Improvements to a Two-Dimensional Model for Pneumatic and Non-Pneumatic Tires

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IMPROVEMENTS TO A TWO-DIMENSIONAL MODEL FOR PNEUMATIC AND NON-PNEUMATIC TIRES

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mechanical Engineering

by
Erin Leigh McPillan
May 2017

Accepted by:
Dr. Paul Joseph, Committee Chair
Dr. Tim Rhyne
Dr. Lonny Thompson
ABSTRACT

This thesis is a continuation of the model development work by Amir Gasmi in 2011 and Tim Lewis in 2012. The model was originally developed by Gasmi in 2011 to represent non-pneumatic tires. It centers on a thin, circular beam in contact with a rigid surface, which represents the belt package of the non-pneumatic tire. The beam is connected to a wheel with spokes. Gasmi developed a method to approximate these discrete spokes with a continuous load on the belt package. The results of this non-pneumatic tire model agreed very well with non-linear finite element analysis of a non-pneumatic tire.

This model was adapted for pneumatic tires by Lewis in 2012. Lewis introduced a superposition scheme to reduce errors introduced by tension in the belts at the edge of contact. The sidewalls were approximated as circular in order to calculate sidewall stiffness. The results presented by Lewis for force versus deflection and counterdeflection agreed reasonably well with experimental measurements, but used stiffness values that were more from curve fitting than from a mechanics analysis of the tire geometry and material properties.

The model presented in this thesis was developed directly from Lewis’ model. This model includes more representative sidewall modeling based on membrane theory, and more representative belt properties based on composites theory. Unlike in the results presented by Lewis, the sidewall stiffness values (denoted as $K_r$ and $K_\theta$) are not considered to be inputs to the model; rather, the tire’s geometrical design parameters
(such as outer radius and sidewall width) are the inputs, and the stiffness values are calculated as part of the model. This allows for a more direct connection to tire design.

The model has been improved to take into account a stiffness property called pre-tensioning, which allows the sidewalls (or in the case of a non-pneumatic tire, spokes) to maintain their stiffness in tension at small levels of negative deflection.

This thesis will present a thorough investigation of the solutions with and without superposition, at a range of conditions. Also presented is an investigation of adjustments to input parameters to account for assumptions made in the model, as well as a sensitivity analysis.
ACKNOWLEDGMENTS

I would like to thank many people who supported me as I completed my coursework and this thesis while working full time at Michelin. I had several managers who allowed me the flexibility to balance my work with classes, homework, and research over several years: Dave Hall, Anton Thomas, Vasanti Gharpuray, Patrick Rawlinson, and Jim Frady. At Clemson, my advisor and committee chair Dr. Paul Joseph was very understanding and accommodating of my unusual schedule.

I extend my thanks to Tim Lewis and Amir Gasmi, on whose work this thesis is based. Tim especially left very detailed documentation that helped me understand and continue his work.

I thank my parents, David and Vicki McMullin, for their words of encouragement and understanding when my estimated graduation got pushed back.

Lastly, I was extremely fortunate to have had my husband Charles McPillan as a math tutor, algebra double-checker, and personal cheerleader during the last several years. I thank him for his understanding when I had to dedicate evenings and weekends to my schoolwork, and for his unwavering support.
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CHAPTER ONE

PREVIOUS WORK AND INTRODUCTION

1.1 Literature Review

The objective of this study is to determine and validate the appropriate tire stiffness parameters for a two-dimensional, static pneumatic tire model. The difficulty of this task is the highly nonlinear behavior of a tire and the existence of multiple solutions, not all of which are physically relevant. Ideally each stiffness parameter should be obtained by independent analysis, and the resulting model should predict the correct vertical displacement, counterdeflection and contact pressure. If a model only predicts two of these three responses, it clearly does not capture the correct physics. Similarly, it is not guaranteed that a model that predicts all three is correct. In trying to achieve the correct pneumatic tire model, there are two important parts of modeling a tire: the influence of the tire sidewall on the belts, and the contact of the tire with the ground.

1.1.1 Tire Sidewall Modeling

The shape of the inflated tire has long been studied as a way to increase the tire’s life. During the course of the development of the pneumatic tire, it was determined that if the tire is cured in the shape it will naturally take when inflated, then there would be a reduction in the stresses that exist in the body of the tire after curing. As analysis progressed, later models included structural behavior to describe the forces created by the tire under deflection. As described by Miller, 1985, models of the structural behavior of tires can be grouped into three categories: membrane, shell, and 3D structural. Membrane models are simplest, but lack any representation of bending stiffness of the
sidewall. Shell models include bending stiffness, which can be important in areas of the
tire where the structure is thicker, such as the edge of the belt, or the part of the tire near
the rim. However, there are still some types of deformation that shell models do not
capture (such as shear between layers of cords), so these models typically over-estimate
tire stiffness. Three-dimensional models have the ability to include the most modes of
deformation, but are complicated and are usually evaluated with finite element analysis.
This review will focus on membrane models.

Purdy, 1963, is one of the first publications that described the shape of the tire
using the assumption that the sidewall functions as a membrane; that is, all loads are
carried in the plane of the surface, and the structure does not resist bending. Most of this
work focuses on the path that cords take in bias ply tires having two non-radial plies that
run from bead to bead. The load-carrying capacity of the tire is calculated using cord
tension. The tire stiffness is addressed briefly, but no equations to predict the stiffness
were developed.

Robecchi, 1973 also analyzed the tire in a “membrane state of stress.” This paper
begins with a description of the tire as a surface of rotation with two radii of curvature.
Equations to describe the shape of the tire are derived from geometry and from
equilibrium, as shown in Chapter 2 of thesis. This publication includes the development
of the relationship between the angle of the membrane shape to the horizontal (ϕ) and the
general shape of the membrane curve (Rs and Re). These quantities are used in this thesis
to determine the radial stiffness of the sidewall, although the form shown by Robecchi is
for bias ply tires, as opposed to radial tires. This publication also goes into more depth
regarding the design of tires using these descriptors of the membrane shape, but does not
address the stiffness of the tire.

In the 1980s, Akasaka and Yamazaki modeled tire stiffnesses based on the shape
of the membrane sidewall (Akasaka, 1986; Yamazaki, 1987). These derivations directly
showed the impact of tire geometry on stiffness, and included a pneumatic component
(due to the tension in the sidewall) and a structural component (due to the deformation of
the sidewall). However, these stiffnesses were “pure” (one-dimensional) stiffnesses, and
did not directly correspond to deformations that the tire undergoes during normal use.
For example, the radial stiffness calculation (Yamazaki, 1987) was validated compared to
experiments in which a load was applied to the outside of the tire using a rigid ring, as
opposed to the load applied to the tire by a flat surface such as a road.

Rhyne, 2005 developed a model for vertical tire stiffness based on the geometry
of the deformed tire, which corresponded to how the tire deforms during loading while in
use. Previous models of stiffness, such as Koutny, 1976, had relied on a thermodynamic
analysis of the air contained in the tire. The model developed by Rhyne provided a
“master curve” for vertical stiffness, and demonstrated that a wide range of tire
dimensions fall on or very close to this master curve. However, this does not provide
insight into the impact of tire design changes for a given dimension, such as sidewall
width or belt properties. In The Pneumatic Tire, published by NHTSA in 2006, Padula
expanded upon the work of Rhyne to put the equation for stiffness in terms of more
standard tire size descriptors: aspect ratio, section width, and rim diameter. This
approach still uses overall tire dimension as the only inputs, and does not allow for prediction of the effects of tuning within a given tire dimension.

More recent models of the sidewall include Kim, 2008. The model developed here included circumferential stiffness calculations similar to Akasaka and Yamazaki (and those used in this thesis), but again modeled a “pure” stiffness that does not represent tire usage. Another example of a recent sidewall model is Muhkin, 2013. Although this model did include multiple belts, it was focused on bias ply rather than radial tires.

1.1.2 Tire Contact Modeling

Another important part of modeling the loaded tire is the contact with the ground. Of interest for the problem at hand is a curved beam (the tire belt package) deformed against a flat surface (the road).

In 1965, Wu and Plunkett modeled curved beams in contact, but did not take into account shear stress within the rings. Wu, 1971, analyzed a cylindrical membrane in contact, using a life raft as an example. This is not directly applicable to radial tires in contact with the ground, because the part of the tire that acts as a membrane (the sidewall) is not in contact.

In the field of application to tires, an early reference for modeling contact pressure between a loaded radial tire and flat ground is Yamagishi, 1980 (“The Circumferential Contact Problem for a Belted Radial Tire”). This model ignored shear deformations and took into account the stiffness of the tread rubber, but was found to under-predict the contact pressure. Further work by Yamagishi in the same year (“Singular Perturbation
Solutions of the Circumferential Contact Problem for the Belted Radial Truck and Bus Tire”) explored this model further. In a manner similar to the solutions of Gasmi and Lewis, which are described below, the tire is divided into regions circumferentially to aid solution development. Interestingly, rather than division depending on the structure of the tire (such as how Gasmi’s regions depend on when the non-pneumatic tire spokes go into compression), the regions in this model depend on what mechanisms are most important. For example, in the non-contact region (analogous to Gasmi and Lewis’ tension and compression regions), the bending of the belts is important in the region closest to contact, but the sidewall stiffness is most important far from contact.

Recent advances in modeling this type of contact have been made by Gasmi, et. al., 2010 and 2011, by analytically modeling the contact of a curved beam with a flat surface. The 2010 publication modeled a circular Timoshenko beam in contact between two flat surfaces. The contact pressure was accounted for analytically by using a Taylor series expansion to relate the radial and circumferential displacement of the beam with its vertical displacement. The 2011 publication made use of this contact model in the context of a non-pneumatic tire. Both models showed good agreement with finite element analysis modeling of force and deflection, but the contact pressure results were not compared with experimental data.

Lewis, 2012, made use of the Gasmi beam and contact model and expanded it to pneumatic tires, as described below. More recently, the curved beam modeling including contact has been used for static analysis of various loadings (Wang, et. al., 2015) and dynamic analysis of vibrations of the ring (Wang, et. al., 2016).
1.2 Introduction to Current Model

This thesis is a continuation of the model development work by Amir Gasmi in 2011 and Tim Lewis in 2012. The objective is to develop a tire model that can directly relate tire design parameters to the vertical stiffness of the tire.

This model was originally developed by Gasmi in 2011 for application to non-pneumatic tires, such as the Tweel™ non-pneumatic tire designed and manufactured by Michelin North America. The model centers on a thin, circular beam in contact with a rigid surface, which represents the belt package of the non-pneumatic tire. The beam is connected to a wheel with spokes; like a sidewall in a pneumatic tire, these spokes are in tension at the top of the tire and in compression at the bottom of the tire. Gasmi developed a method to approximate these discrete spokes with a continuous load on the belt package, which is convenient for later application to a pneumatic tire.

For solution, the tire was separated into three regions circumferentially: the region in contact with the ground, the region in which the spokes are in tension, and between them the region where the spokes are not in contact and also not in tension (“free” region). The non-pneumatic tire spokes modeled by Gasmi buckle in compression; thus in the contact region and the free region, the spokes do not exert any force on the belt beam. The spoke stiffness is linear, so in the tension region, the spokes apply a force that increases linearly with radial displacement. The normal pressure in the contact patch is approximated as a sum of cosine functions; because the tire and boundary conditions are symmetric, the contact pressure must be an even function (Gasmi, 2012).
The results of this non-pneumatic tire model were shown to agree very well with linear and non-linear finite element modeling of a non-pneumatic tire. The continuous spoke approximation was also shown to be justified.

This model was adapted for pneumatic tires by Lewis in 2012. Primary differences between the Gasmi model and that of Lewis are a more complex radial spring, the inflation pressure and the addition of sidewall shear stress. The latter increases the order of the differential equation. Lewis approximated the sidewall as circular in order to calculate the stiffness values, which are different in each of the three circumferential regions. Lewis also introduced the superposition scheme to the solution procedure, which is described in detail in Section 4.3. Superposition helps to address both nonlinear effects for large deflection and the effect of inflation pressure on the contact stress. The results presented by Lewis for force versus deflection and force versus counterdeflection agreed reasonably well with experimental measurements, but were based on stiffness values that were more from curve fitting than from a mechanics analysis of the tire geometry and material properties.

The model presented in this thesis was developed directly from Lewis’ model. This model includes more representative sidewall modeling based on membrane theory. This involves predicting the shape of the sidewall in order to predict the radial component of sidewall tension, and modeling of the structural stiffness in the radial and circumferential directions. This also includes modeling of the torsional stiffness of the tire based on membrane theory, and consideration of the slope of the belts in the transverse direction.
This model also makes use of more representative belt properties based on composites theory. Unlike in the results presented by Lewis, the sidewall stiffness values (denoted as $K_r$ and $K_θ$) are not considered to be inputs to the model; rather, the tire’s geometrical design parameters (such as outer radius and sidewall width) are the inputs, and the stiffness values are calculated as part of the model.

The model has been improved to take into account a stiffness property called pre-tensioning, which allows the sidewalls (or in the case of a non-pneumatic tire, spokes) to maintain their stiffness in tension at small levels of negative deflection. This is called pre-tensioning because in a non-pneumatic tire, this behavior would be achieved by building the tire with spokes that are in tension in the unloaded state. A parameter has been added to the model to represent this new flexibility in modeling the stiffness of the sidewall or spokes.

This thesis will present a thorough investigation of the solutions with and without superposition, at a range of conditions. Also presented is an investigation of adjustments to input parameters to account for assumptions made in the model, and a sensitivity analysis.
CHAPTER TWO
MEMBRANE THEORY OVERVIEW

This section follows course material from AUE 8290, taught at Clemson University by Dr. Timothy Rhyne.

2.1 General Membrane Theory for Tires

A membrane is a thin shell that does not resist bending, and thus does not experience any stress due to bending. This assumption implies that the shell has zero thickness. The sidewall of an inflated tire can be modeled using a rotationally-symmetric membrane, such as described in Timoshenko, 1959.

First, we consider a section of the membrane defined by the angles d\(\theta\) and d\(\phi\). In the figure below, \(r_1\) is the radius of curvature of the membrane in the plane containing the axis of rotation and the radial direction. The \(\theta\) direction is circumferential, the \(x\) direction is the tire’s axis of rotation, and the \(r\) direction is perpendicular the \(x\) axis. \(N_\theta\) and \(N_\phi\) represent the tension per unit length in the sidewall in their respective directions.

