August 2020

A Study of Quasi-Birth-Death Processes and Markovian Bitcoin Models

Kayla Doris Javier
Clemson University, kjavier@clemson.edu

A Study of Quasi-Birth-Death Processes and Markovian Bitcoin Models

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical and Statistical Sciences

by
Kayla Javier
August 2020

Accepted by:
Dr. Brian Fralix, Committee Chair
Dr. Jeff Kharoufeh
Dr. Peter Kiessler
Dr. Xin Liu

Abstract

In this dissertation we study a variety of continuous-time Markov chains (CTMCs) and present new formulas that can be used to find the stationary distribution and the Laplace transforms of the transition functions. Our first set of results involve a level-dependent Quasi-Birth-Death (QBD) processes. We study the distribution of the state and the associated running maximum level at a fixed time t. We present new expressions for the Laplace transforms of the transition functions containing this information. This work involves making use of a collection of $\mathbf{R}$-matrices often found in matrix analytic literature. We also show how our methods can be used to study the joint distribution of the running minimum level and state of a level-dependent Markov process of $M/G/1$-type. Our next set of results are based on a homogeneous QBD proccess. These results involve first computing a new class of $\mathbf{R}$ and $\mathbf{G}$-matrices that can be used to find the Laplace transforms of the transition functions associated with a homogeneous QBD process with finitely many levels. Our final set of results are based on two CTMCs studied in Göbel et al. [16], which were created to model the interactions between a small pool of miners and a larger collection of miners within the Bitcoin network. We use the random-product technique, introduced by Buckingham and Fralix [5], to find the stationary distribution of this model when all miners are honest and when the small pool of miners implement the Selfish Mining strategy introduced by Eyal and Sirer in [8]. We also study the Laplace transforms of the transition functions associated with these CTMCs and other performance measures such as the expected time it takes for a "fork" in the blockchain to be resolved.

Acknowledgments

I am tremendously grateful to my dissertation advisor, Dr. Brian Fralix, without whom this work would not have been possible. I thank him for helping me navigate graduate school and the research process. I would also like to thank the other members of my dissertation committee, Drs. Jeff Kharoufeh, Peter Kiessler, and Xin Liu for their advice and support throughout my dissertation.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The focus of this dissertation is to study a collection of stochastic models. Many structures we encounter in our daily lives can naturally be modeled using stochastic processes. Classical examples include the number of customers in a bank waiting to be served, the value of a stock, and the amount of inventory held at a warehouse. Continuous-time Markov chains (CTMCs) are often used to model random phenomena in a way that can be studied mathematically. Applications of CTMCs extend to many fields such as finance, biology, computer science, operations research, and statistics. In this dissertation we study a variety of CTMCs using methods that fall under the matrix analytic umbrella, which will be formally introduced later in this chapter.

The rest of this introduction is laid out as follows: in Section 1.1 we give an introduction to CTMCs and describe some of their properties. We also define some important quantities that will be studied throughout this dissertation. Furthermore, we introduce a class of CTMCs referred to as block structured processes. In Section 1.2 we discuss the transient behavior of a CTMC as well as its stationary distribution, when it exists. These two topics are covered extensively in each of the following three chapters. Section 1.3 discusses the two main tools we use in our derivations: the random product technique and a collection of **R**-matrices and **G**-matrices. These matrices will be familiar to those who work in matrix analytic methods. Lastly, Section 1.4 gives a brief summary of the rest of this dissertation.

1.1 Continuous-time Markov Chains

A stochastic process $\{X(t); t \geq 0\}$, having countable state space S is said to be a CTMC if it satisfies the Markov property. Namely, for all real numbers $s, t \geq 0$ and all states $x, y \in S$

$$\mathbb{P}(X(s+t) = y \mid X(s) = x, X(u), 0 \leq u \leq s) = \mathbb{P}(X(s+t) = y \mid X(s) = x),$$

where $\{X(u) : 0 \leq u \leq s\}$ denotes the history of $\{X(t); t \geq 0\}$ up to time s. Associated with $\{X(t); t \geq 0\}$ is its transition rate matrix, or generator, $\mathbf{Q} := [q(x,y)]_{x,y \in S}$ such that for each $x \neq y \in S$, $q(x,y)$ is the rate that X makes a transition from state x to state y. Furthermore, for each state $x \in S, q(x,x) := -\sum_{y \neq x} q(x,y)$. For ease of notation we define $q(x) := -q(x,x)$, which is the rate corresponding to each exponential sojourn in state x.

Further associated with X is the set of transition times $\{T_n\}_{n \geq 0}$ where T_n represents the nth transition time of X and $T_0 := 0$. We often denote $X(T_n)$ as X_n and the process $\{X_n\}_{n \geq 0}$ is called the embedded discrete-time Markov chain (DTMC) associated with $\{X(t); t \geq 0\}$. There is also a collection of hitting-time random variables linked to X, where for set $T \subset S$ we define τ_T as

$$\tau_T := \inf\{t > 0 : X(t-) \neq X(t) \in T\},$$

which represents the first time X makes a transition into the set T. We also define hitting times for the embedded DTMC where for each $T \subset S$, we define η_T as

$$\eta_T := \inf\{n \geq 1 : X_n \in T\}.$$

If T is a singleton, namely if $T = \{x\}, x \in S$ we often denote the hitting times $\tau_{\{x\}}$ and $\eta_{\{x\}}$ as τ_x and η_x instead, to simplify notation.

An important property of τ_T and η_T is that they are stopping times, allowing us to use the strong Markov property. A random variable η is a stopping time with respect to $\{X_n\}_{n \geq 0}$, if, for each integer $n \geq 0$, there exists a function $g_n : \mathbf{R}^{n+1} \to \{0,1\}$ satisfying

$$\mathbf{1}(\eta = n) = g_n(X_0, X_1, \ldots, X_n).$$

In words, this means that the indicator $\mathbf{1}(\eta = n)$ can be expressed a function of $X_0, X_1, \ldots, X_n$.

In the continuous-time setting, a random variable τ is a stopping time with respect to the process $\{X(t); t \geq 0\}$ if for each $t \geq 0$, the set $\{\tau \leq t\}$ depends only on the collection of random variables $\{X(s); 0 \leq s \leq t\}$. The strong Markov property, which is often used in the proofs of this dissertation generalizes the Markov property in the following way: if τ is a stopping time with respect to $\{X(t); t \geq 0\}$, and $X(\tau) = x$ for some $x \in S$, then for each real number $s > 0$ and each state $y \in S$,

$$\mathbb{P}(X(\tau + s) = y \mid \mathcal{F}_\tau) = \mathbb{P}(X(s) = y \mid X(0) = x),$$

where $\mathcal{F}_\tau$ denotes the history of $\{X(t); t \geq 0\}$ up to the time τ.

We now introduce a special class of CTMCs: block structured Markov processes. Consider the CTMC $\{Y(t); t \geq 0\}$ whose state space is given by S, which is partitioned into levels as

$$S = \bigcup_{n \geq 0} L_n$$

where for each integer $n \geq 0$,

$$L_n := \{(n, 1), (n, 2), \dots (n, d_n)\}$$

for $d_n < \infty$, which is allowed to vary with n. Ordering the states lexicographically, the generator of $\{Y(t); t \geq 0\}$ is given by

$$\mathbf{Q} := \begin{bmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & \mathbf{A}_{1,4} & \cdots \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & \mathbf{A}_{1,4} & \cdots \\ \mathbf{A}_{2,0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{A}_{2,4} & \cdots \\ \mathbf{A}_{3,0} & \mathbf{A}_{3,1} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Each matrix $\mathbf{A}_{i,j}$ is a $d_i \times d_j$ matrix containing the rates of moving from a state in level L_i to a state in level L_j. The process $\{Y(t); t \geq 0\}$ is referred to as a block structured Markov process due to the nature of its generator. The second and third chapters of this dissertation focus primarily on Quasi-Birth-Death (QBD) processes, which is a special type of block structured Markov process where $\mathbf{A}_{i,j} = \mathbf{0}$ if $|i - j| > 1$. Markov processes of M/G/1 type, which is a special case of the block structure Markov process where $\mathbf{A}_{i,j} = \mathbf{0}$ if $j < i-1$, will also be studied in Chapter 2. QBD processes and their variants have many applications as they allow us to move beyond exponential distributions while still applying Markovian reasoning.

1.2 Transient behavior and stationary distributions

Throughout this dissertation we are interested in studying the transient, or time-dependent behavior of certain CTMCs. Namely, we are interested in devising ways to calculate $\mathbb{P}(X(t) = y \mid X(0) = x)$ for any $x, y \in S$ and any $t \geq 0$. These transition probabilities are often instead denoted using the notation $p_{x,y}(t)$. These transition probabilities are usually very difficult to obtain so instead we focus on studying their Laplace transforms: for each $x, y \in S$, the Laplace transform $\pi_{x,y}$ of $p_{x,y}$ is defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$, the set of all complex numbers having positive real part, as

$$\pi_{x,y}(\alpha) := \int_0^\infty e^{-\alpha t} p_{x,y}(t) dt.$$

Once these quantities are known, the transition probabilities can be computed using numerical inversion techniques. Abate and Whitt give an algorithm that outlines this process in [1].

Another quantity of interest is the stationary distribution, provided it exists. The stationary distribution, which describes the long-run behavior of X, is denoted by $\mathbf{p} = [p_y]_{y \in S}$ where

$$p_y := \lim_{t \to \infty} \mathbb{P}(X(t) = y \mid X(0) = x), \quad x \in S.$$

The stationary distribution is known to exist whenever $\{X(t); t \geq 0\}$ is ergodic, i.e. irreducible and positive recurrent. A CTMC is said to be irreducible if its embedded DTMC is irreducible. The DTMC $\{X_n\}_{n \geq 0}$ is said to be irreducible if for each $x, y \in S$, $\mathbb{P}(X_n = y \mid X_0 = x) > 0$ and $\mathbb{P}(X_m = x \mid X_0 = y) > 0$ for some n and m. We say that $\{X(t); t \geq 0\}$ is positive recurrent if for every state $x \in S$, $\mathbb{E}_x[\tau_x] < \infty$. Here, the notation $\mathbb{E}_x[\cdot]$ is shorthand for $\mathbb{E}[\cdot \mid X(0) = x]$. Similarly, we often denote the conditional probability $\mathbb{P}(\cdot \mid X(0) = x)$ as $\mathbb{P}_x(\cdot)$.

1.3 Random product method and matrix analytic methods

There are a few methods that we will use time and time again in our proofs. The random product method, introduced by Buckingham and Fralix in [5] is used briefly in the second and third chapters but is featured heavily in the fourth chapter. We will briefly describe the random product technique here. Given a CTMC $\{X(t); t \geq 0\}$ with state space S and generator $\mathbf{Q}$, we construct another CTMC $\{\tilde{X}(t); t \geq 0\}$ whose state space is also S. Its generator $\tilde{\mathbf{Q}}$ must satisfy two properties: (i) for each pair of distinct states $x, y \in S$,

$$\tilde{q}(x,y) > 0 \text{ if and only if } q(y,x) > 0,$$

and (ii) for each state $x \in S$

$$\tilde{q}(x) = q(x).$$

Associated with $\{\tilde{X}(t); t \geq 0\}$ is its collection of transition times $\{\tilde{T}_n\}_{n\geq 0}$ and hitting times $\{\tilde{\eta}_x\}_{x\in S}$ and $\{\tilde{\tau}_x\}_{x\in S}$. The random product technique gives the following: suppose X is an ergodic CTMC and fix a state $x \in S$. Its stationary distribution $\mathbf{p}$ satisfies for each state $y \neq x$

$$p_y = p_x \mathbb{E}_y \left[\mathbf{1}(\tilde{\eta}_x < \infty) \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right].$$

The random product technique also states if X is a CTMC with $X(0) = x$ for some $x \in S$, then the Laplace transform $\pi_{x,y}$ satisfies, for each $y \neq x$

$$\pi_{x,y}(\alpha) = \pi_{x,x}(\alpha) \mathbb{E}_y \left[\mathbf{1}(\tilde{\eta}_x < \infty) e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right].$$

These two equations give us a way to calculate the stationary distribution and the Laplace transforms of the transition functions outside of the usual approach of using the balance equations and the forward equations respectively.

Another method used throughout this dissertation involves applying the following identity: suppose $\{X(t); t \geq 0\}$ is a continuous-time Markov chain with state space S. For a non-empty set

$T \subset S$ with $T \neq S$, for each state $x \in T^c$ and each $y \in T$

$$\pi_{x,y}(\alpha) = \sum_{z \in T^c} \pi_{x,z}(\alpha)(q(z) + \alpha)\mathbb{E}_z \left[\int_0^{\tau_{T^c}} e^{-\alpha t} \mathbf{1}(Y(t) = y)) dt \right].$$

While this result is not new, we prove this formally in Chapter 2. This identity gives us another way to calculate the Laplace transforms.

Additionally, throughout the second and third chapters we use a collection of **R** and **G**-matrices that are often used in matrix analytic methods. Theses matrices can be used as a tool to help calculate the Laplace transforms of the transition functions. The elements of the **R**-matrices are made up of expectations. We avoid providing a general definition of these **R**-matrices, as we will use many different types of **R**-matrices throughout this dissertation. Closely related to the **R**-matrices are **G**-matrices, which will contain the the probabilities of a QBD process moving down (or up) a certain number of levels. These **G**-matrices matrices will be helpful when deriving the **R**-matrices. Chapters 2 and 3 will heavily feature these matrices.

1.4 Summary

The rest of this dissertation is organized as follows. Chapter 2 contains a study of the joint distribution of the state of a level-dependent Quasi-Birth-Death process, as well as its running maximum level, at a fixed time t. We derive expressions for the Laplace transforms of the transition functions that contain this information. Additionally, we derive expressions for the Laplace transforms of the transition functions that contain the state as well the running minimum of a level-dependent Markov process of M/G/1 type. The contents of Chapter 2 can be found in [22], which has been submitted for publication.

Chapter 3 contains a study of level independent Quasi-Birth-Death processes. We provide a new study of the time-dependent behavior of a QBD processes that has two boundary levels. Through completely probabilistic methods, we study the distribution of the amount of time it takes such a QBD process to move from one level to another level. We also show how the Laplace transforms of the transition functions of such a QBD process can be expressed in terms of simpler **R**-matrices that appear in the Laplace transforms of the transition functions of two different, but related, QBD processes having infinitely many levels. The contents of this chapter can be found in

[23], which has been submitted for publication.

Chapter 4 contains a study of two different continuous-time Markov chain models recently studied in Göbel et al. [16], which were created to model the interactions between a small pool of miners, and a larger collection of miners, within the Bitcoin network. The first model we discuss represents the case where all miners behave honestly and follow the Bitcoin protocol, while the second model represents the case where the smaller pool of miners use the Selfish Mining strategy of Eyal and Sirer [8]. We give a new derivation of the stationary distribution of the process in the honest mining case and further build on the results of Göbel et al. by showing that the normalizing constant can be expressed in closed-form. We also use similar techniques to derive expressions for the Laplace transforms of the transition functions. We then illustrate how these techniques yield expressions for the stationary distribution and the Laplace transforms of the transition functions of the process when the smaller pool implements Selfish Mining. This chapter was recently accepted by Stochastic Models and can be found in [21].

Chapter 2

Level-dependent Quasi-Birth-Death processes

2.1 Introduction and Preliminary Results

Given a real-valued stochastic process $\{X(t); t \geq 0\}$, we can define both the *running maximum* process $\{\overline{X}(t); t \geq 0\}$ and the *running minimum* process $\{\underline{X}(t); t \geq 0\}$, where for each $t \geq 0$,

$$\overline{X}(t) := \sup_{s \in [0,t]} X(s), \qquad \underline{X}(t) := \inf_{s \in [0,t]} X(s).$$

The marginal distributions of these processes are very tractable when $\{X(t); t \geq 0\}$ represents Brownian motion, and they are also well-known to play a prominent role in the theory of Lévy processes: readers seeking an introduction to Lévy processes are referred to Kyprianou [31].

In the recent work of Mandjes and Taylor [37], the authors present a recursive procedure that can be used to calculate the joint distribution of both the state (which tracks level and phase) of a level-dependent Quasi-Birth-Death (QBD) process and its running maximum level, at an independent exponential time: once these distributions can be calculated efficiently, Erlangization can be used to further study, numerically, the joint distribution of the running maximum level, the level, and the phase at each fixed time t. The results contained in [37] were derived 'from scratch' by making clever use of first-step analysis and censoring arguments, as well as sample-path properties satisfied by level-dependent QBD processes. Our objective is to build on the work of [37] by showing how alternative formulas can be derived in an arguably more straightforward manner from theory that has been developed in the matrix-analytic literature. In fact, not only will we analyze level-dependent QBD processes, we will also explain how our results and ideas apply to level-dependent Markov processes of M/G/1-type, assuming of course that we replace the running maximum level process with a running minimum level process.

An important ingredient needed in our analysis is a formula that can be found at the top of page 124 of Latouche and Ramaswami [32], which we now describe in reasonable detail. Suppose $\{Y(t); t \geq 0\}$ is a CTMC having state space S and generator (transition rate matrix) $\mathbf{Q} := [q(x,y)]_{x,y \in S}$, where for each $x \in S$,

$$q(x) := -q(x,x) \geq 0$$

denotes the sojourn rate associated with each exponential sojourn spent in state x by $\{Y(t); t \geq 0\}$. We assume throughout that each CTMC we study satisfies the property that $q(x) < \infty$ for each $x \in S$.

Further associated with $\{Y(t); t \geq 0\}$ is a collection of transition functions $\{p_{x,y}\}_{x,y \in S}$, where for each $x, y \in S$,

$$p_{x,y}(t) := \mathbb{P}_x(Y(t) = y), \qquad t \geq 0$$

where $\mathbb{P}_x(\cdot)$ represents a conditional probability, given $Y(0) = x$. Each transition function $p_{x,y}$ has associated with it a Laplace transform $\pi_{x,y} : \mathbb{C}_+ \to \mathbb{C}$, which is defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$—the set of all complex numbers having positive real part—as

$$\pi_{x,y}(\alpha) := \int_0^\infty e^{-\alpha t} p_{x,y}(t) dt, \qquad \alpha \in \mathbb{C}_+.$$

Readers should recall that two continuous functions defined on $[0, \infty)$ are equal if and only if their Laplace transforms are equal on $\mathbb{C}_+$ (in fact the functions are equal if and only if their Laplace transforms are equal on $[0, \infty)$) and once we can numerically calculate a Laplace transform at each point in $\mathbb{C}_+$, we can use one of many numerical transform inversion algorithms, such as that found in [1], to calculate the underlying continuous function at various points of $[0, \infty)$.

For each subset $T \subset S$, we define

$$\tau_T := \inf\{t \geq 0 : Y(t-) \neq Y(t) \in T\}$$

which represents the first time $\{Y(t); t \geq 0\}$ makes a transition to a state contained in T. Readers should note that $\tau_T > 0$ with probability one, even if $X(0) \in T$, as τ_T represents the first time the chain makes a transition to a state in T, which could have been made from a state $x \in T$ if $X(0) = x$.

Theorem 2.1.1 *(page 124 of Latouche and Ramaswami) Suppose T is a nonempty subset of S, where $T \neq S$. Then for each $x \in T^c$, and each $y \in T$,*

$$p_{x,y}(t) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t p_{x,z}(s) q(z,w) \mathbb{P}_w(Y(t-s) = y, \tau_{T^c} > t - s) ds. \tag{2.1}$$

While this result is obviously known, in [32] the formula appears to be given only with the intention of using it as a tool for deriving the stationary distribution of QBD processes, but we feel that this result deserves its own theorem. The authors of [32] appear to establish the result with a Markov renewal argument, but here is an alternative argument that follows from ideas found in [12].

Proof One way to derive Theorem 2.1.1 involves using the framework from Chapter 9 of Brémaud [4], where a CTMC is thought of as being governed entirely by a countable collection of independent, homogeneous Poisson processes.

Here is a rough sketch of the construction: for each $x, y \in S$ where $x \neq y$, we construct a Poisson process $\{N_{x,y}(t); t \geq 0\}$ with rate $q(x,y)$. Setting now $Y(0) = y_0$—an arbitrarily chosen state—we define the first transition time T_1 of $\{Y(t); t \geq 0\}$ as

$$T_1 := \inf_{y \in S} \inf\{t \geq 0 : N_{y_0,y}(t) = 1\}$$

and we set $Y(t) = y_0$ for $0 \leq t < T_1$, with $Y(T_1) = y_1$ for that state y_1 that attains the infimum (such a state exists with probability one). Next, given $y_1 = Y(T_1)$, set

$$T_2 := \inf_{y \in S} \inf\{t \geq 0 : N_{y_1,y}(t + T_1) - N_{y_1,y}(T_1) = 1\}$$

and again, define $Y(t) = y_1$ for $T_1 \leq t < T_2$, and set $Y(T_2) = y_2$ where y_2 is the state that attains the infimum. From here, one can define $\{Y(t); t \geq 0\}$ inductively over the entire line: note that it is possible for $\{Y(t); t \geq 0\}$ to have infinitely many transitions in a finite time interval, meaning

$$T_\infty := \lim_{n \to \infty} T_n < \infty$$

and in this case we construct an extra 'cemetery state' ∂, and assume the process stays at this cemetery state from the explosion time onward. Readers should find it clear, at least on an intuitive level, that $\{Y(t); t \geq 0\}$ is a CTMC with transition rate matrix $\mathbf{Q}$, but we refer those interested in seeing a rigorous description of this procedure to Chapter 9, Sections 1 and 2 of [4].

Thinking of $\{Y(t); t \geq 0\}$ in this manner, we can observe that for each $x \in T^c$ and each $y \in T$, if $Y(0) = x$ we have

$$\mathbf{1}(Y(t) = y) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t \mathbf{1}(Y(s-) = z, \tau_{T^c}(s) > t, Y(t) = y) N_{z,w}(ds)$$

where $Y(s-)$ is the left-hand-limit of Y at s, and for each $C \subset S$,

$$\tau_C(s) := \inf\{t \geq s : Y(t-) \neq Y(t) \in C\}.$$

Taking the expectation of both sides, while further applying the Campbell-Mecke formula to the right-hand-side, as is done in [12], gives

$$\mathbb{P}_x(Y(t) = y) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t \mathbb{P}_x(Y(s) = z) q(z, w) \mathbb{P}_w(\tau_{T^c} > t - s, Y(t - s) = y) ds$$

which proves the claim. $\diamondsuit$

Remark It is also possible to establish Theorem 2.1.1 via the *random-product technique*. Even though the random-product technique requires less of a technical background in measure-theoretic probability, when using this technique one has to specially treat both absorbing states, as well as states that cannot be reached from any state (meaning the only way the CTMC can visit this state is if it starts there). Such states may appear in a few places of our analysis, so we decided to motivate Theorem 2.1.1 with the line of reasoning given in [12], which uses the point process framework of [4].

The next result is a corollary of Theorem 2.1.1.

Corollary 2.1.1 *Fix a nonempty subset $T \subset S$ where $T \neq S$. Then for each $x \in T^c$, and each $y \in T$,*

$$\pi_{x,y}(\alpha) = \sum_{z \in T^c} \pi_{x,z}(\alpha)(q(z) + \alpha)\mathbb{E}_z\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Y(t) = y)dt\right], \qquad \alpha \in \mathbb{C}_+. \tag{2.2}$$

Proof This result follows from Theorem 2.1.1: simply multiply both sides of (2.1) by $e^{-\alpha t}$, integrate with respect to t over $[0, \infty)$, and apply Fubini's Theorem. $\diamondsuit$

Equation (2.2) can alternatively be stated as

$$\pi_{x,y}(\alpha) = \sum_{z \in T^c} \pi_{x,z}(\alpha) \sum_{w \in T} q(z, w)\mathbb{E}_w\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Y(t) = y)dt\right], \qquad \alpha \in \mathbb{C}_+. \tag{2.3}$$

We will often find it useful to state Equation (2.2) in this manner.

