Consistent Asymptotic Homogenization Method for Lattice Structures Based on the Virtual Power Principle

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CONSISTENT ASYMPTOTIC HOMOGENIZATION METHOD FOR LATTICE STRUCTURES BASED ON THE VIRTUAL POWER PRINCIPLE

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
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Master of Science
Mechanical Engineering

by
David Bracho
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Accepted by:
Dr. Lonny Thompson, Committee Chair
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Abstract

The properties of mechanical meta-materials in the form of a periodic lattice have drawn the attention of researchers in the area of material design. Computational methods for solving elasticity problems that involve a large number of repeating structures, such as the ones present in lattice materials, is often impractical since it needs a considerable amount of computational resources. Homogenization methods aim to facilitate the modeling of lattice materials by defining equivalent effective properties. Various approaches can be found in the literature, but few follow a consistent methodology applicable to any lattice geometry. In the present work, the asymptotic homogenization method developed by Caillerie and later by Dos Reis is examined, which follows a consistent derivation of effective properties by use of the virtual power principle progressively from a beam, to a cell and finally to the cluster of cells forming the lattice. Because of the scale separation between the small scale of the unit cell and the large scale of the lattice domain, the asymptotic homogenization method can be used. It is shown that the virtual power of the continuum resulting from this analysis is that of micropolar elasticity. The method is implemented to obtain effective properties for different lattices: a square, triangular, hexagonal (“honeycomb”), “kagome” and a simple chiral lattice topology. Finally, results are compared with effective properties of similar lattices found in other studies.
Acknowledgments

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Chapter 1

Introduction

1.1 Motivation

A key problem in engineering design is the identification of materials with the most desirable properties for any given application. There is, however, a limited number of possible materials found in nature. The properties that they exhibit, thus, lie within a restricted range of values. The problem is simply explained through an Ashby chart as in Figure 1.1. The chart contrasts Young’s Moduli ($E$) vs. densities ($\rho$) for different materials. Clearly, most materials found in nature present larger $E$ values as their density increases. This leaves empty regions in the chart, some of which represent desirable properties for engineering applications. The main goal of modern material design is to fill these "holes" through different approaches.

A new range of properties can be achieved by the simple combination of the already available materials e.g. composites and sandwiched materials. Furthermore, it is possible to expand the range of physical attributes by arranging the constituent materials in precise patterns. Such structures exhibit characteristics that depend not only on the properties of each component, but on the geometry in which they have been arranged.

This concept is of special interest in the study of meta-materials. Meta-materials are defined as "artificially structured materials used to control and manipulate light, sound, and many
other physical phenomena." The inner structure of meta-materials is often characterized by a repeating arrangement of a unit cell. The scale of the unit cell is significant and depends on the excitation being applied on the material. In general, the size of the unit cell has to be smaller than the wavelength of the aforementioned loading excitation.

The consideration in scale size is of special importance to the aim of the present work. It is assumed that there is a scale in which the cluster of cells responds as a continuous material, which allows for homogenization.

In particular, the properties of mechanical meta-materials built in the form of a lattice has drawn the attention of researchers in the area. In these materials, the unit cell walls enclose a void. Again, the geometry of the cell walls give the material its mechanical properties, while voids endow it with diminished density. It should be noted that an increase in mechanical efficiency through this approach is present in naturally occurring trabecular bone, balsa wood, etc as referenced in [53], which hints at the importance of exploring these material’s capabilities further. These kind of materials have obvious applications in a vast range of industries, such as aerospace and transportation or prosthetics.
1.2 The problem of homogenization

Computational methods for solving elasticity problems that involve large numbers of repeating structures is not practical; it calls for a considerable amount of computational resources. If the object to be modeled contains a vast number of this repeating structure, i.e. the unit cell, it follows that its behavior should approach that of a body consisting of a continuous material. The purpose of the homogenization process is to find the effective properties for the heterogeneous material as if it were a continuous medium. The process is explained graphically in Figure 1.2.

Several approaches have been considered when attempting to homogenize lattice structures. Often mentioned in the literature is the method employed by Gibson and Ashby [23], which consists in studying the modes of deformation of the unit cell walls under specific loads. Effective properties of the material are found by assuming cell walls behave as Euler Bernoulli beams. It has been pointed out by Arabnejad and Pasini [4] that the method gives adequate solutions for simple topologies and loads. However, as the unit cell structure increases in complexity, the solution is not easy to find. Furthermore, understanding how cell topology deforms under loading is necessary and not always obvious.

Therefore, more recent approaches aim to find a more generalized method of homogenization, one that does not require extensive topology analysis and can be equally applied to describe different and more complex geometries. Energy methods, like the one used by Kumar and McDowell [30], try to carry displacement of nodes in the lattice to a continuum via Taylor expansions.
A similar approach is Asymptotic Homogenization, where the displacement is assumed to be an asymptotic expansion expressed in two scales: a macro scale where the lattice behaves as a homogeneous material, and a micro scale defined by the unit cell.

The present work centers on the more generalized methods of homogenization (in contrast to the previously described method developed by Gibson and Ashby in [23]) applied on lattice structures.

1.3 Homogenization on lattice structures: Literature Review

The different methods of homogenization for lattice structures have been summarized in the work by Noor [37]. The paper references various studies that fall into the following categories [37]:

1. Methods that try to match the force and deformation of a small section of the lattice to those in the whole domain, thus obtaining properties for the continuum.

2. Methods that use discrete field analysis.

3. Methods that consider the deformation energy of the unit cell equivalent to that of the continuum through Taylor expansions.

4. Methods that use asymptotic expansions.

Gibson and Ashby’s method falls into the first category, since it employs the characteristics of a deformed cell to derive the effective elastic properties. However, recent studies are trying to develop homogenization via the two final approaches. They appear simple to implement, and are able to analyze a good number of lattice geometries.

The most familiar method in discrete field analysis is the formulation of difference equations, usually applied on structural problems. This method solves for discrete (as opposed to continuous)
variables defined on a pattern of elements, like the ones forming a lattice. In depth discussion of the method is provided by Dean in [14].

Energy methods on the unit cell were specifically referenced by Noor [37] and more recently by Kumar and McDowell [30]. Noor considers lattice behavior as a truss, i.e. a "beam-like structure" where each member is dominated by axial forces (like a simple bar) and nodes are considered as pin joints. Kumar and McDowell go further to consider strain energy density on a unit cell made of bending members i.e. beams. Their work follows earlier papers such as [7, 9] in which homogenization is taken to a micropolar continuum as opposed to a classical elasticity continuum. This becomes necessary since the introduction of beam cross-section slope requires an additional deformation parameter, an independent rotation, that is not found in the classical elastic displacement field.

The fourth category of methods mentioned above involve the use of asymptotic expansions. The mathematical concept and its applications are thoroughly discussed in the work by Papanicolau et al. [39]. The technique considers two levels: one "macro" level where the heterogeneous material is assumed to respond homogeneously, and a "micro" level dictated by the behavior of a unit cell. Asymptotic analysis usually requires the introduction of a small parameter which defines the micro-scale of the problem. This provides a way to analyze the different scales on which the effective properties of a lattice may be applied. Energy methods usually lack this scale distinction. Furthermore, the identification of scale levels clearly differentiates the behavior between a large assemble of cells and a single unit. This is why energy methods do not appear to capture the effects of the lattice as a whole.

There is extensive research applying asymptotic homogenization. The approach is described as being "justified rigorously for many models" [8] and can be applied "to all kinds of processes that occur in periodic media" [8]. A 1D example of the method is described on [48]. This and more details of asymptotic homogenization on a general elasticity problem are given by Hassani and Hinton [27]. Research on asymptotic homogenization for lattice structures has been conducted by Caillerie et al. [11], specifically on the behavior of trusses with pin joints. Their work has been taken further by Ganghoffer et al. [43, 15, 3, 24], assuming a lattice of bending members and using
asymptotic homogenization to micropolar continuum.

Arabnejad and Pasini [4] propose a different method by introducing asymptotic homogenization into the weak form or virtual work expression for an elastic problem. By use of an asymptotic expansion on the displacement field \( u \) they show that the total strain tensor is a sum of a macro strain tensor and a micro strain tensor. The result is then found in a fashion similar to finite element analysis on a non linear problem.

Evidently, asymptotic homogenization is a versatile method, having been implemented on homogenization by direct derivations or numerical analysis. However, deriving the effective properties from a theoretical framework rather than by computational simulations, is usually preferred. (Algorithms require large number of iterations, fine meshes, and comparison of several different cases before adequate conclusions can be made) Thus, if direct derivation is desirable and if bending of the unit-cell walls is to be considered, then asymptotic homogenization to micropolar continua is the most suitable method to examine.

It is clear from the many examples in the literature, that micropolar theory is of special interest to researchers in the area. It appears to fit into most homogenization assumptions while maintaining certain level of simplicity. It has also been throughly studied in the past. Micropolar elasticity lies between the classical elasticity theory and the second gradient theories that form the denominated micromorphic continua. Eringen developed a complete mathematical framework for these higher gradient deformation theories in [18]. However, the Cosserat brothers and other researchers were the first to make significant contributions to the theory [31]. In general, micropolar elasticity introduces a rigid rotation independent of the displacement field which is usually referred to as ”micro rotation”. This allows for the possibility of couples to be introduced at any material point, other than the regular traction vectors, which are defined by a couple-stress tensor.

1.4 Research scope and outline

The present work aims to answer the following research questions:
R. Q. 1: *How can effective properties for lattice materials be derived by use of an asymptotic homogenization method?*

R. Q. 2: *How does this method compare to others in the literature?*

To answer these research questions, the present study follows the work begun by Callerie and Tollenaere [48], and Ganghoffer et al. [15, 43, 3]. Several concepts are essential to understanding their procedure and developing a consistent method: application of the principle of virtual work, asymptotic homogenization, beam theory and micropolar continuum formulations. In particular, this analysis makes use of the Timoshenko beam theory, in place of the Euler Bernoulli beam theory used by the authors. These concepts are described throughout chapter two. The application of the method is discussed in chapter three where it is followed to find effective properties for two cases: the square and hexagonal “honeycomb” lattice geometries. Chapter four discusses the results obtained for these and other geometries, and compares them to results found in the literature.
Chapter 2

Asymptotic Homogenization of Lattice Structures: Method Description

This chapter aims to give a detailed description of the method for asymptotic homogenization of lattice structures. Each of the following sections constitutes a progressive step towards finding the effective properties of the lattice material. All necessary definitions and assumptions are discussed at each stage.

2.1 The lattice and the unit cell

In the context of this study, a lattice is defined as a heterogeneous medium that can be modeled by interlacing beams. It is assumed that the rigidity of this heterogeneous medium is predominantly determined by the material and arrangement of the interlaced beams. That is, the space between beams is considered void.

It is assumed that all beams in the lattice are connected by rigid joints which do not
allow for any independent rotations of each member. In addition, beam members are subject to axial forces, shear forces and bending moments. This is in contrast to trusses, where members are modeled in the way of bars acting only under axial loads.

The entire lattice medium can be reproduced by repetition of a single structure denoted as the unit cell. Thus, periodicity is assumed over the entire lattice, meaning that a physical quantity measured at one cell is the same as the response evaluated at any other cell.

Any of the joints pertaining to the unit cell can be used to track its location in the lattice. Consider a 2D lattice. Any set of integers \( (q^1, q^2) \) indicates the position of a joint and the unit cell it belongs to. (i.e. \( q^1 \) the joint number on the horizontal axis, and \( q^2 \) the joint number on the vertical axis). A cell may contain more than one joint. However, only one of the cell’s joints is selected to pinpoint its location in the lattice. This joint will be referred to as the characteristic joint of the cell. To define any and all joints in the lattice, consider the following, as introduced in [48] and [11]: \( \tilde{n} = (q^1, q^2, n) \), where \( n \in \mathbb{N} \) is the set of joints in the particular cell defined by \( (q^1, q^2) \).

Figure 2.1 shows a typical lattice structure. Two beam members connected through the joint at \((2, 2)\) are part of the unit cell. The cell is such that the entire lattice can be constructed by tessellation of this structure along the direction of \( \hat{e}_q^1 \) and \( \hat{e}_q^2 \). Using a similar expression as with joints, \( \tilde{b} = (q^1, q^2, b) \) represents the any beam in the lattice with \( b \in \mathcal{B} \) being the set of beams in cell \((q^1, q^2)\).

Beams are connected at their ends with joints inside the cell, and joints belonging to neighboring cells. Thus, the position of the neighboring joints relative to a characteristic joint needs to be determined. For each joint a neighboring joint can be defined by \( d = (d^1, d^2) \) where \( d^1, d^2 \) can take the values \(-1, 0, 1\). Therefore, they can be positioned by the set of integers \( (q^1 + d^1, q^2 + d^2) \) on a 2D lattice (Again, \( q^1, q^2 \) define the location of the characteristic joint of interest). The values for \( d^i \) depend on the location of the neighboring joint as shown in figure 2.2.

A comprehensive way to track the position of adjacent joints is required since with this method displacements, rotations, forces, couples are considered in the unit cell around the char-
characteristic joint. Beams connect two joints, usually with only one belonging to the unit cell and the other to a neighboring cell. In homogenization, the length of each beam is assumed to be very small compared to the size of the lattice. Behavior of the second joint can therefore be analyzed in terms of the characteristic joint by use of Taylor expansions. The integers $d^i$ is used to track the position so that the correct function derivative is used when approximating by use of Taylor series. For example, displacement of the characteristic joint can be denoted on the lattice by $u_1(q^1, q^2)$. A beam may connect the characteristic joint to a joint in a cell immediately to the right, as in the lattice of figure 2.2. If displacement needs to be described using only point $(q^1, q^2)$, displacement of the adjacent joint to the right can be denoted as $u_2(q^1 + d^1, q^2 + d^2)$ with $(d^1 = 1, d^2 = 0)$, i.e. $u_2(q^1 + 1, q^2)$.

It is also necessary to identify the origin and end joint of a beam in the lattice. For any such beam, the origin is denoted by $\tilde{A}(\tilde{b}) = (q^1, q^2, A(b))$, with $A(b)$ the origin joint of beam $b \in \mathcal{B}$ and $A(b) \in \mathcal{N}$. The end joint of any beam may be denoted in a similar fashion by $\tilde{E}(\tilde{b}) = (q^1 + d^1_{(b)}, q^2 + d^2_{(b)}, E(b))$. However, here $d^i_{(b)}$ is the value $d^i$ associated to beam $(b)$ if the beam connects to a neighboring cell as was previously discussed.

Separation of scales is key to the homogenization process. Therefore, it is necessary here to define the size of the lattice, which also denotes the large scale. This is accomplished by means
of $L$ as shown in figure 2.3. In contrast, the small scale length is denoted by $l$ and it describes a size equivalent to that of a unit cell. The need for both a large and a small scale is discussed thoroughly in section 2.5. Assuming a lattice with large number of very small cells, the present discussion makes use of the small case $l$ when referring to the length of any element pertaining to a cell i.e. the length of its walls or beams.

In order to determine the position of a cell with respect to a basis’ origin, it is convenient to consider the length of one of the cell’s walls. Henceforth, this length of the cell wall (on a small scale as on a refined lattice) is denoted as $l'$, while $l^b$ is used to denote the length of a beam. Final expressions for the homogenized properties of the lattice will include $l'$, since the length of the beams $l^b$ can be easily expressed in terms of the characteristic length of the unit cell. Subscripts are used to denote different beam and wall lengths for cases where the unit cell so requires, i.e. $l^b_n$ and $l'_n$.

A basis is then defined using the directions on which the unit cell must be repeated in order to construct the entire lattice. Note this basis is not always orthogonal, since periodicity in
the lattice does not always follow two orthogonal directions. As an example, consider a square and hexagonal lattice, as in figure 2.4. To form a lattice, a square unit cell is simply tessellated in two orthogonal directions. In contrast, a honeycomb lattice can be achieved through a different kind of tessellation, across two directions that may be non-orthogonal. Therefore, consider the set of unit vectors $B_q = \{ \hat{e}_q^i \}$ as a non-orthogonal basis for the lattice domain.

Implementing the contravariant basis $\hat{e}_q^i$, it is now possible to define the position vector of a characteristic joint for any cell. Thus, using the variables introduced in this section, the position vector is:

$$\vec{P}_0 = (q^1 l'_1) \hat{e}_q^1 + (q^2 l'_2) \hat{e}_q^2$$

(2.1)

Where $l'_1, l'_2$ are the lengths of the cell walls in the $\hat{e}_q^1, \hat{e}_q^2$ directions respectively.

Figure 2.5 shows this vector for the joint of the cell at $P((q^1, q^2) = (2, 2))$ with respect to the $\hat{e}_q^i$ basis, stated simply as $\vec{P}_0 = (2l'_1) \hat{e}_q^1 + (2l'_2) \hat{e}_q^2$.

Observe that, with the current expressions, $\vec{P}_0$ is not able to map all points on the lattice. Since $q^1, q^2$ are integers, and $l'_1, l'_2$ are given lengths of the unit cell, only the location of each cell’s characteristic joint can be determined. Through homogenization, this joints will become points in
2.2 Virtual power of internal forces for a beam

Essential to the present study is the application of the principle of virtual power to be able to define the necessary internal forces: stress tensors and couple stress tensors to be introduced. For this purpose, the analysis is first performed on a single beam. Assume the beam from figure 2.6 is any sample beam isolated from the lattice. A basis is defined by the two unit vectors $b$ in the continuum. On the other hand, points found inside the cell are taken to the continuum through the use of asymptotic expansions due to the separation of scales as described in section 2.5.
the axial direction and $e_{b^\perp}$ in the direction perpendicular to the beam. Suppose, also, this sample beam is subject to forces and moments exerted by adjacent beams around the end nodes. (This assumption is imposed so that, at a later stage, the lattice may be represented by sole use of its joints.)

The virtual power $P_{AE}^*$ associated to the external forces exerted in the beam is defined as

$$P_{AE}^* = \sum A \cdot \vec{v}_A^* + \sum M_A \cdot \vec{\omega}_A^* + \sum A \cdot \vec{v}_E^* + \sum M_E \cdot \vec{\omega}_E^*$$

(2.2)

where $\vec{v}_A^*$, $\vec{v}_E^*$ are the virtual velocities at A and E, and $\vec{\omega}_A^*$, $\vec{\omega}_E^*$ are the virtual rotation rates at A and E respectively. Recall that these virtual fields are arbitrary, but may be kinematically admissible i.e. comply with the problem’s boundary conditions.

