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Andrew Walton Green

Clemson University, waltong95@gmail.com

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THE UNCERTAINTY PRINCIPLE IN CONTROL THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation
Presented to
the Graduate School of
Clemson Univesity

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematics

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by
Andrew Walton Green
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Accepted by:
Dr. Shitao Liu, Committee Co-Chair
Dr. Mishko Mitkovski, Committee Co-Chair
Dr. Benjamin Jaye
Dr. Jeong-Rock Yoon

Abstract

In this thesis, we will study the interaction between problems in control theory for partial differential equations and inequalities of the uncertainty principle type. The main results will concern the boundary observability of the viscoelastic wave equation and energy decay rates of damped wave equations. In the boundary case, we will prove what may be viewed as a higher dimensional version of Ingham's inequality, replacing the complex exponentials with Laplacian eigenfunctions.

For energy decay rates on the real line, we will use a version of the Paneah-Logvinenko-Sereda theorem for functions with Fourier support contained in multiple intervals. We prove the exact variation which we need and apply it to internal observability as well as decay rates for damped wave equations as well. We also give partial results in higher dimensions and some open problems.

We will also investigate the connection between compactness of localization operators and uncertainty principles from an abstract harmonic analysis perspective. We give some general results which are applied to the wavelet transform.

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Contents

Introduction

The uncertainty principle is the general statement that a function cannot be well-localized in both time and frequency. Various interpretations of "well-localized" may give rise to different mathematical theorems which are often called uncertainty principles themselves, though they are only instances (phenomena) of the uncertainty principle (noumena).

The classical Heisenberg-Pauli-Weyl uncertainty principle measures localization by the variance of a function

$$\mathrm{Var}(f) = \int |f(x) - \int f(y)\, dy|^2\, dx$$

and states that

$$\mathrm{Var}(f)\,\mathrm{Var}(\hat{f}) \geq c\|f\|_{L^2}^4.$$

$\hat{f}$ is the Fourier transform of f, and it is the most common understading of the frequency profile of a function. If $f : \mathbb{R}^d \to \mathbb{C}$, then its Fourier transform, denoted by either $\hat{f}$ or $\mathcal{F}(f)$, is defined as follows.

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx$$

for each $\xi \in \mathbb{R}^d$. In this way, by the classical theorem of Plancherel, $\mathcal{F}$ can be extended to a unitary operator on $L^2(\mathbb{R}^d)$ with $\mathcal{F}^* f(\xi) = \mathcal{F} f(-\xi)$, while the integral above is well defined for all $f \in L^1(\mathbb{R}^d)$.

On the other hand, if $\Omega \subset \mathbb{R}^d$ is smooth and bounded, and $f : \Omega \to \mathbb{C}$ is an element of $L^2(\Omega)$, then its frequency profile can be represented as a discrete function $n \to \langle f, \phi_n \rangle$ where ϕ_n are an orthonormal basis for $L^2(\Omega)$ of Laplacian eigenfunctions.

In this thesis, we will not only apply these ideas to control problems for time evolution partial differential equations, but also formulate new uncertainty principles suggested by problems in control theory.

Roughly speaking, the controllability problem is to try to make a system behave according to our wishes. There are certain parameters (called "control" functions) of the system which may be manipulated in order to achieve a desired state. In this thesis we mainly consider evolution systems–also referred to as distributed systems, namely, the phenomenon is "distributed" in a geometrical domain–which are governed by partial differential equations (PDEs), and we are allowed to act on the trajectories of the systems by means of a boundary or internal force.

The most common technique is to prove an observability inequality for the dual problem. Generally, this states that one can bound the initial or final data by a suitable "observation" which is dual to the mechanism by which the system is "controlled."

The relationship between these two fields historically began with the connection between the so-called "moment method" and the classical problem of independence of nonharmonic Fourier series [53, 2]. In section (1.2) we will give a brief example of how to reformulate the control problem as a *moment problem*: Given a sequence of functions $\{e_n\}_{n=1}^{\infty}$ and a sequence of scalars $\{c_n\}_{n=1}^{\infty}$, does there exists a function f satisfying

$$\langle f, e_n \rangle = c_n \text{ for } n = 1, 2, \ldots? \tag{0.0.1}$$

To solve this infinite-dimensional system of equations, the constraints must be independent is some way. The notion of independence which makes this problem well-posed is given by that of a Riesz sequence (see Section 1.2).

This is the perspective of Chapter 2, in which we study the controllability properties of a viscoelastic wave. For $T > 0$, $M \in H^2(0, T)$, let w satisfy

$$w_{tt}(x,t) + \Delta w(x,t) = \int_0^t M(t-s)\Delta w(x,s)\, ds \tag{0.0.2}$$

for each $(x,t) \in \Omega \times (0,T)$. We establish the exact boundary controllability of (0.0.2) by showing that an appropriate harmonic system forms a Riesz sequence. This part of the thesis is joint work with S. Liu and M. Mitkovski [20]. We will also apply this controllability result to the study of a viscoelastic inverse source problem.

The other connection we rely on in our analysis is between the homogeneous evolution problem $u'(t) = \mathcal{A}u$ and the nonhomogeneous stationary problem $(\mathcal{A} - i\lambda)u = f$ ($\mathcal{A}$ is a suitable differential operator, $\lambda \in \mathbb{R}$) in works such as [51, 11, 52, 8, 42, 9]. In other words, one may study the resolvent of $\mathcal{A}$ on the imaginary axis. In Chapter 3, we will show the explicit connection between observability and certain properties of the resolvent.

Using these ideas, in Chapter 3, we study the fractional Klein-Gordon Equation on $\mathbb{R}^d$. This work is carried out in [17]. Let w satisfy

$$w_{tt}(x,t) + (-\Delta + 1)^{s/2} w(x,t) = 0 \tag{0.0.3}$$

for $(x,t) \in \mathbb{R}^d \times (0,\infty)$. We will prove the observability inequality for w from any relatively dense set E when $d = 1$. Moreover, due the close connection between observability and energy decay, we can apply the same techniques to compute the energy decay rates of the damped Klein-Gordon equation

$$w_{tt}(x,t) + \gamma(x) w_t(x,t) + (-\Delta + 1)^{s/2} w(x,t) = 0$$

under the condition that the measue $\gamma(x)\, dx$ is relatively dense.

The main tool in our study of the Klein-Gordon equation (0.0.3) is an uncertainty principle of the form

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(E)}$$

for functions f with $\hat{f}$ supported in certain annuli $A_\lambda(\delta) = \{\xi \in \mathbb{R}^d : \lambda \leq |\xi| \leq \lambda + \delta\}$. The strategy is to compute the dependence of C on λ and δ for certain classes of sets E. In one dimension, the annuli $A_\lambda(\delta)$ are just two intervals, so the approach is different and thus better results can be obtained than in higher dimensions.

In the final section, we give a framework for uncertainty principles in abstract harmonic analysis. We use this perspective to understand the role of compactness in the study of uncertainty principles. The objects we focus on are the so-called localization operators $L_E : \mathcal{H} \to \mathcal{H}$, for $E \subset X$ a locally compact group, defined by

$$L_E f = \int_E \langle f, k_x \rangle k_x \, d\mu(x)$$

where $\{k_x\}_{x \in X}$ is a Parseval frame satisfying appropriate assumptions. The term localization operator comes from the fact that L_X is the identity, so L_E localizes f to E. The goal to to find an uncertainty principle of the form $\langle f, k_x \rangle = 0$ for x outside of E implies $f = 0$. In other words, no f in $\mathcal{H}$ can be localized on E.

We give two different conditions on the set E which yield the following uncertainty principle: There exists $\alpha > 0$ such that

$$\int_{X \backslash E} |\langle f, k_x \rangle|^2 \, d\mu(x) \geq \alpha \|f\|^2$$

for all $f \in \mathcal{H}$. In particular, if $\langle f, k_x \rangle = 0$ for x in $X \backslash E$, then $f = 0$.

The simplest condition, with minimal restrictions on (X, μ, k_x) is that $\mu(E) < \infty$.

Adding some assumptions, we obtain the result for E which have the property that the Berezin transform $\langle L_E k_x, k_x\rangle \to 0$ as $d(x,1) \to \infty$. These are also the so-called "thin" sets from [15].

Observability on The Real Line

We begin with a simple example to illustrate our ideas. Let $1 \le p < \infty$. Given $f \in L^p(\mathbb{R})$ smooth, we can solve the transport equation:

$$(\partial_x \pm \partial_t)u(x,t) = 0 \quad u(x,0) = f(x) \quad x \in \mathbb{R}, t > 0 \tag{0.0.4}$$

by $u(x,t) = f(x \mp t)$. The observability inequality for the transport equation is immediate: Let μ be a relatively dense measure on $\mathbb{R}$, which means there exists $c, T > 0$ such that

$$c \le \mu([y, y+T]) = \int_y^{y+T} d\mu \tag{0.0.5}$$

for almost every $y \in \mathbb{R}$. Then, setting $y = x \mp t$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} |u(x,t)|^p \, d\mu(x)\, dt &= \int_0^T \int_{\mathbb{R}} |f(y)|^p \, d\mu(y \pm t)\, dt \\ &= (\pm) \int_{\mathbb{R}} |f(y)|^p \int_y^{y \pm T} d\mu(s)\, dy \\ &\ge c\|f\|_{L^p}^p. \end{aligned} \tag{0.0.6}$$

We use this simple result to connect to the wave equation. If w satisfies the wave equation $(\partial_{xx}-\partial_{tt})w = 0$, then w can be broken up two ways. Setting $u^{\pm} = (\partial_x \pm \partial_t)w$, $u^{\pm}$ satisfies the transport equation (0.0.4). Then, applying (0.0.6) to $u^{\pm}$ and setting $w^0(x) = w(x,0)$ and $w^1(x) = w_t(x,0)$, we have

$$\int_0^T \int_{\mathbb{R}} |w_t(x,t) + w_x(x,t)|^2 \, d\mu(x) \, dt \geq c\|w^1 + w_x^0\|^2$$
$$\text{and} \quad \int_0^T \int_{\mathbb{R}} |w_t(x,t) - w_x(x,t)|^2 \, d\mu(x) \, dt \geq c\|w^1 - w_x^0\|^2.$$

So, by the parallelogram identity, we get

$$\int_0^T \int_{\mathbb{R}} |w_t(x,t)|^2 + |w_x(x,t)|^2 \, d\mu(x) \, dt \geq c(\|w^1\|^2 + \|w_x^0\|^2). \tag{0.0.7}$$

Taking $d\mu(x) = \mathbb{1}_E(x)dx$, one obtains the usual observability inequalities for sets E satisfying

$$m(E \cap [y, y+T]) \geq c \quad \text{for all } y \in \mathbb{R}.$$

However, this also yields observability for point sampling. Indeed, let $\Lambda \subset \mathbb{R}$ be a discrete set and

$$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$$

where δ_λ is the point-mass measure defined by $\delta_\lambda(A) = 1$ if $\lambda \in A$ and 0 otherwise. In this case, $\mu([y, y+T]) = \#(\Lambda \cap [y, y+T])$. So, if Λ satisfies $\#(\Lambda \cap [y, y+T]) \geq c$

for all y, then μ is relatively dense and we obtain

$$\sum_{\lambda \in \Lambda} \int_0^T |w_t(\lambda, t)|^2 + |w_x(\lambda, t)|^2 \, dt \geq c(\|w_1\|^2 + \|w_x^0\|^2).$$

We also comment on the sharp observability time T and constant c. Concerning c, all the computations above were equality except in (0.0.6), which could be made equality by taking $\tilde{\mu} \leq \mu$ such that (0.0.5) holds with equality.

Let $T_0 = \inf\{T : \inf_{y \in \mathbb{R}} \mu([y, y+T]) > 0\}$. The above argument shows that (0.0.7) cannot hold for $T < T_0$. Concerning the critical time T_0, if the support of μ has a gap of length T_0, then only for $T > T_0$ can can $\mu([y, y+T])$ be positive. This corresponds to the speed of propogation for this wave being 1. One must wait time T_0 for the data to travel distance T_0.

This is not the end of the discussion though since we will actually prove a stronger observability inequality of the form

$$\int_0^T \int_E |w_t(x,t)|^2 \, dx \, dt \geq c \left(\|w^1\|^2 + \|w_x^0\|^2 + \|w_0\|^2 \right)$$

for the Klein-Gordon equation $w_{tt} - w_{xx} + w = 0$ and E relatively dense. However, we do not obtain the sharpness in the time T or constant c.

Chapter 1

Preliminaries

1.1 Control and Observability

Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subseteq \mathcal{G} \to \mathcal{H}$ be linear (unbounded) operators. We consider the controlled evolution equation:

$$\begin{cases} \dfrac{d}{dt}u(t) + \mathcal{A}u(t) = \mathcal{B}f(t) \quad t \in [0,T] \\ \\ u(0) = u_0 \in \mathcal{H} \end{cases} \tag{1.1.1}$$

where $f \in L^2([0,T];\mathcal{G})$ is the control function and $\mathcal{B}$ represents the mechanism by which the system is controlled. Throughout, we assume that there exists a unique weak solution to (1.1.1) for each u_0, f. The control problem is the following: Given $u_0, u_T \in \mathcal{H}$, does there exist $f \in L^2([0,T];\mathcal{G})$ and $T > 0$ such that u satisfies (1.1.1) with $u(0) = u_0$ and $u(T) = u_T$? We say (1.1.1) is exactly controllable in time T if the answer to this question yes. If we only consider the final state $u_T \equiv 0$, then the system is null controllable.

Alongside the existence of a control, one may wonder about the related stability question, namely if there is a $C > 0$ such that for all u_0, one can find such an f with $\|f\| \leq C\|u_0\|$? This is usually automatic (from the closed graph theorem) once existence is established, so we do not emphasize this point in the results that follow.

Dual to controllability is the notion of observability. It is a property of the dual equation to (1.1.1:

$$\begin{cases} \frac{d}{dt}v(t) - \mathcal{A}^* v(t) = 0 \quad t \in [0,T] \\ \\ v(0) = v_0 \in \mathcal{H} \end{cases} \tag{1.1.2}$$

Definition 1.1.1. The system (1.1.2) is said to be observable in time T if there exists $c > 0$ such that

$$\int_0^T \|\mathcal{B}^* v(t)\|_{\mathcal{G}}^2 \geq c\|v_0\|_{\mathcal{H}}^2 \quad \text{for all } v_0 \in \mathcal{H}. \tag{1.1.3}$$

In most applications, controllablity is equivalent to observability of the dual system. For most of the results in this thesis, only the observability inequality is considered, but there is an associated controllability result. The relationship between control and observability was introduced by D. L. Russell and S. Dolecki in [13]. It was applied to wave equations by J. L. Lions in [36] using the Hilbert Uniqueness Method, showing that observability implies controllability.

Proposition 1.1.2. *Suppose (1.1.1) and (1.1.2) both have weak solutions for all*

$u_0, v_0 \in \mathcal{H}$ and the $\mathcal{B}^*$ satifies the upper regularity inequality $\|\mathcal{B}^* v\|_{L^2([0,T];\mathcal{G})} \leq C\|v_0\|$ for all $v_0 \in \mathcal{H}$. Then, (1.1.1) is null controllable if and only if (1.1.2) is observable.

Proof. Let u, v be solutions of (1.1.1) and (1.1.2) respectively. For each t,

$$\begin{aligned} \frac{d}{dt}\langle u(t), v(t)\rangle &= \langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle \\ &= \langle -\mathcal{A}u(t) + \mathcal{B}f(t), v(t)\rangle + \langle u(t), \mathcal{A}^* v(t)\rangle \\ &= \langle f(t), \mathcal{B}^* v(t)\rangle. \end{aligned}$$

Integrating from 0 to T, we have

$$\langle u(T), v(T)\rangle - \langle u(0), v(0)\rangle = \int_0^T \langle f(t), \mathcal{B}^* v(t)\rangle \, dt.$$

We can see that the null control problem is solved if and only if there is f such that

$$\langle f, \mathcal{B}^* v\rangle_{L^2([0,T];\mathcal{G})} = \langle u_0, v_0\rangle \tag{1.1.4}$$

for all $v_0 \in \mathcal{H}$. To show necessity, null controllability implies for each v_0, there is $f(v_0)$ such that $\|f(v_0)\|_{L^2([0,T];\mathcal{G})} \leq C\|v_0\|$ and (1.1.4) holds with $u_0 = v_0$. Applying Cauchy-Schwarz to (1.1.4) establishes observability.

On the other hand, if (1.1.2) is observable, then the observation operator $\mathcal{W}^* v_0 := \mathcal{B}^* v \in L^2([0,T];\mathcal{G})$ is bounded below (and bounded by the regularity assumption). Therefore $\mathcal{W}$ is surjective and we can find f such that $\mathcal{W}f = u_0$. In this way, f

satisfies (1.1.4) since

$$\langle f, \mathcal{B}^* v\rangle = \langle f, \mathcal{W}^* v_0\rangle = \langle \mathcal{W} f, v_0\rangle = \langle u_0, v_0\rangle$$

for all $v_0 \in \mathcal{H}$. $\square$

1.2 Moment Method

The reasoning in the above proof can serve as an introduction to the moment method. If $\mathcal{A}^*$ has an orthonormal basis of eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$, then null control is equivalent to (1.1.4) holding with $v_0 = \phi_n$ for all n. In this case, $v(t) = e^{\lambda_n t}\phi_n$ and null control is equivalent to the following moment problem: Find $f \in L^2([0,T];\mathcal{G})$ such that

$$\int_0^T e^{\lambda_n t}\langle f(t), \mathcal{B}^*\phi_n\rangle_{\mathcal{G}}\, dt = \langle u_0, \phi_n\rangle \text{ for all } n.$$

In this section, we will see that this is equivalent to the sequence $\{e^{\lambda_n t}\mathcal{B}^*\phi_n\}$ forming a Riesz sequence in $L^2([0,T];\mathcal{G})$, as well as prove some useful facts about moment problems and Riesz sequences.

The idea of viewing the control problem as a "moment problem" was an early development in the field of control theory by D. L. Russell [54]. This method has been extended in different directions [22, 29, 39, 46], but the common feature of most results has been the requirement for the space dimension to be equal to one. This is due to the fact that $\mathcal{B}^*$ is often some kind of restriction or trace operator so in one-dimension, $\mathcal{B}^*\phi_n$ may be just a number. Then the analysis is reduced to an exponential system.

Example 1.2.1 (Wave Equation)**.** Consider the boundary controllability of the wave equation with potential on $(0, 1) \times (0, T)$.

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + V(x)u(x,t) = 0 \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) \\ u(0,t) = f(t) \quad u(1,t) = 0 \end{cases} \tag{1.2.1}$$

Multiply (1.2.1) by $e^{i\lambda_n t}\phi_n(x)$ where

$$(-\partial_x^2 + V)\phi_n = \lambda_n^2 \phi_n \quad \phi_n(0) = \phi_n(1) = 0.$$

Integrating by parts,

$$e^{i\lambda_n t}\langle [u_t(\cdot,t) - i\lambda_n u(\cdot,t)], \phi_n\rangle \Big|_{t=0}^{T} = -\phi_n'(0)\int_0^T f(t)e^{i\lambda_n t}\,dt.$$

So, finding such an f is equivalent to solving the moment problem: For $c \in \ell^2$, find $f \in L^2(0,T)$ such that

$$\int_0^T f(t)e^{i\lambda_n t}\,dt = c_n \quad \text{for all } n.$$

This is equivalent to the classical interpolation problem for analytic functions: Find $g \in L^2(\mathbb{R})$ such that $\operatorname{supp} \hat{g} \subset [0,T]$ with $g(\lambda_n) = c_n$. Then $f = \hat{g}$. Such problems are very well-studied and we refer to the book [58] and the references therein. One can also see [14] for a similar approach regarding the heat equation. In this case, replace u_{tt} with u_t and the exponential system has real frequencies instead of imaginary.

Probably, the most comprehensive treatment, to date, on the use of complex exponentials in control problems is the monograph [2] where, in addition, approximate controllability results (even in higher space dimension) are obtained using complex exponentials in concert with standard uniqueness results. In Section 2.2, we will establish exact boundary controllability of the wave equation in arbitrary dimensions from this perspecitve.

1.2.1 Riesz Sequences

Definition 1.2.2. A sequence $\{e_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a Riesz sequence if there exists $c, C > 0$ such that

$$c\sum |a_n|^2 \leq \left\|\sum a_n e_n\right\|^2 \leq C\sum |a_n|^2$$

for all finite sequences $\{a_n\} \subset \mathbb{C}$. If the lower (upper) inequality holds, then $\{e_n\}$ is said to be a Riesz-Fischer (Bessel) sequence. By finite sequence we mean that $\{a_n\}$ has only finitely many non-zero entries.

Definition 1.2.3. Given $\{e_n\} \subset \mathcal{H}$ and $c \in \ell^2$, we say $f \in \mathcal{H}$ is a solution of the moment problem if

$$\langle f, e_n \rangle = c_n \quad \text{for all } n. \tag{1.2.2}$$

Proposition 1.2.4. *Let $\{e_n\} \subset \mathcal{H}$. The following are equivalent:*

(i) $\{e_n\}$ is a Riesz-Fischer sequence.

(ii) The moment problem has a solution for any $c \in \ell^2$.

(iii) There exists a Bessel sequence $\{f_n\} \subset \mathcal{H}$ such that $\langle e_n, f_m \rangle = \delta_{n,m}$ (In this case we say $\{e_n\}$ and $\{f_n\}$ are biorthogonal).

