12-2016

Boundary Controllability for One-Dimensional Wave and Heat Equations with Potential

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A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mathematical Sciences

by
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December 2016

Accepted by:
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Abstract

In this thesis we investigate the boundary controllability of the wave and heat equations with bounded potential in one dimension. This is done by way of the observability inequality. For the wave equation, we use the Hilbert Uniqueness Method of J. L. Lions to show the observability inequality is sufficient for exact controllability. Observability is shown by the multiplier method when there is no potential and a special Exchange of Variables technique for when potential is present. Due to limitiations of this method we also use a Carleman Estimate which can be extended to higher dimensions. For the heat equation, we use a Variational Method to show observability is sufficient for null controllability. The proof of observability is accomplished by an analogous Carleman Estimate to that for the wave equation.

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1 Introduction

In this thesis we present the boundary controllability problem for the one dimensional wave and heat equations with potential. That is, we study to what extent the wave and heat equations with potential can be controlled from a particular area on the boundary of the space domain where the equations are defined. In order to better demonstrate the ideas and simplify the proofs, we only focus on the one-dimensional case, but for most of the results in this thesis it can be seen how to extend our methods to higher dimensions. Physically speaking, our problems amount to the manipulation of a force or heat flux applied to one end of a string or a rod in order to achieve a specific shape of the string or temperature profile of the rod.

The main goals of this thesis are twofold. First is to understand the relationship between the "controllability" and the "observability" of a given evolution system, in particular for the wave and the heat equation, which are the prototypes of the second-order *hyperbolic* and *parabolic* equations. This relationship then leads to our second goal, which is to establish the observability inequality for the corresponding dual system. As we will see throughout this thesis, there is a major difference between these two types of equations. In addition, the presence of the potential also significantly complicates the methods that are used to achieve the observability inequality.

Roughly speaking, the controllability problem is to try to make a system behave according to our wishes. There are certain parameters (called "control" functions) of the system which may be manipulated in order to achieve a desired state. In this thesis we mainly consider the evolution systems—also referred as distributed systems, namely, the phenomenon is "distributed" in a geometrical domain—which are governed by partial differential equations (PDEs), and we are allowed to act on the trajectories of the systems by means of a suitable boundary condition.

To motivate the ideas behind our approach, let us first look at the classical problem of controlling a finite dimensional ordinary differential equation (ODE) system. Consider the ODE system

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + Bf(t) \quad t \in (0,T) \\ \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, $u(\cdot)$ is the *state variable*, $f(\cdot)$ is the *control variable*, $\mathbb{R}^n$ and $\mathbb{R}^m$ are the *state space* and *control space*, respectively. We say the ODE system (1.1) is **exactly controllable** in time $T > 0$ if for any initial state $u_0 \in \mathbb{R}^n$ and any final state $u_1 \in \mathbb{R}^n$, there exists a control function $f \in L^2(0,T;\mathbb{R}^m)$ such that the solution u satisfies $u(T) = u_1$.

Notice in the above definition of exact controllability any initial datum u_0 is required to be driven to the final datum u_1. Nevertheless, in view of the linearity of the system, without loss of generality, we may assume that $u_1 = 0$. Indeed, if $u_1 \neq 0$, then we may solve the system

$$\begin{cases} \dfrac{d\tilde{u}(t)}{dt} = A\tilde{u}(t) \quad t \in (0,T) \\ \\ \tilde{u}(T) = u_1 \end{cases}$$

backward in time and define the new state $v = u - \tilde{u}$ which satisfies

$$\begin{cases} \dfrac{dv(t)}{dt} = Av(t) + Bf(t) \quad t \in (0,T) \\ \\ v(0) = u_0 - \tilde{u}(0) \end{cases} \tag{1.2}$$

Observe that $u(T) = u_1$ if and only if $v(T) = 0$. Thus steering the solution u of (1.1) from u_0 to u_1 is equivalent to steering the solution v of (1.2) from the initial data $v_0 = u_0 - \tilde{u}(0)$ to 0. This observation leads to the notion of null controllability. More precisely, we say the ODE system (1.1) is **null controllable** in $T > 0$ if for any initial state $u_0 \in \mathbb{R}^n$ there exists a control function $f \in L^2(0, T; \mathbb{R}^m)$ such that the solution u satisfies $u(T) = 0$.

The above argument shows that the exact controllability and the null controllability are equivalent properties for the finite dimensional systems. This result may be extended to time reversible infinite dimensional systems, for example, the wave equation, as we are going to see in Chapter 3. On the other hand, however, this equivalence is not necessarily true for time irreversible infinite dimensional systems. For instance, as we will see in Chapter 4, the heat equation is only null boundary controllable but not exactly boundary controllable.

The controllability problem of the ODE system (1.1) was first studied by R.E.Kalman [4], and it was shown that the system (1.1) is exactly controllable in time $T > 0$ if and only if the rank of the matrix

$$rank[B, AB, \cdots, A^{n-1}B] = n.$$

where n is the dimension of the state space for (1.1). However, this criterion is not applicable for general infinite dimensional systems. Instead, it was noted in [4] that the exact controllability of system (1.1) is closely related to an inequality (see (1.5) below) for the corresponding adjoint homogeneous system. More precisely, let A^* be the adjoint matrix of A. Consider the following homogeneous *adjoint system* of (1.1)

$$\begin{cases} \dfrac{d\varphi}{dt} = -A^*\varphi(t) \quad t \in (0, T) \\ \varphi(T) = \varphi_T \end{cases} \tag{1.3}$$

and define the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$ by

$$J(\varphi_T) = \frac{1}{2}\int_0^T |B^*\varphi|^2 dt + \langle u_0, \varphi(0)\rangle. \tag{1.4}$$

We say the dual system (1.3) is **observable** in time $T > 0$ if there exists $C > 0$ such that

$$\int_0^T |B^*\varphi|^2 dt \geq C|\varphi(0)|^2 \tag{1.5}$$

for all $\varphi_T \in \mathbb{R}^n$, where φ is the corresponding solution of (1.3). The inequality (1.5) is in general called the **observability inequality**. It was shown that the system (1.1) is exactly controllable at time $T > 0$ if and only if the system (1.3) is observable in time $T > 0$. In particular, the observability inequality (1.5) implies the quadratic functional J defined in (1.4) is coercive.

Systems governed by PDEs are typically infinite dimensional. It is natural to seek to extend the above equivalence between the controllability and observability to the infinite dimensional settings. However, such extensions are not very straightforward since the controllability and observability of PDEs depend strongly on the nature of the underlying systems, such as propagation speed of solutions and time reversibility.

Definition Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces and consider the evolution system

$$\begin{cases} \dfrac{dU(t)}{dt} = AU(t) & t \in (0,T) \\ U(0) = U_0 & \\ BU(t) = f(t) & t \in (0,T) \end{cases} \tag{1.6}$$

where $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{H} \to \mathcal{G}$, $U : \mathbb{R}^+ \to \mathcal{H}$ and $f : \mathbb{R}^+ \to \mathcal{G}$.

The system (1.6) is said to be **exactly controllable** in time $T > 0$ if given any initial state $U_0 \in \mathcal{H}$ and final state $U_1 \in \mathcal{H}$, there exists a *control* function $f \in L^2(0,T;\mathcal{G})$ such that the solution U of (1.6) satisfies $U(T) = U_1$.

The system (1.6) is said to be **null controllable** in time $T > 0$ if given any initial state $U_0 \in \mathcal{H}$, there exists a *control* function $f \in L^2(0,T;\mathcal{G})$ such that the solution U of (1.6) satisfies $U(T) = 0$.

Remark Clearly exact controllability is stronger than null controllability.

As in the finite dimensional case, we may transfer the controllability problem to the corresponding observability problem, but the author is not aware of a unified approach. Intuitively, exact controllability may be thought of as surjectivity of a solution operator, say S. It is a well known functional analysis result that if S is bounded and linear, then this is equivalent to the adjoint operator S^* being bounded below.

Additionally, the proposed finite-dimensional duality relationsip between observability and controllability is physically logical. While controllability can be thought of the strength of the influence of the control (the outside force) on the (internal) phenomenon, observability refers to a parameter which can be "observed" (from the outside) which communicates information about the phenomenon. This internal/external paradigm is very intuitive for the specific setting of this thesis which is boundary control.

Thus, it is not surprising that the corresponding *observability inequality* of system (1.6) takes the following general form:

$$\|B^*\varphi\| \geq C\|F\varphi\| \tag{1.7}$$

where φ is the phenomenon governed by the dual system, which is *homogeneous* on the boundary where the control is applied in the original problem (1.6). B^* is called the *observability operator* and F refers to the part of the phenomenon about which the observation communicates information.

In the following chapters, we utilize the Hilbert Uniqueness Method of J.L. Lions for the wave equation and a Variational Method for the heat equation to show that (1.7) is sufficient for either null or exact controllability. In fact, we recast (1.7)—for each evolution equation of concern—into an equivalent PDE estimate: there exists a constant $C > 0$ such that

$$\int_0^T \int_\Gamma (\text{suitable trace of } \varphi)^2 \, d\Gamma dt \geq C E_\varphi(0) \tag{1.8}$$

where $E_\varphi(t)$ is the "energy" associated with the solution φ of the dual, boundary homogeneous problem. Moreover, Γ is the "observed" portion of the boundary of the dual problem. This equivalently says that Γ is the "controlled" portion of the boundary of the original problem, namely, Γ is the part of where controls are applied. Last, $T > 0$ is a universal time for boundary observation of the dual problem, or for the boundary controllability of the original problem.

We note at the outset that the homogeneous boundary condition on φ is 'complimentary' to the trace appearing on the left-hand side of (1.8). For example, in the case of the wave equation, if the boundary controls are applied to the Dirichlet boundary condition on $\Gamma \times (0, T)$, then the dual problem satisfies homogeneous Dirichlet boundary condition on $\Gamma \times (0, T)$, while the trace of φ occurring on the left-hand side of (1.8) is the Neumann trace. We refer to Chapter 3 for more details, and Chapter 4 for the similar situation of the heat equation.

Thus the crux of the entire boundary controllability problem is to establish the validity of the observability inequality (1.8) (possibly with the minimal universal time T, and with a minimal observed/controlled portion of the boundary Γ) with the optimal relationship between the topology of the trace and the topology of the initial energy. We note that the observability inequality (1.8) is an inverse-type inequality: it reconstructs the initial energy of the φ-evolution equation in terms of the information on the boundary. One implication of this is an important unique continuation principle: if what we observe (suitable trace of φ) is zero, then the internal phenomenon (initial energy) must also be zero.

By the end of the 1980's, energy method—with special multipliers—was the main technique to prove the observability inequality in the cases of classical time reversible systems such as the wave and the plate equation (constant coefficients and no energy level terms). However, the multiplier method is inadequate to treat more general models with variable coefficients in the principle part and/or in the energy level terms (such as with potentials). This feature will be specifically shown in Chapter 3 where we consider the exact boundary controllability for the one dimensional wave equation with potential. To deal with the energy level terms in the equations, we introduce an "exchange of variables" method which works exclusively for the one dimensional wave equation (See Section 3.2 below). For general dimensions, one common approach now is to use weighted energy inequalities called the *Carleman estimates* (See Sections 3.3 and 4.2). We first give some brief history of Carleman estimates here.