Considering the lattice of beams, the virtual power $P_{AE}^*$ becomes a virtual power of internal forces for the lattice. Virtual power for all internal forces is obtained by adding the different virtual powers obtained for each beam. This is shown in expression (2.3). As a side note, the present work
only considers 2D lattices, so that the direction of moments and rotation rate fields are described on the axis $e^3$ perpendicular to the plane.

\[
P_{\text{int}}^* = -(T_A \cdot \vec{v}_A^* + M_A \cdot \vec{\omega}_A^* + T_E \cdot \vec{v}_E^* + M_E \cdot \vec{\omega}_E^*) \tag{2.3}
\]

The method as delineated by Dos Reis in [43] intends to describe the cells behavior specifically around the characteristic joints defined in section 2.1. For this purpose, a more useful form of the expression (2.3) is one written in terms of forces and moments exerted at the joints only, gives:

\[
A \begin{bmatrix} \vec{T}_A \\ M_A \end{bmatrix} E \begin{bmatrix} \vec{T}_E \\ M_E \end{bmatrix} = -\vec{T}_A \times \vec{AE} - \vec{T}_A \tag{2.4}
\]

In the expressions above, the moments and forces at A and E are determined in terms of the moments and forces at A. The circled letters denote the joint taken as reference to obtain the moments.
To obtain balanced expressions, and show clearly the relation between the virtual rotation rates at each joint, a third point at the center of the beam $C$ may be defined. (See figure 2.6.) Take a wrench of the moment on $A$ so that it lies on $C$ as follows:

$$
\overrightarrow{MC} = \overrightarrow{MA} + \overrightarrow{T_A} \times \frac{\overrightarrow{AE}}{2}
$$

Replacing $\overrightarrow{ME}$ by its expression in terms of $\overrightarrow{MA}$ in (2.3) gives:

$$
P_{\text{int}}^* = -\overrightarrow{T_A} \cdot (\vec{v}_E^* - \vec{v}_A^*) + \left( -\overrightarrow{MA} - \overrightarrow{T_A} \times \overrightarrow{AE} \right) \cdot (\vec{\omega}_E^* - \vec{\omega}_A^*) - \left( \overrightarrow{T_A} \times \overrightarrow{AE} \right) \cdot (\vec{\omega}_A^*)
$$

This expression (2.4) is now rearranged to group the terms that affect the rotation rate and velocity fields separately. The difference in rotation rates and velocities is emphasized in the following procedure by use of the moment at $C$ center of the beam.

$$
P_{\text{int}}^* = -\overrightarrow{T_A} \cdot (\vec{v}_E^* - \vec{v}_A^*) - \overrightarrow{MA} \cdot (\vec{\omega}_E^* - \vec{\omega}_A^*) - \left( \overrightarrow{T_A} \times \overrightarrow{AE} \right) \cdot (\vec{\omega}_A^* + \vec{\omega}_E^* - \vec{\omega}_A^*)
$$

The form of equation (2.5) is convenient for later homogenization, since it demonstrates distinctly the role of forces and moments. $A$ and $E$ are neighboring joints, the differences that appear in the first two terms can be expressed in terms of the gradients. To simplify the expression further we define $\overrightarrow{T_A}$ by its components: $\overrightarrow{T_A} = n_A e^b + t_A e^{b\perp}$. Since $\overrightarrow{AE} = l_b e^b$, $l_b$ being the beam length as shown in figure 2.6, the moment expressed by the cross product in (2.4) is, $\overrightarrow{AE} \times \overrightarrow{T_A} = l_b t_A e^3$. 

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Replacing in \( (2.5) \) gives:

\[
P_{\text{int}}^* = -n_A \left[ e^b \cdot (\vec{v}_E^* - \vec{v}_A^*) \right] - t_A \left[ e^\perp \cdot (\vec{v}_E^* - \vec{v}_A^*) \right] \\
- (\vec{M}_A^F + \frac{lb}{2} t_A e^3) \cdot (\omega^*_E - \omega^*_A) + (lb t_A e^3) \cdot (\omega^*_A) \quad (2.6)
\]

or keeping \( \vec{M}_C^A \):

\[
P_{\text{int}}^* = -n_A \left[ e^b \cdot (\vec{v}_E^* - \vec{v}_A^*) \right] - t_A \left[ e^\perp \cdot (\vec{v}_E^* - \vec{v}_A^*) \right] \\
- \vec{M}_C^A \cdot (\omega^*_E - \omega^*_A) - (lb t_A e^3) \cdot \left( \frac{\omega^*_A + \omega^*_E}{2} \right) \quad (2.7)
\]

### 2.3 Virtual power of internal forces for the lattice

In section 2.2 equilibrium of the beam is assumed to build expression (2.3), as equilibrium of the lattice is assumed. Consider the equilibrium of the lattice. It is possible to write the principle of virtual power for the lattice as the sum of the of the different expressions obtained for each cell. The virtual power of internal forces for the lattice is expressed as:

\[
P_{\text{int}}^{\text{latt}} = \sum_{b=1}^{B} -\vec{T}_A(b) \cdot (\vec{v}_E(b) - \vec{v}_A(b)) - \vec{M}_C^A(b) \cdot (\omega^*_E(b) - \omega^*_A) - (\vec{T}_A \times \vec{A}_E(b)) \cdot \left( \frac{\omega^*_A(b) + \omega^*_E(b)}{2} \right) \quad (2.8)
\]

Expression (2.8) uses \( b \) to distinguish between \( A \) and \( B \) joints for different beams in the cell, where \( B \) is the number of beams in the unit cell.

Notice that with expression (2.8), the beams themselves are no longer considered, just the forces and moments applied to the nodes. This gives a simplified model, with forces and moments taken to nodes \( A \) and \( E \) for beams in the cell, which is conveyed in figure 2.7. Seeing that the lattice is built by periodic repetition of the unit cell, and again, considering it is in equilibrium, the virtual power principle of the whole lattice can be expressed as a sum over its total number of constituent cells. To do so, the integer values \( q^1, q^2 \) are used to number cells on two directions of
Forces and moments are taken to the A and E joints the lattice plane as defined in Figure 2.1.

\[
P_{\text{int}}^{\text{latt}} = \sum_{q_1, q_2 \in \mathbb{Z}^2} \sum_{b=1}^B -\vec{T}_{A(b)} \cdot (\vec{v}_{E(b)}^* - \vec{v}_{A(b)}^*) \\
- \vec{M}_{C(b)} \cdot (\vec{\omega}_{E(b)}^* - \vec{\omega}_{A(b)}^*) - (\vec{T}_{A(b)} \times \vec{AE}(b)) \cdot \left( \frac{\vec{v}_{A(b)}^* + \vec{v}_{E(b)}^*}{2} \right)
\]

In expression (2.9), \( \mathbb{Z}^2 \) is the set of all integers\( \{(q^1, q^2), q^1, q^2 \in \mathbb{Z}\} \) representing the total number of cells in the lattice. It is worth noting this is no longer an expression for external virtual power, since the external forces on each beam become internal once the lattice is assembled.

### 2.4 Expressions of forces and moments in a beam

Along with expression (2.9), which presents the lattice’s internal virtual power, constitutive relations for the beams must be introduced to find the stresses characterizing the lattice behavior. There are several theories that provide with useful constitutive relations for beams. The well known Euler-Bernoulli beam theory has been used extensively to find effective properties of a lattice, by Dos
Reis and Ganghoffer [15], by Gibson and Ashby using their own method [23], Kumar and McDowell [30], Arebnejad and Pasini [4]. Mention of alternative theories as applied to homogenization is scarce in the literature. Arguments to conclude in favor of Euler-Bernoulli are lacking, or the discussion is simply dismissed, perhaps due to the prevalence of this theory and its simplicity.

The present work aims to apply Timoshenko Beam theory. The rationale behind this particular choice is the hypothesis that Timoshenko’s affinity to the micropolar continuum is superior than that of Euler Bernoulli. Briefly explained, while Euler Bernoulli considers cross-section rotation as deduced from the displacement, Timoshenko considers it an independent variable analogous to the rigid rotation present in micropolar continuum theory. This section concentrates on listing all expressions taken from Timoshenko theory that are relevant to the homogenization method being described. Derivation of such expressions for both Timoshenko and Euler-Bernoulli is discussed in appendix A.

Observe figure 2.8 summarizing the notation used for the geometry of a beam in the lattice. It is assumed that beams have rectangular cross-sections with height $h$ and depth $d_t$. Depth lies on the $e^3$ axis, and is considered to be very long in order to apply a 2D plane strain model.

![Figure 2.8: Beam geometry, with forces and moments around the nodes.](image)

Area of the rectangular beam is simply $A = d_t h$ and moment of inertia $I = \frac{d_t h^3}{12}$.

Define also, displacement $\mathbf{u}_N$ and section rotation $\phi_N$ due to deformation, for nodes $N = A$
and $N = E$ of the beam as shown on figure 2.9.

![Figure 2.9: Displacement and cross-section rotation angle for a beam in the lattice. Adapted from [3, 43]](image)

Replacing the area and moment of inertia into the formulas detailed in (A) the following relations at nodes A and E of a beam give the expressions of the forces and moments in terms of the displacements and notations at the two ends. Note that the force vector $\mathbf{T}_A = n_A e^b + t_A e^{b\perp}$, and $M_A$, $M_E$, $M_C^{(2)}$ are moments around the $e^3$ axis perpendicular to the lattice plane.

$$n_A = -\frac{d_i h E}{l_b} (e^b \cdot [\mathbf{u}_E - \mathbf{u}_A]) \quad (2.10)$$

$$t_A = -\frac{E d_i h^3}{(l_b)^3} \left(1 + \frac{E h^2}{\alpha_f (l_b)^2}\right) (e^{b\perp} \cdot [\mathbf{u}_E - \mathbf{u}_A]) \quad (2.11)$$
\[ M_A = -\frac{Ed_t h^3}{2(l_b)^2 \left(1 + \frac{E h^2}{G_f (l_b)^2}\right)} \left(e^{b_L} \cdot [\mathbf{u}_E - \mathbf{u}_A]\right) \]

\[ + \frac{Ed_t h^3}{2(l_b)^2} \left[1 + \left(\frac{E h^2}{G_f (l_b)^2}\right)\right] \left[ \frac{(\phi_A + \phi_E)}{2}\right] - \frac{Ed_t h^3}{6(l_b)} \left[ \frac{(\phi_E - \phi_A)}{2}\right] \quad (2.12) \]

\[ M_E = -\frac{Ed_t h^3}{2(l_b)^2 \left(1 + \frac{E h^2}{G_f (l_b)^2}\right)} \left(e^{b_L} \cdot [\mathbf{u}_E - \mathbf{u}_A]\right) \]

\[ + \frac{Ed_t h^3}{2(l_b)^2} \left[1 + \left(\frac{E h^2}{G_f (l_b)^2}\right)\right] \left[ \frac{(\phi_A + \phi_E)}{2}\right] + \frac{Ed_t h^3}{6(l_b)} \left[ \frac{(\phi_E - \phi_A)}{2}\right] \quad (2.13) \]

The moment \( \overline{M_A}^C \), denoting the equivalent moment \( \overline{M_A}^C \) as taken from the center of the beam \( C \), is of particular interest since it appears in the virtual power formulation described earlier.

\[ \overline{M_A}^C = -\frac{Ed_t h^3}{6(l_b)} \left[ \frac{(\phi_E - \phi_A)}{2}\right] \quad (2.14) \]

Expressions (2.10-2.14) use \( E \) to denote the Young’s modulus for the beam material, \( G \) its shear modulus, and \( f_s \) the correction factor for shear force distribution across the beam cross section (details in Appendix A).

Equations (2.10) to (2.13) include several terms that refer to the geometry of the beam, i.e. its height \( h \), length \( l_b \) and depth \( d_t \) on the \( e^3 \) direction. Depending on the problem, either plane stress or strain, \( d_t \) is assumed respectively to be very small or very large. Therefore, its effect on the behavior of the homogenized lattice is considered independent of the geometry on the 2D plane. This is not necessarily the case for dimension \( h \), a dimension that lies on the plane. To make an accurate analysis of \( h \) and \( l_b \), the term \( \eta \) is introduced in the expressions, where \( \eta = \frac{h}{l_b} \). In this case
$h$ and $l^b$ can be considered as of same order of magnitude. Using $\eta$, equations $\text{(2.10)}$ to $\text{(2.14)}$ give:

\begin{align*}
    n_A &= -d_t \eta E (e^b \cdot [\bar{u}_E - \bar{u}_A]) \\
    t_A &= -\frac{Ed_t \eta^3}{1 + \frac{E_{\eta}^2}{G_f}} (e^{b \perp} \cdot [\bar{u}_E - \bar{u}_A]) \\
    &+ \frac{Ed_t \eta^3 (l^b)}{(1 + \frac{E_{\eta}^2}{G_f})} \left[ (\phi_A + \phi_E) \right] \\
    &+ \frac{Ed_t \eta^3 (l^b)^2}{(1 + \frac{E_{\eta}^2}{G_f})} \left[ (\phi_A + \phi_E) \right] \\
&+ \frac{Ed_t \eta^3 (l^b)^2}{6} \left[ (\phi_E - \phi_A) \right] \\
    M_A &= -\frac{Ed_t \eta^3 (l^b)}{2(1 + \frac{E_{\eta}^2}{G_f})} (e^{b \perp} \cdot [\bar{u}_E - \bar{u}_A]) \\
    &+ \frac{Ed_t \eta^3 (l^b)^2}{2 \left[ 1 + \left( \frac{E_{\eta}^2}{G_f} \right) \right]} \left[ (\phi_A + \phi_E) \right] \\
&+ \frac{Ed_t \eta^3 (l^b)^2}{6} \left[ (\phi_E - \phi_A) \right] \\
    M_E &= -\frac{Ed_t \eta^3 (l^b)}{2(1 + \frac{E_{\eta}^2}{G_f})} (e^{b \perp} \cdot [\bar{u}_E - \bar{u}_A]) \\
    &+ \frac{Ed_t \eta^3 (l^b)^2}{2 \left[ 1 + \left( \frac{E_{\eta}^2}{G_f} \right) \right]} \left[ (\phi_A + \phi_E) \right] \\
&+ \frac{Ed_t \eta^3 (l^b)^2}{6} \left[ (\phi_E - \phi_A) \right] \\
    M_C &= -\frac{Ed_t \eta^3 (l^b)^2}{12} (\phi_E - \phi_A)
\end{align*}

2.5 Asymptotic homogenization

An intuitive explanation of the use of asymptotic expansions in homogenization is given in the work by Hassini and Hinton [27]. Assume $f(x)$ is a function describing some physical quantity of the heterogeneous media. Also, consider period $Q$, which describes the appearance of its repeating constituents (i.e. cells in the case of a lattice), the quantity exhibits a global behavior that is noticeable on a large scale. However, the media’s heterogeneities cause this quantity to vary inside a small region of close values of $x$. It is possible to describe this small region through a second scale.
of a size comparable to period \( Q \). The small scale is usually described in the literature by use of the parameter \( \varepsilon \). Consider a periodic lattice of a large number of cells \( N_c \) spread over the 2D domain \( \Omega \). The number of cells is large and \( N_c^{1/2} \) can be chosen as the small parameter \( \varepsilon \). A sequence of lattices is built starting with a unit cell of finite size, applying a homothetic transformation with said factor \( \varepsilon \) and rescaling the domain \( \Omega \). The domain \( \Omega \) is now filled with the repetitive periodic small cell.

It was previously noted that the unit cell and the lattice may introduce two different cell wall lengths (\( l'_1 \) and \( l'_2 \)), and different beam sizes inside the unit cell (\( l^{b=1,2,...} \)). Still, the parameter \( \varepsilon \) is meant to describe the scale of all lengths in the unit cell. It is assumed that the walls in the unit cell and the beams inside it have the same comparable size and work at the same level as defined by \( \varepsilon \). That is, in both the large and small scales, the relationship between length walls and beams is preserved. Due to the homothety, if a cell wall of length \( L'_1 \) is 1.2 times the length of the cell wall in another axis, \( L'_2 \), the relation on the large scale \( L'_1 = 1.2L'_2 \) is kept in the small scale such that \( l'_1 = 1.2l'_2 \). This makes it convenient, later on, to express formulas in terms of only one cell wall length. Dimension of other walls and beams are multiples of the reference length of choice. An example to illustrate this particular convention is given in figure 2.10.

Figure 2.11 illustrates the effect of scaling using the small parameter \( \varepsilon \). As explained in \([27]\), it is possible to imagine \( f(x) \) as both a function of \( x \) and a function of the scaled variable \( x/\varepsilon \), so that \( f(x, x/\varepsilon) \). In other words, \( 1/\varepsilon \) makes it possible to magnify and distinguish the effect of the small constituents of the heterogeneous media. If \( \tilde{f}(x) \) is a vector function describing any physical quantity, because of the separation of scales, \( \tilde{f}^\varepsilon(x) \) can be considered as:

\[
\tilde{f}^\varepsilon(x) = \tilde{f}^0(x) + \varepsilon \tilde{f}^1(x, \frac{x}{\varepsilon}) + \varepsilon^2 \tilde{f}^2(x, \frac{x}{\varepsilon}) + ... \tag{2.20}
\]

in order to show its dependence on the macroscopic variable \( x \) and the microscopic variable \( y = (\frac{x}{\varepsilon}) \).

In a discrete case, as the lattice For the purpose of homogenization, the distance between joints, namely the length of the walls of the unit cell \( L' \), is \( \varepsilon L'_i(i = 1, 2) \). The domain is covered by repetitive cells becoming smaller and smaller as \( \varepsilon \) tends to zero. The nodes become neighboring
points and can be considered as material points of a continuum. If, then, one were to magnify the lattice by means of the scaling factor $\frac{1}{\varepsilon}$, the effects of $L'$ would become apparent.

Thus, following the procedure in [48], the small parameter $\varepsilon$ is first introduced into expression (2.1) to properly define any joint position in the lattice. This is accomplished by replacing $l' = \varepsilon L'$ into expression (2.1)

$$\vec{P}_0 = (q^1 \varepsilon L'_1) \vec{e}_1^\lambda + (q^2 \varepsilon L'_2) \vec{e}_2^\lambda$$ (2.21)

As $\varepsilon \to 0$, joints on the lattice “approximate” a continuum. The joints are defined by the Lagrange variables $(\lambda^1, \lambda^2)$ such that $\lambda^1 = \varepsilon q^1$. A basis for the continuum is defined as $e_i^\lambda = L'_i \vec{e}_i^\lambda$ where the vectors are not necessarily unit vectors and not necessarily orthogonal. The position of any point in this continuum is described simply by the vector:

$$\vec{P}_0(\lambda^1, \lambda^2) = \lambda^1 e_1^\lambda + \lambda^2 e_2^\lambda$$ (2.22)
Using unit vectors one may use the expression:

\[
\mathbf{P}_0(\lambda^1, \lambda^2) = (\lambda^1 L'_1) \hat{e}_1^\lambda + (\lambda^2 L'_2) \hat{e}_2^\lambda
\]  

(2.23)

Or, \( \mathbf{e}_i^\lambda \rightarrow L'_i \hat{e}_1^\lambda \) which simply describes a scaling of \( (\lambda^1, \lambda^2) \) by the length of the periodic unit cell.

Again, \( \lambda_i \) can only take discrete values i.e. multiples of \( \varepsilon L' \). Vector \( \mathbf{P}_0 \) holds no information about the points inside the cells. Carrying information from the cell into the continuum is the role of the higher order terms of \( \varepsilon \) in the asymptotic expansion. For example, to describe the position of any given point \( N \) inside the cell on \( (\lambda^1, \lambda^2) \), the position vector in the \( \mathbf{e}_i^\lambda \) basis (the continuum described by the beam joints in the lattice) takes the form 

\[
\mathbf{P}_N(\lambda^1, \lambda^2) = \mathbf{P}_0(\lambda^1, \lambda^2) + \varepsilon \mathbf{P}_1\mathbf{e}_N(\lambda^1, \lambda^2) + \varepsilon^2 \mathbf{P}_2\mathbf{e}_N(\lambda^1, \lambda^2) + ...
\]

This position vector to point \( N \) is depicted in figure 2.12.

