Numerical methods and analysis for continuous data assimilation in fluid models

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NUMERICAL METHODS AND ANALYSIS FOR CONTINUOUS DATA ASSIMILATION IN FLUID MODELS

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

by
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Abstract

Modeling fluid flow arises in many applications of science and engineering, including the design of aircrafts, prediction of weather, and oceanography. It is vital that these models are both computationally efficient and accurate. In order to obtain good results from these models, one must have accurate and complete initial and boundary conditions. In many real-world applications, these conditions may be unknown, only partially known, or contain error. In order to overcome the issue of unknown or incomplete initial conditions, mathematicians and scientists have been studying different ways to incorporate data into fluid flow models to improve accuracy and/or speed up convergence to the true solution.

In this thesis, we are studying one specific data assimilation technique to apply to finite element discretizations of fluid flow models, known as continuous data assimilation. Continuous data assimilation adds a penalty term to the differential equation to nudge coarse spatial scales of the algorithm solution to coarse spatial scales of the true solution (the data). We apply continuous data assimilation to different algorithms of fluid flow, and perform numerical analysis and tests of the algorithms.
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Chapter 1

Introduction

The need to model fluid flow arises across the spectrum of science and engineering, including in weather prediction, oceanography, and aircraft design. These vast practical applications demand models and solvers that are both efficient and accurate. Computational fluid dynamics (CFD) uses mathematical and numerical analysis to solve and analyze these problems. In order for these CFD models to accurately represent real-world phenomena, we need accurate initial and boundary conditions to use with the complex nonlinear differential equations that govern the evolution of such phenomena. Unfortunately, in most situations these conditions are not entirely known, and it may be impossible to determine them completely. This often leads to poor results from the CFD model, especially over long time intervals.

In order to overcome this issue, researchers have recently been developing ways to incorporate data into models. This idea is broadly known as Data Assimilation (DA), and covers a wide range of techniques, including the Kalman filter, 3D/4D Var, and others [17, 20, 58, 65, 11], which we discuss briefly below. Different techniques of DA are being used in weather modeling, climate science, and hydrological and environmental forecasting [58].
The focus of this thesis is on one specific type of DA recently pioneered by Azouani, Olson, and Titi in 2014, which we call continuous DA [4, 5, 14, 46, 80]. We study this technique applied to systems of time dependent partial differential equations (PDE) in discrete settings (in particular, the finite element method and reduced order models), which to date has not been extensively studied. Practical applications of continuous DA include the climate sciences, where it can be used to perform weather, environmental, and hydrological forecasting [57]. These algorithms incorporate new and real-time data into numerical simulations, making the computed solutions better reflect the current state of the system. DA algorithms can be particularly useful when the initial condition of the governing system is unknown, since convergence to the true solution is proven for arbitrary initial conditions. Throughout this thesis, we investigate important properties and efficient implementation of continuous DA applied to different formulations of the Navier Stokes Equations (NSE) and fluid transport equations, which we introduce next.

1.1 The Navier Stokes Equations

Incompressible viscous flow of a Newtonian fluid is modeled by the Navier Stokes equations (NSE): In a bounded, connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $3$) with smooth boundary $\partial \Omega$,

\begin{align}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \nu \Delta u &= f, \\
\nabla \cdot u &= 0,
\end{align}

(1.1) \hspace{1cm} (1.2)
where \( u := u(x,t) \) represents velocity, \( p := p(x,t) \) pressure, \( f := f(x,t) \) external forcing, and \( \nu > 0 \) represents the kinematic viscosity. These equations represent the conservation of linear momentum and mass, respectively. In order to be solvable, the system (1.1)-(1.2) must be equipped with appropriate boundary and initial conditions. The NSE are challenging for many reasons, mathematically and computationally, both of which we discuss below.

The kinematic viscosity depends on the Reynolds number, \( Re \), which describes whether a flow is laminar or turbulent. The Reynolds number is a dimensionless number defined by

\[
Re = \frac{\text{interal forces}}{\text{viscous forces}} = \frac{VD}{\mu},
\]

where \( V \) is the characteristic velocity, and \( D \) the characteristic linear dimension, and \( \mu \) is the viscosity. High Reynolds numbers correspond to more complex flows and even turbulence. Laminar flow is described by smooth fluid motion and corresponds to small Reynolds numbers.

The NSE still lack complete mathematical theory. In 2D, the existence and uniqueness of solutions have been proven by Ladyzhenskaya [61], but this is yet to be shown in 3D. The difference between these systems is the presence of the vortex stretching term, \(((\text{rot } u) \cdot \nabla)u\), which vanishes in 2D. Proof (or disproof) of existence and uniqueness of solutions in 3D is currently a one million dollar Clay Prize problem [28]. We note that these mathematical difficulties exist even in the “easiest” case of periodic boundaries; in the case of physical boundaries, things become even more complicated.

Leray, and others since him, made progress in understanding weak formulations of the NSE, which are derived from multiplying the equations by a test function, integrating over the domain, then applying integration by parts so that the derivatives fall on
the test function. The weak formulation of (1.1) can thus be written as

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u \cdot \frac{d\chi}{dt} dtdx - \sum_{i,j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_i u_j \frac{\partial \chi}{\partial x_j} dtdx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nu u \cdot \nabla \chi + f \cdot \chi - p \cdot (\nabla \cdot \chi) \right) dtdx,
\]

where \( \chi = (\chi_i(x,t))_{1 \leq i \leq 3} \) is the test function that is compactly supported in \( \mathbb{R}^3 \times (0, \infty) \). In this weak formulation, we can see that we now need \( u \in L^2 \), and \( f, p \in L^1 \), whereas (1.1) required \( u \) to be twice differentiable in \( x \). There has been some success in trying to prove existence and regularity of weak solutions of the NSE. In [71], Leray showed that the NSE in 3D always have a weak solution, but the uniqueness of such solutions is unknown. Several years later, Scheffer proved a partial regularity theorem for weak solutions using geometric measure theory [91], Caffarelli–Kohn–Nirenberg expanded upon Scheffer’s results [13], then Lin simplified these results in [72].

Due to the mathematical difficulties of the NSE and the inability to construct exact solutions, we must turn to numerical solvers and CFD. One issue with solving the NSE this way is the computational cost needed to completely capture the physics of the system. In 1941, Kolmogorov showed that \( O(Re^{9/4}) \) mesh points are needed to fully resolve the 3D NSE [31, 68, 59]. This makes direct numerical simulation (DNS) unachievable for many practical applications. As an illustration, the Reynolds number for a plane is \( O(10^8) \) which means \( 10^{18} \) meshpoints are needed. State of the art solvers can currently handle \( 10^{10} \) meshpoints or so. An insufficiently fine mesh is well known to cause failure of the simulation.

For those flows that are within reach, numerical models need to be accurate enough to capture all the physical characteristics of a flow, but still be computationally efficient. This provides the motivation to develop, analyze, and test new models.
1.1.1 Velocity-Vorticity Formulation

There are various formulations of the NSE that each represent the nonlinear term differently. These formulations are equivalent at the continuous level, but can yield different results when discretized. For example, the convective, rotational, conservative, and vorticity-streamfunction formulations have been shown to give different results on certain problems, see [37, 69, 35, 18], which shows that some formulations are more accurate than others in certain situations. The motivation to study velocity-vorticity (VV) formulations of the NSE comes from these findings. The VV formulation has been shown to be more accurate than usual velocity-pressure formulations in vorticity-dominated flow problems [79, 70, 78].

In the VV setting, we restrict our analysis to the 2D case, which reads

\[
\frac{\partial u}{\partial t} - \nu \Delta u + w \times u + \nabla P = f, \\
\nabla \cdot u = 0, \\
\frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w = \text{rot} f. 
\]

where \( w = u_{2x} - u_{1y} \) is the (scalar) vorticity, \( \text{rot} f := f_{2x} - f_{1y} \), and \( w \times u := \langle -u_{2w}, u_{1w} \rangle^T \). Note that in 2D, the vorticity equation reduces to a scalar equation (the \( z \)-component of the 3D vorticity equation) and the vortex stretching term \(-w \cdot \nabla u\) vanishes. We are staying in 2D because estimating the vorticity in 3D requires an estimate of the vortex stretching term, which comes from taking the curl of the nonlinear term in the momentum equation:

\[
\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)w - (w \cdot \nabla)u. 
\]
It seems that such an analysis is out of reach with current analytical tools.

Furthermore, appropriate physical boundary conditions for the vorticity variable are critical because solid wall boundaries are responsible for the production of vorticity and create physical and numerical boundary layers. One common choice of numerical boundary conditions on solid walls is to set $w = \nabla \times u$. This is problematic because $w$ now depends on an unknown velocity field on the solid wall boundaries; applying numerical differentiation to a discrete velocity field on the solid walls may reduce the accuracy of the vorticity solution on all of $\Omega$. In [79], the authors introduced a new kind of boundary condition for vorticity on solid walls,

$$w \cdot n|_{\partial \Omega} = 0, \quad \text{and} \quad \int_{\partial \Omega} (\nabla \times w) \times n \, ds = \int_{\partial \Omega} \nabla p \times n + f \times n \, ds.$$

The second condition can be efficiently implemented as a natural boundary condition in variational methods, and does not impose new constraints on trial and test spaces in the Galerkin method.

### 1.2 Introduction to DA algorithms

Broadly speaking, the term DA refers to schemes that incorporate observational data into simulations to increase the accuracy of solutions and/or obtain better initial conditions. In this section, we will discuss some of the popular DA techniques that have been used to help model physical processes, starting with Continuous DA.
1.2.1 Continuous DA

Continuous DA algorithms were recently pioneered by the mathematicians Azouani, Olson, and Titi [4, 5] in the continuous setting, where they showed that the DA solution converges to the true solution exponentially fast in time provided there is a sufficient number of spatial observations. This approach adds a feedback control term, or nudging term, at the PDE level that nudges the computed solution towards the reference solution corresponding to the observed data.

There has recently been extensive work done on nudging methods. The first works on it assumed noise-free observations, then [8] adapted to the case of noisy data, and [29] then worked on the case in which measurements are obtained discretely in time and may contain error. Computational experiments on the continuous DA algorithms were performed in the cases of the 2D Navier-Stokes equations [34], the 2D Bénard convection equations [2], and the 1D Kuramoto-Sivashinsky equations [63, 73]. In [63], several nonlinear versions of this approach were proposed and studied. Furthermore, a large amount of recent literature has expanded upon this idea; see, e.g., [1, 10, 23, 24, 25, 26, 27, 30, 36, 54, 55, 62, 75].

Previous numerical work on continuous DA algorithms can be found in [9], which studied a continuous-in-time Galerkin approximation of the algorithm, and [49] which studied a Galerkin in space, and explicit in time algorithm for the 2D Navier-Stokes equations (NSE). In this thesis, we analyze discrete numerical algorithms of the Navier-Stokes equations (NSE) with an added data assimilation term. At the contin-
uous level, the corresponding DA algorithm for the NSE is given by the system,

\[ v_t + (v \cdot \nabla) v + \nabla q - \nu \Delta v + \mu I_H (v - u) = f, \]
\[ \nabla \cdot v = 0, \]

where \( v \) is the approximate velocity, \( q \) the pressure of this approximate flow, and \( u \) can be considered the reference (or true) solution that is partially known, i.e. \( I_H (u) \) is known. The viscosity \( \nu > 0 \) and forcing \( f \) are the same as in the NSE. The scalar \( \mu \) is known as the nudging parameter, and \( I_H \) is the interpolation operator, where \( H \) is the resolution of the coarse spatial mesh corresponding to observation points of the true solution data. The added DA term forces (or nudges) the coarse spatial scales of the approximating solution \( v \) to the coarse spatial scales of the true solution \( u \). The initial value of \( v \) is arbitrary.