The idea of introducing suitable exponential weights in estimates for solutions of PDEs goes back to Carleman [1] in 1939, who used these estimates to obtain the *uniqueness* in the Cauchy Problem in two variables. Such idea "dominated all later work in the field" according to Hörmander [3, p.61], who expanded and perfected the idea to more general differential operators.

These theories only referred to solutions which are compactly supported, and as a result the Carleman estimates there involve no boundary terms. However, in our boundary control problem to establish the observability inequalities, a critical role is played precisely by *the traces* of the solutions on the boundary. Thus the classical Carleman estimates, although very strong tools in proving unique continuation, do not give good results when applied directly to boundary value problems.

To the best knowledge of the author, two sources were instrumental in obtaining Carleman–type inequalities for solutions of boundary value problems: D. Tataru in his Ph.D. thesis [14] at the University of Virginia as well as Lavrentev–Romanov–Shishataskii [9] of the Novosibirski school. These two contributions developed independently of each other. Tataru's work referred to a general evolution problem in pseudo-differential form and the resulting Carleman estimates included lower order terms. It inspired the case-by-case treatment via differential multipliers to obtain explicit Carleman estimates for second-order hyperbolic equations in Lasiecka–Triggiani [6]. Their counterparts in the context of a Riemannian manifold, which includes the Euclidean domain case where the Laplacian Δ is replaced by an elliptic operator with variable (in space) coefficients of modest regularity, is given in Lasiecka–Triggiani–Yao [7]. The work of Lavrentev–Romanov–Shishataskii [9] focused instead on second-order hyperbolic equations and yielded first *pointwise* Carleman–type estimates, whose resulting integral form was then localized in an interior set. Motivated by these two sources were the subsequent works of Lasiecka–Triggiani–Zhang [8] yielding the Carleman estimates in pointwise form in the case of general second-order hyperbolic equations, with explicit boundary terms.

The Carleman estimates we derive in this thesis for the one dimensional wave and heat equation with potential are based on the second approach, for which we establish the corresponding pointwise Carleman-type estimates first, and then establish the needed observability inequalities for the corresponding equations.

The rest of the thesis is organized as follows: In Chapter 2 we give the main results of the thesis and provide some of the necessary background on the wave equation. In Chapter 3 we show that exact boundary controllability holds for the one dimensional wave equation via the Hilbert Uniqueness Method (HUM). We establish the observability inequalities using three different methods, depending on whether the potential is present. In Chapter 4 we give some background information on the heat equation and then establish the null controllability for the one dimensional heat equation with potential via the variational method. In particular, we derive the observability inequality from the Carleman estimates.

2 Main Results and Preliminaries

First, we present the main results of this paper, concerning the controllability of these equations and the corresponding observability inequalities, which will proved in later chapters. Second, we state key properties of the wave equation which will be used subsequently. Specifically, we state Hilbert spaces in which the solutions lie, using the following notation, with reference to the classical $L^2(\Omega)$ space for some $\Omega \subseteq \mathbb{R}^n$.

$$
\begin{aligned}
H_0^1(\Omega) &:= \left\{ f \in L^2(\Omega) \text{ s.t. } \nabla f \in L^2(\Omega) \text{ and } f\Big|_{\partial\Omega} = 0 \right\} \\
H^{-1}(\Omega) &:= \left(H_0^1(\Omega) \right)' \\
L^\infty(\Omega) &:= \{ f : \Omega \to \mathbb{R} \text{ s.t. } f \text{ is measurable and } \exists C \geq 0 \text{ s.t. } |f(x)| \leq C \text{ a.e. } x \in \Omega \} \\
C([0,T]; \mathcal{H}) &:= \{ u : [0,T] \to \mathcal{H} \text{ s.t. } u \text{ is continuous} \}
\end{aligned}
$$

where $\mathcal{H}$ is some Hilbert space and $'$ denotes the dual for Hilbert spaces. Additionally, for $f \in H_0^1(\Omega)$, $g \in L^\infty(\Omega)$ define the norms

$$
\begin{aligned}
||f||_{H_0^1(\Omega)} &:= ||f||_{L^2(\Omega)} + ||\nabla f||_{L^2(\Omega)} \\
||g||_\infty &:= \inf\{ C \geq 0 : |f(x)| \leq C \text{ a.e. } x \in \Omega \}
\end{aligned}
$$

Throughout this paper, C represents a strictly positive constant, which may not be consistent from line-to-line. We show some important properties of the wave equation with potential (2.1): an energy estimate (2.8) and an upper bound on the crucial "observation" term (2.10). We conclude with the Hilbert Uniqueness Method (HUM) showing the relationship between observability and exact controllability for the wave equation. The analogous preliminary information concerning the heat equation will be handled in the final chapter (Section 4.1) for the sake of coherence.

2.1 Main Results

Theorem 2.1.1. *Consider the wave equation with bounded potential:*

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0 & \textit{in } [0,L] \times [0,T] \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) & \textit{in } [0,L] \\ u(0,t) = f(t) \quad u(L,t) = 0 & \textit{in } [0,T] \end{cases} \tag{2.1}$$

where $q \in L^\infty[0,L]$ *is given. If* $T > 2L$, *given any* $(u_0, u_1) \in L^2[0,L] \times H^{-1}[0,L]$, *and* $(u_T, u'_T) \in L^2[0,L] \times H^{-1}[0,L]$, *there exists* $f \in L^2[0,T]$ *such that* $u(x,T) = u_T$ *and* $u_t(x,T) = u'_T$. *In other words, (2.1) is exactly boundary controllable from one endpoint of the domain.*

In order to prove this, we show that the following observability inequality is sufficient for showing the exact controllability of (2.1) through the HUM.

Theorem 2.1.2. *For* $T > 2L$, *there exists* $C > 0$ *(dependent only on* T,L, *and* $||q||_\infty$*) such that*

$$||w_0||^2_{H_0^1[0,L]} + ||w_1||^2_{L^2[0,L]} \leq C \int_0^T w_x(0,t)^2 \, dt \tag{2.2}$$

where w is the solution to the *dual problem* to (2.1):

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) + q(x)w(x,t) = 0 & \text{in } [0,L] \times [0,T] \\ w(x,0) = w_0(x) \quad w_t(x,0) = w_1(x) & \text{in } [0,L] \\ w(0,t) = w(L,t) = 0 & \text{in } [0,T] \end{cases} \tag{2.3}$$

for $q \in L^\infty[0,L]$ and $(w_0, w_1) \in H_0^1[0,L] \times L^2[0,L]$. Notice that this dual problem is nearly exactly the same as the original control problem, and this is due to the time reversibility of the wave equation. The heat equation, in contrast, has a very different dual equation since it is strongly irreversible in time. In Chapter 3, we prove inequality (2.2) using a few different methods, most notably the Carleman Estimate technique modeled in [15]. We have similar results for the heat equation:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0 & \text{in } [0,L]\times[0,T] \\ u(x,0) = u_0(x) & \text{in } [0,L] \\ u(0,t) = f(t) \quad u(L,t) = 0 & \text{in } [0,T] \end{cases} \tag{2.4}$$

and its dual:

$$\begin{cases} w_t(x,t) + w_{xx}(x,t) - q(x)u(x,t) = 0 & \text{in } [0,L]\times[0,T] \\ w(x,T) = w_T(x) & \text{in } [0,L] \\ w(0,t) = w(L,t) = 0 & \text{in } [0,T] \end{cases} \tag{2.5}$$

for $u_0, w_T \in L^2[0,L]$, $f \in L^2[0,T]$, and $q \in L^\infty[0,L]$.

Theorem 2.1.3. *Given $u_0 \in L^2[0,L]$, there exists $f \in L^2[0,T]$ such that $u(x,T) = 0$. In other words, (2.4) is null boundary controllable from one end of the domain.*

We show through a variational technique that the following observability inequality is sufficient for the above result:

Theorem 2.1.4. *For w solving (2.5), there exists C (dependent only on T,L, and $||q||_\infty$) such that*

$$||w(\cdot,0)||^2_{L^2[0,L]} \le C\int_0^T w_x(0,t)^2\,dt \tag{2.6}$$

2.2 Wave Equation Preliminaries

First, we refer to [15] to show that (2.1) and (2.3) are well posed, and that the solutions are in the appropriate spaces.

Theorem 2.2.1. *Let $w_0 \in H_0^1[0,L]$, $w_1 \in L^2[0,L]$, and $q \in L^\infty[0,L]$. Then (2.3) has a unique weak solution $w(x,t)$ where*

$$w \in C([0,T]; H_0^1[0,L])$$

$$w_t \in C([0,T]; L^2[0,L])$$

$$w_x(0,\cdot) \in L^2[0,T]$$

We also have the following stronger regularity result.

Theorem 2.2.2. *Let $u_0 \in L^2[0,L]$, $u_1 \in H^{-1}[0,L], q \in L^\infty[0,L]$, and $f \in L^2[0,L]$. Then (2.1) has a unique weak solution $u(x,t)$ where*

$$u \in C([0,T]; L^2[0,L])$$

$$u_t \in C([0,T]; H^{-1}[0,L])$$

2.2.1 Energy Estimate

Broadly speaking, all the inequalities we obtain in this paper fall under the category of Energy Methods. Much of the estimates revolve around an "energy norm" which is usually equation specific. Introduce the following notion:

Definition The *energy* of solution of (2.5), w, is defined by

$$E_w(t) := \frac{1}{2}\int_0^L w_x(x,t)^2 + w_t(x,t)^2\,dx \tag{2.7}$$

Henceforth, we denote the upper bound on the potential q by $M = ||q||_\infty$. We have the following energy estimate

Lemma 2.2.3. *For w solving (2.5) and $E_w(t)$ defined as in (2.7),*

$$e^{-M(L^2/2+1)T}E_w(0) \le E_w(t) \le e^{M(L^2/2+1)T}E_w(0) \quad \forall t \in [0,T] \tag{2.8}$$

In order to show this, we first need the following well-known result. We include the proof to explicitly obtain the constant $(L^2/2)$ in (2.9) below.