It is important to mention that the beam joints forming the lattice in figure 2.12 are close to each other, or \( \varepsilon \rightarrow 0 \), as previously explained. This “continuum” is formed only by the beam joints. In the case of position vectors in a periodic lattice, these terms are \( \mathbf{P}_1, \mathbf{P}_2, ... \) and are independent of the parameters \( \lambda^i \). The position of a point relative to the characteristic joint in any cell is described in the same way for all cells (using the same expressions for \( \mathbf{P}_1, \mathbf{P}_2, ... \)).
To illustrate this, observe figure 2.13. The positions $H1$ and $H2$ are the same relative to their cells’ characteristic joints. However, they lie in two different cells. Their positions are expressed using the same $\vec{B}^1$ but different $\vec{B}^0$ positions of the characteristic joint. For periodic lattices, and small deformations, it is assumed that displacement and rotation fields have similar asymptotic expansions. (Expressions for lattices described as quasi-periodic by Tollenaere and Caillerie in [48] vary slightly, but are not considered in the present analysis.)
2.6 Asymptotic expansion for the displacement field

Using the asymptotic expansion for the position vector, this section aims to derive the expansions for displacement and angular rotation at beam joints in the lattice. To do so, consider figure 2.14 showing beam \([b]\) in the lattice formed between joints \(A\) and \(E\). Joint \(A\), usually set at the cell’s origin, is the characteristic joint defined previously. Joint \(E\) defines the other end of the beam and connects the characteristic joints of two cells.
Using the position vector $\vec{P}^0$ in expression (2.22), the asymptotic expansion for the position of joint $A$ is:

$$\vec{P}^\varepsilon_A(\lambda_1, \lambda_2) = \vec{P}^0(\lambda_1, \lambda_2) + \varepsilon \vec{P}^1_A + \varepsilon^2 \vec{P}^2_A + \ldots \quad (2.24)$$

The asymptotic expansions of different expressions in this section are denoted by the superscript $\varepsilon$.

A Taylor expansion taken around point $A$ is used to express the position of joint $E$. It should be noted that, in a cell, the distance between beam nodes is proportional to $\varepsilon$. Therefore:

$$\vec{P}^\varepsilon_E(\lambda^i) = \vec{P}^\varepsilon_A(\lambda^i + \varepsilon d^i) = \left[ \vec{P}^0(\lambda_i) + \frac{\partial \vec{P}^0}{\partial \lambda^i} \varepsilon d^i + \frac{\partial^2 \vec{P}^0}{\partial \lambda^i \partial \lambda^j} \varepsilon^2 d^i d^j + \ldots \right] + \varepsilon \left[ \vec{P}^1_E \right] + \varepsilon^2 \left[ \vec{P}^2_E \right] + \ldots$$

or:

$$\vec{P}^\varepsilon_E(\lambda^i) = \vec{P}^0(\lambda_i) + \varepsilon \left[ \frac{\partial \vec{P}^0}{\partial \lambda^i} d^i + \vec{P}^1_E \right] + \varepsilon^2 \left[ \frac{\partial^2 \vec{P}^0}{\partial \lambda^i \partial \lambda^j} d^i d^j + \vec{P}^2_E \right] + \ldots \quad (2.25)$$

After deformation, the positions of any node $N$ denoted by the expansions $\vec{P}^\varepsilon_N(\lambda^i)$ become
As mentioned in [48], it is assumed here that the lattice remains periodic after deformation, so that the expansion $\tilde{S}_N(\lambda^i)$ has the same form as $\tilde{P}_N(\lambda^i)$. Therefore, displacement due to deformation is defined by $\tilde{u}_N(\lambda^i) = \tilde{S}_N(\lambda^i) - \tilde{P}_N(\lambda^i)$ (See figure 2.15 below). The asymptotic expansion of the displacement at $A$ is expressed as:

$$\tilde{u}_A(\lambda^i) = \tilde{u}^0(\lambda^i) + \varepsilon \tilde{u}_A^1(\lambda^i) + \varepsilon^2 \tilde{u}_A^2(\lambda^i) + ... \quad (2.26)$$

The same method applied to $E$ with $\tilde{P}_E(\lambda^i), \tilde{S}_E(\lambda^i)$, before and after deformation, yields the displacement of joint $E$, which gives:

$$\tilde{u}_E(\lambda^i + \varepsilon d^i) = \left[ \tilde{u}^0(\lambda^i) + \frac{\partial \tilde{u}^0}{\partial \lambda^i} \varepsilon d^i + \frac{\partial^2 \tilde{u}^0}{\partial \lambda^i \lambda^j} \varepsilon^2 d^i d^j + ... \right] + \varepsilon \left[ \tilde{u}_E^1 \right] + \varepsilon^2 \left[ \tilde{u}_E^2 \right] ...$$

Figure 2.15: Displacement $\tilde{u}^e$ due to deformation. The present work analyzes deformation on the reference configuration under the assumption of small deformations.
The displacement can be expressed as in (2.27):

\[
\mathbf{u}_E^\varepsilon(\lambda^i) = \mathbf{u}^0(\lambda^i) + \varepsilon \left[ \frac{\partial \mathbf{u}^0}{\partial \lambda^i} d^i + \mathbf{u}_E^1 \right] + \varepsilon^2 \left[ \frac{\partial^2 \mathbf{u}^0}{\partial \lambda^i \lambda^j} d^i d^j + \mathbf{u}_E^2 \right] + \ldots
\]  

(2.27)

Additionally, the expressions for beam length and unit vector in direction of the beam may be of importance, particularly in problems that consider large deformations. Following the process in [48] expansion for the length of any beam \([b]\) as in figure [2.14] is found through \(l_{bc} = \| \mathbf{P}_E(\lambda^i) - \mathbf{P}_A(\lambda^i) \| \). \(L^n\) is designated to name the terms of different orders \(n\) in the expansion, analogous to the length of the beam \(L\) defined earlier in the lattice domain on section 2.1. Thus, this expansion in the approximate continuum can be more adequately expressed as:

\[
l_{bc}(\lambda^i) = \varepsilon L_{b0}^0(\lambda^i) + \varepsilon^2 L_{b1}^1(\lambda^i) + \ldots
\]  

(2.28)

Definition of the unit vector for a beam is given by \(e_{bc} = \frac{\mathbf{P}_E(\lambda^i) - \mathbf{P}_A(\lambda^i)}{\| \mathbf{P}_E(\lambda^i) - \mathbf{P}_A(\lambda^i) \|} \), and using the expansion in (2.28):

\[
e_{bc} = \frac{\mathbf{P}_E(\lambda^i) - \mathbf{P}_A(\lambda^i)}{l_{bc}(\lambda^i)}
\]

\[
e_{bc} = \frac{\varepsilon \left[ \frac{\partial \mathbf{P}^0}{\partial \lambda^i} d^i + \mathbf{P}_E^1(\lambda^i) - \mathbf{P}_A^1(\lambda^i) + \ldots \right]}{\varepsilon \left[ L_{b0}^0(\lambda^i) + \varepsilon L_{b1}^1(\lambda^i) + \ldots \right]}
\]

The first term in the expansion above is:

\[
e_{bc} = \frac{\frac{\partial \mathbf{P}^0}{\partial \lambda^i} d^i + \mathbf{P}_E^1(\lambda^i) - \mathbf{P}_A^1(\lambda^i)}{L_{b0}^0(\lambda^i)}
\]  

(2.29)

The expression in (2.29) is suitable when analyzing lattices in large deformations. The present study concentrates on small perturbations, and thus, requires only beam unit vectors explicitly described by the initial lattice configuration. Following a similar notation as in [43],[48],[3], the unit vector in the direction of beam \(b\) used throughout the present study considering only small deformations, is labeled \(e_b\).
2.7 Asymptotic expansion for the rotation field

It is worth recalling that the current method, as described by Dos Reis and Ganghoffer in [3,15], assumes that rigid joints connect the lattice cell elements i.e. beams. Therefore, it is expected that beam deflection i.e. cross-section rotation angle be transferred to the homogenized continuum. This is not straightforward, since different beam models have slightly incompatible ways of dealing with rotation. In Euler Beam theory, the cross-section angle \( \phi \) depends on displacement, specifically \( \phi = \hat{e}^b \cdot [\hat{b} \cdot \nabla u] = \frac{\partial v}{\partial x} \) in case of a straight beam aling the \( x \) axis and with a transverse displacement \( v \). Timoshenko model, on the other hand describes the cross-section rotation \( \phi \) and the transverse displacement \( v \) as independent variables.

The work of Ganghoffer and Dos Reis [15] has already applied Euler-Bernoulli to the method. It was mentioned before that this work concentrates on applying the Timoshenko model expecting a more precise approximation of lattice behavior in case of “short” beams. Furthermore, Timoshenko more closely resembles micropolar elasticity in that they both consider displacement and rotation fields as independent variables. Regardless of the beam model applied, it is necessary to define the rotation field independently first, and later decide, Euler Bernoulli being just a particular case with the rotation expressed in terms of the displacement.

Essential to this section is understanding key aspects of beam deformation models. These models define two different fields. The first one, a displacement field, to describe deformations axial or transverse to the neutral axis of the beam. And a second one, a rotation field, used to describe the rotation of the cross section.

Assume that, at the first order, two expressions define the rotation at beam joints. One is particular to the joint, the other is independent of the joint, but specific to the cell. Noting that this second rotation is independent of the joint, it is therefore interpreted as a rigid rotation. Expansion for the difference of the rotations of \( A \) an \( E \) is associated to this rigid rotation and it is written as:
\[
[\phi_E - \phi_A] = [\phi_E^*(\lambda^i + \varepsilon d^i) - \phi_A^*(\lambda^i)] = \phi_E^0 - \phi_A^0 + \varepsilon \left[ \phi_E^1 - \phi_A^1 + \frac{\partial \phi^0}{\partial \lambda^i} d^i \right] + \ldots \quad (2.30)
\]

The sum of rotations at \( A \) and \( E \) is therefore written as:

\[
[\phi_A + \phi_E] = [\phi_A^*(\lambda^i) + \phi_E^*(\lambda^i + \varepsilon d^i)] = \phi_A^0 + \phi_E^0 + \varepsilon \left[ \phi_A^1 + \phi_E^1 + \frac{\partial \phi^0}{\partial \lambda^i} d^i \right] + \ldots \quad (2.31)
\]

### 2.8 Definition of the virtual fields

Formulation of the virtual power, as determined in section 2.3, is used to define the internal forces needed to be introduced in the lattice. Before applying asymptotic expansions to take into account the lattice heterogeneities on its behavior, it is important to define the virtual fields for the lattice when \( \varepsilon \to 0 \). This is essential, since the virtual power described in section 2.8 is still defined without introduction of the \( \varepsilon \) parameter, i.e. the assumption that lattice joints are close to each other, akin to continuous materials, cannot be enforced.

It must be noted that, in general, virtual velocities and rotation rates are considered continuous, and with continuous derivatives. Since virtual fields can be defined arbitrarily, it is assumed that for all joints \( A \) inside the unit cell, the virtual velocity can be expressed as:

\[
\vec{v}_{A(b)}^* = \vec{v}_0^*(\lambda^i) \quad (2.32)
\]

simply stating that joint \( A \) is taken as a reference point in the \( \varepsilon \) lattice. For joints \( E \) connecting beams to a neighboring cell, the velocity is expressed using a Taylor expansion around (2.32).

\[
\vec{v}_{E(b)}^* = \vec{v}_0^*(\lambda^i + \varepsilon d^{i(b)}) = \vec{v}_0^*(\lambda^i) + \varepsilon \frac{\partial \vec{v}_0^*}{\partial \lambda^j} q^{i(b)} + \ldots \quad (2.33)
\]

It is useful to express these asymptotic expansions in a form as given by (2.9). Therefore,
taking the difference of the virtual velocities on joints $A$ and $E$:

$$\mathbf{v}_E^*(b) - \mathbf{v}_A^*(b) = \mathbf{v}_0^*(\lambda^i + \varepsilon d^{i(b)}) - \mathbf{v}_0^*(\lambda^i) = \varepsilon \frac{\partial \mathbf{v}_0^*}{\partial \lambda^i} d^{i(b)} + ... = \varepsilon \frac{\partial \mathbf{v}_0^*}{\partial \lambda^i} d^{i(b)} + ...$$  \hfill (2.34)

In (2.34) the $0$ subscript is dropped to simplify notation, thus recognizing that $\mathbf{v}^*$ now represents the virtual velocity field of the $\varepsilon \rightarrow 0$ lattice.

In a similar manner, the virtual rotation rate for any joint $A$ inside the cell is taken as reference:

$$\mathbf{\omega}_A^*(b) = \mathbf{\omega}_0^*(\lambda^i)$$  \hfill (2.35)

And again, on joint $E$, the virtual rotation rate is expressed as an approximation using Taylor expansions of the virtual field on $A$.

$$\mathbf{\omega}_E^*(b) = \mathbf{\omega}_0^*(\lambda^i + \varepsilon d^{i(b)}) = \mathbf{\omega}_0^*(\lambda^i) + \varepsilon \frac{\partial \mathbf{\omega}_0^*}{\partial \lambda^i} d^{i(b)} + ...$$  \hfill (2.36)

Looking back at the form of the expression (2.9), the virtual power of the lattice can be defined in terms of the rotation rate $\mathbf{\omega}_A^*(b)$ and the difference of rotation rates $E$ and $A$. Such difference can be expressed as:

$$\mathbf{\omega}_E^*(b) - \mathbf{\omega}_A^*(b) = \mathbf{\omega}_0^*(\lambda^i + \varepsilon d^{i(b)}) - \mathbf{\omega}_0^*(\lambda^i) = \varepsilon \frac{\partial \mathbf{\omega}_0^*}{\partial \lambda^i} d^{i(b)} + ... = \varepsilon \frac{\partial \mathbf{\omega}_0^*}{\partial \lambda^i} d^{i(b)} + ...$$  \hfill (2.37)

Notice that on the virtual power formulated in (2.9) in section 2.3 the moment at $C$ is taken to show a convenient relation of $\mathbf{\omega}_A^*(b)$ and $\mathbf{\omega}_E^*(b)$, namely the semi-sum of these two rotation rates. It is assumed that this semi-sum is equivalent to the rotation rate at the center of the beams $C$, i.e.

$$\frac{\mathbf{\omega}_A^*(b) + \mathbf{\omega}_E^*(b)}{2} = \mathbf{\omega}_C^*(b)$$

The initial fields are null in the boundary $d\Omega$ of $\Omega$ and moments exerted at the nodes.
2.9 Asymptotic expansions for forces

Asymptotic expansions need to be incorporated into the virtual power or weak formulation. The previous section defined such expansions for virtual fields. Similar expressions will be obtained for the internal forces. Asymptotic expansions for internal forces are defined here by inclusion of displacement and rotation expansions derived in section 2.6 and 2.7 into the expressions described in section 2.4.

Terms showing difference of joint displacement are found in all constitutive relations (2.10)...(2.13). The asymptotic expansion of this difference, taking into account the first order terms in the Taylor expansions.

\[
\begin{align*}
[\bar{\mathbf{u}}_E - \bar{\mathbf{u}}_A] = [\bar{\mathbf{u}}_E^r(\lambda^i + \varepsilon d^i) - \bar{\mathbf{u}}_A^r(\lambda^i)] &= \varepsilon \left[ \frac{\partial \bar{\mathbf{u}}_E^0}{\partial \lambda^i} d^i + \bar{\mathbf{u}}_E^1 - \bar{\mathbf{u}}_A^1 \right] \\
&+ \varepsilon^2 [\bar{u}_E^2 - \bar{u}_A^2] + ...
\end{align*}
\]

(2.38)

Terms containing rotation of joints A and E appear in the constitutive expressions. Asymptotic expansions for these terms were found earlier and are listed again below.

\[
\begin{align*}
[\phi_A + \phi_E] &= [\phi_A^r(\lambda^i) + \phi_E^r(\lambda^i + \varepsilon d^i)] = \phi_A^0 + \phi_E^0 + \varepsilon \left[ \phi_A^1 + \phi_E^1 + \frac{\partial \phi_0}{\partial \lambda^i} d^i \right] + ...
\end{align*}
\]

(2.39)

\[
\begin{align*}
[\phi_E - \phi_A] &= [\phi_E^r(\lambda^i + \varepsilon d^i) - \phi_A^r(\lambda^i)] = \phi_E^0 - \phi_A^0 + \varepsilon \left[ \phi_E^1 - \phi_A^1 + \frac{\partial \phi_0}{\partial \lambda^i} d^i \right] + ...
\end{align*}
\]

(2.40)

Next, expressions (2.38) to (2.40) are replaced into the constitutive relations of any beam (2.10)...(2.13).
to obtain their respective asymptotic expansions.

\[ n_A = -d_t \eta E (e^b \cdot [\vec{u}_E - \vec{u}_A]^\varepsilon) \] (2.41)

\[ t_A = -\frac{Ed_t \eta^3}{(1 + \frac{E \eta^2}{\sigma_f r})} (e^{b\perp} \cdot [\vec{u}_E - \vec{u}_A]^\varepsilon) \]
\[ + \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_A + \phi_E]^\varepsilon}{2 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \frac{2}{2} - \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_E - \phi_A]^\varepsilon}{6 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \] (2.42)

\[ M_A = -\frac{Ed_t \eta^3 (c^b \varepsilon L')}{2 \left(1 + \frac{E \eta^2}{\sigma_f r}\right)} (e^{b\perp} \cdot [\vec{u}_E - \vec{u}_A]^\varepsilon) \]
\[ + \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_A + \phi_E]^\varepsilon}{2 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \frac{2}{2} - \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_E - \phi_A]^\varepsilon}{6 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \] (2.43)

\[ M_E = -\frac{Ed_t \eta^3 (c^b \varepsilon L')}{2 \left(1 + \frac{E \eta^2}{\sigma_f r}\right)} (e^{b\perp} \cdot [\vec{u}_E - \vec{u}_A]^\varepsilon) \]
\[ + \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_A + \phi_E]^\varepsilon}{2 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \frac{2}{2} + \frac{Ed_t \eta^3 (c^b \varepsilon L')^2 [\phi_E - \phi_A]^\varepsilon}{6 \left(1 + \left(\frac{E \eta^2}{\sigma_f r}\right)\right)} \] (2.44)

\[ M_{C}^\oplus = -\frac{Ed_t \eta^3 (c^b L')^2}{12} (\phi_E - \phi_A)^\varepsilon \] (2.45)

From now, \( M_{C}^\oplus \) will be written \( M_C \). Notice that the length of the beam \( l^b \) has been replaced with \( (c^b \varepsilon L') \). This is to account for the separation of scales by means of \( \varepsilon \). The study here does not include lattices that have complex periodicity (see Caillerie’s first example in [48]), so a complete expansion of \( l^b \) as in (2.28) is not necessary. The length of the beam is exactly defined by \( \varepsilon = \frac{l^b}{L_{cb}} \). Also, it is deemed convenient to utilize a cell wall length \( l' = c^b \varepsilon \) (explained in section 2.5) instead of \( l^b \). This helps analyze the formulations clearly in terms of unit cell size. Recall that all lengths described inside the cell, beams and walls, are assumed to be on the same order.