Proof. First, we show (ii) implies (iii). Set $Y = \overline{\text{span}}\{e_n\}$. First notice that if the moment problem has a solution, it has a unique solution in Y. Indeed, let f be a solution, then $f = f_1 + f_2$ where $f_1 \in Y$ and $f_2 \in Y^\perp$. Then clearly f_1 is a solution. Moreover it is unique since

$$\langle g, e_n \rangle = 0 \quad \forall n \implies g \in Y^\perp.$$

Define the solution operator $T : \ell^2 \to Y$ which maps $c \in \ell^2$ to the unique element $g \in Y$ such that $\langle g, e_n \rangle = c_n$ for all n. We claim T is closed. Let $c^{(m)} \to c \in \ell^2$ and

$Tc^{(m)} = g_m \to y \in Y$. Then, for $f \in \text{span}\{e_n\}$, $f = \sum a_n e_n$ for some finite sequence $\{a_n\}$ so

$$\langle g_m, f\rangle = \sum a_n c_n^{(m)} \to \sum a_n c_n = \langle g, f\rangle$$

This implies $g_m \overset{\text{w}}{\to} g = Tc$ since g_m is bounded and $\text{span}\{e_n\}$ is dense in Y. But $g_m \overset{\text{w}}{\to} y$ so $y = g$ and T is closed. By the closed graph theorem, T is bounded. This means that not only is there a unique solution in Y, but it is uniformly bounded: there exists $C > 0$ such that

$$||g||^2 \leq C \sum |c_n|^2$$

Define $f_n = T(0, \ldots, 0, 1, 0, \ldots)$ where the 1 is the n-th entry. In this way, $\{f_n\}$ and $\{e_n\}$ are biorthogonal. To show $\{f_n\}$ is a Bessel sequence, take a finite sequence $\{c_n\}$ and one can check that $\sum c_n f_n$ is in Y and solves the moment problem. Since this solution is unique, $\sum c_n f_n = T(c)$ so

$$\|\sum c_n f_n\|^2 \leq C \sum |c_n|^2.$$

To show (iii) implies (i), fix a finite sequence $\{a_n\}$. Then, by the Bessel inequality for $\{f_n\}$,

$$\begin{aligned}\sum |a_n|^2 &= \left\langle \sum a_m f_m, \sum a_n e_n \right\rangle \\ &\le \left\| \sum a_m f_m \right\| \cdot \left\| \sum a_n e_n \right\| \\ &\le \left(C \sum |a_n|^2 \right)^{1/2} \left\| \sum a_n e_n \right\| .\end{aligned}$$

Finally, to show (i) implies (ii), fix $c_n \in \ell^2$. Define the linear functional μ on $Y = \text{span}\{e_n\}$ by

$$\mu \left(\sum a_n e_n \right) = \sum a_n \bar{c}_n$$

for all finite sequences $\{a_n\}$. μ is well-defined since $\{e_n\}$ are linearly independent so the representation $f = \sum a_n e_n$ is unique. Since $\{e_n\}$ is a Riesz-Fischer sequence, μ is a bounded linear functional on Y and can be continuously extended to $\overline{Y}$ (and then to $\mathcal{H}$ by taking $\mu = 0$ on $Y^{\perp}$). By the Riesz representation theorem, there exists $f \in \mathcal{H}$ such that

$$\mu(g) = \langle g, f \rangle$$

for all $g \in \mathcal{H}$. In particular, for g=e_n,

$$\langle f, e_n \rangle = \overline{\mu(e_n)} = c_n.$$

□

Corollary 1.2.5. *Every Riesz sequence has a biorthogonal Riesz sequence.*

Proof. Let $\{e_n\}$ be a Riesz sequence. Then it is a Riesz-Fischer sequence, so it has a biorthogonal Bessel sequence $\{f_n\}$. However, $\{f_n\}$ has a biorthognal Bessel sequence, namely $\{e_n\}$. Therefore $\{f_n\}$ is also a Riesz-Fischer sequence. □

We also have the following stability result for Riesz-Fischer sequences.

Lemma 1.2.6. *Let $\{e_n\} \subset \mathcal{H}$ be a Riesz-Fischer sequence. If there exists $q \in (0,1)$ such that*

$$\left\| \sum a_n(e_n - f_n) \right\| \leq q \left\| \sum_n a_n e_n \right\| \tag{1.2.3}$$

for all finite sequences $\{a_n\}$, then $\{f_n\}$ is also a Riesz-Fischer sequence.

Proof. By the triangle inequality,

$$\left\| \sum a_n f_n \right\| \geq \left\| \sum a_n e_n \right\| - \left\| \sum a_n(e_n - f_n) \right\| \geq (1-q) \left\| \sum a_n e_n \right\|.$$

□

We will also use a weaker notion of independence than that of a Riesz-Fischer sequence.

Definition 1.2.7. A sequence $\{e_n\} \subset \mathcal{H}$ is said to be ℓ^2-independent if the only element $c \in \ell^2$ for which

$$\sum c_n e_n = 0$$

is $c \equiv 0$.

Lemma 1.2.8. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$. If $\{f_n\}_{n\geq N}$ is a Riesz sequence for some $N \in \mathbb{N}$ and $\{f_n\}_{n=1}^{\infty}$ is ℓ^2-independent, then $\{f_n\}_{n=1}^{\infty}$ is a Riesz sequence.*

Proof. Set $Y = \overline{\operatorname{span}}\{f_n\}$. Decompose $Y = \overline{\operatorname{span}}_{n\geq N}\{f_n\} \oplus \overline{\operatorname{span}}_{n\geq N}\{f_n\}^{\perp}$. Let $\{e_n\}$ be an orthonormal basis for Y. Define $T : Y \to Y$ by $T(e_n) = f_n$ for $n \geq N$. Let $\{g_n\}_{n=1}^{N-1}$ be an orthonormal basis for $\overline{\operatorname{span}}_{n\geq N}\{f_n\}^{\perp}$ (This has the same dimension as $\operatorname{span}_{n<N}\{f_n\}$ by ℓ^2-independence). Define $T(e_n) = g_n$ for $n < N$. T can be extended to all of Y and T is bounded above and below since $\{f_n\}_{n\geq N} \cup \{g_n\}_{n<N}$ is a Riesz sequence. Indeed, for $f = \sum a_n e_n$,

$$\|Tf\|^2 = \left\|\sum_{n\geq N} a_n f_n + \sum_{n<N} a_n g_n\right\|^2 = \left\|\sum_{n\geq N} a_n f_n\right\|^2 + \left\|\sum_{n<N} a_n g_n\right\|^2 \tag{1.2.4}$$

$$\geq c\sum_{n\geq N} |a_n|^2 + \sum_{n<N} |a_n|^2 \geq \min\{c, 1\}\|f\|^2$$

So T is invertible. Define $K : Y \to Y$ by $K(e_n) = 0$ for $n \geq N$ and $K(e_n) = f_n - g_n$ for $n < N$. To show $\{f_n\}$ is a Riesz sequence, it suffices to show that $T + K$ is invertible. Since K is of finite rank (and thus compact), by the Fredholm alternative for compact perturbations, we only need to check that $T + K$ is injective. Let $f = \sum a_n e_n$ such that $(T + K)f = 0$. This implies

$$\sum a_n f_n = 0$$

However, since $\{f_n\}$ is ℓ^2-independent, all a_n must be zero so $f = 0$. □

1.2.2 Sequences of Vector Exponentials

We now specify to the case where e_n has the special form $e^{\lambda_n t}\eta_n$ and $\mathcal{H} = L^2([0,T];\mathcal{G})$ where $\mathcal{G}$ is another Hilbert space. We will prove two results, the first showing the stability of the Bessel sequence property.

Lemma 1.2.9. *Let $\mathcal{G}$ be a Hilbert space and $\{e^{\lambda_n t}\eta_n\} \subset L^2([0,T];\mathcal{G})$ a Bessel sequence. Then, $\{e^{\mu_n t}\eta_n\}$ is a Bessel sequence whenever*

$$\sup_n |\mu_n - \lambda_n| < \infty.$$

Proof. Set $\delta_n = \mu_n - \lambda_n$, $r = \sup |\delta_n|$.

$$\left\|\sum a_n e^{\mu_n t}\eta_n\right\|_{L^2([0,T];\mathcal{H})} = \left\|\sum_n a_n \eta_n e^{\lambda_n t} e^{\delta_n t}\right\| = \left\|\sum_n a_n \eta_n e^{\lambda_n t} \sum_{k=0}^{\infty} \frac{(\delta_n)^k}{k!} t^k\right\|$$

$$\leq \sum_{k=0}^{\infty} \frac{T^k}{k!} \left\|\sum_n a_n (\delta_n)^k \eta_n e^{\lambda_n t}\right\| \leq C \sum_{k=0}^{\infty} \frac{(Tr)^k}{k!} \sqrt{\sum_n |a_n|^2} = Ce^{rT}\sqrt{\sum |a_n|^2}.$$

□

The next lemma shows that these vector exponential Riesz sequences (though not orthogonal) still preserve the following property of Fourier series: improved regularity implies improved decay of the coefficients. This is a slight generalization of Lemma 3.3 in [47] which is used in the proof of Proposition 2.1.4.

Lemma 1.2.10. *Let $\mathcal{G}$ be a Hilbert space. Let $\{e^{\lambda_n t}\eta_n\}$ be a Riesz-Fischer sequence in $L^2([0,T_0];\mathcal{G})$ for some $\{\eta_n\} \subseteq \mathcal{G}$ and $\{\lambda_n\} \subset \mathbb{C}$. If there exists $\{a_n\} \in \ell^2$ such that*

$$\frac{d}{dt}\left(\sum a_n e^{\lambda_n t}\eta_n\right) \in L^2([0,T];\mathcal{G})$$

for some $T > T_0$, then $\{a_n \lambda_n\} \in \ell^2$.

Proof. For simplicity, set $F(t) = \sum a_n e^{\lambda_n t}\eta_n$. We can find $h_1 > 0$ such that both

$$\left\| F'(t) - \frac{F(t+|h|) - F(t)}{|h|} \right\|^2_{L^2([0,T_0];\mathcal{G})} \leq 1 \quad \text{and} \quad \left| \frac{e^{ih} - 1}{h} \right|^2 > 1/2$$

in for all $|h| < h_1$ ($h \in \mathbb{C}$). Fix $N \in \mathbb{N}$. There exists $h_0 \in \mathbb{C}$, with $|h_0| < h_1$, such that $|\lambda_n h_0| < h_1$ for $|n| \leq N$. Then, letting c be the lower Riesz sequence constant for $\{e^{i\lambda_n t}\eta_n\}$,

$$\frac{c}{2}\sum_{|n|\leq N} |a_n\lambda_n|^2 \leq c \sum_{|n|\leq N} \left| a_n \frac{e^{\lambda_n |h_0|} - 1}{h_0} \right|^2 \leq \left\| \sum_{n\in\mathbb{Z}_0} a_n \frac{e^{\lambda_n |h_0|} - 1}{h_0} e^{\lambda_n t}\eta_n \right\|^2_{L^2([0,T_0];\mathcal{G})}$$

$$= \left\| \frac{F(t+|h_0|) - F(t)}{|h_0|} \right\|^2_{L^2([0,T_0];\mathcal{G})} \leq \|F'\|^2_{L^2([0,T_0];\mathcal{G})} + 1$$

but N is arbitrary. □

Finally, we include a result which allows us to conclude some orthoganality in the vectors $\{\eta_n\}$ from the properties of $\{e^{\lambda_n t}\eta_n\}$.

Lemma 1.2.11. *Let $\{\lambda_n\} \subset \mathbb{C}$ and $\{e^{\lambda_n t}\eta_n\}$ be a Bessel sequence in $L^2([0,T];\mathcal{G})$.*

There exists $C > 0$ such that for any $\varepsilon \in (0, T]$ and any finite sequence $\{a_n\} \subset \mathbb{C}$,

$$\left\|\sum a_n \eta_n\right\|_{\mathcal{G}}^2 \leq C\left(\varepsilon^{-1} \sum |a_n|^2 + \varepsilon \sum |\lambda_n a_n|^2\right) \tag{1.2.5}$$

Proof. Let $\{a_n\} \subset \mathbb{C}$ be a finite collection of scalars, $\varepsilon \in (0, T]$.

$$\begin{aligned}
\varepsilon \left\|\sum a_n \eta_n\right\|_{\mathcal{G}}^2 &= \int_0^\varepsilon \left\|\sum a_n e^{\lambda_n t} \eta_n - a_n (e^{\lambda_n t} - 1)\eta_n\right\|_{\mathcal{G}}^2 dt \\
&\leq 2\int_0^T \left\|\sum a_n e^{\lambda_n t} \eta_n\right\|^2 dt + 2\int_0^\varepsilon \left\|\sum a_n (e^{\lambda_n t} - 1)\eta_n\right\|^2 dt
\end{aligned}$$

The first term is bounded by $2C \sum |a_n|^2$ by the Bessel inequality. To estimate the second term, since $\int_0^t e^{\lambda_n s}(\lambda_n)\, ds = e^{\lambda_n t} - 1$,

$$\begin{aligned}
\int_0^\varepsilon \left\|\sum a_n (e^{\lambda_n t} - 1)\eta_n\right\|^2 dt &= \int_0^\varepsilon \left\|\int_0^t \sum e^{\lambda_n s}(\lambda_n)\, ds\, a_n \eta_n\right\|^2 dt \\
&\leq \int_0^\varepsilon \left(\int_0^t ds\right)\left(\int_0^t \left\|\sum e^{\lambda_n s}(\lambda_n a_n)\eta_n\right\|^2 ds\right) dt \\
&\leq \frac{\varepsilon^2}{2} \int_0^T \left\|\sum e^{\lambda_n s}(\lambda_n a_n)\eta_n\right\|^2 ds \\
&\leq \frac{C\varepsilon^2}{2} \sum |a_n \lambda_n|^2.
\end{aligned}$$

□

Chapter 2

Observability on Bounded Domains

2.1 Viscoelastic Wave Equation

Let $\Omega \subseteq \mathbb{R}^d$ for $d \geq 1$. We consider the viscoelastic wave equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = \displaystyle\int_0^t M(t-s)\Delta u(x,s)\,ds & \text{in } \Omega \times [0,T] \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \end{cases} \tag{2.1.1}$$

for $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $M \in H^2(0,T)$.

This model is also called the wave-memory equation, since the the system at present time t is influenced by the system at times $s < t$. For this reason, M is called the memory kernel. The usual wave equation ($M = 0$) comes from the elastic stress-strain relation $\sigma = c^2\varepsilon$ where σ is the stress, and ε is the strain. The form in this model comes from modifying this to $\sigma(t) = c^2\varepsilon(t) + \int_0^t M(t-s)\varepsilon(s)\,ds$.

The problem we are interested in is establishing the partial boundary observability inequality of the system (2.1.1), that is: There exists $c > 0$ such that for all $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$c \int_\Omega |\nabla w_0(x)|^2 + |w_1(x)|^2\,dx \leq \int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \nu}(x,t)\right|^2 dS(x)\,dt \tag{2.1.2}$$

where $\Gamma \subseteq \partial\Omega$. In this paper, we only consider Γ satisfying the following geometric condition: There exists $x_0 \in \mathbb{R}^d$ such that

$$\Gamma = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \geq 0\}. \tag{2.1.3}$$

The success in studying the equation (2.1.1) has been mostly limited to the case where the spatial dimension is one. This is largely due the fact that solutions can be approximated by sums of complex exponentials $\{e^{i\lambda_n t}\}$ which are very well-studied [2, 27, 29, 58]. The treatment in [1, 39] follows this approach using the moment method of D. L. Russell [54]. Recently, L. Pandolfi extended this result to $d \leq 3$ [47]. Herein, we complete these results by extending this method to an arbitrary space dimension.

Our main result concerning (2.1.1) is that the following harmonic system forms a Riesz sequence. We will prove below in Proposition 2.1.2 that this is equivalent to the observability inequality. Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ of eigenfunctions of the Dirichlet Laplacian. In other words,

$$\begin{cases} -\Delta\phi_n &= \lambda_n^2\phi_n \quad \text{in } \Omega; \\ \phi_n &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1.4}$$

It is well known that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_n \to \infty$. For simplicity, we set $\lambda_n = \text{sgn}(n)\lambda_{|n|}$ and

$$\psi_n = \frac{1}{\lambda_n}\frac{\partial\phi_{|n|}}{\partial\nu} \quad \text{on } \partial\Omega \tag{2.1.5}$$

for $n \in \mathbb{Z}\backslash\{0\}$ (henceforth $\mathbb{Z}_0$), denoting by $\nu(x)$ the outward normal vector to $\partial\Omega$ at x. To account for the time component of solutions to (2.1.1), we consider the following ordinary differential equation:

$$\begin{cases} z_n''(t) + \lambda_n^2 z_n(t) = -\lambda_n^2\displaystyle\int_0^t M(t-s)z_n(s)\,ds \qquad t \in [0,T]; \\ z_n(0) = 1, \qquad z_n'(0) = i\lambda_n. \end{cases} \tag{2.1.6}$$

We may now state the main result.

Theorem 2.1.1. *Let Ω be a smooth domain in $\mathbb{R}^d$ and Γ, x_0 be defined by (2.1.3). Let $R > 0$ such that $\Omega \subseteq B(x_0, R)$. Then, for $T > 2R$, there exists $C, c > 0$ such that*

$$c\sum |a_n|^2 \leq \int_0^T \int_{\partial\Omega} \left|\sum a_n z_n(t)\psi_n(x)\right|^2 dS(x)\,dt \leq C\sum |a_n|^2 \tag{2.1.7}$$

for all finite sequences of scalars $\{a_n\}$.

Since this model does not fit exactly into the moment method framework from Section (1.2), we first establish the equivalence between the Riesz sequence property and observability.

Proposition 2.1.2. *The observability inequality (2.1.2) holds for all* $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$ *if and only if* $\{z_n \psi_n\}_{n \in \mathbb{Z}_0}$*, defined by (2.1.25) and (2.1.6), is a Riesz-Fischer sequence in* $L^2(\Gamma \times [0,T])$*, i.e. there exists* $c > 0$ *such that*

$$c \sum |a_n|^2 \leq \int_0^T \int_\Gamma \left| \sum a_n z_n(t) \psi_n(x) \right|^2 \, dS(x) \, dt \tag{2.1.8}$$

for all finite sequences of scalars $\{a_n\}$*.*

Proof. Let $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$. We will represent the solution w to (2.1.1) by separation of variables. In the space variable, we expand onto $\{\phi_n\}$. There exist $\{\xi_n\}, \{\eta_n\} \in \ell^2$ such that

$$w_0 = \sum_{n=1}^{\infty} \xi_n \phi_n \quad \text{and} \quad w_1 = \sum_{n=1}^{\infty} \eta_n \phi_n.$$

Since $w_0 \in H_0^1(\Omega)$, by the orthonormality of $\{\phi_n\}$,

$$\int_\Omega |\nabla w_0(x)|^2 \, dx = -\int_\Omega w_0 \Delta w_0 \, dx = \int_\Omega \left(\sum \xi_n \phi_n \right) \left(\sum \lambda_n^2 \xi_n \phi_n \right) \, dx = \sum |\lambda_n \xi_n|^2,$$

therefore $\{\lambda_n \xi_n\} \in \ell^2$. Set $\tilde{\xi}_n = \lambda_n \xi_n$. Then,

$$w_0 = \sum \frac{\tilde{\xi}_n}{\lambda_n} \phi_n.$$

Additionally, we consider the ODE (2.1.6) to account for the time variable. It can then be verified that

$$w(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} \left[\left(\frac{\tilde{\xi}_n}{\lambda_n} - i \frac{\eta_n}{\lambda_n} \right) z_n(t) - \left(\frac{\tilde{\xi}_n}{\lambda_{-n}} + i \frac{\eta_n}{\lambda_{-n}} \right) z_{-n}(t) \right] \phi_n(x) \tag{2.1.9}$$

Then, setting $a_n = \operatorname{sgn}(n)(\tilde{\xi}_{|n|} - i\eta_{|n|})$ for $n \in \mathbb{Z}_0$, the observability inequality (2.1.2) takes the following form:

$$2c \sum_{n=1}^{\infty} |\tilde{\xi}_n|^2 + |\eta_n|^2 = c \sum_{\mathbb{Z}_0} |a_n|^2 \leq \int_0^T \int_\Gamma \left| \sum_{\mathbb{Z}_0} a_n z_n(t) \psi_n(x) \right|^2 dS(x)\, dt \tag{2.1.10}$$

□

Our approach is similar to [1, 39, 47] in the sense that we will argue that $\{z_n \psi_n\}$ is in a certain sense "close" to $\{e^{i\lambda_n t}\psi_n\}$ which is also a Riesz sequence (see Section 2.2). In [39], it is shown that there exists $C_1 > 0$ such that

$$\int_0^T |z_n(t) - e^{(\gamma + i\lambda_n)t}|^2 \, dt \leq \frac{C_1}{\lambda_n^2} \tag{2.1.11}$$

for $\gamma = M(0)/2$ in the special case where $\lambda_n = n$. However, there is no crucial role played by n in the computations so (2.1.11) can be easily verified with general λ_n—see the proof of Proposition 2.1.4, namely equation (2.1.14). The key in [39] is that when $\lambda_n = n$, $\{z_n\}$ and $\{e^{(\gamma+i\lambda_n)t}\}$ are quadratically close, which means

$$\sum_n \int_0^T |z_n(t) - e^{(\gamma+i\lambda_n)t}|^2 \, dt < \infty$$

In [47], the decay (2.1.11) is improved to λ_n^{-4} so quadratically closeness follows from Weyl's lemma when $d \leq 3$. We do not expect to be able to extend the quadratically close property to arbitrary dimensions. Rather, we incorporate the estimates on $\{\psi_n\}$ given below in Lemma 2.1.3 to show that $\{z_n\psi_n\}_{|n|\geq N}$ and $\{e^{(\gamma+i\lambda_n)t}\psi_n\}_{|n|\geq N}$ are equivalent bases for large enough N. We will then invoke the Riesz sequence perturbation results from Section 1.2. In this way, Theorem 2.1.1 will be established once we show three conditions hold:

(i) $\{e^{(\gamma+i\lambda_n)t}\psi_n\}$ is a Riesz sequence.

(ii) There exists $q \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\left\| \sum_{|n|\geq N} a_n \psi_n \left(z_n - e^{(\gamma+i\lambda_n)t} \right) \right\|^2 \leq q \left\| \sum_n a_n \psi_n e^{(\gamma+i\lambda_n)t} \right\|^2$$

for all finite sequences $\{a_n\}$ (Here and henceforth $\|\cdot\|$ denotes the $L^2(\Gamma \times [0,T])$ norm).

(iii) $\{z_n\psi_n\}$ is ℓ^2-independent.

Together (i) and (ii) will establish that $\{z_n\psi_n\}_{|n|\geq N}$ is a Riesz sequence which is then extended to the entire sequence if it is ℓ^2-independent (see Lemmas 1.2.6 and 1.2.8).

(i) is a consequence of the observability of the wave equation ((2.1.1) with $M \equiv 0$) as well as the corresponding upper regularity inequality, which are both well-known [36, 24, 32]. However, in the next section, we will prove this by showing the Riesz sequence property directly (see Theorem 2.2.1). This is then extended to $\{e^{(\gamma+i\lambda_n)t}\psi_n\}$ by noticing that

$$\begin{aligned}\max\{1, e^{\Re(\gamma)T}\} \left\|\sum a_n\psi_n e^{i\lambda_n t}\right\|^2 &\geq \left\|\sum a_n\psi_n e^{(\gamma+i\lambda_n)t}\right\|^2 \\ &\geq \min\{1, e^{\Re(\gamma)T}\} \left\|\sum a_n\psi_n e^{i\lambda_n t}\right\|^2.\end{aligned} \tag{2.1.12}$$

We now give the key lemma in establishing (ii).

Lemma 2.1.3. *Let $\{\psi_n\}$ be defined as in (2.1.25). Then there exists C_α dependent only the domain Ω such that for any finite sequence of scalars $\{a_n\}$,*

$$\int_{\partial\Omega} \left|\sum a_n\psi_n(x)\right|^2 dS(x) \leq C_\alpha \left(\sum |a_n|^2\right)^{1/2} \left(\sum |\lambda_n a_n|^2\right)^{1/2} \tag{2.1.13}$$

The estimate (2.1.13) may be viewed as stating some degree of orthogonality for $\{\psi_n\}$. In proving this, we follow the techniques in [4, 57].