2.2 Level-Dependent QBD Processes

Suppose $\{Y(t); t \geq 0\}$ is a level-dependent QBD process, whose state space S is expressed in terms of a countable union of *levels*:

$$S := \bigcup_{n=0}^{\infty} L_n$$

where, for each integer $n \geq 0$,

$$L_n := \{(n, 1), (n, 2), \ldots, (n, d_n - 1), (n, d_n)\}$$

with d_n being a fixed positive finite integer that is allowed to vary with n. Given the structure of S, it helps, for each $t \geq 0$, to express $Y(t)$ as

$$Y(t) = (X(t), J(t))$$

for each real $t \geq 0$, where $X(t)$ denotes the current level of the process—meaning $X(t) = n$ if and only if $Y(t) \in L_n$—and $J(t)$ represents the current *phase* of the process. We follow the notation scheme from [37] by letting $\mathbf{Q}$ denote the transition rate matrix of $\{Y(t); t \geq 0\}$, where the rows and columns of $\mathbf{Q}$ are ordered in a manner that corresponds to S being ordered lexicographically, so that

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}^{(0)} & \Lambda^{(0)} & \mathbf{0}_{d_0 \times d_2} & \mathbf{0}_{d_0 \times d_3} & \mathbf{0}_{d_0 \times d_4} & \cdots \\ \mathcal{M}^{(1)} & \mathbf{Q}^{(1)} & \Lambda^{(1)} & \mathbf{0}_{d_1 \times d_3} & \mathbf{0}_{d_1 \times d_4} & \cdots \\ \mathbf{0}_{d_2 \times d_0} & \mathcal{M}^{(2)} & \mathbf{Q}^{(2)} & \Lambda^{(2)} & \mathbf{0}_{d_2 \times d_4} & \cdots \\ \mathbf{0}_{d_3 \times d_0} & \mathbf{0}_{d_3 \times d_1} & \mathcal{M}^{(3)} & \mathbf{Q}^{(3)} & \Lambda^{(3)} & \ddots \\ \mathbf{0}_{d_4 \times d_0} & \mathbf{0}_{d_4 \times d_1} & \mathbf{0}_{d_4 \times d_2} & \mathcal{M}^{(4)} & \mathbf{Q}^{(4)} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

where $\mathbf{0}_{m \times n}$ represents the zero matrix with m rows and n columns.

From this description of $\mathbf{Q}$, we can see that the dimensions of $\mathbf{Q}^{(0)}$ and $\Lambda^{(0)}$ are $d_0 \times d_0$ and $d_0 \times d_1$, respectively, while for each integer $n \geq 1$, the dimensions of $\mathcal{M}^{(n)}$, $\mathbf{Q}^{(n)}$, and $\Lambda^{(n)}$ are $d_n \times d_{n-1}$, $d_n \times d_n$, and $d_n \times d_{n+1}$, respectively. Each matrix $\Lambda^{(n)}$ contains transition rates corresponding to transitions made from a state in L_n to a state in L_{n+1}, while each matrix $\mathcal{M}^{(n)}$ contains transition rates corresponding to transitions made from a state in L_n to a state in L_{n-1}. In the interest of avoiding 'nuisance states', we assume throughout that each state $x \in S$ satisfies the following condition: there exist two states $y, z \in S$ (which may depend on x) such that $q(x, y) > 0$ and $q(z, x) > 0$. This is a much more general condition than irreducibility, as we are assuming that $\{Y(t); t \geq 0\}$ has no absorbing states, nor are there states that cannot be reached in one step from any other state in S. This simple assumption will allow us to apply the random-product technique featured in [5, 9, 11] without further comment. Readers should note that in [37], the authors assume the structure of $\mathbf{Q}$ is such that $\{Y(t); t \geq 0\}$ is an irreducible CTMC, which in itself is a harmless assumption to make.

A very important family of matrices associated with $\{Y(t); t \geq 0\}$ is the family of '**R**-matrices' $\{\mathbf{R}_{k+1,k}(\alpha)\}_{k \geq 0}$, where for each integer $k \geq 0$,

$$(\mathbf{R}_{k+1,k}(\alpha))_{i,j} := (-(\mathbf{Q}^{(k+1)})_{i,i} + \alpha)\mathbb{E}_{(k+1,i)}\left[\int_0^{\tau_{D_{k+1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (k,j))dt\right]$$

where for each $k \geq 0$,

$$D_k = \bigcup_{n=k}^{\infty} L_n.$$

Our first lemma shows how to numerically calculate each $\mathbf{R}$-matrix.

Lemma 2.2.1 *The matrices $\mathbf{R}_{k+1,k}(\alpha)$, for $k \geq 0$, satisfy the following recursion: for each integer $k \geq 1$,*

$$\mathbf{R}_{k+1,k}(\alpha) = \mathcal{M}^{(k+1)}[\alpha \mathbf{I}^{(k)} - \mathbf{Q}^{(k)} - \mathbf{R}_{k,k-1}(\alpha)\Lambda^{(k-1)}]^{-1}$$

where $\mathbf{R}_{1,0}(\alpha) = \mathcal{M}^{(0)}(\alpha \mathbf{I}^{(0)} - \mathbf{Q}^{(0)})^{-1}$.

Proof The argument follows with reasoning similar to that described on pages 270 through 272 of [25]: defining, for each $n \geq 1$, and each $m \in \{0, 1, 2, \ldots, n-1\}$, the matrix $\mathbf{R}_{n,m}(\alpha)$ as

$$(\mathbf{R}_{n,m}(\alpha))_{i,j} := (-(\mathbf{Q}^{(n)})_{i,i} + \alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{D_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right]$$

it is not difficult to show, via the random-product technique as is done in [25], that

$$\mathbf{R}_{n,m}(\alpha) = \coprod_{k=n}^{m+1} \mathbf{R}_{k,k-1}(\alpha) := \mathbf{R}_{n,n-1}(\alpha)\mathbf{R}_{n-1,n-2}(\alpha)\cdots\mathbf{R}_{m+1,m}(\alpha).$$

Readers should note our usage of the coproduct symbol $\coprod$: given a collection of matrices $\{H_k\}_{k\geq 0}$, we define

$$\prod_{k=m}^{n} H_k := H_m H_{m+1} \cdots H_n$$

for $m \leq n$, while we define

$$\coprod_{k=m}^{n} H_k := H_m H_{m-1} \cdots H_n$$

for $m \geq n$.

The next step is to establish that $(\alpha \mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})^{-1}$ exists, for each integer $n \geq 1$ and each $\alpha \in \mathbb{C}_+$. Fix an integer $n \geq 1$, and consider an alternative CTMC $\{Y_n(t); t \geq 0\}$

whose state space is given by

$$S_n := \bigcup_{k=0}^{n+1} L_k^{(n)}$$

where $L_k^{(n)} = L_k$ for each integer $k \in \{0, 1, 2, \ldots, n\}$, and $L_{n+1}^{(n)} = \{\Delta\}$, an absorbing state. The transition rate matrix $\mathbf{Q}_n$ is similar to the transition matrix of $\{Y(t); t \geq 0\}$, except that the row corresponding to level $L_{n+1}^{(n)}$ is the zero row, and the rows corresponding to level $L_n^{(n)}$ can be expressed in block partitioned form as

$$[\mathbf{0}_{d_n \times d_0} \quad \mathbf{0}_{d_n \times d_1} \quad \ldots \quad \mathbf{0}_{d_n \times d_{n-2}} \quad \mathcal{M}^{(n)} \quad \mathbf{Q}^{(n)} \quad \Lambda^{(n)} \mathbf{e}_{d_n \times 1}]$$

where $\mathbf{e}_{m \times 1}$ is a column vector with m rows and each element equal to one, and $\mathbf{0}_{m \times n}$ is a zero matrix with m rows and n columns. We also use the notation $\mathbf{e}_{m \times 1}^{(i)}$ to represent the ith *basis vector* in $\mathbb{R}^{m \times 1}$, where the ith component of $\mathbf{e}_{m \times 1}^{(i)}$ is equal to one and all of its other components are equal to zero. Similarly, we let $e_{1 \times n}^{(i)}$ denote the ith basis vector in $\mathbb{R}^{1 \times n}$, which is defined in a completely analogous manner.

We can establish the invertibility of $(\alpha \mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})$ through working with the Laplace transforms of the transition functions of $\{Y_n(t); t \geq 0\}$. Define, for each $0 \leq m_0, m_1 \leq n$, the matrix

$$\Pi_{m_0,m_1}^{(n)}(\alpha) := [\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha)]_{1 \leq i \leq d_{m_0}, 1 \leq j \leq d_{m_1}}$$

where

$$\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha) := \int_0^\infty e^{-\alpha t} \mathbb{P}_{(m_0,i)}(Y_n(t) = (m_1, j)) dt$$

with $\mathbb{P}_{(m_0,i)}$ denoting a conditional probability measure, given $Y_n(0) = (m_0, i)$. Fix $m \in \{0, 1, 2, \ldots, n-1\}$: applying Corollary 2.1.1 where $T = L_m$ yields

$$\Pi_{n,m}^{(n)}(\alpha) = \Pi_{n,n}^{(n)}(\alpha)\mathbf{R}_{n,m}(\alpha) = \Pi_{n,n}^{(n)}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\mathbf{R}_{n-1,n-2}(\alpha) \cdots \mathbf{R}_{m+1,m}(\alpha).$$

Having this fact in mind, if we now write out the Kolmogorov Forward equations associated with

$\{Y_n(t); t \geq 0\}$ in terms of Laplace transforms, we see in particular that

$$\begin{aligned}\alpha \Pi^{(n)}_{n,n}(\alpha) - \mathbf{I}^{(n)} &= \Pi^{(n)}_{n,n-1}(\alpha)\Lambda^{(n-1)} + \Pi^{(n)}_{n,n}(\alpha)\mathbf{Q}^{(n)} \\ &= \Pi^{(n)}_{n,n}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)} + \Pi^{(n)}_{n,n}(\alpha)\mathbf{Q}^{(n)}\end{aligned}$$

which yields

$$\Pi^{(n)}_{n,n}(\alpha)(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)}) = \mathbf{I}^{(n)}$$

proving that the matrix $(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})$ is invertible. Finally, one can use the random-product technique as is done in [25] to show that

$$\alpha\mathbf{R}_{k+1,k}(\alpha) = \mathcal{M}^{(k+1)} + \mathbf{R}_{k+1,k}(\alpha)\mathbf{Q}^{(k)} + \mathbf{R}_{k+1,k}(\alpha)\mathbf{R}_{k,k-1}(\alpha)\Lambda^{(k-1)}$$

meaning we can express $\mathbf{R}_{k+1,k}(\alpha)$ in terms of $\mathbf{R}_{k,k-1}(\alpha)$, thus proving the result. $\diamondsuit$

We are now ready to proceed with the main results of this section. Further associated with $\{Y(t); t \geq 0\}$ is a stochastic process $\{\overline{X}(t); t \geq 0\}$, where for each real $t \geq 0$,

$$\overline{X}(t) := \sup_{0 \leq s \leq t} X(s)$$

which represents the *maximum level* achieved by $\{Y(t); t \geq 0\}$ over the interval $[0, t]$: in [37], the authors refer to $\{\overline{X}(t); t \geq 0\}$ as the *running maximum* process. We can further combine $\overline{X}(t)$ and $Y(t)$ by defining the stochastic process $Z(t) := (\overline{X}(t), X(t), J(t))$, which is clearly also a CTMC, whose state space $\overline{S}$ is

$$\overline{S} = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{n} L_{n,m}$$

where for each integer $n \geq 0$, and each integer $m \in \{0, 1, 2, \ldots, n\}$,

$$L_{n,m} := \{([n,m],1), ([n,m],2), \ldots, ([n,m],d_m - 1), ([n,m],d_m)\}.$$

Observe that state $([n,m],k)$ has level $[n,m]$ and phase k, where $k \in \{1,2,\ldots,d_m\}$.

In Theorem 2.2.1 we study the marginal distributions of $\{Z(t); t \geq 0\}$ by applying Corollary 2.1.1 in various ways. Throughout both this section and the next, we let $\boldsymbol{\pi}_{[n,m]}(\alpha)$ denote a row vector in $\mathbb{C}^{1\times d_m}$ which is of the form

$$\boldsymbol{\pi}_{[n,m]}(\alpha) = [\pi_{([m_0,m_0],i_0),([n,m],1)}(\alpha), \quad \pi_{([m_0,m_0],i_0),([n,m],2)}(\alpha), \quad \cdots \quad \pi_{([m_0,m_0],d_{m_0}),([n,m],d_m)}].$$

Readers should note that the row vector $\boldsymbol{\pi}_{[n,m]}(\alpha)$ depends on $Z(0) = ([m_0,m_0],i_0)$, but we chose to leave this out of the notation in the interest of making the results easier to read. Observe too that we will also occasionally let $\mathbb{P}_{([m_0,m_0],i_0)}$ denote a conditional probability measure, conditioned on $Z(0) = ([m_0,m_0],i_0)$. It will always be clear from the context what is being conditioned on when we write $\mathbb{P}_x$, so we will use this notation throughout the rest of the chapter without further comment.

Theorem 2.2.1 *Suppose $Z(0) = (m_0, m_0, i_0)$. Then*

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}[\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)} - \mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)}]^{-1}. \tag{2.4}$$

Furthermore, for each $n \geq m_0 + 1$,

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[m_0,m_0]}(\alpha)\prod_{\ell=m_0}^{n-1}\Lambda^{(\ell)}[\alpha\mathbf{I}^{(\ell+1)} - \mathbf{Q}^{(\ell+1)} - \mathbf{R}_{\ell+1,\ell}(\alpha)\Lambda^{(\ell)}]^{-1}. \tag{2.5}$$

Finally, for each $n \geq m_0$, and each $m \in \{0,1,2,\ldots,n-1\}$,

$$\boldsymbol{\pi}_{[n,m]}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha)\prod_{\ell=n}^{m+1}\mathbf{R}_{\ell,\ell-1}(\alpha) \tag{2.6}$$

Proof We first prove (2.6). Applying (2.2) to $\{Z(t); t \geq 0\}$ while choosing

$$T = \bigcup_{k=0}^{n-1} L_{n,k}$$

yields, for each state $([n,m],j) \in T$,

$$\boldsymbol{\pi}_{([n,m],j)}(\alpha) = \sum_{i=1}^{d_n}\pi_{([n,n],i)}(\alpha)(q([n,n],i)+\alpha)\mathbb{E}_{([n,n],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,m],j))dt\right]. \tag{2.7}$$

Next, observe that for each $i \in \{1, 2, \ldots, d_n\}$,

$$
\begin{aligned}
&(-(\mathbf{Q}^{(n)})_{i,i} + \alpha)\mathbb{E}_{([n,n],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,m],j))dt\right] \\
&= (-(\mathbf{Q}^{(n)})_{i,i} + \alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{D_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right] \\
&= (\mathbf{R}_{n,m}(\alpha))_{i,j} \qquad (2.8)
\end{aligned}
$$

and applying what was learned in (2.8) to (2.7) yields, upon further simplification,

$$
\boldsymbol{\pi}_{[n,m]}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha)\mathbf{R}_{n,m}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha)\coprod_{\ell=n}^{m+1}\mathbf{R}_{\ell,\ell-1}(\alpha) \qquad (2.9)
$$

proving (2.6).

The next step is to establish (2.5). Applying again (2.2) to $\{Z(t); t \geq 0\}$ while choosing $T = L_{n,n}$ yields, upon simplifying,

$$
\begin{aligned}
\boldsymbol{\pi}_{[n,n]}(\alpha) &= \boldsymbol{\pi}_{[n-1,n-1]}\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} + \boldsymbol{\pi}_{[n,n-1]}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} \\
&= \boldsymbol{\pi}_{[n-1,n-1]}\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} + \boldsymbol{\pi}_{[n,n]}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1}
\end{aligned}
$$

meaning

$$
\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[n-1,n-1]}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})^{-1}
$$

and by repeatedly iterating this equality, we establish (2.5).

It remains to derive (2.4). Thinking now of $\{Z(t); t \geq 0\}$ as being governed by a countable collection of independent, homogeneous Poisson processes as we described in Section 2.1, we observe that for each phase $k \in \{1, 2, \ldots, d_{m_0}\}$, we have that for each $t > 0$,

$$
\begin{aligned}
&\mathbf{1}(Z(t) = ([m_0, m_0], k)) \\
&= \mathbf{1}(Z(t) = ([m_0, m_0], k), \tau_{L^c_{m_0,m_0}} > t) \qquad (2.10) \\
&+ \sum_{j=1}^{d_{m_0-1}}\sum_{\ell=1}^{d_{m_0}}\int_0^t \mathbf{1}(Z(s-) = ([m_0, m_0-1], j), \tau_{L^c_{m_0,m_0}}(s) > t, Z(t) = ([m_0, m_0], k))N_{([m_0,m_0-1],j),([m_0,m_0],\ell)}(ds)
\end{aligned}
$$

Taking expectations of both sides of (2.10), while further applying the Campbell-Mecke formula to the right-hand-side gives

$$
\begin{aligned}
&\mathbb{P}_{([m_0,m_0],i_0)}(Z(t)=([m_0,m_0],k)) \\
&= \mathbb{P}_{([m_0,m_0],i_0)}(Z(t)=([m_0,m_0],k),\tau^c_{L_{m_0,m_0}} > t) \qquad (2.11)\\
&+ \sum_{j=1}^{d_{m_0-1}} \sum_{\ell=1}^{d_{m_0}} \int_0^t \mathbb{P}_{([m_0,m_0],j)}(Z(s)=([m_0,m_0-1],j))\mathbb{P}_{([m_0,m_0],\ell)}(\tau_{L^c_{m_0,m_0}} > t-s, Z(t-s)=([m_0,m_0],k))(\Lambda^{(m_0-1)})_{j,\ell}ds
\end{aligned}
$$

and after multiplying both sides of (2.11) by $e^{-\alpha t}$ and integrating with respect to t over $[0,\infty)$, we get

$$
\begin{aligned}
&\pi_{[m_0,m_0],k}(\alpha) \\
&= \mathbb{E}_{([m_0,m_0],i_0)}\left[\int_0^{\tau_{L^c_{m_0,m_0}}} e^{-\alpha t}\mathbf{1}(Z(t)=([m_0,m_0],k))ds\right] \\
&+ \sum_{j=1}^{d_{m_0-1}} \sum_{\ell=1}^{d_{m_0}} \pi_{([m_0,m_0-1],j)}(\alpha)(\Lambda^{(m_0-1)})_{j,\ell}\mathbb{E}_{([m_0,m_0],\ell)}\left[\int_0^{\tau^c_{L_{m_0,m_0}}} e^{-\alpha t}\mathbf{1}(Z(t)=([m_0,m_0],k))dt\right]
\end{aligned}
$$

which can be stated in matrix form as

$$
\begin{aligned}
\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) &= \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)}-\mathbf{Q}^{(m_0)})^{-1} + \boldsymbol{\pi}_{[m_0,m_0-1]}(\alpha)\Lambda^{(m_0-1)}(\alpha\mathbf{I}^{(m_0)}-\mathbf{Q}^{(m_0)})^{-1} \\
&= \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)}-\mathbf{Q}^{(m_0)})^{-1} + \boldsymbol{\pi}_{[m_0,m_0]}(\alpha)\mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)}(\alpha\mathbf{I}^{(m_0)}-\mathbf{Q}^{(m_0)})^{-1}.
\end{aligned}
$$

Finally, solving for $\boldsymbol{\pi}_{[m_0,m_0]}(\alpha)$ yields

$$
\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)}-\mathbf{Q}^{(m_0)}-\mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)})^{-1}
$$

which proves (2.4), and completes the proof of Theorem 2.2.1. ♢

2.3 Markov Processes of M/G/1 Type

We close by studying the joint distribution of the running minimum level, the level, and the phase of a level-dependent Markov Process of M/G/1-type at a fixed time t. Suppose now that $\{Y(t); t \geq 0\}$ represents a level-dependent Markov process of M/G/1-type whose state space S can be expressed in terms of a countable union of levels:

$$S = \bigcup_{n=0}^{\infty} L_n$$

where, for each integer $n \geq 0$,

$$L_n := \{(n,1), (n,2), \ldots, (n, d_n - 1), (n, d_n)\},$$

where each d_n is a positive integer that varies with n. Just as before, we express $Y(t)$ as $((X(t), J(t))$, where $X(t)$ and $J(t)$ denotes the current level and phase of the process at time t, respectively. We express the transition rate matrix $\mathbf{Q}$ of $\{Y(t); t \geq 0\}$ in block-partitioned form as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \mathbf{A}_{0,2} & \mathbf{A}_{0,3} & \mathbf{A}_{0,4} & \cdots \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & \mathbf{A}_{1,4} & \cdots \\ \mathbf{0}_{d_2 \times d_0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{A}_{2,4} & \cdots \\ \mathbf{0}_{d_3 \times d_0} & \mathbf{0}_{d_3 \times d_1} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Observe that for each integer $i \geq 0$ and each $j \geq i-1$, $\mathbf{A}_{i,j} \in \mathbb{R}^{d_i \times d_j}$ contains the transition rates corresponding to transitions from states in L_i to states in L_j. Again we assume that for each state $x \in S$, there exists two states $y, z \in S$ (that may depend on x) such that $q(x,y) > 0$ and $q(z,x) > 0$.

Just as in Section 2.2, there is an important family of $\mathbf{R}$-matrices $\{\mathbf{R}_{\ell,m}(\alpha)\}_{m \geq 1, 0 \leq \ell < m}$ such that for each integer $m \geq 1$ and each integer $\ell \in \{0, 1, \ldots, m-1\}$

$$(\mathbf{R}_{\ell,m}(\alpha))_{i,j}(\alpha) := (-(\mathbf{A}_{\ell,\ell})_{i,i} + \alpha)\mathbb{E}_{(\ell,i)}\left[\int_0^{\tau_{C_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right]$$

where for each integer $m \geq 1$,

$$C_m = \bigcup_{n=0}^{m} L_n.$$

Our analysis of Markov processes of M/G/1-type also involves a close study of a family of '**G**-matrices' $\{\mathbf{G}_{n,m}(\alpha)\}_{0 \leq n < m}$ where for each integer $n \geq 1$ and each integer $m \in \{0, 1, \ldots, n-1\}$,

$$(\mathbf{G}_{n,m}(\alpha))_{i,j} = \mathbb{E}_{(n,i)}\left[\mathbf{1}(Y(\tau_{L_m}) = (m,j))e^{-\alpha \tau_{L_m}}\right].$$

Our next lemma, Lemma 2.3.1, shows how to express all **R**-matrices in terms of **G**-matrices.