The relation between beam height and length denoted by \( \eta \) (\( \eta = \frac{h}{L_{cb}} \)) is constructed also
under the assumption that $h$ and $l$ are roughly of the same order. This is also in favor of Timoshenko theory, since Euler Bernoulli cannot capture behavior of beams accurately when $h$ and $l$ are roughly of the same order. In other words, Euler Bernoulli gives good approximations only when an additional condition on $\eta$ is imposed, i.e. $\eta \ll 1$. Therefore, a conservative approach requires the use of Timoshenko beam theory, not only to obtain more accurate results, but also in favor of simplicity, to avoid additional analysis of the size of $h$ with respect to the lattice’s large scale.

Separating the terms in increasing orders of $\varepsilon$, expansions of the internal forces can be expressed as follows:

$$n^\varepsilon_A = \varepsilon n^1_A + \varepsilon^2 n^2_A$$

$$n^1_A = -d_t \eta E \left( e^b \cdot \left[ \frac{\partial \bar{u}^0}{\partial \lambda} d^i + \bar{u}^1_E - \bar{u}^1_A \right] \right)$$

$$n^2_A = -d_t \eta E \left( e^{b\perp} \cdot [\bar{u}^2_E - \bar{u}^2_A] \right)$$

(2.46)

$$t^\varepsilon_A = \varepsilon t^1_A + \varepsilon^2 t^2_A$$

$$t^1_A = \frac{Ed_t \eta^3}{1 + \frac{E\eta^2}{Gf_c}} \left( e^{b \perp} \cdot \left[ \frac{\partial \bar{u}^0}{\partial \lambda} d^i + \bar{u}^1_E - \bar{u}^1_A \right] \right)$$

$$+ \frac{Ed_t \eta^3 (c^b L')}{1 + \frac{E\eta^2}{Gf_c}} \left[ \phi^0_A + \phi^0_E \right]$$

$$t^2_A = \frac{Ed_t \eta^3 (c^b L')}{1 + \frac{E\eta^2}{Gf_c}} \left[ \phi^1_A + \phi^1_E + \frac{2}{2} \phi^0 d^i \right]$$

(2.47)
\[ M^2_A = \varepsilon^2 M^2_A + \varepsilon^3 M^3_A \]

\[ M^2_A = -\frac{E_d \eta^3 (c^b L')}{2 \left(1 + \frac{E}{G_f} \right)} \left( e^{b \perp} \cdot \left[ \frac{\partial \bar{u}^{0}_E}{\partial \lambda^i} \cdot \vec{d}^i + \bar{u}^{1}_E - \bar{u}^{1}_A \right] \right) \]

\[ + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^0_A + \phi^0_E \right] + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^0_E - \phi^0_A \right] \]

\[ M^3_A = -\frac{E_d \eta^3 (c^b L')}{2 \left(1 + \frac{E}{G_f} \right)} \left( e^{b \perp} \cdot \left[ \bar{u}^{2}_E - \bar{u}^{2}_A \right] \right) \]

\[ + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^1_A + \phi^1_E + \frac{\partial \phi^0}{\partial \lambda^i} \cdot \vec{d}^i \right] + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^1_E - \phi^1_A + \frac{\partial \phi^0}{\partial \lambda^i} \cdot \vec{d}^i \right] \]

\[ (2.48) \]

\[ M^2_E = \varepsilon^2 M^2_E + \varepsilon^3 M^3_E \]

\[ M^2_E = -\frac{E_d \eta^3 (c^b L')}{2 \left(1 + \frac{E}{G_f} \right)} \left( e^{b \perp} \cdot \left[ \frac{\partial \bar{u}^{0}_E}{\partial \lambda^i} \cdot \vec{d}^i + \bar{u}^{1}_E - \bar{u}^{1}_A \right] \right) \]

\[ + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^0_A + \phi^0_E \right] + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^0_E - \phi^0_A \right] \]

\[ M^3_E = -\frac{E_d \eta^3 (c^b L')}{2 \left(1 + \frac{E}{G_f} \right)} \left( e^{b \perp} \cdot \left[ \bar{u}^{2}_E - \bar{u}^{2}_A \right] \right) \]

\[ + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^1_A + \phi^1_E + \frac{\partial \phi^0}{\partial \lambda^i} \cdot \vec{d}^i \right] + \frac{E_d \eta^3 (c^b L')^2}{2 \left[1 + \left( \frac{E}{G_f} \right) \right]} \left[ \phi^1_E - \phi^1_A + \frac{\partial \phi^0}{\partial \lambda^i} \cdot \vec{d}^i \right] \]

\[ (2.49) \]
\[ M_C^e = \varepsilon^2 M_C^2 + \varepsilon^3 M_C^3 \]

\[ M_C^2 = -\frac{E\rho \eta^3 (c^b L')^2}{12} \left[ \phi_E^0 - \phi_A^0 \right] \]

\[ M_C^3 = -\frac{E\rho \eta^3 (c^b L')^2}{12} \left[ \phi_E^1 - \phi_A^1 + \frac{\partial \phi_A^0}{\partial \lambda^i} d^i \right] \]

(2.50)

### 2.10 Asymptotic expansion for the lattice virtual power of internal forces

Having defined asymptotic expansions for the forces and moments exerted at the joints and the virtual displacement field, it is now possible to formulate the virtual power principle for the lattice at equilibrium. Considering the expression (2.9):

\[ \forall \mathbf{v}_*, \mathbf{\omega}_* \sum_{(q_1, q_2) \in \mathbb{Z}^2} \sum_{b=1}^{B} -\overline{T}_{A(b)} \cdot \left( \varepsilon \frac{\partial \mathbf{\omega}_*}{\partial \lambda^i} d^{i(b)} \right) \]

\[ -\overline{M}_{C(b)} \cdot \left( \varepsilon \frac{\partial \mathbf{\omega}_*}{\partial \lambda^i} d^{i(b)} \right) - (\overline{T}_{A(b)} \times A E(b)) \cdot (\omega_*(\lambda^i)) = 0 \]

For practicality, the expression is expressed as a sum, with no negative terms. Recall that \( \overline{T}_{A} = n_A^e e^{b} + t_A^e e^{b_\perp} \), and replacing above, gives:

\[ \forall \mathbf{v}_*, \mathbf{\omega}_* \sum_{(q_1, q_2) \in \mathbb{Z}^2} \sum_{b=1}^{B} \left( n_{A(b)}^e e^{b} + t_{A(b)}^e e^{b_\perp} \right) \cdot \left( \varepsilon \frac{\partial \mathbf{\omega}_*}{\partial \lambda^i} d^{i(b)} \right) \]

\[ + \overline{M}_{C(b)} \cdot \left( \varepsilon \frac{\partial \mathbf{\omega}_*}{\partial \lambda^i} d^{i(b)} \right) + \left( n_{A(b)}^e e^{b} + t_{A(b)}^e e^{b_\perp} \right) \times \overline{A E(b)} \cdot (\omega_*(\lambda^i)) = 0 \]

Observe that both the first and last terms operate on the vector \( \overline{T}_{A(b)} = n_{A(b)}^e e^{b} + t_{A(b)}^e e^{b_\perp} \).
Therefore, these two may be combined. The vector $\mathbf{AE(b)}$ can be expressed as

$$\mathbf{AE(b)} = t^b e^b = \varepsilon L^b e^b = \varepsilon d^{i(b)} e^\lambda_i$$

and only the term $d^{i(b)}$ depends on the beam $b$.

$$\sum_{(q_1,q_2) \in \mathbb{Z}^2} \sum_{b=1}^{s} \left( n_{A(b)}^e e^b + t_{A(b)}^s e^{b,\perp} \right) \cdot \left( \varepsilon \frac{\partial \mathbf{v}^*}{\partial \lambda^t} d^{i(b)} + \varepsilon d^{i(b)} e^\lambda_i \times \omega^*(\lambda^t) \right)$$

$$+ M_{C(b)}^x \left( \varepsilon \frac{\partial \omega^*}{\partial \lambda^t} d^{i(b)} \right) = 0 \quad (2.51)$$

Looking back at the asymptotic expansions of the constitutive relations, it is evident that not all terms affect a beam’s response at the same order of $\varepsilon$. To understand their effect on the overall lattice, expressions in terms of the orders of $\varepsilon$ (2.46)...(2.49) are introduced into the virtual
power formulation (2.51).

\[
\sum_{(q_1,q_2) \in \mathbb{Z}^2} \sum_{b=1}^{B} \left[ \left( \varepsilon n_{A(b)}^1 + \varepsilon^2 n_{A(b)}^2 \right) e^b + \left( \varepsilon t_{A(b)}^1 + \varepsilon^2 t_{A(b)}^2 \right) e^{b\perp} \right]
\]

\[
\cdot \left[ \varepsilon \frac{\partial \bar{\theta}^*}{\partial \lambda} d^{i(b)} + \varepsilon d^{i(b)} e^\lambda_i \times \omega^*(\lambda^i) \right]
\]

\[
+ \left[ \left( \varepsilon^2 M_{C(b)}^2 + \varepsilon^3 M_{C(b)}^3 \right) e^3 \right] \cdot \left[ \left( \varepsilon \frac{\partial \bar{\omega}^*}{\partial \lambda^i} d^{i(b)} \right) \right] + \ldots
\]

Multiplying the parameters \(\varepsilon\) where possible and factoring out the lowest order of \(\varepsilon\) in the expression i.e. \(\varepsilon^2\), the following is obtained:

\[
(\varepsilon^2) \sum_{(q_1,q_2) \in \mathbb{Z}^2} \sum_{b=1}^{B} \left[ \left( n_{A(b)}^1 + \varepsilon^2 n_{A(b)}^2 \right) e^b + \left( t_{A(b)}^1 + \varepsilon^2 t_{A(b)}^2 \right) e^{b\perp} \right]
\]

\[
\cdot \left[ \varepsilon \frac{\partial \bar{\theta}^*}{\partial \lambda^i} d^{i(b)} - \omega^*(\lambda^i) \times \varepsilon d^{i(b)} e^\lambda_i \right]
\]

\[
+ \left[ \left( \varepsilon M_{C(b)}^2 + \varepsilon^3 M_{C(b)}^3 \right) e^3 \right] \cdot \left[ \left( \varepsilon \frac{\partial \bar{\omega}^*}{\partial \lambda^i} d^{i(b)} \right) \right]
\]

(2.52)

It is worth mentioning here that the virtual power is written entirely for the lattice keeping the beam geometry with \(e^b\) and \(e^{b\perp}\). This will be accomplished in the following sections, and is of particular importance to demonstrate clearly how the terms fit into the micro-polar framework.

### 2.11 Continuum description of the beam lattice

The objective is how to obtain a continuum formulation for the beam lattice and to define the stress tensors of this “equivalent continuum.” The expression (2.52) is the negative of the
virtual power of all the forces exerted on the different joints. To obtain the weak formulation, the terms appearing in (2.52) have to be rewritten. By using classical definitions of the derivatives along a direction and of the cross product,

\[
\frac{\partial \tilde{u}^*}{\partial \lambda^i} = e_i^\lambda \cdot (\nabla \tilde{v}^*_j)_{ji} \\
\frac{\partial \tilde{\omega}^*}{\partial \lambda^i} = e_i^\lambda \cdot (\nabla \tilde{\omega}^*_j)_{ji} \\
\tilde{T} \cdot (\tilde{\omega}^* \times e_i^\lambda) = (\tilde{T} \otimes e_i^\lambda) : (\tilde{T} \cdot \tilde{\omega}^*)
\]

(2.53)

it is possible to rewrite the virtual power as:

\[
(\varepsilon^2) \sum_{(q_1, q_2) \in \mathbb{Z}^2} \left\{ \left( \sum_{b=1}^B S^e_{(b)} \otimes e_i^\lambda \right) : \left[ (\nabla \tilde{v}^*_j)_{ji} - \frac{\partial}{\partial \lambda^i} \tilde{\omega}^* \right] + \left( \sum_{b=1}^B \mu^e_{(b)} \otimes e_i^\lambda \right) : \left( \nabla \tilde{\omega}^*_j \right)_{ji} \right\} + ...
\]

(2.54)

where, for each beam, \( S^e_{(b)} \) and \( \mu^e_{(b)} \) are:

\[
S^e_{(b)} = \left( d^{(b)} \right) \left( n_{1A(b)}^1 + \varepsilon n_{2A(b)}^2 \right) e^b + \left( t_{1A(b)}^1 + \varepsilon t_{2A(b)}^2 \right) e^{b\perp} \\
\mu^e_{(b)} = \left( d^{(b)} \right) \left( \varepsilon M_{C(b)}^2 + \varepsilon^2 M_{C(b)}^3 \right) e^3
\]

(2.55)

If given forces or moments are applied at the joints, the virtual power of all the “forces”
exerted on the joints $\vec{x}_n$ of the lattice is now:

\[
(\varepsilon^2) \sum_{(q_1,q_2) \in \mathbb{Z}^2} \left\{ \mathbf{B} \sum_{b=1}^{N} \sum_{(\lambda_1, \lambda_2)} \mu_{(b)} \otimes \mathbf{e}^\lambda_j \right\} : \left[ \left( \nabla \mathbf{v}^* \right)^*_j - \tilde{\epsilon} \cdot \tilde{\omega}^* \right] + \left( \sum_{b=1}^{N} \mu_{(b)} \otimes \mathbf{e}^\lambda_j \right) : \left( \nabla \mathbf{\omega}^* \right)^*_j + \sum_{(q_1,q_2) \in \mathbb{Z}^2} \sum_{n} \vec{F}^n \cdot \vec{v}^* (\vec{x}_n) + \sum_{(q_1,q_2) \in \mathbb{Z}^2} \sum_{n} \vec{C}(x_n) \cdot \vec{\omega}^* (\vec{x}_n)
\]

(2.56)

where $n$ defines the unit cell nodes.

As $\varepsilon$ tends to zero, the cells are small and the number is large, the surface of each cell is denoted $\varepsilon^2 J$, where $J$ is the surface of the reference cell, which is the Jacobian of the transformation\(^1\) from the $\lambda^i$ space into the $x^i$ space.

\[ J = \det \left| \frac{\partial x^i}{\partial \lambda^j} \right| \]

The sums over the number of cells can then be considered as the Riemann sums of an integral. It can be seen that the forces $\vec{F}^n$ need to be of the order $\varepsilon^2$ and the moments $\vec{C}(x_n)$ of the order $\varepsilon^3$.

\(^1\)Remark: $\frac{\partial \mathbf{P}_0}{\partial \lambda} = \mathbf{e}^\lambda_i$. The complete transformation from $x^i$ to $\lambda^i$ coordinates may be summarized like so:

\[
\begin{bmatrix}
\lambda^1 \\
\lambda^2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{L_1} & 0 \\
0 & \frac{1}{L_2}
\end{bmatrix}
\begin{bmatrix}
x^1 \\
x^2
\end{bmatrix}
\]

curvilinear transformation

So that TR1 is a scaling transformation, and TR2 is a transformation depending on the lattice geometry and direction of periodicity. The resulting transformation $[\mathbf{TTR}] = [\mathbf{TR1}] \cdot [\mathbf{TR2}]$, so that $[\mathbf{TTR}]^{-1} = \frac{\partial x^i}{\partial \lambda^j}$.
The new expression for the virtual power principle is now:

\( \forall \mathbf{v}^*, \mathbf{w}^* \text{ with } \mathbf{v}^* = 0, \mathbf{w}^* = 0 \text{ in the boundary } d\Omega \text{ of the domain } \Omega \)

\[
- \int_{\Omega} \left\{ \frac{1}{J} \left( \sum_{b=1}^{B} S_{(b)}^{\varepsilon_i} \otimes e_i^\lambda \right) : \left[ \nabla \mathbf{v}^* \right]_{ji} - \varepsilon \cdot \mathbf{w}^* \right\} \right. \\
\left. + \frac{1}{J} \left( \sum_{b=1}^{B} \mu_{(b)}^{\varepsilon_i} \otimes e_i^\lambda \right) : \left[ \nabla \mathbf{w}^* \right]_{ji} \right\} \, da \\
+ \int_{\Omega} \left\{ \frac{1}{(\varepsilon^2)J} \sum_n f_n \cdot \mathbf{v}^* (x_n) + \frac{1}{(\varepsilon^2)J} \sum_n \tilde{C}(x_n) \cdot \mathbf{w}^* (x_n) \right\} \, da + ... = 0 \quad (2.57)
\]

It is now possible to define the stresses for the equivalent continuum.

\[
\overline{\sigma} = \frac{1}{J} \left( S^{\varepsilon_i} \otimes e_i^\lambda \right) \\
\overline{\underline{m}} = \frac{1}{J} \left( \mu^{\varepsilon_i} \otimes e_i^\lambda \right)
\]

(2.58)

with:

\[
S^{\varepsilon_i} = \sum_{b=1}^{B} S_{(b)}^{\varepsilon_i} \\
\mu^{\varepsilon_i} = \sum_{b=1}^{B} \mu_{(b)}^{\varepsilon_i}
\]

(2.59)

And the body forces and moments read:

\[
f_n = \frac{1}{J} \sum_n f_n \\
\tilde{C}(x_n) = \frac{1}{J} \sum_n \tilde{C}(x_n)
\]

(2.60)
The virtual power principle can now be written as:

For any \( \vec{v}^* \) and \( \vec{\omega}^* \) null on the boundary \( d\Omega \) of the domain \( \Omega \),

\[
- \int_{\Omega} \left( \bar{\sigma} : \left[ \nabla \vec{v}^* - \bar{\varepsilon} \cdot \vec{\omega}^* \right] + \bar{m} : \nabla \vec{\omega}^* \right) da \\
+ \int_{\Omega} \left( \vec{f}^n \cdot \vec{v}^* + \bar{C}(x_n) \cdot \vec{\omega}^* \right) da = 0 \tag{2.61}
\]

This result shows the equivalence of the two formulations: the lattice or discrete formulation with the introduction of the different beams in the definition of the geometry and the material properties, and the micropolar formulation with the introduction of couple stresses.

The different tensors can be calculated from the lattice as a sum over the beams in the reference cell for \( \bar{\sigma} \) and \( \bar{m} \) or a sum over the nodes in the reference cell for \( \vec{f} \) and \( \bar{C}(x_n) \).

The different contributions \( n, t, M_c \) were written by means of the constitutive equations for the beams discussed in previous sections. Consequently, the force and couple stress tensors are defined in terms of the displacement and rotation expansions in the reference cell, which leads to a straightforward computation of the lattice effective properties.