Proof. Since Ω is smooth and bounded, there exists a smooth vector field α, defined on a neighborhood of $\overline{\Omega}$ such that

$$\alpha(x) \cdot \nu(x) \geq 1$$

for every $x \in \partial\Omega$. Define $V : H_0^1(\Omega) \to L^2(\Omega)$ by $(Vu)(x) = \alpha(x) \cdot \nabla u(x)$. First, since $u = 0$ on $\partial\Omega$, ∇u is a multiple of ν. This implies

$$Vu(x) = (\alpha \cdot \nu)\frac{\partial u}{\partial \nu}(x),\ \forall x \in \partial\Omega.$$

Next we claim that there exists $C_\alpha > 0$ such that

$$\left| \int_\Omega u[V, \Delta]\bar{u}\, dx \right| \leq C_\alpha \|\nabla u\|^2$$

for any $u \in H^3(\Omega) \cap H_0^1(\Omega)$. Indeed, using Einstein notation summing over $i, j = 1, 2, \ldots, d$

$$\begin{aligned}
\Delta V u &= \partial_{ii} \left(\alpha_j (\partial_j u) \right) \\
&= (\partial_{ii}\alpha_j)(\partial_j u) + 2(\partial_i \alpha_j)(\partial_{ij} u) + \alpha_j (\partial_{jii} u) \\
&= V\Delta u + (\partial_{ii}\alpha_j)(\partial_j u) + 2(\partial_i \alpha_j)(\partial_{ij} u)
\end{aligned}$$

Integrating by parts once and applying the Poincaré inequality yields

$$\begin{aligned}\left|\int_\Omega uV\Delta\bar{u} - u\Delta V\bar{u}\,dx\right| &= \left|\int_\Omega u(\partial_{ii}\alpha_j)(\partial_j\bar{u}) + 2u(\partial_i\alpha_j)(\partial_{ij}\bar{u})\,dx\right| \\ &= \left|\int_\Omega u(\partial_{ii}\alpha_j)(\partial_j\bar{u}) - 2\left[(\partial_i\bar{u})(\partial_i\alpha_j) + u(\partial_{ii}\alpha_j)\right](\partial_j\bar{u})\,dx\right| \\ &\leq C_\alpha \int_\Omega |\nabla u|^2\,dx\end{aligned}$$

Take $u = \sum a_n\phi_n\lambda_n^{-1}$ for a finite set of scalars $\{a_n\}$. Notice that $\|\nabla u\|^2 \leq 2\sum|a_n|^2$ and $\|\Delta u\|^2 \leq 2\sum|\lambda_n a_n|^2$ (the factor of 2 is due to the negative indices). Then, using Cauchy-Schwartz and the above estimates on V, we have

$$\begin{aligned}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 dS \leq \int_{\partial\Omega}(\alpha\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 dS &= \int_{\partial\Omega}\frac{\partial u}{\partial\nu}V\bar{u}\,dS \\ &= \int_\Omega \Delta uV\bar{u} - u\Delta V\bar{u}\,dx \\ &= \int_\Omega \Delta uV\bar{u} - uV\Delta\bar{u} + u[V,\Delta]\bar{u}\,dx \\ &= \int_\Omega \Delta uV\bar{u} + (\nabla\cdot\alpha)u\Delta\bar{u} + Vu\Delta\bar{u} + u[V,\Delta]\bar{u}\,dx \\ &\leq C_\alpha\left(\sum|a_n|^2\right)^{1/2}\left(\sum|\lambda_n a_n|^2\right)^{1/2}.\end{aligned}$$

□

Proof of (ii). Let $c_\gamma = (T-2R)\min\{1, e^{\Re(\gamma)T}\}/C_\Gamma$ be the constant from the lower Riesz sequence inequality (2.1.12) for $\{e^{(\gamma+i\lambda_n)t}\psi_n\}$. Since $\lambda_n\to\infty$, there exists

$N \in \mathbb{N}$ such that

$$\frac{c_\gamma^{-1} C_\alpha C_1}{\lambda_N} < 1.$$

Applying Lemma 2.1.3 and then the estimate (2.1.11), we have

$$\begin{aligned}
\int_0^T \int_\Gamma \left| \sum_{|n|\geq N} a_n \psi_n(x) \left(z_n(t) - e^{(\gamma+i\lambda_n)t} \right) \right|^2 &\leq \int_0^T \int_{\partial\Omega} \left| \sum_{|n|\geq N} a_n \psi_n(x) \left(z_n(t) - e^{(\gamma+i\lambda_n)t} \right) \right|^2 \\
&\leq C_\alpha \left(\int_0^T \sum_{|n|\geq N} |a_n (z_n - e^{(\gamma+i\lambda_n)t})|^2 \right)^{1/2} \left(\int_0^T \sum_{|n|\geq N} |\lambda_n a_n (z_n - e^{(\gamma+i\lambda_n)t})|^2 \right)^{1/2} \\
&\leq C_\alpha C_1 \lambda_N^{-1} \sum |a_n|^2 \\
&\leq C_\alpha C_1 c_\gamma^{-1} \lambda_N^{-1} \left\| \sum a_n e^{(\gamma+i\lambda_n)t} \psi_n \right\|^2 .
\end{aligned}$$

□

Establishing the ℓ^2-independence of $\{z_n \psi_n\}$ is the most computationally intensive part of the proof. The general strategy follows [1, 39] with adjustments to account for the additional vectors $\{\psi_n\}$.

Proposition 2.1.4. *For $T > 2R$, the sequence $\{z_n \psi_n\}$ defined by (2.1.25) and (2.1.6) is ℓ^2-independent in $L^2(\Gamma \times [0,T])$, i.e. for any $\{a_n\} \in \ell^2$ s.t. $\sum a_n z_n \psi_n \Big|_{\Gamma \times [0,T]} = 0$, $a_n = 0$ for all n.*

Proof. Set $e_n(t) = z_n(t) - e^{(\gamma+i\lambda_n)t}$. Then,

$$e_n(t) = \frac{\gamma}{\gamma + i2\lambda_n} \left[e^{(\gamma+i\lambda_n)t} - e^{-i\lambda_n t} \right] + \int_0^t K_n(t-s) e_n(s)\, ds + b_n(t) \qquad (2.1.14)$$

where

$$K_n(t) = \int_0^t M'(t-s)\cos(\lambda_n s)\,ds - M(t) + 2\gamma\cos(\lambda_n t),$$

$$b_n(t) = \int_0^t \cos(\lambda_n(t-s)) \int_0^s M'(s-r)e^{(\gamma+i\lambda_n)r}\,dr\,ds - \int_0^t M(t-s)e^{(\gamma+i\lambda_n)s}\,ds.$$

These computations are carried out rigorously in [39] so we do not reproduce them here. Integrating b_n by parts and applying the Gronwall Inequality, $|e_n(t)| \le C\lambda_n^{-1}$, thus establishing (2.1.11).

Now, take $\{a_n\} \in \ell^2$ such that $\sum a_n z_n \psi_n = 0$. Convergence is understood in the $L^2(\Gamma \times [0,T])$ norm. This implies

$$\sum a_n e^{(\gamma+i\lambda_n)t}\psi_n = -\sum a_n e_n \psi_n \quad \text{in } L^2(\Gamma \times [0,T])$$

We claim that

$$\frac{d}{dt}\left(\sum a_n e^{(\gamma+i\lambda_n)t}\psi_n\right) = -\sum a_n e_n' \psi_n.$$

This will be immediate once it is shown that the RHS converges since the derivative is a closed operator. It suffices to show that $\{e_n'\psi_n\}$ is a Bessel sequence (def. 1.2.2). We compute e_n' explicitly from (2.1.14) by

$$e_n'(t) = \frac{\gamma}{\gamma + 2i\lambda_n}\left[(\gamma + i\lambda_n)e^{(\gamma+i\lambda_n)t} + i\lambda_n e^{-i\lambda_n t}\right] + \int_0^t K_n'(t-s)e_n(s)\,ds + b_n'(t)$$

since $K_n(0) = 0$. Noting that

$$K_n'(t) = M'(0) + \int_0^t M''(t-s)\cos(\lambda_n s)\,ds - M'(t) - 2\gamma\lambda_n \sin(\lambda_n t),$$

we see that

$$\begin{aligned}\int_0^t K_n'(t-s)e_n(s)\,ds &= O(\lambda_n^{-1}) - 2\gamma\lambda_n \int_0^t \sin\lambda_n(t-s)e_n(s)\,ds \\ &= O(\lambda_n^{-1}) - 2\gamma e_n(t) + 2\gamma \int_0^t \cos\lambda_n(t-s)e_n'(s)\,ds \\ &= O(\lambda_n^{-1}) + 2\gamma \int_0^t \cos\lambda_n(t-s)e_n'(s)\,ds\end{aligned}$$

since the first three terms of K_n' are bounded, $e_n = O(\lambda_n^{-1})$, and $e_n(0) = 0$. We also compute

$$\begin{aligned}b_n'(t) = &\int_0^t M'(t-r)e^{(\gamma+i\lambda_n)r}\,dr - \lambda_n \int_0^t \sin\lambda_n(t-s)\int_0^s M'(s-r)e^{(\gamma+i\lambda_n)r}\,dr\,ds \\ &- M(0)e^{(\gamma+i\lambda)t} - \int_0^t M'(t-s)e^{(\gamma+i\lambda_n)s}\,ds \\ = &-\int_0^t M'(t-s)e^{(\gamma+i\lambda_n)s}\,ds + \int_0^t \cos\lambda_n(t-s)\int_0^s M''(s-r)e^{(\gamma+i\lambda_n)r}\,dr\,ds \\ &+ M'(0)\int_0^t \cos\lambda_n(t-s)e^{(\gamma+i\lambda_n)s}\,ds - 2\gamma e^{(\gamma+i\lambda_n)t} \\ = &\frac{e^{(\gamma+i\lambda_n)t}}{2\gamma}\left(M'(t) - 2\gamma^2\right) - \frac{e^{i\lambda_n t}}{2\gamma}\left(M'(0) + \int_0^t M''(s)e^{\gamma s}\,ds\right) + O(\lambda_n^{-1}).\end{aligned}$$

If $\gamma = 0$ then the terms containing γ^{-1} will not appear ($e^{(\gamma+i\lambda_n)t} - e^{i\lambda_n t} = 0$). Therefore

$$e_n'(t) = 2\gamma \int_0^t \cos \lambda_n (t-s) e_n'(s)\, ds + D_{1,n}(t) e^{i\lambda_n t} + D_{2,n} e^{-i\lambda_n t} + O(\lambda_n^{-1}) \tag{2.1.15}$$

where

$$D_{1,n}(t) = \left(\frac{\gamma(\gamma + i\lambda_n)}{\gamma + 2i\lambda_n} + \frac{M'(t)}{2\gamma} - \gamma \right) e^{\gamma t} - \frac{1}{2\gamma} \left(M'(0) + \int_0^t M''(s) e^{\gamma s}\, ds \right) \tag{2.1.16}$$

$$\text{and} \quad D_{2,n} = \frac{i\lambda_n \gamma}{\gamma + 2i\lambda_n}. \tag{2.1.17}$$

Notice if $\gamma = 0$, $\{e_n' \psi_n\}$ is a Bessel sequence by Theorem 2.2.1 and Lemma 2.1.3 so we may skip to (2.1.21). Otherwise, we note that the Volterra Equation

$$u(t) = 2\gamma \int_0^t \cos \lambda_n (t-s) u(s)\, ds + v(t)$$

has the unique solution

$$u(t) = v(t) + \int_0^t R_n(t-s) v(s)\, ds$$

where

$$R_n(t) = 2\gamma e^{\gamma t}\left(\cosh \mu_n t + \frac{\gamma}{\mu_n}\sinh \mu_n t\right)$$
$$= \gamma\left(1+\frac{\gamma}{\mu_n}\right)e^{(\gamma+\mu_n)t} + \gamma\left(1+\frac{\gamma}{\mu_n}\right)e^{(\gamma-\mu_n)t} \tag{2.1.18}$$

and $\mu_n = \sqrt{\gamma^2 - \lambda_n^2}$ for $\lambda_n \neq \pm\gamma$. We do not consider the case $\lambda_n = \pm\gamma$ since this only constitutes finitely many elements in the sequence and plays no role in the convergence. This allows us to rewrite (2.1.15) as

$$e_n'(t) = D_{1,n}(t)e^{i\lambda_n t} + D_{2,n}e^{-i\lambda_n t} + O(\lambda_n^{-1}) \tag{2.1.19}$$
$$+ \int_0^t R_n(t-s)[D_{1,n}(s)e^{i\lambda_n s} + D_{2,n}e^{-i\lambda_n s} + O(\lambda_n^{-1})]\, ds.$$

$\{e_n'\psi_n\}$ will be a Bessel sequence if each term on the RHS is when multiplied by ψ_n. The first two terms are since $\{e^{i\lambda_n t}\psi_n\}$ is a Riesz sequence and the $O(\lambda_n^{-1})$ terms are by Lemma 2.1.3 and the fact that $\Re(\mu_n)$ is bounded. So it only remains to show the integral terms with $D_{1,n}, D_{2,n}$ are. We will only estimate the integral term with $D_{1,n}$. $D_{2,n}$ is handled similarly. Notice that $D_{1,n}$ (2.1.16) can be written as

$$D_{1,n}(t) = c_n e^{\gamma t} + D(t).$$

Then, for $\{a_n\} \subset \mathbb{C}$,

$$\sum a_n \psi_n \int_0^t R_n(t-s) D_{1,n}(s) e^{i\lambda_n s}\, ds$$
$$= \sum c_n a_n \psi_n \int_0^t R_n(t-s) e^{(\gamma + i\lambda_n)s}\, ds$$
$$+ \sum a_n \psi_n \int_0^t R_n(t-s) D(s) e^{i\lambda_n s}\, ds.$$

We now consider the final sum in four pieces (the other sum is simpler and can be treated analogously noting that $\{c_n\}$ is bounded):

$$\sum a_n \psi_n \int_0^t R_n(t-s) D(s) e^{i\lambda_n s}\, ds$$
$$= \sum_{n>0} + \sum_{n<0} a_n \psi_n \gamma \left(1 + \frac{\gamma}{\mu_n}\right) e^{(\gamma+\mu_n)t} \int_0^t D(s) e^{(i\lambda_n - \gamma - \mu_n)s}\, ds$$
$$+ \sum_{n>0} + \sum_{n<0} a_n \psi_n \gamma \left(1 - \frac{\gamma}{\mu_n}\right) e^{(\gamma-\mu_n)t} \int_0^t D(s) e^{(i\lambda_n - \gamma + \mu_n)s}\, ds \tag{2.1.20}$$
$$=: S_1 + S_2 + S_3 + S_4.$$

It can be checked that $\sup_{n>0} |\gamma + \mu_n - i\lambda_n| = \sup_{n<0} |\gamma - \mu_n - i\lambda_n| < \infty$. Therefore, $\{e^{\gamma + \mu_n t} \psi_n\}_{n>0}$ and $\{e^{\gamma - \mu_n t} \psi_n\}_{n<0}$ are Bessel sequences by Lemma 1.2.9. Then, for

$\delta_n = (i\lambda_n - \gamma - \mu_n)$, $r = \sup_{n>0} |\delta_n|$,

$$\left(\int_0^T \left\| \sum_{n>0} a_n \psi_n e^{(\gamma+\mu_n)t} \int_0^t D(s) e^{(i\lambda_n - \gamma - \mu_n)s} \, ds \right\|^2 dt \right)^{1/2}$$

$$= \left(\int_0^T \left\| \sum_{n>0} a_n \psi_n e^{(\gamma+\mu_n)t} \int_0^t D(s) \sum_{k=0}^{\infty} \frac{\delta_n^k s^k}{k!} \, ds \right\|^2 dt \right)^{1/2}$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_0^T \left\| \sum_{n>0} a_n \delta_n^k \psi_n e^{(\gamma+\mu_n)t} \int_0^t D(s) s^k \, ds \right\|^2 dt \right)^{1/2}$$

$$\leq C_D \sum_{k=0}^{\infty} \frac{T^k}{k!} \left(\int_0^T \left\| \sum_{n>0} a_n \delta_n^k \psi_n e^{(\gamma+\mu_n)t} \right\|^2 dt \right)^{1/2}$$

$$\leq C_D \sum_{k=0}^{\infty} \frac{T^k}{k!} \left(\sum |a_n \delta_n^k|^2 \right)^{1/2}$$

$$\leq C_D e^{rT} \left(\sum |a_n|^2 \right)^{1/2}.$$

Therefore S_1 (2.1.20) converges and by similar reasoning S_4. To deal with S_2 and S_3, we simply integrate by parts to pick up a factor of $(i\lambda_n - \gamma \pm \mu_n)^{-1}$. Then, since $i\lambda_n - \gamma + \mu_n = O(\lambda_n)$ for $n > 0$ and $i\lambda_n - \gamma - \mu_n = O(\lambda_n)$ for $n < 0$, Lemma 2.1.3 guarantees the convergence of S_2 and S_3.

We now have that $\{e_n' \psi_n\}$ is a Bessel sequence so $\sum a_n e_n' \psi_n$ converges and

$$\sum a_n e_n' \psi_n = \frac{d}{dt} \left(\sum a_n e^{(\gamma + i\lambda_n)t} \psi_n \right) \quad \text{in } L^2(\Gamma \times [0,T]). \tag{2.1.21}$$

Using Lemma 1.2.10, we obtain that $\{a_n \lambda_n\} \in \ell^2$. This follows since $\{e^{(\gamma+i\lambda_n)t}\psi_n\}$ forms a Riesz sequence and $\sum a_n e^{i(\gamma+\lambda_n)t}\psi_n$ has one derivative in time. Thus, $\delta_n :=$

$a_n(\gamma + i\lambda_n) \in \ell^2$.

This process can be repeated since $e_n''(t)$ can be computed from (2.1.19) picking up a factor of λ_n at most. Then, $\{\lambda_n^{-1} e_n'' \psi_n\}$ is a Bessel sequence and we argue as above to obtain

$$\sigma_n := a_n(\gamma + i\lambda_n)^2 \in \ell^2.$$

This now shows that $\sum a_n z_n'' \psi_n$ converges since

$$\sum a_n z_n''(t)\psi_n = \sum \sigma_n e^{(\gamma+i\lambda_n)t}\psi_n + \sum \frac{\delta_n}{\gamma + i\lambda_n} e_n''(t)\psi_n.$$

Now, for simplicity, set $\Psi_n = \sum_{\lambda_m=\lambda_n} a_m \psi_m$ for each $n \in \Lambda := \{-n, n : n \in \mathbb{N}, \lambda_n < \lambda_{n+1}\}$ (i.e. the set of distinct eigenvalues of $-\Delta$). We now claim that the tail $\{a_n\}_{|n|\geq N}$ must be zero. Using (2.1.6),

$$0 = \sum_{\mathbb{Z}_0} a_n z_n''(t)\psi_n = \sum_{\Lambda} \Psi_n z_n''(t) = -\sum_{\Lambda} \lambda_n^2 \Psi_n z_n(t) - \int_0^t M(t-s) \sum_{\Lambda} \lambda_n^2 \Psi_n z_n(t).$$

By standard theory of Volterra integral equations, this implies $\sum \lambda_n^2 \Psi_n z_n(t) = 0$. Now, for each $n \in \Lambda$, set $\Psi_n^{(1)} = (\lambda_1^2 - \lambda_n^2)\Psi_n$. Then, notice that $\Psi_n^{(1)}$ has the following properties:

(a) $\sum \Psi_n^{(1)} z_n = \lambda_1^2 \sum \Psi_n z_n(t) - \sum \lambda_n^2 \Psi_n z_n(t) = 0$.

(b) $\Psi_1^{(1)} = \Psi_{-1}^{(1)} = 0$ but for $|n| > 1$, $\Psi_n^{(1)} = 0 \iff \Psi_n = 0$.

This can be repeated for $m \in \Lambda$, $2 \leq m < N$ by setting $\Psi_n^{(m)} = (\lambda_m^2 - \lambda_n^2)\Psi_n^{(m-1)}$ (Here

$m-1$ means the index in Λ immediately preceding m). Thus, we have constructed

$$\sum_{|n|\geq N} b_n z_n \psi_n = \sum_{\{|n|\geq N\}\cap\Lambda} \Psi_n^{(N-1)} z_n = 0 \quad \text{with} \quad b_n = a_n \prod_{\substack{1\leq k<N,\\ k\in\Lambda}} (\lambda_k^2 - \lambda_n^2).$$

But the subsequence $\{z_n\psi_n\}_{|n|\geq N}$ is a Riesz sequence by (ii) so $b_n = 0$ which implies $a_n = 0$ for $|n| \geq N$. Now we only need to deal with the finite sum

$$\sum_{\{|n|\leq N\}\cap\Lambda} \Psi_n z_n = 0. \tag{2.1.22}$$

In other words we need to show $\{z_n\}_{|n|\leq N\cap\Lambda}$ is linearly independent. If it is not, then there is a smallest linearly dependent subset, indexed by $\{n_k\}_{k=1}^M$, $M \geq 2$, and suitable $\{c_{n_k}\}$ (non-zero) such that

$$\sum_{k=1}^M c_{n_k} z_{n_k}(t) = 0.$$

Then,

$$0 = \sum_{k=1}^M c_{n_k} z''_{n_k}(t) = \sum_{k=1}^M -\lambda_{n_k}^2 c_{n_k} z_{n_k}(t) - \int_0^t M(t-s) \sum_{k=1}^M \lambda_{n_k}^2 c_{n_k} z_{n_k}(s)\, ds$$

so

$$\sum_{k=1}^M \lambda_{n_k}^2 c_{n_k} z_{n_k}(t) = 0.$$

Therefore we have found a smaller linearly dependent collection, namely

$$\sum_{k=1}^{M} (\lambda_{n_M}^2 - \lambda_{n_k}^2) c_{n_k} z_{n_k}(t) = 0$$

where one or at most two of the new coefficients are zero (two only if λ_{n_M} and λ_{-n_M} are in the collection). So, we only need to check that $c_1 z_n + c_2 z_{-n} = 0$ implies $c_1, c_2 = 0$. This follows simply from $z_n(0) = z_{-n}(0)$ but $z_n'(0) = -z_{-n}'(0)$ (see (2.1.6)). Thus $\{z_n\}$ is linearly independent for distinct λ_n. Therefore, (2.1.22) implies that $\Psi_n = 0$. Using Lemma 2.2.2 (below), for each $n \in \Lambda$,

$$0 = \int_\Gamma |\Psi_n(x)|^2 (x - x_0) \cdot \nu(x)\, dS(x) \geq \int_{\partial\Omega} |\Psi_n(x)|^2 (x - x_0) \cdot \nu(x)\, dS(x)$$

$$= \sum_{\lambda_m = \lambda_n} \sum_{\lambda_\ell = \lambda_n} \int_{\partial\Omega} a_m \psi_m(x) \overline{a_\ell \psi_\ell(x)} (x - x_0) \cdot \nu\, dS(x) = 2 \sum_{\lambda_m = \lambda_n} |a_m|^2 \geq 0,$$

so $a_m = 0$ for all m. □

2.1.1 Abstract Viscoelastic System

This strategy is applied in a more general situation in [19]. Let $\mathcal{G}$, $\mathcal{H}$ be Hilbert spaces and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint elliptic operator satisfying appropriate assumptions (A) below. We consider the following viscoelastic system for $w : [0,T] \to \mathcal{H}$:

$$\begin{cases} w''(t) + \mathcal{A}w(t) = \displaystyle\int_0^t M(t-s)\mathcal{A}w(s)\,ds \quad t \in [0,T] \\ \\ w(0) = w_0 \quad w'(0) = w_1 \end{cases} \tag{2.1.23}$$

with the memory kernel $M \in H^2(0,T)$, and w_0, w_1 being the initial conditions. In this abstract setup, the boundary conditions will be contained in $\mathcal{D}(\mathcal{A})$. We also introduce the observation operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{H} \to \mathcal{G}$. We impose the following assumption on $\mathcal{A}$ and $\mathcal{B}$:

(A) Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be self-adjoint, closed with dense range, having compact resolvent and *semibounded*, i.e.,

$$\langle \mathcal{A}u, u\rangle \geq -c\|u\|^2,$$

for some $c > 0$ and all $u \in \mathcal{D}(\mathcal{A})$. Denote by $\mathcal{H}^1$ the completion of $\mathcal{D}(\mathcal{A})$ with respect to the norm $\|x\|_1^2 := \|x\|^2 + \||\mathcal{A}|^{1/2}x\|^2$.