1.2.1.1 Comparison of continuous DA and linear feedback control of flow models

As described above, in continuous DA models, a penalty term is added to the PDE to nudge coarse spatial scales of the computed solutions towards those of the reference solutions, or data. The forcing \( f \) of the reference solution is considered known. In linear feedback control models, the right hand side forcing \( f \) is chosen differently. Instead, given a target flow \( U \), \( f \) is chosen so that the computed solution \( u \) is “close” to the target flow. That is, given the forcing \( F \) of the target flow \( U \), we take

\[ f = F - \gamma (U - u), \]
where \( \gamma > 0 \) is constant; this forcing steers the controlled flow to the target flow over time [40, 41]. While these two techniques are related, the major difference is that in DA, we are nudging towards measurements that come from the true state of the system, when the full state is not known beyond these measurements. With feedback control, we are trying to make our simulation match some known (steady) target flow.

### 1.2.2 Other DA techniques

The Kalman filter was introduced in 1960 by R.E. Kalman in the context of discrete time linear stochastic difference equations. It is a predictor-corrector algorithm that first estimates the future state of a system, which is then adjusted by measurements at that time to help improve the accuracy of the next estimation [94]. This is a well-studied technique that can be found in numerous textbooks, see [20, 58, 65, 17], and the references therein.

Another popular technique is known as 3D/4D Var, which stands for three dimensional (or four dimensional) variational data analysis. These models have been used in numerical weather prediction since the early 2000s. The goal of 3D Var is to find an optimal estimate of the true state of a system by minimizing a cost function that represents the error in the model.

### 1.3 Introduction to reduced order models

In chapter 5, we apply continuous DA to a reduced order model (ROM) for fluid flow. ROMs have been explored for decades [48]. When successful, they can decrease the computational cost of a DNS by orders of magnitude. For fluids dominated by a small
number of recurrent spatial structures, these ROMs are built using the following steps [47, 48, 86]:

1. Generate a collection of snapshots of the flow, either from numerical experiments or from physical data.

2. From these snapshots, select a small number of ROM basis functions. Note that small is a relative term that depends on the problem, in our studies, we considered 8 to 20 functions.

3. Project the equations of motion into this basis.

4. Advance the velocity in time to estimate flows different from the one generating the snapshots.

A major issue in using ROMs in practical fluid problems is their lack of accuracy in complicated domains, over long timer intervals, or when too few ROM bases are used. To overcome this issue, several approaches have been explored. One such approach is Closure Modeling, where a correction term is added to the standard ROM to model the effect of the discarded ROM modes [7, 81, 84, 89]. Another approach is Data-Driven Modeling. In this approach, available numerical or experimental data is used to construct ROM operators [82] or to determine the unknown coefficients in classical ROM operators [7, 32, 84]. Several other approaches can be found in [16, 39, 89, 6, 87, 88, 96, 95, 77].

We will propose another way to improve ROM accuracy in this thesis that uses continuous DA in the ROM setting. This approach is different from other uses of DA for ROMs, e.g., [15, 56, 74, 92], in which the authors use ROMs to speed up classical DA algorithms (e.g., 4D-VAR), whereas in the model proposed here, DA is used to
improve the ROM accuracy.

1.4 Overview

In the next chapter, we introduce the necessary mathematical preliminaries and notation that is used throughout the thesis. In Chapter 3, we propose and analyze an IMEX finite element DA scheme for the NSE, which include convergence analysis and numerical experiments that test the performance of the DA algorithm. Throughout our analysis in Chapter 3, we keep the interpolation operator $I_H$ general (so long as it satisfies certain properties). Next, in Chapters 4 and 5, we apply this new interpolation operator to a Velocity-Vorticity formulation of the NSE and to a Reduced Order Model for fluid flow, respectively. We include numerical analysis results in both chapters as well. Lastly, in Chapter 6 we introduce a new operator to use in Continuous DA algorithms. This operator allows for efficient and simple implementation of DA into existing codes while still providing optimal convergence rates.
Chapter 2

Notation

In this chapter we will introduce the mathematical notation and inequalities used throughout the thesis. We also present preliminaries for the reduced order model in chapter 5, data assimilation, and the numerical experiment of flow past a cylinder.

2.1 Preliminaries and functions spaces

We consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and denote its boundary by $\partial \Omega$. The notation $\| \cdot \|$ and $(\cdot, \cdot)$ is used throughout to denote the $L^2(\Omega)$ norm and inner product. All other norms will be clearly labeled with subscripts. Also throughout, the constant $C$ is used to denote a generic constant, possibly changing at each instance, that is independent of timestep $\Delta t$, meshwidth $h$, and parameters $\mu, \nu$.

We denote the natural velocity and pressure spaces by $X = (H^1_0(\Omega))^d$ and $Q = L^2_0(\Omega)$. We define $H^{-1}(\Omega)$ to be the dual space of $X$. The Poincaré inequality is known to hold in the velocity space: For all $\phi \in X$, there exists a constant $C_P$ depending only
on the size of $\Omega$, satisfying

$$\|\phi\| \leq C_P \|\nabla \phi\|.$$  

We now define the trilinear form $b : X \times X \times X \to \mathbb{R}$ by

$$b(u, v, w) := \frac{1}{2} \left( (u \cdot \nabla v, w) - (u \cdot \nabla w, v) \right).$$

Equivalently, we can write this operator as

$$b(u, v, w) := (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)w, v).$$

This operator is skew symmetric, $b(u, v, v) = 0$ for all $u, v \in X$, and it satisfies the following bounds [66]:

**Lemma 1.** For any $u, v, w \in X$, there exists a constant $C$ depending only on $\Omega$ satisfying

$$|b(u, v, w)| \leq M\|u\|_{L^3}\|\nabla v\|\|\nabla w\|,$$

$$|b(u, v, w)| \leq M\|\nabla u\|\|\nabla v\|\|\nabla w\|,$$

$$|b(u, v, w)| \leq M\|\nabla u\|\|\nabla v\|\|\nabla w\|,$$

$$|b(u, v, w)| \leq M\|u\|_{L^\infty}\|\nabla v\|_{L^3}\|\nabla w\|,$$

$$|b(u, v, w)| \leq M\|u\|_{L^\infty}\|\nabla v\|_{L^3} + \|v\|_{L^\infty}\|\nabla w\|,$$

for all $u, v, w \in X$ for which the norms on the right hand sides are finite.

**Proof.** The first bound is obtained by applying Hölder’s inequality then the Sobolev
inequality for $L^6(\Omega)$,

$$|b(u, v, w)| \leq \frac{1}{2} (\|u\|_{L^3} \|\nabla v\|_{L^6} + \|u\|_{L^3} \|\nabla w\| \|v\|_{L^6})$$

$$\leq C \|u\|_{L^3} \|\nabla v\| \|\nabla w\|.$$

The second bound is obtained by applying the Sobolev bound for $L^3(\Omega)$ and Poincaré inequality to the first bound,

$$\|u\|_{L^3} \leq C \|u\|^{1/2} \|\nabla u\|^{1/2} \leq C \|\nabla u\|.$$

The last two inequalities follow from from Hölder's inequality, Sobolev inequalities, and the Poincaré inequality. ■

2.2 Discretization Preliminaries

We will denote by $\tau_h$ a regular, conforming triangulation of the domain, and let $(X_h, Q_h) \subset (X, Q)$ be an inf-sup stable pair of discrete velocity-pressure spaces. The inf-sup (LBB) stability condition [12] is given by

$$\inf_{Q_h} \sup_{v_h \in X_h} (q_h, \nabla \cdot v_h) \|q_h\| \|\nabla v_h\| \geq \beta > 0,$$

with $\beta$ independent of $h$. To simplify the analysis, denote by $P_k(\tau_h)$ the space of piecewise polynomials of degree $k$ on $\tau_h$, and we let $(X_h, Q_h) = (P_k \cap X, P_{k-1} \cap Q)$ be Taylor-Hood or Scott-Vogelius elements. Note that if the pressure space is discontinuous (for example, with Scott-Vogelius elements), then an appropriate mesh must be chosen for the inf-sup condition to hold [98, 97, 3, 85, 97, 98, 52, 43, 44, 42].
We remark that all of our results are extendable to any inf-sup stable pair of velocity-pressure spaces, with minor modifications.

We assume that the mesh is sufficiently regular so that the inverse inequality holds:
There exists a constant $C$ such that for all $v_h \in X_h$,

$$\|\nabla v_h\| \leq C h^{-1} \|v_h\|.$$  

We next define the discretely divergence free subspace by

$$V_h := \{v_h \in X_h, \; (\nabla \cdot v_h, q_h) = 0 \; \forall q_h \in Q_h\}.$$  

Analysis in the case of a BDF2 approximation to the time derivative term often requires the use of the $G$-norm, where the matrix $G$ is defined as

$$G = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix}.$$  

This matrix then induces the norm $\|x\|_G^2 := (x, Gx)$, which is equivalent to the vector $L^2$ norm: There exists constants $C_l, C_u > 0$ such that

$$C_l \|x\| \leq \|x\|_G \leq C_u \|x\|_G,$$

where $C_l = 3 - 2\sqrt{2}$ and $C_u = 3 + 2\sqrt{2}$. Now set $\chi_v^n := [v^{n-1}, v^n]^T$, then if $v^i \in L^2(\Omega)$, we have the identity

$$\left(\frac{1}{2}(3v^{n+1} - 4v^n + v^{n-1}), v^{n+1}\right)$$

$$= \frac{1}{2}(\|\chi_v^{n+1}\|_G^2 - \|\chi_v^n\|_G^2) + \frac{1}{4}\|v^{n+1} - 2v^n + v^{n-1}\|^2.$$  

(2.1)
2.2.1 DA preliminaries

The DA algorithms have an interpolation operator, which we denote by $I_H : X \to X$. This operator must satisfy

$$\|I_H(\phi) - \phi\| \leq C_I H \|\nabla \phi\|, \quad (2.2)$$

$$\|I_H(\phi)\| \leq C_I \|\phi\|, \quad (2.3)$$

for all $\phi \in X$. Here, $H$ is a characteristic point spacing for the interpolant. The spacing $H$ corresponds in practice to points where (true solution) measurements are taken, so $H$ should be as large as possible while still satisfying (2.2)-(2.3).

Throughout this thesis, the parameter $\mu$ is assumed to satisfy the upper bound

$$\mu \leq C \nu H^{-2}. \quad (2.4)$$

However, we remark that $\mu$ need not scale with $H^{-2}$ as $H \to 0$. The mesh width $H$ may need to be sufficient small for the algorithms to work (as $\mu$ will also be bounded below by a data dependent constant in the theorems to follow), but once $H$ is ‘small enough’ $\mu$ need not increase if $H$ decreases further. In our NSE-type analysis to follow, $\mu$ will also have a data-dependent lower bound, but choosing $H$ small enough will guarantee the existence of an appropriate $\mu$.