### 2.12 Micropolar elasticity framework

This section concentrates on explaining key concepts of the theory and how the effective properties can be derived. Comparisons with classical elasticity are often necessary to help understand micropolar theory and some important differences between the two are mentioned here. Although, Eringen gives the basic concepts of micromorphic elasticity in [18], from which the particular case of micropolar elasticity is derived. Complete derivation of the theory is not detailed here, but introductory remarks by Eringen help understand the fundamentals. Afterwards, equilibrium equations and the formulations concerning micropolar elasticity are presented simply, for application in the present homogenization method.

Start by describing the position of \( P \), a material point, in the reference configuration. In classical elasticity, only a single vector to \( P \) is used to describe its position, which following
Eringen’s notation, is denoted as $X_i$. After deformation, point $P$ moves to the position described by $x_i(X_i, t)$. An important general assumption is that material points are infinitesimal elements of matter in a volume. In micromorphic elasticity theories, these infinitesimal volume points are themselves endowed with deformation capabilities. Hence, a new vector $\Xi_i$ on $P$ is generated to represent said intrinsic deformation. In this case, $P$ can be interpreted as the centroid of the particle. Figure 2.16 shows how this additional vector is defined. In the current configuration, on time $t$ after deformation, the new vector of the particle intrinsic deformation is $\xi_i(X_i, \Xi_i, t)$. However, since particles are assumed infinitesimal and close to each other, the transformation from $\Xi_i \rightarrow \xi_i$ is directly described by a linear approximation. This micromotion, as denoted by Eringen, is thus represented through a matrix of constant elements $\Xi_i$ so that: $\xi_i = \chi_{ij} \Xi_j(X_i, t)$. The final position in the current configuration is described by $x'_i = x_i(X_i, t) + \xi_i(X_i, \Xi_i, t)$.

In classical elasticity, deformation is characterized by the deformation gradient $F_{ij} = \frac{\partial x_i}{\partial X_j}$. This is maintained in the general micromorphic theory, and is now referred to the macromotion gradient, to distinguish it from the micromotion $\chi_{ij}$. Now, micropolar elasticity is a special case of the micromorphic theory, where $\chi_{ij} = \chi_{ji} = 1$. This implies that the vector associated to the particle does not describe any deformation, only a rigid rotation. The rigid rotation of the vector is represented by the angle $\phi$ in figure 2.16. Note that this is different from the rotation assigned to the macromotion gradient, i.e. the macrorotation, represented by $R_{ij}$ in the polar decomposition $F_{ij} = R_{ij} C_{ji}$, where $C_{ji} = F_{ij} F_{ji}$ define the Cauchy deformation tensor. Thus, in a micropolar medium, the points are considered as rigid solids and the motion of any material point is defined by a displacement $\vec{u}$ and a rotation $\vec{\phi}$.

From the given explanation it is easy to deduce that, in addition to the familiar displacements from classical elasticity, micropolar theory introduces the aforementioned microrotation. The form of the strains in micropolar theory may be deduced by observing the virtual power of internal forces, formulated in (??) and presented here using tensor notation:

$$P^*_{int} = -\int_{\Omega} \bar{\sigma} : \left( \nabla \bar{\varepsilon}^* - \bar{\Pi} \cdot \omega^* \right) + \bar{m} : \left( \nabla \omega^* \right) dA$$

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And in index notation depicting the elements involved on each term.

\[
P_{int}^* = - \int_\Omega \sigma_{ij} \left( \frac{\partial v_i^*}{\partial x_i} - \epsilon_{ijk} \omega_j^* k \right) + m_{ij} \left( \frac{\partial \omega_i^*}{\partial x_j} \right) \, dV \quad (2.62)
\]

Figure 2.16: Position of a material point \( P \) before and after deformation in Classical Elasticity as compared to Micropolar Elasticity. \( \phi \) represents the microrotation.
Observe that, in the absence of a virtual rotation field $\omega^*$, expression (2.62) reverts back to the classical internal virtual power formulation

$$ P_{int}^* = - \int_\Omega \bar{\sigma} : \left( \nabla \nu^* \right) dA $$

The same calculations with the velocity and rotation rate can be done and the real work of the internal forces can be also obtained to show the conjugate strains and stresses in small deformations. The strains are defined as:

$$ \bar{\varepsilon} = \nabla \bar{u} - \bar{\varepsilon} \cdot \bar{\phi} $$
$$ \bar{\kappa} = \nabla \bar{\phi} $$

(2.63)

Where $\bar{u}$ is the displacement field and $\phi$ the rotation field. In index notation with commas to denote partial derivatives:

$$ \varepsilon_{ij} = u_{j,i} - \varepsilon_{ijk} \phi_k $$
$$ \kappa_{ji} = \phi_{j,i} $$

(2.64)

Note that the strain is not symmetric. This in contrast to classical elasticity, where the strain may be taken as the symmetric part of the displacement gradient $\varepsilon = \frac{1}{2} \left( \nabla u^T + \nabla u \right)$

The presence of the rotation hinders the possibility of such an approximation in micropolar elasticity.

Consider, now, that external loads consist in forces and couples acting on the surface boundary of the micropolar elastic material. Consider, also, the presence of forces and couples acting on the body. Through the virtual power principle formulation, taking the mentioned surface and body loads into account, and observing that the virtual velocity and rotation rate are independent of each other, it is possible to derive the two governing equilibrium equations for micropolar elasticity.
They are:

$$\nabla \cdot \bar{\sigma} + \bar{f} = 0$$

$$\nabla \cdot \bar{m} + \bar{\epsilon} \cdot \bar{\sigma} + \bar{g} = 0$$

The same are shown here in index notation:

$$\sigma_{ji,j} + f_i = 0$$

$$m_{ji,j} + \epsilon_{iml}\sigma_{lm} + g_i = 0$$

(2.65)

with body forces \(f_i\) and body couples \(g_i\).

In classical elasticity, body or surface couples cannot be modeled, and are nonexistent in the equilibrium equations of the theory. Removing all couples, the equations are those of classical elasticity. The first equation reverts to \(\nabla \cdot \bar{\sigma} + \bar{f} = 0\) and the second shows \(\bar{\sigma}\) is symmetric. The second equations in both theories derive from equilibrium of angular momentum applied on the infinitesimal volume. In micropolar elasticity, this second equation introduces a couple stress tensor, and the symmetry of the force stress tensor can no longer be assumed.

Solution of these partial differential equations can be found under imposed boundary conditions, which are given displacements and rotations, or traction forces and couples, on the boundary. These traction forces and couples are described through the vector \(n_j\) normal to the surface boundary.

$$S_i^t = n_j\sigma_{ji}$$

$$\mu_i^t = n_j\sigma_{ji}$$

(2.66)

The constitutive laws for micropolar elasticity are of special interest, since they provide the
means to find effective properties after homogenization. In three dimensions, fourth order tensors are used to present the properties of the material in the constitutive law: \( A, B, C, D \).

\[
\sigma_{ij} = A_{ijkl}e_{kl} + B_{ijkl}\kappa_{kl}
\]

\[
m_{ij} = C_{ijkl}e_{kl} + D_{ijkl}\kappa_{kl}
\]

(2.67)

Note the existence of a strain energy implies that the elements coupling the stress and couple stress tensors found on \( B_{ijkl} \) and \( C_{ijkl} \) are such that \( B_{ijkl} = C_{klji} \).

Since the method is limited here to the analysis of planar elasticity, the 2D constitutive law may be summarized by use of index notation as shown below:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
\sigma_{21} \\
m_{13} \\
m_{23}
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{12} \\
e_{21} \\
k_{13} \\
k_{23}
\end{bmatrix}
\]

(2.68)

with \( A, B, C, D \) elements organized in a second order tensor of effective properties. In a more illustrative form, using components of the displacement vector \( u_1 = u \) and \( u_2 = v \), and \( x^1 = x \), \( x^2 = y \), the following notations are used for expressing examples and results ahead.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy} \\
\sigma_{yx} \\
m_{xz} \\
m_{yz}
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial x} + \phi \\
\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial y} + \phi
\end{bmatrix}
\]

(2.69)

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An analysis of terms in $A, B, C, D$ gives some insight into a structure that may be used to organize the results found in the previous section. The stress and couple stress tensors can be decomposed into terms containing the strain $\varepsilon$ and curvature $\kappa$ only.

\[
\sigma_{ij} = A_{ijkl}e_{kl} \quad + \quad B_{ijkl}\kappa_{kl}
\]
\[
m_{ij} = C_{ijkl}e_{kl} \quad + \quad D_{ijkl}\kappa_{kl}
\]

(2.70)

Here, expressions from (2.58) are used, separating them in the orders of $\varepsilon$ using superscripts $1, 2, 3$ (to show the terms affecting $\varepsilon^0, \varepsilon^1, \varepsilon^2$ respectively.) This completes the framework in which to describe the homogenized properties for the lattice.

### 2.13 Solving for the unknowns in asymptotic expansions

In sections 2.6 and 2.7, asymptotic expansions where defined for the displacement and rotation fields. In section 2.8, those expansions were inserted into the internal force expressions. Recall that, to generate the expansions, additional displacement and rotation vectors acting on $\varepsilon$ and $\varepsilon^2$ were defined, i.e. $\vec{u}_N^1, \vec{u}_N^2, \phi_N^1,$ and $\phi_N^2$. These additional terms allow for an asymptotic analysis of the various expressions in the internal forces. In other words, they help emphasize which expressions have similar impact on the lattice behaviour. They are part of the constitutive law and need to be expressed in terms of the known terms $\vec{u}^0$ and $\phi^0$. To this aim, systems of equations on each order of $\varepsilon$ will be generated.

Apart from the characteristic joint, the unit cell may have additional joints lying inside. Recall that there is an asymptotic expansion for the displacement and rotation at each joint. This means there are $N$ asymptotic expansions that must be solved, $N$ being the number of joints in...
the cell \( (N = |\mathcal{N}|) \). The task is somewhat simplified, since joints connecting beams to neighboring cells are approximated to the characteristic joint via taylor expansions, as it has been previously mentioned. However, the linear system increases in complexity depending on the pattern formed by the connected beams. At this point, it is important to clarify how joints in the cell are defined in the present method. Figure 2.17 demonstrates the process. A characteristic joint is named 1, and a joint inside the cell is numbered 2. Note that only the characteristic joint and joints inside the cell are numbered. Again, joints connecting to adjacent cells are described in terms of the characteristic joint using \( d^i \).
Different $A$ and $E$ joints for each beam

Only two nodes are defined. Joints connecting to neighboring cells are defined by the characteristic joint 1 and the auxiliary variable $d''$

Figure 2.17: Definition of joints in the cell
Having identified the joints of the cell, equilibrium is assumed on each one. Using the virtual power principle for the cell at equilibrium, considering only expansions of the internal forces i.e.

\[
\sum_{b=1}^{B} \vec{T}^\varepsilon_{A(b)} \cdot \vec{v}^*_A(b) + \vec{M}^\varepsilon_{A(b)} \cdot \vec{\omega}^*_A(b) + \vec{T}^\varepsilon_{E(b)} \cdot \vec{v}^*_E(b) + \vec{M}^\varepsilon_{E(b)} \cdot \vec{\omega}^*_E(b) = 0
\] (2.71)

one may define equilibrium of forces and moments at each node by separating the virtual fields, which are assumed to be independent.

The result is a set of \( N \) equations, of forces and moments, at each order \( O \) of \( \varepsilon \). These completely solves the model, since there are \( O \) unknowns for each expansion order of \( \varepsilon \) and \( N \) expansions defined at each node. The following shows the sets of equations to be solved.
On the first order:

\[
\begin{align*}
\sum_{b=1}^{B} T^{(1)}_{A(b)=1} + T^{(1)}_{E(b)=1} &= 0 \\
\sum_{b=1}^{B} M^{(2)}_{A(b)=1} + M^{(2)}_{E(b)=1} &= 0 \\
\sum_{b=1}^{B} T^{(1)}_{A(b)=2} + T^{(1)}_{E(b)=2} &= 0 \\
\sum_{b=1}^{B} M^{(2)}_{A(b)=2} + M^{(2)}_{E(b)=2} &= 0 \\
\vdots \\
\sum_{b=1}^{B} T^{(1)}_{A(b)=N} + T^{(1)}_{E(b)=N} &= 0 \\
\sum_{b=1}^{B} M^{(2)}_{A(b)=N} + M^{(2)}_{E(b)=N} &= 0
\end{align*}
\]

On the second order:

\[
\begin{align*}
\sum_{b=1}^{B} T^{(2)}_{A(b)=1} + T^{(2)}_{E(b)=1} &= 0 \\
\sum_{b=1}^{B} M^{(3)}_{A(b)=1} + M^{(3)}_{E(b)=1} &= 0 \\
\sum_{b=1}^{B} T^{(2)}_{A(b)=2} + T^{(2)}_{E(b)=2} &= 0 \\
\sum_{b=1}^{B} M^{(3)}_{A(b)=2} + M^{(3)}_{E(b)=2} &= 0 \\
\vdots \\
\sum_{b=1}^{B} T^{(2)}_{A(b)=N} + T^{(2)}_{E(b)=N} &= 0 \\
\sum_{b=1}^{B} M^{(3)}_{A(b)=N} + M^{(3)}_{E(b)=N} &= 0
\end{align*}
\]

To clarify the method, take again the example in figure 2.17. When building the equations on any order for joint \( N = 2 \), only the forces and moments of beams \( b = 3, 4 \) have to be considered, since these are the only ones that comply with the condition \( A(b) = 2 \ E(b) = 2 \), i.e. are the only beams connected to joint 2. This is analogous to the well known method of joints usually applied to trusses, but with the addition of moments at each joint. A depiction of the method is shown.
in figure 2.18. Note, also, that forces and moments coming from beams belonging to neighboring cells are also taken into account. In the example, on node 1, a force $\overrightarrow{T}_{A(b=1)=1}$ is exerted by beam 1. The neighboring beam 1 of the cell to the left also applies a force on node 1, which is $\overrightarrow{T}_{E(b=1)=1}$. These are all considered in the system of equations shown above. Furthermore, this way of presenting the joint method allows for an algorithmic solution to be implemented. In the following chapter, the method is applied on different lattice geometries to analyze each of the steps described to this point.
Chapter 3

Description of the method for square and honeycomb lattices

The procedure to obtain the effective properties was applied to various lattice geometries often mentioned in the literature. Before analyzing all resulting effective properties, a step by step explanation of the method is given for the square and honeycomb cases. The flowchart in figure 3.1 summarizes the steps that were followed to find the results.

Unit cells are analyzed on the large scale as in figure 2.10 i.e. considering expressions in terms of the cell’s wall length $L’$.

3.1 Identification of the periodic unit cell

The square and honeycomb structures are depicted in figure 2.4. The periodic unit cells that generate each of the aforementioned lattices are also shown in the illustration. They are singled out and presented in figure 3.2. Following the direction of periodicity to generate the lattice, a local basis $\{e^i_\lambda\}$ is defined, as shown in the figure above. Note that the length of the unit cell walls are equal $L'_1 = L'_2$ and denoted simply by $L'$ in both cases.

An angle $\alpha$, defined in the figure above, may be useful to determine the transformation from...
the \( \{e^i_\lambda\} \) to the \( \{e^x_i\} \) basis of Cartesian coordinates for any tessellation of parallelogram shaped cells. Take expression (2.22) denoting the position of any cell in \( \lambda \) coordinates

\[
\vec{P}_0(\lambda^1, \lambda^2) = \lambda^1 e^\lambda_1 + \lambda^2 e^\lambda_2
\]

The same may be written in the \( \{e^x_i\} \) basis so that

\[
\vec{P}_0(\lambda^1, \lambda^2) = (L'_1\lambda^1 + L'_2\lambda^2 \cos \alpha) e^x_1 + (L'_2\lambda^2 \sin \alpha) e^x_2
\]

This transformation may be also expressed by means of the matrix \( [\frac{\partial e^x_i}{\partial \lambda^j}] \). Introducing the angle
α, this matrix can be written as:

\[
\left[ \begin{array}{c} \frac{\partial x^j}{\partial \lambda^i} \\ \frac{\partial x^j}{\partial \lambda^i} \end{array} \right] = \begin{bmatrix} L_1' & 0 \\ L_2' \cos \alpha & L_2' \sin \alpha \end{bmatrix}
\]

Using these expressions, the following table summarizes the transformations from the $\lambda^i$ to the $x^i$ coordinates for both the square and honeycomb lattices.
Table 3.1: Transformation from $\lambda^i$ to $x^i$ coordinates for square and honeycomb cell geometries

<table>
<thead>
<tr>
<th></th>
<th>Square Lattice</th>
<th>Honeycomb Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>cell wall length</td>
<td>$L'$</td>
<td>$L'$</td>
</tr>
<tr>
<td>angle $\alpha$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{\pi}{3}$</td>
</tr>
<tr>
<td>$\vec{P}_0^0(\lambda^1, \lambda^2)$</td>
<td>$\vec{P}_x^0(\lambda^1, \lambda^2) = (L'\lambda^1) \mathbf{e}_1^x + (L'\lambda^2) \mathbf{e}_2^x$</td>
<td>$\vec{P}_x^0(\lambda^1, \lambda^2) = \left( L'\lambda^1 + \frac{L'}{2}\lambda^2 \right) \mathbf{e}_1^x + \left( \sqrt{3}\frac{L'}{2}\lambda^2 \right) \mathbf{e}_2^x$</td>
</tr>
</tbody>
</table>

| $\left[ \frac{\partial x^j}{\partial \lambda^i} \right]$ | $\begin{bmatrix} L' & 0 \\ 0 & L' \end{bmatrix}$ | $\begin{bmatrix} L'_1 & 0 \\ \frac{L'}{2} & \frac{3L'}{2} \end{bmatrix}$ |

3.2 Definition of cell joints / Periodicity

The next step in the process is the identification of joints in the cell and their numbering. This process was already described at the end section 2.13 and it is applied here on the square and honeycomb lattices. First, take one joint on a vertex of the unit cell to be the reference joint. This is the only joint pertaining to the cell that lies on its border. Joints on other vertices belong to neighboring cells are considered by way of $d^i$ as in figure 2.2. After identifying this joint, the other joints are numbered inside the cell. Numbers and values for $d^i$ are assigned such that periodicity is taken into account i.e. joints in all cells are defined the same way. The following figure shows how they are defined for the square and honeycomb lattice.
3.3 Definition of beams

After defining the joints in the lattice, beams must be numbered and connection between the cell joints have to be determined. Consider figure 3.4 showing numbering of beams on the square and honeycomb cells.

Figure 3.4: Numbering of beams in square and honeycomb unit cells

Ends $A$ and $E$ of the beams are assigned joint numbers to establish connectivity in the cell.
The table 3.2 summarizing all connections is given as in Dos Reis [43] for the square and honeycomb cases. Additionally, the table contains the unit vectors $e^b$ denoting the unit vector along each beam. These vectors are here described in Cartesian coordinates to facilitate calculations. Recall also that the variable $c^b$ shows the relationship between the length of each beam and the cell wall, i.e. $L^b = c^b L'$. This has also been added to the table 3.2 below.