Lemma 2.3.1 *For each integer $m \geq 1$, and each integer $\ell \in \{0, 1, 2, \ldots, m-1\}$, we have*

$$\mathbf{R}_{\ell,m}(\alpha) = \sum_{k=m}^{\infty} \mathbf{A}_{\ell,k}\mathbf{G}_{k,m}(\alpha)\left[\alpha\mathbf{I}^{(m)} - \sum_{k=m}^{\infty}\mathbf{A}_{m,k}\mathbf{G}_{k,m}(\alpha)\right]^{-1} \tag{2.12}$$

where we follow the convention that $\mathbf{G}_{m,m}(\alpha) := \mathbf{I}^{(m)}$. Furthermore, for each $m \geq 0$, and each $k > m$,

$$\mathbf{G}_{k,m}(\alpha) = \prod_{\ell=k}^{m+1}\mathbf{G}_{\ell,\ell-1}(\alpha) := \mathbf{G}_{k,k-1}(\alpha)\mathbf{G}_{k-1,k-2}(\alpha)\cdots\mathbf{G}_{m+1,m}(\alpha) \tag{2.13}$$

*and the family of **G**-matrices $\{\mathbf{G}_{k+1,k}(\alpha)\}$ satisfy the following recursive scheme: for each integer $k \geq 1$,*

$$\mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k-1}\left[\alpha\mathbf{I}^{(k)} - \mathbf{A}_{k,k} - \sum_{i=k+1}^{\infty}\mathbf{A}_{k,i}\prod_{j=i}^{k+1}\mathbf{G}_{j,j-1}(\alpha)\right]^{-1}. \tag{2.14}$$

Proof We follow the line of reasoning given in the unpublished manuscript [25]. First, we define the collection of matrices $\{\mathbf{N}_m(\alpha)\}_{m\geq 1}$, where for each integer $m \geq 1$, and each integer $i, j \in \{1, 2, \ldots, d_m\}$ (where possibly $i = j$),

$$(\mathbf{N}_m(\alpha))_{i,j} := \mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right].$$

Applying a first-step analysis argument shows that

$$\begin{aligned}(\mathbf{N}_m(\alpha))_{i,j} = {} & \frac{\mathbf{1}(i=j)}{q((m,i))+\alpha} + \sum_{k\neq i}\frac{q((m,i),(m,k))}{q((m,i))+\alpha}(\mathbf{N}_m(\alpha))_{k,j} \\ & + \sum_{k=m+1}^{\infty}\sum_{n=1}^{d_k}\frac{q((m,i),(k,n))}{q((m,i))+\alpha}\mathbb{E}_{(k,n)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t)=(m,j))dt\right].\end{aligned} \tag{2.15}$$

We can use the strong Markov property at the stopping time τ_{L_m} to further simplify the remaining expectations found in (2.15): indeed,

$$
\begin{aligned}
&\mathbb{E}_{(k,n)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t)=(m,j))dt\right] \qquad (2.16)\\
&=\sum_{\ell=1}^{d_m}\mathbb{E}_{(k,n)}[\mathbf{1}(Y(\tau_{L_m})=(m,\ell))e^{-\alpha\tau_{L_m}}]\mathbb{E}_{(m,\ell)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t)=(m,j))dt\right]\\
&=\sum_{\ell=1}^{d_m}(\mathbf{G}_{k,m}(\alpha))_{n,\ell}(\mathbf{N}_m(\alpha))_{\ell,j}.
\end{aligned}
$$

Plugging (2.16) into (2.15), then expressing (2.15) (while remembering that $\mathbf{G}_{m,m}(\alpha)=\mathbf{I}^{(m)}$) we get

$$\alpha\mathbf{N}_m(\alpha)=\mathbf{I}^{(m)}+\sum_{k=m}^{\infty}\mathbf{A}_{m,k}\mathbf{G}_{k,m}(\alpha)\mathbf{N}_m(\alpha), \qquad (2.17)$$

which implies

$$\left[\alpha\mathbf{I}^{(m)}-\sum_{k=m}^{\infty}\mathbf{A}_{m,k}\mathbf{G}_{k,m}(\alpha)\right]\mathbf{N}_m(\alpha)=\mathbf{I}^{(m)}$$

meaning

$$\mathbf{N}_m(\alpha)=\left[\alpha\mathbf{I}^{(m)}-\sum_{k=m}^{\infty}\mathbf{A}_{m,k}\mathbf{G}_{k,m}(\alpha)\right]^{-1}. \qquad (2.18)$$

We are now ready to derive (2.12). From the definition of $\mathbf{R}_{\ell,m}(\alpha)$, we can see from applying both first-step analysis and the strong Markov property that

$$\mathbf{R}_{\ell,m}(\alpha)=\sum_{k=m}^{\infty}\mathbf{A}_{\ell,k}\mathbf{G}_{k,m}(\alpha)\mathbf{N}_m(\alpha). \qquad (2.19)$$

Plugging (2.18) into (2.19) yields (2.12).

The next step is to establish (2.13). Fix an integer $m\geq 0$ and an integer $k>m$. Using again the strong Markov property, we get

$$\mathbf{G}_{k,m}(\alpha)=\mathbf{G}_{k,k-1}(\alpha)\mathbf{G}_{k-1,m}(\alpha),$$

and by a simple induction argument, we get

$$\mathbf{G}_{k,m}(\alpha) = \prod_{\ell=k}^{m+1} \mathbf{G}_{\ell,\ell-1}(\alpha)$$

which establishes (2.13).

It remains to derive (2.14). Fix $i \in \{1, 2, \ldots, d_k\}$ and $j \in \{1, 2, \ldots, d_{k-1}\}$,

$$\begin{aligned}(\mathbf{G}_{k,k-1}(\alpha))_{i,j} = \frac{q((k,i),(k-1,j))}{q((k,i))+\alpha} + \sum_{\ell \neq i} \frac{q((k,i),(k,\ell))}{q((k,i))+\alpha}(\mathbf{G}_{k,k-1}(\alpha))_{\ell,j} \\ + \sum_{m=k+1}^{\infty} \sum_{\ell=1}^{d_m} \frac{q((k,i),(m,\ell))}{q((k,i))+\alpha}(\mathbf{G}_{m,k-1}(\alpha))_{\ell,j}\end{aligned}$$

or, in matrix form,

$$\alpha \mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k-1} + \sum_{m=k}^{\infty} \mathbf{A}_{k,m} \mathbf{G}_{m,k-1}(\alpha). \tag{2.20}$$

Applying (2.13) to (2.20) shows that

$$\alpha \mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k} + \sum_{i=k+1}^{\infty} \mathbf{A}_{k,i} \left(\prod_{j=i}^{k+1} \mathbf{G}_{j,j-1}(\alpha) \right) \mathbf{G}_{k,k-1}(\alpha) \tag{2.21}$$

and solving for $\mathbf{G}_{k,k-1}(\alpha)$ in (2.21) gives

$$\mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k+1,k} \left[\alpha \mathbf{I}^{(k+1)} - \mathbf{A}_{k+1,k+1} - \sum_{i=k+1}^{\infty} \mathbf{A}_{k+1,i} \prod_{j=i}^{k+1} \mathbf{G}_{j,j-1}(\alpha) \right]^{-1}$$

which proves (2.14). $\diamondsuit$

While Lemma 2.3.1 is theoretically interesting, it is only practically useful if the **G**-matrices can be calculated numerically. It is not clear in general if there is a way to calculate these matrices, but they can be calculated if we impose additional assumptions on $\{Y(t); t \geq 0\}$. Suppose, for instance, that there exists an integer $n_0 \geq 1$ large enough such that $\mathbf{A}_{n,k} = \mathbf{A}_{k-n}$ for all $n \geq n_0$ and $k \geq n-1$. Under this additional assumption, one can see that $\mathbf{G}_{n,n-1}(\alpha) = \mathbf{G}(\alpha)$ for each $n \geq n_0$, where

$$\mathbf{G}(\alpha) := \mathbf{G}_{n_0,n_0-1}(\alpha).$$

As explained in [25], the matrix $\mathbf{G}(\alpha)$ is the pointwise limit of a sequence of matrices $\{\mathbf{G}(N,\alpha)\}_{N\geq 0}$, where $\mathbf{G}(0,\alpha) = \mathbf{0}_{d_{n_0}\times d_{n_0}}$, and for each integer $N \geq 0$,

$$\mathbf{G}(N+1,\alpha) = (\alpha\mathbf{I}^{(d_{n_0})} - \mathbf{A}_0)^{-1}\left[\mathbf{A}_{-1} + \sum_{n=1}^{\infty}\mathbf{A}_n\mathbf{G}(N,\alpha)^n\right].$$

We are now ready to set up and establish the main result of this section. We associate with $\{Y(t); t \geq 0\}$ the stochastic process $\{\underline{X}(t); t \geq 0\}$ where for each $t \geq 0$,

$$\underline{X}(t) := \inf_{0\leq s\leq t} X(s)$$

which represents the *running minimum level* achieved by $\{Y(t); t \geq 0\}$ over the interval $[0,t]$. Next, for each $t \geq 0$ we define $Z(t) := (\underline{X}(t), X(t), J(t))$, and just as was the case in the previous section, $\{Z(t); t \geq 0\}$ is a CTMC with state space

$$\underline{S} = \bigcup_{n=0}^{\infty}\bigcup_{m=n}^{\infty} L_{n,m}$$

where for each integer $n \geq 0$ and each integer $m \geq n$,

$$L_{n,m} := \{([n,m],1), ([n,m],2), \ldots, ([n,m],d_m - 1), ([n,m],d_m)\}.$$

In our next result, Theorem 2.3.1, we show how to derive the Laplace transforms of the transition functions associated with $\{Z(t); t \geq 0\}$.

Theorem 2.3.1 *Suppose* $Z(0) = (m_0, m_0, i_0)$. *Then*

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)} - \mathbf{A}_{m_0,m_0} - \mathbf{R}_{m_0,m_0+1}(\alpha)\mathbf{A}_{m_0+1,m_0})^{-1}. \quad (2.22)$$

Furthermore, for each integer $n \in \{0, 1, \ldots, m_0 - 1\}$,

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[m_0,m_0]}(\alpha) \coprod_{\ell=m_0-1}^{n} \mathbf{A}_{\ell+1,\ell} \left[\alpha \mathbf{I}^{(\ell)} - \mathbf{A}_{\ell,\ell} - \mathbf{R}_{\ell,\ell+1}(\alpha)\mathbf{A}_{\ell+1,\ell}\right]^{-1}. \tag{2.23}$$

Finally, for each integer $n \in \{0, 1, \ldots, m_0 - 1, m_0\}$ and each integer $m \geq n$,

$$\boldsymbol{\pi}_{[n,m+1]}(\alpha) = \sum_{k=n}^{m} \boldsymbol{\pi}_{[n,k]}(\alpha)\mathbf{R}_{k,m+1}(\alpha). \tag{2.24}$$

Proof We will first establish invertibility of $(\alpha \mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n})$ for any integer $n \in \{0, 1, \ldots, m_0 - 1, m_0\}$ and each $\alpha \in \mathbb{C}_+$. To do so, we use a strategy similar to that used in Lemma 3.2.1. Fix an integer $n \geq 1$ and consider an alternative CTMC $\{Y_n(t); t \geq 0\}$ whose state space is given by

$$S_n := \bigcup_{k=n-1}^{\infty} L_k^{(n)}$$

where $L_{n-1}^{(n)} = \{\Delta\}$ (a single absorbing state) and for each $k \geq n$, $L_k^{(n)} := L_k$. The transition rate matrix $\mathbf{Q}_n$ of $\{Y_n(t); t \geq 0\}$ is such that the row corresponding to level $L_{n-1}^{(n)}$ is a row containing all zeros, the rows corresponding to $L_n^{(n)}$ can be expressed in block-partitioned form as

$$[\mathbf{A}_{n,n-1}\mathbf{e} \quad \mathbf{A}_{n,n} \quad \mathbf{A}_{n,n+1} \quad \mathbf{A}_{n,n+2} \quad \cdots]$$

and for each $k \geq n + 1$, the rows corresponding to level $L_k^{(n)}$ the same as the rows corresponding to L_k in the transition rate matrix of $\{Y(t); t \geq 0\}$.

Next, we define, for each m_0 and $m_1 \geq n$ the matrix

$$\Pi_{m_0,m_1}^{(n)}(\alpha) := [\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha)]_{1 \leq i \leq d_{m_0}, 1 \leq j \leq d_{m_1}}$$

where

$$\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha) := \int_0^{\infty} e^{-\alpha t}\mathbb{P}_{(m_0,i)}(Y_n(t) = (m_1, j))dt.$$

Observe that by applying Corollary 2.1.1 while choosing

$$T = \bigcup_{k=n+1}^{\infty} L_k$$

yields

$$\Pi_{n,n+1}^{(n)}(\alpha) = \Pi_{n,n}^{(n)}(\alpha)\mathbf{R}_{n,n+1}(\alpha).$$

With this in mind, after writing out the Kolmogorov Forward equations associated with $\{Y_n(t); t \geq 0\}$ in terms of Laplace transforms, we see that

$$\Pi_{n,n}^{(n)}(\alpha)(\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}) - \Pi_{n,n+1}(\alpha)\mathbf{A}_{n+1,n} = \mathbf{I}^{(n)}$$
$$\Pi_{n,n}^{(n)}(\alpha)(\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}) - \Pi_{n,n}(\alpha)\mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n} = \mathbf{I}^{(n)}$$

which yields

$$\Pi_{n,n}^{(n)}(\alpha)(\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n}) = \mathbf{I}^{(n)}$$

proving that the matrix $(\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n})$ is invertible. A similar argument can be made for the case where $n = 0$, but in that case we can establish invertibility with the forward equations of $\{Y(t); t \geq 0\}$.

We now prove (2.24): fix $n \in \{0, 1, \ldots, m_0 - 1, m_0\}$ and suppose $m \geq n$. Applying Corollary 2.1.1 to $\{Z(t); t \geq 0\}$ while choosing

$$T = \bigcup_{\ell=m+1}^{\infty} L_{n,\ell}$$

yields, for each $j \in \{1, 2, \ldots, d_{m+1}\}$,

$$\pi_{([n,m+1],j)}(\alpha) = \sum_{\ell=n}^{m} \sum_{i=1}^{d_\ell} \pi_{([n,\ell],i)}(\alpha)(q([n,\ell],i) + \alpha)\mathbb{E}_{([n,\ell],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n, m+1], j))dt\right]. \tag{2.25}$$

Furthermore, for each $\ell \in \{n, n+1, \ldots, m\}$ and each $i \in \{1, 2, \ldots, d_\ell\}$,

$$\begin{aligned}
&(q([n,\ell],i)+\alpha)\mathbb{E}_{([n,\ell],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t)=([n,m+1],j))dt\right] \\
&= (-(\mathbf{A}_{\ell,\ell})_{i,i}+\alpha)\mathbb{E}_{(\ell,i)}\left[\int_0^{\tau_{C_m}} e^{-\alpha t}\mathbf{1}(Y(t)=(m+1,j))dt\right] \\
&= (\mathbf{R}_{\ell,m}(\alpha))_{i,j}. \qquad (2.26)
\end{aligned}$$

Applying (2.26) to (2.25), then writing (2.25) in matrix form yields

$$\boldsymbol{\pi}_{[n,m+1]}(\alpha) = \sum_{\ell=n}^{m} \boldsymbol{\pi}_{[n,\ell]}(\alpha)\mathbf{R}_{\ell,m+1}(\alpha)$$

which proves (2.24).

We next establish (2.22). for each $j \in \{1, 2, \ldots, d_{m_0}\}$,

$$\begin{aligned}
&\mathbf{1}(Z(t)=(m_0,m_0,j)) \\
&= \mathbf{1}(Z(t)=(m_0,m_0,j), \tau_{L^c_{m_0,m_0}} > t) \qquad (2.27) \\
&+ \sum_{k=1}^{d_{m_0}}\sum_{\ell=1}^{d_{m_0}}\int_0^t \mathbf{1}(Z(s-)=(m_0,m_0+1,k), \tau_{L^c_{m_0,m_0}}(s) > t, Z(t)=(m_0,m_0,j))N_{(m_0,m_0+1,k),(m_0,m_0,\ell)}(ds)
\end{aligned}$$

Just as we did in the proof of Theorem 2.2.1, after taking the expectation of both sides of (2.27), applying the Campbell-Mecke formula to the right-hand-side, multiplying by $e^{-\alpha t}$, integrating, and then simplifying, we get

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)} - \mathbf{A}_{m_0,m_0} - \mathbf{R}_{m_0,m_0+1}(\alpha)\mathbf{A}_{m_0+1,m_0})^{-1}$$

proving (2.22).

It remains to derive (2.23). Fix an integer $n \in \{0, 1, \ldots, m_0 - 1\}$: applying Corollary 2.1.1

to $\{Z(t); t \geq 0\}$ while choosing $T = L_{n,n}$ reveals that

$$\begin{aligned}
&\pi_{[n,n],j}(\alpha) \\
&= \sum_{i=1}^{d_{n+1}} \pi_{[n+1,n+1],i}(\alpha)(q([n+1,n+1],i)+\alpha)\mathbb{E}_{([n+1,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right] \\
&+ \sum_{i=1}^{d_{n+1}} \pi_{[n,n+1],i}(\alpha)(q([n,n+1],i)+\alpha)\mathbb{E}_{([n,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right]. \qquad (2.28)
\end{aligned}$$

Again, for $k \in \{n, n+1\}$, $i \in \{1, 2, \ldots, d_k\}$, and $j \in \{1, 2, \ldots, d_n\}$,

$$\begin{aligned}
&(q([k,n+1],i)+\alpha)\mathbb{E}_{([k,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right] \\
&= (-(\mathbf{A}_{n+1,n+1})_{i,i}+\alpha)\mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{L^c_{n}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right]. \qquad (2.29)
\end{aligned}$$

Plugging (2.29) into (2.28) and simplifying further shows that

$$\boldsymbol{\pi}_{[n,n](\alpha)} = \boldsymbol{\pi}_{[n+1,n+1]}(\alpha)\mathbf{A}_{n+1,n}[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}]^{-1} + \boldsymbol{\pi}_{[n,n+1]}(\alpha)\mathbf{A}_{n+1,n}[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}]^{-1}$$

and by writing $\boldsymbol{\pi}_{[n,n+1]}(\alpha)$ in terms of $\boldsymbol{\pi}_{[n,n]}(\alpha)$ and solving for $\boldsymbol{\pi}_{[n,n]}(\alpha)$ yields

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[n+1,n+1]}(\alpha)\mathbf{A}_{n+1,n}\left[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n}\right]^{-1}.$$

which yields, upon repeated iterations of this equality, (2.23), thus proving Theorem 2.3.1. $\diamondsuit$

Chapter 3

Finite Quasi-Birth-Death processes

3.1 Introduction

In 1982, Hajek [18] showed that the stationary distribution of a homogeneous Quasi-Birth-Death (QBD) process having finitely many levels exhibits its own type of 'matrix-geometric' form that contains two different types of **R**-matrices associated with those that appear in the stationary distribution of homogeneous QBD processes having infinitely many levels and a single boundary level. Fifteen years later, Keilson and Masuda showed in [29] that the Laplace transforms of the transition functions of a homogeneous QBD process also exhibit an analogous type of 'matrix-geometric' form, but to obtain this form the authors make use of what they refer to as a 'compensation method' which is very analytic in flavor, and appears to be quite different from the approach used by Hajek in [18]: this compensation method was also used in Keilson and Zachmann [30] to study stationary distributions associated with these processes. The approach given in [29] also appears to be somewhat incomplete, as they leave open the problem of calculating certain Laplace transforms associated with the boundary levels of the QBD process.

Our objective is to provide a completely probabilistic approach towards deriving such matrix-geometric expressions for the transition functions of a homogeneous QBD process having finitely many levels, while simultaneously showing how to numerically calculate all involved Laplace transforms, including those associated with the boundary levels. It is important to point out that the formulas we derive for the Laplace transforms of the transition functions associated with a homogeneous finite-level QBD process are similar in flavor to quantities recently derived in Dendievel et al. [6], where the authors were instead interested in studying the behavior of a reward function associated with a homogeneous QBD process having finitely many levels, but in the analysis found in [6] the authors rely extensively on the theory of matrix difference equations, whereas our approach avoids usage of this theory entirely.

The key to deriving our main results involves first deriving, through entirely probabilistic methods, the joint distribution of the amount of time it takes a homogeneous QBD process to reach, from a given level n, either a level $a < n$ or a level $b > n$, as well as the level and phase of the process at this random time: these distributions can be fully described with two types of '$\mathbf{G}$-matrices' associated with the QBD process. This distribution can be studied probabilistically with the strong Markov property, through a matrix generalization of an argument found in Doroudi et al. [7] in the context of M/M/1 queues. Interestingly, the same type of proof technique can be used to derive simple expressions for two different types of $\mathbf{R}$-matrices that, given levels a, n, b satisfying $a < n < b$, keep track of the (discounted) expected amount of time spent by the QBD process in state (n, j) for some phase j before the chain revisits either level a or level b, given it starts either at some state in level a, or some state in level b. It may seem strange at first glance that the same proof technique can be used to derive both of these type of matrices, but what makes this possible is the fact that each element of these $\mathbf{R}$-matrices can be expressed in terms of the expected value of a 'random-product' governed by an alternative CTMC related to the original QBD process. Readers wishing to read more about the random-product technique itself should consult [5, 9, 11]: moreover, [25] also shows how the random-product technique can be used to derive the Laplace transforms of the transition functions of a QBD process with a single boundary.

It is also important to observe that our overall approach appears to yield new results that address the amount of time it takes a homogeneous QBD process with finitely many levels to move from one level to another. Properties of these hitting-time distributions have been studied by numerous authors in both the discrete-time and continuous-time context, see e.g. [15], [33], and [39], but to the best of our knowledge our approach appears to yield new expressions for the Laplace-Stieltjes transforms of these hitting-time random variables.

3.2 Two Important Lemmas

Here we state and prove two useful lemmas that provide us with computable expressions for the matrices needed in order to derive our main results.

Suppose $\{F(t); t \geq 0\}$ is an irreducible, homogeneous Quasi-Birth-Death (QBD) process, having a state space S of the form

$$S = \bigcup_{n \in \mathbb{Z}} L_n$$

where for each integer $n \in \mathbb{Z}$, $L_n = \{(n,1),(n,2),\ldots,(n,d)\}$ for some fixed integer $d \geq 1$. The transition rate matrix $\mathbf{Q}$ of $\{F(t); t \geq 0\}$ also exhibits a block-partitioned structure that is constructed using only three matrices $\mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1 \in \mathbb{R}^{d \times d}$, where for each $i, j \in \{1, 2, \ldots, d\}$ (where possibly $i = j$), and each $n \in \mathbb{Z}$,

$$q((n,i),(n-1,j)) = (\mathbf{A}_{-1})_{i,j}, \quad q((n,i),(n,j)) = (\mathbf{A}_0)_{i,j}, \quad q((n,i),(n+1,j)) = (\mathbf{A}_1)_{i,j}$$

and for any two integers n, m satisfying $|n-m| \geq 2$, $q((n,i),(m,j)) = 0$. for each $i, j \in \{1, 2, \ldots, d\}$.

We further associate with $\{F(t); t \geq 0\}$ hitting-time random variables of the form τ_A, where for each $A \subset S$,

$$\tau_A := \inf\{t \geq 0 : F(t) \in A\}.$$

From these hitting times, we construct the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$, where for each $i, j \in \{1, 2, \ldots, d\}$, the (i,j)th element found in the matrix $\mathbf{G}(\alpha)$ is

$$(\mathbf{G}(\alpha))_{i,j} := \mathbb{E}_{(0,i)}[e^{-\alpha \tau_{L_{-1}}} \mathbf{1}(F(\tau_{L_{-1}}) = (-1,j))], \qquad (\hat{\mathbf{G}}(\alpha))_{i,j} := \mathbb{E}_{(0,i)}[e^{-\alpha \tau_{L_1}} \mathbf{1}(F(\tau_{L_1}) = (1,j))].$$

The level-independent structure of $\mathbf{Q}$ reveals that for each integer $n \geq 1$,

$$(\mathbf{G}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha \tau_{L_{n-1}}} \mathbf{1}(F(\tau_{L_{n-1}}) = (n-1,j))],$$

$$(\hat{\mathbf{G}}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha \tau_{L_{n+1}}} \mathbf{1}(F(\tau_{L_{n+1}}) = (n+1,j))].$$

Furthermore, we can use the Strong Markov property to show that for each $a, b \in \mathbb{Z}$ satisfying $a < b$,

$$(\mathbf{G}(\alpha)^{b-a})_{i,j} = \mathbb{E}_{(b,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(F(\tau_{L_a}) = (a,j))], \quad (\hat{\mathbf{G}}(\alpha)^{b-a})_{i,j} = \mathbb{E}_{(a,i)}[e^{-\alpha\tau_{L_b}}\mathbf{1}(F(\tau_{L_b}) = (b,j))].$$

Fix two integers $a, b \in \mathbb{Z}$, where $a < b$. Together the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ can be used to construct the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$, where for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{G}_{n,a,b}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(\tau_{L_a} < \tau_{L_b}, F(\tau_{L_a}) = (a,j))]$$

and

$$(\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_b}}\mathbf{1}(\tau_{L_b} < \tau_{L_a}, F(\tau_{L_b}) = (b,j)].$$

The next lemma, Lemma 3.2.1, shows that both of these matrices can be expressed explicitly in terms of $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$. This lemma is very similar to an exercise found in Karlin and Taylor [26] pertaining to Brownian motion, and it is also similar to a result in the work of Doroudi et al. [7], which addresses analogous hitting-time results associated with a process that is the difference of two independent, homogeneous Poisson processes.