Many of the results in the data assimilation sections will use the following inequality.

**Lemma 2.** Suppose constants $r$ and $B$ satisfy $r > 1$, $B \geq 0$. Then if the sequence of real numbers $\{a_n\}$ satisfies

$$ra_{n+1} \leq a_n + B, \quad (2.5)$$
we have that

\[ a_{n+1} \leq a_0 \left( \frac{1}{r} \right)^{n+1} + \frac{B}{r - 1}. \]

**Proof.** The inequality (2.5) can be written as

\[ a_{n+1} \leq \frac{a_n}{r} + \frac{B}{r}. \]

Recursively, we obtain

\[
\begin{align*}
    a_{n+1} &\leq \frac{1}{r} \left( \frac{a_{n-1}}{r} + \frac{B}{r} \right) + \frac{B}{r} \\
    &= \frac{a_{n-1}}{r^2} + \frac{B}{r} \left( 1 + \frac{1}{r} \right) \\
    &\vdots \\
    &\leq \frac{a_0}{r^{n+1}} + \frac{B}{r} \left( 1 + \frac{1}{r} + \cdots + \frac{1}{r^n} \right).
\end{align*}
\]

Now the resulting finite geometric series is bounded as

\[
\frac{B}{r} \left( 1 + \frac{1}{r} + \cdots + \frac{1}{r^n} \right) = \frac{B}{r} \cdot \frac{1 - (1/r)^{n+1}}{1 - (1/r)} \leq \frac{B}{r} \cdot \frac{1}{1 - (1/r)} \leq \frac{B}{r - 1},
\]

which gives the result. ■

### 2.2.2 Numerical experiment: Flow past a cylinder

Since we perform the 2D flow past a cylinder benchmark test [90] in multiple chapters, we present the setup of this problem now. The domain is a 2.2 × 0.41 rectangular channel with a cylinder of radius 0.05 centered at (0.2, 0.2), see figure 2.1. No-slip boundary conditions are prescribed for the walls and the cylinder, while the inflow
and outflow profiles are given by

\[ u_1(0, y, t) = u_1(2.2, y, t) = \frac{6}{0.41^2} y(0.41 - y), \]
\[ u_2(0, y, t) = u_2(2.2, y, t) = 0. \]

Figure 2.1: Shown above is the domain for the flow past a cylinder test problem.

Lift and drag calculations will be computed and compared to the literature [90, 76], to help verify the accuracy of simulations. For these calculations, we used the formulas

\[
c_{d}(t) = 20 \int_{S} \left( \nu \frac{\partial u_{tS}(t)}{\partial n} n_{y} - p(t) n_{x} \right) dS, \]
\[
c_{l}(t) = 20 \int_{S} \left( \nu \frac{\partial u_{tS}(t)}{\partial n} n_{x} - p(t) n_{y} \right) dS, \]

where \( p(t) \) is the pressure, \( u_{tS} \) the tangential velocity \( S \) the cylinder, and \( n = \langle n_{x}, n_{y} \rangle \) the outward unit normal to the domain. For calculations, we use the global integral formula from [51].
Chapter 3

Data assimilation in IMEX-FEM schemes for the Navier-Stokes equations

In this chapter we perform numerical analysis on an IMEX-FEM DA scheme for the NSE. We analyze stability and convergence results, where, under the assumption that sufficiently regular solutions exist, we are able to prove well-posedness, stability, and long time accuracy, provided the time step size and nudging parameter from the DA term satisfy certain restrictions. The numerical tests included in this chapter illustrate both the theory and highlight how important the choices of nudging parameter and elements can be on the accuracy of results.

We also include a grad-div term in the scheme, which is known to improve mass conservation and reduce the effect of the pressure on the velocity error [52]. Thus,
the continuous DA grad-div algorithm reads

\[ v_t + (v \cdot \nabla)v + \nabla q + \mu I_H (v - u) - \gamma \nabla (\nabla \cdot v) = f, \]
\[ \nabla \cdot v = 0, \]

where \( v \) is the approximate velocity, \( u \) the true velocity, \( q \) the pressure of this approximate flow, and \( f \) the external forcing. The constant \( \gamma = O(1) \) is the grad-div parameter, and we note that at the continuous level, this term is zero. The scalar \( \mu \geq 0 \) is the nudging parameter, and \( I_H \) is the interpolation operator, where \( H \) is the characteristic spacing of the coarse mesh corresponding to data. The initial value of \( v \) can be chosen arbitrarily.

We begin this chapter with the analysis of a second order IMEX-FEM scheme, and then present three numerical experiments that illustrate the theory.

### 3.1 A second order temporal discretization

A BDF2 IMEX scheme for NSE with DA is studied in this section. We prove well-posedness, and global in time stability and convergence. For most common element choices, grad-div stabilization is known to improve mass conservation and reduce the effect of the pressure on the velocity error [52]; a similar effect is observed in the convergence result for this continuous DA scheme, as well as in the numerical tests.

The second order IMEX-FEM DA algorithm is defined as follows.

**Algorithm 3.1.1.** Given any initial conditions \( v_h^0, v_h^1 \in V_h, \) forcing \( f \in L^\infty (0, \infty; L^2(\Omega)), \) and true solution \( u \in L^\infty (0, \infty; L^2(\Omega)) \), find \( (v_h^{n+1}, q_h^{n+1}) \in \)
\((X_h, Q_h)\) for \(n = 1, 2, ...,\) satisfying

\[
\frac{1}{2\Delta t} \left( 3v_h^{n+1} - 4v_h^n + v_h^{n-1}, \chi_h \right) + b(2v_h^n - v_h^{n-1}, v_h^{n+1}, \chi_h) - (q_h^{n+1}, \nabla \cdot \chi_h) \\
+ \gamma(\nabla \cdot v_h^{n+1}, \nabla \cdot \chi_h) + \nu(\nabla v_h^{n+1}, \nabla \chi_h) + \mu(I_H(v_h^{n+1} - u^{n+1}), \chi_h)
\]
\[
= (f^{n+1}, \chi_h), \quad \text{(3.3)}
\]
\[
(\nabla \cdot v_h^{n+1}, r_h) = 0, \quad \text{(3.4)}
\]

for all \((\chi_h, r_h) \in X_h \times Q_h\), with \(I_H\) a given interpolation operator satisfying (2.2)-(2.3).

We note that again the initial conditions can be chosen arbitrarily in \(V_h\), although more accurate initial conditions may result in faster convergence to the true solution. Well-posedness and long time stability of this algorithm use G-stability theory on the time derivative term. Throughout the analysis in this chapter, let \(\alpha := \nu - 2\mu C_I H^2\).

We state and prove the result now.

Lemma 3. Suppose \(\mu\) and \(H\) satisfy

\[
0 < H < \frac{\sqrt{\nu}}{C_I \sqrt{2}} \quad \text{and} \quad 1 \leq \mu < \frac{\nu}{2C_I^2 H^2}.
\]

Then for any time step size \(\Delta t > 0\), Algorithm 3.1.1 is well-posed globally in time, and solutions are nonlinearly long-time stable: for any \(n > 1\),

\[
C_u^{-2} \left( \|v_h^{n+1}\|^2 + \|v_h^n\|^2 \right) + \frac{\alpha \Delta t}{2} \|\nabla v_h^{n+1}\|^2 + \frac{\mu \Delta t}{4} \|v_h^{n+1}\|^2
\]
\[
\leq \left( C_t^{-2}(\|v_h^1\|^2 + \|v_h^0\|^2) + \frac{\alpha \Delta t}{2} \|\nabla v_h^1\|^2 + \frac{\mu \Delta t}{4} \|v_h^1\|^2 \right) \left( 1 + \lambda \Delta t \right)^{n+1}
\]
\[
+ C \lambda^{-1} \nu^{-1} F^2 + C \lambda^{-1} \mu U^2.
\]

where \(\lambda = \min\{2\Delta t^{-1}, \frac{\mu C_I^2}{4}, \frac{\alpha C_I^2 C_P^2}{2}\}\), \(\alpha = \nu - 2\mu C_I^2 H^2\), \(U := \|u\|_{L^2(0,\infty;L^2)}\), and
\[ F := \|f\|_{L^\infty(0,\infty; H^{-1})}. \]

**Proof.** Choose \( \chi_h = v_h^{n+1} \) in (3.3) and use (2.1) to obtain the bound

\[
\frac{1}{2\Delta t} \left( \|[v_h^{n+1}; v_h^n]\|_G^2 \right) + \nu \|\nabla v_h^{n+1}\|^2 + \mu(I_H(v_h^{n+1}), v_h^{n+1}) \\
\leq \frac{1}{2\Delta t} \left( \|[v_h^n; v_h^{n-1}]\|_G^2 \right) + |(f^{n+1}, v_h^{n+1})| + \mu |(I_H(u^{n+1}), v_h^{n+1})|,
\]

noting that we dropped the non-negative terms \( \gamma \|\nabla \cdot v_h^{n+1}\|^2 \) and \( \frac{1}{4\Delta t} \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 \) from the left hand side, and that the nonlinear and pressure terms vanish. Majorize the right hand side nudging and forcing terms, and multiply both sides by \( 2\Delta t \), we reduce the bound to

\[
\|[v_h^{n+1}; v_h^n]\|_G^2 + \alpha \Delta t \|\nabla v_h^{n+1}\|^2 + \mu \Delta t \|v_h^{n+1}\|^2 \leq \|[v_h^n; v_h^{n-1}]\|_G^2 + \Delta t(2\nu^{-1} F^2 + C \mu U^2).
\]

Next, drop the viscous term on the left hand side, and add \( \frac{\mu \Delta t}{4} \|v_h^n\|^2 + \frac{\alpha \Delta t}{2} \|\nabla v_h^n\|^2 \) to both sides. This gives

\[
\|[v_h^{n+1}; v_h^n]\|_G^2 + \frac{\mu \Delta t}{4} \|v_h^{n+1}\|^2 + \frac{\alpha \Delta t}{2} \|\nabla v_h^{n+1}\|^2 + \frac{\mu \Delta t}{4} \left( \|v_h^{n+1}\|^2 + \|v_h^n\|^2 \right) \\
+ \frac{\alpha \Delta t}{2} \left( \|\nabla v_h^{n+1}\|^2 + \|\nabla v_h^n\|^2 \right) + \frac{\mu \Delta t}{2} \|v_h^{n+1}\|^2 + \alpha \Delta t \|\nabla v_h^{n+1}\|^2 \\
\leq \left( \|[v_h^n; v_h^{n-1}]\|_G^2 + \frac{\mu \Delta t}{4} \|v_h^n\|^2 + \frac{\alpha \Delta t}{2} \|\nabla v_h^n\|^2 \right) + \Delta t(2\nu^{-1} F^2 + C \mu U^2),
\]
which reduces using Poincaré’s inequality and G-norm equivalence to

\[ \| [v_h^{n+1}; v^n_h] \|_G^2 + \frac{\mu \Delta t}{4} \| v_h^{n+1} \|^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^{n+1} \|^2 + \frac{\mu \Delta t}{4} \| v_h^{n+1} \|^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^{n+1} \|^2 \]

\[ + \frac{\mu \Delta t C_l^2}{4} \| [v_h^{n+1}; v^n_h] \|_G^2 + \frac{\alpha \Delta t C_f^2}{2} \| [v_h^{n+1}; v^n_h] \|_G^2 + \frac{\mu \Delta t}{2} \| v_h^{n+1} \|^2 + \alpha \Delta t \| \nabla v_h^{n+1} \|^2 \]

\[ \leq \left( \| [v_h^n; v_h^{n-1}] \|_G^2 + \frac{\mu \Delta t}{4} \| v_h^n \|^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^n \|^2 \right) + \Delta t (2\nu^{-1} F^2 + C \mu U^2), \]