(B) Let $\mathcal{B} : \mathcal{D}(\mathcal{B}) \to \mathcal{G}$ be closed with dense range that satisfies the *observability-regularity* inequality: There exists $T_0 \geq 0$ such that for any $T > T_0$, there exists $C > 0$ such that for w satisfying (2.1.23) with $M = 0$,

$$C^{-1}\|(w_0, w_1)\|_{\mathcal{H}^1\times\mathcal{H}} \leq \|\mathcal{B}w\|_{L^2([0,T];\mathcal{G})} \leq C\|(w_0, w_1)\|_{\mathcal{H}^1\times\mathcal{H}} \tag{2.1.24}$$

for all $(w_0, w_1) \in \mathcal{H}^1 \times \mathcal{H}$.

Under these assumptions, we have observability of the viscoelastic system.

Theorem 2.1.5. *Assume conditions (A) and (B) are satisfied. Let $M \in H^2(0,T)$. Then, for any $T > T_0$, there exists $C > 0$ such that (2.1.24) holds for any $(w_0, w_1) \in \mathcal{H}^1 \times \mathcal{H}$.*

There are a few modifications to make to the arguments above. First, we must account for the presence of a non-positive eigenvalues, but the condition (A) only allows for finitely many of them, so they can be neglected in the proofs of (ii) and (iii), with the exception of checking that $\{z_n\}$ remains linearly independent.

By condition (A), $\mathcal{A}$ has an orthonormal basis of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ with eigenvalues $\{\mu_n\}_{n=1}^{\infty} \subset \mathbb{R}$, each of finite multiplicity with $\mu_n \to \infty$. Set $\lambda_n = \operatorname{sgn}(n)\sqrt{\mu_{|n|}}$ for each $n \in \mathbb{Z}_0$. We divide $\{\lambda_n\}_{n \in \mathbb{Z}_0}$ into two classes, indexed by

$$J_0 = \{n \in \mathbb{Z}_0 : \lambda_n = 0\}, \quad J_1 = \mathbb{Z}_0 \backslash J_0.$$

Then, define

$$\psi_n = \begin{cases} \operatorname{sgn}(n)\mathcal{B}\phi_{|n|} & \text{for } n \in J_0, \\ \dfrac{\mathcal{B}\phi_{|n|}}{\lambda_n} & \text{for } n \in J_1. \end{cases} \tag{2.1.25}$$

Define z_n as above, except when $\lambda_n = 0$, $z_n(t) = 1 + i\operatorname{sgn}(n)t$.

By the same proof as Proposition 2.1.2, the inequality (2.1.24) is equivalent to $\{e^{i\lambda_n t}\psi_n\}$ forming a Riesz sequence. From this we can recover the relevant properties of $\{\psi_n\}$. First, we replace Lemma 2.1.3 with Lemma 2.1.13. This is enough to establish (ii). Second, at the end of the proof of the ℓ^2-independence, we used the fact that for each $m \in \mathbb{Z}_0$,

$$\sum_{\lambda_n=\lambda_m} a_n \psi_n = 0$$

implies $a_n = 0$. This follows from the lower Riesz sequence inequality for $\{e^{i\lambda_n t}\psi_n\}$. In fact, we obtain

$$T \max\{1, e^{\Im(\lambda_n)}\} \left\| \sum_{\lambda_n=\lambda_m} a_n \psi_n \right\|^2 \geq \int_0^T \left\| \sum_{\lambda_m=\lambda_n} a_m e^{i\lambda_m t} \psi_m \right\|^2 dt \geq c \sum_{\lambda_n=\lambda_m} |a_m|^2.$$

We briefly summarize two cases in which the conditions (A) and (B) are both satisfied (therefore our Theorems 2.1.5, 2.1.7, and 2.1.8 apply). First, the Dirichlet viscoelastic wave equation defined on an open bounded domain Ω with smooth boundary where $\mathcal{A}$ is a self-adjoint elliptic operator with a bounded potential. Taking $\mathcal{B}$ as the Neumann trace on a suitable portion of boundary $\partial\Omega$ (see [3] for sharp conditions), the condition (B) is well known to be satisfied, see for example [32, 36, 59], with $\mathcal{H} = L^2(\Omega)$.

Another case to which our result applies is the viscoelastic plate equation where $\mathcal{A} = \Delta^2$ with Dirichlet boundary condition. It has been considered in [31] under a smallness assumption on the memory kernel and in [48] in dimension two. The observability-regularity inequality (2.1.24) can be found in [33, Remark 1.3] when $\mathcal{B}$ is the third-order boundary trace and $\mathcal{H} = H_0^1(\Omega)$. As shown in [36, 60], (2.1.24) still holds for the second-order boundary trace with $\mathcal{H} = L^2(\Omega)$. Moreover, in both cases, $T_0 = 0$ so the viscoelastic plate equation we consider is still observable in arbitrary time $T > 0$.

A point of interest is the *Neumann viscoelastic* control and observation problem (e.g., take $\mathcal{A}$ to be the Neumann Laplacian). To the best of our knowledge, this has not been studied in the literature, and it would be a consequence of our Theorem 2.1.5 below, except that it is not known if there are suitable spaces $\mathcal{H}$, $\mathcal{G}$ and operator $\mathcal{B}$ satisfying the condition (B). For the natural choice of $\mathcal{B}$ as the Dirichlet trace in the case of wave equations, the closest to (2.1.24) to our best knowledge is Theorem 2.1.1 in [35] for the lower inequality and Theorem 1.1 in [34] for the upper inequality.

2.1.2 Application to Inverse Source Problem

As a consequence of the controllability result, we study the reconstruction and stability of an unknown source $f \in \mathcal{H}$ from the observed data $\mathcal{B}u \in \mathcal{G}$ in the following system: Let $u : [0, T] \to \mathcal{H}$ satisfy

$$\begin{cases} u''(t) + \mathcal{A}u(t) = \displaystyle\int_0^t M(t-s)\mathcal{A}u(s)\,ds + \sigma(t)f, \quad \text{for } t \text{ in } [0,T]; \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{2.1.26}$$

Solving an inverse problem through the observability/controllability of the underlying system is a well established technique and has produced various methods in inverse problems. In particular, the celebrated Boundary Control method pioneered by Belishev [5] which deals with the so called many measurements formulation [26]. For our inverse problem with a single measurement formulation, we refer to [37] and references therein.

Inverse source problems for partial differential equations have also been studied extensively in the literature [6, 26]. For the viscoelastic inverse problem considered here, [12] and [40] studied more general viscoelastic equations and showed similar stability estimates by means of Carleman estimates. However, their method does not produce the reconstruction formula as we have in Theorem 2.1.8.

First we give the relationship between the systems (2.1.23) and (2.1.26).

Lemma 2.1.6. *Let w satisfy (2.1.23) with $w_0 = 0$, $w_1 = f \in \mathcal{H}$. Then*

$$u(t) = \int_0^t \sigma(t-s)w(s)\,ds \tag{2.1.27}$$

satisfies (2.1.26).

Proof. First notice that for any $v \in C^1(0,T)$, integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_0^t \sigma(t-s)v(s)\,ds &= \int_0^t \sigma'(t-s)v(s)\,ds + \sigma(0)v(t) \qquad (2.1.28)\\ &= -\sigma(t-s)v(s)\Big|_{s=0}^{s=t} + \int_0^t \sigma(t-s)v'(s)\,ds + \sigma(0)v(t)\\ &= \sigma(t)v(0) + \int_0^t \sigma(t-s)v'(s)\,ds. \qquad (2.1.29)\end{aligned}$$

Applying this to (2.1.27), u satisfies the homogeneous initial conditions for (2.1.26) since $w(0) = 0$. Differentiating (2.1.27) with respect to t and applying (2.1.29) twice,

$$\begin{aligned} u''(t) &= \sigma'(t)w(0) + \sigma(t)w'(0) + \int_0^t \sigma(t-s)w''(s)\,ds \\ &= \sigma(t)f + \int_0^t \sigma(t-s)w''(s)\,ds \end{aligned}$$

where we have used the fact that $w(0) = 0$ and $w'(0) = f$. Next, we claim that

$$\begin{aligned} \mathcal{A}u(t) + \int_0^t M(t-s)\mathcal{A}u(s)\,ds &= \int_0^t \sigma(t-s)\mathcal{A}w(s)\,ds \\ &\quad + \int_0^t \int_0^s M(t-s)\sigma(s-r)\mathcal{A}w(r)\,dr\,ds \\ &= \int_0^t \sigma(t-s)\left(\mathcal{A}w(s) + \int_0^s M(s-r)\mathcal{A}w(r)\,dr\right)\,ds. \end{aligned}$$

If this holds, then the lemma is proved. We only need to confirm the last step, establishing that the convolutions commute. Indeed,

$$\int_0^t \int_0^s M(t-s)\sigma(s-r)v(r)\,dr\,ds = \int_0^t \int_r^t M(t-s)\sigma(s-r)\,ds\,v(r)\,dr$$

$$\begin{aligned} = \int_0^t \int_r^t M(\tau-r)\sigma(t-\tau)\,d\tau\,v(r)\,dr &= \int_0^t \int_0^\tau M(\tau-r)\sigma(t-\tau)v(r)\,dr\,d\tau \\ &= \int_0^t \sigma(t-\tau)\int_0^\tau M(\tau-r)v(r)\,dr\,d\tau \end{aligned}$$

for any $v \in C(0,T)$. □

The stability estimate is a simple consequence of this lemma.

Theorem 2.1.7. *Assume conditions (A) and (B) are satisfied, $M \in H^2(0,T)$, $\sigma \in C^1[0,T]$ with $\sigma(0) \neq 0$, and $T > T_0$. Then there exists $C > 0$ such that for any $f \in \mathcal{H}$, u satisfying (2.1.26),*

$$C^{-1}\|f\|_{\mathcal{H}} \leq \|\mathcal{B}u\|_{H^1([0,T];\mathcal{G})} \leq C\|f\|_{\mathcal{H}}. \tag{2.1.30}$$

Proof. As a consequence of Theorem 3.2.1, with $w_0 = 0$, and $w_1 = f$,

$$\|f\|_{\mathcal{H}} \asymp \|\mathcal{B}w\|_{L^2([0,T];\mathcal{G})}. \tag{2.1.31}$$

Then, in light of Lemma 4.2.5,

$$u'(x,t) = \sigma(0)w(t) + \int_0^t \sigma'(t-s)w(s)\,ds. \tag{2.1.32}$$

We first prove the lower inequality in (2.1.30). By standard theory of Volterra equations [55], there exists $K \in C[0,T]$ (which we will henceforth call the resolvent kernel of $\sigma'/\sigma(0)$) such that

$$\sigma(0)w(t) = u'(t) + \int_0^t K(t-s)u'(s)\,ds. \tag{2.1.33}$$

Note that for any $\rho \in C[0,1], v \in L^2(0,T)$,

$$\int_0^T \left| \int_0^t \rho(t-s)v(s)\,ds \right|^2 dt \leq \int_0^T \int_0^t |\rho(t-r)|^2\,dr \int_0^t |v(s)|^2\,ds\,dt$$
$$\leq \frac{T^2\|\rho\|_\infty^2}{2} \int_0^T |v(s)|^2\,ds.$$

Applying this to (2.1.32) and (2.1.33), we obtain

$$\|\mathcal{B}w\|_{L^2([0,T];\mathcal{G})} \asymp \|\mathcal{B}u\|_{H^1([0,T];\mathcal{G})} \tag{2.1.34}$$

Applying (3.2.5) proves the theorem. □

The other component of the inverse problem is to give a reconstruction formula for f, from the observation $\mathcal{B}u$.

Theorem 2.1.8. *Under the assumptions of Theorem 2.1.7, there exists* $\{\theta_n\} \subset L^2([0,T];\mathcal{G})$ *such that*

$$f = \sum_{n=1}^{\infty} \phi_n \left\langle \mathcal{B}u', \theta_n \right\rangle_{L^2([0,T];\mathcal{G})}$$

for u *satisfying (2.1.26).*

Proof. First, since $\{z_n\psi_n\}_{n\in\mathbb{Z}_0}$ is Riesz sequence in $L^2([0,T];\mathcal{G})$, setting $w_n = \frac{z_n - z_{-n}}{2i}$ for $n \in \mathbb{N}$, $\{w_n\psi_n\}_{n\in\mathbb{N}}$ is still a Riesz sequence. Indeed, for a finite sequence $\{a_n\}_{n\in\mathbb{N}} \subset$

$\mathbb{C}$,

$$
\begin{aligned}
2\sum_{n=1}^{\infty}|a_n|^2 &= \sum_{n\in\mathbb{Z}_0}|a_{|n|}|^2 \\
&\asymp \left\|\sum_{n\in\mathbb{Z}_0} a_{|n|}z_n\psi_n\right\|^2 \\
&= \left\|\sum_{n=1}^{\infty} a_n z_n\psi_n + \sum_{n=1}^{\infty} a_n z_{-n}\psi_{-n}\right\|^2 \\
&= \left\|\sum_{n=1}^{\infty} a_n z_n\psi_n - \sum_{n=1}^{\infty} a_n z_{-n}\psi_n\right\|^2 \\
&= \left\|\sum_{n=1}^{\infty} 2ia_n w_n\psi_n\right\|^2 \qquad (2.1.35)
\end{aligned}
$$

By the formula for z_n (2.1.6), for $n \in J_1$, w_n satisfies

$$
\begin{cases}
w_n''(t) + \lambda_n^2 w_n(t) = -\lambda_n^2\int_0^t M(t-s)w_n(s)\,ds \qquad t\in[0,T] \\
w_n(0) = 0 \qquad w_n'(0) = \lambda_n
\end{cases}
\qquad (2.1.36)
$$

and $w_n(t) = t$ for $n \in J_0$. Since $\{w_n\psi_n\}$ is a Riesz sequence, there exists a biorthogonal Riesz sequence (Lemma 1.2.5), say $\{p_k\}$. Next we compute the adjoint of the Volterra operator on $L^2([0,T];\mathcal{G})$, $V_\rho v(t) = \int_0^t \rho(t-s)v(s)\,ds$ for any $\rho \in L^2(0,T)$.

$$
\int_0^T\int_0^t \rho(t-s)v(s)\,dsz(t)\,dt = \int_0^T\int_s^T \rho(t-s)v(s)z(t)\,dt\,ds
$$

$$
= \int_0^T v(t)\int_t^T \rho(s-t)z(s)\,ds\,dt
$$

So $V_\rho^* z(t) = \int_t^T \rho(s-t)\, z(s)\, ds$. We want to find θ_k such that

$$p_k = (\sigma(0) + V_{\sigma'}^*)\theta_k. \tag{2.1.37}$$

Recalling K from (2.1.32) and (2.1.33), we see that $(I + V_K)(\sigma(0) + V_{\sigma'}) = \sigma(0)I$ so if we set $\theta_k = \sigma(0)^{-1}(I + V_K^*)p_k$, then (2.1.37) is satisfied. Indeed,

$$(\sigma_0 + V_{\sigma'}^*)\theta_k = \sigma(0)^{-1}[(I + V_K)(\sigma(0) + V_{\sigma'})]^* p_k = p_k,$$

thus establishing (2.1.37). This gives the reconstruction formula. Indeed,

$$u(t) = \int_0^t \sigma(t-s) \sum_{n=1}^{\infty} a_n w_n(s) \phi_n \, ds \tag{2.1.38}$$

where

$$a_n = \begin{cases} \langle f, \phi_n \rangle & \text{for } n \in J_0 \cap \mathbb{N}, \\ \dfrac{\langle f, \phi_n \rangle}{\lambda_n} & \text{for } n \in J_1 \cap \mathbb{N}, \end{cases}$$

which implies

$$\mathcal{B}u' = \mathcal{B} \sum_{n=1}^{\infty} a_n (\sigma(0) + V_{\sigma'}) w_n \phi_n = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle (\sigma(0) + V_{\sigma'}) w_n \psi_n.$$

Finally, by (2.1.37), for each $k \in \mathbb{N}$

$$\begin{aligned}
\langle \mathcal{B}u', \theta_k \rangle_{L^2([0,T];\mathcal{G})} &= \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_{L^2([0,T];\mathcal{G})} \langle (\sigma(0) + V_{\sigma'}) w_n \psi_n, \theta_k \rangle_{L^2([0,T];\mathcal{G})} \\
&= \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_{L^2([0,T];\mathcal{G})} \langle w_n \psi_n, (\sigma(0) + V_{\sigma'}^*) \theta_k \rangle_{L^2([0,T];\mathcal{G})} \\
&= \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_{L^2([0,T];\mathcal{G})} \langle w_n \psi_n, p_k \rangle_{L^2([0,T];\mathcal{G})} \\
&= \langle f, \phi_k \rangle_{\mathcal{H}}.
\end{aligned}$$

□

Remark 2.1.9. Moreover, $\{\theta_k\}$ is also a Riesz sequence. This follows from the fact that $(\sigma(0) + V_K^*)$ is bounded with a bounded inverse so

$$\left\| \sum a_k \theta_k \right\| = \left\| (\sigma(0) + V_K^*) \sum a_k p_k \right\| \asymp \left\| \sum a_k p_k \right\|$$

and $\{p_k\}$ is a Riesz sequence since it is biorthogonal to a Riesz sequence.

Remark 2.1.10. The $H^1([0,T];\mathcal{G})$-norm in the lower inequality in Theorem 2.1.7 cannot be replaced by $L^2([0,T];\mathcal{G})$.

Proof. Assume the inequality can be improved. Then by (2.1.38), $\{y_n \psi_n\}$ forms a Riesz sequence in $L^2([0,T];\mathcal{G})$ where

$$y_n(t) = \int_0^t \sigma(t-s) w_n(s)\, ds.$$

However, in the case of no memory ($M = 0$ in (2.1.6)), for $n \in J_1$, $w_n(t) = \sin(\lambda_n t)$ in which case

$$\int_0^t \sigma(t-s)\sin(\lambda_n s)\,ds = -\frac{1}{\lambda_n}\left(\sigma(0)\cos(\lambda_n t) + \int_0^t \sigma'(t-s)\cos(\lambda_n s)\,ds\right)$$

so $\|y_n\|_{L^2[0,T]} \leq C|\lambda_n|^{-1}$. Since $\{z_n\psi_n\}$ is also a Riesz sequence, applying the upper inequality, we get ($z_n(t) = e^{i\lambda_n t}$)

$$T\|\psi_m\|_{\mathcal{G}}^2 = \int_0^T \|e^{i\lambda_m t}\psi_m\|_{\mathcal{G}}^2\,dt \leq C.$$

Therefore, $\|\psi_n\| \leq C$ which implies

$$\|y_n\psi_n\|_{L^2([0,T];\mathcal{G})} \leq \frac{C}{|\lambda_n|}. \tag{2.1.39}$$

However, if $\{y_n\psi_n\}$ was a Riesz sequence, then taking a_n to be 1 in the m-th entry and 0 everywhere else,

$$\|y_m\psi_m\|_{L^2([0,T];\mathcal{G})} = \left\|\sum a_n y_n \psi_n\right\|_{L^2([0,T];\mathcal{G})} \geq c$$

which contradicts (2.1.39) since $|\lambda_n| \to \infty$. $\square$

2.2 Wave Equation

Here we give a new proof of the observability of the wave equation which extends the older harmonic analysis method to higher dimensions.

Theorem 2.2.1. *Let $T > 2R$, R such that $\Omega \subseteq B(x_0, R)$ for $x_0 \in \mathbb{R}^d$ defined by (2.1.3). Then there exists $C, c > 0$ such that*

$$c \sum |a_n|^2 \leq \int_0^T \int_\Gamma \left| \sum a_n e^{i\lambda_n t} \psi_n(x) \right|^2 dS(x)\, dt \leq C \sum |a_n|^2$$

for all $\{a_n\} \in \ell^2$. Moreover, $c = (T - 2R)/C_\Gamma$ where $C_\Gamma = \max_\Gamma (x - x_0) \cdot \nu(x)$. In other words, $\{e^{i\lambda_n t}\psi_n\}$ is a Riesz sequence in $L^2(\Gamma \times [0, T])$.

We first state two preliminary lemmas concerning the functions $\{\psi_n\}$ from (2.1.25). Define the following operator $A : H_0^1(\Omega) \to L^2(\Omega)$ which connects the boundary terms ψ_n with the interior eigenfunctions ϕ_n.

$$(Au)(x) = m(x) \cdot \nabla u(x) \quad \text{where} \quad m(x) = x - x_0 \tag{2.2.1}$$

for $u \in H_0^1(\Omega)$ and $x \in \Omega$.