Lemma 3.2.1 *Given $a, n, b \in \mathbb{Z}$ satisfying $a < n < b$, we have*

$$\mathbf{G}_{n,a,b}(\alpha) = [\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-n}\mathbf{G}(\alpha)^{b-n}]\mathbf{G}(\alpha)^{n-a}[\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a}]^{-1}. \tag{3.1}$$

Moreover,

$$\hat{\mathbf{G}}_{n,b,a}(\alpha) = [\mathbf{I} - \mathbf{G}(\alpha)^{n-a}\hat{\mathbf{G}}(\alpha)^{n-a}]\hat{\mathbf{G}}(\alpha)^{b-n}[\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a}]^{-1}. \tag{3.2}$$

Proof Fix $i, j \in \{1, 2, \ldots, d\}$, and observe first that

$$
\begin{aligned}
(\mathbf{G}(\alpha)^{n-a})_{i,j} &= \mathbb{E}_{(n,i)}[e^{-\alpha \tau_{L_a}} \mathbf{1}(F(\tau_{L_a}) = (a, j))] \\
&= \mathbb{E}_{(n,i)}[e^{-\alpha \tau_{L_a}} \mathbf{1}(F(\tau_{L_a}) = (a, j), \tau_{L_a} < \tau_{L_b})] \\
&+ \sum_{\nu=1}^{m} \mathbb{E}_{(n,i)}[e^{-\alpha \tau_{L_a}} \mathbf{1}(F(\tau_{L_a}) = (a, j), F(\tau_{L_b}) = (b, \nu), \tau_{L_b} < \tau_{L_a})] \\
&= (\mathbf{G}_{n,a,b}(\alpha))_{i,j} + \sum_{\nu=1}^{m} (\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,\nu} (\mathbf{G}(\alpha)^{b-a})_{\nu,j}
\end{aligned}
$$

which, in matrix form, is simply

$$
\mathbf{G}(\alpha)^{n-a} = \mathbf{G}_{n,a,b}(\alpha) + \hat{\mathbf{G}}_{n,b,a}(\alpha)\mathbf{G}(\alpha)^{b-a}.
$$

A similar argument further reveals that

$$
\hat{\mathbf{G}}(\alpha)^{b-n} = \mathbf{G}_{n,a,b}(\alpha)\hat{\mathbf{G}}(\alpha)^{b-a} + \hat{\mathbf{G}}_{n,b,a}(\alpha).
$$

Solving this resulting system consisting of two matrix equations with two matrix unknowns, while making use the fact that $(\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a})^{-1}$ and $(\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a})^{-1}$ exist due to both $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ having spectral radius strictly less than one, yields

$$
\mathbf{G}_{n,a,b}(\alpha) = [\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-n}\mathbf{G}(\alpha)^{b-n}]\mathbf{G}(\alpha)^{n-a}[\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a}]^{-1}
$$

and

$$
\hat{\mathbf{G}}_{n,b,a}(\alpha) = [\mathbf{I} - \mathbf{G}(\alpha)^{n-a}\hat{\mathbf{G}}(\alpha)^{n-a}]\hat{\mathbf{G}}(\alpha)^{b-n}[\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a}]^{-1}
$$

which proves the claim. $\diamondsuit$

A similar type of result also holds within the context of **R**-matrices. For each subset $A \subset \mathbb{Z}$, each integer $m \in A$, and each integer $n \in A^c$, we define the matrix $\mathbf{R}_{m,A,n}(\alpha)$ as follows: for each

$i,j \in \{1,2,\ldots,d\}$,

$$(\mathbf{R}_{m,A,n}(\alpha))_{i,j} := (q((m,i))+\alpha)\mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_A}} e^{-\alpha t}\mathbf{1}(F(t)=(n,j))dt\right]$$

where $L_A := \bigcup_{m\in A} L_m$.

Given the homogeneous structure present among the block structure of $\mathbf{Q}$, it is well-known (see e.g. [25]) that for each $m \in \mathbb{Z}$, and each $n > m$, that

$$\mathbf{R}_{m,\{m\},n}(\alpha) = \mathbf{R}_{0,\{0\},1}(\alpha)^{n-m}.$$

Likewise, for each $m \in \mathbb{Z}$ and each $n < m$, we have

$$\mathbf{R}_{m,\{m\},n}(\alpha) = \mathbf{R}_{0,\{0\},-1}(\alpha)^{m-n}$$

so it is useful to define the matrices $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$ as

$$\mathbf{R}(\alpha) := \mathbf{R}_{0,\{0\},1}(\alpha), \quad \hat{\mathbf{R}}(\alpha) := \mathbf{R}_{0,\{0\},-1}(\alpha).$$

Our next result, Lemma 3.2.2, provides us with a way of expressing, for $a < n < b$, the matrices $\mathbf{R}_{a,\{a,b\},n}(\alpha)$ and $\mathbf{R}_{b,\{a,b\},n}(\alpha)$ in terms of $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$. Readers should compare the proof we provide of this result with the proof of Lemma 10.3.1 from [32], which instead addresses the case where $\alpha = 0$ (and instead addresses the discrete-time case).

Lemma 3.2.2 *Fix two integers a, b such that $a < b$. Then for each integer $n \in \{a+1, a+2, \ldots, b-2, b-1\}$, we have*

$$\mathbf{R}_{a,\{a,b\},n}(\alpha) = (\mathbf{I}-\mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-a})^{-1}\mathbf{R}(\alpha)^{n-a} - (\mathbf{I}-\mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-a})^{-1}\mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-n} \tag{3.3}$$

and

$$\mathbf{R}_{b,\{a,b\},n}(\alpha) = -(\mathbf{I}-\hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{b-a})^{-1}\hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{n-a} + (\mathbf{I}-\hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{b-a})^{-1}\hat{\mathbf{R}}(\alpha)^{b-n}. \tag{3.4}$$

Proof This result can be established by using the random-product technique: see [5, 9, 11], as well as [25] for an example of how the random-product technique was first applied to the theory of Markov processes of G/M/1-type. In order to use the random-product technique, we associate with $\{F(t); t \geq 0\}$ an alternative CTMC $\{\tilde{F}(t); t \geq 0\}$ whose generator $\tilde{\mathbf{Q}}$ satisfies two properties:

(i) For each $x, y \in S$ satisfying $x \neq y$, $\tilde{q}(x, y) > 0$ if and only if $q(y, x) > 0$;

(ii) For each $x \in S$, $\sum_{y \neq x} \tilde{q}(x, y) = \sum_{y \neq x} q(x, y)$.

In light of the homogeneous structure of $\mathbf{Q}$, we can choose $\tilde{\mathbf{Q}}$ so that it also exhibits a block-partitioned structure that is constructed using only three matrices $\tilde{\mathbf{A}}_{-1}, \tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1 \in \mathbb{R}^{d \times d}$, where for each $i, j \in \{1, 2, \ldots, d\}$ (where possibly $i = j$), and each $n \in \mathbb{Z}$,

$$\tilde{q}((n,i),(n-1,j)) = (\tilde{\mathbf{A}}_{-1})_{i,j}, \quad \tilde{q}((n,i),(n,j)) = (\tilde{\mathbf{A}}_0)_{i,j}, \quad \tilde{q}((n,i),(n+1,j)) = (\tilde{\mathbf{A}}_1)_{i,j}$$

and for any two integers n, m satisfying $|n-m| \geq 2$, $\tilde{q}((n,i),(m,j)) = 0$. for each $i, j \in \{1, 2, \ldots, d\}$.

We further associate with $\{\tilde{F}(t); t \geq 0\}$ the DTMC $\{\tilde{F}_n\}_{n \geq 0}$, where $\tilde{F}_0 := \tilde{F}(0)$, and for each integer $n \geq 1$, $\tilde{F}_n$ represents the state of $\{F(t); t \geq 0\}$ immediately after its nth transition time. We further associate with both $\{\tilde{F}(t); t \geq 0\}$ and $\{\tilde{F}_n\}_{n \geq 0}$ the following hitting-time random variables: for each subset $A \subset S$,

$$\tilde{\tau}_A := \inf\{t \geq 0 : F(t) \in A\}, \qquad \tilde{\eta}_A := \inf\{n \geq 0 : F_n \in A\}$$

and for each state $x \in S$, we set $\tilde{\tau}_x := \tilde{\tau}_{\{x\}}$ and $\tilde{\eta}_x := \tilde{\eta}_{\{x\}}$.

Next, recall from [25] that for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{R}(\alpha)^{n-a})_{i,j} := \mathbb{E}_{(n,j)} \left[\mathbf{1}(\tilde{\eta}_{L_a} < \infty) \mathbf{1}(\tilde{F}(\tilde{\tau}_{L_a}) = (a,i)) e^{-\tilde{\tau}_{L_a}} \prod_{\ell=1}^{\tilde{\eta}_{L_a}} \frac{q(\tilde{F}_\ell, \tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1}, \tilde{F}_\ell)} \right].$$

We can see from the Strong Markov property that

$$
\begin{aligned}
(\mathbf{R}(\alpha)^{n-a})_{i,j} &= \mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{L_{a,b}} < \infty)\mathbf{1}(\tilde{F}(\tilde{\tau}_{L_{a,b}}) = (a,i))e^{-\tilde{\tau}_{L_{a,b}}}\prod_{\ell=1}^{\tilde{\eta}_{L_{a,b}}}\frac{q(\tilde{F}_\ell,\tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1},\tilde{F}_\ell)}\right] \\
&\quad + \sum_{k=1}^{M}\mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{L_{a,b}} < \infty)\mathbf{1}(\tilde{F}(\tilde{\tau}_{L_{a,b}}) = (b,k))e^{-\tilde{\tau}_{L_{a,b}}}\prod_{\ell=1}^{\tilde{\eta}_{L_{a,b}}}\frac{q(\tilde{F}_\ell,\tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1},\tilde{F}_\ell)}\right](\mathbf{R}(\alpha)^{b-a})_{i,k} \\
&= (\mathbf{R}_{a,\{a,b\},n}(\alpha))_{i,j} + \sum_{k=1}^{M}(\mathbf{R}(\alpha)^{b-a})_{i,k}(\mathbf{R}_{b,\{a,b\},n}(\alpha))_{k,j}
\end{aligned}
$$

which implies

$$\mathbf{R}(\alpha)^{n-a} = \mathbf{R}_{a,\{a,b\},n}(\alpha) + \mathbf{R}(\alpha)^{b-a}\mathbf{R}_{b,\{a,b\},n}(\alpha).$$

A similar argument reveals that

$$\mathbf{R}(\alpha)^{b-n} = \hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}_{a,\{a,b\},n}(\alpha) + \hat{\mathbf{R}}_{b,\{a,b\},n}(\alpha)$$

which proves the claim. $\diamondsuit$

We close this section by noting that the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ can be calculated by using the iterative process explained in [24]. Once this is done, $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$ can be found by noting that

$$\mathbf{R}(\alpha) = \mathbf{A}_1(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1\mathbf{G}(\alpha))^{-1}$$

and

$$\hat{\mathbf{R}}(\alpha) = \mathbf{A}_{-1}(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}(\alpha))^{-1}.$$

These formulas are very well-known for the case where $\alpha = 0$: see e.g. Chapter 8 of [32].

3.3 Homogeneous QBD Processes with Finitely Many Levels

The matrices described in Lemmas 3.2.1 and 3.2.2 of the previous section can be used to study the time-dependent behavior of a homogeneous QBD process with finitely many levels. Suppose $\{Y(t); t \geq 0\}$ is a QBD process whose state space is given by S, where S is decomposed into a finite number of levels $L_0, L_1, \ldots, L_C$ for some integer $C \geq 1$, i.e.

$$S = \bigcup_{n=0}^{C} L_n.$$

We assume each level L_n is defined as

$$L_n := \{(n, 1), (n, 2), \ldots, (n, d)\}$$

for some fixed positive integer d. The transition rate matrix $\mathbf{Q} := [q(x, y)]_{x,y \in S}$ of $\{Y(t); t \geq 0\}$ can be expressed in block-partitioned form as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{-1} & \mathbf{A}_0 & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{-1} & \mathbf{C}_0 \end{pmatrix}$$

where $\mathbf{0} \in \mathbb{R}^{d \times d}$ is the zero matrix, and $\mathbf{B}_0, \mathbf{C}_0, \mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1 \in \mathbb{R}^{d \times d}$ are structured so that $\mathbf{Q}$ satisfies the properties of a generator matrix associated with an irreducible, stable, and conservative continuous-time Markov chain: in other words, each off-diagonal element of $\mathbf{Q}$ is nonnegative, each diagonal element of $\mathbf{Q}$ is strictly negative and finite, and for each fixed row of $\mathbf{Q}$, the elements of that row always sum to zero. Readers should note the word 'stable' used here does not refer to positive recurrence, rather, it refers to the fact that each row sum of $\mathbf{Q}$ is zero: this terminology is commonly used in the literature on continuous-time Markov chains, see for instance the text of Anderson [2]. Having said this, due to S being finite and $\{Y(t); t \geq 0\}$ being irreducible, we may also conclude that $\{Y(t); t \geq 0\}$ is also 'stable' in the sense of positive recurrence. Note too that the number of elements in L_0 and L_C could possibly be different from d, but in the interest of readability we assume throughout that each level contains d states.

The block-partitioned structure exhibited above by $\mathbf{Q}$ corresponds to the way S is decomposed into levels, as the order of the rows and columns of $\mathbf{Q}$ corresponds to the states of S being ordered lexicographically, meaning $(i_1, j_1) < (i_2, j_2)$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. Moreover, for each $i, j \in \{1, 2, \ldots, d\}$, where possibly $i = j$, we have (i)

$$q((0,i),(0,j)) = (\mathbf{B}_0)_{i,j}, \quad q((C,i),(C,j)) = (\mathbf{C}_0)_{i,j};$$

(ii) for each integer $n \in \{0, 1, \ldots, C-1\}$,

$$q((n,i),(n+1,j)) = (\mathbf{A}_1)_{i,j};$$

(iii) for each integer $n \in \{1, 2, \ldots, C\}$,

$$q((n,i),(n-1,j)) = (\mathbf{A}_{-1})_{i,j};$$

and finally (iv) for each integer $n \in \{1, 2, \ldots, C-1\}$,

$$q((n,i),(n,j)) = (\mathbf{A}_0)_{i,j}.$$

We will also need to make use of hitting-time random variables associated with $\{Y(t); t \geq 0\}$. For each subset A of S, we define

$$\tau_A := \inf\{t \geq 0 : Y(t-) \neq Y(t) \in A\}$$

where for each $t > 0$, $Y(t-) := \lim_{s \uparrow t} Y(s)$ is the left-hand-limit of Y at t.

3.3.1 Distribution of the time it takes to reach a level

For each $m, n \in \{0, 1, \ldots, C\}$ satisfying $m > n$, we define the matrix $\mathbf{G}_{m,n}(\alpha)$ as follows: for each $i, j \in \{1, 2, \ldots, M\}$, we have

$$(\mathbf{G}_{m,n}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}[e^{-\alpha\tau_{L_n}}\mathbf{1}(Y(\tau_{L_n}) = (n, j))].$$

Similarly, for each $m, n \in \{0, 1, \ldots, C\}$ satisfying $m < n$, we define the matrix $\hat{\mathbf{G}}_{m,n}(\alpha)$ as

$$(\hat{\mathbf{G}}_{m,n}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}[e^{-\alpha\tau_{L_n}}\mathbf{1}(Y(\tau_{L_n}) = (n, j))].$$

We further define, for $0 \le a < n < b \le C$, the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$ as

$$\begin{aligned}(\mathbf{G}_{n,a,b}(\alpha))_{i,j} &:= \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{a,b}}}\mathbf{1}(Y(\tau_{L_{a,b}}) = (a, j))],\\ (\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,j} &:= \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{a,b}}}\mathbf{1}(Y(\tau_{L_{a,b}}) = (b, j))]\end{aligned}$$

where just as in the previous section, $L_{a,b} := L_a \cup L_b$. It is easy to see that the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$ are equal to the matrices we defined in the previous section.

Our next proposition provides us with the matrices needed in order to derive the Laplace-Stieltjes transform of the amount of time it takes $\{Y(t); t \ge 0\}$ to move from one fixed level to another fixed level.

Proposition 3.3.1 *The matrices* $\{\mathbf{G}_{m,n}(\alpha)\}_{0 \le m,n \le C; m \ne n}$ *are as follows: (i) first,*

$$\hat{\mathbf{G}}_{0,1}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1, \qquad \mathbf{G}_{C,C-1}(\alpha) = (\alpha\mathbf{I} - \mathbf{C}_0)^{-1}\mathbf{A}_{-1}. \tag{3.5}$$

(ii) For each integer $n \ge 2$,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = [\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)]^{-1}\,\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha). \tag{3.6}$$

(iii) For each integer $n \le C - 2$,

$$\mathbf{G}_{C,n}(\alpha) = \left[\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha)\right]^{-1}\mathbf{A}_{-1}\mathbf{G}_{C-1,n,C}(\alpha). \tag{3.7}$$

(iv) For each integer $m \in \{1, 2, \ldots, C-1\}$ and each integer $n \in \{m+1, m+2, \ldots, C\}$,

$$\hat{\mathbf{G}}_{m,n}(\alpha) = \hat{\mathbf{G}}_{m,n,0}(\alpha) + \mathbf{G}_{m,0,n}(\alpha)[\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)]^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha). \tag{3.8}$$

(v) Finally, for each integer $m \in \{1, 2, \ldots, C-1\}$ and each integer $n \in \{0, 1, \ldots, m-1\}$,

$$\mathbf{G}_{m,n}(\alpha) = \mathbf{G}_{m,n,C}(\alpha) + \hat{\mathbf{G}}_{m,C,n}(\alpha)\left[\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha)\right]^{-1}\mathbf{A}_{-1}\mathbf{G}_{C-1,n,C}(\alpha). \tag{3.9}$$

Proof We first show that the matrices $(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))$ are invertible for each integer $n \geq 2$, and the matrices $(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha))$ are invertible for each $n \leq C-2$. Given a subset $A \subset \{0, 1, 2, \ldots, C\}$, we define for each $m \in A^c$ the matrix $\mathbf{N}_{m,A}(\alpha)$, defined as

$$(\mathbf{N}_{m,A}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))\right],$$

where for $A = \{a_1, a_2, \ldots a_k\}$, $\tau_A := \tau_{L_{a_1,a_2,\ldots,a_k}}$.

Fixing $i, j \in \{1, 2, \ldots, M\}$, we observe through a first-step analysis argument that

$$\begin{aligned}\mathbb{E}_{(0,i)}\left[\int_0^{\tau_{L_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (0,j))dt\right] &= \frac{\mathbf{1}(i=j)}{-(\mathbf{B}_0)_{i,i} + \alpha} \\ &+ \sum_{k \neq i} \frac{(\mathbf{B}_0)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha}(\mathbf{N}_{0,\{n\}}(\alpha))_{k,j} \\ &+ \sum_{k} \frac{(\mathbf{A}_1)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha}\mathbb{E}_{(1,i)}\left[\int_0^{\tau_{L_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (0,j))dt\right]\end{aligned}$$

and after applying the Strong Markov property to the remaining expectations and rewriting the equations in terms of matrices, we get

$$\alpha\mathbf{N}_{0,\{n\}}(\alpha) = \mathbf{I} + \mathbf{B}_0\mathbf{N}_{0,\{n\}}(\alpha) + \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\mathbf{N}_{0,\{n\}}(\alpha)$$

which implies

$$(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))\mathbf{N}_{0,\{n\}}(\alpha) = \mathbf{I}$$

which proves $(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))$ is invertible. A similar argument can be used to show that $(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\mathbf{G}_{C-1,C,n}(\alpha))$ is invertible, if we replace $\mathbf{N}_{0,\{n\}}(\alpha)$ with the matrix $\mathbf{N}_{C,\{n\}}(\alpha)$.

It remains to establish statements (3.5)-(3.9). The first equality found in (3.5) can be proven with a first-step analysis argument: for each $i, j \in \{1, 2, \ldots, M\}$, where possibly $i = j$, we have

$$(\hat{\mathbf{G}}_{0,1}(\alpha)))_{i,j} = \sum_{k \neq i} \frac{(\mathbf{B}_0)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha} (\hat{\mathbf{G}}_{0,1}(\alpha))_{k,j} + \frac{(\mathbf{A}_1)_{i,j}}{-(\mathbf{B}_0)_{i,i} + \alpha}$$

and these equations can alternatively be expressed in matrix form as

$$\hat{\mathbf{G}}_{0,1}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1.$$

The other equality found in statement (3.5) follows from an analogous argument.

We next prove statement (3.6): again, a first-step analysis argument can be used to show that for each integer $n \geq 2$,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n}(\alpha).$$

Furthermore,

$$\hat{\mathbf{G}}_{1,n}(\alpha) = \hat{\mathbf{G}}_{1,n,0}(\alpha) + \mathbf{G}_{1,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha).$$

Hence,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha) + (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha)$$

and solving for the single unknown matrix gives

$$\begin{aligned}\hat{\mathbf{G}}_{0,n}(\alpha) &= \left[\mathbf{I} - (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\right]^{-1} (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha) \\ &= \left[\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\right]^{-1} \mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha)\end{aligned}$$

proving (3.6). A similar argument can be used to establish (3.7).

Statement (3.8) follows from (3.6), once we notice that for $0 < m < n \leq C$,

$$\hat{\mathbf{G}}_{m,n}(\alpha) = \hat{\mathbf{G}}_{m,n,0}(\alpha) + \mathbf{G}_{m,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha)$$

and a similar argument can be used to show that (3.9) follows from (3.7). $\diamondsuit$

3.3.2 The Laplace Transforms of the Transition Functions

Together, Lemmas 3.2.1, 3.2.2, and Proposition 3.3.1 can be used to derive what appear to be new, computable expressions for the Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$. We assume throughout (and without loss of generality) that $Y(0) = (n_0, i_0)$ with probability one for some state $(n_0, i_0) \in S$. For each state $(n, j) \in S$, we define the transition function $p_{(n_0,i_0),(n,j)} : [0, \infty) \to [0, 1]$ as

$$p_{(n_0,i_0),(n,j)}(t) := \mathbb{P}(Y(t) = (n, j) \mid Y(0) = (n_0, i_0)), \quad t \geq 0.$$

Associated with $p_{(n_0,i_0),(n,j)}(t)$ is its Laplace transform $\pi_{(n_0,i_0),(n,j)}$ which is defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$ as

$$\pi_{(n_0,i_0),(n,j)}(\alpha) := \int_0^\infty e^{-\alpha t} p_{(n_0,i_0),(n,j)}(t)dt, \quad \alpha \in \mathbb{C}_+.$$

Our next result, Theorem 3.3.1, is stated in [24], and is a Laplace transform interpretation of an unlabeled result found at the top of page 124 of [32].

Theorem 3.3.1 *Suppose T and D are two disjoint subsets of S. Then for each $x \in T$, $y \in D$,*

$$\pi_{x,y}(\alpha) = \sum_{z \in T} \pi_{x,z}(\alpha)(q(z) + \alpha)\mathbb{E}_z \left[\int_0^{\tau_T} e^{-\alpha t} \mathbf{1}(Y(t) = y)dt \right], \quad \alpha \in \mathbb{C}_+$$

We can use Theorem 3.3.1 to establish a result that can be used to find the Laplace transform of the transitions functions of $\{Y(t); t \geq 0\}$. Since our results will be in matrix form, we define

$$\boldsymbol{\pi}_n(\alpha) := [\pi_{(n_0,i_0),(n,1)}(\alpha), \pi_{(n_0,i_0),(n,2)}(\alpha), \ldots, \pi_{(n,M)}(\alpha)], \quad \alpha \in \mathbb{C}_+.$$

We suppress the initial state (n_0, i_0) when we write $\boldsymbol{\pi}_n(\alpha)$, but readers should understand that these vectors depend on the initial state.

The next result, Theorem 3.3.2, provides an expression for the Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$ that is highly analogous to the expressions found in [18] for the stationary distribution of $\{Y(t); t \geq 0\}$, for the case where $\{Y(t); t \geq 0\}$ is non-null recurrent. Throughout, the vector $\mathbf{e}_i$ denotes the ith basis vector in $\mathbb{R}^{1\times d}$, where the ith component of $\mathbf{e}_i$ is equal to one, and all of its other components are equal to zero.