The inflated sidewall is considered to be axisymmetric, so each cross-section in the \(x-r\) plane is identical. This section is called a meridian plane. \(N_\phi\) varies along the membrane, but because the sidewall is axisymmetric, \(N_\theta\) is constant (the same for each meridian).
Figure 2.1.1: Definition of the coordinate system for a tire sidewall

Figure 2.1.2: Two views of the membrane element
The sum of all forces normal to the membrane surface is shown in Equation 2.1.1. The three terms represent, in order: the normal component of the force on the lower side times the length of the lower side; the normal component of force on the outer meridian side times the length of the meridian side; the inflation pressure times the area of the element.

\[
\sum F_{normal} = 0 = -\left[ (N_\phi - \frac{dN_\phi}{d \phi} d\phi) \sin(d\phi) (r - dr) d\theta \right] \\
- [N_\theta d\theta \sin \phi (r_1) d\phi] + [P(r_1 d\phi) (r d\theta)] \tag{2.1.1}
\]

The equation can be simplified by expanding the terms, neglecting the differential terms, rearranging, and eliminating \( \sin(\phi) \) by noting that \( r = r_2 \sin(\phi) \) (see Figure 2.1.2). The final equation relates the tension in the \( \theta \) and \( \phi \) directions to the inflation pressure.

\[
\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = P \tag{2.1.2}
\]

Summing the forces tangent to the membrane yields Equation 2.1.3. The terms, in order, represent: the tangent force on the upper side times the length of the upper side; the tangent component of force on the lower side times the length of the lower side; the tangent component of the force on the outer meridian side times the length of the meridian side.

\[
\sum F_{tangent} = 0 = [N_\phi r d\theta] - \left[ (N_\phi - \frac{dN_\phi}{d \phi} d\phi) \cos(d\phi) (r - dr) d\theta \right] \\
- [(N_\theta d\theta \cos \phi) r_1 d\theta] \tag{2.1.3}
\]

The equation can be simplified by expanding the terms, using small angle assumptions, and removing higher-order differential terms. It can be simplified further.
by using Equations 2.1.4a and 2.1.4b; the exact differential, and a relationship derived
from Figure 2.1.2.

\[
\frac{d}{d\phi} (rN_\phi) - N_\theta r_1 \cos \phi = 0
\]  
(2.1.4a)

\[
d\phi = \frac{dr}{r_1 \cos \phi}
\]  
(2.1.4b)

The final form is shown in Equation 2.1.5.

\[
\frac{d}{dr} (rN_\phi) - N_\theta = 0
\]  
(2.1.5)

This yields a second relationship between \(N_\theta\) and \(N_\phi\). Equations 2.1.2 and 2.1.5
are the membrane equilibrium equations. Using the definition of the radius of curvature
and geometry derived from Figure 2.1.2, the following two expressions for the \(r_1\) and \(r_2\)
radii (in terms of \(x\) and \(r\)) can be developed:

\[
r_1 = \left[1+\left(\frac{dx}{dr}\right)^2\right]^{3/2} \frac{d^2x}{dr^2}
\]  
(2.1.6)

\[
r_2 = r\left[1+\left(\frac{dx}{dr}\right)^2\right]^{1/2} \frac{dx}{dr}
\]  
(2.1.7)

At this point four independent equations (2.1.2, 2.1.5, 2.1.6, 2.1.7) have been
obtained, but there are five dependent variables \((N_\theta, N_\phi, r_1, r_2, x)\). The independent
variable is \(r\). In order to obtain a sufficient number of equations, consider that the
membrane consists of a net of cords. All the loads are carried by this net, and there is no
slip between the cords. As shown in Figure 2.1.3, the angle between the cords and the
circumferential direction is denoted as \(\beta\).
Figure 2.1.3: Tension in the cords within the membrane

The relationships between the cord tension and the membrane tension can be used to develop the following equation relating $N_\theta$ and $N_\phi$, but this equation introduces another variable ($\beta$), so another equation is needed.

\[
\frac{N_\phi}{N_\theta} = \tan^2 \beta
\]  

(2.1.8)

To obtain the additional equation, and following Figure 2.1.4, consider what happens to the net of cords when it is rolled onto a cylindrical drum. $X$ is the length of cord between two intersections with cords running in the other direction, and it remains constant as we have assumed there is no slip between the inextensible cords. At a larger radius ($r_b$), the circumference is larger than at a smaller radius ($r_a$), so the diamond shapes in the cord net are stretched in the circumferential direction. This means that the cord angle $\beta$ is different for the two radii ($\beta_a$ and $\beta_b$). The geometry of the diamond shape can be used to relate the width of the diamonds ($r_ad\theta$ and $r_bd\theta$) to the length of cord between...
intersections (X). Those equations at \( r_a \) and \( r_b \) can be combined to provide a relationship between \( \beta \) and \( r \). (Note: this is referred to as the “conventional cord path.”)

\[
\frac{\cos \beta}{r} = \frac{\cos \beta_s}{r_s} = \text{constant} \tag{2.1.9}
\]

The set of equations is now complete, and are summarized in Table 2.1.1.

---

**Figure 2.1.4: The cord net at two radii on the membrane**

**Summary of Membrane Equations**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = P )</td>
<td>(1)</td>
</tr>
<tr>
<td>( \frac{d}{dr} (rN_\phi) - N_\theta = 0 )</td>
<td>(2)</td>
</tr>
<tr>
<td>( \frac{N_\phi}{N_\theta} = \tan^2 \beta )</td>
<td>(3)</td>
</tr>
<tr>
<td>( r_1 = \left[1 + \left(\frac{dx}{dr}\right)^2\right]^{3/2} )</td>
<td>(4)</td>
</tr>
<tr>
<td>( r_2 = \frac{r\left[1 + \left(\frac{dx}{dr}\right)^2\right]^{1/2}}{\frac{dx}{dr}} )</td>
<td>(5)</td>
</tr>
<tr>
<td>( \frac{\cos \beta}{r} = \text{constant} )</td>
<td>(6)</td>
</tr>
</tbody>
</table>

**Table 2.1.1: Summary of General Membrane Theory equations**
2.2 Radial Sidewall Case

In the case of a radial tire, the cords in the sidewall of the tire run in the radial direction; that is, \( \beta = \pi/2 \), and is constant. Since the cords can only carry load in the direction of their axes, the sidewall cannot support any tension in the circumferential direction, which means that \( N_\theta = 0 \). This can be used to simplify the first and second equations in Table 2.1.1. Furthermore, the equations involving \( \beta \) are no longer needed.

In order to determine the stress in the membrane in terms of known quantities, the expression for \( N_\phi \) in terms of pressure is substituted into the first \( N_\phi \) equation, which is then integrated twice. The known values of \( \phi \) (\( \phi = 0 \) at \( r = r_e \), \( \phi = \pi/2 \) at \( r = r_s \)) are used to determine the values of the resulting constants of integration. The following equations are the result.

\[
\sin \phi = \frac{r^2 - r_e^2}{r_s^2 - r_e^2} \quad (2.2.1)
\]

\[
r_1 = \frac{r_s^2 - r_e^2}{2r} \quad (2.2.2)
\]

From this, we can see that for a given membrane curve (defined by \( r_s \) and \( r_e \)) the term \((r_s^2 - r_e^2)/2\) is constant, and also that the product of the radial location \( r \) and the radius of curvature \( r_1 \) is constant.

\[
rr_1 = \frac{r^2 - r_e^2}{2} = constant \quad (2.2.3)
\]

This relationship is substituted into the equation that relates \( N_\phi \) to inflation pressure to obtain an equation for \( N_\phi \) in terms of known quantities. Recall that \( N_\phi \) is the tension per unit length of the membrane (in this case, the sidewall of the tire). This
expression will be used in Chapter 3 to determine the load exerted by the sidewall on the belt package of the tire.

\[ N_\phi = \frac{P}{r} \left( \frac{r_1^2 - r_0^2}{2} \right) \]  

(2.2.5)

2.3 Membrane Shape

The expressions developed above can also be used to determine the shape of the sidewall. The shape must be known so that the total load exerted on the belt package can be decomposed into its axial and radial components. Because the membrane can only support load tangent to its surface, the shape of the sidewall determines the direction in which the force is exerted.

This derivation begins with the general membrane theory equations (Table 2.1.1). All unknowns except \( x \) must be eliminated from the membrane equations in order to determine the membrane shape. First, Equation 2.1.8 is combined with Equation 2.1.9, and the result is combined Equations 2.1.2 and 2.1.5. The radii of curvature equations (2.1.6 and 2.1.7) are then plugged into the combined equation. The final combined equation is shown below.

\[
\frac{d^2 x}{dr^2} + \frac{dx}{dr} \left[ 1 + \left( \frac{dx}{dr} \right)^2 \right] \left[ \frac{r \cos^2 \beta_s}{(r_1^2 - r^2 \cos^2 \beta_s)} - \frac{2r}{(r^2 - r_0^2)} \right] = 0 \]  

(2.3.1)

In the radial sidewall case where \( \beta_h = 90^\circ \), the term containing \( \cos^2(\beta) \) drops out of the equation, yielding this simplified version.

\[
\frac{d^2 x}{dr^2} + \frac{dx}{dr} \left[ 1 + \left( \frac{dx}{dr} \right)^2 \right] \left[ \frac{2r}{(r^2 - r_0^2)} \right] = 0 \]  

(2.3.2)

To solve this equation, a new variable \( z = dx/dr \) is introduced. The equation is integrated twice to determine an analytical solution for the membrane shape of a radial
sidewall using elliptical integrals. Practically speaking, the membrane curve can be
drawn numerically by calculating the radii of curvature and sweeping out small steps of
the curve from the top radius down to a minimum chosen radius. The process for
determining the membrane shape of a specific tire of known dimension is explained in
more detail in Chapter 3.
CHAPTER THREE

RADIAL AND TORSIONAL STIFFNESS CALCULATIONS

3.1 Radial Stiffness Calculation Methods

This section describes three methods for calculating the radial stiffness of the sidewall of a tire. Part of the stiffness arises from the tension in the sidewall cords. This tension is present because of the inflation pressure, so this part of the stiffness is referred to as the pneumatic part. The pneumatic part is treated first, and three methods are described to calculate it. The goal of each method is to determine the radial load that the sidewalls of the tire exert on the belts, which will be used to calculate the stiffness of the tire. A derivation of the structural part of radial stiffness is presented next.

The geometry of a cross-section of a tire is shown in Figure 3.1.1. The first two methods presented in this section assume that the sidewall shape remains circular as the tire is inflated and loaded. The third method models the sidewall as a membrane with radial cords, as explained in Chapter 2. The membrane method more closely represents a tire sidewall. The circular sidewall methods have been used in previous work (Lewis, 2012) and are presented for comparison.

When using the membrane method, the radius of curvature of the sidewall varies from the point where the sidewall is attached to the rim to where it is attached to the belts. The angles $(\theta/2)$ and $\varphi$ are defined similarly and used in the same way in some calculations, but their values differ due to the different shapes of the sidewall.
Figure 3.1.1: Geometry of the tire sidewall

Figure 3.1.2: Dimensions of the reference tire

<table>
<thead>
<tr>
<th>Reference Tire Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{inf}$</td>
</tr>
<tr>
<td>$L$</td>
</tr>
<tr>
<td>$R_{in}$</td>
</tr>
<tr>
<td>$\theta_{inf}$</td>
</tr>
</tbody>
</table>
3.1.1 Circular Sidewall – Equilibrium Method

The first method described assumes a circular sidewall. The cords in the sidewall are assumed to be inextensible, so the arc length of the sidewall is constant. The subscript “\textit{inf}” refers to the inflated tire with no external loading.

\[ R_{\text{inf}} \theta_{\text{inf}} = R \theta = \text{constant} \quad (3.1.1) \]

The vertical deflection, \( \Delta L \), is defined as the change in radial distance between the wheel attachment point and the belt attachment point. In the equation below, \( L \) is defined geometrically in terms of the radius of curvature of the sidewall and the angle swept by the sidewall. As shown, the vertical deflection can be determined in terms of the inflated dimensions of the tire and the sidewall angle in the deflected state (\( \theta \)). The sidewall angle must be determined numerically for a given level of deflection.

\[ \Delta Z = 2V \sin \left( \frac{\theta}{2} \right) - 2V W \sin \left( \frac{\theta_{\text{inf}}}{2} \right) \quad (3.1.2) \]

The inflated, unloaded condition is taken as the reference condition for the following calculations. Figure 3.1.3 shows the stresses on a section of the sidewall of the inflated tire. There is no stress in the sidewall in the 2-direction because the cords are radial. \( R_{\text{in}} \) is the radius of the rim, and thus the inner radius of the sidewall. \( R_{\text{out}} \) is the inner radius of the belts, and thus the outer radius of the sidewall. \( d\beta \) is the angle in the circumferential direction that defines a section of the sidewall.
Equilibrium of forces in the 3-direction (axial) and of the moment about the rim attachment point yields the following equations, where $t$ is the thickness of the sidewall in the 3-direction.

\[
\sum F_3 = 0 = -P(R_{out} - R_{in}) \frac{R_o + R_i}{2} d\beta + \sigma_{1,out} t R_{out} d\beta \sin \frac{\theta_{inf}}{2} + \sigma_{1,in} t R_{in} d\beta \sin \frac{\theta_{inf}}{2} 
\]  
(3.1.4)

\[
\sum M_{wheel} = 0 = \frac{1}{2} P(R_{out} - R_{in})^2 R_i d\beta + \frac{1}{3} P(R_{out} - R_{in})^3 d\beta - \sigma_{1,out} t R_{out} d\beta \sin \frac{\theta_{inf}}{2} (R_{out} - R_{in}) 
\]  
(3.1.5)

$F_{out}$ is the force per unit length that the sidewall applies to the summit belts. This definition is applied to the moment balance equation.

\[
F_{out} = \sigma_{1,out} t 
\]  
(3.1.6a)

\[
F_{out} R_0 \sin \frac{\theta_{inf}}{2} = \frac{1}{2} P(R_{out} - R_{in}) R_i + \frac{1}{3} P(R_{out} - R_{in})^2 
\]  
(3.1.6b)

\[
\frac{F_{out}}{P} = \frac{\frac{1}{2}(R_{out} - R_{in}) R_{in} + \frac{1}{3}(R_{out} - R_{in})^2}{R_{out} \sin \frac{\theta_{inf}}{2}} 
\]  
(3.1.6c)
\( F_P \) is defined as the radial force per unit length that the sidewalls apply to the summit in the inflated, unloaded state. \( F_P \) is two times the radial component of \( F_{out} \) because each sidewall applies an equal amount of force to the belts.

\[
\frac{F_P}{P} = \left( \frac{1}{2} (R_{out} - R_{in}) R_i + \frac{1}{3} (R_{out} - R_{in}) \right)^2 \left( \frac{2 \cos \frac{\theta_{inf}}{2}}{R_{out} \sin \frac{\theta_{inf}}{2}} \right) \tag{3.1.7}
\]

\( q_r \) is defined as the effective radial pressure that the sidewall applies to the summit, averaged across the width of the tire, \( b \). If the ratio \( q_r/P \) were equal to 1, then the effective sidewall pressure (acting to pull the summit belts towards the axis of rotation) would be equal to the internal pressure (acting to push the summit belts away), which would lead to zero change in radius of the belts when the tire is inflated, compared to the uninflated state.

\[
- \frac{q_{r,inf}}{P} = \left( \frac{F_P}{P} \right) \left( \frac{1}{b} \right) = \left( \frac{1}{2} (R_{out} - R_{in}) R_i + \frac{1}{3} (R_{out} - R_{in}) \right)^2 \left( \frac{2 \cos \frac{\theta_{inf}}{2}}{R_{out} \sin \frac{\theta_{inf}}{2}} \right) \left( \frac{1}{b} \right) \tag{3.1.8}
\]

As the belts are moved with respect to the rim, \( \Delta L \) becomes non-zero, and the force and moment equilibrium equations change as follows. It is noted that \( L = (R_{out} - R_{in}) \).

\[
\sum F_3 = 0 = -P (L + \Delta L) \frac{R_{out} + \Delta L - R_{in}}{2} d\beta + \sigma_{1, out} t (R_{out} + \Delta L) d\beta \sin \frac{\theta}{2} + \sigma_{1, in} t R_{in} d\beta \sin \frac{\theta}{2} \tag{3.1.9}
\]

\[
\sum M_{wheel} = 0 = \frac{1}{2} P (L + \Delta L)^2 R_{in} d\beta + \frac{1}{3} P (L + \Delta L)^3 d\beta - \sigma_{1, out} t (R_{out} + \Delta L) d\beta \sin \frac{\theta}{2} (L + \Delta L) \tag{3.1.10}
\]
Once again, the definition of $F_{out}$ is applied to the moment balance equation to obtain an expression for the normalized effective radial pressure that the sidewalls apply to the summit. For each level of deflection ($\Delta L$), the sidewall angle $\theta$ must be determined numerically, then the normalized radial pressure can be calculated directly.

\[-\frac{q_r}{p} = 2 \frac{F_{out}}{p} \cos \frac{\theta}{2} \left( \frac{1}{b} \right) = \]
\[
\left( \frac{1}{2}(R_{out} - R_{in} + \Delta L)^2 R_i + \frac{1}{3}(R_{out} - R_{in} + \Delta L)^3 \right) \left( 2 \cos \frac{\theta}{2} \left( \frac{1}{b} \right) \right) \quad (3.1.11)
\]

3.1.2 Circular Sidewall – Tension Method

The second method to determine the sidewall stiffness also assumes circular sidewalls, but in this case the tension in the sidewall is approximated using the inflation pressure and the radius of curvature of the sidewall. This section follows Lewis, 2012. Lewis presents another, more complicated method for obtaining the angle of the sidewall, $\theta$, but it obtains the same results as the method described above. Thus, only the alternative method for determining sidewall stiffness is presented here.

The tension per unit length (circumferentially) in a pressurized membrane is approximated by Equation 3.1.12.

\[ T = PR \quad (3.1.12) \]

In this method, this approximation of the tension is used in place of the values obtained through the balance of moments. The expression for the effective radial load is obtained using the radial component of the tension, as follows
\[ q_r b = -T \cos \left( \frac{\theta}{2} \right) \quad (3.1.13) \]

Put into the same terms as the previous method, the normalized effective radial load is shown in Equation 3.1.14.

\[ \frac{q_r}{p} = -\frac{R}{b} \cos \left( \frac{\theta}{2} \right) \quad (3.1.14) \]

This equation holds for any level of deformation, so it can be used to directly calculate the ratio \( F_p/bP \) in the inflated state. Using the reference tire dimensions from the previous section, \( F_p/bP = 0.4063 \), which is 10% higher than the value calculated using the previous method.