Table 3.2: Connectivity tables for Square and Honeycomb Cells

<table>
<thead>
<tr>
<th>SQUARE LATTICE</th>
<th>HONEYCOMB LATTICE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>beam b=[1]</strong></td>
<td><strong>beam b=[1]</strong></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
</tr>
<tr>
<td>d₁</td>
<td>1</td>
</tr>
<tr>
<td>d₂</td>
<td>0</td>
</tr>
<tr>
<td>$e^b$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$c^b$</td>
<td>1</td>
</tr>
<tr>
<td><strong>beam b=[2]</strong></td>
<td><strong>beam b=[2]</strong></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
</tr>
<tr>
<td>d₁</td>
<td>0</td>
</tr>
<tr>
<td>d₂</td>
<td>0</td>
</tr>
<tr>
<td>$e^b$</td>
<td>$\begin{bmatrix} \sqrt{3}/2 &amp; -1/2 \ 1/2 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$c^b$</td>
<td>$\sqrt{3}/3$</td>
</tr>
<tr>
<td><strong>beam b=[3]</strong></td>
<td><strong>beam b=[3]</strong></td>
</tr>
<tr>
<td>A</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
</tr>
<tr>
<td>d₁</td>
<td>1</td>
</tr>
<tr>
<td>d₂</td>
<td>1</td>
</tr>
<tr>
<td>$e^b$</td>
<td>$\begin{bmatrix} \sqrt{3} \sqrt{3} \sqrt{3} \sqrt{3} \end{bmatrix}$</td>
</tr>
<tr>
<td>$c^b$</td>
<td>$\sqrt{3}/3$</td>
</tr>
</tbody>
</table>

3.4 Asymptotic expansions for internal forces

Next step consists in finding the expressions in (2.46) through (2.49) for each beam, that is, the expansions for internal forces given by $n^A(b), t^A(b), M^A(b), M^E(b), M^C(b)$. To do so, the connectivity table of the previous step will be used. For each beam, $E$ and $A$ will be replaced by the corresponding joint number. Recall that the superscripts on the aforementioned expressions denote the term order in the asymptotic expansion of the internal forces. Subscripts, on the other hand, are used to associate the expressions to a joint in the cell.

At this step, to analyze each expression in detail, the gradients of displacement and rotation, originally in the $\{e^i_A\}$ basis, are transformed to the $\{e^i_x\}$ basis of Cartesian coordinates. To this
end, the transformations explained in the first step are conveniently implemented.

\[
\frac{\partial \mathbf{a}^0}{\partial \lambda^i} = \frac{\partial \mathbf{P}_0}{\partial \lambda^i} \cdot \nabla \mathbf{u}_x
\]

\[
\frac{\partial \phi^0}{\partial \lambda^i} = \frac{\partial \mathbf{P}_0}{\partial \lambda^i} \cdot \nabla \phi_x
\]

Note that the values for \( e^b \) and the directions described by \( e^b \) for each beam are applied on the present step as well. Tables 3.3 through 3.8 show all the internal force expressions for the square case and the first order expressions for the honeycomb lattice, to help illustrate the procedure. In order to simplify the expressions, and compare with the results obtained in reference [43], “stiffness” coefficients \( k_f \), \( k_l \), and \( \psi \) have been introduced. They are defined as: \( k_f = E\eta^3 \), \( k_l = E\eta \), and \( \psi = 1 + \frac{E\eta^2}{G_f} \).
Table 3.3: First Order terms for Square Lattice

<table>
<thead>
<tr>
<th>Int. Forces</th>
<th>beam b=1</th>
<th>beam b=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_A^{1(b)}$</td>
<td>$-(∂u/∂x)L'k_l$</td>
<td>$-(∂v/∂y)L'k_l$</td>
</tr>
<tr>
<td>$t_A^{1(b)}$</td>
<td>$L'k_f \frac{(∂v/∂x)−φ_1^0}{ψ}$</td>
<td>$L'k_f \frac{(∂u/∂y)+φ_1^0}{ψ}$</td>
</tr>
<tr>
<td>$M^2_{A(b)}$</td>
<td>$-\frac{L'^2 k_f ((∂v/∂x)−φ_1^0)}{2ψ}$</td>
<td>$\frac{L'^2 k_f ((∂u/∂y)+φ_1^0)}{2ψ}$</td>
</tr>
<tr>
<td>$M^2_{E(b)}$</td>
<td>$-\frac{L'^2 k_f ((∂v/∂x)−φ_1^0)}{2ψ}$</td>
<td>$\frac{L'^2 k_f ((∂u/∂y)+φ_1^0)}{2ψ}$</td>
</tr>
<tr>
<td>$M^2_{C(b)}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
### Table 3.4: Second Order terms for Square Lattice

<table>
<thead>
<tr>
<th>Int. Forces</th>
<th>beam b=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_A^2(b)$</td>
<td>0</td>
</tr>
<tr>
<td>$t_A^2(b)$</td>
<td></td>
</tr>
<tr>
<td>$M_A^3(b)$</td>
<td></td>
</tr>
<tr>
<td>$M_E^3(b)$</td>
<td></td>
</tr>
<tr>
<td>$M_C^3(b)$</td>
<td></td>
</tr>
</tbody>
</table>

**Int. Forces**
- $n_A^2(b)$
- $t_A^2(b)$
- $M_A^3(b)$
- $M_E^3(b)$
- $M_C^3(b)$
Table 3.5: Second Order terms for Square Lattice

<table>
<thead>
<tr>
<th>Int. Forces</th>
<th>beam b=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^2_{A(b)}$</td>
<td>0</td>
</tr>
<tr>
<td>$t^2_{A(b)}$</td>
<td>$L'k_f \left( \phi^0 + \frac{\partial \phi^0}{\partial y}L \right) / \psi$</td>
</tr>
<tr>
<td>$M^3_{A(b)}$</td>
<td>$L'^2 k_f \left( \frac{2 \phi^0 + (\partial \phi^0 / \partial y) L}{4 \psi} \right) - \frac{(\partial \phi^0 / \partial y) L'^3 k_f}{12}$</td>
</tr>
<tr>
<td>$M^3_{E(b)}$</td>
<td>$\frac{(\partial \phi^0 / \partial y) L'^3 k_f}{12} + \frac{L'^2 k_f \left( 2 \phi^0 + (\partial \phi^0 / \partial y) L \right)}{4 \psi}$</td>
</tr>
<tr>
<td>$M^3_{C(b)}$</td>
<td>$- \frac{(\partial \phi^0 / \partial y) L'^3 k_f}{12}$</td>
</tr>
</tbody>
</table>
Table 3.6: First Order terms for Honeycomb Lattice

<table>
<thead>
<tr>
<th>Int. Forces</th>
<th>beam b=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^1_{A(b)}$</td>
<td>$k_l \left( \frac{v_1}{2} - \frac{v_2}{2} + \frac{\sqrt{3} (u_1-u_2)}{2} \right)$</td>
</tr>
<tr>
<td>$t^1_{A(b)}$</td>
<td>$k_f \left( \frac{u_1}{2} - \frac{u_2}{2} + \frac{\sqrt{3} (v_1-v_2)}{2} \right)$ $+ \frac{\sqrt{3} L' k_f (\phi_1^0 + \phi_2^0)}{6 \psi}$</td>
</tr>
<tr>
<td>$M^2_{A(b)}$</td>
<td>$\frac{L'^2 k_f (\phi_1^0 - \phi_2^0)}{36}$ $+ \frac{L' k_f (3 v_1 - 3 v_2 + L' \phi_1^0 + L' \phi_2^0 - \sqrt{3} u_1 + \sqrt{3} u_2)}{12 \psi}$</td>
</tr>
<tr>
<td>$M^2_{E(b)}$</td>
<td>$\frac{L' k_f (3 v_1 - 3 v_2 + L' \phi_1^0 + L' \phi_2^0 - \sqrt{3} u_1 + \sqrt{3} u_2)}{12 \psi}$ $- \frac{L'^2 k_f (\phi_1^0 - \phi_2^0)}{36}$</td>
</tr>
<tr>
<td>$M^2_{C(b)}$</td>
<td>$\frac{L'^2 k_f (\phi_1^0 - \phi_2^0)}{36}$</td>
</tr>
</tbody>
</table>
Table 3.7: First Order terms for Honeycomb Lattice

<table>
<thead>
<tr>
<th>HONEYCOMB LATTICE FIRST ORDER TERMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Int. Forces</td>
</tr>
<tr>
<td>$n^1_{A(b)}$</td>
</tr>
<tr>
<td>$l^1_{A(b)}$</td>
</tr>
<tr>
<td>$M^2_{A(b)}$</td>
</tr>
<tr>
<td>$M^2_{E(b)}$</td>
</tr>
<tr>
<td>$M^2_{C(b)}$</td>
</tr>
</tbody>
</table>
Table 3.8: First Order terms for Honeycomb Lattice

<table>
<thead>
<tr>
<th>Int. Forces</th>
<th>beam b=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^1_{A(b)}$</td>
<td>$-k_l \left( v^1_1 - v^1_2 + \frac{(\partial v/\partial x) L}{2} + \frac{\sqrt{3} (\partial v/\partial y) L}{2} \right)$</td>
</tr>
<tr>
<td>$t^1_{A(b)}$</td>
<td>$k_f \left( u^1_1 - u^1_2 + \frac{(\partial u/\partial x) L}{\psi} + \frac{\sqrt{3} (\partial u/\partial y) L}{\psi} \right) + \frac{\sqrt{3} L' k_f (\phi^0_1 + \phi^0_2)}{6 \psi}$</td>
</tr>
<tr>
<td>$M^2_{A(b)}$</td>
<td>$L' k_f \left( L' \phi^0_1 + L' \phi^0_2 + 2 \sqrt{3} u^1_1 - 2 \sqrt{3} u^1_2 + 3 (\partial u/\partial x) L + \sqrt{3} (\partial u/\partial y) L \right) - \frac{L'^2 k_f (\phi^0_1 - \phi^0_2)}{36}$</td>
</tr>
<tr>
<td>$M^2_{E(b)}$</td>
<td>$\frac{L'^2 k_f (\phi^0_1 - \phi^0_2)}{36} + \frac{L' k_f \left( L' \phi^0_1 + L' \phi^0_2 + 2 \sqrt{3} u^1_1 - 2 \sqrt{3} u^1_2 + 3 (\partial u/\partial y) L + \sqrt{3} (\partial u/\partial x) L \right)}{12 \psi}$</td>
</tr>
<tr>
<td>$M^2_{C(b)}$</td>
<td>$- \frac{L'^2 k_f (\phi^0_1 - \phi^0_2)}{36}$</td>
</tr>
</tbody>
</table>
3.5 Linear system of equations: construction and solution

Next step is the application of the method of joints to solve for the unknown values in the expansions. The unknown values in the first order terms are $\vec{u}_N^1 = [u_N^1, v_N^1], \phi_N^0$. For the second order terms the unknowns are $\vec{u}_N^2 = [u_N^2, v_N^2], \phi_N^1$. Therefore, $N$ sets of three equations are required to solve for the unknowns at each order. Due to periodicity, beams $b$ of one cell are connected to the same beams $b$ of a neighboring cell around a connecting joint on the border. Therefore, only forces and moments of beams $b$ inside the cell can be considered. To illustrate this, observe figure 3.5 showing the only joint in the square lattice. The expression for the force and moment at end $E$ of beam $b = 1'$ are the same as the expressions for forces and moments of beam $b = 1$. Hence, only expressions for beams in the unit cell are required to construct the system of equations.

![SQUARE UNIT CELL](image)

Figure 3.5: The method considers forces from all 4 beams around joint 1 in the square unit cell

For the square lattice, only one joint is required to define the system. Therefore for each
order, this system contains three equations considering the present 2D case: two for the forces on each planar direction, and one for moments, which are all described around the direction perpendicular to the plane. It should be noted that the unknowns are expressed in terms of the gradients of displacement for the first order: \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \), and in terms of the gradients of rotation for the second order: \( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \). The square lattice systems are presented here, as a simple example showing these terms.

**Table 3.9:** First order system of equations for square lattice.

<table>
<thead>
<tr>
<th>SQUARE LATTICE FIRST ORDER SYSTEM OF EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>JOINT: ( N = 1 )</td>
</tr>
</tbody>
</table>
| \[ \begin{align*}
\sum_{b=1}^{B} T^{(1)}_{A(b)=1} + T^{(1)}_{E(b)=1} \\
\sum_{b=1}^{B} M^{(2)}_{A(b)=1} + M^{(2)}_{E(b)=1}
\end{align*} \] = 0 |
| \[ \begin{align*}
0 \\
0 \\
\frac{(\partial u/\partial y)}{\psi} L^2 k_f - \frac{(\partial v/\partial x)}{\psi} L^2 k_f + \frac{2 L^2 k_f \phi_1}{\psi}
\end{align*} \] = 0 |
Table 3.10: Second order system of equations for square lattice.

## SQUARE LATTICE SECOND ORDER SYSTEM OF EQUATIONS

**JOINT: N = 1**

\[
\begin{align*}
\sum_{b=1}^{B} T^{(2)}_A(b) &= 1 + T^{(2)}_E(b) = 1 \\
\sum_{b=1}^{B} M^{(3)}_A(b) &= 1 + M^{(3)}_E(b) = 1 \\
\sum_{b=1}^{B} b &= 1
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
0 \\
0 \\
\frac{(\partial \phi^0 / \partial x)}{2 \psi} L^3 k_f + \frac{(\partial \phi^0 / \partial y)}{2 \psi} L^3 k_f + \frac{2 L^2 k_f \phi^0}{\psi}
\end{align*}
\]

\[= 0\]

Two joints are required to analyze the honeycomb unit cell. Therefore, on each order, the resulting systems have six equations. Table 3.11 shows the ones belonging to the first order to illustrate how results are obtained for a more complex structure.

Expressions for the unknowns are substituted back into the formulations of the internal forces. It is important to remember, at this stage, that the microrotation \( \phi^0 \) is assumed to be the rotation around the characteristic joint. In both cases analyzed here, this means that \( \phi_{1}^{0} = \phi^{0} \). Therefore, \( \phi_{1}^{0} \) is in reality a known value, but treating it as an unknown may give insight into the physical interpretation of the microrotation in the lattice.
Table 3.11: First order system of equations for honeycomb lattice.

**HONEYCOMB LATTICE FIRST ORDER SYSTEM OF EQUATIONS**

<table>
<thead>
<tr>
<th>JOINT: N = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{T}^{(1)}}{A(b)} + \frac{\vec{T}^{(1)}}{E(b)} = 0)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{M}^{(2)}}{A(b)} + \frac{\vec{M}^{(2)}}{E(b)} = 0)</td>
</tr>
<tr>
<td>(\left{ \left(6 k_f u_1^1 - 6 k_f u_2^1 + 6 \psi k_1 u_1^1 - 6 \psi k_1 u_2^1 \right) + \frac{\partial v}{\partial x} \left(\sqrt{3} L k_f - \sqrt{3} \psi L k_1 \right) + \frac{\partial u}{\partial x} \left(3 L k_f + 3 \psi L k_1 \right) + \frac{\sqrt{3} (\partial u / \partial y) L k_f}{2} \right} = 0)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{T}^{(1)}}{A(b)} = 2 + \frac{\vec{T}^{(1)}}{E(b)} = 2)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{M}^{(2)}}{A(b)} = 2 + \frac{\vec{M}^{(2)}}{E(b)} = 2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>JOINT: N = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{T}^{(1)}}{A(b)} + \frac{\vec{T}^{(1)}}{E(b)} = 0)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{M}^{(2)}}{A(b)} + \frac{\vec{M}^{(2)}}{E(b)} = 0)</td>
</tr>
<tr>
<td>(\left{ \left(6 k_f v_1^1 - 6 k_f v_2^1 + 6 \psi k_1 v_1^1 - 6 \psi k_1 v_2^1 \right) - \frac{\partial v}{\partial x} \left(\sqrt{3} L k_f - \sqrt{3} \psi L k_1 \right) - \frac{\partial u}{\partial x} \left(3 L k_f + 3 \psi L k_1 \right) - \frac{\sqrt{3} (\partial u / \partial y) L k_f}{2} \right} = 0)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{T}^{(1)}}{A(b)} = 2 + \frac{\vec{T}^{(1)}}{E(b)} = 2)</td>
</tr>
<tr>
<td>(\sum_{b=1}^{B} \frac{\vec{M}^{(2)}}{A(b)} = 2 + \frac{\vec{M}^{(2)}}{E(b)} = 2)</td>
</tr>
<tr>
<td>(\left{ \frac{6 k_f u_1^2 - 6 k_f u_2^2 + 6 \psi k_1 u_1^2 - 6 \psi k_1 u_2^2}{12 \psi} - \frac{\partial v}{\partial x} \left(\sqrt{3} L k_f - \sqrt{3} \psi L k_1 \right) - \frac{\partial u}{\partial x} \left(3 L k_f + 3 \psi L k_1 \right) - \frac{\sqrt{3} (\partial u / \partial y) L k_f}{2} \right} = 0)</td>
</tr>
</tbody>
</table>
3.6 Stress vectors

To obtain the stress vectors, refer back to expression (2.59), stated again here:

\[
S^{\varepsilon_i} = \sum_{b=1}^{B} S_{(b)}^{\varepsilon_i} \\
\mu^{\varepsilon_i} = \sum_{b=1}^{B} \mu_{(b)}^{\varepsilon_i}
\]

And for each beam considering orders 1 and 2 of \( \varepsilon \), the expressions are found by using the following:

\[
S_{(b)}^{\varepsilon_i} = \left( d_{i(b)}^{1} \right) \left( \left( n_{A(b)}^{1} + \varepsilon n_{A(b)}^{2} \right) e^{b} + \left( t_{A(b)}^{1} + \varepsilon t_{A(b)}^{2} \right) e^{b\perp} \right)
\]
\[
\mu_{(b)}^{\varepsilon_i} = \left( d_{i(b)}^{1} \right) \left( \left( \varepsilon M_{C(b)}^{2} + \varepsilon^{2} M_{C(b)}^{3} \right) e^{3} \right)
\]

(3.1)

This expressions can be found on each order separately, so that two different stress vectors are found, as explained by (2.59). It is important to discuss the purpose of \( d_{1(b)}^{1} \) and \( d_{2(b)}^{2} \), which separates the terms acting on \( e_{1}^{\lambda} \) and on \( e_{2}^{\lambda} \) respectively. Therefore, the first component of the stress vectors \( S^{i} \), \( \mu^{i} \) acts on \( e_{1}^{\lambda} \), while the second component acts on \( e_{2}^{\lambda} \). The following expressions show how to
obtain each component of the force and couple stress vectors:

\[
S^{\varepsilon 1} = \sum_{b=1}^{s} \left( d^{1(b)} \right) \left( n_{A(b)}^1 + \varepsilon n_{A(b)}^2 \right) e^b + \left( t_{A(b)}^1 + \varepsilon t_{A(b)}^2 \right) e^{b\perp}
\]

\[
S^{\varepsilon 2} = \sum_{b=1}^{s} \left( d^{2(b)} \right) \left( n_{A(b)}^1 + \varepsilon n_{A(b)}^2 \right) e^b + \left( t_{A(b)}^1 + \varepsilon t_{A(b)}^2 \right) e^{b\perp}
\]

\[
\mu^{\varepsilon 1} = \sum_{b=1}^{s} \left( d^{1(b)} \right) \left( \varepsilon M_{C(b)}^2 + \varepsilon^2 M_{C(b)}^3 \right) e^3
\]

\[
\mu^{\varepsilon 2} = \sum_{b=1}^{s} \left( d^{2(b)} \right) \left( \varepsilon M_{C(b)}^2 + \varepsilon^2 M_{C(b)}^3 \right) e^3
\]

(3.2)

The components of the stress vectors for the square lattice are here detailed as guide. The superscript denoting the order of \( \varepsilon \) is enclosed in parenthesis to distinguish it from \( i \) that denotes the component. Notice they are also expressed as vectors.