Theorem 3.3.2 *The Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$ are as follows. (i) If $n_0 = 0$, we see that for $1 \leq n \leq C-1$,*

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) &= \left[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\hat{\mathbf{R}}(\alpha)^C\right]\mathbf{R}(\alpha)^n \\ &+ \left[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1}\mathbf{R}(\alpha)^C + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\right]\hat{\mathbf{R}}(\alpha)^{C-n}. \quad (3.10)\end{aligned}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$ and $\boldsymbol{\pi}_C(\alpha)$ satisfy

$$\boldsymbol{\pi}_C(\alpha) = \boldsymbol{\pi}_0(\alpha)\mathbf{A}_1\hat{\mathbf{G}}_{1,C,0}(\alpha)(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,0}(\alpha))^{-1} \quad (3.11)$$

and

$$\boldsymbol{\pi}_0(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0}(\alpha))^{-1}. \quad (3.12)$$

(ii) If $n_0 = C$, we see that for $1 \leq n \leq C-1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) &= \left[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\hat{\mathbf{R}}(\alpha)^C\right]\mathbf{R}(\alpha)^n \\ &+ \left[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1}\mathbf{R}(\alpha)^C + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\right]\hat{\mathbf{R}}(\alpha)^{C-n}. \quad (3.13)\end{aligned}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$ and $\boldsymbol{\pi}_C(\alpha)$ satisfy

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_C(\alpha)\mathbf{A}_{-1}\mathbf{G}_{C-1,0,C}(\alpha)(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,C}(\alpha))^{-1} \quad (3.14)$$

and

$$\boldsymbol{\pi}_C(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C}(\alpha))^{-1} \quad (3.15)$$

(iii) Finally, suppose $n_0 \in \{1, 2, \ldots, C-1\}$*. For* $1 \le n \le n_0 - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) = &\Big[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{n_0}\hat{\mathbf{R}}(\alpha)^{n_0}]^{-1} - \boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{n_0}\mathbf{R}(\alpha)^{n_0}]^{-1}\hat{\mathbf{R}}(\alpha)^{n_0}\Big]\mathbf{R}(\alpha)^n \\ &+ \Big[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{n_0}\hat{\mathbf{R}}(\alpha)^{n_0}]^{-1}\mathbf{R}(\alpha)^{n_0} + \boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{n_0}\mathbf{R}(\alpha)^{n_0}]^{-1}\Big]\hat{\mathbf{R}}(\alpha)^{n_0-n}.\end{aligned} \tag{3.16}$$

For $n_0 + 1 \le n \le C - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) = &\Big[\boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{C-n_0}\hat{\mathbf{R}}(\alpha)^{C-n_0}]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{C-n_0}\mathbf{R}(\alpha)^{C-n_0}]^{-1}\hat{\mathbf{R}}(\alpha)^{C-n_0}\Big]\mathbf{R}(\alpha)^{n-n_0} \\ &+ \Big[-\boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{C-n_0}\hat{\mathbf{R}}(\alpha)^{C-n_0}]^{-1}\mathbf{R}(\alpha)^{C-n_0} + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{C-n_0}\mathbf{R}(\alpha)^{C-n_0}]^{-1}\Big]\hat{\mathbf{R}}(\alpha)^{C-n}.\end{aligned} \tag{3.17}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$*,* $\boldsymbol{\pi}_C(\alpha)$*, and* $\boldsymbol{\pi}_{n_0}(\alpha)$ *satisfy*

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{A}_{-1}\mathbf{G}_{n_0-1,0,n_0}(\alpha)(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n_0}(\alpha))^{-1} \tag{3.18}$$

$$\boldsymbol{\pi}_C(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{A}_1\hat{\mathbf{G}}_{n_0+1,C,n_0}(\alpha)(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n_0}(\alpha))^{-1} \tag{3.19}$$

and

$$\boldsymbol{\pi}_{n_0}(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{n_0-1,n_0}(\alpha) - \mathbf{A}_1\mathbf{G}_{n_0+1,n_0}(\alpha))^{-1}. \tag{3.20}$$

Proof We begin the proof by first setting up some additional notation. For each subset $A \subset \{0, 1, 2, \ldots, C\}$, each $m \in A$, and each $n \in A^c$, define the matrix $\mathbf{R}^{(0,C)}_{m,A,n}(\alpha)$ as follows: for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{R}^{(0,C)}_{m,A,n}(\alpha))_{i,j} := (q((m,i)) + \alpha)\mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_A}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right].$$

It is obvious from the transition structure of both $\{F(t); t \ge 0\}$ and $\{Y(t); t \ge 0\}$ that for $0 \le a < n < b \le C$,

$$\mathbf{R}^{(0,C)}_{a,\{a,b\},n}(\alpha) = \mathbf{R}_{a,\{a,b\},n}(\alpha)$$

and precisely the same can be said for $\mathbf{R}_{b,\{a,b\},n}(\alpha)$ and $\mathbf{R}^{(0,C)}_{b,\{a,b\},n}(\alpha)$.

We focus on case (iii) by establishing the validity of (3.16), (3.17), (3.18), (3.19), and (3.20), which all correspond to the case where $n_0 \in \{1, 2, \ldots, C-1\}$. Fix such an n_0: observe first that for $0 < n < n_0$, an application of Theorem 3.3.1, under the choice $T = L_0 \cup L_{n_0}$ yields

$$\boldsymbol{\pi}_n(\alpha) = \boldsymbol{\pi}_0(\alpha)\mathbf{R}_{0,\{0,n_0\},n}(\alpha) + \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{R}_{n_0,\{0,n_0\},n}(\alpha) \tag{3.21}$$

and applying both (3.3) and (3.4) to (3.21) yields (3.17). A similar argument can be used to establish (3.17) for $n \in \{n_0+1, \ldots, C-1\}$, where in that case we apply Theorem 3.3.1 while instead choosing $T = L_{n_0} \cup L_C$, then again applying Lemma 3.2.2.

The next step is to establish (3.18). Applying Theorem 3.3.1 while choosing $T = L_{n_0}$ gives

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{R}^{(0,C)}_{n_0,\{n_0\},0}(\alpha). \tag{3.22}$$

Conditioning on the first jump and using the strong Markov property we see that

$$\mathbf{R}^{(0,C)}_{n_0,\{n_0\},0}(\alpha) = \mathbf{A}_{-1}\mathbf{G}_{n_0-1,0,n_0}(\alpha)\mathbf{N}_{0,\{n_0\}}(\alpha) \tag{3.23}$$

where for each integer $n \neq n_0$, the matrix $\mathbf{N}_{n,\{n_0\}}(\alpha)$ is defined as

$$(\mathbf{N}_{n,\{n_0\}}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}\left[\int_0^{\tau_{L_{n_0}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right].$$

A first-step analysis argument can be used to show that

$$\mathbf{N}_{0,\{n_0\}}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n_0}(\alpha))^{-1}. \tag{3.24}$$

Plugging (3.24) into (3.23), and plugging that into (3.22) yields (3.18), and the same type of reasoning used to establish (3.18) can be used to establish (3.19).

It remains to derive (3.20). For each $m, n \in \{0, 1, 2, \ldots, C\}$, where possibly $m = n$, we define the matrix $\Pi_{m,n}(\alpha)$ as

$$\Pi_{m,n}(\alpha) := [\pi_{(m,i),(n,j)}(\alpha)]_{1 \le i,j \le M}.$$

From the Forward equations associated with $\{Y(t); t \geq 0\}$, we see that

$$\alpha\Pi_{n_0,n_0}(\alpha) - \mathbf{I} = \Pi_{n_0,n_0-1}(\alpha)\mathbf{A}_1 + \Pi_{n_0,n_0}(\alpha)\mathbf{A}_0 + \Pi_{n_0,n_0+1}(\alpha)\mathbf{A}_{-1}$$

which yields

$$\Pi_{0,0}(\alpha)(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1}) = \mathbf{I}$$

from which we get

$$\boldsymbol{\pi}_0(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1})^{-1}. \tag{3.25}$$

We now claim that

$$\mathbf{G}_{n_0+1,n_0}(\alpha) = \mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1}.$$

One way to show this involves a technique found in Chapter 9 of Brémaud [4], where a CTMC is thought of as being governed entirely by a countable collection of independent homogeneous Poisson processes. In our case, suppose $\{Y(t); t \geq 0\}$ is governed by the collection of Poisson processes $\{N_{x,y}(t)\}_{x,y\in S, x\neq y}$. Define $\theta(t) = 0$ if L_{n_0} has not been visited yet by time t and 1 otherwise. Then, if $Y(0) = (n_0 + 1, i)$,

$$\mathbf{1}(Y(\tau_{L_{n_0}}) = (n_0, j))e^{-\alpha\tau_{L_0}} = \sum_k \int_0^\infty e^{-\alpha t}\mathbf{1}(Y(t-) = (n_0+1, k), \theta(t-) = 0)N_{(n_0+1,k),(n_0,j)}(dt).$$

Taking the expectation of both sides while applying the Campbell-Mecke formula to the right-hand side gives

$$\mathbb{E}_{(n_0+1,i)}[\mathbf{1}(Y(\tau_{L_{n_0}}) = (n_0, j))e^{-\alpha\tau_{L_0}}] = \sum_k \mathbb{E}_{(n_0+1,i)}\left[\int_0^{\tau_{L_{n_0}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n_0+1, k))dt\right](\mathbf{A}_{-1})_{k,j}$$

or, in matrix form,

$$\mathbf{G}_{n_0+1,n_0}(\alpha) = \mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1}.$$

Thus we now have

$$\mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1} = \mathbf{A}_1\mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1} = \mathbf{A}_1\mathbf{G}_{n_0+1,n_0}(\alpha). \tag{3.26}$$

Analogously, we can also show

$$\hat{\mathbf{R}}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 = \mathbf{A}_{-1}\mathbf{N}_{n_0-1,\{n_0\}}(\alpha)\mathbf{A}_1 = \mathbf{A}_{-1}\hat{\mathbf{G}}_{n_0-1,n_0}(\alpha). \tag{3.27}$$

Substituting equations (3.26) and (3.27) into equation (3.25) gives (3.20), which concludes the proof of the theorem. $\diamondsuit$

Chapter 4

Markovian Bitcoin models

4.1 Introduction

Bitcoin is a decentralized digital payment system that allows users within the system to make transactions between one another without using a central authority (e.g. a bank) to manage the exchange of funds. Bitcoin transactions are stored in *blocks* which make up what is known as a *blockchain*, which is managed and updated by a collection of users known as *miners*. Note that technically, there is no single blockchain to speak of: instead, each miner keeps track of its own version of the blockchain, and the miners communicate with each other in order to come to an overall consensus, based on the Bitcoin protocol, on what blocks should be included in a blockchain. Readers interested in an introduction on how Bitcoin works are referred to the survey paper of Tschorsch and Scheuermann [41]: see also Franco [13] for a textbook-level introduction to Bitcoin.

Ideally, all miners will agree on the structure of the blockchain, but due to communication delays or possible deviations from the standard mining protocol, miners may have different versions of the blockchain for a period of time. Such discrepancies are not good if they exist for a relatively large period of time, as disagreements between blockchain versions could possibly lead to fraudulent behavior, such as double-spending attacks. In this chapter, we will study what happens to the Bitcoin network when there are communication delays between a smaller pool of miners and the rest of the network, both (i) when all miners are mining according to the Bitcoin protocol, and (ii) when the smaller pool use a strategy referred to in Eyal and Sirer [8] as Selfish Mining. Under Selfish Mining, a smaller pool of miners working together to mine blocks may choose to withhold information about recently discovered blocks in an attempt to earn more revenue in various ways. Suppose all miners have the same information and the Selfish Mining pool discovers a block. They will inform all others in the pool, but they will not inform others outside of the pool. Once they have established a lead of two or more, the pool can publish a block every time the honest community mines a block and the pool publishes two blocks if their lead has been reduced to one. In this way, the pool allows the honest community to waste their time mining blocks that never had a chance to be included in the blockchains of all miners, as miners will always seek to add blocks to the largest chain within the blockchain: such blocks that are not accepted by other miners are often referred to as *stale blocks*, or *orphan blocks*.

Our objective is to present a detailed study of two CTMC models introduced in Göbel et al. [16], which were introduced in order to better understand how Bitcoin is affected when a smaller pool of miners implement Selfish Mining, in order to gain an advantage over the larger group of miners in the system. In each model we consider, it is assumed that all miners in the smaller pool can communicate instantaneously with one another, all miners in the larger group can communicate instantaneously with one another, but there are communication delays between the smaller pool and the larger group. Readers should keep in mind that these assumptions are still far from realistic, given that in reality, communication delays will exist between individual members of the smaller pool, as well as between individual miners outside of the pool, yet one could argue that these models are still interesting, as they illustrate how Selfish Mining can affect the overall network.

The analysis technique used in both models involves usage of the recently-discovered random product technique introduced in [5]. Interestingly, many of the ideas we use to study the CTMC from [16] that captures Selfish Mining are very similar to ideas often used in the matrix-analytic community: experts in that field will recognize many of these ideas within the proof of Theorem 4.3.1, for example, even though the main ideas are being applied in a somewhat nontraditional manner. Readers interested in seeing how the random product technique from [5] can be used to re-derive many classical results from the area of matrix-analytic methods are referred to [25]. More traditional approaches to the theory of matrix-analytic methods can be found in the textbooks of Latouche and Ramaswami [32] and He [19].

This chapter is organized as follows. Section 4.2 considers first the case where all participants mine in an honest manner: for this model, we present a new derivation of the stationary distribution, and we also present a closed-form expression for $p_{(0,0)}$, which represents the long-run fraction of time both groups completely agree on the structure of the blockchain. Next, we also show how to derive similar expressions for the Laplace transforms of the transition functions of this model, under the assumption that both groups agree on the structure of the blockchain at time zero. In Section 4.3, we derive the stationary distribution of the CTMC introduced in [16] which attempts to model the case where the smaller pool of miners implement Selfish Mining. We also show that similar expressions can be derived for the Laplace transforms of the transition functions as well, if we again assume that both the pool and the group agree on the structure of the blockchain at time zero. One key step in the derivation of the stationary distribution of the Selfish Mining CTMC involves usage of an idea that is very similar to the idea often used to show a quasi-birth-death process has a matrix-geometric stationary distribution, and parts of the algorithm we use for calculating certain elements of the stationary distribution of the Selfish Mining CTMC involve use of a recursion that is similar in structure to Ramaswami's formula. We conclude this chapter in Section 4.4, by briefly discussing other generalizations that can be analyzed with our approach.

We close this introduction by mentioning a few other studies of aspects of Bitcoin that involve the use of models and techniques from applied probability: mentioning all relevant studies of Bitcoin is impossible, considering that the paper of Nakamoto [38] has been cited close to 6000 times as of now. The papers of Kasahara and Kawahara [27] and Kawase and Kasahara [28] use two different variations of the M/G/1 queue with batch services to model the amount of time it takes a new Bitcoin transaction to be included within a mined block. The papers of Li et al. [34, 35] each use a matrix-analytic approach towards modeling transaction-confirmation times, with the model found in [35] being a generalization of the model from [34]. The paper of Huberman et al. [20] studies the behavior of the transaction fees associated with arriving transactions to the network, as well as the behavior of the waiting time of an arbitrary transaction until it is included in a block. In the work of Frolkova and Mandjes [14], a notion of one-sided communication between two miners in the network is modeled with a $G/M/\infty$ queue with synchronized departures, which models instances where one particular miner (miner A) has more information about how many blocks have been mined than another (miner B), and there is a delay in the amount of time it takes miner A to inform miner B of the existence of a block. Expressions for various performance measures of this system are given in [14], and the authors of [14] also show that their system can be approximated with a growth-collapse model under a fluid-scaling. A follow-up study to [14] can be found in [10], which shows that many different generalizations of the $G/M/\infty$ model introduced in [14] can be studied in multiple ways with techniques from the theory of point processes. Finally, in the work of Bowden et al. [3], the authors present various point processes that seek to model the time instances when mined blocks are accepted and added to a blockchain: there they argue that these time instances are not necessarily closely modeled by points from a homogeneous Poisson process.

4.2 When all miners are honest

We first consider the case where both the smaller pool and the larger pool behave honestly, and follow the Bitcoin protocol. In order to model honest mining behavior among both pools, Göbel et al. [16] introduced the CTMC $\{X(t); t \geq 0\}$ whose state space is given by $S := \{(i,j) : i \geq 0, j \geq 0\}$, and whose generator (transition rate matrix) is given by $\mathbf{Q} := [q(x,y)]_{x,y \in S}$. The elements of $\mathbf{Q}$ are defined as follows: given positive rates λ_1, λ_2, and μ, we have that for any two distinct states $(i,j),(k,\ell) \in S$,

$$q((i,j),(k,\ell)) := \begin{cases} \lambda_1, & k = i+1,\, \ell = j; \\ \lambda_2, & k = i,\, \ell = j+1; \\ \mu, & k = \ell = 0,\, i \neq j; \end{cases} \tag{4.1}$$

with all other off-diagonal entries of $\mathbf{Q}$ set equal to zero. The diagonal elements $\{q(x,x)\}_{x \in S}$ of $\mathbf{Q}$ satisfy

$$q(x,x) := -q(x) \tag{4.2}$$

where $q(x)$ is the sojourn rate associated with each exponential sojourn time spent in state x by $\{X(t); t \geq 0\}$. Later it will help to partition S into a collection of diagonal subsets $\{D_k\}_{k \in \mathbb{Z}}$ of S, where for each $k \in \mathbb{Z}$,

$$D_k := \{(i,j) : i \geq 0,\ j \geq 0,\ j - i = k\}. \tag{4.3}$$

This partitioning also makes it easier to describe the diagonal elements of $\mathbf{Q}$: indeed, $q(x) := \lambda_1 + \lambda_2$ for each state $x \in D_0$, while for each integer $k \neq 0$, and each state $x \in D_k$, $q(x) := \lambda_1 + \lambda_2 + \mu$. A picture of the rate diagram can be found in Figure 4.1.

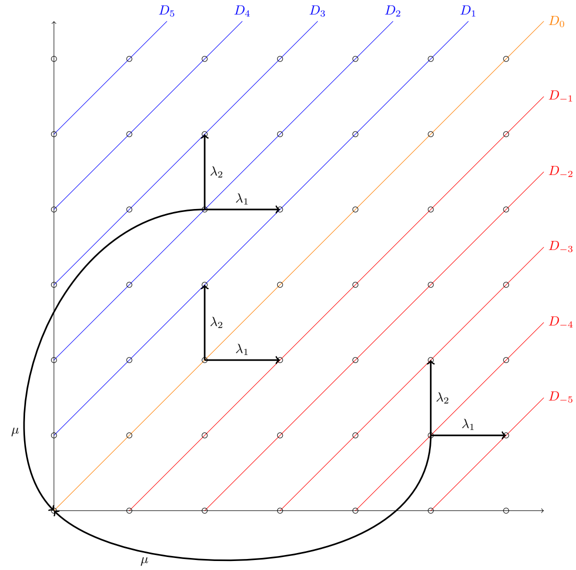

Figure 4.1: Transition rate diagram when all miners are honest

4.2.1 Hitting Times

An important random variable associated with the CTMC $\{X(t); t \geq 0\}$ is the amount of time it takes this chain to reach state $(0, 0)$, as this corresponds to the state where the blockchain versions of each of the two pools agree. For each state $x \in S$, define

$$\tau_x := \inf\{t \geq 0 : X(t-) \neq X(t) = x\} \tag{4.4}$$

where $X(t-) := \lim_{s \uparrow t} X(s)$ is the left-hand limit of X at time t. Observe that when $X(0) \neq x$, τ_x is simply the amount of time it takes $\{X(t); t \geq 0\}$ to reach state x: however, when $X(0) = x$, τ_x is the amount of time it takes $\{X(t); t \geq 0\}$ to *return* to state x. More generally, for each subset A of S, we define

$$\tau_A := \inf\{t \geq 0 : X(t-) \in A^c, X(t) \in A\}$$

which represents the first time $\{X(t); t \geq 0\}$ makes a transition into the set A.

The following proposition shows how to calculate both the first moment, as well as the Laplace-Stieltjes transform (LST) of $\tau_{(0,0)}$, when $X(0) = (i, j)$ for each state $(i, j) \in S$, but before stating this result we first need to define some additional quantities. For each $\alpha \in \mathbb{C}_+ := \{\alpha \in \mathbb{C} :$

$Re(\alpha) > 0\}$, the open halfplane consisting of all complex numbers having a positive real part, let $\phi_1(\alpha)$ denote the Laplace-Stieltjes transform (evaluated at α) of the busy period of an M/M/1 queue whose arrival rate is λ_1, and whose service rate is λ_2. Similarly, let $\phi_2(\alpha)$ denote the Laplace-Stieltjes transform of the busy period of an M/M/1 queue whose arrival rate is λ_2, and whose service rate is λ_1. Recall that for $\alpha \in \mathbb{C}_+$,

$$\phi_1(\alpha) = \frac{\lambda_1 + \lambda_2 + \alpha - \sqrt{(\lambda_1 + \lambda_2 + \alpha)^2 - 4\lambda_1\lambda_2}}{2\lambda_1} \tag{4.5}$$

and furthermore $\phi_2(\alpha) = \lambda_1\phi_1(\alpha)/\lambda_2$, so clearly $\lambda_2\phi_2(\alpha) = \lambda_1\phi_1(\alpha)$.

Proposition 4.2.1 *The law of $\tau_{(0,0)}$ under the probability measure $\mathbb{P}_{(i,j)}$ satisfies the following properties.*

(a) For each integer $i \geq 1$, we have

$$\mathbb{E}_{(i,i)}[e^{-\alpha\tau_{(0,0)}}] = \mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}]. \tag{4.6}$$

(b) For each integer $k \leq -1$, and each state $(i,j) \in D_k$,

$$\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \phi_1(\alpha+\mu)^{i-j}\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\mu}{\mu+\alpha}(1 - \phi_1(\alpha+\mu)^{i-j}). \tag{4.7}$$

(c) For each integer $k \geq 1$, and each state $(i,j) \in D_k$,

$$\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \phi_2(\alpha+\mu)^{j-i}\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\mu}{\mu+\alpha}(1 - \phi_2(\alpha+\mu)^{j-i}). \tag{4.8}$$

Finally,

$$\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\mu}{\mu+\alpha}\left[\frac{\frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}(1-\phi_1(\alpha+\mu)) + \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}(1-\phi_2(\alpha+\mu))}{1 - \left(\frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\phi_1(\alpha+\mu) + \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}\phi_2(\alpha+\mu)\right)}\right] \tag{4.9}$$

and

$$\mathbb{E}_{(0,0)}[\tau_{(0,0)}] = \frac{1 + \frac{1}{\mu}\left(\lambda_1(1-\phi_1(\mu)) + \lambda_2(1-\phi_2(\mu))\right)}{\lambda_1(1-\phi_1(\mu)) + \lambda_2(1-\phi_2(\mu))}. \tag{4.10}$$

Proof We first establish (4.6): for each state $(i,i) \in D_0$, $i \geq 1$, we can show through a 'sum-over-paths from (i,i) to $(0,0)$ approach' that for each $\alpha \in \mathbb{C}_+$,

$$\mathbb{E}_{(i,i)}[e^{-\alpha\tau_{(0,0)}}] = \mathbb{E}[e^{-\alpha\tau_{(0,0)}}]. \tag{4.11}$$

We omit the details of the proof, as the result can be well-understood on an intuitive level, given the structure of $\mathbf{Q}$.

We now establish (4.7): fix a state (i,j) satisfying $i > j$. Given the dynamics of $\{X(t); t \geq 0\}$, observe that under the measure $\mathbb{P}_{(i,j)}$, $\tau_{(0,0)}$ can be expressed as

$$\tau_{(0,0)} = \tau_{D_0} + (\tau_{(0,0)} - \tau_{D_0}) \tag{4.12}$$

where under $\mathbb{P}_{(i,j)}$,

$$\tau_{D_0} := \inf\{t \geq 0 : X(t) \in D_0\} \tag{4.13}$$

represents the amount of time it takes $\{X(t); t \geq 0\}$ to reach the diagonal D_0.

We can use the strong Markov property, applied at the stopping time τ_{D_0} to make the following claim about the joint distribution of τ_{D_0} and $\tau_{(0,0)} - \tau_{D_0}$. Let e_μ and γ_1 denote two independent random variables, where e_μ is exponentially distributed with rate μ, and γ_1 is equal in distribution to the amount of time it takes an M/M/1 queueing system, having arrival rate λ_1 and service rate λ_2, to move from state $i - j$ to state 0. From the transition structure of $\mathbf{Q}$, we can see that

$$\tau_{D_0} \stackrel{d}{=} \min(e_\mu, \gamma_1).$$

To see why, observe that while the chain is in the set $\cup_{k \leq -1} D_k$, each transition to the East corresponds to a movement from a diagonal D_j to a diagonal D_{j-1}, which corresponds to an arrival from an M/M/1 queueing system with arrival rate λ_1. Similarly, each transition to the North corresponds to a movement from a diagonal D_j to a diagonal D_{j+1}, which corresponds to a service completion from an M/M/1 queueing system with service rate λ_2; finally, a transition from a state in $\cup_{k \leq -1} D_k$ to state $(0,0)$ corresponds to an exponential clearing instant (which removes all work from the M/M/1 queue) with rate μ.