**3.1.3 Membrane Sidewall**

The third method for calculating the sidewall stiffness makes use of the radial sidewall as a special case of general bias ply membrane theory, as explained in Chapter 2. The shape of an inflated membrane in equilibrium is described by two parameters: \( R_s \), the top radius of the curve, and \( R_e \), the radius at which the curve is widest. The geometry of the curve is shown in Figure 3.1.4.

Unlike the circular sidewall, the radius of curvature of the sidewall is not constant with radius (radial distance from the axis of rotation). In fact, the product \( (rR) \) is a constant, meaning that the radius of curvature decreases as the radius increases, so that the curvature radius is smallest at the peak of the membrane curve.

In order to fully define the part of the equilibrium curve that is the sidewall, three more parameters are needed to locate the curve laterally and to determine the top and bottom of the sidewall. In this case, the lateral position of the sidewall is determined by
the width of the tire, \( b \), and the top and bottom of the sidewall are determined by the known tire dimensions, \( R_{in} \) and \( R_{out} \). Thus, \( R_s \) and \( R_e \) must be determined for the inflated state.

![Figure 3.1.4: Sidewall dimensions for membrane theory](image)

\( R_s \) and \( R_e \) are important because they mathematically define the shape of the membrane curve. However, these are not easily-specified design parameters for the tire. In order to find \( R_s \) and \( R_e \), two other constraints are placed on the curve, then an optimization is run to find the \( R_s \) and \( R_e \) that best satisfy the set of five constraints. As mentioned above, the first three constraints are determined by known tire dimensions: \( R_{out} \), \( R_{in} \), and the known value for \( x(R_{out}) \) (based on the width of the rim on which the tire will be mounted during use). The remaining two dimensions needed to define the membrane curve were set as objectives in the optimization program. In order to provide a clear comparison to the circular sidewall methods, the two following values for the two
objectives were used: $x(R_{\text{out}}) = x(R_{\text{in}})$, and $W = x(R_{\text{out}}) - x(R_{e}) = 20$ mm. These values match the geometry of the circular sidewall methods. The optimization program determines values of $R_{e}$ and $R_{s}$ that conform to the constraints and best meet the objectives.

The tension per unit length in the sidewall is given by the following equation, which can be easily evaluated after $R_{s}$ and $R_{e}$ have been determined numerically.

$$T = \frac{P}{R} \left( \frac{R_{e}^{2} - R_{s}^{2}}{2} \right) \quad (3.1.15)$$

As in the circular sidewall method, the normalized effective radial load of the sidewalls on the belts is calculated using the radial component of sidewall tension. The equation is evaluated at $r = R_{\text{out}}$ to find the load at the edge of the belts.

$$-\frac{q_r}{P} = \left( \frac{1}{bR_{\text{out}}} \right) \left( \frac{R_{e}^{2} - R_{s}^{2}}{2} \right) \cos \phi \quad (3.1.16)$$

3.1.4 Method Comparison – Inflated Tire

Table 3.1.1 compares results for the three methods. All values shown are the sum of the values for both sidewalls. None of the three methods implicitly include the width of the belts, $b$, in the calculations. Two realistic but arbitrary values of $b$ have been used in all three cases in order to normalize the sidewall load to a non-dimensional ratio. The value of $b$ changes the absolute value of the normalized load because the effect of the inflation pressure depends on the width. The relative comparisons between the methods are not affected because they all have the same dependence on $b$. 

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Table 3.1.1: Sidewall load normalized by pressure in the inflated state, calculated using three methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$F_p/bP = -q_p/P$ at inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b = 200$ mm</td>
</tr>
<tr>
<td>Circular Sidewall, Tension Method</td>
<td>0.4063</td>
</tr>
<tr>
<td>Circular Sidewall, Equilibrium Method</td>
<td>0.3647</td>
</tr>
<tr>
<td>Membrane Sidewall</td>
<td>0.3890</td>
</tr>
</tbody>
</table>

3.1.5 Method Comparison – Loaded Tire

![Figure 3.1.5: Normalized effective radial load of the sidewall due to inflation pressure](image)

Figure 3.1.5 displays the normalized radial load that the sidewalls apply to the summit belts as a function of deflection. The comparison is shown over a range of vertical deflection that is relevant for most loading cases for the tire used in this study. Negative values of deflection indicate that the summit has moved closer to the belts, such as is the case in the contact patch of a loaded tire. Positive values indicate that the summit has moved away from the belts, such as is the case far from the contact region.
(this is referred to as counterdeflection). $\Delta L = 0$ corresponds to the circumferential location where the deflection is equal to the deflection of the inflated, unloaded tire.

Positive values of the normalized radial load indicate that the sidewall is counteracting the inflation pressure (applying a force toward the axis of rotation of the wheel); this is nearly always the case.

![Figure 3.1.6: Sidewall angle at the belt attachment point](image)

The angle of the sidewall at the point where it connects to the belts is very similar between the methods for the inflated state and for negative deflection. The calculated values are significantly different for large values of positive deformation.
<table>
<thead>
<tr>
<th>Sidewall Type</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular Sidewall – Tension Method</td>
<td>[-q_r \frac{P}{R} = 2 \frac{R}{b} \cos \left(\frac{\theta}{2}\right)]</td>
</tr>
</tbody>
</table>
| Circular Sidewall – Equilibrium Method | \[-q_r \frac{P}{P} = \frac{F_{out}}{P} \cos \frac{\theta}{2} \]
| \[= 2 \left( \frac{1}{2} \left( R_{out} - R_{in} + \Delta L \right) R_i + \frac{1}{3} \left( R_{out} - R_{in} + \Delta L \right)^3 \right) \left( \frac{1}{h} \right) \left( \cos \frac{\theta}{2} \right) \] |
| Membrane Sidewall | \[-q_r \frac{P}{P} = 2 \left( \frac{1}{bR_{out}} \right) \left( \frac{R_x^2 - R_e^2}{2} \right) \cos \phi \] |

Table 3.1.2: Summary of radial load equations, including both sidewalls

<table>
<thead>
<tr>
<th>Sidewall Type</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular Sidewall – Tension Method</td>
<td>[-q_y \frac{P}{P} = 2 \frac{R}{h} \sin \left(\frac{\theta}{2}\right)]</td>
</tr>
</tbody>
</table>
| Circular Sidewall – Equilibrium Method | \[-q_y \frac{P}{P} = \frac{F_{out}}{Ph} \sin \frac{\theta}{2} \]
| \[= 2 \left( \frac{1}{2} \left( R_{out} - R_{in} + \Delta L \right) R_i + \frac{1}{3} \left( R_{out} - R_{in} + \Delta L \right)^3 \right) \left( \frac{1}{h} \right) \] |
| Membrane Sidewall | \[-q_y \frac{P}{P} = 2 \left( \frac{1}{hR_{out}} \right) \left( \frac{R_x^2 - R_e^2}{2} \right) \sin \phi \] |

Table 3.1.3: Summary of axial load equations, including both sidewalls

### 3.1.6 Structural Part of Radial Stiffness

This derivation of the structural part of the radial stiffness follows Akasaka 1986, and also uses information from Yamazaki 1987.

The first structural contribution to the radial stiffness is the bending of the rubber in the sidewall. The curvature of the sidewall is denoted as \( \kappa \).
\[
\kappa = \frac{2r \sin \phi}{(R_{out}^2 - R_e^2)^2} \quad (3.1.17)
\]

The incremental change in curvature due to a change in the radius at the belts \((R_{out})\) is given by Equation 3.1.18.

\[
\delta \kappa = \left( \frac{\partial \kappa}{\partial R_e} \left( \delta R_e \right) + \frac{\partial \kappa}{\partial \phi} \left( \frac{\delta \phi}{\delta R_{out}} \right) + \frac{\partial \kappa}{\partial R_{out}} \right) \delta R_{out} \equiv \zeta \delta R_{out} \quad (3.1.18)
\]

The partial derivatives are given by the following expressions.

\[
\frac{\partial \kappa}{\partial R_e} = \frac{4rR_e \sin \phi}{(R_{out}^2 - R_e^2)^2} \quad (3.1.19a)
\]

\[
\frac{\partial \kappa}{\partial \phi} = \frac{2r \cos \phi}{(R_{out}^2 - R_e^2)} \quad (3.1.19b)
\]

\[
\frac{\partial \kappa}{\partial R_{out}} = \frac{-4rR_{out} \sin \phi}{(R_{out}^2 - R_e^2)^2} \quad (3.1.19c)
\]

The changes in \(R_e\) and \(\phi_{out}\) due to a change \(R_{out}\) in were evaluated numerically, using the sidewall shape information calculated in the previous section.

The bending strain energy stored in a fan-shaped segment is given by Equation 3.1.20.

\[
U_B = \frac{2}{2} \int_{R_{in}}^{R_{out}} D_\phi (\delta \kappa)^2 ds \quad (3.1.20)
\]

\(D_\phi\) is the bending rigidity of the sidewall rubber in the meridian direction, and is defined by Equation 3.1.21.

\[
D_\phi = \frac{E_m \ell^3}{3(1-\nu_m^2)} \quad (3.1.21)
\]

\(E_m\) is the elastic modulus of the rubber in the sidewall, and \(\nu_m\) is its Poisson’s ratio. \(\Delta \theta\) is the included angle between two adjacent radial cords in the sidewall, and \(\ell\) is
the thickness of the sidewall. In this model, \( t \) is assumed to be constant with \( r \) for the sake of simplicity.

The second structural contribution to the radial stiffness is the stretching of the rubber between the cords, in the circumferential direction. As the top of the sidewall is displaced radially, the radial location of each point on the sidewall also changes. This means that the circumference at that point on the sidewall changes, leading to stretching of the rubber, which contributes to the radial stiffness of the sidewall. The radial displacement of a point on the sidewall (denoted with the subscript \( p \)) is given by the following expression.

\[
\delta r_p = -\left( \frac{\partial l}{\partial R_e} \frac{\delta R_e}{\delta R_{out}} + \frac{\partial l}{\partial \phi} \frac{\delta \phi}{\delta R_{out}} + \frac{\partial l}{\partial R_{out}} \right) \frac{\delta R_{out}}{\partial r_p} \equiv \zeta_2 \delta R_{out} \tag{3.1.22}
\]

The partial derivatives are given by the following expressions. Note that the expressions are integrated with respect to \( r \) (radius), as opposed to \( s \) (curve length).

\[
\frac{\partial l}{\partial R_e} = -2 R_e \sin^2 \phi \int_{R_{in}}^{R_{out}} \left( \frac{1}{G^{3/2}} \right) (r^2 - R_e^2) (R_{out}^2 - r^2) dr \tag{3.1.23}
\]

\[
\frac{\partial l}{\partial \phi} = (R_{out}^2 - R_e^2) \sin \phi \cos \phi \int_{R_{in}}^{R_p} \left( \frac{1}{G^{3/2}} \right) (r^2 - R_e^2) dr \tag{3.1.24}
\]

\[
\frac{\partial l}{\partial r_p} = \frac{(R_{out}^2 - R_e^2)}{\sqrt{(R_{out}^2 - R_e^2)^2 - \sin^2 \phi (r_p^2 - R_e^2)^2}} \tag{3.1.25}
\]

\[
\frac{\partial l}{\partial R_{out}} = (R_{out}^2 - R_e^2) \left[ (R_{out}^2 - R_e^2)^2 - ((R_{out}^2 - R_e^2) \sin^2 \phi)^2 \right]^{-1/2} \tag{3.1.26}
\]

The strain energy due to the rubber stretching is given by Equation 3.1.27.

\[
U_s = \frac{2}{2} \int_{R_{in}}^{R_{out}} C_\theta (\varepsilon_\theta)^2 ds \tag{3.1.27}
\]
\( C_\theta \), given by Equation 3.1.28, is the extensional rigidity of the rubber sheet in the circumferential direction. \( V_m \) is the volume fraction of the rubber. The expression for radial displacement \( (u) \) can be substituted into the strain energy expression by using this relationship between strain and displacement: \( \varepsilon_\theta = u/r \).

\[
C_\theta = \frac{E_m tr \Delta \theta}{V_m(1-v_m^2)}
\]  

(3.1.28)

The two structural components are combined in \( q_{r,\text{struct}} \), which is added to the pneumatic \( q_r \) calculated in the previous section to obtain the total radial stiffness of the sidewall. This expression from Akasaka has been divided by the width of the belts \( (b) \) in order to put this structural component in the same terms as the pneumatic component.

\[
q_{r,\text{struct}} = \frac{2 \delta R_{\text{out}}}{b R_{\text{in}}} \left[ \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{E_m t^3}{3(1-v_m^2)} \xi_\theta^2 ds + \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{E_m t}{V_m(1-v_m^2)} \xi_\theta^2 \frac{1}{r} ds \right]
\]  

(3.1.29)

### 3.1.7 Total Radial Stiffness versus Deflection

The following figures show the total radial load calculated for a range of radial deflections, for inflation pressures equal to 0, 0.5, 1.0, 2.0, and 4.0 bar. The results are shown in units of pressure, as well as normalized by the inflation pressure (except for 0 bar). The colored lines overlaying the blue lines represent typical ranges of deflection for each pressure.

The radial load curves for all pressure approach the levels of 0 bar at high levels of negative deflection because in this region the angle between the sidewall and the belts approaches horizontal., so the radial component of the sidewall tension also approaches zero. Thus, the radial load in this region is dominated by the structural stiffness. In the 0 bar case, the radial load is due only to structural stiffness for all levels of deflection.
Figure 3.1.7: Total radial load versus deflection

Figure 3.1.8: Total radial load versus deflection, normalized by inflation pressure
3.2 Torsional Stiffness Calculation

The structural part of the torsional stiffness is due to the shearing of the sidewall material. The pneumatic part arises from the tension in the sidewall due to inflation pressure. As with the radial stiffness, the structural and pneumatic parts of torsional stiffness are assumed to be independent, and are derived separately in this section.

3.2.1 Structural Part of Torsional Stiffness

Equilibrium of the moments around the axis of rotation yields the following expression for $\tau_{r\theta}$, which is valid for the whole sidewall (from the rim to the belts).

$$\tau_{r\theta} = \frac{M}{2\pi rt} \quad , \quad R_{in} \leq r \leq R_{out} \tag{3.2.1}$$

This expression for shear stress is substituted into the relationship between stress and strain. It is assumed that radial displacement ($u_r$) is constant around the circumference of the tire (in the $\theta$ direction) because the loading is axisymmetric.

$$\gamma_{t\theta} = \frac{\tau_{t\theta}}{G} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \tag{3.2.2a}$$

$$\frac{du_{\theta}}{dr} = \frac{u_{\theta}}{r} = \frac{M}{2\pi rt^2} \quad \tag{3.2.2b}$$
Integration leads to the following expression for circumferential displacement.

\[ u_{\theta} = Cr - \frac{M}{4\pi Gtr} \quad (3.2.3) \]

The constant of integration can be determined by applying the known boundary condition that the sidewall is attached to the rim, so circumferential displacement is zero.

\[ u_{\theta}(r = R_{in}) = 0 \rightarrow C = \frac{M}{4\pi GtR_{in}^2} \quad (3.2.4) \]

The actual shear modulus and thickness of the sidewall vary along the sidewall, and so effective terms \((G_e\) and \(t_e)\) are used to represent the sidewall.

\[ u_{\theta} = \frac{M}{4\pi(Gt)eR_{in}} \left[ \frac{r^2 - R_{out}^2}{rR_{in}} \right] \quad (3.2.5) \]

Thus, at \( r = R_{out} \) (the radius of the summit belts), the circumferential displacement is given by Equation 3.2.6.

\[ u_{\theta} = \frac{\tau_{\theta}2\pi R_{out}^2 t}{4\pi(Gt)eR_{in}} \left[ \frac{R_{out}^2 - R_{in}^2}{R_{out}R_{in}} \right] = \frac{\tau_{\theta}tR_{out}}{2(Gt)e} \left[ \frac{R_{out}^2 - R_{in}^2}{R_{in}^2} \right] \quad (3.2.6) \]

The uniform shear stress acting on the inner surface of the belts \((r = R_{out})\) is defined as \(q_{\theta}\). Thus, at \( r = R_{out}\),

\[ t\tau_{\theta} = b q\theta_s. \]

\[ u_{\theta} = \frac{b q\theta_s R_{out}}{2(Gt)e} \left[ \frac{R_{out}^2 - R_{in}^2}{R_{in}^2} \right] \quad (3.2.7a) \]

\[ \frac{q\theta_s}{p} = \frac{u_{\theta}}{b R_{out}} \left[ \frac{R_{in}^2}{R_{out}^2 - R_{in}^2} \right] \quad (3.2.7b) \]

When the tire deforms radially, \( R_{in} \) remains constant, because the radius of the rim is fixed. However, \( R_{out} \) changes based on \( u_r \). Therefore, the structural part of the torsional shear stress depends on the radial displacement, through \( R_{out} \). This effect is not included in this model.
3.2.2 Pneumatic Part of Torsional Stiffness

The pneumatic part of the torsional stiffness arises from the circumferential component of the sidewall tension due to inflation pressure. This section uses information from Rhyne 1994.