Table 3.12: First order square lattice components of the stress vector

<table>
<thead>
<tr>
<th>SQUARE LATTICE FIRST ORDER STRESS VECTOR COMPONENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma^{(1)}1 )</td>
</tr>
<tr>
<td>( -\left( \partial u/\partial x \right) L' k_l )</td>
</tr>
<tr>
<td>( \left( \partial v/\partial x \right) L' k_f \phi^0 - \frac{\partial \psi}{\psi} )</td>
</tr>
<tr>
<td>( \Psi )</td>
</tr>
<tr>
<td>( \Sigma^{(1)}2 )</td>
</tr>
<tr>
<td>( -\left( \partial u/\partial y \right) L' k_f \phi^0 - \frac{\partial \psi}{\psi} )</td>
</tr>
<tr>
<td>( \left( \partial v/\partial y \right) L' k_l )</td>
</tr>
<tr>
<td>( \mu^{(2)}1 )</td>
</tr>
<tr>
<td>( \bar{0} )</td>
</tr>
<tr>
<td>( \mu^{(2)}2 )</td>
</tr>
<tr>
<td>( \bar{0} )</td>
</tr>
</tbody>
</table>
### Table 3.13: Second order square lattice components of the stress vector

**SQUARE LATTICE SECOND ORDER STRESS VECTOR COMPONENTS**

| \( S^{(2)1} \) | \[
\begin{pmatrix}
0 \\
L' k_f \\
\psi
\end{pmatrix} + \left( \frac{(\partial \phi^0 / \partial x) L'}{4} - \frac{(\partial \phi^0 / \partial y) L'}{4} \right)
\]
| \( S^{(2)2} \) | \[
\begin{pmatrix}
-L' k_f \phi^0 \\
(\partial u / \partial y) L' k_f \\
- (\partial v / \partial y) L' k_f
\end{pmatrix}
\]
| \( \mu^{(3)1} \) | \[
\begin{pmatrix}
0 \\
0 \\
- \frac{(\partial \phi^0 / \partial x) L'^3 k_f}{12}
\end{pmatrix}
\]
| \( \mu^{(3)2} \) | \[
\begin{pmatrix}
0 \\
0 \\
- \frac{(\partial \phi^0 / \partial y) L'^3 k_f}{12}
\end{pmatrix}
\]

### 3.7 Stress tensors

Stress tensors are formed by performing the tensor products shown in ?? again, the stress tensors may be separated in the different orders of \( \varepsilon \). Each component of the stress vector resulting from the previous step is multiplied by the respective component of \( e^\lambda_i \) in Cartesian coordinates, given by \( \frac{\partial P^0}{\partial x} \). This transformation was set in step 1, and is shown in table 3.1. Therefore, the following operation is performed on the square and honeycomb lattices.

\[
\bar{\sigma} = \frac{1}{J} (S^{e1} \otimes e^\lambda_1 + S^{e2} \otimes e^\lambda_2)
\]

\[
\bar{m} = \frac{1}{J} (\mu^{e1} \otimes e^\lambda_1 + \mu^{e2} \otimes e^\lambda_2)
\]

The above operations require that moments be expressed in vector form. Since they are
described around the axis perpendicular to the plane, the resulting tensor product is a 3x3 matrix, with only two nonzero elements. These are the only two elements that concern the 2D study. Stress tensors for the honeycomb and square lattices are also presented here. They are separated on each corresponding order of $\varepsilon$, as explained by 2.70, to better show how they affect the matrix of effective properties, constructed in the next step.

Table 3.14: Square lattice stress tensors

<table>
<thead>
<tr>
<th>SQUARE LATTICE STRESS TENSORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^{(1)}$</td>
</tr>
<tr>
<td>$m^{(2)}$</td>
</tr>
<tr>
<td>$\sigma^{(2)}$</td>
</tr>
<tr>
<td>$m^{(3)}$</td>
</tr>
</tbody>
</table>
Table 3.15: Stress tensors for Honeycomb Lattice

<table>
<thead>
<tr>
<th>$\sigma^{(1)}$</th>
<th>$\sqrt{3} k_1 (3 (\partial u/\partial x) k_f - (\partial u/\partial y) k_f + (\partial u/\partial x) \psi k_f + (\partial u/\partial y) \psi k_f) 6 (k_f + \psi k_f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2 \sqrt{3} k_f^2 \phi^0 + \sqrt{3} (\partial u/\partial y) k_f^2 - \sqrt{3} (\partial v/\partial x) k_f^2 + 2 \sqrt{3} \psi k_f k_l \phi^0 + 3 \sqrt{3} (\partial u/\partial y) \psi k_f k_l + \sqrt{3} (\partial v/\partial x) \psi k_f k_l$</td>
</tr>
<tr>
<td></td>
<td>$2 \sqrt{3} k_f^2 \phi^0 + \sqrt{3} (\partial u/\partial y) k_f^2 - \sqrt{3} (\partial v/\partial x) k_f^2 + 2 \sqrt{3} \psi k_f k_l \phi^0 - \sqrt{3} (\partial u/\partial y) \psi k_f k_l - 3 \sqrt{3} (\partial v/\partial x) \psi k_f k_l$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{3} k_1 (3 (\partial v/\partial y) k_f + (\partial u/\partial x) k_f + (\partial u/\partial y) \psi k_f + (\partial v/\partial y) \psi k_f) 6 (k_f + \psi k_f)$</td>
</tr>
</tbody>
</table>

| $\sigma^{(2)}$ | $- \frac{(\partial \phi^0 / \partial x) L' k_f k_l}{6 (k_f + \psi k_f)}$ | $\frac{(\partial \phi^0 / \partial y) L' k_f k_l}{6 (k_f + \psi k_f)}$ |
|----------------|------------------------------------------------------------------------------------------------------------------|
|                | $\frac{(\partial \phi^0 / \partial y) L' k_f k_l}{6 (k_f + \psi k_f)}$ | $\frac{(\partial \phi^0 / \partial y) L' k_f k_l}{6 (k_f + \psi k_f)}$ |

<table>
<thead>
<tr>
<th>$\sigma^{(3)}$</th>
<th>$0$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{3} (\partial \phi^0 / \partial x) L'^2 k_f$</td>
<td>$\sqrt{3} (\partial \phi^0 / \partial y) L'^2 k_f$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{108}{108}$</td>
<td>$\frac{108}{108}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
### 3.8 Identification of the effective properties

From the stress tensors shown above, the coefficients for each variable in the micropolar strain tensor are collected and organized as in (2.69). The square lattice’s $K$ matrix is presented below to show how properties are arranged. The dominant terms of the stress tensor in the next chapter, results are shown for various other lattice geometries.

<table>
<thead>
<tr>
<th>SQUARE EFFECTIVE PROPERTIES ON EACH ORDER</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K =$</td>
</tr>
<tr>
<td>$k_l$ 0 0 0 0 0 0</td>
</tr>
<tr>
<td>0 $k_l$ 0 0 0 0</td>
</tr>
<tr>
<td>0 0 $k_f/\psi$ 0 0 0</td>
</tr>
<tr>
<td>0 0 0 $k_f/\psi$ 0 0</td>
</tr>
<tr>
<td>0 0 0 0 $L'^2 k_f/12$ 0</td>
</tr>
<tr>
<td>0 0 0 0 0 $L'^2 k_f/12$</td>
</tr>
</tbody>
</table>

Table 3.16: Effective properties $K$ for the square lattice case
Chapter 4

Results

Effective properties for various lattice geometries found through DosReis’s homogenization method are presented in this section. To help visualize the results, elements of the $K$ matrix are denoted by subscript indices, i.e. $K_{11}, K_{12}...K_{66}$.

4.1 Honeycomb lattice

Definition of nodes and beams for an hexagonal lattice were already defined in steps 3.1, 3.2 and 3.3 described in the previous chapter. Results are shown in table 4.1, keeping the dominant terms of the force stress tensor and the dominant terms of the couple stress tensor.

4.2 Triangular lattice

A lattice constructed by tessellation of equilateral triangles is analyzed here. Figure 4.1 shows how the unit cell is defined and the value for $\alpha$. Table 4.2 shows the definitions required for the triangular lattice. Table 4.3 shows the effective properties resulting from the homogenization method. Again, only the dominant terms of the stress and couple stress tensors found are kept.
Table 4.1: Effective properties for hexagonal lattice

<table>
<thead>
<tr>
<th>HEXAGONAL LATTICE EFFECTIVE PROPERTIES ON EACH ORDER</th>
</tr>
</thead>
</table>
| $K^{(1)} = \begin{pmatrix}
K_{11} & K_{12} & 0 & 0 & 0 & 0 \\
K_{21} & K_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & K_{33} & K_{34} & 0 & 0 \\
0 & 0 & K_{43} & K_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & K_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & K_{66}
\end{pmatrix}$ |

$K_{11} = K_{22} = \frac{\sqrt{3} k_l (3 k_f + \psi k_l)}{6 (k_f + \psi k_l)}$

$K_{12} = K_{21} = -\frac{\sqrt{3} k_l (k_f - \psi k_l)}{6 (k_f + \psi k_l)}$

$K_{33} = K_{44} = \frac{\sqrt{3} k_f^2 + 3 \sqrt{3} \psi k_l k_f}{6 \psi (k_f + \psi k_l)}$

$K_{34} = K_{43} = -\frac{(\sqrt{3} k_f^2 - \sqrt{3} \psi k_f k_l)}{6 \psi (k_f + \psi k_l)}$

$K_{55} = K_{66} = \frac{\sqrt{3} L^2 k_f}{108}$
Figure 4.1: Definition of beams and joints in triangular unit cells

Table 4.2: Triangular lattice beam and joint numbers

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>d₁</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>d₂</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>e^b</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>e^b</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 4.3: Effective properties for triangular lattice

<table>
<thead>
<tr>
<th>TRIANGULAR EFFECTIVE PROPERTIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{pmatrix} K_{11} &amp; K_{12} &amp; 0 &amp; 0 &amp; 0 &amp; 0, \ K_{21} &amp; K_{22} &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; K_{33} &amp; K_{34} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; K_{43} &amp; K_{44} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; K_{55} &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; K_{66} \end{pmatrix} )</td>
</tr>
</tbody>
</table>

\[
K_{11} = K_{22} = \frac{\sqrt{3} \left( k_f + 3 \psi k_l \right)}{4 \psi}
\]

\[
K_{12} = K_{21} = -\frac{\sqrt{3} \left( k_f - \psi k_l \right)}{4 \psi}
\]

\[
K_{33} = K_{44} = \frac{\sqrt{3} \left( 3 k_f + \psi k_l \right)}{4 \psi}
\]

\[
K_{34} = K_{43} = -\frac{\sqrt{3} \left( k_f - \psi k_l \right)}{4 \psi}
\]

\[
K_{55} = K_{66} = \frac{\sqrt{3} L^2 k_f}{12}
\]
4.3 Kagome lattice

Many authors have analyzed the properties of the kagome lattice geometries, which naturally appears in certain crystals. The unit cell of the lattice is shown in 4.2. A table defining beams and joints is given in 4.4 adapted from the definition proposed by DosReis in [43]. Results are given in table 4.5. For this particular lattice all terms are included, since it is difficult to determine the dominant terms on each tensor. The results show characteristics of a completely anisotropic material.

Figure 4.2: Definition of beams and joints in “Kagome” lattice
### Table 4.4: Kagome lattice beam and joint numbers

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>d₁</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d²</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>eᵇ</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>-1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>cᵇ</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

### 4.4 Simple Chiral lattice

In his work, Dos Reis [43] analyzes a chiral geometry denominated “diamond chiral”. Chirality, a term most often used in chemistry, refers to a geometry that is not superposable on its mirror image. It is analyzed here for comparison. The unit cell is defined as in figure 4.3. Beams and joints for the lattice are shown in table 4.6. Finally, effective properties as elements of the compliance matrix $C$ are given in table 4.7. In this particular case, the compliance matrix shows simpler elements than those in the stiffness matrix $K$ allowing for simpler analysis.
Table 4.5: Effective properties for "Kagome" lattice

<table>
<thead>
<tr>
<th>KAGOME EFFECTIVE PROPERTIES</th>
</tr>
</thead>
</table>
| $K = \begin{pmatrix}
  K_{11} & K_{12} & 0 & 0 & 0 & 0 \\
  K_{21} & K_{22} & 0 & 0 & 0 & 0 \\
  K_{31} & K_{32} & K_{33} & K_{34} & 0 & 0 \\
  K_{41} & K_{42} & K_{43} & K_{44} & 0 & 0 \\
  0 & 0 & 0 & K_{54} & K_{55} & 0 \\
  K_{61} & K_{62} & 0 & 0 & 0 & K_{66}
\end{pmatrix}$ |

- $K_{11} = \sqrt{3} k_f + 8 \sqrt{3} k_l + 8 \sqrt{3} \psi k_l$
- $K_{12} = K_{21} = K_{34} = \sqrt{3} k_f - \sqrt{3} k_l + \sqrt{3} \psi k_l$
- $K_{22} = 16 \sqrt{3} k_f + 3 \sqrt{3} k_l + 3 \sqrt{3} \psi k_l$
- $K_{31} = -3 k_f$
- $K_{32} = 6 k_f$
- $K_{33} = 8 \sqrt{3} k_f + 8 \sqrt{3} \psi k_f$
- $K_{41} = 3 k_f$
- $K_{42} = 4 k_f$
- $K_{43} = 8 \sqrt{3} k_f + 16 \sqrt{3} k_f + 8 \sqrt{3} \psi k_f$
- $K_{54} = \sqrt{3} L' k_f$
- $K_{55} = 8 \sqrt{3} L' k_f$
- $K_{61} = -\frac{1}{4} k_f$
- $K_{62} = -\frac{1}{4} \sqrt{3} L' k_f$
- $K_{66} = \frac{1}{3} \sqrt{3} L' k_f$
**Figure 4.3:** Definition of beams and joints in simple chiral lattice

**Table 4.6:** Beam and joint numbers for a lattice with chiral geometry

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>d¹</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>d²</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>e[^b]</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
</tr>
<tr>
<td>e[^b]</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
</tr>
<tr>
<td>c[^b]</td>
<td>(\frac{\sqrt{2}}{4})</td>
<td>(\frac{\sqrt{2}}{4})</td>
<td>(\frac{\sqrt{2}}{4})</td>
<td>(\frac{\sqrt{2}}{4})</td>
<td>(\frac{\sqrt{2}}{4})</td>
<td>(\frac{\sqrt{2}}{4})</td>
</tr>
</tbody>
</table>

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Table 4.7: Compliance matrix for a simple chiral lattice

<table>
<thead>
<tr>
<th>Sample Chiral Lattice Effective Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{C} =$</td>
</tr>
</tbody>
</table>
| $\begin{pmatrix}
C_{11} & C_{13} & 0 & 0 & 0 & 0 \\
0 & C_{22} & C_{23} & C_{24} & 0 & 0 \\
C_{31} & 0 & C_{33} & C_{34} & C_{35} & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{pmatrix}$ | $C_{11} = C_{22} = \frac{3}{2} (k_f + 2k_l + \psi k_l)$, $C_{13} = C_{24} = k_f - 6k_l + 2\psi k_l$, $C_{14} = C_{23} = -\frac{(2k_f + 12k_l + \psi k_l)}{2k_f k_l}$, $C_{31} = C_{42} = -\frac{(k_f + 18k_l - \psi k_l)}{2k_f k_l}$, $C_{33} = C_{44} = \frac{k_f + 18k_l + 4\psi k_l}{2k_f k_l}$, $C_{34} = C_{43} = \frac{2k_f + 36k_l - \psi k_l}{2k_f k_l}$, $C_{55} = C_{66} = \frac{216}{L^2 k_f}$ |
4.5 Result comparison and analysis

According to Gauthier in [44], and used by Yang et. al. [52], in micropolar plane strain, assuming centrosymmetry, the stiffness matrix takes the form:

\[
\begin{bmatrix}
\lambda + \mu^* + \kappa & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + \mu^* + \kappa & 0 & 0 & 0 \\
0 & 0 & \mu^* + \kappa & \mu^* & 0 \\
0 & 0 & \mu^* & \mu^* + \kappa & 0 \\
0 & 0 & 0 & 0 & \gamma
\end{bmatrix}
\]

with \(\lambda, \mu^*, \kappa,\) and \(\gamma\) acting like Lam constants. Taking these into account, it is possible to define micropolar elastic moduli:

\[
E^* = \frac{(2\mu^* + \kappa)(3\lambda + 2\mu^* + \kappa)}{(2\lambda + 2\mu^* + \kappa)}
\]

\[
\nu^* = \frac{\lambda}{(2\lambda + 2\mu^* + \kappa)}
\]

\[
G^* = \frac{2\mu^* + \kappa}{2}
\]

where \(E^*, \nu^*, G^*\), are the Young’s modulus, Poisson’s ratio and shear modulus respectively. However, the Young’s modulus \(E^*\) that will be used here for result comparison is defined as the inverse of the first term of the compliance matrix \(C\), since this definition is employed by Dos Reis [43], Arabnejad and Pasini[4], and Vigliotti[49]. Using the expressions above, moduli resulting from the present study are compared to the results recorded by Arabnejad and Pasini in [4]. Their work assembled the results obtained by several authors. These results are summarized in tables 4.8, 4.9 and 4.10. The method studied here corresponds to Dos Reis et al. [15] using Timoshenko beam theory.

The last result table shows two values that emerge from the comparison between classical and micropolar elastic moduli. "In planar micropolar elasticity theory, the degree of micropolarity
exhibited by a loaded heterogeneous material is quantified by a dimensionless constitutive parameter, the coupling number. According to Cowin, the value for \( N \) ranges from 0 to 1, 0 showing classical elastic behavior, and 1 exhibiting behavior in a special case of micropolar elasticity known as "coupled stress". Note that all lattice geometries, given some small number for \( \eta \), show micropolar behavior, due to how effective properties have been organized using the method. Looking at \( N \), for the square lattice topology, gives \( N = 1/\sqrt{2} \), which is close to 1, showing behavior that is closer to coupled stress.

The value of the characteristic length, also presented in the last table, should give insight into the scale of the unit cell as compared to the lattice domain. Expressions to compute the characteristic length \( l^2_c \) and coupling number \( N^2 \) are given in [36].

From the tables it is clear that the method gives the same results for the simple square lattice, as with other methods. The results in Young’s and shear moduli that most closely resemble are the ones obtained by the method here studied and the energy method applied by Kumar and McDowell. Shear and Young’s moduli found by the different methods for the honeycomb and triangular lattices are very similar when \( \eta \) is close to zero. Taking a closer look at the Young’s moduli, Dos Reis’ method appears to give a larger value than other studies.