Lemma 2.2.4 (Poincaré Inequality)**.** *Let $w(x,t)$ be a solution of (4.1). Then for each $t \in [0,T]$,*

$$\int_0^L w(x,t)^2\,dx \leq \frac{L^2}{2}\int_0^L w_x(y,t)\,dy \tag{2.9}$$

Proof. By Hölder's Inequality and the Fundamental Theorem of Calculus, for $(x,t) \in [0,L]\times[0,T]$,

$$\begin{aligned} w(x,t) = \int_0^x w_x(y,t)\,dy &\leq \left(\int_0^x w_x(y,t)^2\,dy\right)^{1/2} \cdot \left(\int_0^x 1^2\,dy\right)^{1/2} \\ &\leq \left(\int_0^L w_x(y,t)^2\,dy\right)^{1/2} \cdot (x)^{1/2} = \left(x\int_0^L w_x(y,t)^2\,dy\right)^{1/2} \end{aligned}$$

Therefore

$$\int_0^L w(x,t)^2\,dx \leq \int_0^L x\int_0^L w_x(y,t)^2\,dy\,dx = \frac{L^2}{2}\int_0^L w_x(y,t)^2\,dy$$

□

Proof of Lemma 2.2.3. For $t \in [0,T]$,

$$\begin{aligned} E_w'(t) &= \frac{d}{dt}\frac{1}{2}\int_0^L w_x^2 + w_t^2\,dx = \int_0^L w_x w_{xt} + w_t w_{tt}\,dx \\ &= w_x w_t\Big|_{x=0}^{L} - \int_0^L w_{xx}w_t\,dx + \int_0^L w_t w_{tt}\,dx \\ &= \int_0^L (w_{tt} - w_{xx})w_t\,dx = -\int_0^L q(x)ww_t\,dx \\ (*) \qquad &\geq -\frac{M}{2}\int_0^L (w^2 + w_t^2)\,dx \\ &\geq -\frac{M}{2}\int_0^L \left(\frac{L^2}{2}w_x^2 + w_t^2\right)\,dx \\ &\geq \frac{-M(L^2/2+1)}{2}\int_0^L (w_x^2 + w_t^2)\,dx \\ &= -M(L^2/2+1)E_w(t) \end{aligned}$$

Then,

$$\frac{d}{dt}\left(e^{M(L^2/2+1)t}E_w(t)\right) = e^{M(L^2/2+1)t}\left[E_w'(t) + M(L^2/2+1)E_w(t)\right] \geq 0$$

so $e^{M(L^2/2+1)t}E_w(t)$ is a monotone increasing function of t which gives

$$E_w(0) = e^{M(L^2/2+1)t}E_w(t)\Big|_{t=0} \leq e^{M(L^2/2+1)t}E_w(t) \leq e^{M(L^2/2+1)T}E_w(t)$$

for all $t \in [0, T]$, so we have the LHS of (2.8). To attain the RHS, we bound above by M at $(*)$:

$$E_w'(t) \leq \frac{M}{2}\int_0^L (w^2 + w_t^2)\,dx \leq \frac{M(L^2/2+1)}{2}\int_0^L (w_x^2 + w_t^2)\,dx = M(L^2/2+1)E_w(t)$$

Then,

$$\frac{d}{dt}\left(e^{-M(L^2/2+1)t}E_w(t)\right) = e^{-M(L^2/2+1)t}\left[E_w'(t) - M(L^2/2+1)E_w(t)\right] \leq 0$$

thus $e^{-M(L^2/2+1)t}E_w(t)$ is monotone decreasing:

$$E_w(0) \geq e^{-M(L^2/2+1)t}E_w(t) \geq e^{-M(L^2/2+1)T}E_w(t)$$

for all $t \in [0, T]$. Therefore, we have for all $t \in [0, T]$

$$e^{-M(L^2/2+1)T}E_w(0) \leq E_w(t) \leq e^{M(L^2/2+1)T}E_w(0)$$

□

Remark Inequality (2.8) is an estimate on how fast the energy of the wave may change. It is not necessarily increasing or decreasing, but we do know that it cannot grow/shrink faster than the exponential function $e^{M(L^2/2+1)t}$.

Remark As a consequence of the Poincaré Inequality, the initial energy $E_w(0)$ is equivalent to the $H_0^1[0,L] \times L^2[0,L]$ norm of the initial data, (w_0, w_1):

$$c\left(||w_0||^2_{H_0^1[0,L]} + ||w_1||^2_{L^2[0,L]}\right) \leq E_w(0) \leq C\left(||w_0||^2_{H_0^1[0,L]} + ||w_1||^2_{L^2[0,L]}\right)$$

This will transform the observability inequality (2.2) to

$$E_w(0) \leq C\int_0^T w_x(0,t)\,dt$$

which we prove in Chapter 3.

2.2.2 Boundedness of $||w_x(0,\cdot)||_{L^2[0,T]}$

We also have the following regularity result, which may be viewed as a portion of Theorem 2.2.1. It is more explicit though, in that it gives the reason that $w_x(0,\cdot) \in L^2[0,T]$.

Proposition 2.2.5. *Let w be a solution of (2.3). Then, there exists $C > 0$ (depending only on T,L, and M) such that*

$$\int_0^T w_x(0,t)^2\,dt \leq C\left(||w_0||^2_{H^1} + ||w_1||^2_{L^2}\right) \tag{2.10}$$

Proof. Using integration by parts on the following identity,

$$0 = \int_0^T\int_0^L (w_{tt} - w_{xx} + qw)(x-L)w_x\,dx\,dt$$

we obtain

$$\begin{aligned}\frac{L}{2}\int_0^T w_x(0,t)^2\,dt = -\int_0^L (x-L)w_tw_x\Big|_{t=0}^T\,dx + \int_0^T \frac{x-L}{2}w_t^2\Big|_{x=0}^L\,dt \\ - \int_0^T\int_0^L \frac{L}{2}w_t^2 + \frac{1}{2}w_x^2 + (x-L)qww_x\,dx\,dt\end{aligned} \tag{2.11}$$

Since $w(0,t) = w(L,t) = 0$,

$$\int_0^T \frac{x-L}{2} w_t^2 \Big|_{x=0}^{L} dt = 0$$

Now, we continue estimating the RHS of (2.11) by way of (2.9) and (2.8).

$$\begin{aligned}
\frac{L}{2}\int_0^T w_x(0,t)^2\,dt &\le \frac{L}{2}\int_0^L w_t(x,T)^2 + w_x(x,T)^2 + w_t(x,0)^2 + w_x(x,0)^2\,dx \\
&\qquad + \frac{ML+L+1}{2}\int_0^T\int_0^L w_t^2 + w_x^2 + w^2 + w_x^2\,dx\,dt \\
&\le L\left[E_w(T) + E_w(0)\right] + C_0\int_0^T E_w(t)\,dt \\
&\le L\left[e^{M(L^2/2+1)T}E_w(0) + E_w(0)\right] + C_0\int_0^T e^{M(L^2/2+1)T}E_w(0)\,dt \\
&\le C E_w(0)
\end{aligned}$$

$\square$

2.3 Hilbert Uniqueness Method

In order to show that observability (2.2) is a sufficient condition for the exact controllability of (2.1), we employ the Hilbert Uniqueness Method of Lions [11]. Begin by constructing the following *backward problem* from the solution w of (2.3).

$$\begin{cases}
y_{tt}(x,t) - y_{xx}(x,t) + q(x)y(x,t) = 0 & \text{in } [0,L]\times[0,T] \\
y(x,T) = 0 \quad y_t(x,T) = 0 & \text{in } [0,L] \\
y(0,t) = w_x(0,t) \quad y(L,t) = 0 & \text{in } [0,T]
\end{cases} \tag{2.12}$$

To show that (2.12) is well-posed, consider the forward equation:

$$\begin{cases}
z_{tt}(x,t) - z_{xx}(x,t) + q(x)z(x,t) = 0 & \text{in } [0,L]\times[0,T] \\
z(x,0) = 0 \quad z_t(x,0) = 0 & \text{in } [0,L] \\
z(0,t) = w_x(0,T-t) \quad z(L,t) = 0 & \text{in } [0,T]
\end{cases} \tag{2.13}$$

Then, for $y(x,t) = z(x, T-t)$, y solves (2.12). This is the crucial property of the wave equation referred to earlier as time reversibility. Well-posedness of (2.12) follows from the well-posedness of (2.13) and the fact that $w_x(0,\cdot) \in L^2[0,T]$ (see Thm. 2.2.1). Now, define the mapping

$$\begin{aligned}\Lambda : H_0^1[0,L] \times L^2[0,L] &\to H^{-1}[0,L] \times L^2[0,L] \\ (w_0, w_1) &\mapsto (-y_t(\cdot,0), y(\cdot,0))\end{aligned}$$

Then we claim Λ is bounded and linear. Indeed,

$$\begin{aligned}||\Lambda(w_0,w_1)||_{H^{-1}[0,L]\times L^2[0,L]} &= ||(-y_t(x,0), y(x,0))||_{H^{-1}[0,L]\times L^2[0,L]} \\ &\leq C_1 ||w_x(0,t)||_{L^2[0,T]} \\ &\leq C_2 ||(w_0.w_1)||_{H_0^1[0,L]\times L^2[0,L]}\end{aligned}$$

with the first inequality following from the well-posedness of (2.12) and the second from Prop. 2.2.5. Linearity follows from the linearity of the wave equation. Moreover, assuming the observability inequality (2.2), we have that Λ is bounded below on the diagonal. That is, there exists $C > 0$ (depending only on T,L, and $||q||_\infty$) such that

$$\langle \Lambda(w_0,w_1), (w_0,w_1)\rangle \geq C\left(||w_0||^2_{H_0^1} + ||w_1||^2_{L^2}\right) \tag{2.14}$$

for all $(w_0, w_1) \in H_0^1[0,L] \times L^2[0,L]$ where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H_0^1[0,L] \times L^2[0,L])' = H^{-1}[0,L] \times L^2[0,L]$ and $H_0^1[0,L] \times L^2[0,L]$, namely for $(g,\hat{g}) \in H^{-1}[0,L] \times L^2[0,L]$ and $(h,\hat{h}) \in H_0^1[0,L] \times L^2[0,L]$,

$$\begin{aligned}\langle (g,\hat{g}), (h,\hat{h})\rangle &:= \langle (g,\hat{g}), (h,\hat{h})\rangle_{(H^{-1}[0,L]\times L^2[0,L]),(H_0^1[0,L]\times L^2[0,L])} \\ &= \int_0^L g(x)h(x)\,dx + \int_0^L \hat{g}(x)\hat{h}(x)\,dx\end{aligned}$$

To show (2.14), we note the following "duality" relationship between $y(x,t)$, a solution to (2.12) and $w(x,t)$, a solution to (2.3).

$$\begin{aligned}
0 &= \int_0^L \int_0^T (w_{tt} - w_{xx} + q(x)w)y \, dt \, dx \\
&= \int_0^L (w_t y - w y_t)\Big|_{t=0}^{T} dx + \int_0^L \int_0^T w y_{tt} \, dt \, dx \\
&\quad - \left[\int_0^T (w_x y - w y_x)\Big|_{x=0}^{L} dt + \int_0^L \int_0^T w y_{xx} \, dt \, dx \right] + \int_0^L \int_0^T q w y \, dt \, dx \\
&= -\int_0^L w_1 y(x,0) - w_0 y_t(x,0) \, dx + \int_0^T w_x(0,t)^2 \, dt \\
&= -\langle \Lambda(w_0, w_1), (w_0, w_1) \rangle + \int_0^T w_x(0,t)^2 \, dt
\end{aligned}$$

So,

$$\langle \Lambda(w_0, w_1), (w_0, w_1) \rangle = \int_0^T w_x(0,t)^2 \, dt \overset{(2.2)}{\geq} C \left(||w_0||^2_{H_0^1} + ||w_1||^2_{L^2} \right)$$

for all $(w_0, w_1) \in H_0^1[0,L] \times L^2[0,L]$. This is where we invoked the observability inequality (2.2), which we have not proved yet, but devote Chapter 3 to doing so. Then, by a well-known functional analysis result (see Appendix A.1), there exists a unique (w_0^*, w_1^*) such that $\Lambda(w_0^*, w_1^*) = (-u_1, u_0)$. The desired control is then

$$f(t) = w_x^*(0,t)$$

where $w^*(x,t)$ solves (2.3) with initial data (w_0^*, w_1^*). This is clear if we think about Λ as identifying which initial conditions (with the negative of the time derivative) are driven to zero by $w_x(0,t)$. So this choice of (w_0^*, w_1^*) are ones for which $w_x^*(0,t)$ drives (u_0, u_1) to zero, which is the desired null controllability result.