As the rim is rotated around the tire’s axis of rotation with respect to the belts, the angle $\alpha$ represents the change in angle of the cords, compared to their original radial position. The total torque due to the tension in the cords is represented by $\Gamma$.

$$\Gamma = \left(\frac{p(R_{out}^2 - R_{in}^2)}{2R_{out}}\right)(2\pi R_{out})(\cos \phi)(\sin \alpha)(R_{out})(2) \tag{3.2.8}$$

The first term in Equation 3.2.8 is the tension per unit length in the sidewall cords, as described in the membrane sidewall section of the radial stiffness derivation. The second term is the length (circumference) of the sidewall at the interface with the belts. The third term reduces the quantity to the in-plane component of the tension; the other component is acting perpendicular to the plane of the wheel. The fourth term is the
circumferential component of the tension, and the fifth term is the moment arm for the tension. The final 2 is included to account for both sidewalls.

Using a small angle approximation \( \cos(\alpha) \approx 1 \), the change in \( \Gamma \) with respect to \( \alpha \) is represented by the following expression.

\[
\frac{d\Gamma}{d\alpha} = \left( \frac{P(R_{out}^2 - R_{in}^2)}{2R_{out}} \right) \left( 2\pi R_{out} \right) (\cos \phi) (R_{out})(2)
\]  

Again approximating with small angles, we can relate \( \alpha \) to \( u_\theta \) in order to obtain an expression for the change in \( \Gamma \) with respect to \( u_\theta \).

\[
u_\theta = (R_{out} - R_{in}) \tan \alpha
\]  
\[
du_\theta = (R_{out} - R_{in}) \, d\alpha
\]  
\[
\frac{d\Gamma}{du_\theta} = \frac{d\Gamma}{d\alpha} \frac{d\alpha}{du_\theta} = 2\pi P(R_{out}^2 - R_{in}^2) \left( \frac{R_{out}}{R_{out} - R_{in}} \right) (\cos \phi)
\]  

Using the same relation as in the structural section, \( t_\tau_{r\theta} = bq_{\theta,p} \). The “p” subscript denotes pneumatic stiffness.

\[
\tau_{r\theta} = \frac{M}{2\pi r^2 t} = \left( \frac{1}{2\pi R_{out}^2} \right) \left( \frac{d\Gamma}{du_\theta} \right) u_\theta
\]  
\[
\frac{q_{\theta,p}}{P} = \frac{t_\tau_{r\theta}}{P} = \left( \frac{1}{2\pi R_{out}^2 b_P} \right) \left( \frac{d\Gamma}{du_\theta} \right) u_\theta
\]  

### 3.2.3 Total Torsional Stiffness

The stiffness is about 33% pneumatic and 67% structural for the example tire at 2.0 bar inflation pressure.

\[
\frac{q_{\theta}}{P} = \frac{q_{\theta,p}^2 q_{\theta,z}}{P} = \frac{u_\theta}{b} \left[ \left( \frac{R_{out}^2}{R_{out}(R_{out} - R_{in})} \right) + \frac{2(gt)_e}{P} \frac{R_{out}^2}{R_{out} - R_{in}} \right]
\]  

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3.3 Addition of Poisson Effect

The axial loads described above cause the belts in the summit to stretch laterally (in the direction parallel to the axis of rotation), and due to their Poisson ratio, this leads to contraction in the circumferential direction. This phenomenon counteracts radial expansion as the tire is inflated, effectively reducing radial displacement due to inflation pressure.

\[ \varepsilon_{r\theta} = \frac{1}{R+z} \left( \frac{d\theta_0}{d\theta} + u_r + z \frac{d\phi}{d\theta} \right) \]  

(3.3.1)

For pure inflation, \( du_{\theta 0}/d\theta = 0 \) because the inflation is axisymmetric, and \( z(d\phi/d\theta) = 0 \) because there is no rotation of the cross-sections of the belts. We also assume that the thickness of the belts is much smaller than the radius of the tire \((z<<R)\). With those effects, the equation simplifies, as shown in (3.3.2). \( u_{r0} \) is the radial deflection due to pure inflation.

\[ u_{r0} = R \varepsilon_{\theta} \]  

(3.3.2)

By incorporating the relationship between stress and strain, \( u_{r\theta} \) can be expressed in terms of stress.

\[ \varepsilon_{\theta} = \frac{\sigma_{\theta}}{E_{\theta}} - \frac{\nu_{y\theta} \sigma_{y}}{E_{y}} \]  

(3.3.3a)

\[ u_{r0} = R \left( \frac{\sigma_{\theta}}{E_{\theta}} - \frac{\nu_{y\theta} \sigma_{y}}{E_{y}} \right) \]  

(3.3.3b)

The following expressions for stress are substituted into Equation 3.3.3b.

\[ \sigma_{\theta} = \frac{(p - F_P/b) R_0}{h} \]  

(3.3.4)

\[ \sigma_{y} = \frac{F_{axial}}{h} = \frac{T_{belsy} \sin \phi}{h} = \frac{P}{R} \left( \frac{R_0^2 - R^2}{2} \right) \frac{1}{h} \sin \phi \]  

(3.3.5)
\[ u_{r0} = \frac{PR^2b}{(EA)_\theta} \left( 1 - \frac{F_p}{P_b} - \nu_{y\theta} \frac{E_{\theta}}{E_y} \frac{1}{R} \frac{F_{axial}}{p} \right) \]  

(3.3.6)

The axial force calculated from the radial membrane sidewall is substituted in order to obtain the final expression for the radial deflection due to inflation.

\[ u_{r0} = \frac{PR^2b}{(EA)_\theta} \left( 1 - \frac{F_p}{P_b} - \nu_{y\theta} \frac{E_{\theta}}{E_y} \frac{1}{R} \left( \frac{R^2 - R_x^2}{2R} \sin \phi \right) \right) \]  

(3.3.7)

3.4 Determination of Belt Properties

A value of \( EA = 2.70 \times 10^5 \) N for the tire in this study was provided by Dr. Timothy Rhyne. It was calculated using the results shown in McGinty, et. al., 2008. The values of \( EI \) and the properties required for the Poisson effect \( (E_{\theta}, E_y, \nu_{y\theta}) \) were calculated using the method from Walter 1978, as shown below. McGinty, et. al., 2008 had similar results. Table 3.4.1 shows the belt package properties used to calculate the belt properties for the validation tire.

<table>
<thead>
<tr>
<th>Belt Angle (from circumferential) (( \alpha ))</th>
<th>30 deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cord Radius</td>
<td>0.25 mm</td>
</tr>
<tr>
<td>Cord Density</td>
<td>500 cords/m</td>
</tr>
<tr>
<td>Total thickness of the belts</td>
<td>5 mm</td>
</tr>
<tr>
<td>Cord modulus (( E_c ))</td>
<td>( 1.1 \times 10^{11} ) Pa</td>
</tr>
<tr>
<td>Rubber Modulus in Belt Package (( G_r ))</td>
<td>4.0 MPa</td>
</tr>
<tr>
<td>Volume Fraction of Cord (( v_c ))</td>
<td>5%</td>
</tr>
</tbody>
</table>

Table 3.4.1: Belt package characteristics for the validation tire

In Equations 3.4.1 through 3.4.3, the subscript \( r \) denotes a property of the rubber, and \( c \) a property of the cords. \( \theta \) is the circumferential direction, and \( y \) is the axial direction. The volume fraction of the metal cords in the belts is \( v_c \). An expression for \( E_{\theta} \)
can be obtained by substituting \( \left( \frac{\pi}{2} - \alpha \right) \) for \( \alpha \) in Equation 3.4.1, and an expression for \( \nu_{y\theta} \) be obtained by substituting \( \left( \frac{\pi}{2} - \alpha \right) \) for \( \alpha \) in Equation 3.4.2

\[
E_y = E_c \nu_c \cos^2 \left( \frac{\pi}{2} - \alpha \right) + 4G_r (1 - \nu_c)
- \left[ E_c \nu_c \sin^2 \left( \frac{\pi}{2} - \alpha \right) \cos^2 \left( \frac{\pi}{2} - \alpha \right) + 2G_r (1 - \nu_c) \right]^{1/2}
\]

\[
\nu_{y\theta} = \frac{E_c \nu_c \sin^2 \left( \frac{\pi}{2} - \alpha \right) \cos^2 \left( \frac{\pi}{2} - \alpha \right) + 2G_r (1 - \nu_c)}{E_c \nu_c \sin^2 \left( \frac{\pi}{2} - \alpha \right) \cos^2 \left( \frac{\pi}{2} - \alpha \right) + 2G_r (1 - \nu_c)}
\]

\[
(El)_\theta = \frac{(E_h h^3 / 12)}{1 - \nu_{y\theta} \nu_{\theta y}}
\]

An initial value for \( GA \) was calculated by taking a shear modulus of rubber of 4.0 MPa, with the belt width of 182.9 mm and height of 5 mm. This results in a value of \( GA \) of 3658 N, which was used in the initial results shown in Chapter 5. However, it is recognized that this simple calculation is not sufficient to characterize the belt composite. In reality the layer of rubber between the metal plies is smaller. For a belt package with two plies of cords of 0.5 mm diameter and a total height of 5 mm, 4 mm of the height of the belt package (80%) is dominated by rubber. This includes the rubber between the cords, as well as rubber above and below the cords. Thus, an upper estimate of \( GA \) would be 4500 N. As will be shown in Chapter 6, the results are not sensitive to such an increase in \( GA \). It was noted that \( EI \) above is very sensitive to the input belt angle, so a representative value of 0.1 Nm\(^2\) was used for this analysis. This value was obtained by
using Equation 3.4.5, and the relationship $EI = (EA/12)h^2$. Table 3.4.2 shows the resulting properties which are inputs into the tire model.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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</tr>
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<tr>
<td>$EI$</td>
<td>0.1 Nm²</td>
<td>$\nu_{y\theta}$</td>
<td>0.334</td>
</tr>
<tr>
<td>$EA$</td>
<td>$2.7 \times 10^5$ N</td>
<td>$\nu_{0y}$</td>
<td>2.875</td>
</tr>
<tr>
<td>$E_\theta / E_y$</td>
<td>11</td>
<td>$(\nu_{y\theta})(\nu_{0y})$</td>
<td>0.96</td>
</tr>
<tr>
<td>$GA$</td>
<td>3600 N</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4.2: Effective belt properties
4.1 Differential Equations in Three Regions

This section follows Chapter 2 of Lewis, 2012, which is based on Gasmi, 2011.

The governing differential equations for the tire model are based on a Timoshenko beam model of the belts. The cross-section of the belt beam is allowed to rotate and is assumed to remain straight, meaning that the circumferential deformation is linear through the thickness of the belts. Also, it is assumed that radial displacement depends only on circumferential location, meaning that it does not vary through the thickness of the belts. Following these assumptions leads to the following equations for stress and strain in the circular beam.

\[
\varepsilon_{rr} = 0 \quad (4.1.1a)
\]

\[
\varepsilon_{\theta\theta} = \frac{1}{R+z} \left( \frac{du_\theta}{d\theta} + u_r + z \frac{d\phi}{d\theta} \right) \quad (4.1.1b)
\]

\[
\gamma_{\theta r} = \frac{1}{R+z} \left( \frac{du_r}{d\theta} - u_\theta + R\phi \right) \quad (4.1.1c)
\]

\[
\sigma_{\theta\theta} = \frac{E}{R+z} \left( \frac{du_\theta}{d\theta} + u_r + z \frac{d\phi}{d\theta} \right) \quad (4.1.1a)
\]

\[
\tau_{\theta r} = \frac{G}{R+z} \left( \frac{du_r}{d\theta} - u_\theta + R\phi \right) \quad (4.1.1b)
\]

These expressions are used to obtain expressions for the virtual strain energy and virtual potential energy of the ring. The total virtual work, the sum of the virtual strain energy and virtual potential energy, is zero. Simplifying the virtual work equation provides the following differential equations, with the boundary conditions given in Equations 4.1.3.
In order to find solutions for these differential equations, they can be decoupled into three circumferential regions of the tire; the tension region, the compression region, and the contact region. The boundary between the tension and compression regions is the angle at which the radial deformation is equal to a chosen level of radial deformation, as explained below ($\theta_S$). The boundary between the compression and contact regions is the angle at which the tire begins to contact the ground ($\theta_L$).

As calculated in Chapter 3, the radial load applied to the belt package by the sidewall varies with radial deflection. This curve can be characterized by three stiffnesses, one for each of the three regions of the tire. The boundary point between $K_r^T$ and $K_r^C$ is $u_r(\theta_S)$, and the boundary point between $K_r^C$ and $K_r^G$ is $u_r(\theta_L)$. The end points used for the fits of $K_r^G$ and $K_r^T$ are the maximum negative radial deflection (which occurs at $\theta = 0$ degrees), and maximum positive radial deflection (which occurs at $\theta = 180$ degrees).
degrees), respectively. These values are determined iteratively during the solution process.

Thus, the following piecewise function is used to represent $q_r$.

$$-b q_r = \begin{cases} 
K^T_r (u_r - u^l_{r0}) + F_p, & u_r \geq u^l_{r0} \\
K^C_r (u_r - u^l_{r0}) + F_p, & u^l_{r0} \leq u_r \leq u^l_{r0} \\
K^C_r (u_r - u^l_{r0}) + F_p - K^C_r (u^l_{r0} - u^l_{r0}), & u_r \leq u^l_{r0}
\end{cases} \tag{4.1.4}$$

Figure 4.1.1: Piecewise function representing $q_r$
Figure 4.1.2: An example of how the three $K_r$ stiffnesses represent the normalized radial load of the sidewall on the belts. In this graph the $K_r$ stiffness values have been normalized by pressure and the width of the belts.

In Lewis’ model, $u^I_{r0}$ was forced to occur at $(u_r - u_{r0} = 0)$, which is the radial deflection of the inflated, unloaded tire. In the current model $u^I_{r0}$ can be chosen to be a value not equal to the inflated deflection. This means that in the current model, $\theta_S$ does not necessarily correspond to $u_{r0}$. Choosing $u^I_{r0} < 0$ allows for a better fit of the $q_r$ versus deflection curve, which should more accurately represent actual tire behavior. The chosen value of $u^I_{r0}$ is denoted as $u^*_r$.

The piecewise expressions for $q_r$ can be substituted into the three differential equations above. The superscripts for $K^T_r$, $K^C_r$, and $K^G_r$ have been dropped for simplicity.
These equations are the same for each region of the tire; only the value of $K_r$ changes.

