partial \nu}, \frac{\partial \beta}{\partial \nu}\right)|_{\Gamma_1\times[0,T]} = (0,0,0,0,0). \tag{4.18}$$

Proof. We will prove the shorter

$$\left(\frac{\partial \psi}{\partial \nu}, \frac{\partial \phi}{\partial \nu}\right)|_{\Gamma_1\times[0,T]} = (0,0,0) \tag{4.19}$$

since $\frac{\partial w}{\partial \nu} = 0$ follows immediately from the given assumptions and this, paired with (4.19), readily implies (4.18) from the definitions in (4.8). Thus, under consideration of the assumptions in Proposition 4.1.1 notice we can rewrite the final two boundary conditions in (4.7) as

$$\begin{cases} \nabla\psi \cdot (\nu_1, \frac{1-\mu}{2}\nu_2) + \nabla\phi \cdot (\frac{1-\mu}{2}\nu_2, \mu\nu_1) = 0 \\ \nabla\psi \cdot (\mu\nu_2, \frac{1-\mu}{2}\nu_1) + \nabla\phi \cdot (\frac{1-\mu}{2}\nu_1, \nu_2) = 0 \quad . \end{cases} \tag{4.20}$$

Moreover, $\psi = 0$ implies $\nabla_{\tan}\psi = 0$ on $\Sigma_1 \triangleq [0,T] \times \Gamma_1$, hence,

$$\left|\frac{\partial \psi}{\partial \nu}\right| = |\nabla \psi| \Rightarrow \frac{\nabla \psi}{|\nabla \psi|} = \pm \nu \Rightarrow \nabla \psi = \pm |\nabla \psi| \nu = \pm \left|\frac{\partial \psi}{\partial \nu}\right| \nu.$$

Similarly, we also have $\nabla \phi = \pm \left|\frac{\partial \phi}{\partial \nu}\right| \nu$.

Case I: Consider the same sign case where we assume $\nabla \psi = \left|\frac{\partial \psi}{\partial \nu}\right| \nu$, $\nabla \phi = \left|\frac{\partial \phi}{\partial \nu}\right| \nu$ (the case where $\nabla \psi = -\left|\frac{\partial \psi}{\partial \nu}\right| \nu$, $\nabla \phi = -\left|\frac{\partial \phi}{\partial \nu}\right| \nu$ is similar). Then (4.20) becomes

$$\begin{cases} \left|\frac{\partial \psi}{\partial \nu}\right| (\nu_1^2 + \frac{1-\mu}{2}\nu_2^2) + \left|\frac{\partial \phi}{\partial \nu}\right| (\frac{1+\mu}{2}\nu_1\nu_2) = 0 \\ \left|\frac{\partial \psi}{\partial \nu}\right| (\frac{1+\mu}{2}\nu_1\nu_2) + \left|\frac{\partial \phi}{\partial \nu}\right| (\frac{1-\mu}{2}\nu_1^2 + \nu_2^2) = 0 \end{cases}.$$

Solving for $\left|\frac{\partial \psi}{\partial \nu}\right|, \left|\frac{\partial \phi}{\partial \nu}\right|$ gives the coefficient matrix

$$A = \begin{pmatrix} \nu_1^2 + \frac{1-\mu}{2}\nu_2^2 & \frac{1+\mu}{2}\nu_1\nu_2 \\ \frac{1+\mu}{2}\nu_1\nu_2 & \frac{1-\mu}{2}\nu_1^2 + \nu_2^2 \end{pmatrix}$$

and we have $\det A = \frac{1-\mu}{2}(\nu_1^2 + \nu^2) \neq 0$ which results in $\left|\frac{\partial \psi}{\partial \nu}\right| = \left|\frac{\partial \phi}{\partial \nu}\right| = 0$.

Case II: Consider the mixed sign case $\nabla \psi = \left|\frac{\partial \psi}{\partial \nu}\right| \nu$, $\nabla \phi = -\left|\frac{\partial \phi}{\partial \nu}\right| \nu$. (the case where $\nabla \psi = -\left|\frac{\partial \psi}{\partial \nu}\right| \nu$, $\nabla \phi = \left|\frac{\partial \phi}{\partial \nu}\right| \nu$ is similar). Again in a similar manner (4.20) becomes

$$\begin{cases} \left|\frac{\partial \psi}{\partial \nu}\right| (\nu_1^2 + \frac{1-\mu}{2}\nu_2^2) - \left|\frac{\partial \phi}{\partial \nu}\right| (\frac{1+\mu}{2}\nu_1\nu_2) = 0 \\ \left|\frac{\partial \psi}{\partial \nu}\right| (\frac{1+\mu}{2}\nu_1\nu_2) - \left|\frac{\partial \phi}{\partial \nu}\right| (\frac{1-\mu}{2}\nu_1^2 + \nu_2^2) = 0 \end{cases}.$$

Solving for $\left|\frac{\partial \psi}{\partial \nu}\right|, \left|\frac{\partial \phi}{\partial \nu}\right|$ gives the coefficient matrix

$$A = \begin{pmatrix} \nu_1^2 + \frac{1-\mu}{2}\nu_2^2 & -\frac{1+\mu}{2}\nu_1\nu_2 \\ \frac{1+\mu}{2}\nu_1\nu_2 & -\frac{1-\mu}{2}\nu_1^2 + \nu_2^2 \end{pmatrix}$$

where once more we have $\det A = -\frac{1-\mu}{2}(\nu_1^2 + \nu^2) \neq 0$ with the implication being $\left|\frac{\partial \psi}{\partial \nu}\right| = \left|\frac{\partial \phi}{\partial \nu}\right| = 0$.

Thus, under all considerations we have the desired result. $\square$

4.2 Geometrical Assumptions and Main Results

Throughout this paper we assume the boundary of the domain consists of the closure of two disjoint parts Γ_0 and Γ_1, both relatively open in Γ, with Γ_1 being the observed part (where the measurement is taken) that was used in the formulation of the inverse problem. In other words, we have $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. In addition, we claim the following geometrical assumptions for the triple $\{\Omega, \Gamma_0, \Gamma_1\}$:

(A.1) There exists a function $d : \overline{\Omega} \to \mathbb{R}$, which is of class $C^3(\overline{\Omega})$ and is strictly convex in the metric $g = \rho(x, y)dxdy$, that satisfy the following properties (through translation and rescaling if necessary):

(i)

$$\frac{\partial d}{\partial \nu} = \langle Dd(x, y), \nu(x, y)\rangle \geq 0 \text{ for all } (x, y) \in \Gamma_1$$

(ii)

$$D^2 d(X, X) \equiv \langle D_X(Dd), X\rangle_g \geq 2|X|_g^2, \forall X \in M_p, \ \min_{\overline{\Omega}} d(x, y) \equiv m > 0$$

where $Dd = \nabla_g d$ is a vector field on Ω and $D^2 d$ is the Hessian of d (a second-order tensor) and M_p is the tangent space at $p = (x, y) \in \Omega$.

(A.2) $d(x, y)$ has no critical point on $\overline{\Omega}$, namely:

$$\inf_{(x,y)\in\Omega} |Dd| = q > 0, \text{ so that we can take } \inf_{(x,y)\in\overline{\Omega}} \frac{|Dd|^2}{d} > 4.$$

These geometrical assumptions permit the construction of a vector field that enables a pseudo-convex function necessary for allowing a Carleman estimate containing no lower order terms for the wave equation. These assumptions are first formulated in [25] under the framework of a Euclidean metric, with [44] employing them under the more general Riemannian framework. For examples and detailed illustrations of large general classes of domains satisfying the aforementioned assumptions we refer to [44, Appendix B].

Remark 4.2.1. *Since the MT model is a system of hyperbolic equations behaving like wave equations, we must take into account the variety of wave speeds when selecting the metric g. It can be observed that the choice of metric $g = \rho(x,y)dxdy$ corresponds to the slowest wave speed $\sqrt{\frac{1}{\rho}}$ in the diagonalized system derived in (4.16). We further detail the reason behind this choice in Remark 4.2.2 below.*

4.2.1 Carleman estimates for Riemannian wave equations.

In this section we present the Carleman estimate without lower-order terms for the general Riemannian wave equation as show in [44]. Only pertinent results are shown as needed for the proof of the main theorems and we refer to [44] for further details. Let us also mention that even though [44] works with any finite dimension, here we only focus on the two dimensional setting.

Consider a Riemannian metric $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and squared norm $|X|^2 = g(X,X)$, on a smooth two dimensional manifold $\mathcal{M}$. Thus we have the Riemannian manifold $(\mathcal{M}, g)$ where we define Ω as an open bounded, connected, compact set of $\mathcal{M}$ with smooth boundary Γ. Let ν denote the unit outward normal field along the boundary Γ. Further, we denote by Δ_g the Laplace–Beltrami operator on the manifold $\mathcal{M}$ and by D the Levi–Civita connection on $\mathcal{M}$ [11]. Consider the following second-order hyperbolic equation with energy level terms on Ω:

$$u_{tt}(x,y,t) - \Delta_g u(x,y,t) = F(u) + G(x,y,t), \qquad (x,y,t) \in Q = \Omega \times [-T,T] \tag{4.21}$$

where the forcing term

$$G(x,y,t) \in L^2(Q), \qquad \int_Q G^2 dQ < \infty$$

with $dQ = d\Omega dt$, and $d\Omega$ is the volume element of the manifold $\mathcal{M}$ in its Riemann metric g and the energy level differential term $F(u)$ is given by

$$F(u) = \langle P(x,y,t), Du \rangle + p_1(x,y,t)u_t + p_0(x,y,t)u,$$

where functions p_0 and p_1 are defined on $\Omega \times [-T,T]$, and $P(t)$ is a vector field on $\mathcal{M}$ for $t > 0$. We assume the differential term satisfies the following estimate: There exists some constant $C_T > 0$ such that