Lemma 2.2.2. *Let A and m be defined by (2.2.1). Then, for all $j, k \in \mathbb{Z}_0$,*

$$\int_{\partial\Omega} (m\cdot\nu)\psi_j\overline{\psi_k}\,dS = \begin{cases} \dfrac{\lambda_j^2-\lambda_k^2}{\lambda_j\lambda_k}\displaystyle\int_\Omega A\phi_{|j|}\overline{\phi_{|k|}}\,dx & \text{if } |j|\neq|k|; \\ 2 & \text{if } j=k; \\ -2 & \text{if } j=-k. \end{cases} \tag{2.2.2}$$

Proof. We use the fact that

$$A\phi_j(x) = (m\cdot\nu)\frac{\partial\phi_j}{\partial\nu}(x),\ \forall x\in\partial\Omega,\ j\in\mathbb{N}$$

as in Lemma 2.1.3 since $\phi_j = 0$ on $\partial\Omega$. Notice also that $[\Delta, A] = 2\Delta$. Indeed, for each i,

$$\begin{aligned}\sum_j \partial_{jj}(x_i - x_{0,i})\partial_i &= \sum_{j\neq i}(x_i - x_0)\partial_{jj}\partial_i + \partial_i[(x_i - x_{0,i})\partial_{ii} + \partial_i] \\ &= \sum_j (x_i - x_0)\partial_i\partial_{jj} + 2\partial_{ii}.\end{aligned}$$

Applying these facts along with Green's Theorem,

$$\begin{aligned}\int_{\partial\Omega}(m\cdot\nu)\psi_j(x)\overline{\psi_k(x)}\;dS &= \frac{1}{\lambda_j\lambda_k}\int_{\partial\Omega} A\phi_{|j|}\frac{\partial\overline{\phi_{|k|}}}{\partial\nu}\;dS \\ &= \frac{1}{\lambda_j\lambda_k}\int_\Omega A\phi_{|j|}\Delta\overline{\phi_{|k|}} - \Delta(A\phi_{|j|})\overline{\phi_{|k|}}\;dx\end{aligned}$$

$$= \begin{cases} \dfrac{\lambda_j^2 - \lambda_k^2}{\lambda_j \lambda_k} \displaystyle\int_\Omega A\phi_{|j|}\overline{\phi_{|k|}}\, dx & \text{if } |j| \neq |k|; \\ \dfrac{1}{\lambda_j \lambda_k} \displaystyle\int_\Omega 2\lambda_j^2 |\phi_{|j|}|^2\, dx = \pm 2 & \text{if } |j| = |k|. \end{cases}$$

□

Lemma 2.2.3. *The sequence* $\{\lambda_j^{-1} A\phi_{|j|}\}_{j\in\mathbb{Z}_0}$ *is a Bessel sequence in* $L^2(\Omega)$*. More precisely, for all* $u \in \ell^2(\mathbb{Z}_0)$*,*

$$\int_\Omega \left| \sum_j u_j \frac{A\phi_{|j|}}{\lambda_j} \right|^2 \leq R^2 \sum_j \left(|u_j|^2 - u_j \bar{u}_{-j} \right). \tag{2.2.3}$$

Secondly,

$$\int_\Omega A\phi_{|j|}\phi_{|k|} = -\int_\Omega \phi_{|j|} A\phi_{|k|} \tag{2.2.4}$$

for $|j| \neq |k|$*.*

Proof. Notice that the system $\{\lambda_j^{-1}\nabla\phi_{|j|}\}_{j\in\mathbb{Z}_0}$ has some sense of orthogonality. Indeed, for each $j, k \in \mathbb{Z}_0$,

$$\int_\Omega \frac{\nabla\phi_{|j|} \cdot \nabla\overline{\phi}_{|k|}}{\lambda_j \lambda_k} = -\int_\Omega \frac{\phi_{|j|}\Delta\overline{\phi_{|k|}}}{\lambda_j \lambda_k} = \begin{cases} 0 & \text{if } |j| \neq |k|; \\ 1 & \text{if } j = k; \\ -1 & \text{if } j = -k. \end{cases}$$

Then, using the definition of A in (2.2.1) and the Cauchy-Schwarz Inequality, we obtain for $\{u_j\} \in \ell^2(\mathbb{Z}_0)$,

$$\int_\Omega \left|\sum_j u_j \frac{A\phi_j}{\lambda_j}\right|^2 \le R^2 \int_\Omega \left|\sum_j u_j \frac{\nabla\phi_j}{\lambda_j}\right|^2 = R^2 \left(\sum_j |u_j|^2 - \sum_j u_j \bar{u}_{-j}\right).$$

Now we proceed to the second statement in the lemma. Recalling m from (2.2.1),

$$m_i \partial_i \phi_j \overline{\phi}_k = \partial_i (m_i \phi_j \overline{\phi}_k) - (\partial_i m_i)\phi_j \phi_k - m_i \phi_j \partial_i \overline{\phi_k}.$$

Summing over $i = 1, \ldots, d$ (recall d is the dimension of the space) and integrating over Ω yields

$$\int_\Omega A\phi_j \overline{\phi_k} = \int_\Omega \nabla \cdot (m\phi_j \overline{\phi_k}) - d\int_\Omega \phi_j \overline{\phi_k} - \int_\Omega \phi_j A\overline{\phi_k} \tag{2.2.5}$$

which gives the desired identity since $\phi_j = 0$ on $\partial\Omega$ and $\{\phi_j\}$ are orthonormal. □

Now we give the proof of the lower inequality in Theorem 2.2.1. The upper inequality follows by a similar argument but with A replaced by V from the proof of Lemma 2.1.3. To be concise, all sums are assumed to be taken over $\mathbb{Z}_0$ unless otherwise stated.

Proof of Theorem 2.2.1. For $C_\Gamma := \max_{x\in\Gamma}[m(x)\cdot\nu(x)] \le R$, we have the following estimate using Lemma 2.2.2.

$$C_\Gamma \int_\Gamma \int_0^T \left|\sum_j a_j e^{i\lambda_j t}\psi_j(x)\right|^2 \, dt\, dS \ge \int_\Gamma \int_0^T \left|\sum_j a_j e^{i\lambda_j t}\psi_j(x)\right|^2 m(x)\cdot\nu(x)\; dt\, dS$$

$$\geq \sum_j \sum_k a_j \bar{a}_k \int_0^T e^{i(\lambda_j - \lambda_k)t} \int_{\partial\Omega} (m \cdot \nu)\psi_j \overline{\psi_k}\, dS$$

$$= 2T \sum_j |a_j|^2 - \underbrace{\sum_j a_j \bar{a}_{-j} \frac{e^{i2\lambda_j T} - 1}{i\lambda_j}}_{\text{I}}$$

$$+ \underbrace{\sum_j \sum_{k \neq \pm j} a_j \bar{a}_k \left(\frac{1}{i\lambda_j} + \frac{1}{i\lambda_k} \right) \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega A\phi_j \overline{\phi_k}\, dx}_{\text{II}}\,. \tag{2.2.6}$$

First, notice we can rewrite expression II as:

$$\begin{aligned}
\text{II} &= \sum_j \sum_{k \neq \pm j} a_j \bar{a}_k \frac{1}{i\lambda_j} \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega A\phi_j \overline{\phi_k}\, dx \\
&\quad + \sum_j \sum_{k \neq \pm j} a_j \bar{a}_k \frac{1}{i\lambda_k} \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega A\phi_j \overline{\phi_k}\, dx \\
&= \sum_j \sum_{k \neq \pm j} a_j \bar{a}_k \frac{1}{i\lambda_j} \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega A\phi_j \overline{\phi_k}\, dx \\
&\quad + \sum_k \sum_{j \neq \pm k} \overline{\bar{a}_j a_k \frac{1}{-i\lambda_k} \left(e^{i(\lambda_k - \lambda_j)T} - 1 \right) \left(-\int_\Omega \phi_j A\overline{\phi_k}\, dx \right)} \\
&= 2\,\Re \left(\sum_j \sum_{k \neq -j} a_j \bar{a}_k \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega \frac{A\phi_j}{i\lambda_j} \overline{\phi_k}\, dx \right).
\end{aligned}$$

where in the second equality we have exchanged the sums and applied the second statement in Lemma 2.2.3. To obtain the final equality, simply notice that when $k = j$, the summands are zero so we may include them at no cost.

Now, we will include the terms when $k = -j$. By (2.2.5), $2\,\Re \int_\Omega A\phi_{|j|}\bar{\phi}_{|-j|} = -d$.

Thus we can rewrite the second two terms in the original inequality (2.2.6) as

$$\text{I+II} = (d-1)\,\Re\left(\sum_j a_j\bar{a}_{-j}\frac{e^{i2\lambda_j T}-1}{i\lambda_j}\right) + 2\,\Re\left(\sum_j\sum_k a_j\bar{a}_k\left(e^{i(\lambda_j-\lambda_k)T}-1\right)\int_\Omega \frac{A\phi_j}{i\lambda_j}\overline{\phi_k}\,dx\right). \tag{2.2.7}$$

Additionally, we have the following identity for the first sum in (2.2.7):

$$\begin{aligned}&\sum_j a_j\bar{a}_{-j}\frac{e^{i2\lambda_j T}-1}{i\lambda_j}\\ &\quad=\int_\Omega\left(\sum_j a_j e^{i\lambda_j T}\frac{\phi_{|j|}}{i\lambda_j}\right)\overline{\left(\sum_k a_k e^{i\lambda_k T}\phi_{|k|}\right)} - \left(\sum_j a_j\frac{\phi_{|j|}}{i\lambda_j}\right)\overline{\left(\sum_k a_k\phi_{|k|}\right)}dx.\end{aligned}$$

Then we split the double sum from (2.2.7) into two terms (one with $e^{i(\lambda_j-\lambda_k)T}$ and one with -1) and estimate each with the corresponding portion in the above identity.

$$\begin{aligned}&\left|\int_\Omega (d-1)\left(\sum_j a_j e^{i\lambda_j T}\frac{\phi_{|j|}}{i\lambda_j}\right)\overline{\left(\sum_k a_k e^{i\lambda_k T}\phi_{|k|}\right)} + 2\sum_j\sum_k a_j\bar{a}_k e^{i(\lambda_j-\lambda_k)T}\int_\Omega \frac{A\phi_{|j|}}{i\lambda_j}\overline{\phi_{|k|}}\right|\\ &\quad=\left|\int_\Omega\left((d-1)\sum_j a_j e^{i\lambda_j T}\frac{\phi_{|j|}}{\lambda_j} + 2\sum_j a_j e^{i\lambda_j T}\frac{A\phi_{|j|}}{\lambda_j}\right)\overline{\left(\sum_k a_k e^{i\lambda_k T}\phi_{|k|}\right)}\right|\\ &\quad\le \frac{1}{4R}\int_\Omega\left|(d-1)\sum_j a_j e^{i\lambda_j T}\frac{\phi_{|j|}}{\lambda_j} + 2\sum_j a_j e^{i\lambda_j T}\frac{A\phi_{|j|}}{\lambda_j}\right|^2 + R\left|\sum_k a_k e^{i\lambda_k T}\phi_{|k|}\right|^2.\end{aligned} \tag{2.2.8}$$

Note that by (2.2.5), for any $u \in H_0^1(\Omega)$,

$$\begin{aligned}\|(d-1)u + 2Au\|^2 &= (d-1)^2\|u\|^2 + 4(d-1)\,\Re(u, Au) + 4\|Au\|^2 \\ &= (-1-d)(d-1)\|u\|^2 + 4\|Au\|^2 \\ &\le 4\|Au\|^2,\end{aligned}$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(\Omega)$ norm and inner product. Apply this to (2.2.8) with $u = \sum a_j e^{i\lambda_j T}\phi_{|j|}\lambda_j^{-1}$. Then, applying (2.2.3) from Lemma 2.2.3, we have

$$\frac{1}{R}\int_\Omega \left|\sum_j a_j e^{i\lambda_j T}\frac{A\phi_{|j|}}{\lambda_j}\right|^2 + R\int_\Omega \left|\sum_k a_k e^{i\lambda_k T}\phi_{|k|}\right|^2 \tag{2.2.9}$$

$$\le R\sum_j \left(|a_j|^2 - a_j\bar{a}_{-j}e^{i2\lambda_j T}\right) + R\sum_k \left(|a_k|^2 + a_k\bar{a}_{-k}e^{i2\lambda_k T}\right) = 2R\sum_j |a_j|^2.$$

The other term (with $e^{i(\lambda_j-\lambda_k)T}$ replaced by -1) is estimated in a manner similar to (2.2.8) and (2.2.9). Substituting (2.2.9) and the corresponding estimate for -1 into the original inequality gives the desired result:

$$
\begin{aligned}
C_\Gamma \int_0^T \int_{\partial\Omega} \left| \sum_j a_j e^{i\lambda_j t} \psi_j(x) \right|^2 \, dS \, dt \\
&\geq 2T \sum_j |a_j|^2 + \sum_j a_j \bar{a}_{-j} \frac{e^{i2\lambda_j T} - 1}{i\lambda_j} \\
&\quad + \sum_j \sum_{k \neq \pm j} a_j \bar{a}_k \left(\frac{1}{i\lambda_j} + \frac{1}{i\lambda_k} \right) \left(e^{i(\lambda_j - \lambda_k)T} - 1 \right) \int_\Omega A\phi_{|j|} \overline{\phi_{|k|}} \\
&\geq 2T \sum_j |a_j|^2 - 4R \sum_j |a_j|^2 = 2\,(T - 2R) \sum_j |a_j|^2.
\end{aligned}
$$

□

Chapter 3

Resolvent Estimates

Consider the abstract evolution system

$$\begin{cases} i\dfrac{d}{dt}u(t) = \mathcal{A}u(t) & t > 0 \\ u(0) = u_0 \end{cases} \tag{3.0.1}$$

where $\mathcal{H}$ is a Hilbert space, $u : \mathbb{R}^+ \to \mathcal{H}$ and $\mathcal{A} : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a possibly unbounded linear operator.

We will relate to certain problems in control theory for this system by studying the resolvent of $\mathcal{A}$. First, we give the relationship between the resolvent and the observability inequality. The following proposition is inspired by [42, 11], but we provide a simplified proof which suffices for our applications.

Proposition 3.0.1. *Let $\mathcal{A}, \mathcal{B} : \mathcal{H} \to \mathcal{H}$ be linear operators on a Hilbert space. Asu-*

ume that $\mathcal{A}$ is self-adjoint and $\mathcal{B}$ bounded. There exists $c > 0$ such that

$$c\|f\|^2 \leq \|(\mathcal{A} - \lambda)f\|^2 + \|\mathcal{B}f\|^2 \tag{3.0.2}$$

for all $f \in \mathcal{D}(\mathcal{A})$, $\lambda \in \mathbb{R}$ if and only if there exists $C, T > 0$ such that

$$\|u_0\| \leq \int_0^T \|\mathcal{B}u(t)\|^2 \, dt \tag{3.0.3}$$

for all $u_0 \in \mathcal{D}(\mathcal{A})$ and u satisfying (4.2.3).

Proof. We first prove the forward direction. Let $T > 0$ and $u_0 \in \mathcal{D}(\mathcal{A})$ and $u = e^{it\mathcal{A}}u_0$ be the corresponding strong solution to (4.2.3). Then, for $\psi \in C_0^\infty[0, 1]$, set $v(t) = \psi(t/T)u(t)$. Then,

$$iv'(t) = \frac{i}{T}\psi'(t/T)u(t) + i\psi(t/T)u'(t) = \frac{i}{T}\psi'(t/T)u(t) - \mathcal{A}v(t).$$

By construction, $v \in C_0^\infty([0, 1]; \mathcal{H})$ so integrating by parts, $i\widehat{v'}(\tau) = i(2\pi)^{-d/2} \int_{\mathbb{R}} v'(t)e^{-it\tau} \, dt = -i \int_{\mathbb{R}} v(t)(-i\tau)e^{-it\tau} \, dt = -\tau\hat{v}(\tau)$. Therefore,

$$(\mathcal{A} - \tau)\hat{v}(\tau) = \frac{i}{T}\widehat{\psi'(\cdot/T)u}(\tau).$$

Applying (3.0.2) for $\lambda = \tau$, we have

$$c\|\hat{v}(\tau)\|^2 \leq \frac{1}{T^2}\|\widehat{\psi'(\cdot/T)u}(\tau)\|^2 + \|\mathcal{B}\hat{v}(\tau)\|^2. \tag{3.0.4}$$

Integrating in τ and using Plancherel's theorem, we obtain

$$c\int_{\mathbb{R}} \|\psi(t/T)u(t)\|^2\,dt \le \frac{1}{T^2}\int_{\mathbb{R}} \|\psi'(t/T)u(t)\|^2\,dt + \int_{\mathbb{R}} \|\psi(t/T)\mathcal{B}u(t)\|^2\,dt. \quad (3.0.5)$$

Now, for any $\phi \in L^2(\mathbb{R})$, since $\|u(t)\| = \|e^{it\mathcal{A}}u_0\| = \|u_0\|$ for all t,

$$\int_{\mathbb{R}} \|\phi(t/T)u(t)\|^2\,dt = \int_{\mathbb{R}} |\phi(t/T)|^2 \|e^{it\mathcal{A}}u_0\|^2\,dt = T\|\phi\|^2_{L^2(\mathbb{R})}\|u_0\|^2.$$

Applying this to the first two terms in (3.0.5), we obtain

$$cT\|\psi\|^2_{L^2(0,1)}\|u_0\|^2 \le \frac{1}{T}\|\psi'\|^2_{L^2(0,1)}\|u_0\|^2 + \int_0^T \|\mathcal{B}u(t)\|^2\,dt.$$

So, for T large enough, there exists C_T such that

$$\|u_0\|^2 \le C_T \int_0^T \|\mathcal{B}u(t)\|^2\,dt$$

for any $u_0 \in \mathcal{D}(\mathcal{A})$.

For the other direction, we use the fact that

$$\frac{d}{dt}e^{i\lambda t}u(t) = e^{i\lambda t}(i\lambda u(t) + u'(t)) = e^{i\lambda t}(i\lambda - i\mathcal{A})u(t) = -ie^{i\lambda t}e^{i\mathcal{A}t}(\mathcal{A}-\lambda)u_0.$$

Therefore,

$$\|e^{i\lambda t}u(t) - u_0\| = \left\| \int_0^t -ie^{i\lambda s}e^{i\mathcal{A}s}(\mathcal{A} - \lambda)u_0 \, ds \right\| \leq t\|(\mathcal{A} - \lambda)u_0\|.$$

Now, assuming (3.0.3), we have

$$C^{-1}\|u_0\|^2 \leq \int_0^T \|\mathcal{B}u(t)\|^2 \, dt \leq 2\int_0^T \|\mathcal{B}(e^{i\lambda t}u(t) - u_0)\|^2 \, dt + 2T\|\mathcal{B}u_0\|^2.$$

Since $\mathcal{B}$ is bounded, $\|\mathcal{B}(e^{i\lambda t}u(t) - u_0)\|^2 \leq Ct^2\|(\mathcal{A} - \lambda)u_0\|^2$ which proves (3.0.2) with f replaced by u_0. □

3.1 Uncertainty Principle

We will derive our resolvent estimates from inequalities of the uncertainty principle type. The particular form of the uncertainty principle we will use is the Paneah-Logvinenko-Sereda Theorem. One definition is needed before stating the result.

Definition 3.1.1. A set $E \subset \mathbb{R}^d$ is said to be relatively dense if there exists $R, \gamma > 0$ such that

$$m(E \cap B(x, R)) \geq \gamma m(B(x, R))$$

for all x in $\mathbb{R}^d$.

Theorem 3.1.2 (PLS Theorem)**.** *Let E be relatively dense and $\sigma > 0$. There exists*

$C > 0$ *such that*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)}$$

for all $f \in L^2(\mathbb{R}^d)$ *with* $\operatorname{supp} \hat{f} \subset B(0, \sigma)$.

This result has been proved many times and inspired many variations [49, 38, 23, 30, 45, 43, 28]. The precise statements we will need come from the work of Kovrijkine in [30]. However, we will not need the full strength of these results so we prove the version we will use in Theorem 3.1.6. Relative density is also necessary for the inequality above to hold. This will more or less be proved in Section 3.2.2.

We begin with a simple version of the Turàn Lemma (see [44] for the general version).

Lemma 3.1.3. *Let* $p(x) = ce^{iax} + de^{ibx}$ *for* $a, b \in \mathbb{R}$, $c, d \in \mathbb{C}$. *Then for any* $E \subset [0, 2\pi]$,

$$\sup_{x \in [0,2\pi]} |p(x)| \leq \frac{8\pi^2}{|E|^2} \sup_{x \in E} |p(x)|.$$

Proof. It is enough to show $\mu\{x \in [0, 2\pi] : |1 + re^{isx}| < \varepsilon\} \leq \pi\sqrt{\varepsilon}$ for $\varepsilon, |r| \leq 1$ and $s \in \mathbb{R}$. If so, then (assume $|c| \geq |d|$)

$$\sup_{x \in [0,2\pi]} |ce^{iax} + de^{ibx}| \leq |c| + |d| \leq 2|c|$$

However, $|p(x)| = |c||1 + d/ce^{i(b-a)x}|$. Taking $\sqrt{\varepsilon} = |E|/(2\pi) \leq 1$, there must be a point $x_0 \in E$ such that $|1 + re^{isx}| \geq |E|^2/(2\pi)^2$. Therefore,

$$\sup_{x \in E} |p(x)| \geq |p(x_0)| \geq \frac{|c| \cdot |E|^2}{4\pi^2} \geq \frac{|E|^2}{8\pi^2} \sup_{x \in [0,2\pi]} |p(x)|.$$

We now turn to estimating the measure of the level set. We claim that $f(x) := 1 - (\frac{2x}{\pi})^2 - \cos(x) \geq 0$. Indeed, $f'(x) = \sin(x) - \frac{8}{\pi^2}x$ has at most two zeros in the interval $[0, \frac{\pi}{2}]$ since the sine function is concave on $[0, \frac{\pi}{2}]$ and a concave function intersects a straight line at most twice. $f'(0) = 0$. There must be another zero in the interval, let us say at γ, since $f(0) = f(\frac{\pi}{2}) = 0$. Therefore, $f' \geq 0$ on $[0, \gamma]$ and $f' \leq 0$ on $[\gamma, \frac{\pi}{2}]$. This proves that $f \geq 0$. So, if $\cos(x) \geq 1 - \beta$, then $1 - (\frac{2x}{\pi})^2 \geq 1 - \beta$ which is equivalent to $x \leq \frac{\pi}{2}\sqrt{\beta}$. Therefore

$$\mu\{x \in [0, \pi/2] : \cos(x) \geq 1 - \beta\} \leq \frac{\pi}{2}\sqrt{\beta}.$$

Now we can prove the first claim, if $|r| \leq 1 - \varepsilon$, then $|1 + re^{-sx}| \geq 1 - |r| \geq \varepsilon$ so there is nothing to prove. On the other hand, $|1 + re^{isx}|^2 = 1 + |r|^2 + 2\rho\cos(sx)$ where $\rho = \Re r$. Then

$$\{|1 + re^{isx}| \leq \varepsilon\} = \{\cos(sx) \leq (\varepsilon^2 - (1 + |r|^2))/2\rho\}$$

$$\subset \{\cos(sx) \leq -(1 - \varepsilon)/\rho\} \subset \{\cos(sx) \leq -(1 - \varepsilon)\}$$

using the fact that $|r| \geq 1 - \varepsilon$ implies $\varepsilon^2 - (1 + |r|^2) \leq 2\varepsilon - 2$ and $\rho \leq 1$. Letting k

be the smallest integer such that $|s| \le k$, we have

$$\begin{aligned}
|s|\mu\{x \in [0, 2\pi] : \cos(sx) \le -(1-\varepsilon)\} &= \mu\{y \in [0, 2\pi|s|] : \cos(y) \le -(1-\varepsilon)\} \\
&\le \sum_{j=1}^{k} \{y \in [2\pi(j-1), 2\pi j] : \cos(y) \le -(1-\varepsilon)\} \\
&= (k-1)\{y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] : \cos(y) \ge 1-\varepsilon\} \\
&\le (k-1)\pi\sqrt{\varepsilon} \\
&\le |s|\pi\sqrt{\varepsilon}.
\end{aligned}$$

□

Scaling and shifting, this easily extends this to any interval $I \in \mathbb{R}$.

Corollary 3.1.4. *Let I be an interval in $\mathbb{R}$ and $E \subset I$.*

$$\sup_{x \in I} |p(x)| \le 2 \left(\frac{|I|}{|E|} \right)^2 \sup_{x \in E} |p(x)|.$$

We can also extend to the L^q case.