However, for the wave equation we have something stronger due to the time reversibility. In this case, null controllability is equivalent to exact controllability. It is already clear that exact controllability implies null controllability, so here we show the converse.

Proposition 2.3.1. *If (2.1) is null controllable, then (2.1) is exactly controllable.*

Proof. Let (u_0, u_1) be the initial states and (u_T, u'_T) be the desired final states. Assuming (2.1) is null controllable, there exists f such that u solves

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0 & \text{in } [0,L] \times [0,T] \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) & \text{in } [0,L] \\ u(0,t) = f(t) \quad u(L,t) = 0 & \text{in } [0,T] \end{cases}$$

with the property $u(x,T) = 0 = u_t(x,T)$. Also, there exists g such that $\tilde{u}$ solves

$$\begin{cases} \tilde{u}_{tt}(x,t) - \tilde{u}_{xx}(x,t) + q(x)\tilde{u}(x,t) = 0 & \text{in } [0,L] \times [0,T] \\ \tilde{u}(x,0) = u_T(x) \quad \tilde{u}_t(x,0) = u'_T(x) & \text{in } [0,L] \\ \tilde{u}(0,t) = g(t) \quad \tilde{u}(L,t) = 0 & \text{in } [0,T] \end{cases}$$

with the propoerty $\tilde{u}(x,T) = 0 = \tilde{u}_t(x,T)$. Then, for $v(x,t) = u(x,t) + \tilde{u}(x,T-t)$, v solves

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) + q(x)v(x,t) = 0 & \text{in } [0,L] \times [0,T] \\ v(x,0) = u_0(x) \quad v_t(x,0) = u_1(x) & \text{in } [0,L] \\ v(0,t) = f(t) + g(T-t) \quad u(L,t) = 0 & \text{in } [0,T] \end{cases}$$

with $v(x,T) = u_T$ and $v_t(x,T) = u'_T$. Therefore, (2.1) is exactly controllable. □

3 Exact Controllabillity of the Wave Equation

Now we move on to showing the observability of the dual problem. If $q = 0$, then this inequality is greatly simplified. The control problem is simplified to the original wave equation, and observability can be shown by the following method.

3.1 Multiplier Method

In this simplified case, the dual problem is

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0 & \text{in } [0,L] \times [0,T] \\ w(x,0) = w_0(x) \quad w_t(x,0) = w_1(x) & \text{in } [0,L] \\ w(0,t) = w(L,t) = 0 & \text{in } [0,T] \end{cases} \tag{3.1}$$

This problem is simpler largely due to the fact that the energy is conserved (i.e. $E_w(t)$ is a constant function of t):

$$\frac{d}{dt}E_w(t) = \frac{d}{dt}\int_0^L w_x^2 + w_x^2\,dx = \int_0^L w_x w_{xt} + w_t w_{tt} = \int_0^L -w_{xx}w_t + w_t w_{tt} = 0$$

The name of this method comes from the fact that we begin with an identity which is the equation multiplied by $(x-L)w_x$, which is called the "multiplier"

$$0 = \int_0^L \int_0^T (w_{tt} - w_{xx})(x-L)w_x\,dt\,dx$$

Using integration by parts, we get

$$\frac{1}{2}\int_0^T \int_0^L w_t^2 + w_x^2 \, dx \, dt = -\int_0^L (x-L) w_x w_t \Big|_{t=0}^{T} dx + \frac{1}{2}\int_0^T (x-L) w_x^2 \Big|_{x=0}^{L} dt$$

Then,

$$\begin{aligned}
\int_0^T E_w(t)\, dt &= -\int_0^L (x-L) w_x w_t \Big|_{t=0}^{T} dx + \frac{L}{2}\int_0^T w_x^2(0,t)\, dt \\
TE_w(0) &\leq L\int_0^L \frac{1}{2}\left[w_x^2(x,T) + w_t^2(x,T) + w_x^2(x,0) + w_x^2(x,0)\right] dx \\
&\quad + \frac{L}{2}\int_0^T w_x^2(0,t)\, dt \\
&= L[E_w(T) + E_w(0)] + \frac{L}{2}\int_0^T w_x^2(0,t)\, dt
\end{aligned}$$

Thus, we have the desired result (2.2),

$$\frac{2(T-2L)}{L} E_w(0) \leq \int_0^T w_x^2(0,t)\, dt$$

if $T > 2L$ to guarantee that the constant is strictly positive. However, this condition on T has a physical interpretation. Since we are observing from $x = 0$, we must wait some time for the wave to travel from the boundary $x = 0$ to $x = L$ and then back for us to "observe" anything. The wave speed for (3.1) is 1 so this takes time $2L$.

3.2 Exchange of Variables

When potential is present, the multiplier method fails, largely due to the fact that energy is no longer conserved. Instead, we have the energy estimate from Lemma 2.2.3. So we explore two alternative methods. The first alternative method is what will be called Exchange of Variables. We introduce another equation which can be thought of as exchanging the space variable and the time variable:

$$\begin{cases} \tilde{w}_{xx}(t,x) - \tilde{w}_{tt}(t,x) - q(x)\tilde{w}(t,x) = 0 & \text{in } [0,T]\times[0,L] \\ \tilde{w}(t,0) = 0 \quad \tilde{w}_x(t,0) = w_x(0,t) & \text{in } [0,T] \\ \tilde{w}(0,x) = 0 \quad \tilde{w}(T,x) = 0 & \text{in } [0,L] \end{cases} \tag{3.2}$$

Notice that $z(x,t) = w(x,t) - \tilde{w}(t,x)$ solves the following:

$$\begin{cases} z_{xx}(x,t) - z_{tt}(x,t) - q(x)z(x,t) = 0 & \text{in } [0,L]\times[0,T] \\ z(0,t) = 0 \quad z_x(0,t) = 0 & \text{in } [0,T] \end{cases}$$

If we view $z(0,t) = z_x(0,t) = 0$ as an "initial" condition (again by exchanging the roles of x and t), then by standard domain of dependence arguement for wave equations, $z = 0$ inside the region

$$R := \{(x,t)\,|\,0 \le x \le L, x \le t \le T - x\}$$

Thus, $\tilde{w}(t,x) = w(x,t)$ in R. Also, since $\tilde{w}$ can be thought of as exchanging the time and space variables, we consider its energy as a function of x:

$$E_{\tilde{w}}(x) := \frac{1}{2}\int_0^T \tilde{w}_t(t,x)^2 + \tilde{w}_x(t,x)^2\,dt$$

The energy estimate (2.7) holds here since $\tilde{w}$ has a zero "boundary" condition. Now,

$$\begin{aligned} \int_0^T w_x(0,t)^2\,dt &= \int_0^T \tilde{w}_x(t,0)^2\,dt = E_{\tilde{w}}(0) = \frac{1}{L}\int_0^L E_{\tilde{w}}(0)\,dx \\ &\ge \frac{1}{L}e^{-M(T^2/2+1)L}\int_0^L E_{\tilde{w}}(x)\,dx = C_1/2\int_0^L\int_0^T \tilde{w}_x^2 + \tilde{w}_t^2\,dt\,dx \\ (**) \qquad &\ge C_1/2\int_0^L\int_L^{T-L} \tilde{w}_x^2 + \tilde{w}_t^2\,dt\,dx = C_1/2\int_0^L\int_L^{T-L} w_x^2 + w_t^2\,dt\,dx \\ &= C_1\int_L^{T-L} E_w(t)\,dt \ge C_1\int_L^{T-L} e^{-M(L^2/2+1)T}E_w(0)\,dt \\ &= CE_w(0) \end{aligned}$$

which is the observability inequality (2.2) we set out to prove. Notice that at $(**)$ is where the condition $T > 2L$ is employed. The key to this technique was that these equations are in one dimension, so the time and space variables are arbitrarily assigned. Due to this, it cannot be extended to higher dimensions.

3.3 Carleman Estimate

In light of the very limited nature of the previous method, we turn to the final method, the Carleman Estimate. Though we will only shown this in one-dimension, this strategy can clearly be extended to higher dimensions. We begin with the following pointwise estimate:

Lemma 3.3.1. *Let w, ψ be in $C^2([0,L] \times [0,T])$ and set $v = e^{\lambda\psi} w$. Then, for all (x,t) in $[0,L] \times [0,T]$,*

$$\begin{aligned} e^{2\lambda\psi}|w_{tt} - w_{xx}|^2 \geq \frac{\partial}{\partial t}&(4\lambda\psi_x v_x v_t - 2\lambda A\psi_t v^2 - 2\lambda\psi_t v_t^2 - 2\lambda\psi_t v_x^2) \\ &+ \frac{\partial}{\partial x}(2\lambda A\psi_x v^2 - 2\lambda\psi_x v_t^2 + 4\lambda\psi_t v_t v_x - 2\lambda\psi_x v_x^2) \\ &+ (2\lambda\psi_{xx} + 2\lambda\psi_{tt})v_x^2 + (2\lambda\psi_{tt} + 2\lambda\psi_{xx})v_t^2 \\ &- 8\lambda\psi_{xt} v_x v_t + Bv^2 \end{aligned} \tag{3.3}$$

where

$$A = (\lambda^2\psi_t^2 - \lambda\psi_{tt} - \lambda^2\psi_x^2 + \lambda\psi_{xx}) \tag{3.4}$$

$$B = 2\lambda(A_t\psi_t + A\psi_{tt} - A_x\psi_x - A\psi_{xx}) \tag{3.5}$$

Proof. First, we simply compute the LHS of (3.3), called the principle part, in terms of v and ψ using $w = e^{-\lambda\psi} v$ along with the product and chain rules:

$$w_t = \partial_t(e^{-\lambda\psi} v) = e^{-\lambda\psi}(-\lambda\psi_t v + v_t)$$

$$
\begin{aligned}
w_{tt} &= \partial_t[e^{-\lambda\psi}(-\lambda\psi_t v + v_t)] \\
&= -\lambda\psi_t e^{-\lambda\psi}(-\lambda\psi_t v + v_t) + e^{-\lambda\psi}(-\lambda\psi_{tt} v - \lambda\psi_t v_t + v_{tt}) \\
&= e^{-\lambda\psi}(\lambda^2\psi_t^2 v - 2\lambda\psi_t v_t - \lambda\psi_{tt} v + v_{tt})
\end{aligned}
$$

In a similar fashion,

$$
\begin{aligned}
w_x &= \partial_x(e^{-\lambda\psi} v) = e^{-\lambda\psi}(-\lambda\psi_x v + v_x) \\
w_{xx} &= \partial_x[e^{-\lambda\psi}(-\lambda\psi_x v + v_x)] \\
&= -\lambda\psi_x e^{-\lambda\psi}(-\lambda\psi_x v + v_x) + e^{-\lambda\psi}(-\lambda\psi_{xx} v - \lambda\psi_x v_x + v_{xx}) \\
&= e^{-\lambda\psi}(\lambda^2\psi_x^2 v - 2\lambda\psi_x v_x - \lambda\psi_{xx} v + v_{xx})
\end{aligned}
$$