$$|F(u)| \leq C_T[u^2 + u_t^2 + |Du|^2], \qquad (x,y,t) \in Q$$

Pseudo-convex function. Given the existence of the strongly convex function $d(x,y)$ satisfying the geometrical condition (A.1), we can define a psuedo-convex function, $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ of class C^3 by setting

$$\Phi(x,y,t) = d(x,y) - ct^2; \;\; (x,y) \in \Omega, \;\; t \in [-T,T], \tag{4.22}$$

with $T > 0$ and $c \in (0,1)$ selected following the process in [44]. The threshold time $T_0 > 0$ is defined by setting

$$T_0^2 \equiv 4 \max_{(x,y)\in\bar{\Omega}} d(x,y). \tag{4.23}$$

Let $T > T_0$ then there exists $\delta > 0$ fixed and satisfying

$$cT^2 > 4(\max_{x\in\bar{\Omega}} d(x,y) + \delta)$$

so that

$$\Phi(x,y,-T) \equiv \Phi(x,y,T) \leq -\delta$$

holds uniformly for $(x,y) \in \Omega$. Additionally, for some $\sigma > 0$ we have, for t_0, t_1 chosen symmetrically about 0 where $-T < t_0 < 0 < t_1 < T$, the property

$$\min_{(x,y)\in\bar{\Omega},t\in[t_0,t_1]} \Phi(x,y,t) \geq \sigma, \;\text{ where } 0 < \sigma < \min_{x\in\bar{\Omega}} d(x,y)$$

holds since $\Phi(x,y,0) = d(x,y) \geq m > 0$. This generates the region

$$Q(\sigma) \equiv \{(x,y,t) : (x,y) \in \Omega, \, t \in [-T,T], \, \Phi(x,y,t) \geq \sigma > 0\},$$

Thus giving

$$\Omega \times [t_0, t_1] \subset Q(\sigma) \subset \Omega \times [-T,T]. \tag{4.24}$$

Remark 4.2.2. *Choosing the metric $g = \rho(x,y)dxdy$ corresponding to the slowest speed in the system* (4.16) *allows the use of the same choice of pseudo-convex function $\Phi(x,y,t)$ and, ultimately, establishes the region $Q(\sigma)$ as a common region for each equation in the model. The Carleman estimate used within contains integral terms over this region so maintaining its commonality is desirable for multiple applications of the Carleman-type estimates to the different equations in the* *diagonalized system (4.16). This is due to the combining of these estimates as a necessary step in the proofs contained herein.*

Carleman estimate for Riemannian wave equations at the $H^1 \times L^2$-level. Consider the solutions $u(x, y, t)$ to the general wave equation, as expressed in (4.21), in the class

$$\begin{cases} u \in H^1(Q) \equiv L^2(-T, T; H^1(\Omega)) \cap H^1(-T, T; L^2(\Omega)); \\ u_t, \frac{\partial u}{\partial \nu} \in L^2(-T, T; L^2(\Gamma)). \end{cases} \tag{4.25}$$

Then, under the assumptions (A.1) and (A.2) on Ω the following Carleman-type estimate holds for these solutions:

$$\begin{aligned} BT(u) + 2\int_Q e^{2\tau\Phi} |G|^2 \, dQ + Ce^{2\tau\sigma} \int_Q u^2 \, dQ + C\tau^3 e^{-2\tau\delta}[E_u(0) + E_u(T)] \\ \geq \tau \int_Q e^{2\tau\Phi}[u_t^2 + |Du|^2] \, dQ + \tau^3 \int_{Q(\sigma)} e^{2\tau\Phi} u^2 \, dxdt \end{aligned} \tag{4.26}$$

where $\delta > 0$, and $\sigma > 0$ are the constants defined in the previous section and $\tau > 0$ is a sufficiently large free parameter. The energy function $E_u(t)$ is defined as

$$E_u(t) = \int_\Omega [u_t^2 + |Du|^2 + u^2] \, d\Omega.$$

Moreover, the boundary terms $BT(u)$ on $\Gamma \times [-T, T]$, can also be given explicitly and, in particular, by the assumption (A.1) we have

$$BT(u) = 2\tau \int_{-T}^{T} \int_{\Gamma_0} e^{2\tau\Phi} |Du|^2 \langle Dd, \nu \rangle \leq 0 \quad \text{if} \quad u|_{\Gamma \times [-T,T]} = \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1 \times [-T,T]} = 0. \tag{4.27}$$

Last, for the estimate (4.26) and all estimates henceforth, C denotes a generic constant which may depend on the parameters in the problem, but not on the large free parameter τ.

4.2.2 Statement of main results

We are now ready to state the main theorems that will be proved in the next section. We start with the result of the inverse source problem.

Theorem 4.2.3. *Under the geometrical assumptions (A.1) and (A.2), let $T > T_0$ as defined in*

(4.23) and $f \in H^1(\Omega)$. *Additionally, let us also assume the following regularity and positivity properties hold for the fixed functions* $R_1(x,y,t), R_2(x,y,t)$ *and* $R_3(x,y,t)$

$$R_i, R_{it}, R_{itt} \in W^{\ell,\infty}(\Omega \times [0,T]) \ \textit{for} \ i = 1,2,3; \tag{4.28}$$

where $\ell = 0$ *for* $i = 1$ *and* $\ell = 1$ *when* $i = 2,3$, *and*

$$|R_1(x,y,0)| \geq r_0 > 0, \ \max\{|R_2(x,y,0)|, |R_3(x,y,0)|\} \geq r_0 > 0, \quad \textit{for some constant } r_0. \tag{4.29}$$

Then, if the solutions $\{w,\psi,\phi\}$ *of* (4.5) *have the homogeneous boundary measurement of*

$$\frac{\partial w}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 \quad on \quad \Gamma_1 \times [0,T],$$

we have

$$f(x,y) = 0, \quad a.e. \ (x,y) \in \Omega.$$

Next for the primary uniqueness result on recovering $\rho(x,y)$ for the original inverse problem, let $(\bar{w},\bar{\psi},\bar{\phi})(\rho)$ be the set of functions that solve (4.1) with the density function $\rho(x,y)$. Then, if we consider another set $(\tilde{w},\tilde{\psi},\tilde{\phi})(\tilde{\rho})$ satisfying (4.1) with respect to the different density $\tilde{\rho}(x,y)$, we have the following result.

Theorem 4.2.4. *Under the geometrical assumptions (A.1) and (A.2), let* $T > T_0$ *as defined in (4.23) and* ρ *satisfy* (4.2). *Assume further the solution* $(\tilde{w},\tilde{\psi},\tilde{\phi})(\tilde{\rho})$ *satisfies the following regularity conditions*

$$\tilde{w}, \tilde{w}_t, \tilde{w}_{tt} \in L^\infty(\Omega \times [0,T]), \ \tilde{\psi}, \tilde{\psi}_t, \tilde{\psi}_{tt}, \tilde{\phi}, \tilde{\phi}_t, \tilde{\phi}_{tt} \in W^{1,\infty}(\Omega \times [0,T]) \tag{4.30}$$

and the following positivity condition at initial time $t = 0$

$$|\tilde{w}_{tt}(x,y,0)| \geq r_0 > 0, \ \max\left\{|\tilde{\psi}_{tt}(x,y,0)|, |\tilde{\phi}_{tt}(x,y,0)|\right\} \geq r_0 > 0, \ \textit{for some constant } r_0 > 0. \tag{4.31}$$

Then, if the solutions $(\bar{w},\bar{\psi},\bar{\phi})(\rho)$ *and* $(\tilde{w},\tilde{\psi},\tilde{\phi})(\tilde{\rho})$ *have the same boundary measurement of*

$$\frac{\partial \bar{w}}{\partial \nu} = \frac{\partial \tilde{w}}{\partial \nu}, \ \frac{\partial \bar{\psi}}{\partial \nu} = \frac{\partial \tilde{\psi}}{\partial \nu}, \ \frac{\partial \bar{\phi}}{\partial \nu} = \frac{\partial \tilde{\phi}}{\partial \nu} \quad on \quad \Gamma_1 \times [0,T],$$

we have the two density functions in fact coincide, i.e.,

$$\rho(x,y) = \tilde{\rho}(x,y), \quad a.e.\ (x,y) \in \Omega.$$

4.3 Main Proofs

In this section, we provide the proofs for the main uniqueness results established in the previous sections. We focus initially on proving Theorem 4.2.3 for the linearized inverse problem and then, from which we may establish the uniqueness result for the original inverse problem stated in Theorem 4.2.4 via the relationships in (4.4).

4.3.1 Proof of Theorem 4.2.3

To begin the proof of the uniqueness statement for the inverse source problem, we do a natural even extension of the time interval to the three equations in (4.5) to $[-T, 0]$, This results in the equations in the diagonalized system (4.16) being extended in the same manner. Next, using the homogeneous Dirichlet boundary condition and Neumann boundary measurement, we have the following overdetermined problem

$$\begin{cases}
\rho w_{tt} - \Delta w - (\psi_x + \phi_y) = fR_1 & \Omega \times [-T,T] \quad (4.32a) \\
\frac{\rho}{12}\psi_{tt} - \Delta\psi - (1-a)\beta_y + (\psi + w_x) = \frac{1}{12} fR_2 & \Omega \times [-T,T] \quad (4.32b) \\
\frac{\rho}{12}\phi_{tt} - \Delta\phi + (1-a)\beta_x + (\phi + w_y) = \frac{1}{12} fR_3 & \Omega \times [-T,T] \quad (4.32c) \\
\frac{\rho}{12}\alpha_{tt} - \Delta\alpha + \frac{\rho_x}{\rho}\alpha_x + \frac{\rho_y}{\rho}\alpha_y + F_1(\beta, w, \psi, \phi) = G_1(f, \rho, R_1, R_2, R_3) & \Omega \times [-T,T] \quad (4.32d) \\
\frac{\rho}{12}\beta_{tt} - a\Delta\beta + \frac{a\rho_x}{\rho}\beta_x + \frac{a\rho_y}{\rho}\beta_y + F_2(\alpha, w, \psi, \phi) = G_2(f, \rho, R_2, R_3) & \Omega \times [-T,T] \quad (4.32e) \\
(w,\psi,\phi,\alpha,\beta)|_{t=0} = (w_t,\psi_t,\phi_t,\alpha_t,\beta_t)|_{t=0} = (0,0,0,0,0) = 0 & \Omega \quad (4.32f) \\
(w,\psi,\phi,\alpha,\beta)|_{\Gamma\times[-T,T]} = (0,0,0,0,0) & \Gamma \times [-T,T] \quad (4.32g) \\
\left(\frac{\partial w}{\partial \nu}, \frac{\partial \psi}{\partial \nu}, \frac{\partial \phi}{\partial \nu}, \frac{\partial \alpha}{\partial \nu}, \frac{\partial \beta}{\partial \nu}\right)|_{\Gamma_1\times[-T,T]} = (0,0,0,0,0) & \Gamma_1 \times [-T,T] \quad (4.32h)
\end{cases}$$