Corollary 3.1.5.

$$\|p\|_{L^q(I)} \le 2^{1/q} 8 \left(\frac{|I|}{|E|} \right)^{2+1/q} \|p\|_{L^q(E)}.$$

Proof. Let $E \subset I$. Setting $F = \{x \in E : |p(x)| \ge t\}$, $\mu(F) \le t^{-q} \int_E |p|^q$. So, taking $t^q = 2|E|^{-1} \int_E |p|^q$, we have $\mu(E \backslash F) = \mu(E) - \mu(F) \ge \mu(E)/2$. Applying Corollary

3.1.4 to the set $E\backslash F$, we get

$$|I|^{-1/q}\|p\|_{L^q(I)} \leq \sup_{x\in I} |p(x)| \leq 8\left(\frac{|I|}{|E|}\right)^2 \left(\frac{2}{|E|}\right)^{1/q} \|p\|_{L^q(E)}.$$

□

We will now prove the following "annulus" version of the Paneah-Logvinenko-Sereda Theorem. As we will see below, to apply such inequalities to wave equations, we need $\operatorname{supp}\hat{f}$ to be contained in an annulus. However, in one dimension, an annulus is just the union of two intervals, and we will exploit this fact to obtain a better result for $d = 1$ than for $d \geq 2$. See the discussion around Theorem (3.2.1) below.

Theorem 3.1.6. *Let $f \in L^2(\mathbb{R})$ with $\operatorname{supp}\hat{f} \subset B(a,\sigma) \cup B(b,\sigma)$ and $E \subset \mathbb{R}$ be relatively dense. If $\sigma \leq \frac{\gamma^{5/2}}{2^{11/2}R}$, then*

$$\int |f|^2 \leq \frac{2^{10}}{\gamma^5}\int_E |f|^2.$$

Proof. First, we need the Bernstein Inequality. We will only use the L^2 version. So, for $\operatorname{supp}\hat{g} \subset B(0,\sigma)$,

$$\|g'\|^2_{L^2(\mathbb{R})} = \int_{B(0,\sigma)} |\xi\hat{g}(\xi)|^2\, d\xi \leq \sigma^2\|g\|^2_{L^2(\mathbb{R})}.$$

Let $f \in L^2(\mathbb{R})$ with $\operatorname{supp}\hat{f} \subset B(a,\sigma) \cup B(b,\sigma)$. Then, $f = e^{iax}f_1(x) + e^{ibx}f_2(x)$ for $\operatorname{supp}\hat{f}_i \subset B(0,\sigma)$. First we prove the special case for sets E satisfying $|E \cap I| \geq \gamma$

for all intervals I of length 1. Let I be such an interval. For any $x, y \in I$, $f_i(x) = \int_x^y f_i'(t)\,dt + f_i(y)$. Set $p(x,y) = e^{iax}f_1(y) + e^{ibx}f_2(y)$ and $R(x,y) = \int_x^y [e^{iax}f_1'(t) + e^{ibx}f_2'(t)]\,dt$ so that $f(x) = p(x,y) + R(x,y)$. Applying Corollary 3.1.5 with $q = 2$, we obtain with $C_E = 2^7/\gamma^5$

$$\begin{aligned}\int_I |f(x)|^2\,dx &\leq 2\int_I |p(x,y)|^2\,dx + 2\int_I |R(x,y)|^2\,dx \\ &\leq 2C_E\int_{E\cap I} |p(x,y)|^p\,dx + 2\int_I |R(x,y)|^p\,dx \\ &\leq 4C_E\int_{E\cap I} |f(x)|^p\,dx + 2(2C_E+1)\int_I |R(x,y)|^p\,dx.\end{aligned}$$

Now, since $\int_I |\int_x^y g(t)\,dt|^2\,dx \leq \int_I |x-y| \int_I |g(t)|^2\,dt\,dx \leq |I|^2 \int_I |g(t)|^2\,dt$ for any $x, y \in I$ and $g \in L^2(I)$,

$$\int_I |R(x,y)|^2\,dx \leq \int_I |f_1'(t)|^2 + |f_2'(t)|^2\,dt.$$

Summing over all the intervals and applying Bernstein's inequality,

$$\begin{aligned}\int_{\mathbb{R}} |f|^2 &\leq 4C_E\int_E |f|^2 + 2(2C_E+1)\int_{\mathbb{R}} |f_1'|^2 + \int_{\mathbb{R}} |f_2'|^2 \\ &\leq 4C_E\int_E |f|^2 + 2(2C_E+1)\sigma^2\int_{\mathbb{R}} |f|^2.\end{aligned}$$

So, if $\sigma^2 \leq \frac{1}{16C_E} \leq \frac{1}{4(2C_E+1)}$, the final term can absorbed and we obtain $\|f\|_{L^2(\mathbb{R})} \leq \sqrt{8C_E}\|f\|_{L^2(E)} = 2^5/\gamma^{5/2}\|f\|_{L^2(E)}$. We scale to get the more general case. Suppose there exists $R > 0$ such that $m(E \cap I) \geq \gamma R$ for all intervals of length R. Setting

$\Xi = E/R$, for any interval J of length 1,

$$m(\Xi \cap J) = R^{-1} m(E \cap RJ) \geq \gamma.$$

$\int_E |f(x)|^2\, dx = R \int_\Xi |g(y)|^2\, dy$ where $g(y) = f(yR)$. In this case $\hat{g}(\xi) = R^{-1}\hat{f}(\xi/R)$ so if $\operatorname{supp}\hat{f} \subset B(a,\sigma) \cup B(b,\sigma)$, then $\operatorname{supp}\hat{g} \subset B(Ra, R\sigma) \cup B(Rb, R\sigma)$. So, applying the above result to g and Ξ, we get, for $\sigma^2 \leq 1/(16CR^2)$

$$\|f\| = R^{1/2}\|g\| \leq \frac{2^5 R}{\gamma^{5/2}}\|g\|_{L^2(\Xi)} = \frac{2^5}{\gamma^{5/2}}\|f\|_{L^2(E)}.$$

□

Proposition 3.1.7. *Let $\Omega \subset \mathbb{R}$ be relatively dense, $s > 0$. There exists $c > 0$ (depending on Ω, s) such that for all $f \in L^2(\mathbb{R})$, $\lambda \geq 0$.*

$$c\|f\|^2_{L^2(\mathbb{R})} \leq (1+\lambda)^{\frac{2}{s}-2}\|((-\Delta+1)^{s/2} - \lambda)f\|^2_{L^2(\mathbb{R})} + \|f\|^2_{L^2(\Omega)}. \tag{3.1.1}$$

The constant c depends polynomially on γ, R from Definition 3.1.1. The operator $(-\Delta+1)^{s/2}$ is understood as a strictly positive Fourier multiplier:

$$(-\Delta+1)^{s/2} f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (|\xi|^2+1)^{s/2} \hat{f}(\xi) e^{ix\xi}\, d\xi.$$

Throughout, we denote by $\|\cdot\|$ the norm $\|\cdot\|_{L^2(\mathbb{R})}$. We begin with the following algebraic lemma.

Lemma 3.1.8. *Let $s > 0$. There exists $c_s > 0$ such that*

$$|\tau^s - \lambda| \geq c_s(1+\lambda)^{1-1/s}$$

for all $\tau, \lambda \geq 0$ in the region $|\tau - \lambda^{1/s}| > 1$.

Proof. First, for any $s > 0$, there exists $d_s, D_s > 0$ such that

$$d_s \max(x,y)^{s-1}|x-y| \leq |x^s - y^s| \leq D_s \max(x,y)^{s-1}|x-y| \tag{3.1.2}$$

for all $x, y \in \mathbb{R}_+$. Indeed, consider the function $g(z) = (1-z^s)/(1-z)$, $z \in [0,1)$. $g(0) = 1$ and $\lim_{z\to 1^-} g(z) = s$. Defining $g(1) = s$, g is continuous and always positive on $[0,1]$ so it has a minimum and a maximum, say d_s and D_s. It can also be shown that $d_s = \min(s,1)$ and $D_s = \max(s,1)$. Take $z = x/y$ for $x \geq y$ to obtain (3.1.2).

Next, consider two cases.

(i) If $\tau \geq \lambda^{1/s} + 1$, then

$$|\tau^s - \lambda| \geq d_s \max(\tau, \lambda^{1/s})^{s-1}|\tau - \lambda^{1/s}| = d_s\tau^{s-1}|\tau - \lambda^{1/s}|.$$

The function $x \mapsto x^{s-1}(x-\mu)$ is positive and increasing for $x > \mu + 1$, so we can bound the final term from below by its value at $\tau = \lambda^{1/s} + 1$ which yields

$$|\tau^s - \lambda| \geq d_s(\lambda^{1/s}+1)^{s-1}.$$

(ii) If $\tau \leq \lambda^{1/s} - 1$, then

$$|\tau^s - \lambda| \geq d_s \max(\tau, \lambda^{1/s})^{s-1} \cdot 1 = d_s(\lambda^{1/s})^{s-1}.$$

If $s < 1$, then $s - 1 < 0$ so $(\lambda^{1/s})^{s-1} \geq (\lambda^{1/s} + 1)^{s-1}$. Since $0 \leq \tau \leq \lambda^{1/s} - 1$, $\lambda \geq 1$. So, for $s \geq 1$,

$$(\lambda^{1/s})^{s-1} = \frac{(\lambda^{1/s} + \lambda^{1/s})^{s-1}}{2^{s-1}} \geq \frac{(\lambda^{1/s} + 1)^{s-1}}{2^{s-1}}.$$

Therefore, there exists c_s such that

$$|\tau^s - \lambda| \geq c_s(\lambda^{1/s} + 1)^{s-1} \geq c_s(\lambda + 1)^{1-\frac{1}{s}}$$

where in the final step, we have used the fact that for $p \leq q$, $(x^q + y^q)^{1/q} \leq (x^p + y^p)^{1/p}$. □

Proof of Proposition 3.1.7. Let $\lambda \geq 0$, $s > 0$ and $g \in L^2(\mathbb{R})$ such that $\operatorname{supp} \hat{g} \subset A_\lambda := \{\xi \in \mathbb{R} : \left|(|\xi|^2 + 1)^{1/2} - \lambda^{1/s}\right| \leq \delta\}$. We will take δ small so we assume $\delta < \frac{1}{2}$. Notice that if $\lambda^{1/s} \leq 1 + \delta$, then $A_\lambda \subset [-\sqrt{5\delta}, \sqrt{5\delta}]$. If $\lambda^{1/s} \geq 1 + \delta$, then A_λ is the union of the two intervals

$$\pm\left[\sqrt{(\lambda^{1/s} - \delta)^2 - 1}, \sqrt{(\lambda^{1/s} + \delta)^2 - 1}\right].$$

The width of these intervals is also no more than $\sqrt{5\delta}$. Indeed, the width is a decreasing function of λ, so we may get an upper bound evaluating it at $\lambda^{1/s} = 1 + \delta$. Therefore,

$$|\sqrt{(\lambda^{1/s} - \delta)^2 - 1} - \sqrt{(\lambda^{1/s} + \delta)^2 - 1}| \leq \sqrt{(1 + 2\delta)^2 - 1} = \sqrt{4\delta + 2\delta^2} \leq \sqrt{5\delta}.$$

Therefore, for $\Omega \subset \mathbb{R}$ which is relatively dense, if δ is small enough, by Theorem 3.1.6, there exists $C > 0$ (independent of λ and g) such that

$$\|g\| \leq C\|g\|_{L^2(\Omega)}.$$

Denote by P_λ the projection $P_\lambda f = \mathcal{F}^{-1}(\mathbb{1}_{A_\lambda}\mathcal{F}(f))$. Then, for $f \in L^2(\mathbb{R})$,

$$\begin{aligned}
\|f\|^2 &= \|P_\lambda f\|^2 + \|(I - P_\lambda)f\|^2 \\
&\leq C\|P_\lambda f\|^2_{L^2(\Omega)} + \|(I - P_\lambda)f\|^2 \\
&= C\|f - (I - P_\lambda)f\|^2_{L^2(\Omega)} + \|(I - P_\lambda)f\|^2 \\
&\leq 2C\|f\|^2_{L^2(\Omega)} + 2C\|(I - P_\lambda)f\|^2_{L^2(\Omega)} + \|(I - P_\lambda)f\|^2 \\
&\leq 2C\|f\|^2_{L^2(\Omega)} + (2C + 1)\|(I - P_\lambda)f\|^2.
\end{aligned}$$

It remains to estimate the final term. Lemma 4.2.4 can be scaled so that if $|\tau - \lambda^{1/s}| \geq$

δ, then $|\tau^s - \lambda| \geq c_s(\delta^s + \lambda)^{1-1/s}$. Taking $\tau = (|\xi|^2 + 1)^{1/2}$, we obtain

$$\begin{aligned}
\|((-\Delta + 1)^{s/2} - \lambda)f\|^2 &= \int [(|\xi|^2 + 1)^{s/2} - \lambda]^2 |\hat{f}(\xi)|^2 \, d\xi \\
&\geq \int_{A_\lambda^c} [(|\xi|^2 + 1)^{s/2} - \lambda]^2 |\hat{f}(\xi)|^2 \, d\xi \\
&\geq c_s(\delta^s + \lambda)^{2-\frac{2}{s}} \int_{A_\lambda^c} |\hat{f}(\xi)|^2 \, d\xi \\
&= c_s(\delta^s + \lambda)^{2-\frac{2}{s}} \|(I - P_\lambda)f\|^2.
\end{aligned}$$

□

3.2 Klein-Gordon Equation

We will apply the results of the previous section to the damped fractional Klein-Gordon equation recently introduced by Malhi and Stanislavova in [41]. For $(x,t) \in \mathbb{R}^d \times \mathbb{R}_{\geq 0}$, let w satisfy

$$w_{tt}(x,t) + \gamma(x) w_t(x,t) + (-\Delta + 1)^{s/2} w(x,t) = 0. \tag{3.2.1}$$

The damping force is represented by γw_t. Herein, we study the decay rate of the energy of w, defined by

$$E(t) = \|(w(t), w_t(t))\|_{H^{s/2} \times L^2} = \left(\int_{\mathbb{R}^d} |(-\Delta + 1)^{s/4} w(x,t)|^2 + |w_t(x,t)|^2 \, dx \right)^{1/2}.$$

The rate of change of the energy can be related explicitly to γ through the following identity.

$$\begin{aligned}\frac{d}{dt}E(t)^2 &= 2\,\Re\int_{\mathbb{R}} \overline{(-\Delta+1)^{s/4}w}(-\Delta+1)^{s/4}w_t + w_t\bar{w}_{tt} \\ &= 2\,\Re\int_{\mathbb{R}} \overline{((-\Delta+1)^{s/2}w + w_{tt})}w_t \\ &= -2\int_{\mathbb{R}} \gamma|w_t|^2\end{aligned}$$

In particular, if $\gamma = 0$, then the energy is conserved, i.e. there is no decay.

Theorem 3.2.1. *Let $0 \le \gamma \in L^\infty(\mathbb{R})$. There exists $R > 0$ such that*

$$\inf_{a\in\mathbb{R}} \int_{a-R}^{a+R} \gamma(x)\,dx > 0 \tag{3.2.2}$$

if and only if there exists $C, \omega > 0$ such that

$$E(t) \le \begin{cases} C(1+t)^{\frac{-s}{4-2s}}\|w(0), w_t(0)\|_{H^s\times H^{s/2}} & \text{if} \quad 0 < s < 2 \\ \\ Ce^{-\omega t}E(0) & \text{if} \quad s \ge 2 \end{cases}$$

for all $t > 0$ whenever the right-hand side is finite.

Note that for γ bounded, the condition (3.2.2) is equivalent to $\{x \in \mathbb{R} : \gamma(x) \ge \varepsilon\}$ being a relatively dense set (Definition 3.1.1) for ε small enough. However, if γ is unbounded, then (3.2.2) is the weaker condition.

The above result does not say anything about the optimality of the rates. However, we can answer the question posed in [41] concerning the value of the threshold between exponential and polynomial decay. We will show that exponential decay neccesitates that s be greater than 2 (as long as γ is not bounded away from zero), thus establishing $s = 2$ as the threshold.

Theorem 3.2.2. *Let $0 \leq \gamma \in L^\infty(\mathbb{R})$ and $s > 0$. Suppose*

(i) $m(\{\gamma = 0\}) > 0$.

(ii) There exists $C, \omega > 0$ such that $E(t) \leq Ce^{-\omega t} E(0)$ for all $t > 0$.

Then $s \geq 2$.

The main ingredient in our proof is the resolvent estimate just proved in Proposition 3.1.7 for the fractional Laplacian. In order to conclude the polynomial or exponential decay in Theorem 3.2.1, we will use (as a black box) the following two results on semigroups which connect resolvent bounds for the generator to the decay of the semigroup. For exponential decay, there is the following characterization from [25, Theorem 3] (See also [16, 51]).

Theorem 3.2.3 (Gearhart-Pruss Test)**.** *Let e^{tA} be a C_0-semigroup in a Hilbert space $\mathcal{H}$ and assume there exists $M > 0$ such that $\|e^{tA}\| \leq M$ for all $t \geq 0$. Then, there exists $C, \omega > 0$ such that*

$$\|e^{tA}\| \leq Ce^{-\omega t}$$

if and only if $i\mathbb{R} \subset \rho(A)$ and $\sup_{\lambda \in \mathbb{R}} \|(A - i\lambda)^{-1}\| < \infty$.

For the polynomial decay, we use the following result from [8, Theorem 2.4]:

Theorem 3.2.4 (Borichev-Tomilov)**.** *Let e^{tA} be a C_0-semigroup on a Hilbert space $\mathcal{H}$. Assume there exists $M > 0$ such that $\|e^{tA}\| \leq M$ for all $t \geq 0$ and $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$,*

$$\|e^{tA}A^{-1}\| = O(t^{-1/\alpha}) \text{ as } t \to \infty$$

if and only if $\|(A - i\lambda)^{-1}\| = O(\lambda^{\alpha})$ as $\lambda \to \infty$.

3.2.1 Proof of Energy Decay Rates

To apply (3.1.1) to the wave equation (4.2.3), we first represent the wave equation as a semigroup: Setting $W(t) = (w(t), w_t(t))$, we see that (4.2.3) is equivalent to

$$\frac{d}{dt}W(t) = \mathcal{A}_\gamma W(t)$$

where $\mathcal{A}_\gamma : H^s \times H^{s/2} \to H^{s/2} \times L^2$ is densely defined by $A_\gamma(u_1, u_2) = (u_2, -(-\Delta + 1)^{s/2}u_1 - \gamma u_2)$. The Sobolev space H^r for $r > 0$ is defined by the decay of the Fourier transform:

$$H^r := \left\{ u \in L^2 : \|u\|_{H^r}^2 = \int_{\mathbb{R}} (|\xi|^2 + 1)^r |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}, \quad \langle u, v\rangle_{H^r} := \langle (\Delta + 1)^r u, v\rangle_{L^2}.$$

The definition above is more convenient for our setting so that $\|u\|_{H^{s/2}} = \|(-\Delta + 1)^{s/4}u\| = \langle(-\Delta+1)^{s/2}u, u\rangle_{L^2}$, but the multiplier is equivalent to the usual multiplier $(|\xi|+1)^{2r}$. It can be easily checked that $\mathcal{A}_0$ is a closed skew-adjoint operator on $H^{s/2} \times L^2$ therefore $e^{t\mathcal{A}_0}$ is a semigroup of unitary operators. Then, since $\gamma \geq 0$, for $U = (u_1, u_2) \in H^s \times H^{s/2}$,

$$\Re\langle\mathcal{A}_\gamma^* U, U\rangle_{H^{s/2}\times L^2} = \Re\langle\mathcal{A}_\gamma U, U\rangle_{H^{s/2}\times L^2}$$

$$= \Re\langle\mathcal{A}_0 U, U\rangle_{H^{s/2}\times L^2} - \langle\gamma u_2, u_2\rangle_{L^2} = -\langle\gamma u_2, u_2\rangle_{L^2} \leq 0.$$

Moreover, since $\gamma \in L^\infty(\mathbb{R})$, the domain of $\mathcal{A}_\gamma$ is the same as $\mathcal{A}_0$. So, by classical semigroup theory [50] e^{tA_γ} is a C_0-semigroup of contractions. We now apply Proposition 3.1.7 to $\mathcal{A}_0$ and $\mathcal{A}_\gamma$. The first step is an observability inequality for the undamped wave equation (4.2.3).

Proposition 3.2.5. *Let $\Omega \subset \mathbb{R}$ be relatively dense, $s > 0$. Then, there exists $c > 0$ such that*

$$c\|U\|^2_{H^{s/2}\times L^2} \leq (|\lambda|+1)^{\frac{4}{s}-2}\|(\mathcal{A}_0 - i\lambda)U\|^2_{H^{s/2}\times L^2} + \|u_2\|^2_{L^2(\Omega)}$$

for all $U = (u_1, u_2) \in H^s \times H^{s/2}$ and $\lambda \in \mathbb{R}$.