Substitutiing these for w_{tt} and w_{xx} yields

$$
\begin{aligned}
e^{2\lambda\psi}|w_{tt} - w_{xx}|^2 &= |(\lambda^2\psi_t^2 v - 2\lambda\psi_t v_t - \lambda\psi_{tt} v + v_{tt}) \\
&\qquad - (\lambda^2\psi_x^2 v - 2\lambda\psi_x v_x - \lambda\psi_{xx} v + v_{xx})|^2 \\
&= \big|\big[(\lambda^2\psi_t^2 - \lambda\psi_{tt} - \lambda^2\psi_x^2 + \lambda\psi_{xx})v + v_{tt} - v_{xx}\big] \\
&\qquad + (2\lambda\psi_x v_x - 2\lambda\psi_t v_t)\big|^2 \\
&= |(Av + v_{tt} - v_{xx}) + (2\lambda\psi_x v_x - 2\lambda\psi_t v_t)|^2 \\
&\geq 2(Av + v_{tt} - v_{xx})(2\lambda\psi_x v_x - 2\lambda\psi_t v_t) \\
&= 4\lambda A\psi_x v v_x - \lambda A\psi_t v v_t + 4\lambda\psi_x v_x v_{tt} \\
&\qquad - 4\lambda\psi_t v_t v_{tt} - 4\lambda\psi_x v_x v_{xx} + 4\lambda\psi_t v_t v_{xx}
\end{aligned}
$$

Then, each term can be rewritten as follows:

$$
\begin{aligned}
4\lambda A\psi_x v v_x &= \frac{\partial}{\partial x}(2\lambda A\psi_x v^2) - 2\lambda A_x\psi_x v^2 - 2\lambda A\psi_{xx} v^2 \\
-4\lambda A\psi_t v v_t &= \frac{\partial}{\partial t}(-2\lambda A\psi_t v^2) + 2\lambda A_t\psi_t v^2 + 2\lambda A\psi_{tt} v^2
\end{aligned}
$$

$$\begin{aligned}
4\lambda\psi_x v_x v_{tt} &= \frac{\partial}{\partial t}(4\lambda\psi_x v_x v_t) - 4\lambda\psi_{xt} v_x v_t - 4\lambda\psi_x v_{xt} v_t \\
&= \frac{\partial}{\partial t}(4\lambda\psi_x v_x v_t) - 4\lambda\psi_{xt} v_x v_t + \frac{\partial}{\partial x}(-2\lambda\psi_x v_t^2) + 2\lambda\psi_{xx} v_t^2 \\
-4\lambda\psi_t v_t v_{tt} &= \frac{\partial}{\partial t}(-2\lambda\psi_t v_t^2) + 2\lambda\psi_{tt} v_t^2 \\
-4\lambda\psi_x v_x v_{xx} &= \frac{\partial}{\partial x}(-2\lambda\psi_x v_x^2) + 2\lambda\psi_{xx} v_x^2 \\
4\lambda\psi_t v_t v_{xx} &= \frac{\partial}{\partial x}(4\lambda\psi_t v_t v_x) - 4\lambda\psi_{tx} v_t v_x + \frac{\partial}{\partial t}(-2\lambda\psi_t v_x^2) + 2\lambda\psi_{tt} v_x^2
\end{aligned}$$

Combining these and grouping the like terms, we have

$$\begin{aligned}
e^{2\lambda\psi}|w_{tt} - w_{xx}|^2 \geq \frac{\partial}{\partial t}&(4\lambda\psi_x v_x v_t - 2\lambda A\psi_t v^2 - 2\lambda\psi_t v_t^2 - 2\lambda\psi_t v_x^2) \\
&+ \frac{\partial}{\partial x}(2\lambda A\psi_x v^2 - 2\lambda\psi_x v_t^2 + 4\lambda\psi_t v_t v_x - 2\lambda\psi_x v_x^2) \\
&+ (2\lambda\psi_{xx} + 2\lambda\psi_{tt})v_x^2 + (2\lambda\psi_{tt} + 2\lambda\psi_{xx})v_t^2 \\
&- 8\lambda\psi_{xt} v_x v_t + 2\lambda(A_t\psi_t + A\psi_{tt} - A_x\psi_x - A\psi_{xx})v^2
\end{aligned}$$

□

Now we turn to the proof of (2.2).

Proof of Thm 2.1.2. Begin by specifying ψ for the wave equation. In order to do so, recall the condition from both the multiplier method and exchange of variable method: $T > 2L$. This is needed in order to pick $\alpha \in (0,1)$ such that

$$L^2 < \alpha \left(\frac{T}{2}\right)^2 \tag{3.6}$$

and $\epsilon \in (0,1)$ such that ,

$$(L+\epsilon)^2 < \alpha \left(\frac{T}{2}\right)^2 \tag{3.7}$$

These parameters will be crucial to ensuring key properties of ψ which we define to be

$$\psi(x,t) := \tfrac{1}{2}\left[(x - L - \epsilon)^2 - \alpha(t - T/2)^2\right] \tag{3.8}$$

for all $(x,t) \in [0,L] \times [0,T]$. By (3.7), we have the following property

$$\psi(x,0) = \psi(x,T) = \frac{1}{2}\left[(x-L-\epsilon)^2 - \alpha(T/2)^2\right] \leq \frac{1}{2}\left[(-L-\epsilon)^2 - \alpha(T/2)^2\right] < 0$$

for all $x \in [0,L]$ so there exists $T_1 \in (0,T/2)$ and $T_1' \in (T/2,T)$ such that $\psi(x,t) < 0$ for all $t \leq T_1$ and $t \geq T_1'$. Also,

$$\psi(x,T/2) = \frac{1}{2}\left[(x-L-\epsilon)^2 - \alpha(T/2-T/2)^2\right] \geq \epsilon^2/2$$

for all $x \in [0,L]$ so there exists $T_0 \in (0,T/2)$ and $T_0' \in (T/2,T)$ such that $\psi(x,t) > \epsilon^2/3$ for all $T_0 \leq t \leq T_0'$. Then, define

$$\Lambda := \{(x,t) \in [0,L] \times [0,T] : \psi(x,t) \geq \epsilon^2/4\} \tag{3.9}$$

Notice Λ has the following relationship to the previously definely domains:

$$[0,L] \times [T_0,T_0'] \subset \Lambda \subset [0,L] \times [T_1,T_1'] \tag{3.10}$$

Now we will restate (3.3) with ψ as defined in (3.8) substituted in the definition of the coefficients A and B, as well as in the v_x^2 and v_t^2 terms in the inequality:

$$\begin{aligned} e^{2\lambda\psi}|w_{tt} - w_{xx}|^2 \geq & \frac{\partial}{\partial t}(4\lambda\psi_x v_x v_t - 2\lambda A\psi_t v^2 - 2\lambda\psi_t v_t^2 - 2\lambda\psi_t v_x^2) \\ & + \frac{\partial}{\partial x}(2\lambda A\psi_x v^2 - 2\lambda\psi_x v_t^2 + 4\lambda\psi_t v_t v_x - 2\lambda\psi_x v_x^2) \\ & + 2\lambda(1-\alpha)v_x^2 + 2\lambda(1-\alpha)v_t^2 + Bv^2 \end{aligned} \tag{3.11}$$

$$A = \lambda^2\left[\alpha^2(t-T/2)^2 - (x-L-\epsilon)^2\right] + \lambda(\alpha+1) \tag{3.12}$$

$$B = \lambda^3\left[(6+2\alpha^2)(x-L-\epsilon)^2 - \alpha^2(2+6\alpha)(t-T/2)^2\right] + 2\lambda^2(1-\alpha)(1+\alpha) \tag{3.13}$$

Integrate both sides of (3.11) from 0 to L in x and τ to τ' in t where $\tau \in [0, T_1]$ and $\tau' \in [T_1', T]$.

$$\begin{aligned} 2\lambda(1-\alpha)\int_\tau^{\tau'}\int_0^L (v_x^2+v_t^2)\,dx\,dt + \int_\tau^{\tau'}\int_0^L Bv^2\,dx\,dt \le& \qquad (3.14)\\ 2\lambda\int_0^L \left[-2\psi_x v_x v_t + \psi_t(Av^2+v_t^2+v_x^2)\right]\Big|_{t=\tau}^{\tau'} dx& \\ + 2\lambda\int_\tau^{\tau'} \left[-2\psi_t v_t v_x + \psi_x(v_x^2+v_t^2-Av^2)\right]\Big|_{x=0}^{L} dt& \\ + \int_\tau^{\tau'}\int_0^L e^{2\lambda\psi}|w_{tt}-w_{xx}|^2\,dx\,dt& \\ =: I_1 + I_2 + I_3& \end{aligned}$$

Formally, (3.14) is what we call the Carleman Estimate. Now by (3.6),

$$\begin{aligned} |A(x,t)| &\le \lambda^2\left[\alpha^2(T/2)^2 + (L+\epsilon)^2\right] + \lambda(\alpha+1) \\ &\le 2\lambda^2\alpha^2(T/2)^2 + \lambda(\alpha+1) \\ &\le \lambda^2(2\alpha^2(T/2)^2 + \alpha + 1) \end{aligned}$$

for $\lambda \ge 1$.

$$\begin{aligned} I_1 &\le 2\lambda\int_0^L \left[(L+\epsilon)(v_x^2+v_t^2) + \alpha T/2(|A|v^2+v_x^2+v_t^2)\right]\Big|_{t=\tau} \\ &\qquad + \left[(L+\epsilon)(v_x^2+v_t^2) + \alpha T/2(|A|v^2+v_x^2+v_t^2)\right]\Big|_{t=\tau'} dx \\ &\le \lambda^3\alpha T(\alpha^2T^2/2+2)\int_0^L v(x,\tau)^2 + v_t(x,\tau)^2 + v_x(x,\tau)^2 + v(x,\tau')^2 + v_t(x,\tau')^2 + v_x(x,\tau')^2\,dx \\ &\le \lambda^3\alpha T(\alpha^2T^2/2+2)\int_0^L e^{2\lambda\psi(x,\tau)}\left[w(x,\tau)^2 + \lambda^2\psi_t^2 w(x,\tau)^2 + w_t(x,\tau)^2 + \lambda^2\psi_x^2 w(x,\tau)^2 + w_x(x,\tau)^2\right] \\ &\qquad + e^{2\lambda\psi(x,\tau')}\left[w(x,\tau')^2 + \lambda^2\psi_t^2 w(x,\tau')^2 + w_t(x,\tau')^2 + \lambda^2\psi_x^2 w(x,\tau')^2 + w_x(x,\tau')^2\right] dx \\ &\le C_1\lambda^5\int_0^L w(x,\tau)^2 + w_t(x,\tau)^2 + w_x(x,\tau)^2 + w(x,\tau')^2 + w_t(x,\tau')^2 + w_x(x,\tau')^2\,dx \end{aligned}$$

for $\lambda > 1$, where $C_1 = \alpha T(\alpha^2T^2/2+2)(1+\alpha^2T^2/2)$.