Using the strong Markov property, we can see that under the measure $\mathbb{P}_{(i,j)}$, if $\{X(t); t \geq 0\}$ reaches $D_0 \backslash \{(0,0)\}$ before it reaches state $(0,0)$, then $\tau_{(0,0)} - \tau_{D_0}$ is equal in distribution to a random variable Z, which, by (4.6), is equal in distribution to $\tau_{(0,0)}$ under the law $\mathbb{P}_{(0,0)}$, and independent of the process up to the stopping time τ_{D_0}. Otherwise, if $\{X(t); t \geq 0\}$ reaches $(0,0)$ before $D_0 \backslash \{(0,0)\}$, then we set $\tau_{(0,0)} - \tau_{D_0}$ to be zero. Thus,

$$\tau_{(0,0)} - \tau_{D_0} \stackrel{d}{=} \mathbf{1}(e_\mu > \gamma_1) Z$$

and moreover,

$$(\tau_{D_0}, \tau_{(0,0)} - \tau_{D_0}) \stackrel{d}{=} (\min(e_\mu, \gamma_1), \mathbf{1}(e_\mu > \gamma_1) Z) \tag{4.14}$$

where e_μ, γ_1, and Z are all independent of each other.

The next step is to express $\mathbb{E}_{(i,j)} [e^{-\alpha \tau_{(0,0)}}]$ in terms of $\mathbb{E}_{(0,0)}[e^{-\alpha \tau_{(0,0)}}]$. Using (4.12) and(4.14), we get

$$\mathbb{E}_{(i,j)}[e^{-\alpha \tau_{(0,0)}}] = \mathbb{E}\left[e^{-\alpha(\min(e_\mu, \gamma_1) + \mathbf{1}(\gamma_1 < e_\mu) Z)}\right] \tag{4.15}$$

and this Laplace-Stieltjes transform has a closed-form representation: first, observe that conditioning on both e_μ and γ_1 gives

$$\begin{aligned}
\mathbb{E}\left[e^{-\alpha(\min(e_\mu, \gamma_1) + \mathbf{1}(\gamma_1 < e_\mu) Z)}\right] &= \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} e^{-\alpha \mathbf{1}(\gamma_1 < e_\mu) Z}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} e^{-\alpha \mathbf{1}(\gamma_1 < e_\mu) Z} \mid e_\mu, \gamma_1\right]\right] \\
&= \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbb{E}\left[e^{-\alpha \mathbf{1}(\gamma_1 < e_\mu) Z} \mid e_\mu, \gamma_1\right]\right].
\end{aligned} \tag{4.16}$$

Second, we simplify the inner conditional expectation within (4.16) by summing over indicator functions in the following manner:

$$
\begin{aligned}
\mathbb{E}\left[e^{-\alpha \mathbf{1}(\gamma_1<e_\mu)Z} \mid e_\mu, \gamma_1\right] &= \mathbb{E}\left[e^{-\alpha \mathbf{1}(\gamma_1<e_\mu)Z} \mid e_\mu, \gamma_1\right] \mathbf{1}(e_\mu < \gamma_1) \\
&+ \mathbb{E}\left[e^{-\alpha \mathbf{1}(\gamma_1<e_\mu)Z} \mid e_\mu, \gamma_1\right] \mathbf{1}(\gamma_1 < e_\mu) \\
&= \mathbf{1}(e_\mu < \gamma_1) + \mathbb{E}[e^{-\alpha Z} \mid \gamma_1, e_\mu] \mathbf{1}(\gamma_1 < e_\mu) \\
&= \mathbf{1}(e_\mu < \gamma_1) + \mathbb{E}_{(0,0)}\left[e^{-\alpha \tau_{(0,0)}}\right] \mathbf{1}(\gamma_1 < e_\mu). \qquad (4.17)
\end{aligned}
$$

After plugging (4.17) into (4.16), we conclude that

$$
\begin{aligned}
\mathbb{E}_{(i,j)}\left[e^{-\alpha \tau_{(0,0)}}\right] &= \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbf{1}(e_\mu < \gamma_1)\right] \\
&+ \mathbb{E}_{(0,0)}\left[e^{-\alpha \tau_{(0,0)}}\right] \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbf{1}(\gamma_1 < e_\mu)\right]. \qquad (4.18)
\end{aligned}
$$

The next step is to simplify the two unknown expectations appearing in (4.18) that are expressed in terms of e_μ and γ_1. First,

$$
\begin{aligned}
\mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbf{1}(\gamma_1 < e_\mu)\right] &= \mathbb{E}\left[e^{-\alpha \gamma_1} \mathbf{1}(\gamma_1 < e_\mu)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-\alpha \gamma_1} \mathbf{1}(\gamma_1 < e_\mu) \mid \gamma_1\right]\right] \\
&= \mathbb{E}\left[e^{-\alpha \gamma_1} e^{-\mu \gamma_1}\right] \\
&= \mathbb{E}\left[e^{-(\mu+\alpha)\gamma_1}\right] = \phi_1(\mu+\alpha)^{i-j}. \qquad (4.19)
\end{aligned}
$$

The other unknown expectation satisfies

$$
\begin{aligned}
\mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbf{1}(e_\mu < \gamma_1)\right] &= \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)}\right] - \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)} \mathbf{1}(\gamma_1 < e_\mu)\right] \\
&= \mathbb{E}\left[e^{-\alpha \min(e_\mu, \gamma_1)}\right] - \phi_1(\mu+\alpha)^{i-j} \qquad (4.20)
\end{aligned}
$$

and by using Fubini's Theorem,

$$
\begin{aligned}
\mathbb{E}\left[e^{-\alpha\min(e_\mu,\gamma_1)}\right] &= \mathbb{E}\left[1-\left(1-e^{-\alpha\min(e_\mu,\gamma_1)}\right)\right] \\
&= 1-\mathbb{E}\left[\int_0^{\min(e_\mu,\gamma_1)}\alpha e^{-\alpha y}dy\right] \\
&= 1-\alpha\int_0^\infty e^{-\alpha y}\mathbb{P}(e_\mu > y, \gamma_1 > y)dy \\
&= 1-\alpha\int_0^\infty e^{-(\alpha+\mu)y}\mathbb{P}(\gamma_1 > y)dy \\
&= 1-\frac{\alpha}{\mu+\alpha}(1-\phi_1(\mu+\alpha)^{i-j}) \qquad (4.21)
\end{aligned}
$$

which means that

$$
\begin{aligned}
\mathbb{E}\left[e^{-\alpha\min(e_\mu,\gamma_1)}\mathbf{1}(e_\mu < \gamma_1)\right] &= 1-\frac{\alpha}{\mu+\alpha}(1-\phi_1(\mu+\alpha)^{i-j})-\phi_1(\mu+\alpha)^{i-j} \\
&= \frac{\mu}{\mu+\alpha}(1-\phi_1(\mu+\alpha)^{i-j}). \qquad (4.22)
\end{aligned}
$$

Plugging both (4.19) and (4.22) into (4.18) gives

$$
\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\mu}{\mu+\alpha}(1-\phi_1(\mu+\alpha)^{i-j})+\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}]\phi_1(\mu+\alpha)^{i-j} \qquad (4.23)
$$

which establishes (4.7). Furthermore, due to the symmetry present in the transition structure of $\{X(t); t \geq 0\}$, we can clearly see that for each state (i,j) satisfying $i < j$,

$$
\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\mu}{\mu+\alpha}(1-\phi_2(\mu+\alpha)^{j-i})+\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}]\phi_2(\mu+\alpha)^{j-i}. \qquad (4.24)
$$

thus proving (4.8).

It remains to show both (4.9) and (4.10). Conditioning on the first transition shows that

$$
\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\mathbb{E}_{(1,0)}[e^{-\alpha\tau_{(0,0)}}]+\frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}\mathbb{E}_{(0,1)}[e^{-\alpha\tau_{(0,0)}}]. \qquad (4.25)
$$

Plugging both (4.7) and (4.8) into the right-hand-side of (4.25) and solving for the single unknown $\mathbb{E}[e^{-\alpha\tau_{(0,0)}}]$ yields (4.9), while (4.10) follows immediately from (4.9) by taking derivatives and setting $\alpha = 0$. $\diamondsuit$

4.2.2 Deriving the stationary distribution

Our first task is to present a new derivation of the stationary distribution $\mathbf{p} := [p_y]_{y \in S}$ of this CTMC, which exists when λ_1, λ_2, and μ are all positive. We derive $\mathbf{p}$ by making use of a lattice path counting technique from the recent paper [36], which itself involves usage of the random-product technique introduced in [5]. Given our CTMC $\{X(t); t \geq 0\}$ with state space S and generator $\mathbf{Q}$, we construct another CTMC $\{\tilde{X}(t); t \geq 0\}$ whose state space is also S, but whose generator $\tilde{\mathbf{Q}}$ satisfies the following two properties: (i) for each pair of distinct states $x, y \in S$,

$$\tilde{q}(x,y) > 0 \quad \text{if and only if} \quad q(y,x) > 0 \tag{4.26}$$

and (ii) for each state $x \in S$,

$$\sum_{y \neq x} \tilde{q}(x,y) = \sum_{y \neq x} q(x,y). \tag{4.27}$$

Observe that one possible choice for $\tilde{\mathbf{Q}}$ is the generator of the time-reversal of $\{X(t); t \geq 0\}$, but choosing this generator requires knowledge of the stationary distribution $\mathbf{p}$, which we do not know. Fortunately, our analysis will not require us to choose a specific $\tilde{\mathbf{Q}}$: what is important here is the structure of the transition diagram of $\tilde{\mathbf{Q}}$—which is completely determined by the structure of the transition diagram of $\mathbf{Q}$—not the actual values of the transition rates within $\tilde{\mathbf{Q}}$.

Further associated with $\{\tilde{X}(t); t \geq 0\}$ is its collection of transition times $\{\tilde{T}_n\}_{n \geq 0}$, where $\tilde{T}_0 := 0$ and for each integer $n \geq 1$, $\tilde{T}_n$ denotes the nth transition time of $\{\tilde{X}(t); t \geq 0\}$. From these transition times, we define the embedded discrete-time Markov chain (DTMC) $\{\tilde{X}_n\}_{n \geq 0}$, where $\tilde{X}_n := \tilde{X}(T_n)$ represents the state of the CTMC immediately after its nth transition. Finally, for each state $x \in S$ we define the hitting-time random variables

$$\tilde{\eta}_x := \inf\{n \geq 0 : \tilde{X}_n = x\}, \qquad \tilde{\tau}_x := \inf\{t \geq 0 : \tilde{X}(t) = x\}. \tag{4.28}$$

The following result, Theorem 4.2.1, was established in [5].

Theorem 4.2.1 *Suppose $\{X(t); t \geq 0\}$ is an ergodic CTMC, and fix a state $x \in S$. Then its* *stationary distribution* $\mathbf{p}$ *satisfies, for each state* $y \in S$,

$$p_y = p_x w_y \tag{4.29}$$

where $w_x := 1$*, and for each state* $y \neq x$,

$$w_y := \mathbb{E}_y \left[\mathbf{1}(\tilde{\eta}_x < \infty) \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right]. \tag{4.30}$$

We will occasionally refer to the fixed state x within Theorem 4.2.1 as the reference point. In order to derive the stationary distribution $\mathbf{p}$ of $\{X(t); t \geq 0\}$, we will find it useful to set $x := (0, 0)$.

Theorem 4.2.1 can be used to establish the following result, which provides a closed-form expression for each element of $\mathbf{p}$.

Theorem 4.2.2 *The stationary distribution of the honest mining CTMC is as follows: for* $(i, j) \neq (0, 0)$,

$$p_{(i,j)} = p_{(0,0)} \sum_{x=0}^{\min(i,j)} \left[\frac{2^x (x + |i-j|)}{i+j-x} \binom{i+j-x}{\max(i,j)} \right] \frac{\lambda_1^i \lambda_2^j}{(\lambda_1 + \lambda_2)^x (\lambda_1 + \lambda_2 + \mu)^{i+j-x}}. \tag{4.31}$$

Furthermore,

$$p_{(0,0)} = \frac{1 - \frac{2\lambda_1}{\lambda_1 + \lambda_2} \phi_1(\mu)}{1 + \frac{\lambda_1 + \lambda_2}{\mu} - \frac{2\lambda_1}{\mu} \phi_1(\mu)}. \tag{4.32}$$

Formula (4.31) was derived in the work of Göbel et al. [16] by verifying that (4.31) satisfies the global balance equations of $\{X(t); t \geq 0\}$. Not only do we give a different approach for deriving this formula, we further build on the results found in [16] by establishing (4.32), which shows that the stationary probability $p_{(0,0)}$ can be expressed explicitly in terms of λ_1, λ_2, and μ.

Proof We begin our proof of Theorem 4.2.2 by proving (4.32), but this follows immediately from applying (4.10) to the well-known formula

$$p_{(0,0)} = \frac{1}{q((0,0))\mathbb{E}_{(0,0)}[\tau_{(0,0)}]} = \frac{1}{(\lambda_1 + \lambda_2)\mathbb{E}_{(0,0)}[\tau_{(0,0)}]}. \tag{4.33}$$

It remains to establish (4.31) for each state $(i, j) \neq (0, 0)$, but this can be done via Theorem 4.2.1 by simplifying $w_{(i,j)}$, where we choose state $(0, 0)$ to be the reference point. Readers should note that the steps we use to simplify $w_{(i,j)}$ are very similar to the lattice path counting technique introduced in [36] to study both the M/E$_r$/1 and E$_r$/M/1 queueing systems, but since the lattice path counting technique we invoke here does not technically fall within the framework of [36], we present a detailed proof.

Given a fixed state $(i,j) \neq (0,0)$, define, for each integer $n \geq 1$, $\mathcal{C}_n$ as the set of all feasible paths $(x_0, x_1, \ldots, x_n)$ with respect to $\tilde{\mathbf{Q}}$ that satisfy (i) $x_0 = (i,j)$, (ii) $x_n = (0,0)$, and (iii) $x_1, x_2, \ldots, x_{n-1} \neq (0,0)$. Then

$$\begin{aligned} w_{(i,j)} &= \sum_{n=1}^{\infty} \sum_{x_0,x_1,\ldots,x_n \in \mathcal{C}_n} \left[\prod_{\ell=1}^{n} \frac{q(x_n, x_{n-1})}{\tilde{q}(x_{n-1}, x_n)} \right] \left[\prod_{\ell=1}^{n} \frac{\tilde{q}(x_{n-1}, x_n)}{\tilde{q}(x_{n-1})} \right] \\ &= \sum_{n=1}^{\infty} \sum_{x_0,x_1,\ldots,x_n \in \mathcal{C}_n} \left[\prod_{\ell=1}^{n} \frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} \right] \end{aligned} \tag{4.34}$$

where the second equality follows from the fact that the diagonal terms of $\tilde{\mathbf{Q}}$ and $\mathbf{Q}$ agree.

Next, note that every transition made by $\tilde{\mathbf{Q}}$ is always either (i) to the West, or (ii) to the South, until the process reaches state $(0,0)$. If a feasible step $(x_{\ell-1}, x_\ell)$ is a Western transition, then its corresponding term in the product is

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu} \tag{4.35}$$

if $x_{\ell-1}$ is not an element of D_0. If $x_{\ell-1} \in D_0$, then

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \tag{4.36}$$

A similar statement can be made when $(x_{\ell-1}, x_\ell)$ represents a transition to the South: when $x_{\ell-1}$ is not an element of D_0,

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu} \tag{4.37}$$

and when $x_{\ell-1} \in D_0$,

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} = \frac{\lambda_2}{\lambda_1 + \lambda_2}. \tag{4.38}$$

Observe also that every feasible path from state (i, j) to state $(0, 0)$ must consist of exactly $i + j$ steps, which implies

$$w_{(i,j)} = \sum_{x_0, x_1, \ldots, x_{i+j} \in C_{i+j}} \prod_{\ell=1}^{i+j} \frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})} \tag{4.39}$$

and in each feasible path found in C_{i+j}, exactly i transitions are to the West, and j are to the South. Each term within the product corresponding to each feasible path from state (i, j) to state $(0, 0)$ has to take one of the four values found in (4.35), (4.36), (4.37), and (4.38), and from the structure of these products, we can see that in order to simplify $w_{(i,j)}$ completely, we only need to keep track of the number of times a transition is made from D_0. For instance, each Western transition made by $\tilde{X}$ yields a product term whose numerator is λ_1, but whose denominator is either $(\lambda_1 + \lambda_2)$ or $(\lambda_1 + \lambda_2 + \mu)$, depending on whether or not the transition was made from a state in D_0, and a similar statement may be made with regard to Southern transitions. Let $d_x(i, j)$ denote the number of feasible paths under $\tilde{\mathbf{Q}}$ that start at (i, j), end at $(0, 0)$, and make a transition from a state in D_0 exactly x times: then

$$w_{(i,j)} = \sum_{x=0}^{\min(i,j)} d_x(i, j) \frac{\lambda_1^i \lambda_2^j}{(\lambda_1 + \lambda_2)^x (\lambda_1 + \lambda_2 + \mu)^{i+j-x}}. \tag{4.40}$$

It remains to compute $d_x(i, j)$ for each $x \geq 0$. These terms were stated correctly for the case where $i > j$ in [16], but here we choose to derive them explicitly, as this will help us later. Clearly $d_0(i, i) = 0$ whenever $i \geq 1$, because in order for $\tilde{X}$ to move from state (i, i) to state $(0, 0)$, it must make a transition from diagonal D_0 at least once. For $i, j \geq 0$ satisfying $i \neq j$,

$$d_0(i, j) = \frac{|j - i|}{j + i} \binom{j + i}{i} \tag{4.41}$$

which follows from the classical Ballot Theorem: see e.g. Renault [40].

We are now ready to calculate $d_x(i, i)$, for each integer $x \geq 1$ and each integer $i \geq 1$. Using (4.41), notice that for $i \geq 1$,

$$\begin{aligned} d_1(i, i) &= d_0(i - 1, i) + d_0(i, i - 1) \\ &= \frac{1}{2i - 1} \binom{2i - 1}{i} + \frac{1}{2i - 1} \binom{2i - 1}{i} = \frac{2}{2i - 1} \binom{2i - 1}{i}. \end{aligned} \tag{4.42}$$

Next, note that by (4.42), for $i \geq 2$,

$$\begin{aligned}
d_2(i,i) &= \sum_{\ell=1}^{i-1} d_1(\ell,\ell) d_1(i-\ell, i-\ell) \\
&= \sum_{\ell=1}^{i-1} \frac{2}{2\ell-1}\binom{2\ell-1}{\ell}\frac{2}{2(i-\ell)-1}\binom{2(i-\ell)-1}{i-\ell-1} \\
&= 4\sum_{\ell=0}^{i-2} \frac{1}{2\ell+1}\binom{2\ell+1}{\ell}\frac{1}{2(i-2-\ell)+1}\binom{2(i-2-\ell)+1}{i-2-\ell} \\
&= \frac{4(2)}{2(i-2)+2}\binom{2(i-2)+2}{i-2} \\
&= \frac{(2)2^2}{2i-2}\binom{2i-2}{i} \qquad (4.43)
\end{aligned}$$

where the fourth equality follows from an application of Identity (5.63) on page 202 of Graham et al. [17]; this identity is sometimes known as the Rothe-Hagen identity. From here, one can use (4.43) combined with induction to verify that for $x \geq 1$, $i \geq x$,

$$d_x(i,i) = \frac{x2^x}{2i-x}\binom{2i-x}{i}. \qquad (4.44)$$

A similar argument can be used to derive $d_x(i,j)$ for the case where $i \neq j$: it suffices to consider only the case where $j > i$. Observe that for $x \geq 1$, $i \geq x$,

$$\begin{aligned}
d_x(i,j) &= \sum_{\ell=x}^{i} d_x(\ell,\ell) d_0(i-\ell, j-\ell) \\
&= \sum_{\ell=x}^{i} \frac{x2^x}{2\ell-x}\binom{2\ell-x}{\ell-x}\frac{j-i}{i+j-2\ell}\binom{i+j-2\ell}{i-\ell} \\
&= \sum_{\ell=0}^{i-x} \frac{x2^x}{2\ell+x}\binom{2\ell+x}{\ell}\frac{j-i}{i+j-2\ell-2x}\binom{i+j-2\ell-2x}{i-\ell-x} \\
&= 2^x\sum_{\ell=0}^{i-x} \frac{x}{2\ell+x}\binom{2\ell+x}{\ell}\frac{j-i}{2(i-x-\ell)+j-i}\binom{2(i-x-\ell)+j-i}{i-x-\ell} \\
&= \frac{2^x(x+j-i)}{i+j-x}\binom{i+j-x}{j} \qquad (4.45)
\end{aligned}$$

where again, the third equality follows from Identity (5.63) on page 202 of Graham et al. [17]. A similar argument shows further that when $j < i$, we have for $j \geq x$,

$$d_x(i,j) = \frac{2^x(x+i-j)}{i+j-x}\binom{i+j-x}{i} \tag{4.46}$$

meaning we can conclude that for $x \geq 0$,

$$d_x(i,j) = \frac{2^x(x+|i-j|)}{i+j-x}\binom{i+j-x}{\max(i,j)}. \tag{4.47}$$

This establishes (4.31), as well as the proof of Theorem 4.2.2. $\diamondsuit$

4.2.3 Calculating the Laplace transforms of the transition functions

It is also possible to express the Laplace transforms of the transition functions of $\{X(t); t \geq 0\}$ in closed-form, if we further assume that $X(0) = (0,0)$ with probability one. Recall that for each state $(i,j) \in S$, the transition function $p_{(i,j)} : [0,\infty) \to [0,1]$ is defined as

$$p_{(i,j)}(t) := \mathbb{P}(X(t) = (i,j) \mid X(0) = (0,0)), \qquad t \geq 0 \tag{4.48}$$

and associated with $p_{(i,j)}$ is its Laplace transform $\pi_{(i,j)}$, which is defined on $\mathbb{C}_+$ as

$$\pi_{(i,j)}(\alpha) := \int_0^\infty e^{-\alpha t} p_{(i,j)}(t) dt, \qquad \alpha \in \mathbb{C}. \tag{4.49}$$

These transforms can be evaluated with the random-product technique as well, thanks to Theorem 4.2.3. This theorem was first given in [5] for the case where $\alpha > 0$, and later extended to $\mathbb{C}_+$ in [9].

Theorem 4.2.3 *Each Laplace transform $\pi_{(i,j)}$, for $(i,j) \in S$, satisfies*

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha) w_{(i,j)}(\alpha), \qquad \alpha \in \mathbb{C}_+ \tag{4.50}$$

where $w_{(0,0)}(\alpha) = 1$ *on* $\mathbb{C}_+$, *and for each state* $(i,j) \neq (0,0)$,

$$w_{(i,j)}(\alpha) := \mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(0,0)} < \infty) e^{-\alpha \tilde{\tau}_{(0,0)}} \prod_{\ell=1}^{\tilde{\eta}_{(0,0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right], \qquad \alpha \in \mathbb{C}_+. \tag{4.51}$$

Using Theorem 4.2.3, we can make use of another lattice path counting procedure to establish the following result.