where

$$\begin{cases} F_1(\beta, w, \psi, \phi) = \frac{\rho_x}{\rho}\left(\frac{1}{11}w_x - a\beta_y - \psi\right) + \frac{\rho_y}{\rho}\left(\frac{1}{11}w_y + a\beta_x - \phi\right) \\ F_2(\alpha, w, \psi, \phi) = (\phi_x - \psi_y) + \frac{\rho_x}{\rho}\left(\frac{1}{11}w_y + \alpha_y - \phi\right) - \frac{\rho_y}{\rho}\left(\frac{1}{11}w_x + \alpha_x - \psi\right) \\ G_1(f, \rho, R_1, R_2, R_3) = f\left[\frac{1}{12}\left(R_{2x} - \frac{\rho_x}{\rho}R_2\right) + \frac{1}{12}\left(R_{3y} - \frac{\rho_y}{\rho}R_3\right) - \frac{1}{11}R_1\right] + \frac{f_x}{12}R_2 + \frac{f_y}{12}R_3 \\ G_2(f, \rho, R_2, R_3) = \frac{f}{12}\left(\frac{\rho_y}{\rho}R_2 - \frac{\rho_x}{\rho}R_3 + R_{3x} - R_{2y}\right) - \frac{f_y}{12}R_2 + \frac{f_x}{12}R_3. \end{cases} \tag{4.33}$$

Note each equation in the above system (4.32) can be written as a Riemannian wave equation with respect to the common metric $g = \rho(x,y)dxdy$, modulo first-order terms (LOT) [47], thus yielding the following system

$$\begin{cases} w_{tt} - \Delta_g w + LOT_w = fR_1 + (\psi_x + \phi_y) & \Omega \times [-T,T] & (4.34a) \\ \psi_{tt} - 12\Delta_g \psi + LOT_\psi = fR_2 + (1-a)\beta_y - w_x & \Omega \times [-T,T] & (4.34b) \\ \phi_{tt} - 12\Delta_g \phi + LOT_\phi = fR_3 - (1-a)\beta_x - w_y & \Omega \times [-T,T] & (4.34c) \\ \alpha_{tt} - 12\Delta_g \alpha + LOT_\alpha = G_1(f, \rho, R_1, R_2, R_3) - F_1(\beta, w, \psi, \phi) & \Omega \times [-T,T] & (4.34d) \\ \beta_{tt} - 12a\Delta_g \beta + LOT_\beta = G_2(f, \rho, R_2, R_3) - F_2(\alpha, w, \psi, \phi) & \Omega \times [-T,T] & (4.34e) \\ (w, \psi, \phi, \alpha, \beta)|_{t=0} = (w_t, \psi_t, \phi_t, \alpha_t, \beta_t)|_{t=0} = (0,0,0,0,0) = 0 & \Omega & (4.34f) \\ (w, \psi, \phi, \alpha, \beta)|_{\Gamma \times [-T,T]} = (0,0,0,0,0) & \Gamma \times [-T,T] & (4.34g) \\ \left(\frac{\partial w}{\partial \nu}, \frac{\partial \psi}{\partial \nu}, \frac{\partial \phi}{\partial \nu}, \frac{\partial \alpha}{\partial \nu}, \frac{\partial \beta}{\partial \nu}\right)|_{\Gamma_1 \times [-T,T]} = (0,0,0,0,0) & \Gamma_1 \times [-T,T] & (4.34h) \end{cases}$$

with F_1, F_2, G_1, G_2 as defined in (4.33).

Thus, under the regularity assumptions for R_1, R_2, R_3 in (4.28) in combination with the boundary conditions in (4.34g-h), we can apply the Carleman estimates (4.26) to each of the solutions w, ψ, ϕ, α, and β of class (4.25), taking their sum and dropping the non-positive boundary terms to yield, the estimate

$$C \int_Q e^{2\tau\Phi}(f^2 + f_x^2 + f_y^2 + |Dw|^2 + |D\psi|^2 + |D\phi|^2 + |D\alpha|^2 + |D\beta|^2 + \psi^2 + \phi^2)\, dQ + Ce^{2\tau\sigma}$$
$$\geq \tau \int_Q e^{2\tau\Phi}(w_t^2 + \psi_t^2 + \phi_t^2 + \alpha_t^2 + \beta_t^2 + |Dw|^2 + |D\psi|^2 + |D\phi|^2 + |D\alpha|^2 + |D\beta|^2)\, dQ$$
$$+ \tau^3 \int_{Q(\sigma)} e^{2\tau\Phi}(w^2 + \psi^2 + \phi^2 + \alpha^2 + \beta^2)\, dxdydt \tag{4.35}$$

where on the larger side we have also used the basic algebraic inequalities, recalling again the regularity on R_1, R_2, R_3 and ρ, and the definition of F_1, F_2, G_1, G_2 (4.33)

$$|fR_1|^2 + |fR_2|^2 + |fR_3|^2 + |G_1|^2 + |G_2|^2 \leq C(f^2 + f_x^2 + f_y^2)$$

$$|\psi_x + \phi_y|^2 + |(1-a)\beta_y - w_x|^2 + |-(1-a)\beta_x - w_y|^2 + |F_1|^2 + |F_2|^2$$
$$\leq C\left(|Dw|^2 + |D\psi|^2 + |D\phi|^2 + |D\alpha|^2 + |D\beta|^2 + \psi^2 + \phi^2\right).$$

Absorb the like terms in (4.35) with large enough τ, write

$$\int_Q e^{2\tau\Phi}(\psi^2 + \phi^2)\, dQ = \int_{Q(\sigma)} e^{2\tau\Phi}(\psi^2 + \phi^2)\, dxdydt + \int_{Q\setminus Q(\sigma)} e^{2\tau\Phi}(\psi^2 + \phi^2)\, dxdydt$$

and use the fact that $e^{2\tau\Phi} < e^{2\tau\sigma}$ on $Q \setminus Q(\sigma)$, we have

$$C \int_Q e^{2\tau\Phi}(f^2 + f_x^2 + f_y^2)\, dQ + Ce^{2\tau\sigma}$$
$$\geq \tau \int_Q e^{2\tau\Phi}(w_t^2 + \psi_t^2 + \phi_t^2 + \alpha_t^2 + \beta_t^2 + |Dw|^2 + |D\psi|^2 + |D\phi|^2 + |D\alpha|^2 + |D\beta|^2)\, dQ$$
$$+ \tau^3 \int_{Q(\sigma)} e^{2\tau\Phi}(w^2 + \psi^2 + \phi^2 + \alpha^2 + \beta^2)\, dxdydt \tag{4.36}$$

where again C is a generic constant that may depend on Ω, R_1, R_2, R_3, ρ, a, w, ψ, ϕ, α, β, but *not* on the large parameter τ.

Remark 4.3.1. *The system in* (4.34) *contains four Riemannian wave equations (4.34b-e) with a* *constant coefficient on the term including the Laplace-Beltrami operator, Δ_g. This implies a wave speed for these equations, in particular a constant wave speed larger than one since the slowest speed was chose for the Riemannian metric g. While not shown here, one can verify by referencing the proof in [44] that these constants can be easily absorbed into the estimate.*

Next we differentiate (4.34a-e) in time t, and therefore have the following system

$$\begin{cases} (w_t)_{tt} - \Delta_g(w_t) + LOT_{w_t} = fR_{1t} + ((\psi_t)_x + (\phi_t)_y) & \Omega \times [-T,T] \quad (4.37a) \\ (\psi_t)_{tt} - 12\Delta_g(\psi_t) + LOT_{\psi_t} = fR_{2t} + (1-a)(\beta_t)_y - (w_t)_x & \Omega \times [-T,T] \quad (4.37b) \\ (\phi_t)_{tt} - 12\Delta_g(\phi_t) + LOT_{\phi_t} = fR_{3t} - (1-a)(\beta_t)_x - (w_t)_y & \Omega \times [-T,T] \quad (4.37c) \\ (\alpha_t)_{tt} - 12\Delta_g(\alpha_t) + LOT_{\alpha_t} = G_{1t} - F_{1t} & \Omega \times [-T,T] \quad (4.37d) \\ (\beta_t)_{tt} - 12a\Delta_g(\beta_t) + LOT_{\beta_t} = G_{2t} - F_{2t} & \Omega \times [-T,T] \quad (4.37e) \end{cases}$$

with initial and boundary conditions

$$\begin{cases} w_t(\cdot,0) = \psi_t(\cdot,0) = \phi_t(\cdot,0) = \alpha_t(\cdot,0) = \beta_t(\cdot,0) = 0 & \text{in } \Omega \\ w_{tt}(\cdot,0) = fR_1(\cdot,0),\ \psi_{tt}(\cdot,0) = fR_2(\cdot,0),\ \phi_{tt}(\cdot,0) = fR_3(\cdot,0), & \\ \alpha_{tt}(\cdot,0) = fR_{2x}(\cdot,0) + fR_{3y}(\cdot,0) - \frac{12}{11}R_1(\cdot,0),\ \beta_{tt}(\cdot,0) = fR_{3x}(\cdot,0) - fR_{2y}(\cdot,0) & \text{in } \Omega \\ w_t = \psi_t = \phi_t = \alpha_t = \beta_t = 0 & \text{in } \Gamma \times [-T,T] \\ \dfrac{\partial w_t}{\partial \nu} = \dfrac{\partial \psi_t}{\partial \nu} = \dfrac{\partial \phi_t}{\partial \nu} = \dfrac{\partial \alpha_t}{\partial \nu} = \dfrac{\partial \beta_t}{\partial \nu} = 0 & \text{in } \Gamma_1 \times [-T,T] \end{cases} \tag{4.38}$$