The boundary conditions are given in Equation 4.1.5.

\[
EA \frac{d^2 u_{\theta 0}}{d \theta^2} - (GA + R^2 K_0) u_{\theta 0} + (EA + GA) \frac{du_r}{d \theta} + RGA \phi = -R^2 b q^*_\theta 
\]

\[
-GA \frac{d^2 u_r}{d \theta^2} + (EA + R^2 K_r) u_r + (EA + GA) \frac{du_{\theta 0}}{d \theta} - RGA \frac{d \phi}{d \theta} = R^2 b q^*_r 
\]

\[
EI \frac{d^2 \phi}{d \theta^2} - R^2 GA \phi - RGA \frac{du_r}{d \theta} + RGA u_{\theta 0} = 0
\]

These equations can be manipulated to obtain three decoupled differential equations.

\[
\frac{d^6 u_r}{d \theta^6} + d_4 \frac{d^4 u_r}{d \theta^4} + d_2 \frac{d^2 u_r}{d \theta^2} + d_0 u_r = f 
\]

\[
\frac{du_{\theta 0}}{d \theta} = -Q_1 \left( \frac{d^4 u_r}{d \theta^4} + \frac{d^2 u_r}{d \theta^2} \right) - Q_2 EA u_r - Q_2 R^2 b \left( \frac{K_r u_r}{b} - q^*_r \right) 
\]

\[
+ Q_1 R^2 b \frac{d^2}{d \theta^2} \left( \frac{K_r u_r}{b} - q^*_r \right) - Q_1 \frac{R^2 b (EA + GA) d \dot{q}_\theta}{EA GA} 
\]

\[
\phi = \frac{1}{R} \left( u_{\theta 0} - \left( 1 + \frac{EA}{GA} \frac{du_r}{d \theta} \right) - \frac{EA}{R GA} \frac{d^2 u_{\theta 0}}{d \theta^2} - \frac{R b}{GA} \left( q^*_\theta - \frac{K_\theta u_{\theta 0}}{b} \right) \right) 
\]

Terms in the decoupled differential equations are given in Equations 4.1.7.

\[
d_0 = - \left( 1 + \frac{R^2 K_r}{EA} \right) \frac{R^4 K_\theta}{EI} 
\]

\[
d_2 = - \left( 1 + \frac{R^2 K_\theta}{GA} \right) \left( 1 + \frac{R^2 K_r}{EA} \right) + \frac{R^4 K_r}{EI} 
\]

\[
d_4 = 2 - R^2 \left( \frac{K_r}{GA} + \frac{K_\theta}{EA} \right) 
\]
In Lewis, 2012, the solution to these equations is found by first considering the inflated, unloaded tire. This state is important because the sidewall stiffness values are defined relative to this condition \( (u_r, u_\theta) \). The tension and compression regions are considered next, followed by the contact region. The approximate method for determining contact pressure described by Gasmi, 2011, is used in the contact region. The final expressions for \( u_r, u_\theta, \varphi, N, V, \) and \( M \) are given on the following pages.

\[
f = -\frac{R^2 b}{G A} \frac{d^4 q_\theta^*}{d\theta^4} + R^2 b \left( \frac{R^2 E I K_\theta + E I G A + R^2 E A G A}{E A E I G A} \right) \frac{d^2 q_r^*}{d\theta^2} - \frac{R^6 b K_\theta q_r^*}{E A E I}
\]

\[
Q_1 = \frac{E A G A E I}{E A (\frac{E A I A + G A E I}{E A} + R^2 E A G A) - (E A + G A) R^2 E I K_\theta}
\]

\[
Q_2 = \frac{E A E I + G A E I + R^2 E A G A}{E A (\frac{E A I A + G A E I}{E A} + R^2 E A G A) - (E A + G A) R^2 E I K_\theta}
\]
Tension region expressions are shown in Equations 4.1.8.

\[ u^T_r(\theta) = u_{r_0} + c_2 \cos(A_2^T \theta) + c_3 \sin(A_2^T \theta) \]
\[ + c_4 \cos(\theta(A_3^T + A_4^T)) + c_5 \sin(\theta(A_3^T + A_4^T)) \]
\[ + c_6 \cos(\theta(A_3^T - A_4^T)) + c_7 \sin(\theta(A_3^T - A_4^T)) \] (4.1.8a)

\[ u^T_\theta(\theta) = B_2^T (c_2 \sin(A_2^T \theta) - c_3 \cos(A_2^T \theta)) \]
\[ + B_3^T (c_4 \sin(\theta(A_3^T + A_4^T)) - c_5 \cos(\theta(A_3^T + A_4^T))) \]
\[ + B_4^T (c_6 \sin(\theta(A_3^T - A_4^T)) - c_7 \cos(\theta(A_3^T - A_4^T))) \] (4.1.8b)

\[ \phi^T(\theta) = C_2^T (c_2 \sin(A_2^T \theta) + c_3 \cos(A_2^T \theta)) \]
\[ + C_3^T (c_4 \sin(\theta(A_3^T + A_4^T)) + c_5 \cos(\theta(A_3^T + A_4^T))) \]
\[ + C_4^T (c_6 \sin(\theta(A_3^T - A_4^T)) + c_7 \cos(\theta(A_3^T - A_4^T))) \] (4.1.8c)

\[ V^T(\theta) = -\frac{GA}{R} \left( (A_2^T + B_2^T - RC_2^T)(c_2 \sin(A_2^T \theta) - c_3 \cos(A_2^T \theta)) \right. \]
\[ + \left. (A_3^T + A_4^T + B_3^T - RC_3^T)(c_4 \sin(\theta(A_3^T + A_4^T)) - c_5 \cos(\theta(A_3^T + A_4^T))) \right. \]
\[ + \left. (A_3^T - A_4^T + B_4^T - RC_4^T)(c_6 \sin(\theta(A_3^T - A_4^T)) - c_7 \cos(\theta(A_3^T - A_4^T))) \right) \] (4.1.8d)

\[ N^T(\theta) = \frac{EA}{R} \left( u_{r_0} + (1 + A_2^T B_2^T)(c_2 \cos(A_2^T \theta) + c_3 \sin(A_2^T \theta)) \right. \]
\[ + \left. (1 + (A_3^T + A_4^T)B_3^T)(c_4 \cos(\theta(A_3^T + A_4^T)) + c_5 \sin(\theta(A_3^T + A_4^T))) \right. \]
\[ + \left. (1 + (A_3^T - A_4^T)B_4^T)(c_6 \cos(\theta(A_3^T - A_4^T)) + c_7 \sin(\theta(A_3^T - A_4^T))) \right) \]
\[ M^T(\theta) = \frac{EI}{R} \left( C_2^T A_2^T (c_2 \cos(A_2^T \theta) + c_3 \sin(A_2^T \theta)) \right. \]
\[ + C_3^T (A_3^T + A_4^T) (c_4 \cos(\theta(A_3^T + A_4^T)) + c_5 \sin(\theta(A_3^T + A_4^T))) \]
\[ + C_4^T (A_3^T - A_4^T) (c_6 \cos(\theta(A_3^T - A_4^T)) + c_7 \sin(\theta(A_3^T - A_4^T))) \left. \right) \]

(4.1.8e)

Compression region expressions are shown in Equation 4.1.9.

\[ u_r^C(\theta) = u_{r0} + c_9 \cos(A_2^C \theta) + c_{10} \sin(A_2^C \theta) \]
\[ + c_{11} \cos(\theta(A_3^C + A_4^C)) + c_{12} \sin(\theta(A_3^C + A_4^C)) \]
\[ + c_{13} \cos(\theta(A_3^C - A_4^C)) + c_{14} \sin(\theta(A_3^C - A_4^C)) \]  

(4.1.9a)

\[ u_{\theta}^C(\theta) = B_2^C (c_9 \sin(A_2^C \theta) - c_{10} \cos(A_2^C \theta)) \]
\[ + B_3^C \left( c_{11} \sin(\theta(A_3^C + A_4^C)) - c_{12} \cos(\theta(A_3^C + A_4^C)) \right) \]
\[ + B_4^C \left( c_{13} \sin(\theta(A_3^C - A_4^C)) - c_{14} \cos(\theta(A_3^C - A_4^C)) \right) \]

(4.1.9b)

\[ \phi^C(\theta) = C_2^C (c_9 \sin(A_2^C \theta) + c_{10} \cos(A_2^C \theta)) \]
\[ + C_3^C \left( c_{11} \sin(\theta(A_3^C + A_4^C)) + c_{12} \cos(\theta(A_3^C + A_4^C)) \right) \]
\[ + C_4^C \left( c_{13} \sin(\theta(A_3^C - A_4^C)) + c_{14} \cos(\theta(A_3^C - A_4^C)) \right) \]

(4.1.9c)
\[
V^c(\theta) = -\frac{GA}{R} \left( (A_2^c + B_2^c - RC_2^c)(c_9 \sin(A_2^c \theta) - c_{10} \cos(A_2^c \theta)) + (A_3^c + A_4^c + B_3^c - RC_3^c) \left( c_{11} \sin(\theta(A_3^c + A_4^c)) - c_{12} \cos(\theta(A_3^c + A_4^c)) \right) + (A_3^c - A_4^c + B_4^c - RC_4^c) \left( c_{13} \sin(\theta(A_3^c - A_4^c)) - c_{14} \cos(\theta(A_3^c - A_4^c)) \right) \right)
\]

\[
N^c(\theta) = \frac{EA}{R} \left( u_{r0} + (1 + A_2^c B_2^c)(c_9 \cos(A_2^c \theta) + c_{10} \sin(A_2^c \theta)) + (1 + (A_3^c + A_4^c) B_3^c) \left( c_{11} \cos(\theta(A_3^c + A_4^c)) + c_{12} \sin(\theta(A_3^c + A_4^c)) \right) + (1 + (A_3^c - A_4^c) B_4^c) \left( c_{13} \cos(\theta(A_3^c - A_4^c)) + c_{14} \sin(\theta(A_3^c - A_4^c)) \right) \right)
\]

\[
M^c(\theta) = \frac{EI}{R} \left( C_2^c A_2^c (c_9 \cos(A_2^c \theta) + c_{10} \sin(A_2^c \theta)) + C_3^c (A_3^c + A_4^c) \left( c_{11} \cos(\theta(A_3^c + A_4^c)) + c_{12} \sin(\theta(A_3^c + A_4^c)) \right) + C_4^c (A_3^c - A_4^c) \left( c_{13} \cos(\theta(A_3^c - A_4^c)) + c_{14} \sin(\theta(A_3^c - A_4^c)) \right) \right)
\]

Contact region expressions are shown in Equation 4.1.10.

\[
u_r^c(\theta) = u_{r0} + c_{16} \cos(A_2^c \theta) + c_{17} \sin(A_2^c \theta) + c_{18} \cos(\theta(A_3^c + A_4^c)) + c_{19} \sin(\theta(A_3^c + A_4^c)) + c_{20} \cos(\theta(A_3^c - A_4^c)) + c_{21} \sin(\theta(A_3^c - A_4^c)) + q_1 (D_{11} \cos(2\theta) + D_{21}) + \sum_{n=1}^{m} q_n (D_{1n} \cos(\theta(an + 1)) + D_{2n} \cos(\theta(an - 1)))
\]
\[ u^C_\theta(\theta) = B^C_2 (c_9 \sin(A^C_2 \theta) - c_{10} \cos(A^C_2 \theta)) \\
\quad + B^C_3 \left(c_{11} \sin(\theta(A^C_3 + A^C_4)) - c_{12} \cos(\theta(A^C_3 + A^C_4))\right) \\
\quad + B^C_4 \left(c_{13} \sin(\theta(A^C_3 - A^C_4)) - c_{14} \cos(\theta(A^C_3 - A^C_4))\right) \\
\quad + q_1 (E_{11} \sin(2\theta) + E_{21} \theta) \\
\quad + \sum_{n=1}^m q_n (E_{1n} \sin(\theta(an + 1)) + E_{2n} \sin(\theta(an - 1))) \tag{4.1.10b} \]

\[ \phi^G(\theta) = C^C_2 (c_9 \sin(A^C_2 \theta) + c_{10} \cos(A^C_2 \theta)) \\
\quad + C^C_3 \left(c_{11} \sin(\theta(A^C_3 + A^C_4)) + c_{12} \cos(\theta(A^C_3 + A^C_4))\right) \\
\quad + C^C_4 \left(c_{13} \sin(\theta(A^C_3 - A^C_4)) + c_{14} \cos(\theta(A^C_3 - A^C_4))\right) \\
\quad + q_1 (F_{11} \sin(2\theta) + F_{21} \theta) \\
\quad + \sum_{n=1}^m q_n (F_{1n} \sin(\theta(an + 1)) + F_{2n} \sin(\theta(an - 1))) \tag{4.1.10c} \]
$$V^G(\theta) = -\frac{GA}{R} \left( (A_2^C + B_2^C - RC_2^C)(c_9 \sin(A_2^C\theta) - c_{10} \cos(A_2^C\theta)) 
+ (A_3^C + A_4^C + B_3^C - RC_3^C)(c_{11} \sin(\theta(A_3^C + A_4^C)) 
- c_{12} \cos(\theta(A_3^C + A_4^C))) 
+ (A_3^C - A_4^C + B_4^C - RC_4^C)(c_{13} \sin(\theta(A_3^C - A_4^C)) 
- c_{14} \cos(\theta(A_3^C - A_4^C))) 
+ q_1((2D_{11} + E_{11} - RF_{11}) \sin(2\theta) + (E_{21} - RF_{21})\theta) 
+ \sum_{n=0}^{m} q_n \left( (D_{1n}(an + 1) + E_{1n} - RF_{1n}) \sin(\theta(an + 1)) 
+ (D_{2n}(an - 1) + E_{2n} - RF_{2n}) \sin(\theta(an - 1)) \right) \right)$$

(4.1.10d)
\[ N^G(\theta) = \frac{EA}{R} \left( u_{r0} + (1 + A_2^G B_2^G)(c_9 \cos(A_2^G \theta) + c_{10} \sin(A_2^G \theta)) \right. \]
\[ \quad + (1 + (A_3^G + A_4^G) B_3^G) \left( c_{11} \cos\left(\theta(A_3^G + A_4^G)\right) + c_{12} \sin\left(\theta(A_3^G + A_4^G)\right) \right) \]
\[ \quad + (1 + (A_3^G - A_4^G) B_4^G) \left( c_{13} \cos\left(\theta(A_3^G - A_4^G)\right) + c_{14} \sin\left(\theta(A_3^G - A_4^G)\right) \right) \]
\[ \quad + q_1 \left( (D_{11} + 2E_{11}) \cos(2\theta) + D_{21} + E_{21} \right) \]
\[ \quad + \sum_{n=0}^{m} q_n \left( (D_{1n} + E_{1n}(an + 1)) \cos(\theta(an + 1)) \right) \]
\[ \quad + \left( D_{2n} + E_{2n}(an - 1) \cos(\theta(an - 1)) \right) \right) \]
\[ (4.1.10e) \]

\[ M^G(\theta) = \frac{EI}{R} \left( C_2^G A_2^G(c_9 \cos(A_2^G \theta) + c_{10} \sin(A_2^G \theta)) \right. \]
\[ \quad + C_3^G (A_3^G + A_4^G) \left( c_{11} \cos\left(\theta(A_3^G + A_4^G)\right) + c_{12} \sin\left(\theta(A_3^G + A_4^G)\right) \right) \]
\[ \quad + C_4^G (A_3^G - A_4^G) \left( c_{13} \cos\left(\theta(A_3^G - A_4^G)\right) + c_{14} \sin\left(\theta(A_3^G - A_4^G)\right) \right) \]
\[ \quad + q_1(2F_{11} \cos(2\theta) + F_{21}) \]
\[ \quad + \sum_{n=0}^{m} q_n \left( F_{1n}(an + 1) \cos(\theta(an + 1)) \right) \]
\[ \quad + F_{1n}(an - 1) \cos(\theta(an - 1)) \right) \]
Below are the coefficients used in the preceding equations. The $A$, $B$, and $C$ expressions are written below for the tension region. They are the same in the other regions, with the exception of the superscripts ($T$, $C$, $G$).