Before proving this, we mention that by Proposition 3.0.1, this implies that for Ω relatively dense, there exists $C, T > 0$ (again with polynomial dependence on the parameters $\gamma, R)$ such that

$$\|(u_0, u_1)\|_{H^{s/2}\times L^2} \leq C \int_0^T \int_\Omega |u_t(x,t)|^2\, dx\, dt \tag{3.2.3}$$

for all solutions u to the undamped fractional Klein-Gordon equation with $s \geq 2$ for $(x,t) \in \mathbb{R} \times (0,\infty)$:

$$u_{tt}(x,t) + (-\Delta+1)^{s/2}u(x,t) = 0, \quad u(x,0) = u_0, \quad u_t(x,0) = u_1.$$

Proof. For $U = (u_1, u_2) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$, set $w_1 = (-\Delta+1)^{s/4}u_1 - iu_2$ and $w_2 = (-\Delta+1)^{s/4}u_1 + iu_2$. First, by the parallelogram identity,

$$\|w_1\|^2_{L^2(\mathbb{R})} + \|w_2\|^2_{L^2(\mathbb{R})} = 2\|(-\Delta+1)^{s/4}u_1\|^2 + 2\|u_2\|^2 = 2\|U\|^2_{H^{s/2}\times L^2}.$$

Second,

$$\begin{aligned}
\|(\mathcal{A}_0 - \lambda I)U\|^2_{H^{s/2}\times L^2} &= \|(-\Delta+1)^{s/4}(-\lambda u_1 + u_2)\|^2 + \|-(-\Delta+1)^{s/2}u_1 - \lambda u_2\|^2 \\
&= \|-\lambda\frac{w_1+w_2}{2} + i(-\Delta+1)^{s/4}\frac{w_1-w_2}{2}\|^2 \\
&\quad + \|-(-\Delta+1)^{s/4}\frac{w_1+w_2}{2} - i\lambda\frac{w_1-w_2}{2}\|^2 \\
&= \|-i\lambda w_1 - (-\Delta+1)^{s/4}w_1\|^2 + \|-i\lambda w_2 + (-\Delta+1)^{s/4}w_2\|^2.
\end{aligned}$$

So, applying Proposition 3.1.7 to w_1 with s replaced by $s/2$, we have, for $\lambda \geq 0$,

$$
\begin{aligned}
2c\|U\|^2_{H^{s/2}\times L^2} &= c(\|w_1\|^2 + \|w_2\|^2) \\
&\leq (|\lambda|+1)^{\frac{4}{s}-2}\|((-\Delta+1)^{s/4}-\lambda)w_1\|^2 + \|w_1\|^2_{L^2(\Omega)} + c\|w_2\|^2 \\
&\leq (|\lambda|+1)^{\frac{4}{s}-2}\|((-\Delta+1)^{s/4}-\lambda)w_1\|^2 + 2\|w_1-w_2\|^2_{L^2(\Omega)} + (c+2)\|w_2\|^2 \\
&\leq (|\lambda|+1)^{\frac{4}{s}-2}\|((-\Delta+1)^{s/4}-\lambda)w_1\|^2 + 8\|u_2\|^2_{L^2(\Omega)} + \frac{c+2}{(|\lambda|+1)^2}\|((-\Delta+1)^{s/4}+\lambda)w_2\|^2 \\
&\leq (c+2)(|\lambda|+1)^{\frac{4}{s}-2}\|(\mathcal{A}_0 - i\lambda I)U\|^2_{H^{s/2}\times L^2} + 8\|u_2\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We get the case $\lambda < 0$ by exchanging the roles of w_1 and w_2. □

Finally we extend this to $\mathcal{A}_\gamma - i\lambda I$ and prove Theorem 3.2.1. First notice that for any $R, \varepsilon > 0$, $a \in \mathbb{R}$,

$$
\int_{a-R}^{a+R} \gamma(x)\,dx \leq \|\gamma\|_\infty m(\{\gamma \geq \varepsilon\} \cap [a-R, a+R]) + 2R\varepsilon.
$$

So, (3.2.2) implies that $\{\gamma > \varepsilon\}$ is relatively dense for ε small enough. Therefore, taking $\Omega = \{\gamma \geq \varepsilon\}$ and applying Proposition 3.2.5,

$$
\begin{aligned}
c\|U\|^2_{H^{s/2}\times L^2} &\leq (|\lambda|+1)^{\frac{4}{s}-2}\|(\mathcal{A} - i\lambda I)U\|^2_{H^{s/2}\times L^2} + \|u_2\|^2_{L^2(\Omega)} \\
&\leq 2(|\lambda|+1)^{\frac{4}{s}-2}\|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} + \left[2(|\lambda|+1)^{\frac{4}{s}-2} + \varepsilon^{-2}\right]\|\gamma u_2\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.2.4}
$$

We estimate the final term. Since $\mathcal{A}_0$ is skew-adjoint,

$$\Re\langle(\mathcal{A}_\gamma - i\lambda I)U, U\rangle = \Re\langle(\mathcal{A}_0 - i\lambda I)U, U\rangle - \langle\gamma u_2, u_2\rangle = -\|\sqrt{\gamma}u_2\|^2$$

which implies

$$D\|\gamma u_2\|^2 \leq D\|\gamma\|_\infty\|\sqrt{\gamma}u_2\|^2 \leq \frac{D^2\|\gamma\|_\infty^2\|(\mathcal{A}_\gamma - i\lambda)U\|^2}{\delta} + \delta\|U\|^2$$

for any $D, \delta > 0$. Choosing $D = 2(|\lambda|+1)^{\frac{4}{s}-2} + \varepsilon^{-2}$ and $\delta = c/2$, from (3.2.4) we obtain

$$c\|U\|_{H^{s/2}\times L^2}^2 \leq C\left[(|\lambda|+1)^{\frac{4}{s}-2} + (|\lambda|+1)^{\frac{8}{s}-4} + 1\right]\|(\mathcal{A}_\gamma - i\lambda I)U\|_{H^{s/2}\times L^2}^2 + \frac{c}{2}\|U\|_{H^{s/2}\times L^2}^2.$$

Thus, we have proved the following estimate for $(\mathcal{A}_\gamma - i\lambda I)^{-1}$:

$$\|(\mathcal{A}_\gamma - i\lambda I)^{-1}\|_{H^{s/2}\times L^2 \to H^{s/2}\times L^2} \leq \begin{cases} C(|\lambda|+1)^{\frac{4}{s}-2} & 0 < s < 2 \\ C & s \geq 2. \end{cases} \tag{3.2.5}$$

Applying the Theorems 3.2.3 and 3.2.4 allows one to conclude the decay rates in Theorem 3.2.1 from (3.2.5).

3.2.2 Neccessity of (3.2.2)

We now prove the converse in Theorem 3.2.1. By the Gearhart-Pruss Test (Theorem 3.2.3) and Borichev-Tomilov (Theorem 3.2.4), the decay rates of the energy in Theorem 3.2.1 imply

$$c\|U\|^2_{H^{s/2}\times L^2} \leq \|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} \tag{3.2.6}$$

for some $c = c(s,\lambda) > 0$ and for all $U \in H^{s/2}\times L^2$ and all $\lambda \in \mathbb{R}$. Taking $U = ((-\Delta+1)^{-s/4}u, iu)$ for $u \in L^2(\mathbb{R})$, we have

$$2c\|u\|^2 \leq \|(-\lambda + (-\Delta+1)^{s/4})u\|^2 + \|(-(-\Delta+1)^{s/4} - i\gamma + \lambda)u\|^2$$

$$\leq 3\|((-\Delta+1)^{s/4} - \lambda)u\|^2 + 2\|\gamma u\|^2. \tag{3.2.7}$$

Now, we only consider the special case $\lambda = 1$. Let $u \in L^2(\mathbb{R})$ such that $\operatorname{supp}\hat{u} \subset [-D, D]$ for some $D > 0$ to be fixed later. For such u,

$$\|((-\Delta+1)^{s/4} - 1)u\|^2 = \int_{-D}^{D} [(|\xi|^2+1)^{s/4} - 1]^2 |\hat{u}(\xi)|^2\, d\xi \leq [(D^2+1)^{s/4} - 1]^2\|u\|^2.$$

So, taking D small enough, we obtain that there exists $C > 0$ such that

$$\|u\|^2 \leq C\|\gamma u\|^2 \tag{3.2.8}$$

for all $u \in L^2(\mathbb{R})$ satisfying $\operatorname{supp} \hat{u} \subset [-D, D]$. Set $f(x) = \frac{\sin(Dx)}{Dx}$. Then, $\operatorname{supp} \hat{f} \subset [-D, D]$. For each $a \in \mathbb{R}$, set $f_a(x) = f(x - a)$. Of course, $\operatorname{supp} \hat{f}_a \subset [-D, D]$ and $\|f_a\| = \|f\|$. Thus, for any $R > 0$,

$$\|f\|^2 = \|f_a\|^2 \leq C\|\gamma f_a\|^2 = C \int_{[a-R,a+R]} + \int_{[a-R,a+R]^c} |\gamma(x) f_a(x)|^2 \, dx$$

The second integral goes to 0 (uniformly in a) as $R \to \infty$ since γ is bounded and $f \in L^2$. The first integral becomes

$$\int_{a-R}^{a+R} |\gamma(x) f_a(x)|^2 \, dx \leq \|\gamma\|_\infty \int_{a-R}^{a+R} \gamma(x) \, dx$$

since f is bounded by 1. Thus there exists R large such that (3.2.2) holds.

We remark that to prove the neccessity of the condition (3.2.2), the decay rates from Theorem 3.2.1 can be replaced by an a priori weaker condition, namely that there exists $\lambda \geq 1$ such that $i\lambda$ is in the resolvent of $\mathcal{A}_\gamma$ and $\mathcal{A}_\gamma - i\lambda$ has closed range. Then, setting $\mu = \sqrt{\lambda^{2/s} - 1}$, we obtain (3.2.8) for $\operatorname{supp} \hat{u} \subset [\mu - D, \mu + D]$ (D small enough). The proof is completed analogously by taking $f(x) = e^{i\mu x} \frac{\sin(Dx)}{Dx}$.

3.2.3 Proof of Theorem 3.2.2

To prove the threshold value (Theorem 3.2.2), we use the fact that exponential decay yields (3.2.6) with c independent of λ, from which (3.2.7) follows. Suppose that $s < 2$. We will derive a contradiction. In this case, we take $\operatorname{supp} \hat{u} \subset \{\xi \in \mathbb{R} :$

$\big|(|\xi|^2+1)^{s/4}-\lambda\big| \le K\} =: A_\lambda(K)$ for K to be chosen later. Then, we have

$$\|((-\Delta+1)^{s/4}-\lambda)u\|^2 = \int_{A_\lambda(K)} [(|\xi|^2+1)^{s/4}-\lambda]^2 |\hat{u}(\xi)|^2 \, d\xi \le K^2 \|u\|^2.$$

So taking K small enough, we have, as above,

$$c\|u\| \le \|\gamma u\| \tag{3.2.9}$$

whenever $\operatorname{supp} \hat{u} \subset A_\lambda(K)$, $\lambda \in \mathbb{R}$. $A_\lambda(K)$ is the union of the two intervals

$$\pm\left[\sqrt{(\lambda-K)^{4/s}-1}, \sqrt{(\lambda+K)^{4/s}-1}\right]$$

and we notice that the length of these intervals is increasing if $s < 2$. Indeed,

$$\lim_{\lambda\to\infty} \sqrt{(\lambda+K)^{4/s}-1} - \sqrt{(\lambda-K)^{4/s}-1} = \lim_{\lambda\to\infty} \frac{\lambda^{4/s-1}}{\lambda^{2/s}}$$

which is ∞ if $s < 2$. Thus, (3.2.9) holds for $\operatorname{supp} \hat{u}$ contained in any ball since (3.2.9) does not see modulation of u (translation of $\hat{u}$).

We demonstrate that this is a violation of the uncertainty principle. Let $f(x) = \mathbb{1}_{\{\gamma=0\}}(x)\phi(x)$, where ϕ is some positive L^2 function so that $f \in L^2$ and $\gamma f = 0$. Then, $\hat{f} \in L^2$ so setting $g_R = \mathcal{F}^{-1}(\mathbb{1}_{B(0,R)}\hat{f})$, g_R converges to f in the L^2 norm. Therefore, since $\operatorname{supp} \hat{g}_R \subset B(0,R)$, by (3.2.9),

$$c\|g_R\| \leq \|\gamma g_R\| \leq \|\gamma f\| + \|\gamma(g_R - f)\| \leq \|\gamma\|_\infty \|g_R - f\|.$$

The LHS goes to $c\|f\| > 0$ (f is nonzero since $m(\{\gamma = 0\}) > 0$) while the RHS appoaches zero as $R \to \infty$ which is a contradiction.

3.3 Energy Decay in Higher Dimensions

3.3.1 Geometric Control Condition

A natural question is whether the results of the previous section hold in higher dimensions. The first step is to find the appropriate generalization of relative density to higher dimensions.

Definition 3.3.1. A set $E \subset \mathbb{R}^d$ is said to satisfy the Geometric Control Condition (GCC) if there exists $L, c > 0$ such that

$$m_1(\ell \cap E) \geq L \cdot c$$

for all line segments $\ell \subset \mathbb{R}^d$ of length L.

m_1 is the one-dimensional Hausdorff measure, i.e. the Lebesgue measure on the line containing ℓ. In this way, we recover relative density when $d = 1$.

The only part where we used the dimension was in proving the uncertainty principle Theorem 3.1.6. So, we pose the question of whether a higher dimensional analogue holds:

Question 3.3.2. Suppose that $E \subset \mathbb{R}^d$ satisfies the GCC. Does there exist $C, \delta > 0$ such that for every $\lambda > 0$,

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)} \tag{3.3.1}$$

for every $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subset A_\lambda = \{\xi \in \mathbb{R}^d : \lambda \leq |\xi| \leq \lambda + \delta\}$?

We emphasize that the inequality (3.3.1) holds independent of λ. This is analagous to Theorem 3.1.6 holding for arbitrary a and b.

We have made partial progress towards this problem in the paper [18] in the form of two theorems. The first is an analogue of the Paneah-Logvinenko-Sereda Theorem for functions whose Fourier transform is supported in a strip (by a strip of width β, we mean any translation and rotation of $[0, \beta] \times \mathbb{R}^{d-1}$).

Theorem 3.3.3. *Let $E \subset \mathbb{R}^d$. E satisfies the GCC if and only if for any $\beta > 0$, there exists $C > 0$ such that*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)} \tag{3.3.2}$$

whenever $\operatorname{supp} \hat{f}$ *is contained in a strip of width β.*

Applying this, one can give an affirmative answer to the question above if one replaces E in (3.3.1) with a δ-neighborhood of E, E_δ.

Theorem 3.3.4. *Let $E \subset \mathbb{R}^d$ satisfy the GCC. For any $\beta, \delta > 0$, there exists $C > 0$ such that*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E_\delta)} \tag{3.3.3}$$

whenever $f \in L^2(\mathbb{R}^d)$ satisfies $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : \lambda \leq |\xi| \leq \lambda + \beta\}$ *for some $\lambda > 0$.*

Consequently, Theorem 3.2.1 holds in higher dimensions with the additional assumption that γ is uniformly continuous. The observability inequality (3.2.3) also holds under the assumption that E is a δ-neighborhood of a GCC set.

3.3.2 Relative Density

Relaxing the Geometric Control Condition to relative density, some decay does persist, though it is not exponential. For example, if the damping is positive on a $\mathbb{Z}^d$-periodic open set, then Wunsch has shown in [56] that the energy decays polynomially. Burq and Joly in [10] have shown logarithmic decay when the damping is positive on a relatively dense union of balls. This result can be extended to any relatively dense damping using the sharp constant in the PLS theorem in [30].

One could improve this to polynomial decay by proving the following uncertainty principle.

Question 3.3.5. Let $E \subset \mathbb{R}^d$ be relatively dense. Does there exists $\delta, m, C > 0$ such that for all $\lambda > 0$,

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)}$$

for all $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : \big| |\xi|^m - \lambda^m \big| \leq \delta\}$?

m controls the rate at which the widths of the annuli degenerate to 0. For $m = 1$, they have bounded width δ. For $m = d$, they have bounded area $\sim \delta$. With an affirmative answer to this question, one could apply the same strategy as the previous section to get exponential decay for $s \geq 2m$ and polynomial for $s < 2m$. In particular, when $s = 2$, the wave equation would decay polynomially.

Chapter 4

Uncertainty Principle and Localization Operators

4.1 Motivation: Pseudodifferential Operators

Recall Benedicks Theorem [7] which states that if $m(\operatorname{supp} f)m(\operatorname{supp} \hat{f}) < \infty$ then $f = 0$. This can be reformulated as saying the pseudo-differential operator

$$U_{E\times F}f(x) = \mathbb{1}_E(x)\int_{\mathbb{R}^d} \mathbb{1}_F(\xi)\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

does not have an eigenvalue at 1 when $m(E)m(F) < \infty$. This can be generalized to any $\Omega \subset \mathbb{R}^{2d}$ with $m(\Omega) < \infty$:

$$U_\Omega f(x) = \int_{\mathbb{R}^d} \mathbb{1}_\Omega(x,\xi)\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

with a slight modification.

Theorem 4.1.1. *Let* $\Omega \subset \mathbb{R}^{2d}$ *with* $m(\Omega) < \infty$*. If* $U_{\Omega'} f = f$ *for all* $\Omega' \supset \Omega$*, then* $f = 0$.

Proof. Of course if Ω has finite measure, so does any subset. Therefore, replace Ω with the smallest Ω with the above property. U_Ω is compact since its kernel $\mathbb{1}_\Omega$ is square integrable. Indeed,

$$(\tilde{U} f)(x) := \int \mathbb{1}_\Omega(x, y) f(y) e^{2\pi x \cdot y} \, dy$$

is compact thus, $U_\Omega f = \tilde{U}\hat{f}$ is also. Suppose there exists $f \neq 0$ such that $U_{\Omega'} f = f$ for every $\Omega' \supset \Omega$.

Since the Lebesgue measure is continuous w.r.t. translation, we can pick $\varepsilon_k \in \mathbb{R}^d_+$ such that

$$|\Omega \backslash (\Omega - (\varepsilon_k, 0))| \leq 2^{-k}.$$

Then, set $\delta_k = \sum_{j=1}^k (\varepsilon_j, 0)$ and $f_k(x) := f(x + \delta_k)$. Set $\Omega_N = \cup_{j=1}^N (\Omega - \delta_j)$ for each $N \in \mathbb{N}$. In this way $U_{\Omega_N} f_k = f_k$ for $k \leq N$ but not for $k > N$. Now we claim $\{f_k\}$ are linearly independent. Indeed, for any $\sum_{k=1}^N c_k f_k = 0$,

$$0 = (U_{\Omega_N} - U_{\Omega_{N-1}}) \sum_{k=1}^N c_k f_k = c_N (f_N - U_{\Omega_{N-1}} f_N).$$

However, $f_N \neq U_{\Omega_{N-1}} f_N$ therefore $c_N = 0$. This process can be repeated to show all c_k's are zero. Now, consider the operator $U_{\cup_N \Omega_N}$. This operator is also compact since ε_k were chosen so that $\cup_N \Omega_N$ has finite measure. Therefore the eigenspace corresponding to the eigenvlue 1 must be finite dimiensional. However, $U_{\cup_N \Omega_N} f_k = f_k$ for all k, so $\{f_k\}_{k=1}^{\infty} \subset K$. Since $\{f_k\}$ are linearly independent, $\dim K = \infty$, which is a contradiction. □

4.2 Generalized Parseval Frames

We can implement this program in the setting of abstract harmonic analysis. Let X be a locally compact group with left Haar measure μ and left invariant metric d. Such groups will be indexing sets for the following representation of a Hilbert space $\mathcal{H}$.

Definition 4.2.1. A collection of vectors $\{k_x\}_{x \in X} \subset \mathcal{H}$ is said to be a generalized Parseval frame for $\mathcal{H}$ if

$$f = \int_X \langle f, k_x \rangle k_x \, d\mu(x)$$

for each $f, g \in \mathcal{H}$.

We will require that the frames respect the topology and algebra of the group X in the following sense.

(a) $|\langle k_{xz}, k_{xy} \rangle| = |\langle k_z, k_y \rangle|$ for all $x, y, z \in X$.

(b) The function $x \mapsto k_x$ is continuous.

For $E \subset X$ measurable with $\mu(E) > 0$, define the localization operator $L_E : \mathcal{H} \to \mathcal{H}$ by

$$L_E f = \int_E \langle f, k_x \rangle k_x \, d\mu(x).$$

For each f in $\mathcal{H}$, define the spectrum of f, denoted by spec(f), to be $\{x \in X : \langle f, k_x \rangle \neq 0\}$. In this way, $\operatorname{spec} f \subset E$ is equivalent to $L_E f = f$.

We will say a set $E \subset X$ is an annihilating set if $\operatorname{spec} f \subset E$ implies $f = 0$. We consider two classes of sets. In the first, we consider the most general case, for which we impose the following conditions on the group X:

(i) There exists $x \in X$ such that, setting $x_0 = x$, for each $m = 1, 2, \ldots$, there exists x_m such that $x_m^2 = x_{m-1}$.

(ii) For each m, $d(1, x_m^k) \to \infty$ as $k \to \infty$ (k is an integer).

(iii) $x_m \to 1$ as $m \to \infty$.

Under these assumptions, we prove that if $\mu(E) < \infty$, then

1. L_E is compact.
2. If $\operatorname{spec} f \subset E$, then $f = 0$.
3. There exists $c > 0$ such that $\int_{X \setminus E} |\langle f, k_x \rangle|^2 \, d\mu(x) \geq c\|f\|^2$.

We can obtain the same results (1.–3.) for a more general class of sets E under additional assumptions on X and $\{k_x\}$.

Definition 4.2.2. A set $E \subset X$ is said to be thin if

$$\lim_{d(1,y)\to\infty} \int_E |\langle k_x, k_y \rangle|^2 \, d\mu(x) = 0 \tag{4.2.1}$$

The quantity above is $\langle L_E k_y, k_y \rangle$, the so-called Berezin transform of L_E. A useful relationship is $\langle L_{y^{-1}E} k_1, k_1 \rangle = \langle L_E k_y, k_y \rangle$. We will also use the finite measure $\nu(A) = \langle L_A k_1, k_1 \rangle$ so thinness is characterized by $\nu(yE) \to 0$ as $d(1, y) \to \infty$. We impose the following additional assumptions on X and $\{k_x\}_{x\in X}$.

(iv) (X, d) is a Heine-Borel metric space (i.e. every ball is totally bounded).

(v) $\int_X |\langle k_1, k_x \rangle| \, d\mu(x) < \infty$.

(vi) $|\langle k_x, k_y \rangle| \to 0$ as $d(x, y) \to \infty$.

The assumption (iv) is useful is giving an equivalent definition of thinness.

Proposition 4.2.3. *Let (X, d) be a Heine-Borel metric space. The following are equivalent*

(i) E is thin.

(ii) $\lim_{d(y,1)\to\infty} \mu(E \cap B(y, R)) = 0$ *for some* $R > 0$.

(iii) $\lim_{d(y,1)\to\infty} \mu(E \cap B(y, R)) = 0$ *for all* $R > 0$.

Proof. (ii) implies (iii) by the Heine-Borel property since a ball can be covered by finitely many balls of any radius.

Also notice that by the invariance of μ and d, $\mu(E \cap B(y,R)) = \mu(y^{-1}E \cap B(1,R))$. Consider the function $h(x) = \langle k_1, k_x \rangle$. $h(1) = \|k_1\| > 0$ and h is continuous so there exists $\delta > 0$ such that $|h(x)| \geq h(1)/2$ for $x \in B(1,\delta)$. Thus,

$$\frac{h(1)^2}{4}\mu(y^{-1}E \cap B(1,\delta)) \leq \int_{y^{-1}E \cap B(1,\delta)} |h(x)|^2 \, d\mu(x)$$

$$\leq \langle L_{y^{-1}E} k_1, k_1 \rangle = \langle L_E k_y, k_y \rangle.$$

Therefore (i) implies (ii). To show (iii) implies (i), let $\varepsilon > 0$. We can find $R > 0$ such that $\int_{B(1,R)^c} |h(x)|^2 \, d\mu(x) < \varepsilon/2$. For this R, by (iii), there exists $N > 0$ such that for $d(1,y) \geq N$, $\mu(y^{-1}E \cap B(1,R)) \leq \varepsilon/(2\|k_1\|^4)$. Therefore

$$\langle L_E k_y, k_y \rangle = \int_{y^{-1}E} |h(x)|^2 \, d\mu(x) = \int_{y^{-1}E \cap B(0,R)} + \int_{y^{-1}E \cap B(0,R)^c} |h(x)|^2 \, d\mu(x)$$

$$\leq \|k_1\|^4 \mu(y^{-1}E \cap B(0,R)) + \int_{B(0,R)^c} |h(x)|^2 \, d\mu(x) \leq \varepsilon.$$

We used the fact that $|h(x)| \leq \|k_1\| \cdot \|k_x\|$ but $\|k_x\| = \|k_1\|$. Indeed, by (a),

$$\|k_x\|^2 = |\langle k_x, k_x \rangle| = |\langle k_1, k_1 \rangle| = \|k_1\|^2.$$

$\square$

4.2.1 Translations of Thin Sets

We begin with two technical lemmas concerning the sequence $\{x_m\}$ constructed above in (i). The first states that repeated translations of E by elements of this sequence are in some sense independent.

Lemma 4.2.4. *Let E be thin, $L_E \neq 0$, and $\{x_m\}$ satisfy (i) and (ii) above. Then for any subsequence $\{x_{m_j}\}$ of $\{x_m\}$,*

$$\left(\prod_{j=0}^{n} x_{m_j}\right) E \not\subset \bigcup_{j=0}^{n-1} x_{m_j} x_{m_{j-1}} \cdots x_{m_2} x_{m_1} x_{m_0} E. \tag{4.2.2}$$

for each $n = 1, 2, 3, \ldots$.