Again, by the regularity assumptions in (4.28) for R_1, R_2, R_3, as well as $f \in H^1(\Omega)$ and under regularity assumptions on ρ, as well as the boundary conditions in (4.38), we can apply the Carleman estimates (4.26) to the solutions $w_t, \phi_t, \psi_t, \alpha_t, \beta_t$ of class (4.25), absorb the like terms for large enough τ and have the following estimate similar to (4.36):

$$\begin{aligned} C\int_Q e^{2\tau\Phi}(f^2 + f_x^2 + f_y^2)\, dQ + Ce^{2\tau\sigma} \\ \geq \tau \int_Q e^{2\tau\Phi}(w_{tt}^2 + \psi_{tt}^2 + \phi_{tt}^2 + \alpha_{tt}^2 + \beta_{tt}^2 + |Dw_t|^2 + |D\psi_t|^2 + |D\phi_t|^2 + |D\alpha_t|^2 + |D\beta_t|^2)\, dQ \\ + \tau^3 \int_{Q(\sigma)} e^{2\tau\Phi}(w_t^2 + \psi_t^2 + \phi_t^2 + \alpha_t^2 + \beta_t^2)\, dxdydt. \end{aligned} \tag{4.39}$$

Continuing in this fashion, we differentiate (4.37) with respect to time t, and get the corresponding system

$$\begin{cases} (w_{tt})_{tt} - \Delta_g(w_{tt}) + LOT_{w_{tt}} = fR_{1tt} + ((\psi_{tt})_x + (\phi_{tt})_y) & \Omega \times [-T,T] \quad (4.40a) \\ (\psi_{tt})_{tt} - 12\Delta_g(\psi_{tt}) + LOT_{\psi_{tt}} = fR_{2tt} + (1-a)(\beta_{tt})_y - (w_{tt})_x & \Omega \times [-T,T] \quad (4.40b) \\ (\phi_{tt})_{tt} - 12\Delta_g(\phi_{tt}) + LOT_{\phi_{tt}} = fR_{3tt} - (1-a)(\beta_{tt})_x - (w_{tt})_y & \Omega \times [-T,T] \quad (4.40c) \\ (\alpha_{tt})_{tt} - 12\Delta_g(\alpha_{tt}) + LOT_{\alpha_{tt}} = G_{1tt} - F_{1tt} & \Omega \times [-T,T] \quad (4.40d) \\ (\beta_{tt})_{tt} - 12a\Delta_g(\beta_{tt}) + LOT_{\beta_{tt}} = G_{2tt} - F_{2tt} & \Omega \times [-T,T] \quad (4.40e) \end{cases}$$

with initial and boundary conditions

$$\begin{cases} w_{tt}(\cdot,0) = fR_1(\cdot,0),\ \psi_{tt}(\cdot,0) = fR_2(\cdot,0),\ \phi_{tt}(\cdot,0) = fR_3(\cdot,0), & \\ \alpha_{tt}(\cdot,0) = fR_{2x}(\cdot,0) + fR_{3y}(\cdot,0) - \frac{12}{11}R_1(\cdot,0),\ \beta_{tt}(\cdot,0) = fR_{3x}(\cdot,0) - fR_{2y}(\cdot,0) & \text{in } \Omega \\ w_{ttt}(\cdot,0) = fR_{1t}(\cdot,0),\ \psi_{ttt}(\cdot,0) = fR_{2t}(\cdot,0),\ \phi_{ttt}(\cdot,0) = fR_{3t}(\cdot,0), & \\ \alpha_{ttt}(\cdot,0) = fR_{2tx}(\cdot,0) + fR_{3ty}(\cdot,0) - \frac{12}{11}R_{1t}(\cdot,0),\ \beta_{ttt}(\cdot,0) = fR_{3tx}(\cdot,0) - fR_{2ty}(\cdot,0) & \text{in } \Omega. \\ w_{tt} = \psi_{tt} = \phi_{tt} = \alpha_{tt} = \beta_{tt} = 0 & \text{in } \Gamma \times [-T,T] \\ \dfrac{\partial w_{tt}}{\partial \nu} = \dfrac{\partial \psi_{tt}}{\partial \nu} = \dfrac{\partial \phi_{tt}}{\partial \nu} = \dfrac{\partial \alpha_{tt}}{\partial \nu} = \dfrac{\partial \beta_{tt}}{\partial \nu} = 0 & \text{in } \Gamma_1 \times [-T,T]. \end{cases} \tag{4.41}$$

Once more, by the regularity assumptions in (4.28) for R_1, R_2, R_3, as well as $f \in H^1(\Omega)$ and under regularity assumptions on ρ, as well as the boundary conditions in (4.38), we can apply the Carleman estimates (4.26) to the solutions $w_{tt}, \phi_{tt}, \psi_{tt}, \alpha_{tt}$, and β_{tt} of class (4.25), absorb the like terms for large enough τ and have the following estimate similar to (4.36) and (4.39):

$$\begin{aligned} &C\int_Q e^{2\tau\Phi}(f^2 + f_x^2 + f_y^2)\,dQ + Ce^{2\tau\sigma} \\ &\qquad \geq \tau \int_Q e^{2\tau\Phi}(w_{ttt}^2 + \psi_{ttt}^2 + \phi_{ttt}^2 + \alpha_{ttt}^2 + \beta_{ttt}^2 + |Dw_{tt}|^2 + |D\psi_{tt}|^2 + |D\phi_{tt}|^2 + |D\alpha_{tt}|^2 + |D\beta_{tt}|^2)\,dQ \\ &\qquad\quad + \tau^3 \int_{Q(\sigma)} e^{2\tau\Phi}(w_{tt}^2 + \psi_{tt}^2 + \phi_{tt}^2 + \alpha_{tt}^2 + \beta_{tt}^2)\,dxdydt. \end{aligned} \tag{4.42}$$

Taking the sum of the three inequalities (4.36), (4.39), and (4.42), we arrive at

$$
\begin{aligned}
&C\int_Q e^{2\tau\Phi}(f^2+f_x^2+f_y^2)\,dQ + Ce^{2\tau\sigma} \\
&\geq \tau\int_Q e^{2\tau\Phi}(w_t^2+\psi_t^2+\phi_t^2+\alpha_t^2+\beta_t^2+|Dw|^2+|D\psi|^2+|D\phi|^2+|D\alpha|^2+|D\beta|^2)\,dQ \\
&+\tau\int_Q e^{2\tau\Phi}(w_{tt}^2+\psi_{tt}^2+\phi_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2+|Dw_t|^2+|D\psi_t|^2+|D\phi_t|^2+|D\alpha_t|^2+|D\beta_t|^2)\,dQ
\end{aligned}
\tag{4.43}
$$

$$
\begin{aligned}
&+\tau\int_Q e^{2\tau\Phi}(w_{ttt}^2+\psi_{ttt}^2+\phi_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2+|Dw_{tt}|^2+|D\psi_{tt}|^2+|D\phi_{tt}|^2+|D\alpha_{tt}|^2+|D\beta_{tt}|^2)\,dQ \\
&+\tau^3\int_{Q(\sigma)} e^{2\tau\Phi}(w^2+\psi^2+\phi^2++\alpha^2+\beta^2+w_t^2+\psi_t^2+\phi_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\psi_{tt}^2+\phi_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt.
\end{aligned}
$$

We now focus on the integral term $\int_Q e^{2\tau\Phi}(f^2+f_x^2+f_y^2)\,dQ$ appearing on the larger side of this estimate. Specifically, consider the w, α and β equations in (4.34a,d,e) at initial time $t=0$, which yields the matrix system below:

$$
\begin{pmatrix} w_{tt}(x,y,0) \\ \alpha_{tt}(x,y,0) \\ \beta_{tt}(x,y,0) \end{pmatrix} = \underbrace{\begin{pmatrix} R_1(x,y,0) & 0 & 0 \\ M & R_2(x,y,0) & R_3(x,y,0) \\ N & R_3(x,y,0) & -R_2(x,y,0) \end{pmatrix}}_{A} \begin{pmatrix} f(x,y) \\ f_x(x,y) \\ f_y(x,y) \end{pmatrix} \tag{4.44}
$$

where we define

$$
\begin{aligned}
M &= \frac{1}{12}\left(R_{2x}(x,y,0) - \frac{\rho_x}{\rho}R_2(x,y,0) + R_{3y}(x,y,0) - \frac{\rho_y}{\rho}R_3(x,y,0) - \frac{12}{11}R_1(x,y,0)\right) \\
N &= \frac{1}{12}\left(\frac{\rho_y}{\rho}R_2(x,y,0) - R_{2y}(x,y,0) + R_{3x}(x,y,0) - \frac{\rho_x}{\rho}R_3(x,y,0)\right).
\end{aligned}
$$

Notice for the matrix A,

$$
\det(A) = -R_1(x,y,0)\left(R_2^2(x,y,0)+R_3^2(x,y,0)\right) \neq 0,\ \forall (x,y)\in\Omega \tag{4.45}
$$

under the positivity assumptions in (4.29). Hence A is invertible with uniformly norm-bounded inverse for $(x,y)\in\Omega$ by (4.28) and recalling ρ in (4.2), resulting in the following inequality

$$f^2+f_x^2+f_y^2\le C\left(|w_{tt}(x,y,0)|^2+|\alpha_{tt}(x,y,0)|^2+|\beta_{tt}(x,y,0)|^2\right). \tag{4.46}$$