Thus there exists \( \lambda = \min \{ 2\Delta t^{-1}, \frac{\mu C_l^2}{4}, \frac{\alpha C_f^2 C_l^2}{2} \} \) such that

\[ (1 + \lambda \Delta t) \left( \| [v_h^{n+1}; v^n_h] \|_G^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^{n+1} \|^2 + \frac{\mu \Delta t}{4} \| v_h^{n+1} \|^2 \right) \]

\[ \leq \left( \| [v_h^n; v_h^{n-1}] \|_G^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^n \|^2 + \frac{\mu \Delta t}{4} \| v_h^n \|^2 \right) + \Delta t (2\nu^{-1} F^2 + C \mu U^2), \]

and so by Lemma 2,

\[ \left( \| [v_h^{n+1}; v^n_h] \|_G^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^{n+1} \|^2 + \frac{\mu \Delta t}{4} \| v_h^{n+1} \|^2 \right) \]

\[ \leq \left( \| [v_h^n; v_h^0] \|_G^2 + \frac{\alpha \Delta t}{2} \| \nabla v_h^0 \|^2 + \frac{\mu \Delta t}{4} \| v_h^0 \|^2 \right) \left( \frac{1}{1 + \lambda \Delta t} \right)^{n+1} \]

\[ + C \lambda^{-1} (\nu^{-1} F^2 + \mu U^2). \]

Applying the G-norm equivalence completes the proof of stability.

Since the scheme is linear and finite dimensional at each time step, this uniform in \( n \) stability result gives existence and uniqueness of the algorithm at every time step. □

Proving a long time accuracy result for Algorithm 3.1.1 also uses the G-norm on the time derivative terms, which we handle with the G-stability theory in a manner similar to the stability proof.

**Theorem 1.** Let \( u, p \) solve the NSE (1.1)-(1.2) with given \( f \in L^\infty(0, \infty; L^2(\Omega)) \) and
$u_0 \in L^2(\Omega)$, with $u \in L^\infty(0, \infty; H^{k+1}(\Omega))$, $p \in L^\infty(0, \infty; H^k(\Omega))$ ($k \geq 1$), $u_{tt} \in L^\infty(0, \infty; L^2(\Omega))$, and $u_{ttt} \in L^\infty(0, \infty; H^1(\Omega))$. Denote $U := |u|_{L^\infty(0, \infty; H^{k+1})}$ and $P := |p|_{L^\infty(0, \infty; H^k)}$. Assume the time step size satisfies

$$\Delta t < CM^2 \nu^{-1} \left( h^{2k-3} U^2 + \| \nabla u^{n+1} \|_{L^2}^2 + \| u^{n+1} \|_{L^\infty}^2 \right)^{-1},$$

and the parameter $\mu$ satisfies

$$CM^2 \nu^{-1} \left( h^{2k-3} U^2 + \| \nabla u^{n+1} \|_{L^2}^2 + \| u^{n+1} \|_{L^\infty}^2 \right) < \mu < \frac{2 \nu}{C^2 H^2}.$$ 

Then there exists a $\lambda > 0$ (independent of $h$ and $\Delta t$) such that the error in solutions to Algorithm 3.1.1 satisfies, for any $n$,

$$\| u^n - v^n_h \|^2 \leq \left( \frac{1}{1 + \lambda \Delta t} \right)^n \| u^0 - v^0_h \|^2 + \frac{R}{\lambda},$$

where $R = C \nu^{-1} (1 + M^2) \Delta t^4 + C h^{2k} \left( \gamma^{-1} P^2 + (\nu + \gamma + M^2 \nu^{-1} + M^2 \nu^{-1} h^{2k} U^2 + \nu C_I^{-2}) U^2 \right)$.

**Remark 2.** For the case of Taylor-Hood ($P_2, P_1$) or Scott-Vogelius ($P_2, P_1^{disc}$) elements and $0$ initial condition in the DA algorithm, the result of the theorem reduces to

$$\| u^n - v^n_h \| \leq C \left( \left( \frac{1}{1 + \lambda \Delta t} \right)^{n/2} \| u^0 \| + \Delta t^2 + h^2 \right),$$

where $C$ depends on problem data and the true solution, not $\Delta t$ or $h$.

**Remark 3.** The time step restriction is a consequence of the IMEX time stepping. If we instead consider the fully nonlinear scheme, then no $\Delta t$ restriction is required for a similar result to hold. However, there is seemingly a time step restriction necessary
for solution uniqueness for the nonlinear scheme.

Remark 4. Grad-div stabilization reduces the effect of the pressure on the $L^2(\Omega)$ DA solution error. With grad-div, the contribution of the error term is $h^k \gamma^{-1/2} |p|_{L^\infty(0,\infty; H^k)}$, but without it, the $\gamma^{-1/2}$ would be replaced by a $\nu^{-1/2}$. If divergence-free elements were used, then this term would completely vanish. We show in numerical experiment 2 below that a DA simulation will fail with Taylor-Hood elements with $\gamma = 0, 1, 10$, but will work very well with Scott-Vogelius elements.

Proof. Throughout this proof, the constant $C$ will denote a generic constant, possibly changing from line to line, that is independent of $h$, $\mu$, and $\Delta t$.

Using Taylor’s theorem, the NSE (true) solution satisfies, for all $\chi_h \in X_h$,

\[
\frac{1}{2\Delta t} \left( 3u^{n+1} - 4u^n + u^{n-1}, \chi_h \right) + b(2u^n - u^{n-1}, u^{n+1}, \chi_h) - (p^{n+1}, \nabla \cdot \chi_h) + \gamma(\nabla \cdot u^{n+1}, \nabla \cdot \chi_h) + \nu(\nabla u^{n+1}, \nabla \chi_h) \\
= (f^{n+1}, \chi_h) + \frac{\Delta t^2}{3} (u_{tt}(t^*), \chi_h) + \Delta t^2 b(u_{tt}(t^{**}), u^{n+1}, \chi_h), \quad (3.5)
\]

where $t^*, t^{**} \in [t^{n-1}, t^{n+1}]$. Subtracting (3.3) from (3.5) yields the following difference equation, with $e^n := u^n - v^n$:

\[
\frac{1}{2\Delta t} \left( 3e^{n+1} - 4e^n + e^{n-1}, \chi_h \right) + \nu(\nabla e^{n+1}, \nabla \chi_h) + \mu(I_H(e^{n+1}), \chi_h) + \gamma(\nabla \cdot e^{n+1}, \nabla \cdot \chi_h) \\
= \frac{\Delta t^2}{3} (u_{tt}(t^*), \chi_h) + \Delta t^2 (u_{tt}(t^{**}) \cdot \nabla u^{n+1}, \chi_h) - (p^{n+1}, \nabla \cdot \chi_h) \\
+ b(2v^n - v^{n-1}, e^{n+1}, \chi_h) + b(2e^n - e^{n-1}, u^{n+1}, \chi_h).
\]

We decompose the error into a piece inside the discrete space $V_h$ and one outside of it by adding and subtracting $P_{Vh}^2(u^n)$. Denote $\eta^n := u^n - P_{Vh}^2(u^n)$ and $\phi^n_h :=$
Then $e^n = \eta^n + \phi^n_h$ with $\phi^n_h \in V_h$, and we choose $\chi_h = \phi^{n+1}_h$. Using identity (2.1) with $\psi_{\phi} := (\phi^n_h, \phi^{n+1}_h)^T$, the difference equation becomes

$$
\frac{1}{2\Delta t} [\|\psi_{\phi}^{n+1} \|^2_G - \|\psi_{\phi}^n \|^2_G] + \frac{1}{4\Delta t} \|\phi^{n+1}_h - 2\phi^n_h + \phi^{n-1}_h\|^2 + \nu \|\nabla \phi^{n+1}_h\|^2 + \mu \|\phi^{n+1}_h\|^2 \\
+ \gamma \|\nabla \cdot \phi^{n+1}_h\|^2 \\
= \frac{\Delta t^2}{3} (u_{ttt}(t^*), \phi^{n+1}_h) + \Delta t^2(u_{tt}(t^{**}) \cdot \nabla u^{n+1}, \phi^{n+1}_h) - (p^{n+1}, \nabla \cdot \phi^{n+1}_h) \\
b(2\phi^n_h - \phi^{n-1}_h, \phi^{n+1}_h) + b(2\eta^n - \eta^{n-1}, u^{n+1}, \phi^{n+1}_h) \\
b(2\eta^n - v^{n-1}, \phi^{n+1}_h) - \nu(\nabla \eta^{n+1}, \nabla \phi^{n+1}_h) - \mu(I_H \phi^{n+1}_h - \phi^{n+1}_h, \phi^{n+1}_h) \\
- \mu(I_H \eta^{n+1}, \phi^{n+1}_h) - \gamma(\nabla \cdot \eta^{n+1}, \nabla \cdot \phi^{n+1}_h),
$$

where we have added and subtracted $\phi^{n+1}_h$ in the interpolation term on the left hand side. We can now bound the right hand side of (3.6). Many of these terms are bounded in a similar manner as in the case of BDF2 FEM for NSE, for example as in [35, 66, 93]. We will use these techniques (which mainly consist of carefully constructed Young and Cauchy-Schwarz inequalities and Lemma 1) to bound all terms except the nonlinear and nudging terms. For the first nonlinear term in (3.6), we add and subtract $\phi^{n+1}_h$ in the first argument to obtain

$$
b(2\phi^n_h - \phi^{n-1}_h, u^{n+1}, \phi^{n+1}_h) \\
= b(\phi^{n+1}_h, u^{n+1}, \phi^{n+1}_h) - b(\phi^{n+1}_h - 2\phi^n_h + \phi^{n-1}_h, u^{n+1}, \phi^{n+1}_h).
$$

(3.7)

We bound the two resulting terms using Lemma 1 and Young’s inequality, via

$$
b(\phi^{n+1}_h, u^{n+1}, \phi^{n+1}_h) \leq CM\nu^{-1}(\|\nabla u^{n+1}\|^2_{L^2} + \|u^{n+1}\|^2_{L^\infty})\|\phi^{n+1}_h\|^2 + \frac{\nu}{16} \|\nabla \phi^{n+1}_h\|^2,
$$
and
\[
b(\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}, u^{n+1}, \phi_h^{n+1}) \leq CM\nu^{-1}(\|\nabla u^{n+1}\|_{L^3}^2 + \|u^{n+1}\|_{L^\infty}^2)\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \frac{\nu}{16}\|\nabla \phi_h^{n+1}\|^2.
\]

The second nonlinear term in (3.6) is bounded with this same technique:
\[
b(2\eta^n - \eta^{n-1}, u^{n+1}, \phi_h^{n+1}) \leq CM^2\nu^{-1}(\|\nabla u^{n+1}\|_{L^3}^2 + \|u^{n+1}\|_{L^\infty}^2)\|\eta^n - \eta^{n-1}\|^2 + \frac{\nu}{16}\|\nabla \phi_h^{n+1}\|^2.
\]