Theorem 4.2.4 *The Laplace-Stieltjes transforms of this CTMC is as follows: for* $(i,j) \neq (0,0)$,

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha) \sum_{x=0}^{\min(i,j)} d_x(i,j) \frac{\lambda_1^i \lambda_2^j}{(\lambda_1 + \lambda_2 + \alpha)^x (\lambda_1 + \lambda_2 + \mu + \alpha)^{i+j-x}}. \tag{4.52}$$

Furthermore,

$$\pi_{(0,0)}(\alpha) = \frac{(\mu + \alpha)\left[1 - \frac{2\lambda_1}{\lambda_1 + \lambda_2 + \alpha}\phi_1(\mu + \alpha)\right]}{\alpha\mu\left[1 + \frac{\lambda_1 + \lambda_2 + \alpha}{\mu} - \frac{2\lambda_1 \phi_1(\mu + \alpha)}{\mu}\right]}. \tag{4.53}$$

Proof This argument is analogous to the argument we use to establish Theorem 4.2.2. First, we calculate the Laplace transform $\pi_{(0,0)}$, and once that has been found we then show how to express every other Laplace transform $\pi_{(i,j)}$, for $(i,j) \neq (0,0)$, in terms of $\pi_{(0,0)}$. Observe first that for each $\alpha \in \mathbb{C}_+$ (see e.g. Corollary 2.1 of [9])

$$\pi_{(0,0)}(\alpha) = \frac{1}{(\lambda_1 + \lambda_2 + \alpha)\left(1 - \mathbb{E}_{(0,0)}\left[e^{-\alpha \tau_{(0,0)}}\right]\right)}. \tag{4.54}$$

Plugging (4.9) into (4.54) and simplifying yields, after some algebra, (4.53).

It remains to establish (4.52), but to do so it suffices, given Theorem 4.2.3, to calculate $w_{(i,j)}(\alpha)$ for each state (i,j) satisfying $i \neq j$. Letting the set of paths $\mathcal{C}_n$ be defined as before, we observe that

$$w_{(i,j)}(\alpha) \quad = \quad \sum_{n=1}^{\infty} \sum_{(x_0, x_1, \ldots, x_n) \in \mathcal{C}_n} \prod_{\ell=1}^{n} \frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1}) + \alpha}. \tag{4.55}$$

Similar to what we saw in the proof of Theorem 4.2.2, if a feasible step $(x_{\ell-1}, x_\ell)$ is a western transition, then its corresponding term in the product is

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})+\alpha} = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu+\alpha} \tag{4.56}$$

if $x_{\ell-1}$ is not in D_0: if $x_{\ell-1} \in D_0$, then

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})+\alpha} = \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}. \tag{4.57}$$

Similarly, for transitions to the South, when $x_{\ell-1}$ is not in D_0,

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})+\alpha} = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu+\alpha} \tag{4.58}$$

and when $x_{\ell-1} \in D_0$,

$$\frac{q(x_\ell, x_{\ell-1})}{q(x_{\ell-1})+\alpha} = \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}. \tag{4.59}$$

Applying observations (4.56), (4.57), (4.58) and (4.59) as necessary yields

$$w_{(i,j)}(\alpha) = \sum_{x=0}^{\min(i,j)} d_x(i,j) \frac{\lambda_1^i \lambda_2^j}{(\lambda_1+\lambda_2+\alpha)^x(\lambda_1+\lambda_2+\mu+\alpha)^{i+j-x}} \tag{4.60}$$

which implies, due to (4.47),

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha) \sum_{x=0}^{\min(i,j)} d_x(i,j) \frac{\lambda_1^i \lambda_2^j}{(\lambda_1+\lambda_2+\alpha)^x(\lambda_1+\lambda_2+\mu+\alpha)^{i+j-x}} \tag{4.61}$$

thus proving (4.52). This completes the proof of Theorem 4.2.4. $\diamondsuit$

4.3 When a pool of miners implement a 'Selfish Mining' strategy

We now observe what happens when a portion of the pool implements a Selfish Mining strategy. In order to model Selfish Mining behavior, Göbel et al. [16] introduced the CTMC

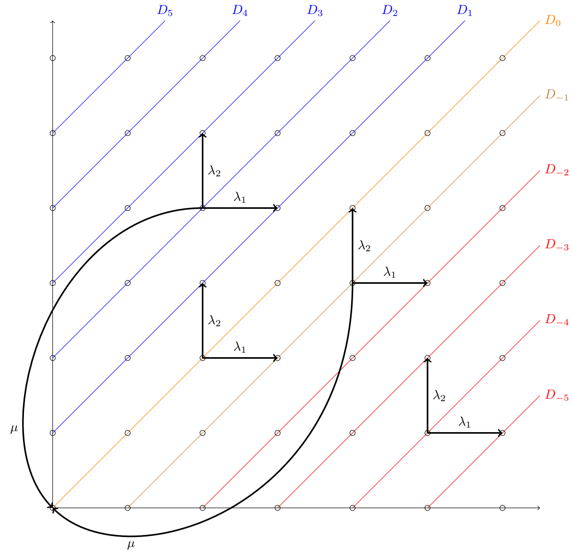

Figure 4.2: Transition rate diagram under Selfish Mining

$\{X(t); t \geq 0\}$ whose state space is given by $S := \{(i,j) : i \geq 0, j \geq 0\}$ and whose generator is given by $\mathbf{Q} := [q(x,y)]_{x,y \in S}$, where the elements of $\mathbf{Q}$ are defined as follows: given possible rates λ_1, λ_2, and μ we define

$$q((i,j),(k,\ell)) = \begin{cases} \lambda_1 & k = i+1, \ell = j; \\ \lambda_2 & k = i, \ell = j+1; \\ \mu & k = \ell = 0 \text{ with } i < j \text{ or } j = i-1, i \geq 2; \end{cases}$$

with all other off-diagonal rates set equal to zero. Just as before, each diagonal element $q(x,x)$, $x \in S$, satisfies $q(x,x) = -q(x)$, where $q(x)$ is the rate corresponding to each exponential sojourn time spent by the CTMC in state x. A picture of the rate diagram can be found in Figure 4.2.

4.3.1 Hitting Times

We can study the behavior of $\{X(t); t \geq 0\}$ by using an approach analogous to the one used in the previous section to study the behavior of the honest mining CTMC. Just as in Section 4.2, our first step consists of showing that the Laplace-Stieltjes transforms of $\tau_{(0,0)}$, under each probability measure $\mathbb{P}_{(i,j)}$, can be calculated numerically.

Proposition 4.3.1 *The law of the hitting time $\tau_{(0,0)}$ under the probability measure $\mathbb{P}_{(i,j)}$ satisfies the following properties.*

(a) For each integer $i \geq 1$, we have

$$\mathbb{E}_{(i,i)}[e^{-\alpha\tau_{(0,0)}}] = \mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}] \tag{4.62}$$

and

$$\mathbb{E}_{(i+1,i)}[e^{-\alpha\tau_{(0,0)}}] = \mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}]. \tag{4.63}$$

(b) For each integer $k \geq 1$, and each $(i,j) \in D_k$,

$$\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \phi_2(\alpha+\mu)^{j-i}\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\mu}{\mu+\alpha}(1-\phi_2(\alpha+\mu)^{j-i}) \tag{4.64}$$

and moreover,

$$\mathbb{E}_{(i,j)}[\tau_{(0,0)}] = \frac{1-\phi_2(\mu)^{j-i}}{\mu} + \phi_2(\mu)^{j-i}\mathbb{E}_{(1,1)}[\tau_{(0,0)}]. \tag{4.65}$$

(c) For each integer $k \leq -2$, and each $(i,j) \in D_k$,

$$\mathbb{E}_{(i,j)}[e^{-\alpha\tau_{(0,0)}}] = \phi_1(\alpha)^{i-j-1}\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}] \tag{4.66}$$

and moreover,

$$\mathbb{E}_{(i,j)}[\tau_{(0,0)}] = \frac{i-j-1}{\lambda_2-\lambda_1} + \mathbb{E}_{(2,1)}[\tau_{(0,0)}]. \tag{4.67}$$

(d) The Laplace-Stieltjes transforms $\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}]$ and $\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}]$ satisfy the linear system

$$\left[1-\frac{\lambda_1\phi_1(\alpha)}{\lambda_1+\lambda_2+\mu+\alpha}\right]\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu+\alpha}\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\mu}{\lambda_1+\lambda_2+\mu+\alpha} \tag{4.68}$$

$$\left[1-\frac{\lambda_2\phi_2(\alpha+\mu)}{\lambda_1+\lambda_2+\alpha}\right]\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\mu}{\mu+\alpha}\frac{\lambda_2(1-\phi_2(\alpha+\mu))}{\lambda_1+\lambda_2+\alpha}. \tag{4.69}$$

Moreover, the expected values $\mathbb{E}_{(2,1)}[\tau_{(0,0)}]$ *and* $\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}]$ *satisfy the linear system*

$$(\lambda_2+\mu)\mathbb{E}_{(2,1)}[\tau_{(0,0)}] = \lambda_2\mathbb{E}_{(1,1)}[\tau_{(0,0)}] + \frac{\lambda_2}{\lambda_2-\lambda_1} \tag{4.70}$$

$$\left[1-\frac{\lambda_2\phi_2(\mu)}{\lambda_1+\lambda_2}\right]\mathbb{E}_{(1,1)}[\tau_{(0,0)}] = \frac{\lambda_1}{\lambda_1+\lambda_2}\mathbb{E}_{(2,1)}[\tau_{(0,0)}] + \frac{1}{\lambda_1+\lambda_2}\left[1+\frac{\lambda_2(1-\phi_2(\mu))}{\mu}\right]. \tag{4.71}$$

(e) The Laplace-Stieltjes transform $\mathbb{E}_{(1,0)}[e^{-\alpha\tau_{(0,0)}}]$ *satisfies*

$$\mathbb{E}_{(1,0)}[e^{-\alpha\tau_{(0,0)}}] = \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\phi_1(\alpha)\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}] + \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}] \tag{4.72}$$

and furthermore,

$$\mathbb{E}_{(1,0)}[\tau_{(0,0)}] = \frac{\lambda_2}{\lambda_2^2-\lambda_1^2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\mathbb{E}_{(1,1)}[\tau_{(0,0)}] + \frac{\lambda_1}{\lambda_1+\lambda_2}\mathbb{E}_{(2,1)}[\tau_{(0,0)}] \tag{4.73}$$

(f) Finally, the Laplace-Stieltjes transform $\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}]$ *satisfies*

$$\begin{aligned}\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}] &= \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}\left[\frac{1-\phi_2(\alpha+\mu)}{\mu+\alpha}\right] \\ &+ \frac{\lambda_1^2}{(\lambda_1+\lambda_2+\alpha)^2}\phi_1(\alpha)\mathbb{E}_{(2,1)}[e^{-\alpha\tau_{(0,0)}}] \\ &+ \frac{\lambda_2}{(\lambda_1+\lambda_2+\alpha)}\left[\frac{\lambda_1}{(\lambda_1+\lambda_2+\alpha)}+\phi_2(\alpha+\mu)\right]\mathbb{E}_{(1,1)}[e^{-\alpha\tau_{(0,0)}}]\end{aligned} \tag{4.74}$$

and similarly,

$$\begin{aligned}\mathbb{E}_{(0,0)}[\tau_{(0,0)}] &= \frac{1}{\lambda_1+\lambda_2}\left[1+\frac{\lambda_1\lambda_2}{\lambda_2^2-\lambda_1^2}+\frac{\lambda_2(1-\phi_2(\mu))}{\mu}\right] \\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2}\left[\frac{\lambda_1}{\lambda_1+\lambda_2}+\phi_2(\mu)\right]\mathbb{E}_{(1,1)}[\tau_{(0,0)}] + \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2\mathbb{E}_{(2,1)}[\tau_{(0,0)}].\end{aligned} \tag{4.75}$$

Proof Statements (4.62) and (4.63) of Proposition 4.3.1 can be established using a 'sum-over-paths' approach: again, we omit the details since the result is intuitively obvious, given the structure of the transition diagram. Next, (4.65) follows from taking derivatives of both sides of (4.64) and setting $\alpha = 0$, and observe from the form of the transition diagram that (4.64) follows from the argument used to establish (4.7) of Proposition 4.2.1.

The next step is to prove (4.66). Assuming $X(0) = (i,j) \in D_k$ for some $k \geq 2$, we can see from (4.63) that, under the probability measure $\mathbb{P}_{(i,j)}$, $\tau_{(0,0)}$ is equal in distribution to the convolution of the amount of time it takes an M/M/1 queue with arrival rate λ_1, service rate λ_2 to move from state $i-j-1$ to state 0, and the law of $\tau_{(0,0)}$ under the probability measure $\mathbb{P}_{(2,1)}$. Once this has been observed, (4.67) quickly follows from (4.66) by taking derivatives of both sides, and setting $\alpha = 0$.

Next, note that (4.68) and (4.69) follow from applying a first-step analysis argument, then applying (4.64) and (4.66), and an analogous argument can be used to establish (4.70) and (4.71). The rest of the statements contained in Proposition 4.3.1 follow from first-step analysis and substitution in an analogous manner: we omit the details. $\diamondsuit$

4.3.2 Calculating the stationary distribution

Our next task is to find the stationary distribution $\mathbf{p}$ of this model, which exists when $0 < \lambda_1 < \lambda_2$ and $\mu > 0$. This is done in Theorem 4.3.1.

Theorem 4.3.1 *The stationary distribution* $\mathbf{p}$ *of* $\{X(t); t \geq 0\}$ *satisfies the following properties:*
(a) the long-run fraction of time $p_{(0,0)}$ *satisfies*

$$p_{(0,0)} \quad = \quad \frac{1}{(\lambda_1 + \lambda_2)\mathbb{E}_{(0,0)}[\tau_{(0,0)}]} \tag{4.76}$$

where $\mathbb{E}_{(0,0)}[\tau_{(0,0)}]$ *can be calculated using Proposition 4.3.1.*
(b) For each integer $k \geq 1$*, and each state* $(i,j) \in D_k$*, we have*

$$p_{(i,j)} = \sum_{\ell=0}^{i} \frac{j-i}{i+j-2\ell} \binom{i+j-2\ell}{j-\ell} \frac{\lambda_1^{i-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2+\mu)^{i+j-2\ell}} p_{(\ell,\ell)}. \tag{4.77}$$

(c) For each integer $k \leq -2$*, and each state* $(i,j) \in D_k$*, we have*

$$p_{(i,j)} = \sum_{\ell=0}^{j} \frac{i-j-1}{i+j-2\ell-1} \binom{i+j-2\ell-1}{i-\ell-1} \frac{\lambda_1^{i-1-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2)^{i+j-2\ell-1}} p_{(\ell+1,\ell)}. \tag{4.78}$$

(d) Next,

$$p_{(1,0)} = \frac{\lambda_1}{\lambda_1+\lambda_2} p_{(0,0)} \tag{4.79}$$

and for each integer $\ell \geq 1$*,*

$$\begin{aligned} p_{(\ell+1,\ell)} &= \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} \sum_{k=0}^{\ell-1} \frac{1}{2\ell-2k-1} \binom{2\ell-2k-1}{\ell-k} \frac{\lambda_1^{\ell-k}\lambda_2^{\ell-1-k}}{(\lambda_1+\lambda_2)^{2\ell-2k-1}} p_{(k+1,k)} \\ &+ \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} p_{(\ell,\ell)}. \end{aligned} \tag{4.80}$$

(e) Finally, for each integer $\ell \geq 1$*,*

$$\begin{aligned} p_{(\ell,\ell)} &= \frac{\lambda_1}{\lambda_1+\lambda_2} \sum_{k=0}^{\ell-1} \frac{1}{2\ell-2k-1} \binom{2\ell-2k-1}{\ell-k} \frac{\lambda_1^{\ell-1-k}\lambda_2^{\ell-k}}{(\lambda_1+\lambda_2+\mu)^{2\ell-2k-1}} p_{(k,k)} \\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2} p_{(\ell,\ell-1)}. \end{aligned} \tag{4.81}$$

From Theorem 4.3.1, we can see that in order to calculate, for example, $p_{(i,j)}$ for $i<j$, we first need to find $p_{(0,0)}$, then use the recursions given in Theorem 4.3.1 to find $p_{(1,0)}$, $p_{(1,1)}$, $p_{(2,1)}$, $p_{(2,2)}$, etc., up to $p_{(i,i)}$.

Proof Statement (4.76) is obvious, but we state it formally within Theorem 4.3.1 to remind readers that Proposition 4.3.1 can be used to compute $p_{(0,0)}$.

Our next task is to use Theorem 4.2.1, where state $(0,0)$ is used as the reference point, to establish both (4.77) and (4.78). Consider first the case where $(i,j) \in D_k$ for some integer $k \geq 1$, meaning $i<j$. Setting

$$\tilde{\eta}_{D_0} := \inf\{n \geq 1 : \tilde{X}_n \in D_i\}$$

we can apply the strong Markov property at the stopping time $\tilde{\eta}_{D_0}$ to express $w_{(i,j)}$ as follows:

$$\begin{aligned} w_{(i,j)} &= \sum_{\ell=0}^{i} \mathbb{E}_{(i,j)} \left[\mathbf{1}(\tilde{\eta}_{D_0} < \infty, \tilde{X}_{\tilde{\eta}_{D_{0,i}}} = (\ell,\ell)) \prod_{k=1}^{\tilde{\eta}_{D_0}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)} \mathbf{1}(\tilde{\eta}_{(0,0)} < \infty) \prod_{k=\tilde{\eta}_{D_0}+1}^{\tilde{\eta}_{(0,0)}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)} \right] \\ &= \sum_{\ell=0}^{i} w_{(\ell,\ell)} \mathbb{E}_{(i,j)} \left[\mathbf{1}(\tilde{\eta}_{D_0} < \infty, \tilde{X}_{\tilde{\eta}_{D_0}} = (\ell,\ell)) \prod_{k=1}^{\tilde{\eta}_{D_0}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)} \right]. \end{aligned} \tag{4.82}$$

The next step is to simplify, for each integer $\ell \in \{0, 1, \ldots, i\}$, the expected value

$$\mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{D_0} < \infty, \tilde{X}_{\tilde{\eta}_{D_0}} = (\ell,\ell)) \prod_{k=1}^{\tilde{\eta}_{D_0}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)}\right]. \tag{4.83}$$

Recall that starting in state (i,j), the tilde process can only make transitions to the West or to the South until it reaches state (ℓ,ℓ). While the process is above D_0, transitions to the West have corresponding product term

$$\frac{q(x_k, x_{k-1})}{q(x_{k-1})} = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu}, \tag{4.84}$$

while transitions to the South have corresponding product term

$$\frac{q(x_k, x_{k-1})}{q(x_{k-1})} = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu}. \tag{4.85}$$

Moreover, in order for $\{\tilde{X}_n\}_{n\geq 0}$ to move from state (i,j) to state (ℓ,ℓ), it must make exactly $i-\ell$ Western transitions, and exactly $j-\ell$ Southern transitions. Define $\mathcal{C}_{i+j-2\ell}$ to be the set of all paths of the form $(x_0, x_1, \ldots, x_{i+j-2\ell})$, where $x_0 = (i,j), x_{i+j-2\ell} = (\ell,\ell)$, and for each $k = 1, \ldots, i+j-2\ell-1, x_k \notin D_{0,i}$. Then,

$$\begin{aligned}
&\mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{D_0} < \infty, \tilde{X}_{\tilde{\eta}_{D_0}} = (\ell,\ell)) \prod_{k=1}^{\tilde{\eta}_{D_0}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)}\right] \\
= &\sum_{x_0,\ldots,x_{i+j-2\ell} \in C_{i+j-2\ell}} \prod_{k=1}^{i+j-2\ell} \frac{q(\tilde{x}_k, \tilde{x}_{k-1})}{q(x_{k-1})}
\end{aligned} \tag{4.86}$$

and for each path $(x_0, x_1, \ldots, x_{i+j-2\ell}) \in C_{i+j-2\ell}$, each term in its corresponding product has to take one of two values found in (4.84) and (4.85). Note too that the number of paths in $\mathcal{C}_{i+j-2\ell}$ is simply $d_0(i-\ell, j-\ell)$, which has been derived previously. Thus,

$$\begin{aligned}
w_{(i,j)} &= \sum_{\ell=0}^{i} w_{(\ell,\ell)} \mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{D_0} < \infty, \tilde{X}_{\tilde{\eta}_{D_0}} = (\ell,\ell)) \prod_{k=1}^{\tilde{\eta}_{D_0}} \frac{q(\tilde{X}_k, \tilde{X}_{k-1})}{\tilde{q}(\tilde{X}_{k-1}, \tilde{X}_k)}\right] \\
&= \sum_{\ell=0}^{i} \frac{j-i}{i+j-2\ell} \binom{i+j-2\ell}{j-\ell} \frac{\lambda_1^{i-\ell} \lambda_2^{j-\ell}}{(\lambda_1 + \lambda_2 + \mu)^{i+j-2\ell}} w_{(\ell,\ell)}
\end{aligned} \tag{4.87}$$

and after multiplying both sides by $p_{(0,0)}$, we have

$$p_{(i,j)} = \sum_{\ell=0}^{i} \frac{j-i}{i+j-2\ell} \binom{i+j-2\ell}{j-\ell} \frac{\lambda_1^{i-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2+\mu)^{i+j-2\ell}} p_{(\ell,\ell)} \tag{4.88}$$

which establishes (4.77). A similar argument can be used to establish (4.78) for the case where state $(i,j) \in D_k$ for some integer $k \leq -2$: in that case, we need to keep track of how $\{\tilde{X}_n\}_{n\geq 0}$ first reaches the set D_{-1} when it starts in state (i,j).

It remains to establish (4.79), (4.80), and (4.81). Recall that since $(0,0)$ is the reference node, $w_{(0,0)} = 1$. Next, a simple first-step analysis argument shows that

$$w_{(1,0)} = \frac{\lambda_1}{\lambda_1+\lambda_2} w_{(0,0)} = \frac{\lambda_1}{\lambda_1+\lambda_2} \tag{4.89}$$

and multiplying both sides of (4.89) by $p_{(0,0)}$ yields (4.79).

We can show that the remaining $w_{(\ell,\ell)}$ and $w_{(\ell,\ell+1)}$ terms, for $\ell \geq 1$, satisfy a simple recursion. Using first-step analysis, combined with (4.77) gives

$$\begin{aligned} w_{(\ell,\ell)} &= \frac{\lambda_1}{\lambda_1+\lambda_2} w_{(\ell-1,\ell)} + \frac{\lambda_2}{\lambda_1+\lambda_2} w_{(\ell,\ell-1)} \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2} \sum_{k=0}^{\ell-1} \frac{1}{2\ell-2k-1} \binom{2\ell-2k-1}{\ell-k} \frac{\lambda_1^{\ell-1-k}\lambda_2^{\ell-k}}{(\lambda_1+\lambda_2+\mu)^{2\ell-2k-1}} w_{(k,k)} \\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2} w_{(\ell,\ell-1)} \end{aligned} \tag{4.90}$$

and similarly, using first-step analysis combined with (4.78) gives

$$\begin{aligned} w_{(\ell+1,\ell)} &= \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} w_{(\ell,\ell)} + \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} w_{(\ell+1,\ell-1)} \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} \sum_{k=0}^{\ell-1} \frac{1}{2\ell-2k-1} \binom{2\ell-2k-1}{\ell-k} \frac{\lambda_1^{\ell-k}\lambda_2^{\ell-1-k}}{(\lambda_1+\lambda_2)^{2\ell-2k-1}} w_{(k+1,k)} \\ &+ \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} w_{(\ell,\ell)} \end{aligned} \tag{4.91}$$

Multiplying both sides of (4.90) and (4.91) by $p_{(0,0)}$ yields (4.81) and (4.80), respectively, which proves Theorem 4.3.1. ◊

4.3.3 Calculating the Laplace transforms of the transition functions

Not surprisingly, we can also calculate the Laplace transform $\pi_{(i,j)}$ associated with each transition function $p_{(i,j)}(t)$, if we further assume that $X(0) = (0,0)$. Theorem 4.3.2 shows how to calculate these transforms: since the proof of Theorem 4.3.2 is similar to the proof of Theorem 4.3.1 in a way analogous to how the proof of Theorem 4.2.4 is similar to the proof of Theorem 4.2.2, in the interest of saving space we omit the details of the proof.

Theorem 4.3.2 *Suppose $X(0) = (0,0)$ with probability one. Then the Laplace transforms $\pi_{(i,j)}$ of the transition functions satisfy the following properties.*

(a) First,

$$\pi_{(0,0)}(\alpha) = \frac{1}{(\lambda_1 + \lambda_2 + \alpha)\left(1 - \mathbb{E}_{(0,0)}\left[e^{-\alpha\tau_{(0,0)}}\right]\right)} \tag{4.92}$$

where $\mathbb{E}_{(0,0)}[e^{-\alpha\tau_{(0,0)}}]$ can be calculated using Proposition 4.3.1.