Using (4.46) we can then derive the following estimate,

$$\begin{aligned}
\int_Q e^{2\tau\Phi}(f^2+f_x^2+f_y^2)\,dQ \le& C\int_{-T}^{T}\int_\Omega e^{2\tau\Phi(x,y,0)}(|w_{tt}(x,y,0)|^2+|\alpha_{tt}(x,y,0)|^2+|\beta_{tt}(x,y,0)|^2)\,d\Omega dt\\
\le& C\left(\int_\Omega\int_{-T}^{0}\frac{d}{ds}\left[e^{2\tau\Phi(x,y,s)}(|w_{tt}(x,y,s)|^2+|\alpha_{tt}(x,y,s)|^2+|\beta_{tt}(x,y,s)|^2)\right]\,dsd\Omega\right.\\
&\left.+\int_\Omega e^{2\tau\Phi(x,y,-T)}(|w_{tt}(x,y,-T)|^2+|\alpha_{tt}(x,y,-T)|^2+|\beta_{tt}(x,y,-T)|^2)\,d\Omega\right)\\
\le& C\left(4Tc\tau\int_\Omega\int_{-T}^{0}e^{2\tau\Phi(x,y,s)}(|w_{tt}(x,y,s)|^2+|\alpha_{tt}(x,y,s)|^2+|\beta_{tt}(x,y,s)|^2)\,dsd\Omega\right.\\
&+2\int_\Omega\int_{-T}^{0}e^{2\tau\Phi(x,y,s)}(|w_{tt}(x,y,s)|\,|w_{ttt}(x,y,s)|+|\alpha_{tt}(x,y,s)|\,|\alpha_{ttt}(x,y,s)|\\
&\qquad+|\beta_{tt}(x,y,s)|\,|\beta_{ttt}(x,y,s)|)\,dsd\Omega\\
&\left.+\int_\Omega e^{2\tau\Phi(x,y,-T)}(w_{tt}^2(x,y,-T)+\alpha_{tt}^2(x,y,-T)+\beta_{tt}^2(x,y,-T))\,d\Omega\right)\\
\le& C\left(\tau\int_Q e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dQ+\int_Q e^{2\tau\Phi}(w_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2)\,dQ\right)
\end{aligned} \tag{4.47}$$

where specific constants are absorbed into the arbitrary constant between steps and we have used the properties of the pseudo-convex function, Φ, and the Cauchy–Schwartz inequality. Replacing the larger side of (4.43) with (4.47), we then get, for τ sufficiently large,

$$\begin{aligned}
C&\left(\tau\int_Q e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dQ+\int_Q e^{2\tau\Phi}(w_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2)\,dQ\right)+Ce^{2\tau\sigma}\ge\\
&\tau\int_Q e^{2\tau\Phi}(w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2+w_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2\,dQ\\
&+\tau^3\int_{Q(\sigma)}e^{2\tau\Phi}(w^2+\alpha^2+\beta^2+w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt
\end{aligned} \tag{4.48}$$

where unnecessary terms were dropped from the smaller side. Recall again that $e^{2\tau\Phi}<e^{2\tau\sigma}$ on

$Q\backslash Q(\sigma)$ allowing us to rewrite the first integral term on the larger side of the estimate in (4.48) as

$$\int_Q e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dQ = \int_{Q(\sigma)} e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt + \int_{Q\backslash Q(\sigma)} e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt$$
$$\leq \int_{Q(\sigma)} e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt + e^{2\tau\sigma}\int_{Q\backslash Q(\sigma)} (w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt.$$

Applying this to the estimate in (4.48) produces

$$C\tau\int_{Q(\sigma)} e^{2\tau\Phi}(w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dQ + C\int_Q e^{2\tau\Phi}(w_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2)\,dQ + C\tau e^{2\tau\sigma} \geq$$
$$\tau\int_Q e^{2\tau\Phi}(w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2+w_{ttt}^2+\alpha_{ttt}^2+\beta_{ttt}^2\,dQ$$
$$+\tau^3\int_{Q(\sigma)} e^{2\tau\Phi}(w^2+\alpha^2+\beta^2+w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt. \quad (4.49)$$

Thus, for τ large enough, the first term on the left hand side of (4.49) can be absorbed into the second term on the right hand side, while the second term on the left hand side can be absorbed into the first term on the right hand side. This yields the simplified estimate

$$C\tau e^{2\tau\sigma} \geq \tau^3\int_{Q(\sigma)} e^{2\tau\Phi}(w^2+\alpha^2+\beta^2+w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt.$$

Again, recall $\Phi \geq \sigma$ on $Q(\sigma)$ and the above simplifies to

$$C \geq \tau^2\int_{Q(\sigma)} (w^2+\alpha^2+\beta^2+w_t^2+\alpha_t^2+\beta_t^2+w_{tt}^2+\alpha_{tt}^2+\beta_{tt}^2)\,dxdydt.$$

Hence, as τ is a free parameter and C is a constant not depending on τ, it must be $w_{tt}\equiv\alpha_{tt}\equiv\beta_{tt}\equiv 0$ on the region $Q(\sigma)$. Since we have $Q(\sigma)\supset\Omega\times[t_0,t_1]$ where $t_0<0<t_1$, then using (4.46) we have

$$f^2+f_x^2+f_y^2 \leq C\left(|w_{tt}(x,y,0)|^2+|\alpha_{tt}(x,y,0)|^2+|\beta_{tt}(x,y,0)|^2\right)\equiv 0$$

implying $f\equiv 0$ for all $(x,y)\in\Omega$. Thus the proof of Theorem 4.2.3 is complete.

4.3.2 Proof of Theorem 4.2.4

The proof of Theorem 4.2.4 follows directly from the proof of Theorem 4.2.3 per relations shown in (4.4) between the inverse source problem and the original inverse problem.

Remark 4.3.2. *We remark that it is possible to achieve the regularity and positivity assumptions in* (4.30) *and* (4.31) *through putting appropriate assumptions on the initial and boundary conditions of the MT system (3). For example, if we assume* $(w_0, w_1), (\psi_0, \psi_1), (\phi_0, \psi_1) \in H^{l+1}(\Omega) \times H^l(\Omega))$ *and* $g_1, g_2, g_3 \in H^{l+1}(\Gamma \times [0, T])$*, for* $l > 3$*, then the solution* $\tilde{w}, \tilde{\psi}, \tilde{\phi}$ *would satisfy* $\tilde{w}_{tt}, \tilde{\psi}_{tt}, \tilde{\phi}_{tt} \in C([0, T]; H^{l-1}(\Omega))$ *and hence the required regularity follows from the embedding* $H^{l-1}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$.

Chapter 5

Conclusions

To summarize, the results presented within have, for the first time, shown the exact (interior) controllability for the Mindlin-Timoshenko system (2.3), and demonstrated the density coefficient to be uniquely recoverable. Both results required navigating the difficulty presented by the presence of coupling terms at the principal order within the system. These outcomes are as presented in Chapters 3 and 4 respectively.

5.1 Ongoing Research

The following is a cursory glance of continuing research by the author that directly stems from or is impacted by the work presented within this dissertation.

5.1.1 Inverse Problem

A matter of continuing research for the inverse problem is developing a diagonalization for the system that would enable recovery of both ρ and a variable h for the MT system. This means assuming the plate is of a non-constant thickness. Once found, the current set up with the single measurement on the boundary is enough to accommodate the recovery of another variable within the system.

5.1.2 Control Theory

Besides the current work with control theory, another problem to consider is the indirect control of a general system of coupled hyperbolic coefficients including first order terms of both components in both equations. This would be an improvement for what is currently known, but would require a new type of Carleman estimate. This estimate would need to have the norm on the larger side in a Sobolev space of negative order and include lower order terms on the smaller side in the instance of a wave equation whose principal operator acting on the component is in a negative Sobolev space, but whose solution is still in $H_0^1(Q)$. The difficulty here would be in deriving such an estimate.

Another possible problem arising from the current result would be to establish a pure boundary exact controllability result. The current result requires observation from a region close to the boundary, but within the domain. The idea then is that this result could be generalized to indirect exact boundary controllability of a coupled systems of hyperbolic equations, which includes some first order terms and does not place a smallness constraint on the zero order terms.

Appendices

Appendix A

Wave Equation with Constant Speed: Carleman type estimate

This part of the appendix provides the proof for Theorem 1.1.3. First we establish an initial pointwise inequality, restate the inequality after specializations that rely on the pseudo-convex function as defined in (1.4), followed by two corollaries that use the assumptions stated in the introduction to improve the RHS of the estimate so it is stated in terms of constants multiplied by lower order terms. This sets the stage for the Carleman estimate. Note that the derivation of boundary terms is included in the statement of the final result.

A.1 Initial Pointwise Inequality

We first derive the following pointwise inequality for the wave equation of fixed wave speed, which leads to an improved Carleman type estimate.