The last nonlinear term in (3.6) requires a bit more work, and we start by adding and subtracting \(2u^n - u^{n-1}\) in the first component, which yields
\[
b(2v^n - v_h^{n-1}, \eta^{n+1}) = b(2u^n - u^{n-1}, \eta^{n+1}) + b(2\phi^n - \phi_h^{n-1}, \phi_h^{n+1})
\]
\[
= b(2u^n - u^{n-1}, \eta^{n+1}) + b(2\phi^n - \phi_h^{n-1}, \phi_h^{n+1}) + b(2\eta^n - \eta^{n-1}, \eta^{n+1}). \tag{3.8}
\]

The first and third terms on the right hand side of (3.8) are bounded in the same way, using Lemma 1 and Young’s inequality, we find
\[
b(2u^n - u^{n-1}, \eta^{n+1}, \phi_h^{n+1}) \leq C\nu^{-1}M^2\|\nabla (2u^n - u^{n-1})\|_{L^3}^2\|\nabla \eta^{n+1}\|^2 + \frac{\nu}{16}\|\nabla \phi_h^{n+1}\|^2,
\]
\[
b(2\eta^n - \eta^{n-1}, \eta^{n+1}, \phi_h^{n+1}) \leq C\nu^{-1}M^2\|\nabla (2\eta^n - \eta^{n-1})\|_{L^3}^2\|\nabla \eta^{n+1}\|^2 + \frac{\nu}{16}\|\nabla \phi_h^{n+1}\|^2.
\]

For the second term in (3.8) we first add \(\phi_h^{n+1}\) to the first argument to obtain
\[
b(2\phi^n - \phi_h^{n-1}, \eta^{n+1}, \phi_h^{n+1}) = b(\phi^{n+1} - 2\phi_h^n + \phi_h^{n-1}, \eta^{n+1}, \phi_h^{n+1}) + b(\phi_h^{n+1}, \eta^{n+1}, \phi_h^{n+1}),
\]

27
and then bound each resulting term using Lemma 1 and Young’s inequality:

\[
b(\phi_h^{n+1}, \eta^{n+1}, \phi_h^{n+1}) \leq CM^2 \nu^{-1}(\|\eta^{n+1}\|_{L^\infty}^2 + \|\nabla \eta^{n+1}\|_{L^3}^2)\|\phi_h^{n+1}\|^2 \\
+ \frac{\nu}{16} \|\nabla \phi_h^{n+1}\|^2,
\]

\[
b(\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}, \eta^{n+1}, \phi_h^{n+1}) \\
\leq CM^2 \nu^{-1}(\|\eta^{n+1}\|_{L^\infty}^2 + \|\nabla \eta^{n+1}\|_{L^3}^2)\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
+ \frac{\nu}{16} \|\nabla \phi_h^{n+1}\|^2.
\]

For the first nudging term in (3.6), we apply Cauchy-Schwarz and Young’s inequalities and (2.2) to obtain

\[
\mu \left\| (I_H(\phi_h^{n+1}) - \phi_h^{n+1}, \phi_h^{n+1}) \right\| \leq \mu \|I_H(\phi_h^{n+1}) - \phi_h^{n+1}\| \|\phi_h^{n+1}\| \\
\leq \mu C_H \|\nabla \phi_h^{n+1}\| \|\phi_h^{n+1}\| \\
\leq \mu C_H^2 H^2 \|\nabla \phi_h^{n+1}\|^2 + \frac{\mu}{4} \|\phi_h^{n+1}\|^2.
\]

Finally, for the last nudging term in (3.6), we employ Cauchy-Schwarz and Young inequalities, along with (2.3), to obtain

\[
\mu(I_H(\eta^{n+1}), \phi_h^{n+1}) \leq \mu \|I_H(\eta^{n+1})\| \|\phi_h^{n+1}\| \\
\leq C \|\eta^{n+1}\|^2 + \frac{\mu}{4} \|\phi_h^{n+1}\|^2.
\]
Collecting the above bounds, we reduce (3.6) to

\[
\frac{1}{2\Delta t} \left[ \|\psi_\phi^{n+1}\|_G^2 - \|\psi_\phi^n\|_G^2 \right] + \frac{9\nu}{16} \|\nabla \phi_h^{n+1}\|^2 + \gamma \|\nabla \cdot \phi_h^{n+1}\|^2 \\
+ \left( \frac{1}{4\Delta t} - CM^2\nu^{-1}(\|\eta^{n+1}\|_L^2 + \|\nabla \eta^{n+1}\|^2 + \|u^{n+1}\|_L^2 + \|\nabla u^{n+1}\|^2) \right) \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
+ \left( \mu - CM^2\nu^{-1}(\|\eta^{n+1}\|_L^2 + \|\nabla \eta^{n+1}\|^2 + \|u^{n+1}\|_L^2 + \|\nabla u^{n+1}\|^2) \right) \|\phi_h^{n+1}\|^2 \\
\leq C\Delta t^2 \|u_{ttt}\|_{L^\infty(t_{n-1},t_{n+1},L^2)} \|\phi_h^{n+1}\| + \Delta t^2((u_{ttt}(t^{**}) \cdot \nabla u^{n+1}, \phi_h^{n+1}) + |(p^{n+1} - r_h, \nabla \cdot \phi_h^{n+1})| \\
+ \nu|\nabla \eta^{n+1} \cdot \nabla \phi_h^{n+1}| + \mu|\Gamma \phi_h^{n+1} - \phi_h^{n+1}, \phi_h^{n+1}| + \mu|\Gamma \eta^{n+1}, \phi_h^{n+1}| \\
+ CM^2\nu^{-1}(\|u^{n+1}\|_{L^3}^3 + \|\nabla u^{n+1}\|_{L^\infty}^2) \|\nabla \eta^{n+1}\|^2 + C\nu^{-1}M^2 \|\nabla (2\eta^n - \eta^{n-1})\|^2 \|\nabla \eta^{n+1}\|^2 \\
+ CM^2\nu^{-1}(\|u^{n+1}\|_{L^3}^3 + \|\nabla u^{n+1}\|_{L^\infty}^2) \|2\eta^n - \eta^{n-1}\|^2 + \gamma|\nabla \cdot \eta^{n+1} \cdot \nabla \cdot \phi_h^{n+1}|, \tag{3.9}
\]

where \(r_h \in Q_h\) is chosen arbitrarily, see e.g. [12]. Now using interpolation estimates (and implicitly also the inverse inequality) along with regularity assumptions, we obtain

\[
\frac{1}{2\Delta t} \left[ \|\psi_\phi^{n+1}\|_G^2 - \|\psi_\phi^n\|_G^2 \right] + \frac{9\nu}{16} \|\nabla \phi_h^{n+1}\|^2 + \gamma \|\nabla \cdot \phi_h^{n+1}\|^2 \\
+ \left( \frac{1}{4\Delta t} - CM^2\nu^{-1}(h^{2k-3}U^2 + \|u^{n+1}\|_L^2 + \|\nabla u^{n+1}\|_{L^3}^2) \right) \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
+ \left( \mu - CM^2\nu^{-1}(h^{2k-3}U^2 + \|u^{n+1}\|_L^2 + \|\nabla u^{n+1}\|_{L^3}^2) \right) \|\phi_h^{n+1}\|^2 \\
\leq C\Delta t^2 \|u_{ttt}\|_{L^\infty(t_{n-1},t_{n+1},L^2)} \|\phi_h^{n+1}\| + \Delta t^2((u_{ttt}(t^{**}) \cdot \nabla u^{n+1}, \phi_h^{n+1}) + |(p^{n+1} - r_h, \nabla \cdot \phi_h^{n+1})| \\
+ \nu|\nabla \eta^{n+1} \cdot \nabla \phi_h^{n+1}| + \mu|\Gamma \phi_h^{n+1} - \phi_h^{n+1}, \phi_h^{n+1}| + \mu|\Gamma \eta^{n+1}, \phi_h^{n+1}| \\
+ CM^2\nu^{-1}h^{2k}U^2(1 + h^{2k}U^2) + \gamma|\nabla \cdot \eta^{n+1} \cdot \nabla \cdot \phi_h^{n+1}|.
\]

Next we use the assumptions on \(\Delta t\) and \(\mu\), and apply bounds to the remaining right
hand side terms similar to NSE convergence analyses in [66] to find

\[
\frac{1}{2\Delta t} \left( \|\psi_n^{n+1}\|_G^2 - \|\psi_n^n\|_G^2 \right) + \alpha \|\nabla \phi_h^{n+1}\|^2 + \frac{\gamma}{2} \|\nabla \cdot \phi_{h}^{n+1}\|^2 \\
\leq C\nu^{-1}(1 + M^2)\Delta t^4 + Ch^2k \left( \gamma^{-1} P^2 + (\nu + \gamma + M^2\nu^{-1} + M^2\nu^{-1}h^{2k}U^2 + \nu C_I^{-2})U^2 \right) \\
=: R. 
\]

This implies, with Poincaré’s inequality that

\[
\|\psi_n^{n+1}\|_G^2 + 2C^2l^2\Delta t\alpha C_P^{-2}\|\phi_{h}^{n+1}\|^2 \leq \|\psi_n^n\|_G^2 + \Delta tR. 
\]

From here, we can proceed just as in to the BDF2 long time stability proof above to obtain

\[
\|\psi_n^{n+1}\|_G^2 \leq \|\psi_0^n\|_G^2 \left( \frac{1}{1 + \lambda \Delta t} \right)^{n+1} + \frac{R}{\lambda}, 
\]

where \( \lambda = 2C^2l^2\alpha C_P^{-2} \). Now the triangle inequality and G-norm equivalence complete the proof. \( \blacksquare \)

### 3.2 Numerical Experiments

We now present results of three numerical tests that illustrate the theory above, and also show the importance of a careful choice of discretization. That is, while the DA theory at the PDE level is critical, we show above that in implementations, convergence to the true solution is only up to discretization parameters \( h \) and \( \Delta t \) to some powers, but which are scaled by constants. In some situations, in particular see numerical experiment 2 below, these constants can be sufficiently large to make DA (essentially) fail if care is not taken with the discretization.
3.2.1 Numerical Experiment 1: Convergence to an analytical solution

For our first experiment, we illustrate the convergence theory above for Algorithm 3.1.1 to a chosen analytical solution

\[ u(x, y, t) = (\cos(y + t), \sin(x - t))^T, \]
\[ p(x, y, t) = \sin(2\pi(x + t)). \]

We take \( \nu = 0.01 \), the forcing function \( f \) is calculated from the continuous NSE, \( \nu \), and the solution, and the initial velocity is taken to be \( u_0 = u(x, y, 0) \).

On the domain \( \Omega = (0, 1)^2 \), we compute on a uniform mesh with Taylor-Hood elements using \( h = \frac{1}{32} \), time step size \( \Delta t = 0.01 \), and with varying values of the nudging parameter \( \mu \). For simplicity we take \( \gamma = 0 \), since the grad-div stabilization has little effect in this test problem. The interpolation operator \( I_H \) is chosen to be the nodal interpolant onto constant functions on the same mesh used for velocity and pressure, and the initial condition for the DA algorithm is zero.