Consider figure 4.4 analyzing the expressions for the Young’s Moduli found by the different methods for the “kagome” topology. \( \psi = 1 \) is assumed for the present study. Recall, also, that the relative density \( \rho \) is \( \sqrt{3} \eta \) for this particular lattice geometry and, therefore, only values ranging from \( \rho = 0 \) to \( \rho = \sqrt{3} \) are examined. All different methods analyzed concur when \( \eta \) is small (less than 0.3) i.e. following Euler Bernoulli’s assumption of long slender beams.

The same analysis is shown in figure 4.5 for the shear moduli found in the aforementioned studies. The expression resulting from the present study is, in comparison, larger. The works in [49] and [51] show similar results when \( \eta \) is small.

The Poisson’s ratios found for the “kagome” lattice is shown in figure 4.6. There is a slight discrepancy in the values when \( \eta \approx 0 \) found by the present study in comparison to the other studies, around 8%.

It is worth mentioning that, by looking at the “kagome” micropolar stiffness matrix in
<table>
<thead>
<tr>
<th>Lattice Geometry</th>
<th>Study</th>
<th>Relative density ( \rho )</th>
<th>Young’s modulus ratio ( \frac{E^*}{E} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SQUARE</strong></td>
<td>Wang and McDowell [50]</td>
<td>( 2\eta )</td>
<td>( \frac{\rho}{2} )</td>
</tr>
<tr>
<td></td>
<td>Kumar and McDowell [30]</td>
<td></td>
<td>( \frac{\rho}{2} )</td>
</tr>
<tr>
<td></td>
<td>DosReis and Ganghofer E-B [43]</td>
<td></td>
<td>( \frac{\rho}{2} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td></td>
<td>( \frac{\rho}{2} )</td>
</tr>
<tr>
<td><strong>HEXAGONAL</strong></td>
<td>Gibson and Ashby [23]</td>
<td>( \frac{3\rho^3}{2} )</td>
<td>( \frac{6\rho^3}{9\rho^2+4} )</td>
</tr>
<tr>
<td>“HONEYCOMB”</td>
<td>Viglotti and Pasini [49]</td>
<td>( \frac{2\rho^3}{\sqrt{3}} )</td>
<td>( \frac{6\rho^3}{9\rho^2+4} )</td>
</tr>
<tr>
<td></td>
<td>DosReis and Ganghofer E-B [43]</td>
<td></td>
<td>( \frac{6\rho^3}{9\rho^2+4} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td></td>
<td>( \frac{6\rho^3}{9\rho^2+4} )</td>
</tr>
<tr>
<td><strong>“KAGOME”</strong></td>
<td>Wang and McDowell [50]</td>
<td>( \sqrt{3}\eta )</td>
<td>( \frac{\rho}{3} )</td>
</tr>
<tr>
<td></td>
<td>Viglotti and Pasini [49], ElSayed [16]</td>
<td></td>
<td>( \frac{\rho(9+2\psi)}{18+\rho^2} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td></td>
<td>( \frac{\rho}{3} ) (( \frac{16\rho^2+24\psi+24}{16\rho^2+9\psi+9} ))</td>
</tr>
<tr>
<td><strong>TRIANGULAR</strong></td>
<td>Wang and McDowell [50]</td>
<td>( 2\sqrt{3}\eta )</td>
<td>( \frac{\rho}{3} )</td>
</tr>
<tr>
<td></td>
<td>Kumar and McDowell [30]</td>
<td></td>
<td>( \frac{\rho^3+12\psi}{\rho^2+36} )</td>
</tr>
<tr>
<td></td>
<td>Viglotti and Pasini [49], ElSayed [16]</td>
<td></td>
<td>( \frac{\rho^3+12\psi}{\rho^2+36} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td></td>
<td>( \frac{\rho^3+12\psi}{\rho^2+36} )</td>
</tr>
</tbody>
</table>
### Table 4.9: Shear Modulus and Poisson’s ratio. Table adapted from Arebnajad and Pasini [4]

<table>
<thead>
<tr>
<th>Lattice Geometry</th>
<th>Study</th>
<th>Relative density ( \rho )</th>
<th>Shear modulus ratio ( \frac{G^*}{E} )</th>
<th>relative Poisson ratio ( \frac{\nu^*}{\nu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SQUARE</strong></td>
<td>Wang and McDowell [50]</td>
<td>( \frac{3}{16} )</td>
<td>2( \eta )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td></td>
<td>Kumar and McDowell [39]</td>
<td>( \frac{3}{16} )</td>
<td>( \frac{2}{15} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>DosReis and Ganghoffer E-B [43]</td>
<td>( \frac{3}{16} )</td>
<td>( \frac{2}{15} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td>( \frac{3}{16} )</td>
<td>( \frac{2}{15} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td><strong>HEXAGONAL</strong></td>
<td>Gibson and Ashby [23]</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>“HONEYCOMB”</td>
<td>Vigliotti and Pasini [49]</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>DosReis and Ganghoffer E-B [43]</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td><strong>“KAGOME”</strong></td>
<td>Wang and McDowell [50]</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td></td>
<td>Vigliotti and Pasini [49], Elsayed [16]</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td><strong>TRIANGULAR</strong></td>
<td>Wang and McDowell [50]</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{1}{12} )</td>
</tr>
<tr>
<td></td>
<td>Kumar and McDowell [39]</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{1}{12} )</td>
</tr>
<tr>
<td></td>
<td>Vigliotti and Pasini [49], Elsayed [16]</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{1}{12} )</td>
</tr>
<tr>
<td></td>
<td>Present study Timoshenko</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{5}{48} )</td>
<td>( \frac{1}{12} )</td>
</tr>
</tbody>
</table>
Table 4.10: Coupling number and characteristic length. Table adapted from Arebnajad and Pasini [4]

<table>
<thead>
<tr>
<th>Lattice Geometry</th>
<th>Study</th>
<th>Relative density $\rho$</th>
<th>$N^2$ (Micropolar only)</th>
<th>characteristic length $l_c^2$ (Micropolar only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SQUARE</td>
<td>Wang and McDowell[50]</td>
<td>$2\eta$</td>
<td>$1/2$</td>
<td>$L^2$</td>
</tr>
<tr>
<td></td>
<td>Kumar and McDowell[30]</td>
<td>$1/2$</td>
<td>$L^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>DosReis and Ganghoffer E-B[43]</td>
<td>$2\eta$</td>
<td>$L^2$</td>
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<td>$1/2$</td>
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<td>$\frac{2\eta}{\sqrt{3}}$</td>
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<td>$2\sqrt{3} \eta$</td>
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Figure 4.4: Comparison between Young’s Modulus ratios found for the “Kagome” lattice in different studies

Figure 4.5: Comparison between Shear Modulus ratios found for the “Kagome” lattice in different studies
Table 4.5 shows that there are elements that cannot be accounted for by the planar centro symmetric matrix explained at the beginning of this section. Thus, the elastic moduli found for this particular lattice topology may not fit the centro symmetrical model properly. Still, the results are close to those found by other studies, which suggests the assumption of centro symmetry is not entirely incorrect.
Chapter 5

Conclusion

With the method studied here, developed first by Caillerie [11] [18], and continued by Ganghoffer et.al. [15], effective properties are determined starting from simple assumptions on the beams pertaining to a lattice. This method is based on the virtual power principle applied to a beam and to the cells in the domain. Through asymptotic expansions the virtual power of a large number of cells forming the lattice leads to the homogeneous continuum. In the first section of the second chapter, the system is described; beams, joints, cells are defined. Application of the virtual power principle on the beam, the cell and the lattice was carefully and rationally analyzed in the remaining sections of the second chapter. Micropolar elasticity was found to naturally emerge in the present derivation as a good framework to describe lattice effective properties. Chapter three follows the derivation of effective properties applying the method on the square and honeycomb lattice topologies. The method appears to work well on simple geometries, given the similarities found with other studies as described in chapter four. However, for more complex geometries, definition of the cell’s beams and joints in accordance to periodicity in the lattice poses a challenge. In particular, significant differences were found for the “kagome” geometry when compared with other studies. Finally, it is important to note that Timoshenko beam theory is applied in the present study as opposed to Euler Beam theory used by Dos Reis. This is of particular importance in cases where the lattice beam members have a height-to-length ratio $\eta > \frac{1}{4}$ [11]. For short beams, the
constant $\psi$ used here shows clearly the effect of the theory’s assumptions on the resulting effective properties. When $\psi = 1$ the results revert to those found by Dos Reis’.

In general, benefits of the method include: the possibility to derive material constants from the microstructure, and to obtain homogenous Cosserat models for the heterogeneous or composite materials. Some of the strengths of the present method become evident when looking at other works. Take the energy method developed by Kumar and McDowell in \[30\], which also makes use of the micropolar framework to describe effective properties. Kumar’s method defines cells with a single central joint. The strain energy of beams inside the cell are then used to obtain micropolar effective properties. The method is based on a Taylor’s expansion around the central joint and the terms to be considered are not well defined. Also, if a lattice cannot be defined by cells with a single joint, it is not clear how to proceed. Dos Reis’ method, in contrast, makes use of the beam’s internal forces allowing the unit cell to be considered as a small frame structure. The method used an asymptotic expansion introducing unknowns, which are determined due to compatibility conditions imposed by the geometry. Therefore, with Dos Reis’ homogenization procedure, it is possible to analyze a vast number of lattice topologies.

However, there is a number of limitations to the method. “It is not certain that the present approach would be valid for non centrosymmetrical lattices.” \[15\]. Although analysis of complex geometries is possible, it is not clear if the characteristics of the geometry translate well to the homogenized continuum. Take the square lattice as an example. Second order terms for the stress tensor are found using the method, as can be seen in \[3.14\]. In this case it is evident that the square lattice has centrosymmetry and, because asymptotic methods are used to find the dominant terms, they are used in the effective properties to show no coupling between the force stress tensor and the micropolar curvature. More analysis is needed to determine if such second order terms are important depending on the type of load applied or on the complexity of the geometry. Another problem mentioned by Dos Reis revolves around the “questionable hypothesis about the zeroth order term of rotations” \[15\]. In his work, the author assumes that the microrotation will always take the form:

$$\phi^0 = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$
This problem is solved here simply by assigning the rotation of the characteristic joint in the cell to be its micro rotation.

Validation of the method is an interesting extension to the present study. It is not clear, for example, if there is an optimal way of defining the unit cell, or if slight variations alter results significantly. Nevertheless, effective properties for complex geometries, such as the “kagome” lattice analyzed in the previous chapter, or chiral shapes, can easily be obtained through the application of this method. The resulting algebraic expressions can be used to solve problems for beam lattices. In order to validate the results, a problem can be defined using finite element analysis and effective properties, and the two solutions have to be compared.

A few experimental studies have been done to find micropolar effects in lattices. In a study conducted by McGregor and Wheel [36], rings of an acrylic polymer were perforated in an arrangement resembling a quadrilateral lattice. The authors also use results from previous experimentation performed on rings with voids in hexagonal arrangement. Through tensile testing, the study finds the micropolar coupling numbers and characteristic lengths for the rings. They conclude that micropolar models could not accurately predict the rings response. This, however, may also have been due to problems in the experimentation since micropolar effects appear only under excitation with small characteristic length. To validate the models, more experiments are required, showing the micropolar effects.

Techniques to obtain effective properties tackling non-centrosymmetrical geometries are scarce. Due to its flexibility, the asymptotic homogenization method is a good first approach. Its main problem lies in the correct translation of the geometrical connectivities and the cell’s shape. In the present study, the steps have been delineated in detail. A next step should pursue the inclusion of slight variations into this existing framework to ensure that non centrosymmetry is modeled properly.
Appendices
Appendix A  Constitutive relations for the beam

Consider the beam in figure 1 lying on the plane $e^b, e^{b\perp}$. The beam is in equilibrium, with external moments and forces acting on joint $A$ and $E$. Assume the external forces $\vec{T}_A, \vec{T}_E$ act on the plane of the beam and external moments $\vec{M}_A, \vec{M}_E$ act as described by the vector perpendicular to the same plane. The beam may be described by a material curve identified as its neutral axis, where, at each point, a planar cross-section can be defined. Body forces and moments are neglected in this analysis.

Figure 1: Considerations on a 2D beam with forces and moments around the nodes.

Consider now the force at any point of the beam to be described by its components so that $\vec{T}_A = n_A e^b + t_A e^{b\perp}$. The governing equations for the beam may be found by application of the principle of virtual power on the beam. Consider any point $P$ on the neutral axis with its associated planar cross-section. Also, a vector from $P$ to a point $Q$ is used to define the cross-section rotation. Thus, two virtual motions are used to construct the virtual power at any point $P$: a virtual velocity $\vec{v}^*$ and a virtual rotation rate $\vec{\omega}^*$ associated to the cross-section rotation. The virtual power principle, taking into account both external and internal forces gives:
\[ P_{ext}^* + P_{int}^* = 0 \]
\[ \downarrow \downarrow \downarrow \]
\[ \bar{T}_A \cdot \bar{v}_A^* + \bar{M}_A \cdot \bar{\omega}_A^* + \bar{T}_E \cdot \bar{v}_E^* + \bar{M}_E \cdot \bar{\omega}_E^* - \int_A^E \bar{p}^* \, dx = 0 \]
\[ -p_{int}^* = \bar{T} \cdot \left( \frac{d\bar{v}^*}{dx} + e^b \times \bar{\omega}^* \right) + \bar{M} \cdot \frac{d\bar{\omega}^*}{dx} \]

In expression (1), \( p^* \) is a linear function of the virtual motions and their gradients \( \frac{d\bar{v}^*}{dx} \), \( \frac{d\bar{\omega}^*}{dx} \). Thus, the governing equations read:

\[
\frac{dn}{dx} = 0 \\
\frac{dt}{dx} = 0 \\
\frac{dM}{dx} + t = 0
\]

Where the vector force at any point has been separated in its components \( \bar{T}(x) = n(x)e^b + t(x)e^b_\perp \), and the moment \( \bar{M}(x) = M(x)e^3 \). Boundary conditions on \( A \) and \( E \) are given by:

\[
\begin{align*}
x = 0 & \\
\quad n(0) &= -n_A & \quad n(l^b) &= n_E \\
\quad t(0) &= -n_A & \quad t(l^b) &= n_E \\
\quad M(0) &= -M_A & \quad M(l^b) &= M_E
\end{align*}
\]
Constitutive relations are necessary to describe deformation of the beam under said loads. In the present work, the constitutive relations in linear elasticity are considered. In order to understand their derivation, observe figure 2 considering $\mathbf{u}(x) = u(x)e^b + v(x)e^{b\perp}$. The cross section rotation $\phi$ is not perpendicular to the neutral axis after deformation. Instead, shear forces on the beam cause it to curve an angle $\gamma = \frac{dv}{dx} - \phi$. Bending moments are only responsible for the $\phi$ cross-section rotation.

Figure 2: Considerations on a 2D beam with forces and moments around the nodes.

Therefore, with shear force $t$, axial force $n$ and bending moment $M$ the constitutive relations
for Timoshenko theory read:

\[ n = EA \frac{du}{dx} \]

\[ t = Gf_s A \left( \frac{dv}{dx} - \phi \right) \]

\[ M = EI \frac{d\phi}{dx} \]

(3)

Where \( f_s \) is a correction factor, to better approximate the shear force distribution along the cross-section of the beam. For rectangular beams \( f_s = \frac{5}{6} \). To solve the given differential equations, the following boundary conditions are considered. Observe that \( \vec{u}(x) = u(x)e^b + v(x)e^{b\perp} \).

\[ u(x = A) = u_A \quad u(x = E) = u_E \]
\[ v(x = A) = v_A \quad v(x = E) = v_E \]
\[ \phi(x = A) = \phi_A \quad \phi(x = E) = \phi_E \]

Now consider the length of the beam \( l^b = \| \vec{AE} \| \), which is equivalent to consider point \( A \) at the origin so that \( x = A = 0 \) and \( x = E = l^b \). Solving using boundary conditions on \( A \), the governing equations give the following solutions:

\[ n(x) = -n_A = n_E \]
\[ t(x) = -t_A = t_E \]
\[ M(x) = t_A x - M_A \]

(4)

Replacing into the constitutive equations on \( 3 \) and solving using boundary conditions at \( x = A \), it
is now possible to find expressions for displacements $u(x), v(x),$ and rotation $\phi(x)$.

$$u(x) = u_A - \frac{n_A}{EA} x$$

$$v(x) = v_A + \frac{t_A x^3}{EI \ 6} - \frac{M_A x^2}{EI \ 2} + \phi_A x - \frac{t_A}{Gf_s A} x$$

$$\phi(x) = \phi_A + \frac{t_A x^2}{EI \ 2} - \frac{M_A}{EI} x$$

\( (5) \)

The homogenization method discussed in the present work requires moments at the ends of the beam be expressed in terms of displacements at both ends of the beam. To this effect, it is necessary to follow the given boundary conditions $u(x = l^b) = u_E, \ v(x = l^b) = v_E, \ \phi(x = l^b) = \phi_E.$ Therefore, with $x = l^b$ in (5), the following expressions for the axial force, shear force and moment at A are determined. Again, note that $\overline{u}(x) = u(x)e^b + v(x)e^{b\perp}$.

$$n_A = -(u_E - u_A) \frac{EA}{l^b}$$

$$t_A = -\frac{12EI}{(l^b)^3 \left(1 + \frac{12EI}{Gf_s A(l^b)^2}\right)} [v_E - v_A] + \frac{12EI}{(l^b)^2 \left(1 + \frac{12EI}{Gf_s A(l^b)^2}\right)} \left[ \frac{(\phi_A + \phi_E)}{2} \right]$$

$$M_A = -\frac{6EI}{(l^b)^2 \left(1 + \frac{12EI}{Gf_s A(l^b)^2}\right)} [v_E - v_A] + \frac{6EI}{(l^b) \left[1 + \left(\frac{12EI}{Gf_s A(l^b)^2}\right)\right]} \left[ \frac{(\phi_A + \phi_E)}{2} \right]$$

$$- \frac{2EI}{(l^b)} \left[ \frac{(\phi_E - \phi_A)}{2} \right]$$

\( (6) \)

From the expressions shown above, it is also possible to deduce the moment at E, and the moment at A taken from the center of the beam C. These additional expressions are useful in the application of the method as detailed in the body chapters.
\[ M_E = - \frac{6EI}{(lb)^2 \left( 1 + \frac{12EI}{Gf_A(lb)^2} \right)} \left[ v_E - v_A \right] + \frac{6EI}{(lb) \left[ 1 + \left( \frac{12EI}{Gf_A(lb)^2} \right) \right]} \left[ \frac{\phi_A + \phi_E}{2} \right] + \frac{2EI}{(lb)} \left[ \frac{\phi_E - \phi_A}{2} \right] \]

\[ M_C = - \frac{2EI}{(lb)} \left[ \frac{\phi_E - \phi_A}{2} \right] \]

(7)
Bibliography


[31] Rod(University of Wisconsin) Lakes. Cosserat elasticity; micropolar elasticity.