(b) For each integer $k \geq 1$, and each $(i,j) \in D_k$,

$$\pi_{(i,j)}(\alpha) = \sum_{\ell=0}^{i} \frac{j-i}{i+j-2\ell}\binom{i+j-2\ell}{j-\ell}\frac{\lambda_1^{i-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2+\mu+\alpha)^{i+j-2\ell}}\pi_{(\ell,\ell)}(\alpha). \tag{4.93}$$

(c) For each integer $k \leq -2$, and each $(i,j) \in D_k$,

$$\pi_{(i,j)}(\alpha) = \sum_{\ell=0}^{j} \frac{i-j-1}{i+j-2\ell-1}\binom{i+j-2\ell-1}{i-\ell-1}\frac{\lambda_1^{i-1-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2+\alpha)^{i+j-2\ell-1}}\pi_{(\ell+1,\ell)}(\alpha). \tag{4.94}$$

(d) Next,

$$\pi_{(1,0)}(\alpha) = \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\pi_{(0,0)}(\alpha) \tag{4.95}$$

and for each $\ell \geq 1$,

$$\begin{aligned}\pi_{(\ell+1,\ell)}(\alpha) \quad &= \quad \frac{\lambda_2}{\lambda_1+\lambda_2+\mu+\alpha}\sum_{k=0}^{\ell-1}\frac{1}{2\ell-2k-1}\binom{2\ell-2k-1}{\ell-k}\frac{\lambda_1^{\ell-k}\lambda_2^{\ell-1-k}}{(\lambda_1+\lambda_2+\alpha)^{2\ell-2k-1}}\pi_{(k+1,k)}(\alpha) \\ &+ \quad \frac{\lambda_1}{\lambda_1+\lambda_2+\mu+\alpha}\pi_{(\ell,\ell)}(\alpha).\end{aligned} \tag{4.96}$$

(e) Finally, for each integer $\ell \geq 1$,

$$\begin{aligned} \pi_{(\ell,\ell)}(\alpha) &= \frac{\lambda_1}{\lambda_1+\lambda_2+\alpha}\sum_{k=0}^{\ell-1}\frac{1}{2\ell-2k-1}\binom{2\ell-2k-1}{\ell-k}\frac{\lambda_1^{\ell-1-k}\lambda_2^{\ell-k}}{(\lambda_1+\lambda_2+\mu+\alpha)^{2\ell-2k-1}}\pi_{(k,k)}(\alpha) \\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2+\alpha}\pi_{(\ell,\ell-1)}(\alpha). \end{aligned} \tag{4.97}$$

4.4 Extensions

Our methods can also be used to analyze other similar models. For instance, suppose that the greedy miners use a different strategy, similar to Selfish Mining, except that once their lead is reduced to m, $m > 1$, they will publish all their blocks. The generator of this new CTMC is given by $\mathbf{Q}$ where the elements of $\mathbf{Q}$ are

$$q((i,j),(k,\ell)) = \begin{cases} \lambda_1 & k = i+1, \ell = j; \\ \lambda_2 & k = i, \ell = j+1; \\ \mu & k = \ell = 0 \text{ with } i < j, \text{ or with } j \geq 1, (i,j) \in \cup_{\ell=-m}^{-1} D_k \end{cases}$$

and all other rates are equal to zero. This type of model could represent a (very crude!) way of adjusting for the fact that in a real system, there is a non-negligible delay in not only the amount of time it takes the pool to communicate blocks to the larger group, but also the amount of time it takes individual members of the larger group to communicate with other members of the larger group: hence, in light of this additional delay, the pool may choose to disclose all of the blocks they have mined in secret once their lead has been reduced to m, in order to ensure that enough miners in the larger pool actaully mine on their longer, previously secret, portion of the chain. Granted, under current conditions this is extremely unlikely to happen, but it could possibly happen if a pool that wishes to implement Selfish Mining has a large enough proportion of computing resources.

The following theorem shows how the stationary distribution of this CTMC can be calculated: we omit the proof, as the derivation of the stationary distribution is a fairly straightforward extension of the derivation used in the previous section.

Theorem 4.4.1 *The stationary distribution $\mathbf{p}$ of $\{X(t) : t \geq 0\}$ satisfies the following properties:*

(a) the long-run fraction of time satisfies

$$p_{(0,0)} = \frac{1}{(\lambda_1+\lambda_2)\mathbb{E}_{(0,0)}[\tau_{(0,0)}]} \tag{4.98}$$

where $\mathbb{E}_{(0,0)}[\tau_{(0,0)}]$ can be computed by first calculating $\mathbb{E}_{(i,1)}[\tau_{(0,0)}]$, for $1 \le i \le m+1$, which can be found by solving a linear system consisting of $m+1$ equations and $m+1$ unknowns.

(b) For each integer $\ell \ge 1$, we have

$$\begin{aligned} p_{(\ell+m,\ell)} &= \frac{\lambda_2}{\lambda_1+\lambda_2+\mu}\sum_{k=0}^{\ell-1}\frac{1}{2\ell-2k-1}\binom{2\ell-2k-1}{\ell-k}\frac{\lambda_1^{\ell-k}\lambda_2^{\ell-k-1}}{(\lambda_1+\lambda_2)^{2\ell-2k-1}}p_{(k+m,k)} \\ &+ \frac{\lambda_1}{\lambda_1+\lambda_2+\mu}p_{(\ell+m-1,\ell)}, \end{aligned} \tag{4.99}$$

and

$$\begin{aligned} p_{(\ell,\ell)} &= \frac{\lambda_1}{\lambda_1+\lambda_2}\sum_{k=0}^{\ell-1}\frac{1}{2\ell-2k-1}\binom{2\ell-1-2k}{\ell-k}\frac{\lambda_2^{\ell-1-k}\lambda_2^{\ell-k}}{(\lambda_1+\lambda_2+\mu)^{2\ell-1-2k}}p_{(k,k)} \\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2}p_{(\ell,\ell-1)}. \end{aligned} \tag{4.100}$$

(c) For $k = 1,\dots,m-1$ and for each integer $\ell \ge 1$, we have,

$$p_{(\ell+k,\ell)} = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu}p_{(\ell+k-1,\ell)} + \frac{\lambda_2}{\lambda_1+\lambda_2+\mu}p_{(\ell+k,\ell-1)}. \tag{4.101}$$

(d) For $k = 1,\dots,m$, we have,

$$p_{(k,0)} = \frac{\lambda_1}{\lambda_1+\lambda_2}p_{(k-1,0)}. \tag{4.102}$$

(e) Next, for each integer $k \ge 1$ and each state $(i,j) \in D_k$, we have,

$$p_{(i,j)} = \sum_{\ell=0}^{i}\frac{j-i}{i+j-2\ell}\binom{i+j-2\ell}{j-\ell}\frac{\lambda_1^{i-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2+\mu)^{i+j-2\ell}}p_{(\ell,\ell)}. \tag{4.103}$$

(f) Finally, for each integer $k \leq -(m+1)$, and each state $(i,j) \in D_k$

$$p_{(i,j)} = \sum_{\ell=0}^{j} \frac{i-m-j}{i+j-2\ell-m} \binom{i+j-2\ell-m}{i-\ell-m} \frac{\lambda_1^{i-m-\ell}\lambda_2^{j-\ell}}{(\lambda_1+\lambda_2)^{i+j-2\ell-m}} p_{(\ell+m,\ell)}. \qquad (4.104)$$

Equations (4.99)-(4.102) can also be represented in matrix form, which may be useful for computational purposes. We will only consider the case when $m = 2n, n \in \mathbb{N}$, but the case where m is odd can be expressed in a similar manner.

First we define some notation. For each integer $k \geq 0$, and each integer $\ell > k$, define

$$r_{\ell,k} := \frac{\lambda_1}{\lambda_1+\lambda_2} \frac{1}{2\ell-2k-1} \binom{2\ell-1-2k}{\ell-k} \frac{\lambda_2^{\ell-1-k}\lambda_2^{\ell-k}}{(\lambda_1+\lambda_2+\mu)^{2\ell-1-2k}},$$

and

$$s_{\ell,k} := \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} \frac{1}{2\ell-2k-1} \binom{2\ell-2k-1}{\ell-k} \frac{\lambda_1^{\ell-k}\lambda_2^{\ell-k-1}}{(\lambda_1+\lambda_2)^{2\ell-2k-1}}.$$

Next, for each integer $0 \leq k \leq n$, we define the diagonal set

$$E_{2n-2k} := \{(2n-2k,0), (2n-2k-1,1), \ldots, (n-k,n-k)\}$$

and its corresponding row vector

$$\mathbf{p}_{2n-2k} := [p_{(2n-2k,0)}, p_{(2n-2k-1,1)}, \ldots, p_{(n-k,n-k)}].$$

For each integer $0 \leq k \leq n-1$, we define the diagonal set

$$E_{2n-2k-1} := \{(2n-2k-1,0), (2n-2k-2,1), \ldots, (n-k,n-k-1)\}$$

and its corresponding row vector

$$\mathbf{p}_{2n-2k-1} := [p_{(2n-2k-1,0)}, p_{(2n-2k-2,1)}, \ldots, p_{(n-k,n-k-1)}].$$

Similarly, for each integer $k \geq 1$ we define

$$E_{2n+2k} := \{(2n+k,k), (2n+k-1,k+1), \ldots, (n+k,n+k)\}$$

and

$$\mathbf{p}_{2n+2k} := [p_{(2n+k,k)}, p_{(2n+k-1,k+1)}, \ldots, p_{(n+k,n+k)}].$$

Finally, we define

$$E_{2n+2k-1} := \{(2n+k-1,k), (2n+k-2,k+1), \ldots, (n+k,n+k-1)\},$$

and

$$\mathbf{p}_{2n+2k-1} := [p_{(2n+k-1,k)}, p_{(2n+k-2,k+1)}, \ldots, p_{(n+k,n+k-1)}].$$

Next, for each $i, j \in \mathbb{N}$

$$\mathbf{A}_{i,j} := \left[\frac{q(x,y)}{q(y)}\right]_{x \in E_i, y \in E_j}.$$

Further, for each even $i \le m$, and each even $j > i$, we define $\mathbf{B}_{i,j}$ as

$$\mathbf{B}_{i,j} := \left[b^{(i,j)}_{x,y}\right]_{x \in E_i, y \in E_j}$$

which is a matrix whose only non-zero entry corresponds to the ordered pair (x_{nw}, y_{nw}), where x_{nw} and y_{nw} are the northwestern-most states in E_i and E_j, respectively: this entry in $\mathbf{B}_{i,j}$ is equal to $r_{j/2,i/2}$. Lastly, for each even $i > m$, and each even $j > i$, we define $\mathbf{C}_{i,j}$ as

$$\mathbf{C}_{i,j} := \left[c^{(i,j)}_{x,y}\right]_{x \in E_i, y \in E_j}$$

which is a matrix whose only non-zero entries correspond to the ordered pairs (x_{nw}, y_{nw}) and (x_{se}, y_{se}), where

$$c^{(i,j)}_{(x_{se},y_{se})} = s_{j/2-n,i/2-n}, \qquad c^{(i,j)}_{(x_{nw},y_{nw})} = r_{j/2,i/2}.$$

We are now ready to present formulas that allow us to compute $p_{(i,j)}$ for $(i,j) \in \cup_{k=0}^{n} D_{-k}$. For each integer $0 \le k \le n-1$

$$\mathbf{p}_{2n-2k} = \mathbf{p}_{2n-2k-1}\mathbf{A}_{2n-2k-1,2n-2k} + \sum_{j=1}^{n-k} \mathbf{p}_{2n-2k-2j}\mathbf{B}_{2n-2k-2j,2n-2k}$$

and

$$\mathbf{p}_{2n-2k-1} = \mathbf{p}_{2n-2k-2}\mathbf{A}_{2n-2k-2,2n-2k-1}.$$

The remaining row vectors can be calculated as follows: for each integer $k \geq 1$

$$\mathbf{p}_{2n+2k-1} = \mathbf{p}_{2n+2k-2}\mathbf{A}_{2n+2k-2,2n+2k-1}$$

and

$$\mathbf{p}_{2n+2k} = \mathbf{p}_{2n+2k-1}\mathbf{A}_{2n+2k-1,2n+2k} + \sum_{j=1}^{n}\mathbf{p}_{2n-2j}\mathbf{B}_{2n-2j,2n+2k} + \sum_{j=1}^{k}\mathbf{p}_{2n+2k-2j}\mathbf{C}_{2n+2k-2j,2n+2k}.$$

4.5 Numerical Examples

Here we provide some numerical results. We will first cover the case where all miners are honest. As done in [16], we set $\lambda_1 = 0.6$/h and $\lambda_2 = 5.4$/h, which corresponds to the smaller pool having 10% of the computing power, and blocks being discovered once every 10 minutes. Furthermore, we set $\mu = 285$/h, which corresponds to an average 12.5s communication delay. These parameters are used since they are reasonable in the blockchain setting but we note that under these parameters the chains described in models 1 and 2 essentially always stays in the set of states (i, j) such that $0 \leq i, j \leq 3$.

In Table 4.1 we calculate the stationary probabilities associated with the states (i, j) such that $0 \leq i, j \leq 3$ using our equations from Theorem 4.2.2. In order to this we must first use equation (4.32) to calculate $p_{(0,0)}$. Once this is done we use (4.31) to calculate $p_{(i,j)}$. In Table 4.2 we present the table found in [16], where they calculated these stationary probabilities using the balance equations along with normalization. Notice here that our results match the results from Göbel et al. aside from two states: state (0,0) and state (1,1), both of which are off in the fourth decimal place. This can be expected since Göbel et al. use normalization techniques to calculate the stationary probability associated with state (0,0), which in turn affects the stationary probability of state (1,1). Observe here that when all miners are honest, both groups agree roughly 98% of the time and the smaller pool of miners only have the lead 0.2% of the time.

Under the Selfish Mining technique, we see some drastic differences. The stationary probabilities calculated using our methods are shown in Table 4.3. Using our methods we use the equations given in Theorem 4.3.1 to calculate $p_{(0,0)}$ and $p_{(1,0)}$. Once this is done we use the recursion to calculate $p_{(1,1)}, p_{(2,1)}, p_{(2,2)}, p_{(3,2)}$, and $p_{(3,3)}$. From there we calculate the remaining $p_{(i,j)}$. Table 4.4, which contains the stationary probabilities given in [16], differ from our results for about half of the states listed. These differences are likely a consequence of truncating the state space. Due to the parameters set forth by Göbel et al. to model the blockchain, it is reasonable to truncate the state space in this way. However, if λ_1 and λ_2 are chosen so that they are closer to the value of μ this would not be a reasonable assumption. Our methods would still work in this case provided $\lambda_1 < \lambda_2$.

Our Results				
(i,j)	0	1	2	3
0	0.9758	0.0181	0.0003	0.0000
1	0.0020	0.0036	0.0001	0.0000
2	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000

Table 4.1: The stationary probabilities when calculated using our methods when all miners are honest

Göbel et al.				
(i,j)	0	1	2	3
0	0.9757	0.0181	0.0003	0.0000
1	0.0020	0.0037	0.0001	0.0000
2	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000

Table 4.2: The stationary probabilities when calculated using the methods specified by Göbel et al. in [16] when all miners are honest

Our Results				
(i,j)	0	1	2	3
0	0.8156	0.0181	0.0003	0.0000
1	0.0816	0.0752	0.0013	0.0000
2	0.0082	0.0003	0.0004	0.0000
3	0.0008	0.0008	0.0000	0.0000

Table 4.3: The stationary probabilities when calculated using our methods under Selfish Mining

Göbel et al.				
(i,j)	0	1	2	3
0	0.8177	0.0121	0.0002	0.0000
1	0.0818	0.0749	0.0011	0.0000
2	0.0082	0.0002	0.0003	0.0000
3	0.0008	0.0008	0.0000	0.0000

Table 4.4: The stationary probabilities when calculated using the methods specified by Göbel et al. in [16] under Selfish Mining

Chapter 5

Conclusions and future work

This dissertation provides a thorough study of several classes of Markov chains. Chapter 2 outlines how to analyze the joint distribution of the running maximum level and the state of a level-dependent QBD processes by providing formulas that can be used to calculate the Laplace transforms of the transition functions using matrix analytic methods. Additionally we study the time-dependent behavior of the running minimum level and the state of a level-dependent Markov process of M/G/1-type. In future work, we would like to apply the methods used in this chapter to study the joint distribution of the running minimum level, running maximum level, and state of a level-dependent QBD process.

Chapter 3 provides a study of the time-dependent behavior of two homogeneous QBD processes. We study a collection of **R**-matrices and **G**-matrices associated with a processes with infinitely many levels and we use this work to study a collection of **G**-matrices associated with a homogeneous process with finitely many levels. Combining these three collections of matrices we are able to derive the Laplace transforms of the transition functions associated with the QBD process containing finitely many levels. The results found in Chapters 2 and 3 could be used in the future to study the time-dependent behavior of other block-structured processes. For example, it would be interesting to see how these methods could be used to study hysteretic QBD processes and other Markov processes having transition diagrams that consist of homoegoeous QBD processes pasted together.

Chapter 4 studies two Markovian bitcoin models. In this Chapter we use the random-product technique to study the stationary distribution as well as the Laplace transforms of the transition functions associated with each model. In the future we would like to see how the random-product technique could be used to study other bitcoin models.

Bibliography

[1] J. Abate and W. Whitt. Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal of Computing*, 7:36–43, 1995.

[2] W.J. Anderson. *Continuous-time Markov chains: An applications-oriented approach*. Springer-Verlag, New York, 1991.

[3] R. Bowden, H. P. Keeler, A. E. Krzesinski, and P. G. Taylor. Block arrivals in the Bitcoin blockchain, 2018. Preprint accessible through the arXiv at https://arxiv.org/abs/1801.07447.

[4] P. Brémaud. *Markov Chains: Gibbs Fields, Monte Carlo Simulation and Queues*. Springer, New York, 1999.

[5] P. Buckingham and B. Fralix. Some new insights into Kolmogorov's criterion, with applications to hysteretic queues. *Markov Processes and Related Fields*, 21:339–368, 01 2015.

[6] S. Dendievel, S. Hautphenne, G. Latouche, and P. G. Taylor. The time-dependent expected reward and deviation matrix of a finite QBD process. *Linear Algebra and its Applications*, 570:61–92, 2019.

[7] S. Doroudi, B. Fralix, and M. Harchol-Balter. Clearing analysis on phases: Exact limiting probabilities for skip-free, unidirectional, quasi-birth-death processes. *Stochastic Systems*, 6:420–458, 2016.

[8] I. Eyal and E.G. Sirer. Majority is not enough: Bitcoin mining is vulnerable. In *Financial Cryptography and Data Security*, pages 436–454. Springer, Berlin, 2014.

[9] B. Fralix. When are two Markov chains similar? *Statistics & Probability Letters*, 107:199–203, 2015.

[10] B. Fralix. On classes of Bitcoin-inspired infinite-server queueing systems. *Queueing Systems*, 95:29–52, 2020.

[11] B. Fralix, F. Hasankhani, and A. Khademi. The role of the random-product technique in the theory of Markov chains on a countable state space., 2020. Submitted for publication: a draft is available at http://bfralix.people.clemson.edu/preprints.htm.

[12] B. Fralix, J.S.H. Van Leeuwaarden, and O. J. Boxma. Factorization identities for a general class of reflected processes. *Journal of Applied Probability*, 50:632–653, 2013.

[13] P. Franco. *Understanding Bitcoin: Cryptography, Engineering and Economics*. John Wiley & Sons, Ltd, Chichester, UK, 2015.

[14] M. Frolkova and M. Mandjes. A Bitcoin-inspired infinite-server model with a random fluid limit. *Stochastic Models*, 35:1–32, 2019.

[15] D. P. Gaver, P. A. Jacobs, and G. Latouche. Finite birth-and-death models in randomly changing environments. *Advances in Applied Probability*, 16:715–731, 1984.

[16] J. Göbel, H. P. Keeler, A. E. Krzesinski, and P. G. Taylor. Bitcoin blockchain dynamics: The selfish-mine strategy in the presence of propagation delay. *Performance Evaluation*, 104:23–41, 2016.

[17] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley Longman Publishing Co., Inc., 1994.

[18] B. Hajek. Birth-and-death processes on the integers with phases and general boundaries. *Journal of Applied Probability*, 19:488–499, 1982.

[19] Q. He. *Fundamentals of Matrix-Analytic Methods*. Springer Publishing Company, Incorporated, New York, 2014.

[20] G. Huberman, J. D. Leshno, and C. Moallemi. An economic analysis of the Bitcoin payment system, 2019. Draft accessible at https://moallemi.com/ciamac/papers/bitcoin-2017.pdf.

[21] K. Javier and B. Fralix. A further study of some Markovian Bitcoin models from Göbel et al. *Stochastic Models*, 36:223–250, 2020.

[22] K. Javier and B. Fralix. On the study of the running maximum and minimum level of level-dependent quasi-birth-death processes and related models, 2020. Submitted for publication: a draft of this paper is accessible at http://bfralix.people.clemson.edu/preprints.htm.

[23] K. Javier and B. Fralix. On the time-dependent behavior of homogeneous Quasi-Birth-Death processes with finitely many levels, 2020. Submitted for publication: a draft of this paper is accessible at http://bfralix.people.clemson.edu/preprints.htm.

[24] J. Joyner. A new look at block-structured Markov processes, 2016. Unpublished manuscript: available at http://bfralix.people.clemson.edu/preprints.htm.

[25] J. Joyner and B. Fralix. A new look at Markov processes of g/m/1-type. *Stochastic Models*, 32:253–274, 2016.

[26] S. Karlin and H.M. Taylor. *A First Course in Stochastic Processes*. Academic Press, New York, 1975.

[27] S. Kasahara and J. Kawahara. Effect of Bitcoin fee on transaction-confirmation process. *Journal of Industrial and Management Optimization*, 15:365–386, 2019.

[28] Y. Kawase and S. Kasahara. Transaction-confirmation time for Bitcoin: A queueing analytical approach to blockchain mechanism. In *Queueing Theory and Network Applications*, volume 10591, pages 75–88, 2017.

[29] J. Keilson and Y. Masuda. Transient probabilities of homogeneous row-continuous bivariate Markov chains with one or two boundaries. *Journal of the Operations Research Society of Japan*, 40:390–400, 1997.

[30] J. Keilson and M. Zachmann. Homogeneous row-continuous bivariate Markov chains with boundaries. *Journal of Applied Probability*, 25A:237–256, 1988.

[31] A. Kyprianou. *Introduction to the Fluctuation Theory of Lévy Processes*. Springer, New York, 2006.

[32] G. Latouche and V. Ramaswami. *Introduction to Matrix Analytic Methods in Stochastic Modeling.* Society for Industrial and Applied Mathematics, Philadelphia, USA, 1999.

[33] G. Latouche and Ramaswami V. Expected passage times in homogeneous quasi-birth-and-death processes. *Stochastic Models*, 11:103–122, 1995.

[34] Q. Li, J. Ma, and Y. Chang. Blockchain queueing theory, 2018. Preprint accessible on arXiv at https://arxiv.org/abs/1808.01795.

[35] Q. Li, J. Ma, Y. Chang, F. Ma, and H. Yu. Markov processes in blockchain systems, 2019. Preprint accessible on arXiv at https://arxiv.org/abs/1904.03598.

[36] X. Liu and B. Fralix. On lattice path counting and the random product representation, with applications to the Er/M/1 queue and the M/Er/1 queue. *Methodology and Computing in Applied Probability*, 21:1119–1149, 2019.

[37] M. Mandjes and P. Taylor. The running maximum of a level-dependent quasi-birth-death process. *Probability in the Engineering and Informational Sciences*, 30:212–223, 2016.

[38] S. Nakamoto. Bitcoin: A peer-to-peer electronic cash system, 2008. Paper available at https://bitcoin.org/bitcoin.pdf.

[39] M. Remiche. Time to congestion in homogeneous quasi-birth-death processes. *Opsearch*, 35:169–192, 1998.

[40] M. Renault. Four proofs of the ballot theorem. *Mathematics Magazine*, 80:345–352, 2007.

[41] F. Tschorsch and B. Scheuermann. Bitcoin and beyond: A technical survey on decentralized digital currencies. *IEEE Communications Surveys and Tutorials*, 18:2084–2123, 2016.