Since $w(0,t) = w(L,t) = 0$,

$$
\begin{aligned}
I_2 &= 2\lambda \int_{\tau}^{\tau'} \psi_x(L,t) v_x(L,t)^2 - \psi_x(0,t) v_x(0,t)^2 \, dt \\
&= -2\lambda \int_{\tau}^{\tau'} \epsilon v_x(L,t)^2 \, dt + 2\lambda (L+\epsilon) \int_{\tau}^{\tau'} v_x(L,t)^2 \, dt \\
&\le -2\lambda\epsilon \int_{T_1}^{T_1'} v_x(L,t)^2 \, dt + 2\lambda (L+\epsilon) \int_0^T v_x(0,t)^2 \, dt \\
&\le 2\lambda (L+\epsilon) \int_0^T v_x(0,t)^2 \, dt \\
&= 2\lambda (L+\epsilon) \int_0^T e^{2\lambda\psi(0,t)} [\lambda \psi_x(0,t) w(0,t)^2 + w_x(0,t)]^2 \, dt \\
&= 2\lambda (L+\epsilon) \int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2 \, dt
\end{aligned}
$$

Here we begin utilizing the properties of the region Λ defined in (3.9) by considering I_3 on Λ and outside Λ. For $Q := [0,T] \times [0,L]$

$$
\begin{aligned}
I_3 &= \int_{\tau}^{\tau'} \int_0^L e^{2\lambda\psi} |-q(x)w|^2 \, dx \, dt \\
&\le M^2 \iint_{Q \setminus \Lambda} + \iint_{\Lambda} e^{2\lambda\psi} w^2 \, dx \, dt \\
&\le M^2 e^{\lambda\epsilon^2/2} \iint_Q w^2 \, dx \, dt + M^2 \iint_{\Lambda} v^2 \, dx \, dt
\end{aligned}
$$

Now we will handle the Bv^2 term in a similar fashion to I_3. We will consider Bv^2 on Λ and outside Λ. Let $Q_{\tau}^{\tau'} := [\tau, \tau'] \times [0,L]$. Then,

$$
\begin{aligned}
\iint_{Q_{\tau}^{\tau'} \setminus \Lambda} Bv^2 \, dx \, dt &\ge -5T^2\lambda^3 \iint_{Q_{\tau}^{\tau'} \setminus \Lambda} v^2 \, dx \, dt \\
&\ge -5T^2\lambda^3 e^{\lambda\epsilon^2/2} \iint_{Q_{\tau}^{\tau'} \setminus \Lambda} w^2 \, dx \, dt \\
&\ge -5T^2\lambda^3 e^{\lambda\epsilon^2/2} \iint_Q w^2 \, dx \, dt
\end{aligned}
$$

On the other hand, for $(x,t) \in \Lambda$

$$\begin{aligned} B &= \lambda^3 \left[(8+2\alpha^2)(x-L-\epsilon)^2 - (6\alpha^3+4\alpha^2)(t-T/2)^2\right] + 2\lambda^2(\alpha^2-\alpha-2) \\ &\geq \lambda^3(8+2\alpha^2)\psi(x,t) + 2\lambda^2(\alpha^2-\alpha-2) \\ &\geq \lambda^3(4+\alpha^2)\epsilon^2/4 + 2\lambda^2(\alpha^2-\alpha-2) \end{aligned}$$

by (3.9). Thus, there exists some $0 < c_B \leq (4+\alpha^2)\epsilon^2/2$ and λ_1 such that

$$\iint_\Lambda Bv^2 \geq \lambda^3 c_B \iint_\Lambda v^2 \qquad \forall\lambda > \lambda_1 \tag{3.15}$$

We restate (3.14) with this information and we restrict the domain of integration for the first term from $Q_\tau^{\tau'}$ to Λ.

$$\begin{aligned} 2\lambda(1-\alpha)\iint_\Lambda (v_x^2+v_t^2)\,dx\,dt + \lambda^3 c_B \iint_\Lambda v^2\,dx\,dt - \lambda^3 5T^2 \iint_Q w^2\,dx\,dt \\ \leq C_1\lambda^5 \int_0^L (w^2+w_t^2+w_x^2)\Big|_{t=\tau} + (w^2+w_t^2+w_x^2)\Big|_{t=\tau'}\,dx \\ + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt \\ + M^2 e^{\lambda\epsilon^2/2}\iint_Q w^2\,dx\,dt + M^2\iint_\Lambda v^2\,dx\,dt \end{aligned} \tag{3.16}$$

for all $\lambda > \lambda_1$. Integrate (3.16) from 0 to T_1 $d\tau$ and T_1' to T $d\tau'$. The only term containing τ or τ' is

$$\begin{aligned} \int_0^{T_1}\int_{T_1'}^T C_1\lambda^5 \int_0^L (w^2+w_t^2+w_x^2)\Big|_{t=\tau} + (w^2+w_t^2+w_x^2)\Big|_{t=\tau'}\,dx\,d\tau\,d\tau' \\ = (T-T_1')C_1\lambda^5 \int_0^{T_1}\int_0^L w^2+w_t^2+w_x^2\,dx\,dt \\ + T_1 C_1\lambda^5 \int_{T_1'}^T\int_0^L w^2+w_t^2+w_x^2\,dx\,dt \\ \leq (T-T_1'+T_1)C_1\lambda^5 \iint_Q w^2+w_t^2+w_x^2\,dx\,dt \end{aligned}$$

All the remaining terms pick up the constant $T_1(T-T_1')$ which can be divided through so we let

$$C_2 = \frac{(T - T_1' + T_1)C_1}{T_1(T - T_1')}$$

Thus, (3.16) becomes

$$\begin{aligned} 2\lambda(1-\alpha) \iint_\Lambda (v_x^2 + v_t^2)\, dx\, dt + \lambda^3 c_B \iint_\Lambda v^2\, dx\, dt - M^2 \iint_\Lambda v^2\, dx\, dt \\ \le \lambda^5 C_2 \iint_Q w^2 + w_t^2 + w_x^2\, dx\, dt + 2\lambda(L+\epsilon) \int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\, dt \quad (3.17) \\ + e^{\lambda\epsilon^2/2}(c_B\lambda^3 + M^2) \iint_Q w^2\, dx\, dt \end{aligned}$$

for all $\lambda > \lambda_1$. We will substitute $v = e^{\lambda\psi} w$ into the first term of (3.17) in order to obtain $E_w(t)$. Notice first that

$$\begin{aligned} v_t &= e^{\lambda\psi} w_t + \lambda\psi_t e^{\lambda\psi} w \\ e^{2\lambda\psi} w_t^2 &= (v_t - \lambda\psi_t e^{\lambda\psi} w)^2 \le 2v_t^2 + 2\lambda^2 e^{2\lambda\psi} \psi_t^2 w^2 \end{aligned}$$

which implies

$$v_t^2 \ge \tfrac{1}{2} e^{2\lambda\psi} w_t^2 - \lambda^2 e^{2\lambda\psi} \psi_t^2 w^2 \ge \tfrac{1}{2} e^{2\lambda\psi} w_t^2 - \lambda^2 \alpha^2 (T/2)^2 v^2$$

and likewise for v_x^2. So, for $\beta \in (0,1)$,

$$\begin{aligned} 2\lambda(1-\alpha) \iint_\Lambda v_t^2 + v_x^2\, dx\, dt &\ge 2\lambda(1-\alpha)\beta \iint_\Lambda v_t^2 + v_x^2\, dx\, dt \\ &\ge \lambda(1-\alpha)\beta \iint_\Lambda e^{2\lambda\psi}(w_t^2 + w_x^2)\, dx\, dt \\ &\quad - \beta\lambda^3(1-\alpha)\alpha^2 T^2/2 \iint_\Lambda v^2\, dx\, dt \end{aligned}$$

Specify β such that $0 < \beta(1-\alpha)\alpha^2 T^2/2 < \min\{c_B, 1\}$ and set

$$c_0 := \min\{c_B, 1\} - \beta(1-\alpha)\alpha^2 T^2/2 > 0 \quad , \quad \lambda_2 := \max\{\lambda_1, \left(M^2/c_0\right)^{1/3}\} \qquad (3.18)$$

Then for all $\lambda > \lambda_2$, the RHS of (3.17) becomes

$$\begin{aligned}
\lambda(1-\alpha)\beta \iint_\Lambda e^{2\lambda\psi}(w_t^2 + w_x^2)\,dx\,dt + (\lambda^3 c_0 - M^2) \iint_\Lambda v^2\,dx\,dt \\
&\geq \lambda(1-\alpha)\beta \iint_\Lambda e^{2\lambda\psi}(w_t^2 + w_x^2)\,dx\,dt \\
&\geq \lambda(1-\alpha)\beta \int_{T_0}^{T_0'} \int_0^L e^{2\lambda\psi}(w_t^2 + w_x^2)\,dx\,dt \\
&\geq 2\lambda(1-\alpha)\beta e^{\lambda 2\epsilon^2/3} \int_{T_0}^{T_0'} E_w(t)\,dt \\
&\geq \lambda c_E e^{\lambda 2\epsilon^2/3} E_w(0)
\end{aligned}$$

where $c_E = 2(1-\alpha)\beta(T_0' - T_0)e^{-M(L^2/2+1)T}$. Restating (3.17),

$$\begin{aligned}
\lambda c_E e^{\lambda 2\epsilon^2/3} E_w(0) &\leq \lambda^5 C_2 \iint_Q w^2 + w_t^2 + w_x^2\,dx\,dt + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt \\
&\quad + e^{\lambda\epsilon^2/2}(c_B\lambda^3 + M^2)\iint_Q w^2\,dx\,dt \\
&\leq \lambda^5 C_2(L^2+2)\int_0^T E_w(t)\,dt + e^{\lambda\epsilon^2/2}(c_B\lambda^3 + M^2)L^2 \int_0^T E_w(t)\,dx\,dt \\
&\quad + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt \\
&\leq e^{\lambda\epsilon^2/2}\left[(c_B\lambda^3 + M^2)L^2 + C_2(L^2+2)\right]\int_0^T E_w(t)\,dt \\
&\quad + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt \\
&\leq e^{\lambda\epsilon^2/2}(c_B\lambda^3 + M^2 + 2C_2)L^2 T e^{MT(L^2+1)} E_w(0) \\
&\quad + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt \\
&\leq \lambda^3 e^{\lambda\epsilon^2/2} C_3 E_w(0) + 2\lambda(L+\epsilon)\int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2\,dt
\end{aligned}$$

for all $\lambda > \lambda_3 > \lambda_2$ where $C_3 = (c_B + M^2 + 2C_2)L^2 T e^{MT(L^2+1)}$ and λ_3 is chosen such that

$$\lambda^5 \leq e^{\lambda\epsilon^2/2} \quad \forall \lambda > \lambda_3$$

Now, we have a similar problem as in the multipler method (Section 3.1), where we wish to combine all the energy terms on the LHS, but we must guarantee the constant remains positive. This is possible by the design of the regions (3.10), specifically $[T_0, T_0'] \times [0, L]$ so that the exponential weight on the LHS ($e^{\lambda 2\epsilon^2/3}$) has a larger coefficient than the one on the RHS ($e^{\lambda\epsilon^2/2}$). Therefore, we can find $\lambda_4 > \lambda_3$ such that $e^{\lambda 2\epsilon^2/3} c_E > \lambda^2 e^{\epsilon^2/2} C_3$ for all $\lambda > \lambda_4$ and

$$\begin{aligned} 2(L+\epsilon) \int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2 \, dt &\geq (e^{\lambda 2\epsilon^2/3} c_E - \lambda^2 e^{\lambda\epsilon^2/2} C_3) E_w(0) \\ &\geq (e^{\lambda\epsilon^2/6} c_E - \lambda^2 C_3) e^{\lambda\epsilon^2/2} E_w(0) \geq c_4 E_w(0) \end{aligned}$$

Therefore,

$$E_w(0) \leq \frac{2(L+\epsilon)}{c_4} \int_0^T e^{2\lambda\psi(0,t)} w_x(0,t)^2 \, dt \leq C e^{\lambda(L+\epsilon)^2} \int_0^T w_x(0,t)^2 \, dt \tag{3.19}$$

for all $\lambda > \lambda_4$ which is the desired result (2.2) $\square$

Though there is likely much information contained in this constant $Ce^{\lambda(L+\epsilon)^2}$, one notable fact is that this constant depends exponentially on $M^{2/3}$. To see this, recall the condition on λ from (3.18).