Lemma A.1.1. *Let $w(t,x) \in C^2(\mathbb{R}_t \times \mathbb{R}_x^2)$; $\ell(t,x) \in C^3(\mathbb{R}_t \times \mathbb{R}_x^2)$; $\zeta(t,x) \in C^2$ in t and C^1 in x be given. Then set $\theta(t,x) = e^{\ell(t,x)}$ so that θ is the exponential weight function. For further ease of notation, set*

$$v(t,x) = \theta(t,x)w(t,x) = e^{\ell(t,x)}w(t,x)$$

$$A = (\ell_t^2 - \ell_{tt}) - c\left(|\nabla \ell|^2 + \Delta \ell\right) - \zeta.$$

Then, letting $\epsilon > 0$ be arbitrary, we have the following pointwise inequality

$$\theta^2[w_{tt} - c\Delta w]^2 - \frac{\partial M}{\partial t} + \boldsymbol{div} V \geq -8cv_t \nabla \ell_t \cdot \nabla v + 2(c\Delta\ell + \ell_{tt} - \zeta)v_t^2 + 2c\left(\zeta - \frac{\epsilon}{2} - c\Delta\ell + \ell_{tt}\right)|\nabla v|^2 + 4c^2 \sum_{i,j=1}^{2} \ell_{x_i x_j} v_{x_i} v_{x_j} + \tilde{B}v^2 \tag{A.1}$$

where

$$\begin{aligned} M :=& \theta^2 \left\{-2\ell_t(w_t^2 + c|\nabla w|^2) + 4c\nabla \cdot \nabla w w_t + 2(-2\ell_t^2 + 2c|\nabla \ell|^2 = \zeta)w_t w \right. \\ & \left. + (-2A\ell_t - 2\ell_t^3 + 2c\ell_t|\nabla|^2 - \zeta_t)w^2\right\} \\ V :=& 2\theta^2 \left\{\ell_{x_i}(cw_t^2 - c^2|\nabla w|^2) - 2w_{x_i}(c\ell_t w_t - c^2\nabla \cdot \nabla w) \right. \\ & \left. + 2\left(c^2|\nabla\ell|^2 - c\ell_t^2 + c\frac{\zeta}{2}\right) w_{x_i} w + \ell_{x_i}(c^2|\nabla\ell|^2 - c\ell_t^2 - cA)w^2\right\} \\ \tilde{B} :=& \left\{2A\zeta - 2\left[\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(c(A+\zeta)\ell_{x_i}) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{c}{\epsilon}|\nabla\zeta|^2 + \zeta_{tt}\right\}. \end{aligned}$$

Proof. **Step 1**

Applying the specializations, $v(t,x) = \theta(t,x)w(t,x) = e^{\ell(t,x)}w(t,x)$ thus, we obtain $w(t,x) = e^{-\ell(t,x)}v(t,x)$. Now,by differentiation, notice

$$\begin{cases} \theta w_{tt} = v_{tt} - 2\ell_t v_t + (\ell_t^2 - \ell_{tt})v \\ \theta w_{x_i x_i} = v_{x_i x_i} + (\ell_{x_i}^2 - \ell_{x_i x_i})v \Rightarrow \theta c\Delta w = c\Delta v - 2c\nabla\ell \cdot \nabla v + c(|\nabla\ell|^2 - \Delta\ell)v. \end{cases} \tag{A.2}$$

Multiplying the principal part of the wave equation in (1.1) by the exponential weight, squaring and making the appropriate substitutions from (A.2) yields

$$\begin{aligned} \theta^2[w_{tt} - c\Delta w]^2 =& \left(v_{tt} - 2\ell_t v_t + (\ell_t^2 - \ell_{tt})v - c\Delta v + 2c\nabla\ell \cdot \nabla v - c(|\nabla\ell|^2 - \Delta\ell)v\right)^2 \\ =& |I_1 + I_2 + I_3|^2 \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} I_1 =& v_{tt} - c\Delta v + Av, \\ I_2 =& -2\ell_t v_t + 2c\nabla\ell \cdot \nabla v, \\ I_3 =& \zeta v, \\ A =& (\ell_t^2 - \ell_{tt}) - c\left|\nabla \ell\right|^2 - c\Delta\ell - \zeta. \end{aligned} \tag{A.4}$$

Remark A.1.2. *The reason for the introduction of the function ζ will become more obvious in later sections when specializations are made. A major part of the role of ζ is to guarantee certain geometrical conditions are met.*

Thus by a basic inequality, (A.3), and (A.4) we have the following estimate on the principal part of (1.1)

$$\theta^2 \left[w_{tt} - c\Delta w\right]^2 \geq 2(I_1 I_2 + I_2 I_3 + I_1 I_3). \tag{A.5}$$

The remainder of the proof will be devoted to refining this pointwise estimate by working with the terms on the RHS of (A.5).

Step 2

Dealing with each term on the RHS of (A.4), this step will prove the below equality regarding the first term:

$$\begin{aligned} 2I_1 I_2 =& \frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2 + Av^2 + c|\nabla v|^2\right) + 4cv_t \nabla\ell \cdot \nabla v\right] \\ & - 2\left\{\sum_{i=1}^{2} \frac{\partial}{\partial x_i}[2c^2 v_{x_i} \nabla\ell \cdot \nabla v - c^2 \ell_{x_i} |\nabla v|^2 - 2c\ell_t v_t v_{x_i} + c\ell_{x_j} v_t^2 - cA\ell_{x_i} v^2\right\} \\ & - 8cv_t \nabla\ell_t \cdot \nabla v_t + 2v_t^2(c\Delta\ell + \ell_{tt}) + 4c^2 \sum_{i,j=1}^{2} \ell_{x_i x_j} v_{x_i} v_{x_j} \\ & - 2(c^2\Delta\ell - c\ell_{tt})|\nabla v|^2 - 2\left[c\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(A\ell_{x_i}) - \frac{\partial}{\partial t}(A\ell_t)\right] v^2. \end{aligned} \tag{A.6}$$

Proof. By direct computation, and substitution from the expressions in (A.4), we have

$$\begin{aligned} 2I_1I_2 &= 2[v_{tt} - c\Delta v + Av][-2\ell_t v_t + 2c\nabla\ell\cdot\nabla v] \\ &= -2\ell_t\frac{\partial}{\partial t}(v_t^2) - 2A\ell_t\frac{\partial}{\partial t}(v^2) + 2cA\ell\cdot\nabla(v^2) \\ &\quad + \underbrace{4cv_{tt}\nabla\ell\cdot\nabla v}_{1} + \underbrace{4c\Delta v\ell_t v_t}_{2} - \underbrace{4c^2\Delta v\nabla\ell\cdot\nabla v}_{3}. \end{aligned} \tag{A.7}$$

We can next rewrite the last three terms of (A.7), in the order in which they are numbered, as follows:

1.

$$\begin{aligned} 4cv_{tt}\nabla\ell\cdot\nabla v &= 4c\frac{\partial}{\partial t}\left(v_t\nabla\ell\cdot\nabla v\right) - 4cv_t\nabla\ell_t\cdot\nabla v - 4cv_t\nabla\ell\cdot\nabla v_t \\ &= 4c\sum_{i=1}^{2}\frac{\partial}{\partial t}\left(\ell_{x_i}v_{x_i}v_t\right) - 4cv_t\nabla\ell_t\cdot\nabla v - 2c\nabla\ell\cdot\nabla\left(v_t^2\right) \end{aligned}$$

2.

$$4c\Delta v\ell_t v_t = 4c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(v_t\ell_t v_{x_i}\right) - 2c\ell_t\frac{\partial}{\partial t}\left(|\nabla v|^2\right) - 4cv_t\nabla\ell_t\cdot\nabla v$$

3.

$$\begin{aligned} -4c^2\Delta v\nabla\ell\cdot\nabla v &= -4c^2\sum_{i,j=1}^{2}v_{x_ix_i}\ell_{x_j}v_{x_j} \\ &= -4c^2\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(v_{x_i}\ell_{x_j}v_{x_j}\right) + 4c^2\sum_{i,j=1}^{2}\ell_{x_jx_i}v_{x_j}v_{x_i} + 2c^2\sum_{i,j=1}^{2}\ell_{x_j}\frac{\partial}{\partial x_j}(v_{x_i}^2). \end{aligned}$$

Making the appropriate substitutions of the expressions derived in 1, 2, and 3 into (A.7) we get

$$
\begin{aligned}
2I_1I_2 = &- 2\ell_t\frac{\partial}{\partial t}(v_t^2) - 2A\ell_t\frac{\partial}{\partial t}(v^2) + 4c\frac{\partial}{\partial t}\left(v_t\nabla\ell\cdot\nabla v\right) - 2c\ell_t\frac{\partial}{\partial t}\left(|\nabla v|^2\right) \\
&+ 2cA\sum_{i=1}^{2}\ell_{x_i}\frac{\partial}{\partial x_i}(v^2) - 2c\sum_{i=1}^{2}\ell_{x_i}\left(v_t^2\right) + 4c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(v_t\ell_t v_{x_i}\right) \\
&- 4c^2\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(v_{x_i}\ell_{x_j}v_{x_j}\right) + 2c^2\sum_{i,j=1}^{2}\ell_{x_j}\frac{\partial}{\partial x_j}(v_{x_i}^2) - 4cv_t\nabla\ell_t\cdot\nabla v \\
&- 4c^2v_t\nabla\ell_t\cdot\nabla v + 4c^2\sum_{i,j=1}^{2}\ell_{x_jx_i}v_{x_j}v_{x_i}.
\end{aligned}
$$

Rearranging and grouping certain partial differential terms yields the final result in (A.6). □

Step 3

Applying the substitutions in (A.4) to the third term of the RHS of (A.5) we shall prove for all $\epsilon > 0$

$$
\begin{aligned}
2I_1I_3 \geq &\frac{\partial}{\partial t}(2\zeta vv_t - \zeta_t v^2) + \left[\zeta_{tt} + 2A\zeta - \frac{c}{\epsilon}|\nabla\zeta|^2\right]v^2 - 2\zeta v_t^2 \\
&+ (2c\zeta - c\epsilon)|\nabla v|^2 - 2c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\zeta v_{x_i}v\right).
\end{aligned} \tag{A.8}
$$

Proof. Multiplying the substituted expression for I_1 and I_3 results in the following three term expression

$$
\begin{aligned}
2I_1I_3 =& 2[v_{tt} - c\Delta v + Av]\zeta v \\
=& \underbrace{2\zeta vv_{tt}}_{1} - \underbrace{2c\Delta v\zeta v}_{2} + 2A\zeta v^2.
\end{aligned} \tag{A.9}
$$

Taking the first two terms from the RHS of (A.9) and computing them separately gives the following simplification

1.

$$
\begin{aligned}
2\zeta vv_{tt} &= 2\frac{\partial}{\partial t}(\zeta vv_t) - 2v_t\frac{\partial}{\partial t}(\zeta v) \\
&= 2\frac{\partial}{\partial t}(\zeta vv_t) - 2v_t\zeta_t v - 2v_t^2\zeta \\
&= 2\frac{\partial}{\partial t}(\zeta vv_t) - \frac{\partial}{\partial t}(\zeta_t v^2) + \zeta_{tt}v^2 - 2v_t^2\zeta
\end{aligned}
$$