Results of the tests are shown in figure 3.1, for \( \mu \) ranging from \( 10^{-5} \) to \( 10^7 \), by plotting the \( L^2 \) difference between the DA computed solution and the true solution versus time.

We observe convergence up to about \( 10^{-4} \), which is the level of the discretization error for the chosen discretization. We observe that for larger choices of \( \mu \), convergence to the true solution (up to discretization error) is much faster.

Table 3.1 gives the convergence rates of the solutions to Algorithm 3.1.1 to the true solution; error is calculated using the \( L^2 \) norm. For these calculations, we take \( \mu = 10 \) and \( \gamma = 1.0 \) and run out to \( t = 4.0 \). When observing the spatial convergence rates,
Figure 3.1: Shown above are log-linear plots of convergence of the DA computed solutions to the true solution with increasing time $t$, for varying choices of the nudging parameter $\mu$. On the left is convergence for $\mu \leq 10$, and on the right is convergence for $\mu \geq 10$.

we fix $\Delta t = 0.001$, while for the temporal error, $h$ is fixed at $1/64$. We observe second order convergence in each case, which is consistent with the analysis. Note that when calculating temporal error, the spatial error becomes dominate when $\Delta t = 1/32$, we do not see the rate drop below 2 when $\Delta t = 1/32$ and $h = 1/128$.

<table>
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<th>Rate</th>
<th>$\Delta t$</th>
<th>Error</th>
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<table>
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<th>Error</th>
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<td>2.001324</td>
</tr>
</tbody>
</table>

Table 3.1: Convergence rates of Algorithm 3.1.1 to the true solution as we cut mesh size $h$, timestep $\Delta t$, then both.

3.2.2 Numerical Experiment 2: The no-flow test and pressure-robustness

For our second test, we show how pressure robustness of the discretization can have a dramatic impact on the DA solution. The test problem we consider is the so-called
‘no-flow test’, where the forcing function of the NSE is given by \( Ra(0, y)^T \), where \( Ra \) is a constant (the Rayleigh number), and with \( Pr \) denoting the Prandtl number:

\[
\frac{1}{Pr} (u_t + u \cdot \nabla u) + \nabla p - \Delta u = Ra(0, y)^T, \quad (3.10)
\]

\[
\nabla \cdot u = 0, \quad (3.11)
\]

\[
u|_{\partial \Omega} = 0. \quad (3.12)
\]

This test problem corresponds to the physical situations of temperature driven flow (i.e. the Boussinesq system), with the temperature \( \theta \) profile specified to be stratified, i.e. \( f = Ra\theta e_2 \) with \( \theta = y \). Linear stratification is a natural steady state temperature profile.

Since the forcing is potential, the solution to the system (3.10)-(3.12) with \( u_0 = 0 \) initial condition is given by

\[
u = 0, \quad p = \frac{Ra}{2} y^2,
\]

for any \( Pr > 0 \), hence the name no-flow.

We consider now the second order DA algorithm, Algorithm 3.1.1, applied to the no-flow test with \( Pr = 1 \) and \( Ra = 10^5 \) (although this may seem like a large choice of a constant, for Boussinesq problems of practical interest, this choice of \( Ra \) is actually quite small). We use both Scott-Vogelius (SV) and Taylor-Hood (TH) elements on a barycenter refined uniform discretization of the unit square with \( h = \frac{1}{32} \), \( I_H \) to be the nodal interpolant in \( X_h \), and nudging parameter \( \mu = 0.1 \). With TH elements, we use \( \gamma = 0, 1, 10 \). The time step size is chosen to be \( \Delta t = 0.025 \), and solutions are computed up to end time \( T = 0.5 \), using the \( X_h \) interpolant of \( (x \cos y, -\sin y)^T \) as the initial condition \( v_0^h \).
Results of the simulations are shown in figure 3.2, as $L^2$ error versus time, and we observe a dramatic difference between solutions from the two element choices. For TH elements, the results are poor due to the large pressure, which adversely affects the velocity error, even using a very accurate interpolant. With $\gamma = 10$, the TH solution is a little better, however, it is still on the order of $10^{-1}$ accuracy, which is not good. The SV solution, on the other hand, is excellent. Its error decays rapidly in time until it reaches a level around $10^{-8}$ and stays there. Thus we observe here that in DA algorithms, element choice can be critical for finding good results in Boussinesq type simulations.

3.2.3 Numerical Experiment 3: 2D channel flow past a cylinder

For our last experiment, we consider Algorithm 3.1.1 applied to the common benchmark problem of 2D channel flow past a cylinder with Reynolds number 100 [90]. There is no external forcing, the kinematic viscosity is taken to be $\nu = 0.001$, and no-slip boundary conditions are prescribed for the walls and the cylinder.

Since we do not have access to a true solution for this problem, we instead use a
computed solution. It is obtained using the same BDF2-IMEX-FEM scheme as in Algorithm 3.1.1 but without nudging (i.e. $\mu = 0$), using $(P_2, P_1^{\text{disc}})$ SV elements on a barycenter refined Delaunay mesh that provides 8,658 elements and 60,994 total degrees of freedom, a time step of $\Delta t = 0.002$, and with the simulation starting from rest ($u_h^0 = u_h^{-1} = 0$). We will refer to this solution as the DNS solution. Lift and drag calculations were performed for the computed solution and compared to the literature [90, 76], which verified the accuracy of the DNS.

For the DA algorithm, we start from $v_h^1 = v_h^0 = 0$, use the same spatial and temporal discretization parameters as the DNS, and start assimilation with the $t = 5$ DNS solution (i.e., time 0 for DA corresponds to $t = 5$ for the DNS). We define $I_H$ to be the $L^2$ projection onto constant functions on coarser meshes, and we compute with 3 different coarse meshes: the Delaunay mesh without the barycenter refinement which provided 2,886 elements, and further coarsening to a 181 element mesh and a 15 element mesh. The average mesh width for these meshes is $H = 0.012, 0.046, 0.162$. The simulation is run on $[0,5]$ (so the corresponding times for the DNS would be $[5,10]$), with varying $\mu$ for each case of $I_H$.

![Figure 3.3](image)

Figure 3.3: Shown above is the $L^2$ difference between the DA and DNS solutions versus time, for varying $\mu$ and $H$.

Results are shown in Figures 3.3-3.5. Figure 3.3 shows the $L^2$ error with time in each simulation. We observe that for each $H$, if $\mu \leq 1$ the DA solution does not sufficiently
converge to the DNS solution by $t=5$, and does not show signs of converging in any time soon after. For $\mu \geq 10$, convergence of the AOT-DA solution to the DNS solution is observed for each $H$. However, we also observe the AOT-DA solution still converges to the DNS solution even for very large $\mu$, in fact there seems to be no negative impact on the convergence when taking $\mu = 10^8$ in the simulations. This does not contradict our theory, which guarantees convergence under the sufficient condition $C(data, u) < \mu < \frac{2\nu}{C^2H^2}$, but does suggest an alternative convergence analysis may be possible for $\mu$ outside this range.

Figure 3.4 shows convergence of the lift and drag coefficients, for the simulation using $H = 0.012$ and $\mu = 10$. The lift coefficient converges fairly rapidly, with the DA and DNS plots matching closely by $t=1.5$. The drag coefficients are not in synch until about $t=3$. For this same simulation, we also show the convergence of the DA solution to the DNS solution in the speed contour plots in Figure 3.5. Here, at $t=0$ there is a major difference, since the DA simulation starts from rest. The accuracy of DA is seen to increase by $t=0.5$ and further by $t=1$, and finally by $t=2$ there is only very slight differences observable between DA and DNS plots. By $t=5$, there is no visual difference between DA and DNS solutions.
Figure 3.5: Contour plots of DA and DNS velocity magnitudes at times 0, 0.5, 1, 2, and 5.
Chapter 4

Continuous DA applied to a velocity-vorticity formulation of NSE

In this chapter we consider continuous DA applied to a velocity-vorticity (VV) formulations of the NSE. In a first order linear scheme, we prove optimal $L^2$ and $H^1$ convergence rates for both velocity and vorticity, provided some restrictions on the coarse mesh width and nudging parameter.
4.1 Analysis of the VV Formulation

Our analysis of DA-VV schemes is restricted to 2D, where the VV scheme takes the form

\[
\begin{align*}
    u_t - \nu \Delta u + w \times u + \nabla p &= f, \\
    \nabla \cdot u &= 0, \\
    w_t - \nu \Delta w + (u \cdot \nabla w) &= \text{rot } f,
\end{align*}
\]

with \( w = (u_2)_x - (u_1)_y \) representing the scalar vorticity, \( \text{rot } f := (f_2)_x - (f_1)_y \), and \( w \times u := (-u_2w, u_1w)^T \).

In this section, we propose a first order discrete scheme for (4.1)-(4.3), where the spatial discretization is the finite element method. We show that the discrete schemes converge to the true NSE solution for both velocity and vorticity variables.

4.1.1 Additional notation & preliminaries

We use the following velocity, pressure, and (scalar) vorticity spaces in this chapter,

\[
\begin{align*}
    X &= (H^1_{\#}(\Omega))^2 = \left\{ v \in (H^1(\Omega))^2 \mid v \text{ periodic on } \Omega, \int_{\Omega} v dx = 0 \right\}, \\
    Q &= L^2_{\#}(\Omega) = \left\{ q \in L^2(\Omega) \mid q \text{ periodic on } \Omega, \int_{\Omega} q dx = 0 \right\}, \\
    W &= H^1_{\#}(\Omega) = \left\{ w \in H^1(\Omega) \mid w \text{ periodic on } \Omega, \int_{\Omega} w dx = 0 \right\}.
\end{align*}
\]
For the finite element discretization, we let $\tau_h$ be a regular, conforming triangulation of $\Omega$, and set

$$(X_h, Q_h, W_h) := (P_k(\tau_h) \cap X, P_{k-1}(\tau_h) \cap Q, P_k(\tau_h) \cap W).$$

The discretely divergence free subspace is again defined by

$$V_h := \{ v_h \in X_h, (\nabla \cdot v_h, q_h) = 0 \ \forall q_h \in Q_h \}.$$ 

The discrete Laplace operator is defined as: For $u \in H^1(\Omega)$, find $\Delta_h u \in X_h$ such that for all $v_h \in X_h$,

$$(\Delta_h u, v_h) = -(\nabla u, \nabla v_h).$$

The Stokes operator is defined as: Find $A_h u \in V_h$ such that for all $v_h \in V_h$,

$$(A_h u, v_h) = -(\nabla u, \nabla v_h).$$

Poincaré’s inequality leads to the following bounds:

$$\| \nabla \phi_h \| \leq C \| A_h \phi_h \| \quad \forall \phi_h \in V_h,$$

$$\| \nabla \psi_h \| \leq C \| \Delta_h \psi_h \| \quad \forall \psi_h \in W_h.$$ 

The discrete Agmon inequalities are given by

$$\| \phi_h \|_{L^\infty} \leq C \| \phi_h \|^{1/2} \| A_h \phi_h \|^{1/2} \quad \forall \phi_h \in V_h,$$

$$\| \psi_h \|_{L^\infty} \leq C \| \psi_h \|^{1/2} \| \Delta_h \psi_h \|^{1/2} \quad \forall \psi_h \in W_h.$$
We define the rot operator of a 2D vector, \( f = (f_1, f_2) \) as the \( z \)-component of the usual curl operator applied to \( (f_1, f_2, 0) \):

\[
\text{rot } f := (\nabla \times (f_1, f_2, 0)) \cdot e_3 = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.
\]