4 Null Controllability of the Heat Equation

For the heat equation, we are not aware of a simple multiplier method. Also, some sort of variable exchange is not feasible since the heat equation does not have the symmetry of variables like the wave equation. Others have taken approaches to the control problem using spectral theory [10], which also incorporates some Carleman–type inequalities. However, we will show the observability inequality in order to acheive null controllability. We begin with some important properties of the heat equation with potential, and a proof of the sufficiency of observability via a Variational Method.

4.1 Heat Equation Preliminaries

One may think that the formulation would be the same for deriving a sufficient condition for the null controllability of the heat equation, but this is not so clear. First note some problems with the HUM. We exploited the Hilbert space structure of the solutions to the wave equation strongly in showing the consequence of the duality pairing being bounded from below. We do not immediately have this nice structure for the heat equation. This can be overcome though by restricting the initial conditions to create a Hilbert space duality relationship, but there are additional problems. Most notably, the equivalent dual equation and backward equation introduced in the HUM are both ill-posed for the heat equation [11]. We also mention that the lower bound we attain from the observability inequality (2.6) is in the L^2-norm which is not the same Hilbert space structure as the solution space for the control equation (2.4).

However, there is a formulation which shows that the null controllability of the heat equation is implied by the observability of the following *dual problem.*

$$\begin{cases} w_t(x,t) + w_{xx}(x,t) - q(x)w(x,t) = 0 & \text{in } [0,L]\times[0,T] \\ w(x,T) = w_T(x) & \text{in } [0,L] \\ w(0,t) = w(L,t) = 0 & \text{in } [0,T] \end{cases} \tag{4.1}$$

for $w_T \in L^2[0,L]$ and $q \in L^\infty[0,L]$ (let $M := ||q||_\infty$). (4.1) is more easily seen to be the dual system (1.3) mentioned in the introduction (If $A = -\Delta$, then $-A^* = \Delta$, and it is "generated" at final time T) than (2.3), the dual of the wave equation. (2.3) did not take this clear form because the wave operator is symmetric, and the time reversability allows for generation at $t = 0$.

As for the wave equation, we have an energy estimate which will be used in the proof of observability, but it is only a one-sided inequality.

Definition For w, solution of (4.1), the *energy* of w is

$$E_w(t) := \frac{1}{2}\int_0^L w(x,t)^2\,dx \tag{4.2}$$

Lemma 4.1.1 (Energy Estimate)**.** *For For w solving (4.1) and $E_w(t)$ defined by (4.1.1)*

$$e^{2MT}E_w(t) \geq E_w(0)$$

for all t in $[0,T]$.

Proof.

$$\begin{aligned} \frac{d}{dt}E_w(t) &= \int_0^L ww_t\,dx = -\int_0^L ww_{xx}\,dx + \int_0^L qw^2\,dx \\ &= ww_x\Big|_0^L + \int_0^L w_x^2\,dx + \int_0^L qw^2\,dx \\ &\geq 0 + 0 - M\int_0^L w^2 = -2ME(t) \end{aligned}$$

By the same monotonicity argument as in (2.8), the result folllows. □

Remark $E_w(0) = ||w(\cdot,0)||^2_{L^2[0,L]}$ so the desired observability inequality (2.6) becomes

$$E_w(0) \leq C \int_0^T w_x(0,t)^2\,dt \tag{4.3}$$

4.1.1 Variational Method

In light of these differences, we turn to a variational method used in [12] and [13]. In [12], this method is also used for the wave equation, yet we presented the HUM to show an alternative. To begin, there is the following necessary and sufficient condition for $u(x,T) = 0$ (null controllability), given an initial state u_0.

Lemma 4.1.2. *Let u be a solution to (2.4) with initial data $u_0 \in L^2[0,L]$. There exists $f \in L^2[0,T]$ such that $u(\cdot,T) = 0$ if and only if there exists $f \in L^2[0,T]$ such that*

$$\int_0^T w_x(0,t)f(t)\,dt + \int_0^L u_0(x)w(x,0) = 0\,dx \tag{4.4}$$

for all $w_T \in L^2[0,L]$ where w is the solution of (4.1).

Proof. Let u, w be solutions to (2.4) and (4.1) respectively with initial data u_0 and w_T. Then,

$$\begin{aligned}
0 &= \int_0^T \int_0^L (u_t - u_{xx} + qu)w\,dx\,dt \\
&= \int_0^L uw\Big|_{t=0}^T dx - \int_0^L \int_0^T uw_t\,dt\,dx - \int_0^T (u_x w - uw_x)\Big|_{x=0}^L dt \\
&\qquad - \int_0^L \int_0^T uw_{xx}\,dt\,dx + \int_0^L \int_0^T quw\,dt\,dx \\
&= \int_0^L u(x,T)w_T(x)\,dx - u_0(x)w(x,0)\,dx - \int_0^T f(t)w_x(0,t)\,dt
\end{aligned}$$

So, if $u(\cdot,T) = 0$, then (4.4) holds. Conversely, if (4.4) holds, then $\langle u(\cdot,T), w_T\rangle_{L^2[0,L]} = 0$ for all $w_T \in L^2[0,L]$ so $u(\cdot,T) = 0$. □

Now, define the functional:

$$\mathcal{J}(w_T) = \frac{1}{2}\int_0^T w_x(0,t)^2\,dt + \int_0^L u_0(x)w(x,0)\,dx$$

for $w_T \in L^2[0,L]$. $\mathcal{J}$ is continuous since it is the sum of two continuous functionals. Clearly the second term ($L^2[0,L]$ inner product) is continuous. Moreover, the quadratic term is continuous due to a regularity result [5]. Also, $\mathcal{J}$ is strictly convex by virtue of the quadratic term and the linearity of the derivative. The key property of $\mathcal{J}$, coercivity, comes from the observability inequality (2.6):

$$\mathcal{J}(w_T) \geq c||w(\cdot,0)||_{L^2}^2 - ||u_0||_{L^2}||w(\cdot,0)||_{L^2}$$

so $\mathcal{J} \to \infty$ as $w(\cdot,0) \to \infty$. However, to show coercivity, we need that $\mathcal{J} \to \infty$ as $w_T \to \infty$, which follows from some standard properties of the heat equation ($w(\cdot,0) \to \infty$ as $w_T \to \infty$). Now with these three conditions, $\mathcal{J}$ has a unique minimum (see Appendix A.2 for general functional analysis result). Let w^* be this minimum of $\mathcal{J}$, then we have the following additional property.

$$\begin{aligned}
0 &= \lim_{h\to 0}\frac{\mathcal{J}(w_T^* + hw_T) - \mathcal{J}(w_T^*)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2}\int_0^T \left[w_x^*(0,t) + hw_x(0,t)\right]^2\,dt + \int_0^L u_0(x)\left[w^*(x,0) + hw(x,0)\right]\,dx \right.\\
&\qquad\qquad \left. - \frac{1}{2}\int_0^T w_x^*(0,t)^2\,dt - \int_0^L u_0(x)w^*(x,0)\,dx\right) \\
&= \int_0^T w_x^*(0,t)w_x(0,t)\,dt + \int_0^L u_0(x)w(x,0)\,dx
\end{aligned}$$

for all $w_T \in L^2[0,L]$.

Therefore, by Lemma 4.1.2, $f(t) = w_x^*(0,t)$ is a control function such that $u(\cdot,T) = 0$. It is interesting to note that not only do the observability inequalities take the same form for both equations, but the control functions are the same term from the dual equations.

Despite these similarites, there does not appear to be a unified method of showing that observability implies controllability. Dolecki and Russell [2] lay out the general Banch space operator duality equivalence, but formulating specific problems in the proper solution spaces is a delicate task, as seen here and in the HUM. However, in an abstract sense, it can be shown that solving the equation

$$\Lambda x = y$$

as in HUM, is equivalent to minimizing the functional

$$\mathcal{J}(x) = \langle \Lambda x, x \rangle - \langle y, x \rangle$$

as in this Variational Method (with certain retriction on Λ, e.g. self-adjoint). So it appears that these two methods are equivalent, and there should some unified abstract setting in which the signficance of the observability inequality can be shown.

4.2 Carleman Estimate

We have laid out the framework and the significance of the observability inequalty which will now be presented using the Carleman Estimate. The strategy is the same as for the previous wave equation, but note certain differences along the way. Recall, the inequality we want to show is (4.3) where w is a solution of (4.1). In doing so, we prove Theorem 2.1.4. Again, we start with a pointwise estimate.

Lemma 4.2.1. *Let w, ψ be in $C([0, L] \times [0, T])$ and set $v = e^{\lambda\psi}$. Then, for all (x, t) in $[0, L] \times [0, T]$,*

$$e^{2\lambda\psi}|w_t + w_{xx}|^2 \geq \frac{\partial}{\partial t}(Av^2 - v_x^2) + \frac{\partial}{\partial x}(2v_x v_t - 2\lambda\psi_x v_x^2 - 2\lambda A\psi_x v^2) + 2\lambda\psi_{xx}v_x^2 + Bv^2 \tag{4.5}$$

where

$$A = \lambda^2\psi_x^2 - \lambda\psi_{xx} - \lambda\psi_t \tag{4.6}$$

$$B = 2\lambda A_x\psi_x + 2\lambda A\psi_{xx} - A_t \tag{4.7}$$

The proof of this lemma is very similar to the proof of Lemma 3.3.1, so it is omitted here. We move on to proving the observability inequality (4.3).

Proof of Thm. 2.1.4. First, specify ψ to be

$$\psi(x,t) := \frac{e^x - e^{2L} - xe^{2L}}{t(T-t)} \tag{4.8}$$

for all (x,t) in $[0,L]\times[0,T]$ and observe the following properties:

$$\begin{aligned} \psi(x,t) &\le (e^L - e^{2L})(4/T^2) < 0 \\ \psi_x(x,t) &\le (e^L - e^{2L})(4/T^2) < 0 \\ \psi_{xx}(x,t) &= e^x(t(T-t))^{-1} > 0 \\ \lim_{t\to 0}\psi(x,t) &= -\infty = \lim_{t\to T}\psi(x,t) \end{aligned} \tag{4.9}$$

Notice that this ψ is radically different from the one chosen in (3.8). Now, integrate (4.5) over $Q := [0,L]\times[0,T]$.

$$\begin{aligned} 2\lambda \iint_Q \psi_{xx}v_x^2\,dx\,dt + \iint_Q Bv^2\,dx\,dt + \int_0^L (Av^2 - v_x^2)\Big|_{t=0}^T dx \le \\ \int_0^T (2v_xv_t + 2\lambda\psi_xv_x^2 + 2\lambda A\psi_xv^2)\Big|_{x=0}^L dt + \iint_Q e^{2\lambda\psi}|w_t + w_{xx}|^2\,dx\,dt \\ =: I_1 + I_2 \end{aligned} \tag{4.10}$$

As in the analagous step for the wave equation, we note that (4.10) is what we term the Carleman Estimate.