2.

$$
\begin{aligned}
-2\zeta\Delta v v &= -2c\sum_{i=1}^{2}\zeta v v_{x_i x_i} \\
&= -2c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\zeta v v_{x_i}\right) + 2cv\nabla\zeta\cdot\nabla v + 2c\zeta\left|\nabla v\right|^2.
\end{aligned}
$$

Substituting these derived expressions for the first two terms in (A.9) and using

$$
2cv\nabla\zeta\cdot\nabla v \geq -c\epsilon\left|\nabla v\right|^2 - \frac{c}{\epsilon}\left|\nabla\zeta\right|^2 v^2,
$$

where $\epsilon > 0$ is arbitrary and recalling $c > 0$, we have the result in (A.8). □

Step 4

In Step 4 we shall prove that

$$
2I_2I_3 = \frac{\partial}{\partial t}(-2\zeta\ell_t v^2) + \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(2c\zeta\ell_{x_i}v^2\right) + 2\left[\frac{\partial}{\partial t}(\ell_t\zeta) - c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\zeta\ell_{x_i})\right]v^2 \tag{A.10}
$$

Proof. By applying the definitions in (A.4) we have

$$
\begin{aligned}
2I_2I_3 &= 2\left(-2\ell_t v_t + 2c\nabla\ell\cdot\nabla v\right)\zeta v \\
&= -2\ell_t\zeta\frac{\partial}{\partial t}(v^2) + 2c\zeta\nabla\ell\cdot\nabla(v^2)
\end{aligned}
\tag{A.11}
$$

where the resulting two terms can be simplified as follows:

$$
-2\ell_t\zeta\frac{\partial}{\partial t}(v^2) = \frac{\partial}{\partial t}(-2\ell_t\zeta v^2) + \left[\frac{\partial}{\partial t}\left(2\ell_t\zeta\right)\right]v^2
$$

$$
\begin{aligned}
2c\zeta\nabla\ell\cdot\nabla(v^2) &= 2c\sum_{i=1}^{2}\zeta\ell_{x_i}\frac{\partial}{\partial x_i}(v^2) \\
&= 2c\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\zeta\ell_{x_i}v^2\right) - 2c\sum_{i=1}^{2}\left[\frac{\partial}{\partial x_i}\left(\zeta\ell_{x_i}\right)\right]v^2.
\end{aligned}
$$

Replacing these derived results for the two expressions in (A.11) gives the result in (A.10). □

Step 5

Substituting (A.6) for $2I_1I_2$, (A.8) for $2I_1I_3$, and (A.10) for $2I_2I_3$ into (A.5) and grouping certain terms we obtain the inequality

$$\begin{aligned}\theta^2[w_{tt}-c\Delta w]^2 \geq\ & \frac{\partial}{\partial t}\left\{-2\ell_t(v_t^2+c\,|\nabla v|^2)+4cv_t\nabla\ell\cdot\nabla v+2\zeta v_t v-2\ell_t(A+\zeta)v^2-\zeta_t v^2\right\} \\ & -2\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left[2c^2 v_{x_i}\nabla\ell\cdot\nabla v-c^2\ell_{x_i}|\nabla v|^2-2c\ell_t v_t v_{x_i}+c\ell_{x_i}v_t^2+c\zeta v_{x_i}v-c(A+\zeta)\ell_{x_i}v^2\right] \\ & -8cv_t\nabla\ell_t\cdot\nabla v+2(c\Delta\ell+\ell_{tt}-\zeta)v_t^2 \\ & +2\left(c\zeta-\frac{c\epsilon}{2}-c^2\Delta\ell+c\ell_{tt}\right)|\nabla v|^2+4c^2\sum_{i,j=1}^{2}\ell_{x_jx_i}v_{x_j}v_{x_i} \\ & +\left\{2A\zeta-2\left[\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(c(A+\zeta)\ell_{x_i}\right)-\frac{\partial}{\partial t}\left((A+\zeta)\ell_t\right)\right]-\frac{c}{\epsilon}|\nabla\zeta|^2+\zeta_{tt}\right\}v^2\end{aligned} \tag{A.12}$$

Step 6

We particularize (A.12) with

$$v=\theta w=e^{\ell}w \Rightarrow \begin{cases} v_t=\theta(w_t+\ell_t w),\ \ v_{x_i}=\theta(w_{x_i}+\ell_{x_i}w) \text{ for } i=1,2 \\ |\nabla v|^2=\theta^2\sum_{i=1}^{2}\left(w_{x_i}+\ell_{x_i}w\right)^2. \end{cases} \tag{A.13}$$

Thus, regarding the above specialization, the terms under $\frac{\partial}{\partial t}$ in (A.12) become

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{-2\ell_t(v_t^2+c\,|\nabla v|^2)+4cv_t\nabla\ell\cdot\nabla v+2\zeta v_t v-2\ell_t(A+\zeta)v^2-\zeta_t v^2\right\} \\ &\quad=\frac{\partial}{\partial t}\left\{\theta^2\left[-2\ell_t(w_t^2+c|\nabla w|^2)+4c\nabla\ell\cdot\nabla w w_t+2(2c|\nabla\ell|^2-2\ell_t^2+\zeta)w_t w\right.\right. \\ &\qquad\left.\left.+(2c\ell_t|\nabla\ell|^2-2\ell_t^3-2A\ell_t-\zeta_t)w^2\right]\right\}.\end{aligned} \tag{A.14}$$

Proof. Making the relevant substitutions we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2+c|\nabla v|^2\right)+4cv_t\nabla\ell\cdot\nabla v+2\zeta vv_t-2\ell_t(A+\zeta)v^2-\zeta_t v^2\right]\\
&=\frac{\partial}{\partial t}\left\{\theta^2\left[-2\ell_t\left((w_t+\ell_t w)^2+c\sum_{i=1}^{2}(w_{x_i}+\ell_{x_i}w)^2\right)+4c\sum_{i=1}^{2}\ell_{x_i}(w_{x_i}+\ell_{x_i}w)(w_t+\ell_t w)\right.\right.\\
&\qquad\left.\left.+2\zeta w(w_t+\ell_t w)-2\ell_t(A+\zeta)w^2-\zeta_t w^2\right]\right\}\\
&=\frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(w_t^2+2\ell_t ww_t+\ell_t^2w^2+c|\nabla w|^2+2cw\nabla w\cdot\nabla\ell+c|\nabla\ell|^2w^2)\right.\\
&\qquad+4c\nabla\ell\cdot\nabla w(w_t+\ell_t w)+4c|\nabla\ell|^2w(w_t+\ell_t w)+2\zeta ww_t+2\zeta\ell_t w^2\\
&\qquad\left.-2\ell_t Aw^2-2\ell_t\zeta w^2-\zeta_t w^2]\right\}\\
&=\frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(w_t^2+c|\nabla w|^2)-4w_t\ell_t^2w-2\ell_t^3w^2-4c\ell_t\nabla w\cdot\nabla\ell w-2\ell_t c|\nabla\ell|^2w^2\right.\\
&\qquad+4c\nabla\ell\cdot\nabla ww_t+4c\nabla\ell\cdot\nabla\ell_t w+4c|\nabla\ell|^2ww_t+4c|\nabla\ell|^2w^2\ell_t+2\zeta w_t w+2\zeta\ell_t w^2\\
&\qquad\left.-2\ell_t w^2A-2\ell_t\zeta w^2-\zeta_t w^2\right\}.
\end{aligned}
$$

Finally, simplifying and grouping terms yields the desired result in (A.14). □

Step 7

Under the same specializations in (A.13) the divergence terms in (A.12) becomes

$$
\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left[2c^2v_{x_i}\nabla\ell\cdot\nabla v-c^2\ell_{x_i}|\nabla v|^2-2c\ell_t v_t v_{x_i}+c\ell_{x_i}v_t^2+c\zeta v_{x_i}v-c(A+\zeta)\ell_{x_i}v^2\right]\\
&\qquad=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left\{\theta^2\left[2c^2w_{x_i}\nabla\ell\cdot\nabla w-c^2\ell_{x_i}|\nabla w|^2-2c\ell_t w_{x_i}w_t+c\ell_{x_i}w_t^2\right.\right.\\
&\qquad\qquad\left.\left.+2(c^2|\nabla\ell|^2-c\ell_t^2+c\tfrac{\zeta}{2})w_{x_i}w+\ell_{x_i}(c^2|\nabla\ell|^2-c\ell_t^2-cA)w^2\right]\right\}
\end{aligned}
\tag{A.15}
$$

Proof. Simplifying the first term on the LHS of (A.15), via (A.13) we have

$$\begin{aligned} 2c^2 v_{x_i} \nabla\ell \cdot \nabla v &= 2c^2 \left(w_{x_i} + \ell_{x_i} w\right) \nabla\ell \cdot \nabla v \\ &= 2c^2\theta \left(w_{x_i} + \ell_{x_i} w\right) \sum_{j=1}^{2} \ell_{x_j} v_{x_j} \\ &= 2c^2\theta \left(w_{x_i} + \ell_{x_i} w\right) \sum_{j=1}^{2} \ell_{x_j} \left(w_{x_j} + \ell_{x_j} w\right) \\ &= 2c^2\theta^2 \left(w_{x_i} + \ell_{x_i} w\right) \left(\nabla\ell \cdot \nabla w + |\nabla\ell|^2 w\right) \\ &= 2c^2\theta^2 \left[(w_{x_i} + \ell_{x_i} w)\nabla\ell \cdot \nabla w + \left(w_{x_i} + \ell_{x_i} w\right)|\nabla\ell|^2 w\right]. \end{aligned}$$