$$A^T_2 = \sqrt{\frac{12d_2^T-(2d_4^T-r_1^T)^2-2r_4^T d_4^T}{6r_1^T}}$$  \hspace{1cm} (4.1.11a)$$

$$A^T_3 = \frac{1}{24r_1^T} \left( \frac{(2d_4^T-r_1^T)^2-12d_2^T-(12d_2^T-(2d_4^T-r_1^T)^2)^2+3((r_1^T)^2-4(d_4^T)^2+12d_2^T)^2}{\sqrt{(12d_2^T-(2d_4^T-r_1^T)^2)^2+3((r_1^T)^2-4(d_4^T)^2+12d_2^T)^2}} \right)$$  \hspace{1cm} (4.1.11b)$$

$$A^T_3 = \frac{1}{24r_1^T} \left( \frac{(2d_4^T-r_1^T)^2-12d_2^T+(12d_2^T-(2d_4^T-r_1^T)^2)^2+3((r_1^T)^2-4(d_4^T)^2+12d_2^T)^2}{\sqrt{(12d_2^T-(2d_4^T-r_1^T)^2)^2+3((r_1^T)^2-4(d_4^T)^2+12d_2^T)^2}} \right)$$  \hspace{1cm} (4.1.11c)$$

$$B^T_2 = -\frac{(A^T_2)^2 Q_1 \left( \left( (A^T_2)^2 - 1 \right) GA + R^2 K_2^T \right) + GAQ_2 (EA + R^2 K_2^T)}{GA A_2^T}$$  \hspace{1cm} (4.1.12a)$$

$$B^T_3 = -\frac{(A^T_2+A^T_4)^2 Q_1 \left( \left( (A^T_2+A^T_4)^2 - 1 \right) GA + R^2 K_2^T \right) + GAQ_2 (EA + R^2 K_2^T)}{GA (A^T_2+A^T_4)}$$  \hspace{1cm} (4.1.12b)$$

$$B^T_3 = -\frac{(A^T_2-A^T_4)^2 Q_1 \left( \left( (A^T_2-A^T_4)^2 - 1 \right) GA + R^2 K_2^T \right) + GAQ_2 (EA + R^2 K_2^T)}{GA (A^T_2-A^T_4)}$$  \hspace{1cm} (4.1.12c)$$

$$C_1 = GA + R^2 K_2^T$$  \hspace{1cm} (4.1.13a)$$

$$C^T_2 = \frac{A_2^T (EA+GA)+B_2^T \left((A^T_2)^2 EA + GA + R^2 K_2^T \right)}{RGA}$$  \hspace{1cm} (4.1.13b)$$

$$C^T_2 = \frac{(A^T_2+A^T_4)^2 (EA+GA)+B_2^T \left((A^T_2+A^T_4)^2 EA + GA + R^2 K_2^T \right)}{RGA}$$  \hspace{1cm} (4.1.13c)$$

$$C^T_2 = \frac{(A^T_2-A^T_4)^2 (EA+GA)+B_2^T \left((A^T_2-A^T_4)^2 EA + GA + R^2 K_2^T \right)}{RGA}$$  \hspace{1cm} (4.1.13d)$$

$$D_{1n} = \frac{(EAEI(n+1)^2 + EK GA (n+1) + R^2 GA EA) (n+1) (n+2) + R^2 K_2^T D_+}{GA E A E (n+1)^2 + 2 R^2 \left(EI (n+1)^2 (K_2 (n+1)^2 + K_2^T) + D_+ (K_2 (R^2 K_2^T + EA) + EAK_2^T (n+1)^2)) \right)^{2}}$$  \hspace{1cm} (4.1.14a)$$

$$D_+ = (R^2 GA + EI(n+1)^2)$$  \hspace{1cm} (4.1.14b)
The known symmetry of the solutions for radial displacement in the tension and contact regions allows for the elimination of three asymmetric integration constants. The quantities \( u_r, u_\theta, \varphi, N, V, \) and \( M \) must be continuous across the regions of the tire; for example, \( u_r^T(\theta_a) = u_r^C(\theta_a) \). This continuity requirement yields 12 continuity equations (six quantities, each continuous across two boundaries), which are used to determine the

\[
D_{2n} = \frac{(EAE(n-1) + R^n G(n+1)+ R^n G(n+1) + R^n G(n+1)) (n-1) (n+1) + R^n K_\theta D_n}{GAE(1+2)(n-1)^2 + R^n (EIGA(n-1) + K_\theta (n-1) + K_\theta^C)} \frac{R^n b}{2}
\]  
(4.1.14c)

\[
D_- = (R^2 G + E(n-1)^2)
\]  
(4.1.14d)

\[
E_{1n,a} = \frac{(EAE(n+2)(n+2) + EAE(n+2) + R^n K_\theta (EAE(n+2) + EAE(n+2) + GAE))}{E(n+2) + E(n+2) + E(n+2) + E(n+2)} \frac{D_{1n}}{(n+1)}
\]  
(4.1.15a)

\[
E_{1n,b} = \frac{E(n+2) + E(n+2) + E(n+2) + E(n+2)}{E(n+2) + E(n+2) + E(n+2) + E(n+2)} \frac{R^n b}{2(n+1)}
\]  
(4.1.15b)

\[
E_{1n} = E_{1n,a} + E_{1n,b}
\]  
(4.1.15c)

\[
E_{2n,a} = \frac{(E(n-2)(n-2) + E(n-2) + E(n-2) + E(n-2))}{E(n-2) + E(n-2) + E(n-2) + E(n-2)} \frac{D_{2n}}{(n-1)}
\]  
(4.1.15d)

\[
E_{2n,b} = \frac{E(n-2) + E(n-2) + E(n-2) + E(n-2)}{E(n-2) + E(n-2) + E(n-2) + E(n-2)} \frac{R^n b}{2(n-1)}
\]  
(4.1.15e)

\[
E_{2n} = E_{2n,a} + E_{2n,b}
\]  
(4.1.15f)

\[
F_{in} = \frac{1}{RGA} \left( (n+1)(E+A)D_{1n} + (E+A(n+1)^2 + R^n K_\theta)E_{in} + \frac{R^n b}{2} \right)
\]  
(4.1.16a)

\[
F_{in} = \frac{1}{RGA} \left( (n-1)(E+A)D_{1n} + (E+A(n-1)^2 + R^n K_\theta)E_{in} + \frac{R^n b}{2} \right)
\]  
(4.1.16b)
12 integration constants (c coefficients). A Taylor expansion of the relationship between vertical deflection, the radius of the tire, and the contact angle is used to calculate equations to determine the coefficients for contact pressure. The number of terms in the expansion is chosen to provide the appropriate number of equations for the number of terms chosen in the pressure approximation. Matching the coefficients of terms with the same order of theta in the Taylor expansion and in the contact pressure approximation provides the equations needed to solve for the contact pressure coefficients.

These equations are assembled into the matrices in Equation 4.1.17, from Lewis, 2012. A solution to the system, which provides the coefficients listed in array \( \{cq\} \), allows for the evaluation of \( u_r, u_\theta, \phi, N, V, \) and \( M \) at any angle \( \theta \) around the tire, and ground pressure in the contact region.
\[
\begin{align*}
\text{Integration Constants for Each Region} & \quad \text{Pressure Coefficients} \\
\text{Tension} & \quad \text{Compression} & \quad \text{Contact} & \quad \delta & \quad \theta \leq \theta_1 \\
\begin{pmatrix}
G_{1,1} \\
G_{1,2}
\end{pmatrix}
& \begin{pmatrix}
G_{1,1} \\
G_{1,2}
\end{pmatrix} & \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix}
\end{align*}
\]

Continuity Equations for \( \theta_5 \)

\[
\begin{pmatrix}
G_{2,1} \\
G_{2,2}
\end{pmatrix} & \begin{pmatrix}
G_{2,1} \\
G_{2,2}
\end{pmatrix} & \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix}
\end{pmatrix}
\]

Continuity Equations for \( \theta_4 \)

\[
\begin{pmatrix}
G_{3,1} \\
G_{3,2}
\end{pmatrix} & \begin{pmatrix}
G_{3,1} \\
G_{3,2}
\end{pmatrix} & \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix} & \begin{pmatrix}
0 \\
0
\end{pmatrix}
\end{pmatrix}
\]

Coefficients on \( \theta_5 \ldots \theta_n \) from Contact Condition

\[
\{H\} = \\
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad \{cq\} = \\
\begin{pmatrix}
c_2 \\
c_4 \\
c_6 \\
c_8 \\
c_{10} \\
c_{11} \\
c_{12} \\
c_{13} \\
c_{14} \\
c_{15} \\
c_{16} \\
c_{18} \\
c_{20} \\
\delta \\
q_0 \\
q_1 \\
q_2 \\
q_3 \\
q_4 \cdots \\
q_{m-1} \\
q_m
\end{pmatrix} \quad \Rightarrow \quad \{cq\} = [G]^{-1} \{H\}
\]

(4.1.17)
4.2 Iterative Procedure for Stiffness Fitting
The values of sidewall stiffness $K_r^T$ and $K_r^C$ can vary significantly depending on the end points chosen for the fitting. $K_r^C$ also varies depending on the level of radial deflection at the edge of contact. In order to obtain the most relevant values for these stiffnesses, the system of equations is solved twice for each condition simulated. The first iteration uses values of $K_r^T$ and $K_r^C$ determined by using estimates for the maximum positive deflection and the deflection at the edge of contact; $K_r^G$ is set to zero for the first iteration. The results of this first iteration provide the radial deflection at the edge of contact, as well as the maximum negative deflection in contact, and the maximum positive deflection at the top of the tire. These values are used to fit $K_r^T$, $K_r^C$, and $K_r^G$ from the $q_r$ curve. Experiments showed that in further iterations, the key levels of radial deflection (maximum negative, maximum positive, and edge of contact) changed less than 1 mm, so one iteration is sufficient to provide relevant values of the stiffness parameters.

4.3 Superposition Method of Solution
Using this linear solution, the normal force in the belts at the edge of contact can cause large errors. Lewis, 2012, describes a method to avoid this problem using the principle of superposition. The desired inflation pressure of the tire is denoted as $P$. The linear solution for the loaded tire is calculated for an inflation pressure $P^*$, at which the normal force in the belts at the edge of contact is zero. The final solution is the addition of this loaded solution at $P^*$ and the inflation-only solution at an inflation pressure of $(P-P^*)$. As part of the addition of the solutions, the normal force $(P-P^*)bL$ is added to the
normal force calculated for $P^*$ (where $b$ is the width of the tire, and $L$ is the contact length calculated at $P^*$). In the contact solution for the pressure $P^*$, the stiffness parameters associated with the full pressure, $P$, are used. In the superposed problem with pressure $P - P^*$, the structure is assumed to be rigid, (since all deformation has already been taken into account). This procedure not only addresses the error in vertical force equilibrium associated with large deformation, it also more correctly accounts for the contribution of the inflation pressure to the contact pressure. Associated with this procedure for all pressures, $P > 0$, is one specific of displacement for which $P = P^*$ and superposition in fact does not have to be used.

4.3.1 Inflation with the Poisson Effect

Lewis’ solution did not account for the reduction in belt radius due to the lateral component of sidewall tension, caused by the Poisson ratio of the belt package. As calculated in Chapter 3, the sidewall exerts an axial load on the belt package, causing the belts to stretch in the axial direction, and contract in the circumferential direction. The axial load is nearly constant with respect to radial deflection (Figure 4.3.1), so the radial contraction due to the Poisson effect is assumed to be constant. This effect is included in the inflation step of the superposition procedure.

$$u_{r0} = \frac{R_0^2 P b}{E A} \left[ 1 - \frac{P_0}{P b} - \nu \theta_y \left( \frac{E_0}{E_y} \left( \frac{1}{R_0} \right) \left( -\frac{q_y b}{P} \right) \right) \right]$$ (4.3.1)
4.3.2 Determination of $P^*$

The first step of the superposition procedure is to calculate $u_{r0}$ as shown in the previous section. The expanded radius ($R_{input} + u_{r0}$) is used as the radius ($R$) for the entries into the $\{G\}$ matrix calculated previously in this chapter.

The next step is to find $P^*$, the inflation pressure needed to achieve $N(\theta_L) \sin \theta_L = 0$. The tire deformation problem is solved (that is, the system is solved to find the coefficients in $\{cq\}$) first at a pressure of 0 bar, then at a second higher pressure. The function $N(\theta_L) \sin \theta_L$ versus inflation pressure was found to be linear, so interpolation between the first two pressures provides $P^*$. The deformation problem is solved a final time with this value of $P^*$. With this final solution, the quantities $u_r$, $u_\theta$, $\varphi$, $N$, $V$, $M$, and ground pressure due to loading at an inflation pressure of $P^*$ are known.
4.3.3 Superposition of Inflated and Loaded States

The final step of the procedure is to superpose the two sets of results. It is assumed that the additional inflation pressure \((P-P^*)\) only affects three quantities: \(N\), \(u_0\), and ground pressure \(q\). These quantities are adjusted as shown in Equations 4.3.2.

\[
N = \left(\frac{EA}{R}\right)u_{r0} + N(\theta)_{structural} \quad (4.3.2a)
\]

\[
u_r = u_{r0} + u_r(\theta)_{structural} \quad (4.3.2b)
\]

\[
q = (P - P^*) + q(\theta)_{structural} \quad (4.3.2c)
\]

Equation 4.3.2c results in an addition to the total vertical force of \((P - P^*)(Lb)\), where \(L\) is the calculated contact length.
One important part of this procedure is the implicit assumption that the “extra” inflation pressure \((P-P^*)\) does not affect the way that the tire deforms under loading. This is justified by two reasons: first, the tire has already been expanded due to the full inflation pressure \(P\); and second, the stiffness quantities for the full pressure have been used. The radial expansion is applied uniformly around the circumference of the tire. The quantities not mentioned in Equations 4.3.2 are unaffected by the superposition of the inflated state \((u_\theta, \phi, V, M)\).

### 4.4 Assumptions and Simplifications of the Model

The solution method presented in this chapter is based on several assumptions and simplifications. Some of these have been stated above, but all are presented together here for completeness. Some of these assumptions will be treated in Chapter 5 through adjustment of inputs to the model.

**Neglecting belt “crown” deformation:** the membrane theory model only considers the sidewalls, and assumes there is a discontinuity in slope between the sidewall and the belts at the edge of the belts. In reality, there are two phenomena that this assumption neglects:

- The belts are slightly curved in the inflated tire (sometimes called the crown radius), and they flatten when in contact with the ground.
- In a real tire, there is no discontinuity of slope at the edge of the belts in the inflated tire. This intuitive assertion is confirmed by study of elastic bodies in other fields, such as in Wager, 2013.
These phenomena would impact the sidewall angle at the edge of the belts (ϕ), which would change the radial component of the sidewall stiffness.

**Frictionless contact:** friction in the contact patch is not considered in the model. This friction would counteract the expansion or contraction of the ring in the circumferential direction, which would appear to make $E$ higher in the contact region. Friction also affects displacement in the axial direction which opposes the assumption of plane stress, which again would appear to make $E$ higher in the contact region. It is important to note that the normal load in the circumferential direction ($N_θ$) is most interesting in and near the contact region.

**Poisson effect:** the Poisson effect of the belts has been taken into account by assuming the axial force of the sidewalls on the belts is constant, but this is a simplification.

**Torsional stiffness:** the torsional stiffness of the sidewall has been calculated using a two-dimensional model. In reality, of course, the sidewall cords are not straight, which would tend to reduce the torsional stiffness by allowing for additional degrees of displacement freedom.

These assumptions will be addressed via adjustments to the input parameters, in Chapter 5.
CHAPTER FIVE

VALIDATION COMPARED TO EXPERIMENTAL RESULTS

The results provided by the model will be validated by comparison to experimental data provided by Dr. Timothy Rhyne and Michelin North America. The data provided are vertical deflection versus load and counterdeflection versus load. Vertical deflection (henceforth referred to as “deflection”) is the distance that the center of the tire moves downward when the tire is loaded. Counterdeflection is the increase in radius at the top of the tire under loading. Deflection ($\delta$) and counterdeflection ($\lambda$) are illustrated in Figure 5.1.

![Figure 5.1: Illustration of deflection and counterdeflection](image)

The tire used in the validation is a typical all-season tire of dimension 225/60R16. Load versus deflection and load versus counterdeflection are provided for five levels of inflation pressure (0, 0.5, 1, 2, and 4 bar). This data was also referenced in Rhyne, 2005.
It is noted that the force versus deflection and force versus counterdeflection curves are nearly linear for all inflation pressures measured. Also, the experimental results show that near zero load, there is locally higher counterdeflection stiffness. This has not been studied in detail, but is hypothesized to come from initial deformation of the belt package of the tire when the tire is first loaded.

Figure 5.2: Experimental results for validation
5.1 Membrane Sidewall without Superposition

As a first step in the validation of the model, the most accurate results available from the current approach were examined and compared to the experimental data. These results are limited to a pressure $P^*$ where the axial force in the circumferential direction is zero at the edge of contact, i.e., $\text{NGL} = N(\theta_L) \sin \theta_L = 0$. These results do not require superposition, since the solutions are only for $P - P^* = 0$ and $P - P^*$ is the coefficient of the superposed term. The limitation of this approach is that $P^*$ will be equal to a given pressure of interest at only one loading condition. To summarize, for a given contact angle (or a given vertical load), there is one level of inflation pressure ($P^*$) that will result in $\text{NGL} = 0$.

5.1.1 Pressure Matching for Given Contact Angle

First, the value $P^*$ was determined iteratively for contact angles ranging from 0 to 24 degrees. This encompasses the range of contact angles seen during normal tire use, and produces $P^*$ values from 0 to about 3.5 bar. In these results it is necessary to use the tire stiffness parameters that exist at the value of $P^*$. The resulting relationship between $P^*$ and contact angle is shown in Figure 5.1.1. Because contact angle is not available from the experimental data, the vertical load was used to match the model’s results to experimental.

As shown in Figure 5.2, the experimental data is only available at discrete values of pressure, but can easily be interpolated to obtain values at any pressure. To accomplish this, the deflection and counterdeflection versus load curves were first fit with third-order polynomials. Three-point Lagrange interpolation was used to calculate load and counterdeflection curves (in the form of third-order polynomials) for each value.
of $P^*$ from the model results. The values of deflection and counterdeflection at the vertical load predicted by the model were compared to the values of deflection and counterdeflection calculated by the model.