Notice

$$\int_0^L (Av^2 - v_x^2)\Big|_{t=0}^{T} dx = 0$$

since $v = e^{\lambda\psi}w$ and $\psi(x,0) = \psi(x,T) = -\infty$. I_1 becomes the observation term, the left-hand side of (4.3):

$$\begin{aligned}
I_1 &= \int_0^T (2v_x v_t + 2\lambda\psi_x v_x^2 + 2\lambda A\psi_x v^2)\Big|_{x=0}^{L} dt \\
&= 2\lambda^2 \int_0^T \left[\psi_x(L,t)v_x(L,t)^2 - \psi_x(0,t)v_x(0,t)^2\right] dt \\
&\leq 2\lambda^2 \int_0^T -\psi_x(0,t)e^{2\lambda\psi(0,t)}w_x(0,t)^2\, dt \\
&\leq 2\lambda^2 e^{c\lambda} \int_0^T w_x(0,t)^2\, dt
\end{aligned}$$

Indeed, $-\psi_x(0,t)e^{2\lambda\psi(0,t)}$ can be bounded above by $e^{c\lambda}$ for some $c > 0$: First, $-\psi_x(0,t)e^{2\lambda\psi(0,t)}$ is positive and continuous on $(0,T)$ and second,

$$\lim_{t\to 0} -\psi_x(0,t)e^{2\lambda\psi(0,t)} = 0 = \lim_{t\to T} -\psi_x(0,t)e^{2\lambda\psi(0,t)}$$

Therefore, it is continuous over a compact interval $[0,T]$ thus bounded. Next, we bound the principle part.

$$I_2 = \iint_Q e^{2\lambda\psi}|w_t + w_{xx}|^2\, dx\, dt = \iint_Q e^{2\lambda\psi}|qw|^2\, dx\, dt \leq M^2 \iint_Q v^2\, dx\, dt$$

The key step in this method is to bound B below so that we may absorb the $M^2 \iint_Q v^2$ term as in (3.15). Using (4.6),(4.7), and (4.8),

$$B \geq \frac{\lambda^3\psi_{xx}\psi_x^2 - C_2\lambda^2 - C_1\lambda - C_0}{t^3(T-t)^3}$$

for some $C_i \geq 0$ where C_i is dependent on T and L for $i = 0, 1, 2$. Since $\psi_{xx} > 0$ by (4.9), there exists $c_B > 0$ and $\lambda_1 \geq 1$ such that for all $\lambda > \lambda_1$

$$\iint_Q Bv^2\,dx\,dt \geq c_B\lambda^3 \iint_Q v^2\,dx\,dt$$

We bound the first term of (4.10) below as well using (4.9) and the Poincaré Inequality

$$2\lambda \iint_Q \psi_{xx} v_x^2\,dx\,dt \geq 2\lambda(4/T^2) \iint_Q v_x^2\,dx\,dt \geq 2\lambda C_4 \iint_Q v^2\,dx\,dt$$

Restating (4.10) with the preceding information, we now have

$$2\lambda C_4 \iint_Q v^2\,dx\,dt + \lambda^3 c_B \iint_Q v^2\,dx\,dt \leq 2\lambda^2 e^{c\lambda} \int_0^T w_x(0,t)^2\,dx\,dt + M^2 \iint_Q v^2\,dx\,dt \quad (4.11)$$

for all $\lambda > \lambda_1$. In order to overcome the $M^2 \iint_Q v^2$ term in (4.11), set

$$\lambda_2 := \max\{\lambda_1, (M^2/c_B)^{1/3}\}$$

Then, for all $\lambda > \lambda_2$

$$\lambda^3 c_B - M^2 > 0$$

We simply drop these terms from (4.11) in the following manner yielding

$$\begin{aligned} 2\lambda C_4 \iint_Q v^2\,dx\,dt &\leq 2\lambda C_4 \iint_Q v^2\,dx\,dt + (\lambda^3 c_B - M^2) \iint_Q v^2\,dx\,dt \\ &\leq 2\lambda^2 e^{c\lambda} \int_0^T w_x(0,t)^2\,dx\,dt \end{aligned}$$

for all $\lambda > \lambda_2$. The LHS is very close to the energy of w, $E_w(t)$. We simply need to substitute $v = e^{\lambda\psi} w$. Fix $\tau \in (0, T/2)$.

$$\begin{aligned} \lambda e^{c\lambda}/C_4 \int_0^T w_x(0,t)^2\,dx\,dt &\geq \iint_Q v^2\,dx\,dt \geq \int_\tau^{T-\tau} \int_0^L e^{2\lambda\psi} w^2\,dx\,dt \\ &\geq 2e^{-c_\tau\lambda} \int_\tau^{T-\tau} E_w(t)\,dx\,dt \geq 2e^{-c_\tau\lambda}(T-2\tau)e^{-2MT} E_w(0) \end{aligned}$$

by Lemma 4.1.1 for some $c_\tau > 0$, depending on T, L, M, and τ. Therefore,

$$\frac{\lambda e^{(c+c_\tau)\lambda+2MT}}{4(L/T)^2(T-2\tau)}\int_0^T w_x(0,t)^2\,dt \geq E_w(0) \tag{4.12}$$

for $\lambda > \lambda_2 > M^{2/3}/c_B$. □

(4.12) is the observability inequality (2.6) we sought to prove, which yields the null controllability result (Theorem 2.1.3) for the heat equation. One important fact derived from the constant is that $T > 0$ is the only requirement on T. This confirms the intuition that the infinite speed of propogation for the heat equation would require no time condition like $T > 2L$ for the wave equation.

There are a few other differences between the Carleman Estiamte methods for the heat equation and the wave equation. The choice of ψ is radically different, and as a result, there was no need to split up the domain Q. However, we still needed to bound the B term below which required special proprties of ψ. These modifications on ψ also necessitated integration of (4.5) over the whole domain Q (as opposed to $Q_\tau^{\tau'}$) in order for certain boundary terms to be zero.

Thus, we have proved the main theorems showing exact/null controllability for the equations considered herein by exploiting the relationship between control of the original system and observability of the dual system. We demonstrated specifically the advantages of Carleman Estimates in acheiving these observability inequalities.

Appendix: Some Functional Analysis Results

Theorem A.1. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}'$ be bounded and linear. If there exists $c > 0$ such that for all $x \in \mathcal{H}$*

$$\langle Tx, x\rangle_{\mathcal{H}',\mathcal{H}} \geq c||x||_{\mathcal{H}}^2$$

then T is invertible. (i.e. $Tx = y$ has a unique solution $x \in \mathcal{H}$ for each $y \in \mathcal{H}'$).

Proof. First, notice that T is bounded below:

$$c||x||_{\mathcal{H}}^2 \leq \langle Tx, x\rangle_{\mathcal{H}',\mathcal{H}} \leq ||Tx||_{\mathcal{H}'}||x||_{\mathcal{H}}$$

so, $||Tx||_{\mathcal{H}'} \geq c||x||_{\mathcal{H}}$ for all $x \in \mathcal{H}$. Now, we show one-to-one. Let $x_1, x_2 \in \mathcal{H}$ such that $Tx_1 = Tx_2$. Then,

$$0 = ||T(x_1 - x_2)||_{\mathcal{H}'} \geq c||x_1 - x_2||_{\mathcal{H}} \geq 0$$

so $x_1 = x_2$. Next, we show T has a closed range. Let $x_n \in \mathcal{H}$ such that $Tx_n \to y$ for some $y \in \mathcal{H}'$. Then,

$$||Tx_n - Tx_m||_{\mathcal{H}'} \geq c||x_n - x_m||_{\mathcal{H}}$$

for all $n, m \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. Thus, there exists $x \in \mathcal{H}$ such that $x_n \to x$. Since T is bounded,

$$y = \lim_{n\to\infty} Tx_n = Tx$$

so $y \in \text{Ran}T$. Finally, we show T is onto. Recall, for $z \in \mathcal{H}'$,

$$d(z, \text{Ran}T) = \max_{\substack{w \in (\text{Ran}T)^\perp \\ ||w|| \leq 1}} |\langle z, w \rangle_{\mathcal{H}', \mathcal{H}}|$$

However, for $w \in (\text{Ran}T)^\perp \subseteq \mathcal{H}$

$$\langle Tv, w \rangle_{\mathcal{H}', \mathcal{H}} = 0$$

for all $v \in \mathcal{H}$. In particular, for $v = w$,

$$\langle Tw, w \rangle_{\mathcal{H}', \mathcal{H}} = 0$$

which implies $w = 0$. Therefore, $(\text{Ran}T)^\perp = \{0\}$ so $d(z, \text{Ran}T) = 0$ and $z \in \overline{\text{Ran}T} = \text{Ran}T$ for all $z \in \mathcal{H}'$. □

Theorem A.2. *Let $\mathcal{H}$ be a Hilbert space and $J : \mathcal{H} \to \mathbb{R}$ be a functional which is*

(i) continuous

(ii) (strictly) convex

(iii) coercive (i.e. $\lim_{||x|| \to \infty} J(x) = \infty$)

Then there exists (a unique) $x^* \in \mathcal{H}$ such that

$$J(x^*) = \inf_{x \in \mathcal{H}} J(x)$$

Proof. Let $\{x_n\} \subseteq \mathcal{H}$ such that $J(x_n) \to \inf J(x)$. Then, $\{J(x_n)\}$ is bounded which implies $\{x_n\}$ is bounded (if not, then $J(x_n) \to \infty$ by coercivity). This means $\{x_n\}$ has a weakly convergent subsequence $x_{n_k} \xrightarrow{w} x^* \in \mathcal{H}$. Now if J is weakly lower semicontinuous, we have that

$$J(x^*) \leq \liminf_{n \to \infty} J(x_{n_k}) = \lim J(x_n) = \inf J(x)$$

in which case x^* is a minimizer of J. We show that this is indeed the case. Since J is continuous, the set

$$S_a := \{x \in \mathcal{H} : J(x) \leq a\}$$

is closed for all $a \in \mathbb{R}$. S_a is also convex. Indeed, let $x, y \in S_a$ and $\lambda \in (0, 1)$. Then,

$$J(\lambda x + (1 - \lambda)y) \leq \lambda J(x) + (1 - \lambda)J(y) \leq a$$

Since S_a is closed and convex, it is weakly closed by Mazur's lemma. For $\{z_n\} \subseteq \mathcal{H}$ which is weakly convergent to $z \in \mathcal{H}$, suppose $\liminf_{n\to\infty} J(z_n) < J(z)$. Then, for $a = J(z)$, there exists a subsequence $\{z_{n_k}\}$ which is entirely contained in S_a and which converges weakly to z. This means $z \in S_a$ which is a contradiction. Thus, J is weakly lower semicontinuous. $\square$

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