Replacing the first term with the above, and making the remaining substitutions from (A.13) into the divergence terms yields

$$\begin{aligned} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \Big\{\theta^2 \Big[&2c^2 w_{x_i} \nabla\ell \cdot \nabla w + 2c^2 \ell_{x_i} \nabla\ell \cdot \nabla w w + 2c^2 w_{x_i} |\nabla\ell|^2 w + 2c^2 \ell_{x_i} |\nabla\ell|^2 w^2 \\ &- c^2 \ell_{x_i} |\nabla w|^2 - 2c^2 \ell_{x_i} \nabla \cdot \nabla\ell w - c^2 \ell_{x_i} |\nabla\ell|^2 w^2 - 2c\ell_t w_{x_i} w_t - 2c\ell_t^2 w_{x_i} w \\ &- 2c\ell_t \ell_{x_i} w w_t - 2c\ell_t^2 \ell_{x_i} w^2 + c\ell_{x_i} w_t^2 + 2c\ell_{x_i} w_t \ell_t w + c\ell_{x_i} \ell_t^2 w^2 \\ &+ c\zeta w_{x_i} w + c\zeta \ell_{x_i} w^2 - cA\ell_{x_i} w^2 - c\zeta \ell_{x_i} w^2\Big]\Big\}. \end{aligned}$$

After cancelling and grouping terms, we have the result in (A.15). □

Step 8

Finally, inserting the expressions (A.14) and (A.15) into (A.12), and recalling how M, V, A, $\tilde{B}$ are defined in (A.1.1) we have

$$\begin{aligned} \theta^2[w_{tt} - c\Delta w]^2 - \frac{\partial M}{\partial t} + \mathbf{div} V \geq &- 8cv_t \nabla\ell_t \cdot \nabla v + 2(c\Delta\ell + \ell_{tt} - \zeta)v_t^2 \\ &+ 2c(\zeta - \frac{\epsilon}{2} - c\Delta\ell + \ell_{tt})|\nabla v|^2 + 4c^2 \sum_{i,j=1}^{2} \ell_{x_i x_j} v_{x_i} v_{x_j} + \tilde{B} v^2 \end{aligned}$$

which is the desired result and hence completes the proof to Lemma (A.1.1). □

A.2 Pointwise Inequality After Specializations

This section makes appropriate choices for the exponential weight function in the multiplier of Lemma A.1.1. An order of τ is also introduced, which is desirable as a certain large quantity. It is a restatement of Theorem 4.1 in [25] as it applies to a wave equation of constant wave speed not necessarily equal to one. Since there are no particular nuances to the case of constant speed that need be addressed, the theorem is presented here without proof.

Theorem A.2.1. *Let $w(t,x)$ be as defined in Lemma A.1.1, and consider functions $d(x) \in C^3(\mathbb{R}^2_x)$, $\alpha(x) \in C^1(\mathbb{R}^2_x)$ and parameter $\tau > 0$. Further consider the specializations*

$$\ell_t(t,x) \equiv \tau \left[d(x) - k\left(t - \frac{T}{2}\right)\right] \equiv \tau\varphi(t,x) \tag{A.16}$$

$$\zeta(x) \equiv \tau\alpha(x) \tag{A.17}$$

thus implying $\theta(t,x) = e^{\tau\varphi(t,x)}$. Moreover, under (A.16) *and* (A.17)*, the pointwise estimate in lemma* (A.1.1) *then becomes*

$$\theta^2[w_{tt} - c\Delta w]^2 - \frac{\partial M}{\partial t} + \boldsymbol{div}\, V \geq 2\tau(c\Delta d - 2k - \alpha)v_t^2 + 2c\tau(\alpha - \frac{\epsilon}{2\tau} - c\Delta d - 2k)|\nabla v|^2$$
$$+ 4\tau c^2 \sum_{i,j=1}^{2} d_{x_ix_j} v_{x_i} v_{x_j} + \tilde{B}v^2. \tag{A.18}$$

where M and V are as defined in Lemma A.1.1 except represented here via the specializations of (A.16) *and* (A.17)*. Similarly, A and $\tilde{B}$, as a result of* (A.16) *and* (A.17)*, become*

$$A = \tau^2 \left[4k^2\left(t - \frac{T}{2}\right)^2 - c|\nabla d|^2\right] + \tau[2k + c\Delta d - \alpha] \tag{A.19}$$

$$\tilde{B} = 2\tau^3 \left\{(2k + c\Delta d - \alpha)c|\nabla d|^2 + 2c^2\mathcal{H}_d \nabla d \cdot \nabla d - (6k + c\Delta d - \alpha)4k^2\left(t - \frac{T}{2}\right)^2\right\} + \mathcal{O}(\tau^2). \tag{A.20}$$

Remark A.2.2. *We differ from the notation in [25] in that the constant k introduced earlier is represented in [25] by c. The change is due to concerns of overloading notation since this symbol is assigned to be the coefficient on the Laplacian in* (1.1)*.*

Remark A.2.3. *The psuedo-convex function, φ need only satisfy the conditions specified in Section*

1.1 of the Introduction. The constant wave speed impacts the choice of the constant k (resp. c in [25]), but there will still exist such a value $k \in (0,1)$ so that φ satisfies all necessary properties to support Theorem A.2.1.

Two important corollaries follow Theorem A.2.1, labeled Corollary 4.2 and Corollary 4.3 in [25]. To establish these corollaries for the wave equation of constant speed, the constraints on the constants must differ slightly from the original proof, $\rho > 0$, $\tilde{\beta} > 0$, $\rho_0 > 0$ as follows:

$$c\Delta d - 2k - \alpha \geq \rho;\ \forall x \in \bar{\Omega} \tag{H.1}$$

$$\begin{bmatrix} d_{x_1x_1} + \gamma & d_{x_1x_2} \\ d_{x_2x_1} & d_{x_2x_2} + \gamma \end{bmatrix} \geq \rho I \tag{H.2}$$

$$(2k+c\Delta d-\alpha)c|\nabla d|^2+2c^2\mathcal{H}_d\nabla d\cdot\nabla d-(6k+c\Delta d-\alpha)4k^2\left(t-\frac{T}{2}\right)^2 \geq \tilde{\beta} > 0,\ \forall (t,x) \in Q^*(\sigma^*), \tag{H.3}$$

where $d(x) \in C^3(\mathbb{R}^2_x)$, and $\alpha \in C^1(\mathbb{R}^2_x)$ and the function $\gamma(x)$ is defined as $\gamma(x) = \alpha(x) - c\Delta d(x) - 2k$.

Recalling (A.20) we then have from (H.3) the estimate

$$\tilde{B}v^2 \geq [2\tau^3\tilde{\beta} + \mathcal{O}(\tau^2)]v^2, \qquad \forall (t,x) \in Q^*(\sigma^*). \tag{A.21}$$

Moreover, if the assumptions (A) and (B) (which differs from (B) in [25]) hold, we have the above satisfied where $\rho = 2\rho_0 + \gamma$. Hence it is easily verified that $\rho > 0$ resulting in the following estimates:

$$2\tau[c\Delta d - 2k - \alpha]v_t^2 \geq 2\rho\tau v_t^2 \tag{A.22}$$

$$\begin{aligned} 2\tau c\left(\alpha - \frac{\epsilon}{2\tau} - c\Delta d - 2k\right)|\nabla v|^2 + 2\tau c(\nabla v)^T 2c\mathcal{H}_d \cdot \nabla v \geq& 2\tau c\left[\gamma + 2\rho_0\right]|\nabla v|^2 \\ =& 2\tau c\left[\gamma + (\rho - \gamma)\right]|\nabla v|^2 \\ =& 2c\tau\rho|\nabla v|^2. \end{aligned} \tag{A.23}$$

From (A.22) and (A.23) we immediately obtain the following corollary (the analog of Corollary 4.2 in [25]):

Corollary A.2.3.1. *Suppose we have a strictly convex function* $d(x) \in (\mathbb{R}_x^2)$ *satisfying the assumptions (A.1) and (A.2), with* $k \in (0,1)$ *as specified 1.1.8b-d of [25] (satisfying* (1.6)*), then (H.1), (H.2), and (H.3) hold where* $\gamma(x) = \alpha(x) - c\Delta d(x) - 2k$ *and* $\rho = 2\rho_0 + \gamma$ *yielding estimates* (A.22) *and* (A.23) *so that Theorem A.2.1 becomes*

$$\theta^2(w_{tt} - c\Delta w)^2 - \frac{\partial M}{\partial t} + \boldsymbol{div} V \geq 2\tau\rho[v_t^2 + c|\nabla v|^2] + \tilde{B}v^2, \qquad t \in [0,T],\ x \in \bar{\Omega} \tag{A.24}$$

assuming $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x^2)$.

Corollary A.2.3.2. *Given the same conditions as presented in Corollary A.2.3.1, then via the specializations in Theorem A.2.1 and recalling* $v \equiv \theta w$*, we have, for any* $0 < \epsilon < 1$*,*

$$\theta^2(w_{tt} - c\Delta w)^2 - \frac{\partial M}{\partial t} + \boldsymbol{div} V \geq \epsilon\tau\rho\theta^2[w_t^2 + c|\nabla w|^2] + \theta^2 B w^2, \qquad t \in [0,T], x \in \bar{\Omega} \tag{A.25}$$

where

$$B \equiv \tilde{B} - 2\epsilon\tau^3\rho\left[\varphi_t^2 + c|\nabla d|^2\right]v^2 \geq \tilde{B} - 2\epsilon\rho\tau^3 \max_{(t,x)\in\bar{Q}}\left\{\varphi_t^2 + c|\nabla d|^2\right\} \tag{A.26}$$

recalling $\nabla\varphi(t,x) = \nabla d(x)$ *for all values of* t *by construction.*

Proof. The proof of Corollary A.2.3.2 follows the proof of Corollary 4.3 in [25] almost exactly with the exception of the constant, c, preceding the $|\nabla v|^2$ term. This is easily verified by the reader. □

There is a small step left to prove Theorem 1.1.3 from Corollary A.2.3.2 and for this we refer the reader to [25, p. 34].

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