### 4.1.2 First order linear scheme

Define \( w^n_h := \text{rot } v^n_h \), let \( u^n \) denote the true NSE velocity solution at time \( t^n \), and set \( \omega^n + \text{rot } u^n \) denote the vorticity. Then the decoupled VV formulation of the fully discrete data assimilation algorithm using a backward Euler approximation to the time derivative reads:

**Algorithm 4.1.1.** Given a forcing \( f \), find \((v^{n+1}_h, w^{n+1}_h, q^{n+1}_h) \in (X_h, W_h, Q_h)\) for \( n = 0, 1, 2, ...\), satisfying

\[
\begin{align*}
\frac{1}{\Delta t} (v^{n+1}_h - v^n_h, \chi_h) + (w^n_h \times v^{n+1}_h, \chi_h) &- (q^{n+1}_h, \nabla \cdot \chi_h) + \nu (\nabla v^{n+1}_h, \nabla \chi_h) \\
&+ \mu_1 (I_H(v^{n+1}_h - u^{n+1}), I_H \chi_h) = (f^{n+1}, \chi_h), \quad (4.4) \\
(\nabla \cdot v^{n+1}_h, r_h) &= 0, \quad (4.5)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\Delta t} (w^{n+1}_h - w^n_h, \psi_h) + b(v^{n+1}_h, w^{n+1}_h, \psi_h) &+ \nu (\nabla w^{n+1}_h, \nabla \psi_h) \\
&+ \mu_2 (I_H(w^{n+1}_h - \text{rot } u^{n+1}), I_H \psi_h) = (\text{rot } f^{n+1}, \psi_h), \quad (4.6)
\end{align*}
\]

for all \((\chi_h, \psi_h, r_h) \in (X_h, W_h, Q_h)\), with \( v^0 \in X \) and \( I_H(u^{n+1}), I_H(\text{rot } u^{n+1}) \) given.

We first show that solutions to algorithm 4.1.1 are long time stable.

**Lemma 4.** Let the forcing be such that \( f, \text{rot } f \in L^\infty(0, \infty; L^2) \), and the true solution \( u \in L^\infty(0, \infty; L^2) \). Then solutions to algorithm 4.1.1 are unconditionally long time
\[ \| v^n_h \|^2 \leq (1 + \lambda \Delta t)^{-n} \| v^0_h \|^2 + \frac{C}{\lambda} \left( \nu^{-1} \Delta t \| f \|_{L^\infty(0,\infty;L^2)}^2 + \mu_1 \Delta t \| u \|_{L^\infty(0,\infty;L^2)}^2 \right), \]

\[ \| w^n_h \|^2 \leq (1 + \lambda \Delta t)^{-n} \| w^0_h \|^2 + \frac{C}{\lambda} \left( \nu^{-1} \Delta t \| \text{rot } f \|_{L^\infty(0,\infty;L^2)}^2 + \mu_2 \Delta t \| \text{rot } u \|_{L^\infty(0,\infty;L^2)}^2 \right), \]

where \( \lambda := \nu C_p^{-2} > 0 \).

**Proof.** Starting with the velocity equation, choose \( \chi_h = v^{n+1}_h \) in equation (4.4), which vanishes the pressure and nonlinear terms, yielding

\[
\frac{1}{2\Delta t} \left( \| v^{n+1}_h \|^2 - \| v^n_h \|^2 + \| v^{n+1}_h - v^n_h \|^2 \right) + \nu \| \nabla v^{n+1}_h \|^2 + \mu_1 \| I_H v^{n+1}_h \|^2
\]

\[ = (f^{n+1}, v^{n+1}_h) + \mu_1 (I_H u^{n+1}, I_H v^{n+1}_h) \]

\[ \leq C \nu^{-1} \| f \|_{L^\infty(0,\infty;L^2)}^2 + \frac{\nu}{2} \| \nabla v^{n+1}_h \|^2 + C \mu_1 \| u^{n+1} \|^2 + \frac{\mu_1}{2} \| I_H v^{n+1}_h \|^2, \tag{4.7} \]

where we have applied Cauchy-Schwarz, Poincaré, and Young’s inequalities to bound the right hand side terms. After reducing (4.7), dropping the non-negative terms \( \| v^{n+1}_h - v^n_h \|^2 \) and \( \frac{\mu_1}{2} \| I_H v^{n+1}_h \|^2 \) from the left hand side, and multiplying by \( 2\Delta t \), we obtain

\[ \| v^{n+1}_h \|^2 + \nu \Delta t \| \nabla v^{n+1}_h \| \leq \| v^n_h \|^2 + C \nu^{-1} \Delta t \| f \|_{L^\infty(0,\infty;L^2)}^2 + C \mu_1 \Delta t \| u^{n+1} \|^2. \]

Lastly, take \( \lambda = \nu C_p^{-2} \) to get

\[ (1 + \lambda \Delta t) \| v^{n+1}_h \|^2 \leq \| v^n_h \|^2 + C \nu^{-1} \Delta t \| f \|_{L^\infty(0,\infty;L^2)}^2 + C \mu_1 \Delta t \| u^{n+1} \|^2. \]

Applying lemma 2 completes the proof for the velocity equation. The result for the vorticity equation follows equivalent arguments after choosing \( \psi_h = w^{n+1}_h \) in (4.6). ■
We now consider the difference between the solutions of (4.4) - (4.6) to the NSE solution. We show that the difference between the NSE solution and the algorithm solution converge up to a $\Delta t + h^k$ dependent constant, independent of the initial condition, provided some assumptions on data hold.

**Theorem 5.** Suppose true solutions $u \in L^\infty(0, \infty; X)$, $\omega \in L^\infty(0, \infty; W)$ and $u_{tt}, \omega_{tt} \in L^\infty(0, \infty; L^2(\Omega))$, the time step $\Delta t$ is sufficiently small, and that $\mu_1, \mu_2$ satisfy

$$C\nu^{-1} \|\eta^{n+1}\|_{L^\infty}^2 \leq \mu_1 \leq \frac{C\nu}{H^2},$$

$$C\nu^{-1}(\|\eta_v^{n+1}\|^2 + \|u^{n+1}\|^2_{L^3} + \|\nabla\eta_w^{n+1}\|^2_{L^\infty} + \|\eta_w\|_{L^\infty}^2 + \|\nabla\omega^{n+1}\|^2) \leq \mu_2 \leq \frac{C\nu}{H^2},$$

where $H$ is chosen sufficiently small so that these inequalities hold. Then for any time $t_n$, $n = 0, 1, 2, ...$, we have

$$\|v_h^n - u^n\|^2 + \|w_h^n - \text{rot } u^n\|^2 \leq (\|v_h^0 - u^0\|^2 + \|w_h^0 - \text{rot } u^0\|^2)(1 + \lambda \Delta t)^{-n} + C(\Delta t^2 + h^{2k}),$$

with $C$ independent of $\Delta t$, $H$, and $h$, and $\lambda > 0$.

**Proof.** First, the true NSE solution satisfies the VV system

$$\frac{1}{\Delta t}(u^{n+1} - u^n) + \omega^n \times u^{n+1} + \nabla P^{n+1} - \nu \Delta u^{n+1} = f^{n+1} - \Delta t u_{tt}(t^*)$$

$$+ (\omega^n - \omega^{n+1}) \times u^{n+1},$$

$$\nabla \cdot u^{n+1} = 0,$$

$$\frac{1}{\Delta t}(\omega^{n+1} - \omega^n) + u^{n+1} \cdot \nabla \omega^{n+1} - \nu \Delta \omega^{n+1} = \text{rot } f^{n+1} - \Delta t \omega_{tt}(t^{**}),$$

where $u^n$ is the velocity at time $t^n$, $P^n$ the pressure, $\omega^n := \text{rot } u^n$, and $t^*, t^{**} \in$
Note that from Taylor expansion, we can write \( \omega^n - \omega^{n+1} = -\Delta t \omega_t(s^*) \) for some \( s^* \in [t^n, t^{n+1}] \). Defining the difference between the velocity solutions as \( e_v^n := u^n - v^n_h \) and the vorticity difference as \( e_w^n := \omega^n - w^n_h \), we obtain the difference equations

\[
\frac{1}{\Delta t} (e_v^{n+1} - e_v^n, \chi_h) + \nu(\nabla e_v^{n+1}, \nabla \chi_h) + \mu_1(I_H e_v^{n+1}, I_H \chi_h) = -\Delta t (u_{tt}(t^*), \chi_h) - \Delta t (\omega_t(s^*) \times u^{n+1}, \chi_h) - (e_w^n \times v^{n+1}_h, \chi_h) - (\omega^n \times e_v^{n+1}, \chi_h) - (P^{n+1}, \nabla \cdot \chi_h),
\]

and

\[
\frac{1}{\Delta t} (e_w^{n+1} - e_w^n, \psi_h) + \nu(\nabla e_w^{n+1}, \nabla \psi_h) + \mu_2(I_H e_w^{n+1}, I_H \psi_h) = -\Delta t (\omega_{tt}(t^{**}), \psi_h) - b(e_w^{n+1}, w^{n+1}_h, \psi_h) - b(u^{n+1}, e_w^{n+1}, \psi_h).
\]

We now decompose the errors into a term that lies in the discrete space \( V_h \) and one outside the space. To do so, add and subtract the \( L^2 \) projection of \( u^n \) onto \( V_h \), denoted \( s^n_h \), to \( e_v^n \) and let \( \eta^n_v := s^n_h - u^n \), \( \phi^n_{h,v} := v^n_h - s^n_h \). Then \( e_v^n = \phi^n_{h,v} + \eta^n_v \) and \( \phi^n_{h,v} \in V_h \).