![Diagram of $P^*$ result versus contact patch angle](image)

**Figure 5.1.1: $P^*$ result versus contact patch angle**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EA$</td>
<td>$2.70 \times 10^5$</td>
<td>N</td>
</tr>
<tr>
<td>$EI$</td>
<td>0.10</td>
<td>Nm$^2$</td>
</tr>
<tr>
<td>$GA$</td>
<td>3600</td>
<td>N</td>
</tr>
<tr>
<td>$u_{r0}$</td>
<td>0</td>
<td>mm</td>
</tr>
<tr>
<td>$W$</td>
<td>28</td>
<td>mm</td>
</tr>
<tr>
<td>Poisson Term</td>
<td>3</td>
<td>--</td>
</tr>
<tr>
<td>$G_e$ (sidewall)</td>
<td>2.35</td>
<td>MPa</td>
</tr>
<tr>
<td>$t_c$ (sidewall)</td>
<td>2</td>
<td>mm</td>
</tr>
<tr>
<td>$R_o$</td>
<td>326.3</td>
<td>mm</td>
</tr>
<tr>
<td>$R_i$</td>
<td>236.3</td>
<td>mm</td>
</tr>
<tr>
<td>$b$</td>
<td>182.9</td>
<td>mm</td>
</tr>
</tbody>
</table>

**Table 5.1.1: Inputs used for validation without superposition**

This analysis showed very promising results for deflection and counterdeflection. As shown in Figure 5.1.2, the force versus deflection and force versus counterdeflection
results agree well with experimental at low deflection. The pressures for which $P=P^*$ for these low levels of deflection are low, so it is not known if the results are better due to low pressure, low deflection or both. Figure 5.1.3 shows that the contact pressure is between $P\times110\%$ and $P+0.2\text{bar}$ for low deflection, but is too low for high deflection.

Figure 5.1.4 shows the vertical pressure $q$ in the contact patch, for the $P=P^*$ match condition at $\theta_L = 3$ degrees. The black line represents the inflation pressure, which is 0.0332 bar.

This type of analysis continues in the next section.
Figure 5.1.3: Contact pressure versus deflection using membrane theory stiffnesses with $P=P^*$, with interpolated experimental results

The following figure shows the contact pressure distribution in the ground region, for two levels of vertical deflection at the $P=P^*$ condition. The solid black line represents the contact pressure. The solid red line shows the contact pressure distribution, and the dashed red line represents the average contact pressure in contact. As shown in Figure 5.1.3, the contact pressure is above inflation pressure for small deflection, which corresponds to a small inflation pressure for the $P=P^*$ condition. At higher levels of deflection, corresponding to higher inflation pressures, the contact pressure is too low.
5.1.2 Contact Angle Matching for Given Inflation Pressure

Another way to perform the validation at the point where \( NGL = 0 \) is to find the correct contact angle for each inflation pressure for which experimental data is available. This has the benefit of being more easily comparable to the experimental results.

For each experimental inflation pressure, the correct contact angle was found for which \( NGL = 0 \). This is equivalent to finding the point for each inflation pressure where \( P^* \) is equal to the inflation pressure, \( P \). The stiffness slopes (force versus deflection, force versus counterdeflection, force versus contact length) were calculated using the point \((0,0)\), and the point where \( P^* = P \). In this section, the red lines represent those slopes. It is important to note that the lines do not represent the results for levels of force between zero and the point where \( P^* = P \). However, since the displacement curves in Figure 5.2 are all very close to linear, this method allows for the prediction of a representative slope.
In all results graphs in this section, the black line with dots represents a reference result. For force versus deflection and force versus counterdeflection, the black line represents measured data; the red line should be as close as possible to these results. Measured values of contact length were not available, so for force versus contact length, the black line represents the relationship between force versus contact length if the contact pressure were equal to inflation pressure. The results (red line) should be about 10% above the black line. It is clear from these results that this latter condition is problematic.

Figure 5.1.5: Stiffness (top) and contact length (bottom) for 0.5 bar, using P=P* solution
For inflation pressures up to 2.0 bar, the slopes of deflection and counterdeflection stiffness agree well with the experimental data (neglecting the initial increased stiffness near zero). However, the force versus contact length graphs show that the contact length is too long, leading to an average contact pressure that is too low.
Figure 5.1.7: Stiffness (top) and contact length (bottom) for 2.0 bar, using P=P* solution
5.1.3 Full Deflection Sweep with No Superposition

The second step of the validation was to perform a sweep of levels of deflection of each experimental pressure without using superposition. For each level of deflection, the structural problem was solved at the inflation pressure, with no regard for the level of tension in the belts at the edge of contact. The contact pressure and force were not adjusted. These results are presented as a numerical study, not as a proposed solution.

Figure 5.1.8: Stiffness (top) and contact length (bottom) for 4.0 bar, using P=P* solution
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EA$</td>
<td>$2.70 \times 10^5$</td>
<td>N</td>
</tr>
<tr>
<td>$EI$</td>
<td>0.10</td>
<td>Nm²</td>
</tr>
<tr>
<td>$GA$</td>
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</tr>
<tr>
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<td>3</td>
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<td>236.3</td>
<td>mm</td>
</tr>
<tr>
<td>$b$</td>
<td>182.9</td>
<td>mm</td>
</tr>
</tbody>
</table>

Table 5.1.2: Inputs used for validation without superposition

The results are shown in the following three graphs. The force versus deflection curves are less linear than experimental, but the overall level of force and deflection is near experimental. The force versus counterdeflection stiffnesses are very close to experimental, especially for pressures of 2.0 bar and below.

The force versus deflection curves do not pass through (0,0) at higher pressures due to issues with the solution without superposition at very low deflection. These issues are less apparent at lower pressures because the belt tension, which is caused by inflation pressure, is lower at lower pressure. Also, at lower pressure, the inflation pressure is closer to $P^*$ (as discussed in Section 4.3), so errors caused by not applying the superposition method are smaller.
Figure 5.1.9: Force versus deflection using membrane theory stiffnesses, without superposition

Figure 5.1.10: Force versus counterdeflection using membrane theory stiffnesses, without superposition
Although the deflection and counterdeflection stiﬀnesses align well with the experimental data, the predicted contact length is too long, as shown in Figure 5.1.10. The red lines represent the predicted results, while the black lines represent the relationship between force and contact length if the contact pressure were equal to inflation pressure. For a given level of force, the predicted contact length is too long; this leads to under-prediction of the average contact pressure. It is understood that a linear solution is not expected to predict accurately the solution for large deflection.

Figure 5.1.10 indicates that the contact pressure comes closer to the expected result at higher levels of force. This occurs as the solution approaches the point where $P^* = P$, as discussed in the previous section.
5.1.4 High Deflection without Superposition

As indicated by Figure 5.1.10 in the previous section, at high levels of deflection, the contact pressure increases and approaches the expected level, which is about 10% above inflation pressure. In fact, there is a contact angle for each pressure for which the contact pressure results are as expected. This contact angle roughly corresponds to the point where $\theta_L = \theta_S$. These contact angles are presented in Table 5.1.3. At this point, the compression region is non-existent. Although it is possible to force $\theta_L = \theta_S$ at lower levels of deflection, that does not produce the expected contact pressure results.

<table>
<thead>
<tr>
<th>Inflation Pressure</th>
<th>$\theta_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 bar</td>
<td>22 deg</td>
</tr>
<tr>
<td>1.0 bar</td>
<td>30 deg</td>
</tr>
<tr>
<td>2.0 bar</td>
<td>34 deg</td>
</tr>
<tr>
<td>4.0 bar</td>
<td>39 deg</td>
</tr>
</tbody>
</table>

Table 5.1.3: Contact angles for which the contact pressure without superposition is about 10% higher than inflation pressure

As with the $P = P^*$ solution, the stiffness slopes (force versus deflection, force versus counterdeflection, force versus contact length) were calculated using the point (0,0), and the point defined by the contact angle shown in Table 5.1.3. The following figures show that these stiffness values agree well with experimental results. However, these results are presented as a numerical study, not as a proposed solution.

It is noted that the expansion due to inflation (before loading) agree well with a known rule of thumb; which is that the tire expands about 1 mm per 1 bar of inflation pressure. Additionally, these results respect the near-inextensibility of the belts in the circumferential direction. The circumference of the belts increases less than 1% from the inflated state to the loaded state.
Figure 5.1.12: Stiffness (top) and contact length (bottom) for 0.5 bar, high deflection without superposition
Figure 5.1.13: Stiffness (top) and contact length (bottom) for 1.0 bar, high deflection without superposition
Figure 5.1.14: Stiffness (top) and contact length (bottom) for 2.0 bar, high deflection without superposition
Figure 5.1.15: Stiffness (top) and contact length (bottom) for 4.0 bar, high deflection without superposition
5.1.5 Validation without Superposition Summary

Solutions without superposition were first explored at the point where $P^*=P$ (where the solutions with and without superposition are identical), then at various levels of deflection. Generally, the results without superposition for force versus deflection and force versus counterdeflection align with experimental, but the predicted contact pressure is too low for inflation pressures above 0 bar. This is likely due to an over-prediction of contact length, but this cannot be verified due to a lack of measured data for contact length. It is noted that force from the linear model cannot be expected to be accurate due to large deformation, so this alignment is likely coincidental.

The results without superposition are improved when the model is run at a high level of deflection, such that $\theta_L = \theta_S$. The results at that point are connected to (0,0) to calculate stiffnessess for force versus deflection, force versus counterdeflection, and force versus contact length. Running the simulation at this high level of vertical deflection provides good results for all three stiffness values. For inflation pressures of 0.5 bar through 4.0 bar, these stiffness values are very close to experimental (for deflection and counterdeflection) or to the expected value (contact length).

Because the force from the linear model cannot be expected to be accurate, it is likely that this alignment with experimental results is coincidental. Experimental data is only currently available for one tire. Application of the same method to another tire and comparison with experimental data would clarify the results.

For inflation pressure of 0 bar, the model without superposition attains good results for low deflection (as shown in Section 5.1.3).
5.2 Membrane Sidewall with Superposition

In this section, the superposition scheme described in Section 4.3 is applied. For each level of deflection that is simulated, the structural problem is solved at $P^*$ (determined such that $NGL = N(\theta_L) \sin \theta_L = 0$), then the contact pressure and total vertical force are corrected with $(P-P^*)$ and $(P-P^*)(Lb)$, respectively.

5.2.1 Membrane Theory without Adjustments, Zero Pressure

As a first assessment of the performance of the model, the results at zero inflation pressure are compared to the previous non-pneumatic tire model results, including the models of Gasmi and Lewis, and the finite-element simulations performed by Gasmi. The membrane theory model presented in Chapter 2 does not apply to the non-pneumatic tire, so the radial stiffness values from Gasmi and Lewis are used directly as inputs.

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</tr>
<tr>
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<td>N</td>
</tr>
<tr>
<td>$u_{r0}$</td>
<td>0</td>
<td>mm</td>
</tr>
<tr>
<td>$W$</td>
<td>N/A</td>
<td>mm</td>
</tr>
<tr>
<td>Poisson Term</td>
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<td>--</td>
</tr>
<tr>
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<td>MPa</td>
</tr>
<tr>
<td>$t_e$ (sidewall)</td>
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<td>mm</td>
</tr>
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</tr>
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</tr>
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<td>0</td>
<td>Pa</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0</td>
<td>N/m</td>
</tr>
</tbody>
</table>

Table 5.2.1: Inputs used for non-pneumatic tire validation
The following figure shows that the current model with superposition recovers the non-pneumatic tire result. These results also show that the model with superposition is more reliable than the pure linear approach, especially for large deformation. While this result is for an inflation pressure of zero, a non-zero pressure $P^*$ is used in the superposition procedure. Neither counterdeflection nor contact length is available from the finite element analysis. Further investigation of the non-pneumatic tire is presented in Section 6.2.

**Figure 5.2.1: Force versus deflection for non-pneumatic tire**

5.2.2 Membrane Theory without Adjustments, All Pressures

The first full validation of the model with superposition is applied using the sidewall stiffness values and belt properties calculated in previous chapters, with no adjustments. The results are shown in the following five figures.
<table>
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<tr>
<td>$EI$</td>
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<td>Nm²</td>
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<td>N</td>
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<td>$u_{r0}$</td>
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</tr>
<tr>
<td>$W$</td>
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<td>mm</td>
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<tr>
<td>Poisson Term</td>
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<td>--</td>
</tr>
<tr>
<td>$G_e$ (sidewall)</td>
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<td>$t_e$ (sidewall)</td>
<td>2</td>
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</tr>
<tr>
<td>$R_o$</td>
<td>326.3</td>
<td>mm</td>
</tr>
<tr>
<td>$R_i$</td>
<td>236.3</td>
<td>mm</td>
</tr>
<tr>
<td>$b$</td>
<td>182.9</td>
<td>mm</td>
</tr>
</tbody>
</table>

Table 5.2.2: Inputs used for validation with superposition

Both force versus deflection and force versus counterdeflection are much too stiff for low levels of deflection. At these low levels of deflection, the value of $P^*$ is small, which leads to a large load correction $(P-P^*)(Lb)$ being added to the structural results. As deflection increases, there is a point for each pressure where the predicted results cross the line prescribed by the experimental data; this point is close to the point where $P^*=P$, as described in Section 5.1. As deflection passes the $P^*=P$ point for each pressure, the $(P-P^*)(Lb)$ becomes negative, meaning that force is subtracted from the structural solution. This leads to the curves falling off at high deflection.
Figure 5.2.2: Force versus deflection using membrane theory stiffnesses, with superposition

Figure 5.2.3: Force versus counterdeflection using membrane theory stiffnesses, with superposition
Although the force versus deflection and counterdeflection graphs are too stiff, the contact pressure is predicted well when using superposition, for low to medium levels of deflection, as shown in Figure 5.2.4. For contact length up to about 150 mm, which is representative of typical use for this tire, the contact pressure is correctly predicted.

![Load vs. Contact Length](image)

**Figure 5.2.4: Force versus contact length using membrane theory stiffnesses, with superposition**

To summarize; the two extreme points of view are that either the method of superposition does not work or the stiffness parameters used are incorrect.

In order to better understand the behavior of these curves, the force from the structural solution only (before the superposition adjustment) can also be examined compared to the experimental results, as shown in the following two figures. These results are very similar to the results without superposition, shown in Section 5.1. The difference in these simulations is that the structural problem has been solved at $P^*$, whereas it was solved at $P$ (inflation pressure) in Section 5.1.
Figure 5.2.5: Force from structural problem (no superposition correction) versus deflection using membrane theory stiffnesses

Figure 5.2.6: Force from structural problem (no superposition correction) versus counterdeflection using membrane theory stiffnesses
5.2.3 Belt Slope Matching

As mentioned in Section 4.4, one of the biggest simplifications made in the membrane theory model is that the changing crown radius of the tire is not considered. The analysis in this section considers that deformation separately from the model; the results are then combined with the membrane theory results to define $q_r$.

A slice of the belts ($d\theta$ thick) is considered as an Euler beam of length $b$, as shown in Figure 5.2.7. $P$ is the inflation pressure, and $\alpha$ is the angle between the sidewalls and horizontal ($\phi = 90 - \alpha$). $F$ is the force per unit length due to the sidewall, and $\sigma_\theta$ is the tensile stress in the circumferential direction, given by Equation 5.2.1. The inner radius of the belts is $R_i$, and $h$ is the height of the belts.

$$\sigma_\theta = \left[ P - \frac{2F \sin \alpha}{b} \right] \frac{R_i}{h} \quad (5.2.1)$$

![Figure 5.2.7: Slice of belt beam](image1)

![Figure 5.2.8: Forces and moments acting on the beam](image2)
The forces and moments acting on the beam are shown in Figure 5.2.8. The outward radial force per unit length due to the inflation pressure is defined as \( w_p \), and the inward radial force per unit length due to \( \sigma_0 \) is defined as \( w_\theta \).

\[
\begin{align*}
w_p &= PR_id\theta & (5.2.2a) \\
w_\theta &= \sigma_0hd\theta & (5.2.2b) \\
w_p - w_\theta &= \frac{2F \sin \alpha}{b} (R_id\theta) & (5.2.2c)
\end{align*}
\]

The quantity \( d \) is introduced because the force due to the cords is applied at the inner radius and has to be transferred to the centroid of the cross section, which adds a couple to the loading. \( d \) is the distance from in the inner radius \( (R_i) \) to the centroid of the cross section. For this standard beam loading, which includes a uniform upward distributed load and an applied couple at the ends, the slope at the left edge is given by Equation 5.2.3, where \( y \) is the axial direction, and \( y=0 \) is the edge of the belt beam.

\[
\left. \frac{du_y}{dy} \right|_{y=0} = \frac{b^2}{24EI} \left[ b(w_p - w_\theta) - \frac{12F \cos \alpha R_i d\theta}{b} \right] & (5.2.3)
\]

The radial component force per unit length along the sidewall is normalized by the width in order to determine \( q_r \). Note that the force is multiplied by two to account for both sidewalls of the tire.

\[
q_r = \frac{2F \sin \alpha}{b} & (5.2.4)
\]

Substituting this into the equation for the slope of the beam at its edge yields the following equation.

\[
\left. \frac{du_y}{dy} \right|_{y=0} = \frac{q_r R_i b^2}{24EI(f/d\theta)} \left[ b - \frac{6d}{\tan \alpha} \right] & (5.2.5)
\]
In order for the slope of the belt at its edge to match the angle of the sidewalls that are pulling on the belts, Equation 5.2.6 must be satisfied.

\[ \tan \alpha = \left. \frac{du_y}{dy} \right|_{y=0} \quad (5.2.6) \]

Combining the previous two equations and solving for \( q_r \) yields Equation 5.2.7.

\[ q_r = \frac{\tan^2(\alpha)24E(\frac{l}{4b^2})}{R_1b^2(b \tan \alpha - 6d)} \quad (5.2.7) \]

This relationship between \( q_r \) and the sidewall angle must be satisfied in order to avoid a discontinuity of slope at the point where the sidewall meets the belts. This relationship is shown graphically in Figure 5.2.9. Note that the result has been graphed against the angle \( \phi \) (90-\( \alpha \)) to be consistent with previous chapters.