Proof. For ease of notation, set $y_j = x_{m_j}$. Let n be a positive integer and suppose the containment (4.2.2) does hold. We claim that

$$y_n^k y_{n-1} \cdots y_1 y_0 E \subset \bigcup_{I \in P_{m_n}} \left(\prod_{i \in I} x_i\right) E =: E_n \tag{4.2.3}$$

for any $k \geq 0$. P_n denotes the power set of $\{0, 1, 2, \ldots, n\}$. We prove (4.2.3) by induction. It is immediate for $k = 1$. Suppose it is true for all $j < k$. Then, for y in $y_n^k y_{n-1} \cdots y_1 y_0 E$, by the assumption that (4.2.2) fails,

$$y \in y_n^{k-1} y_N y_{N-1} \cdots y_1 y_0 E \tag{4.2.4}$$

for some $N \leq n - 1$. If $N = n - 1$, we are done by the induction hypothesis. If $N <$

$n-1$, we can write the missing elements $y_{n-1}\cdots y_{N+1} = y_n^\ell$ for $0 < \ell \le \sum_{i=m_N}^{m_n-1} 2^{n-i}$.

If $k > \ell$, then the set from (4.2.4) equals $y_n^{k-\ell} y_{n-1} y_{n-2} \cdots y_1 y_0 E$ so by the induction hypothesis, we are done. If $k \le \ell$, then there exists $I \subset \{m_N, \ldots, m_n - 1\}$ such that

$$k = \sum_{i \in I} 2^{n-i}.$$

Therefore, $y_n^k y_N y_{N-1} \cdots y_1 y_0 = \left(\prod_{i\in I} x_i\right) x_{m_N} x_{m_{N-1}} \cdots x_{m_1} x_{m_2} E \subset E_n$. This completes the proof of (4.2.3).

We use this to derive a contradiction in order to establish (4.2.2). First notice that if E is thin then so is E_n since it is a finite union of translates of E. Defining the measure $\nu(A) = \langle L_A k_1, k_1\rangle$, note that thinness is equivalent to $\nu(y^{-1}E) \to 0$. By (4.2.3), setting $w_n = y_{n-1} \cdots y_1 y_0$, we have $E = (y_n^k w_n)^{-1} y_n^k w_n E \subset w_n^{-1} y_n^{-k} E_n$. By (ii), $d(1, y_n^k) \to \infty$ as $k \to \infty$ so $d(1, (y_n^k w_n y)^{-1}) \to \infty$ for any $y \in X$. Since E_n is thin,

$$\nu(y^{-1}E) \le \nu((y_n^k w_n y)^{-1} E_n) \to 0 \text{ as } k \to \infty,$$

which implies $0 = \nu(y^{-1}E) = \langle L_E k_y, k_y\rangle$ for every $y \in X$. This implies that for any $f \in \mathcal{H}$,

$$|\langle L_E f, f\rangle| = \int_E |\langle f, k_x\rangle|^2 \, d\mu(x) = \int_E \left| \int_X \langle f, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 \, d\mu(x)$$

$$\le \int_E \int_X |\langle f, k_y\rangle|^2 \, d\mu(y) \int_X |\langle k_y, k_x\rangle|^2 \, d\mu(y) \, d\mu(x)$$

$$= \|f\|^2 \int_X \int_E |\langle k_y, k_x\rangle|^2 \, d\mu(x) \, d\mu(y) = 0$$

so $L_E = 0$ which is the contradiction. □

Next, we want to show that the infinite union of small enough translations still satisfies (4.2.1).

Lemma 4.2.5. *Let E be thin and $y_n \to 1$. There exists a subsequence y_{n_k} such that*

$$F = \bigcup_{N=1}^{\infty} \left(\prod_{k=1}^{N} y_{n_k} \right) E \quad \textit{is thin.}$$

Proof. A key property we will use is the continuity of any outer regular measure $\nu(E)$ under the group operation in the sense that $\lim_{x\to 1} \nu(E \backslash xE) = 0$ whenever $\nu(E) < \infty$. We first prove this for E open. For each $y \in E$, there is a neighborhood of y, say U, which is still contained in E. Since $y \mapsto x^{-1}y$ is continuous, there exists a neighborhood of 1, say V, such that $x^{-1}y \in U$ whenever $x \in V$. This shows that $\chi_{E\backslash(xE)}(y) = \chi_E(y)(1 - \chi_E(x^{-1}y)) = 0$ for such x. This shows that $\chi_{E\backslash xE} \to 0$ pointwise as $x \to 1$. Therefore, by dominated convergence,

$$\lim_{x\to 1} \nu(E\backslash xE) = \lim_{x\to 1} \int_X \chi_{E\backslash xE}(y) \, d\nu(y) = 0.$$

Then, one can extend to any measurable set by approximation, since ν is outer regular.

The measure we will use is $\nu(E) := \int_E |\langle k_x, k_1\rangle|^2 \, d\mu(x)$. ν is finite since $\|k_1\|^2 = \nu(X)$ and ν is outer regular since it is a Radon measure. Therefore ν enjoys the continuity property above. ν is not invariant under the group operation, but we will prove that for any $\varepsilon > 0$ there exists a neighborhood around 1, V such that

$$|\nu(A) - \nu(zA)| \leq \varepsilon \tag{4.2.5}$$

for all $z \in V$, and $A \subset X$. Indeed,

$$|\nu(A) - \nu(zA)| = |\langle L_A k_1 | k_1 \rangle - \langle L_A k_z | k_z \rangle|$$

$$= |\langle L_A(k_1 - k_z) | k_1 \rangle + \langle L_A k_z | (k_1 - k_z) \rangle| \leq C \|k_1 - k_z\|.$$

So the uniform continuity follows from the continuity (and boundedness) of the frames.

We are now ready to construct the subsequence in two steps. First, since $d(y_m, 1) \to 0$, there is a subsequence such that $\sum_{i=0}^{\infty} d(y_{m_i}, 1) < \infty$. Set

$$Y = \left\{ \prod_{i \in I} y_{m_i} : I \subset \mathbb{N}, |I| < \infty \right\} \tag{4.2.6}$$

In this case, Y is bounded. Indeed, setting $I_\ell = I \cap \{\ell + 1, \ell + 2, \ldots\}$

$$d\left(\prod_{i \in I} y_{m_i}, 1\right) \leq \sum_{\ell=0}^{\infty} d\left(\prod_{i \in I_{\ell+1}} y_{m_i}, \prod_{i \in I_\ell} y_{m_i}\right) \leq \sum_{\ell=0}^{\infty} d(y_{m_\ell}, 1).$$

We will now take a subsequence of $\{y_{m_i}\}_{i=1}^{\infty}$ which we denote by $\{y_{n_k}\}_{k=1}^{\infty}$. It is constructed inductively along with a sequence of sets F_k. Set $F_0 = E$. Then, having y_{n_ℓ} for $1 \le \ell \le k$, set $F_k = \cup_{i=1}^k \prod_{j=1}^i y_{n_j} E$. Pick $y_{n_{k+1}}$ as follows: There exists $R_k > 0$ such that if $d(z,1) > R_k$, then $\nu(zE) < 2^{-(k+1)}$. So, since Y is bounded, let us say contained in the ball $B(1,r)$, if $d(z,1) \ge R_k + r$, then $d(zz_I,1) \ge d(z,1) - d(zz_I,z) = d(z,1) - d(z_I,1) \ge R_k + r - r = R_k$ so

$$\nu(zz_IE) \le \tfrac{1}{2^k}$$

for any $z_I \in Y$. Now, the ball $B(1, R_k + r)$ can be covered with $L(k)$ balls of radius $2^{-(k+1)}$ with centers $\{z_j\}_{j=1}^L$ (This can be done since X has the Heine-Borel property) so by (4.2.5),

$$\nu(zA) \le \sup_{j=1,\dots,L} \nu(z_jA) + \tfrac{1}{2^{k+1}}$$

for all $z \in B(1, R_k + r)$. Therefore, since the measures $\nu(z_j\cdot)$ are finite measures, we can pick $y_{n_{k+1}}$ such that

$$\nu(z(y_{n_{k+1}}F_k \backslash F_k)) \le \tfrac{1}{2^{k+1}}.$$

for all $z \in B(1, R_k + r)$. Since $F_{k+1} = F_k \cup (y_{n_{k+1}}F_k)$, we have $\nu(z(F_{k+1}\backslash F_k)) \le \frac{1}{2^k}$ for any $z \in X$.

Now we are ready to prove that $F = \cup_k F_k$ is thin. Let $\varepsilon > 0$ and pick j such that $\sum_{k=j}^{\infty} 2^{-k} \le \frac{\varepsilon}{2}$. Let $R > 0$ such that $\nu(yE) \le \frac{\varepsilon}{2^{j+1}}$ for $d(1,y) \ge R$. Then, for $d(1,y) \ge R + r$, $\nu(yz_IE) \le \frac{\varepsilon}{2^{j+1}}$ since $d(yz_I,1) \ge d(y,1) - d(yz_I,y) \ge R$ for all

$z_I \in Y \subset B(1,r)$. Then, by the fact that $|P_j| = 2^j$, we have

$$\nu(yF_j) \leq \sum_{I \in P_j} \nu(yz_I E) \leq 2^j (\tfrac{\varepsilon}{2^{j+1}}) = \tfrac{\varepsilon}{2}.$$

By construction of y_{n_k}, $\nu(y(F_k \backslash F_{k+1})) \leq 2^{-k}$. Therefore, since $F = \cup_k F_k$ and $F_k \subset F_{k+1}$,

$$\nu(yF) \leq \nu(yF_j) + \sum_{k=j}^{\infty} \nu(y(F_{k+1} \backslash F_k)) \leq \tfrac{\varepsilon}{2} + \sum_{k=j}^{\infty} 2^{-k} \leq \varepsilon.$$

□

Corollary 4.2.6. *Let $\mu(E) < \infty$ and $y_n \to 1$. There exists a subsequence $\{y_{n_k}\}$ such that $\mu(F) < \infty$.*

Proof. Since $\mu(E) < \infty$, $\mu(F_k) < \infty$ so one can pick $\{y_{n_k}\}$ such that $\mu(F_{k+1} \backslash F_k) \leq 2^{-k}$. Therefore,

$$\mu(F) \leq \mu(F_0) + \sum_{k=0}^{\infty} \mu(F_{k+1} \backslash F_k) \leq \mu(E) + 2.$$

□

4.2.2 Compactness

The most general sufficient condition for compactness of L_E is finite measure of E.

Proposition 4.2.7. *If $\mu(E) < \infty$, then L_E is compact.*

Proof. Let f_n converge to 0 weakly in $\mathcal{H}$. $\{f_n\}$ is bounded. Then, $\langle L_E f_n, f_n \rangle = \int_E |\langle f_n, k_x \rangle|^2 \, d\mu(x) \to 0$ by dominated convergence (dominated by a constant function). Since L_E is self-adjoint and (weakly) continuous, this implies $\|L_E f_n\| \to 0$ using the identity

$$\langle T(g + Tg), (g + Tg) \rangle = \langle Tg, g \rangle + \langle T(Tg), Tg \rangle + 2\|Tg\|^2$$

for T self adjoint, $g \in \mathcal{H}$. $\square$

With the extra assumptions (iv)-(vi), we can show thinness is equivalent to compactness in the form of two propositions. We separate them so the importance of each condition (iv)-(vi) is clear.

Proposition 4.2.8. *Let* $|\langle k_x, k_y \rangle| \to 0$ *as* $d(x, y) \to \infty$. *If* L_E *is compact, then* E *is thin.*

Proof. This is immediate if $k_y \xrightarrow{w} 0$ as $d(1, y) \to \infty$. Then $L_E k_y \to 0$ and $|\langle L_E k_{y_n}, k_{y_n} \rangle| \leq \|L_E k_{y_n}\| \to 0$ which is equivalent to (4.2.1). Now, we show that (vi) implies $k_y \xrightarrow{w} 0$. (vi) implies that $\langle k_y, k_x \rangle \to 0$ for each $x \in X$ as $d(1, y) \to \infty$. Moreover, $\overline{\text{span}}\{k_x\} = \mathcal{H}$ since $f = \int c_x k_x$ and the integral is a limit of simple functions, which correspond to linear combinations. Therefore, for any $f \in \mathcal{H}$, there exists $f_n \in \text{span}\{k_x\}$ such that $f_n \to f$. Therefore, since $\|k_y\| = \|k_1\|$ for all y, for any $\varepsilon > 0$ there exists N such that $\|f - f_N\| \leq \varepsilon/(2\|k_1\|)$. Morevoer, there exists M such that if $d(1, y) > M$ then $|\langle k_y, f_N \rangle| \leq \varepsilon/2$. Thus,

$$|\langle k_y, f \rangle| \leq \|k_1\| \cdot \|f - f_N\| + |\langle k_y, f_N \rangle| \leq \varepsilon$$

whenever $d(1, y) > M$. $\square$

Proposition 4.2.9. *Suppose (X, d) is Heine-Borel and $\int |\langle k_x, k_1\rangle| \, d\mu(x) < \infty$. If E is thin, then L_E is compact.*

Proof. Let $\varepsilon > 0$. Since $|\langle k_1, k_z\rangle|$ is integrable, there exists $R > 0$ such that

$$\int_{B(1,R)^c} |\langle k_1, k_z\rangle| \, d\mu(z) \leq \varepsilon.$$

Then, for any $x \in X$, using the group structure

$$\int_{B(x,R)^c} |\langle k_x, k_y\rangle| \, d\mu(y) = \int_{B(x,R)^c} |\langle k_1, k_{x^{-1}y}\rangle| d\mu(y)$$

$$= \int_{B(0,R)^c} |\langle k_1, k_z\rangle| \, d\mu(z) \leq \varepsilon$$

where in the last step we used the identity $x^{-1}B(x, R) = \{x^{-1}y : d(y, x) \leq R\} = \{z : d(xz, x) \leq R\} = \{z : d(z, 1) \leq R\} = B(0, R)$. In the same way, since $x^{-1}X = X$,

$$\int_X |\langle k_x, k_y\rangle| \, d\mu(y) = \int_X |\langle k_1, k_z\rangle| \, d\mu(z) =: M < \infty.$$

First we estimate the "tails" using the Schur property of $\{k_x\}$ and thinness of E. For any $f \in \mathcal{H}$,

$$\int_X \left| \int_{E \cap B(x,R)^c} \langle f, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 d\mu(x)$$

$$\leq \int_X \int_{B(x,R)^c} |\langle f, k_y\rangle|^2 |\langle k_y, k_x\rangle| \, d\mu(y) \int_{B(x,R)^c} |\langle k_y, k_x\rangle| \, d\mu(y) \, d\mu(x)$$

$$\leq M\varepsilon\|f\|^2.$$

Now, since E is thin, by Proposition 4.2.3, there exists $S > 0$ such that for $d(1, y) \geq S$, $\mu(E \cap B(y, R)) \leq \varepsilon$. Thus,

$$\int_{B(0,S)^c} \left| \int_{E\cap B(x,R)} \langle f, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 d\mu(x)$$

$$\leq \int_{B(0,S)^c} \mu(E \cap B(x, R)) \int_{E\cap B(x,R)} |\langle f, k_y\rangle|^2 |\langle k_x, k_y\rangle|^2 \, d\mu(y) \, d\mu(x)$$
$$\leq \varepsilon \int_X \int_X |\langle f, k_y\rangle|^2 |\langle k_x, k_y\rangle|^2 \, d\mu(y) \, d\mu(x) = \varepsilon\|k_1\|^2\|f\|^2.$$

Now, let $f_n \xrightarrow{\text{w}} 0$. $\|f_n\| \leq C$ for all n.

$$\begin{aligned}
\|L_E f_n\|^2 &= \int_X |\langle L_E f_n, k_x\rangle|^2 \, d\mu(x) \\
&= \int_X \left| \int_{E\cap B(x,R)} + \int_{E\cap B(x,R)^c} \langle f_n, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 d\mu(x) \\
&\leq \int_{B(0,S)} + \int_{B(0,S)^c} 2 \left| \int_{E\cap B(x,r)} \langle f_n, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 d\mu(x) + 2C^2 M\varepsilon \\
&\leq 2 \int_{B(0,S)} \left| \int_{E\cap B(x,r)} \langle f_n, k_y\rangle \langle k_y, k_x\rangle \, d\mu(y) \right|^2 d\mu(x) + 2C^2(M + \|k_1\|^2)\varepsilon.
\end{aligned}$$

Since the remaining integrals are both over finite areas and the integrand goes to zero pointwise,

$$\limsup_{n\to\infty} \|L_E f_n\|^2 \leq C'\varepsilon.$$

But ε is arbitrary so $\lim_{n\to\infty} L_E f_n = 0$. Therefore L_E is compact.

□

4.2.3 Annihilating Sets

We are now ready to prove the main result namely that no nontrivial functions can have a thin spectrum.

Proposition 4.2.10. *Let* $f \in \mathcal{H}$ *with* $\operatorname{spec} f$ *thin. Then* $f = 0$.

Proof. Suppose $f \neq 0$ and set $E = \operatorname{spec} f$. $L_E f = \int_E \langle f, k_x \rangle k_x \, d\mu(x) = f$ since $\langle f, k_x \rangle = 0$ for x outside E. Let x_n be a sequence satisfying (i)-(iii). Let $\{y_n\}$ be a subsequence of $\{x_n\}$ such that

$$F = \bigcup_{N=1}^{\infty} \prod_{k=1}^{N} y_n E$$

is thin (such a subsequence exists by Lemma 4.2.5). Set $w_n = \prod_{k=1}^{n} y_k$. Set $f_n = \int_E \langle f, k_x \rangle k_{w_n x} \, d\mu(x)$. In this way,

$$\langle f_n, k_z \rangle = \int_X \langle f, k_x \rangle \langle k_{w_n x}, k_z \rangle \, d\mu(x) = \langle f, k_{w_n^{-1} z} \rangle$$

so $\operatorname{spec} f_n \subset w_n E$. By Lemma 4.2.4, the functions $\{f_n\}_{n=1}^{\infty}$ are linearly independent. Indeed, suppose $\sum_{j=1}^{N} c_j f_j = 0$. There exists $x \in w_N E \setminus \cup_{k=1}^{N-1} w_k E$ in which case $0 = \langle \sum_{j=1}^{N} c_j f_j, k_x \rangle = c_N \langle f_N, k_x \rangle$ so $c_N = 0$. This can be repeated for $N-1, \ldots, 2, 1$ to show all c_n are zero. By Proposition 4.2.9, L_F is compact so its eigenspace corresponding to the eigenvalue 1, denoted by K must be finite dimensional. However, since $\operatorname{spec} f_n \subset w_n E \subset F$,

$$L_F f_n = \int_F \langle f_n, k_x \rangle k_x \, d\mu(x) = \int_X \langle f_n, k_x \rangle k_x \, d\mu(x) = f_n$$

for all n, so $\dim K = \infty$ which is a contradiction. Therefore $f = 0$. $\square$

Corollary 4.2.11. *Let E be thin. There exists $c > 0$ such that $\langle L_{X \setminus E} f, f \rangle \geq c\|f\|^2$ for all $f \in \mathcal{H}$.*

Proof. Since L_E is compact, positive, and self-adjoint, $\|L_E\| \leq \alpha$ where α is the largest eigenvalue. Since $\|L_E\| \leq 1$, $\alpha \leq 1$, but $\alpha \neq 1$ as seen in the previous proof. Therefore,

$$\langle L_{X \setminus E} f, f \rangle = \|f\|^2 - \langle L_E f, f \rangle \geq (1 - \alpha)\|f\|^2.$$

$\square$

4.2.4 Sets of Finite Measure

First, $\mu(E) < \infty$ implies E is thin since $\mu(E) < \infty$ implies (iii) holds in Proposition 4.2.3. Therefore, the conclusion of Lemma 4.2.4 holds for E. Replacing the use of Lemma 4.2.5 with Corollary 4.2.6 in the proof of Proposition 4.2.10 yields the following result in the more general case.

Proposition 4.2.12. *Let $\mu(E) < \infty$. There exists $c > 0$ such that $\langle L_{X \setminus E} f, f \rangle \geq c\|f\|^2$ for all $f \in \mathcal{H}$.*

4.2.5 Application to Wavelet Transform

We give an uncertainty principle for the Wavelet Transform as a consequence.

Consider the group of dilations and translations on $\mathbb{R}^d$ which is $X = \mathbb{R}_{>0} \times \mathbb{R}^d$ with the noncommutative operation $(a, b)(c, d) = (ac, ad + b)$. This is the so-called Affine Group. It has the a Haar measure given by

$$d\mu(a, b) = \frac{da\, db}{a^{d+1}}$$

where da and db are the Lebesgue measures on $\mathbb{R}_{>0}$ and $\mathbb{R}^d$. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^d)$ and the frames are

$$k_{(a,b)}(x) = \phi(\tfrac{x-b}{a})a^{-d/2} =: \pi(a, b)\phi(x)$$

for an admissible wavelet $\phi \in L^2$ satisfying

$$\int_0^\infty |\hat{\phi}(s\rho)|^2 \frac{ds}{s} = 1 \text{ for each } \rho \in \mathbb{S}^{d-1}.$$

This is indeed a Parseval frame satisfying (a) and (b) since the Wavelet transform

$$W_\phi f(a, b) = \langle f, \pi(a, b)\phi\rangle = \int_{\mathbb{R}^d} f(x)\pi(a, b)\phi(x)\, dx$$

obeys the inversion formula (see [21, Theorem 10.2])

$$f = \int_X W_\phi f(a,b)\pi(a,b)\phi\, d\mu(a,b).$$

The decay conditions (v) and (vi) hold if we impose the additional assumptions that

$$\int_{\mathbb{R}_{>0}\times\mathbb{R}^d} |W_\phi\phi(a,b)|\frac{dadb}{a^{d+1}} < \infty$$

and $|W_\phi\phi(a,b)| \to 0$ as $(a,b) \to \infty$. It only remains to verify the conditions on the group.

(i) The square root is well-defined by $\sqrt{(a,b)} = (\sqrt{a}, \frac{b}{\sqrt{a}+1})$.

(ii) $(a,b)^k = (a^k, (\sum_{m=0}^k a^m)b)$. So, if $a > 1$, then $a^k \to \infty$ in the usual topology on $\mathbb{R}$ so $d((a,b)^k, (1,0)) \to \infty$.

(iii) The invariant metric on this group also has the property that if $a \to 1$ and $b \to 0$ in the usual topologies of $\mathbb{R}$ and $\mathbb{R}^d$, then $d((a,b),(1,0)) \to 0$. One can see from the square root formula, that under repeated square roots, the first component converges to 1 and the second to 0.

(iv) Since $\mathbb{R}^{d+1}$ is Heine-Borel, this follows from the fact that balls in X are balls in $\mathbb{R}^{d+1}$ with different radius.

We can now state the result for these functions.

Theorem 4.2.13. *Let $E \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^d$ such that for some $r > 0$,*

$$\mu(E \cap B((a,b),r)) \to 0$$

as $(a,b) \to \infty$. There exists $\alpha > 0$ such that

$$\int_{(\mathbb{R}_{>0} \times \mathbb{R}^d) \setminus E} |W_\phi f(a,b)|^2 \frac{dadb}{a^{d+1}} \geq \alpha \|f\|^2$$

for all $f \in L^2(\mathbb{R}^d)$.

One can also recover a similar result for the Short-Time Fourier Transform as in [15].

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