![Figure 5.2.9: Relationship between \( q_r \) and sidewall angle from belt-slope matching (\( W=20\text{mm} \))](image-url)
The membrane theory model of the sidewall can be used to translate this relationship between $q_r$ and the sidewall angle into terms of radial deflection, which is needed to define $K_r$. The membrane model is used to determine a relationship between radial deflection and sidewall angle, then the belt-slope matching result is used to determine $q_r$ based on the sidewall angle at each level of deflection. For this analysis, a sidewall width of 20 mm has been used for the membrane theory model.

Figure 5.2.10: Normalized radial force from membrane theory alone, and from membrane theory combined with belt slope matching ($W=20$ mm)

The resulting $q_r$ curve shown in Figure 5.2.10 was used to calculate new values of $K_r^T$, $K_r^C$, and $K_r^G$. The results of the model using these new stiffness values for inflation pressure of 2.0 bar are shown in Figures 5.2.11 and 5.2.12.
Figure 5.2.11: Stiffness (top) and contact length (bottom) for belt-slope matching case (P=2 bar, W=20mm)

Figure 5.2.12: Stiffness results (structural force only) for belt-slope matching case (W=20mm)
Figure 5.2.12 above shows that the structural-only force (force calculated before the superposition adjustment) versus deflection and counterdeflection aligns with experimental data, as shown in the full validation with superposition in section 5.2.2.

5.2.4 Superposition with Adjustments

Section 4.4 introduced several other assumptions used in the creation of this model, which may be addressed through the adjustment of the input parameters. Two of these are addressed in this section, with more adjustments addressed in Section 5.3. First, the value of $EA$ was increased by multiplying by 1000 (an extreme value), to counteract the assumption of frictionless contact. Also, the value of $K_f^G$ was set to zero, meaning the sidewall does not pull or push on the belts in the radial direction in contact.

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</tr>
<tr>
<td>$K_f^G$</td>
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<td>Pa</td>
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</tbody>
</table>

Table 5.2.3: Inputs used for validation with superposition, with belt-slope matching and EA and $K_f^G$ adjustments
The figure above shows the stiffness and contact length results with these adjustments for inflation pressure of 2.0 bar. While the adjustments did marginally improve the slopes for deflection and counterdeflection stiffness, the contact length is further from the reference result.

### 5.2.5 Validation with Superposition Summary

The results presented in this section show that the model with the superposition scheme very accurately predicts contact pressure for a range of inflation pressures, for
low to moderate levels of vertical deflection. The predicted force versus deflection and force versus counterdeflection stiffnesses are too stiff compared to experimental data, but the forces before the superposition correction align with experimental data. The inherent contradiction in these results is that the force results are good before applying the superposition correction, and the contact pressure results are good after the correction has been applied.

5.3 Validation with Reduced Sidewall Stiffness and Increased Belt Stiffness

Section 4.4 introduced several other assumptions used in the creation of this model, which may be addressed through the adjustment of the input parameters. Several of these are addressed in this section. To account for the neglect of the belt crown and continuity between the belts and the sidewall, the sidewall radial stiffnesses are reduced. To account for friction in contact and the Poisson effect, the modulus of the belts ($E$) is increased. To account for the two-dimensional assumption when calculating the torsional stiffness, the torsional stiffness is reduced.

The amount of reduction was determined such that the results at each inflation pressure best matched the experimental data provided, with emphasis on 2.0 bar, as the most typical usage condition for this tire. The reductions made to the radial stiffness values were higher at higher inflation pressures, because the pneumatic part of radial stiffness is higher.
In Table 5.3.1, the value of each adjusted parameter is shown for each experimental inflation pressure, with the exception of 0 bar. Each parameter was calculated as described in Chapter 4, then multiplied by the “Multiplier” value to obtain the final value. In order to simplify the problem, and because the stiffness values were already being changed significantly, the iterative solution method described in Section 4.2 was not used in these cases; the values are from a contact angle of 4 degrees for all pressures.

Although it is clear that reduction of the stiffness input values is the correct direction to adjust for the assumptions made in the model development, it is acknowledged that the reduction shown below (multiplication by 0.15 for 2.0 bar, for example) is a very large adjustment, and may not represent the physical tire.

<table>
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Table 5.3.1: Inputs used for validation with superposition, with stiffness adjustments
Figure 5.3.1: Force versus deflection using membrane theory stiffnesses, with superposition and stiffness reduction.

Figure 5.3.2: Force versus counterdeflection using membrane theory stiffnesses, with superposition and stiffness reduction.
The force versus deflection and force versus counterdeflection results align well with experimental data, especially for 2.0 bar and lower, although there is some initial non-linearity in the experimental data is not captured by the model. The amount of counterdeflection for a given force is over-predicted for lower pressures, and under-predicted for higher pressures.

The contact pressure is above inflation pressure for all pressures, as expected. The difference between the contact pressure and the inflation pressure is between 10% of inflation pressure and 0.2 bar up to 1.0 bar inflation pressure; this is consistent with the assumption that the structural part of the tire’s stiffness is approximately constant with pressure, and follows the +10% rule of thumb for 2.0 bar. The inflation pressure is slightly too high for 2.0 bar and 4.0 bar inflation pressure.
The following figure shows the contact pressure distribution in the ground region. The solid black line represents the contact pressure. The solid red line shows the contact pressure distribution, and the dashed red line represents the average contact pressure in contact.

![Ground Pressure in the Contact Region](image)

**Figure 5.3.4**: Contact pressure in the contact patch, for vertical deflection ($\delta$) of 13 mm, with superposition and stiffness reduction

This solution set will be used as the basis of sensitivity studies in Chapter 6.
CHAPTER SIX
SENSITIVITY STUDIES

6.1 Pneumatic Tire

The solution with superposition with reduced sidewall stiffness and increased belt stiffness was used as the basis for these sensitivity studies. The base values for the sidewall and belt stiffnesses are shown in Table 5.3.1, and the base values for all other inputs are shown in Table 6.1.1. The variant values used in each case are noted on the figures in each sub-section. All studies in this chapter are at an inflation pressure of 2.0 bar. It is important to emphasize that these results are very sensitive to the set of parameters that are used as the baseline. For example, Lewis, 2012, showed very different sensitivity to some parameters, mainly due to the fact that the starting point was different.

<table>
<thead>
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<td>$W$</td>
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<td>mm</td>
</tr>
<tr>
<td>$b$</td>
<td>182.9</td>
<td>mm</td>
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</table>

Table 6.1.1: Inputs used for sensitivity studies, with superposition
6.1.1 EA Variation

Neither increasing nor decreasing $EA$ from the nominal value had a significant effect on the results. The deflection, counterdeflection, and contact length stiffnesses did not change significantly even when $EA$ was multiplied by 10 or by 0.1.

Figure 6.1.1: Force versus deflection, $EA$ variation with superposition
Figure 6.1.2: Force versus counterdeflection, $EA$ variation with superposition

Figure 6.1.3: Force versus contact length, $EA$ variation with superposition
6.1.2 EI Variation

A reduction in $EI$ increased the deflection and counterdeflection stiffnesses, while slightly decreasing the contact length stiffness. In these simulation cases the value of $P^*$ is actually slightly negative, meaning that the $(P - P^*)$ superposition correction is slightly greater than $P$. A reduction in EI caused $P^*$ to decrease, which led to a higher final vertical force.

Figure 6.1.4: Force versus deflection, $EI$ variation with superposition
Figure 6.1.5: Force versus counterdeflection, $EI$ variation with superposition

Figure 6.1.6: Force versus contact length, $EI$ variation with superposition
6.1.3 GA Variation

As with $EI$, a reduction in $GA$ increased the deflection and counterdeflection stiffnesses, while slightly decreasing the contact length stiffness. An increased $GA$ improved agreement with the vertical deflection experimental data, but the contact pressure was too high.

![Total Vertical Force vs. Vertical Deflection](image)

Figure 6.1.7: Force versus deflection, $GA$ variation with superposition
Figure 6.1.8: Force versus counterdeflection, $GA$ variation with superposition

Figure 6.1.9: Force versus contact length, $GA$ variation with superposition
6.1.5 $K_T^T$ Variation

The results are sensitive to changes in $K_T^T$. An increase in $K_T^T$ increases the initial stiffness of the tire, but leads to an increase in $P^*$ as deflection increases. As $P^*$ increases, the superposition force correction decreases. This phenomenon is what causes the force curve to curve downward at higher deflection.

![Total Vertical Force vs. Vertical Deflection](image)

Figure 6.1.10: Force versus deflection, $K_T^T$ variation with superposition
Figure 6.1.11: Force versus counterdeflection, $K_T^r$ variation with superposition

Figure 6.1.12: Force versus contact length, $K_T^r$ variation with superposition
6.1.6 $K^C_t$ Variation

A reduction in $K^C_t$ decreases the deflection and counterdeflection stiffnesses of the tire. The compression region is a much smaller portion of the tire, so $K^C_t$ does not have a large impact on the initial stiffness of the tire. An increase in $K^C_t$ caused a large increase in $P^*$, causing the superposition adjustment to become negative, leading to the force decreasing at higher levels of vertical deflection.

Figure 6.1.13: Force versus deflection, $K^C_t$ variation with superposition
Figure 6.1.14: Force versus counterdeflection, $K^C_r$ variation with superposition

Figure 6.1.15: Force versus contact length, $K^C_r$ variation with superposition
6.1.7 $K_r^G$ Variation

Increasing $K_r^G$ changes the vertical force for a given level of deflection. The relationship between contact angle and deflection does not change, so all cases below show the same range of deflection. $K_r^G$ causes a larger increase in force at higher deflection, as the contact region becomes a larger part of the tire.

![Figure 6.1.16: Force versus deflection, $K_r^G$ variation with superposition](image)

Figure 6.1.16: Force versus deflection, $K_r^G$ variation with superposition
Figure 6.1.17: Force versus counterdeflection, $K_r^G$ variation with superposition

Figure 6.1.18: Force versus contact length, $K_r^G$ variation with superposition
6.1.8 $K_\theta$ Variation

The following graphs show that the main impact of changing $K_\theta$ is changing the amount of counterdeflection. Too high of a value of $K_\theta$ greatly reduces the amount of counterdeflection without significantly reducing deflection. This moves the ratio of counterdeflection to deflection further from the expected value of about 10%. The deflection and contact length values are not significantly impacted.

![Total Vertical Force vs. Vertical Deflection](image)

*Figure 6.1.19: Force versus deflection, $K_\theta$ variation with superposition*
Figure 6.1.20: Force versus counterdeflection, $K_\theta$ variation with superposition

Figure 6.1.21: Force versus contact length, $K_\theta$ variation with superposition
6.2 Non-pneumatic Tire

The main parameter of interest for the non-pneumatic tire is pre-tensioning, which allows the modeling of spokes which are in tension in the non-loaded state. Spokes without pre-tensioning buckle at any level of negative radial deflection, meaning that the spoke no longer exerts force on the inside of the belt package. A pre-tensioned spoke maintains force on the belt package for small levels of negative deflection before buckling.

In order to model pre-tensioning, the model has been updated to support a variable split point between the tension and compression regions of the tire. Choosing this split point as a negative value models pre-tension by extending the tension region into areas with negative deflection. The split point in the model is defined in terms of radial deflection.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EA$</td>
<td>$9.0 \times 10^6$</td>
<td>N</td>
</tr>
<tr>
<td>$EI$</td>
<td>168.75</td>
<td>Nm²</td>
</tr>
<tr>
<td>$GA$</td>
<td>3600</td>
<td>N</td>
</tr>
<tr>
<td>$ur_0$</td>
<td>0</td>
<td>mm</td>
</tr>
<tr>
<td>Poisson Term</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$R_0$</td>
<td>200</td>
<td>mm</td>
</tr>
<tr>
<td>$b$</td>
<td>60</td>
<td>mm</td>
</tr>
<tr>
<td>$K_{r,T}$</td>
<td>$1.0 \times 10^6$</td>
<td>Pa</td>
</tr>
<tr>
<td>$K_r, K_r^G, K_r^\theta$</td>
<td>0</td>
<td>Pa</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0</td>
<td>N/m</td>
</tr>
<tr>
<td>$W, G_{c, t_c, R_i}$</td>
<td>Not applicable</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2.1: Inputs used for non-pneumatic tire sensitivity study
Adding pre-tension to the non-pneumatic tire by decreasing $u_{r0}^l$ gave rise to a kink in the deflection and counterdeflection curves at the level vertical deflection equal to $-u_{r0}^l$. When the deflection is less than the magnitude of $u_{r0}^l$, the spokes remain in tension around the entire circumference of the tire, making the tire stiffer. Above this point the tire becomes less stiff, as the spokes begin to buckle in contact. The force response of the non-pneumatic tire is linear above and below the kink point.

![Figure 6.2.1: Force versus deflection, $u_{r0}^l$ variation with superposition, non-pneumatic tire](image.png)
Figure 6.2.2: Force versus counterdeflection, $u'_{r0}$ variation with superposition, non-pneumatic tire
CHAPTER SEVEN

CONCLUSIONS AND FUTURE WORK

7.1 Conclusions
The model presented here predicts force versus deflection, force versus counterdeflection, and force versus contact length for a pneumatic and a non-pneumatic tire.

The non-pneumatic predictions align very well with finite element analysis results, and the improved flexibility in spoke stiffness modeling can be used to explore the impact of pre-tensioning the spokes on the stiffness results.

The pneumatic tire predictions align with experimental results when the tire is simulated in a load and pressure condition such that the vertical component of belt tension is zero at the edge of contact (referred to as the $P=P^*$ condition), for low levels of inflation pressure.

The simulation also aligns with experimental data when it is run at high levels of vertical deflection, without using the superposition scheme. However, there is no reason to believe that this method would correctly predict the force results when the vertical component of belt tension at the edge of the contact patch is large, so this is believed to be coincidental agreement.

A brief study of the bending of the belts under load from the sidewall indicated that the radial stiffness calculated using membrane theory was too high. It was shown that significantly reducing the radial and circumferential stiffnesses from membrane theory
while increasing the modulus of the belts improves the results, but the reduction required to achieve these results may be more than is physically accurate.

### 7.2 Future Work

To investigate and develop this model without further testing, the first step would be to create a finite element model of the tire used in the validation, and simulate loading. Such an analysis would provide details about the tire’s condition that cannot be easily measured, including profiles of deformation and stress around the tire, and contact angle. If a full three-dimensional tire model were available, then the sidewall shape and angle at the edge of the belts could also be examined. This analysis should improve understanding of the current model by pinpointing which outputs do or do not agree with the finite element results.

After finite element modeling, the next step in further developing this model would be to obtain the characterizing parameters and experimental data for another tire, preferably of a significantly different dimension than the tire used in this study. This data should include force, deflection, and counterdeflection as included herein, but also contact length and average contact pressure. For the sake of simplicity, a slick tire (a tire without any tread pattern) should be used if possible, to avoid the complications of locally different contact pressure due to the tread pattern features. If possible, contact angle measurements would also be helpful for better understanding the relationship between contact angle, deflection, and contact length; however, that type of data is difficult to obtain experimentally.
The predicted results for this new tire should be compared to the experimental results first using the stiffnesses calculated using membrane theory and composites theory, then with the adjustments proposed in this study. The prediction capability for force versus deflection and counterdeflection would help determine whether the proposed adjustments are applicable in general. The contact length data would provide another means for validating the model, and in case of discrepancy between predicted and experimental, would be helpful for determining the cause.
REFERENCES