In a similar manner, this time taking the \( L^2 \) projection of \( \text{rot} \ u^n \) onto \( V_h \), we obtain \( e_w^n = \phi^n_{h,w} + \eta^n_w \) with \( \phi^n_{h,w} \in V_h \). Now choose \( \chi_h = \phi^{n+1}_{h,v} \) and \( \psi_h = \phi^{n+1}_{h,w} \). Then the
difference equations become

\[
\frac{1}{2\Delta t} \left[ \| \phi_{h,v}^{n+1} \|^2 - \| \phi_{h,v}^n \|^2 + \| \phi_{h,v}^n - \phi_{h,v}^{n+1} \|^2 \right] + \nu \| \nabla \phi_{h,v}^{n+1} \|^2 + \mu_1 \| \phi_{h,v}^{n+1} \|^2 \\
= -\Delta t(u_t(t^*), \phi_{h,v}^{n+1}) - \Delta t(\omega(t^*) \times u_{h,v}^{n+1}, \phi_{h,v}^{n+1}) - (e^t_{h,v} \times v_{h,v}^{n+1}, \phi_{h,v}^{n+1}) \\
- (\omega^n \times \eta_{h,v}^{n+1}, \phi_{h,v}^{n+1}) - (P^{n+1}, \nabla \cdot \phi_{h,v}^{n+1}) - \nu(\nabla \eta_{h,v}^{n+1}, \nabla \phi_{h,v}^{n+1}) \\
- 2\mu_1(I_H(\phi_{h,v}^{n+1}) - \phi_{h,v}^{n+1}, \phi_{h,v}^{n+1}) - \mu_1 \| I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1} \|^2 \\
- \mu_1(I_{H_p}^{n+1}, I_{H_p}^{n+1}), \quad (4.8)
\]

and

\[
\frac{1}{2\Delta t} \left[ \| \phi_{h,w}^{n+1} \|^2 - \| \phi_{h,w}^n \|^2 + \| \phi_{h,w}^n - \phi_{h,w}^{n+1} \|^2 \right] + \nu \| \nabla \phi_{h,w}^{n+1} \|^2 + \mu_2 \| \phi_{h,w}^{n+1} \|^2 \\
= -\Delta t(\omega_t(t^*), \phi_{h,w}^{n+1}) - b(e^t_{h,w} + u_{h,w}^{n+1}, \phi_{h,w}^{n+1}) - b(u_{h,w}^{n+1}, \eta_{h,w}^{n+1}, \phi_{h,w}^{n+1}) \\
- \nu(\nabla \eta_{h,w}^{n+1}, \nabla \phi_{h,w}^{n+1}) - 2\mu_2(I_H(\phi_{h,w}^{n+1}) - \phi_{h,w}^{n+1}, \phi_{h,w}^{n+1}) - \mu_2 \| I_H \phi_{h,w}^{n+1} - \phi_{h,w}^{n+1} \|^2 \\
- \mu_2(I_{H_p}^{n+1}, I_{H_p}^{n+1}), \quad (4.9)
\]

In the interpolation terms found in the velocity and vorticity equations above, we have added and subtracted \( \phi_{h,v}^{n+1} \) (respectively \( \phi_{h,w}^{n+1} \)) to write them in the forms found above. To see this,

\[
\mu_1(I_{H_p}^{n+1}, I_{H_p}^{n+1}) = \mu_1(I_{H_p}^{n+1}, I_{H_p}^{n+1}) + \mu_1(I_{H_p}^{n+1}, I_{H_p}^{n+1}) \\
= \mu_1(I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}, I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}) + \mu_1(I_{H_p}^{n+1}, I_{H_p}^{n+1}) \\
= \mu_1 \| \phi_{h,v}^{n+1} \|^2 + \mu_1(I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}) + \mu_1(I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}) \\
+ \mu_1(I_{H_p}^{n+1} + \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}) + \mu_1(I_{H_p}^{n+1} + \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}) \\
= \mu_1 \| \phi_{h,v}^{n+1} \|^2 + 2\mu_1(I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}) + \mu_1(I_{H_p}^{n+1} - \phi_{h,v}^{n+1} + \phi_{h,v}^{n+1}).
\]
We can now bound the right hand sides of the above equations. Starting with the velocity difference equation (4.8), the first term on the right hand side will be bounded using Cauchy-Schwarz and Young’s inequalities.

\[ \Delta t(u_{tt}(t^*), \phi_{h,v}^{n+1}) \leq \Delta t\|u_{tt}\|_{L^\infty(0,\infty;L^2(\Omega))}\|\phi_{h,v}^{n+1}\| \]
\[ \leq C\Delta t^2\mu_1^{-1}\|u_{tt}\|_{L^\infty(0,\infty;L^2(\Omega))}^2 + \frac{H_1}{16}\|\phi_{h,v}^{n+1}\|^2. \]

For the second time derivative term, we will apply H"older’s and Young’s inequalities to obtain

\[ \Delta t(\omega_{tt}(s^*) \times u^{n+1}, \phi_{h,v}^{n+1}) \leq C\Delta t\|\omega_{tt}\|_{L^\infty(0,\infty;L^2(\Omega))}\|u^{n+1}\|_{L^\infty}\|\phi_{h,v}^{n+1}\| \]
\[ \leq C\mu_1^{-1}\Delta t^2\|\omega_{tt}\|_{L^\infty(0,\infty;L^2(\Omega))}^2\|u^{n+1}\|_{L^\infty}^2 + \frac{H_1}{16}\|\phi_{h,v}^{n+1}\|^2. \]

In the next term, we will add and subtract \(e_{w}^{n+1}\) in the first component, and \(u^{n+1}\) in the second component to obtain

\[ (\epsilon_{w}^{n} \times v_{h}^{n+1}, \phi_{h,v}^{n+1}) = ((\epsilon_{w}^{n} - \epsilon_{w}^{n+1}) \times \epsilon_{v}^{n+1}, \phi_{h,v}^{n+1}) + (\epsilon_{w}^{n+1} \times \epsilon_{v}^{n+1}, \phi_{h,v}^{n+1}) \]
\[ + ((\epsilon_{w}^{n} - \epsilon_{w}^{n+1}) \times u^{n+1}, \phi_{h,v}^{n+1}) + (\epsilon_{w}^{n+1} \times u^{n+1}, \phi_{h,v}^{n+1}). \quad (4.10) \]

Now, for the first resulting term, we will decompose the vorticity error term then apply Hölder’s and Young’s inequalities to get

\[ ((\epsilon_{w}^{n} - \epsilon_{w}^{n+1}) \times v_{h}^{n+1}, \phi_{h,v}^{n+1}) \leq C\|\epsilon_{h,w}^{n} - \epsilon_{h,w}^{n+1}\|_{L^4}\|\eta_{h}^{n+1}\|_{L^4}\|\phi_{h,v}^{n+1}\|_{L^4} \]
\[ + C\|\eta_{h}^{n} - \eta_{h}^{n+1}\|_{L^\infty}\|\eta_{h}^{n} - \eta_{h}^{n+1}\|_{L^\infty}\|\phi_{h,v}^{n+1}\| \]
\[ \leq C\nu^{-1}\|\nabla \epsilon_{h,w}^{n+1}\|^2\|\phi_{h,v}^{n+1}\|^2 + \frac{\nu}{32}\|\nabla \phi_{h,v}^{n+1}\|^2 \]
\[ + C\mu_1^{-1}\|\eta_{h}^{n} - \eta_{h}^{n+1}\|_{L^\infty}\|\eta_{h}^{n} - \eta_{h}^{n+1}\|_{L^\infty}^2 + \frac{H_1}{32}\|\phi_{h,v}^{n+1}\|^2. \]
We will apply similar arguments to the second resulting term to get

\[
(e^{n+1}_w \times e^{n+1}_v, \phi^{n+1}_{h,v}) \leq C\|\phi_{h,w}^{n+1}\|\|\eta_{v}^{n+1}\|_{L^3}\|\phi_{h,v}^{n+1}\|_{L^6} + C\|\eta_{w}^{n+1}\|\|\eta_{v}^{n+1}\|_{L^\infty}\|\phi_{h,v}^{n+1}\|
\]

\[
\leq C\nu^{-1}\|\phi_{h,w}^{n+1}\|^2\|\eta_{v}^{n+1}\|_{L^3}^2 + \frac{\nu}{32}\|\nabla \phi_{h,v}^{n+1}\|^2
\]

\[
+ C\mu_1^{-1}\|\eta_{w}^{n+1}\|^2\|\eta_{v}^{n+1}\|_{L^\infty}^2 + \frac{\mu_1}{32}\|\phi_{h,v}^{n+1}\|^2.
\]  

\[\text{(4.11)}\]

For the third resulting term, we will decompose the vorticity error term in the first component, then apply Hölder’s inequality to get

\[
((e^n_w - e^{n+1}_w) \times u^{n+1}, \phi^{n+1}_{h,v}) \leq C\|\phi_{h,w}^n - \phi_{h,w}^{n+1}\|\|u^{n+1}\|_{L^3}\|\phi_{h,v}^{n+1}\|_{L^6}
\]

\[
+ C\|\eta_{w}^n - \eta_{w}^{n+1}\|\|u^{n+1}\|_{L^\infty}\|\phi_{h,v}^{n+1}\|
\]

\[
\leq C\nu^{-1}\|u^{n+1}\|_{L^3}^2\|\phi_{h,w}^n - \phi_{h,w}^{n+1}\|^2 + \frac{\nu}{32}\|\nabla \phi_{h,v}^{n+1}\|^2
\]

\[
+ C\mu_1^{-1}\|\eta_{w}^n - \eta_{w}^{n+1}\|^2\|u^{n+1}\|_{L^\infty}^2 + \frac{\mu_1}{32}\|\phi_{h,v}^{n+1}\|^2.
\]

\[\text{(4.12)}\]

Lastly, we have

\[
(e^{n+1}_w \times u^{n+1}, \phi^{n+1}_{h,v}) \leq \|\phi_{h,w}^{n+1}\|\|u^{n+1}\|_{L^3}\|\phi_{h,v}^{n+1}\|_{L^6} + \|\eta_{w}^{n+1}\|\|u^{n+1}\|_{L^\infty}\|\phi_{h,v}^{n+1}\|
\]

\[
\leq C\nu^{-1}\|\phi_{h,w}^{n+1}\|^2\|u^{n+1}\|_{L^3}^2 + \frac{\nu}{32}\|\nabla \phi_{h,v}^{n+1}\|^2
\]

\[
+ C\mu_1^{-1}\|\eta_{w}^{n+1}\|^2\|u^{n+1}\|_{L^\infty}^2 + \frac{\mu_1}{32}\|\phi_{h,v}^{n+1}\|^2.
\]  

\[\text{(4.12)}\]
Therefore, (4.10) is bounded as
\[
\left(\epsilon_w^n \times \nu_h^{n+1}, \phi_h^{n+1}\right) \leq \frac{\nu}{8} \left\| \nabla \phi_h^{n+1} \right\|^2 + \frac{\mu_1}{8} \left\| \phi_h^{n+1} \right\|^2 + C\nu^{-1} \left\| \nabla \eta_v^{n+1} \right\|^2 \left\| \phi_h^{n+1} - \phi_h^n \right\|^2 \\
+ C\mu_1^{-1} \left\| \eta_w^n - \eta_w^{n+1} \right\|^2 \left\| \eta_v^{n+1} \right\|^2 + C\nu^{-1} \left\| \eta_v^{n+1} \right\|^2 \left\| \phi_h^{n+1} \right\|^2 \\
+ C\mu_1^{-1} \left\| \eta_v^{n+1} \right\|^2 \left\| \eta_v^{n+1} \right\|^2 \left\| \phi_h^{n+1} \right\|^2 + C\nu^{-1} \left\| u^n + 1 \right\|^2 \left\| \phi_h^{n+1} \right\|^2 \\
+ C\mu_1^{-1} \left\| \eta_v^{n+1} \right\|^2 \left\| u^n + 1 \right\|^2 \left\| \phi_h^{n+1} \right\|^2 + C\nu^{-1} \left\| u^n + 1 \right\|^2 \left\| \phi_h^{n+1} \right\|^2 \\
+ C\mu_1^{-1} \left\| \eta_v^{n+1} \right\|^2 \left\| u^n + 1 \right\|^2 \left\| \phi_h^{n+1} \right\|^2.
\]

Moving on to the next nonlinear term, we will apply Hölder’s, Poincaré, and Young’s inequalities to obtain
\[
\left(\omega_v^{n+1} \times \eta_v^{n+1}, \phi_h^{n+1}\right) \leq \left\| \omega_v^{n+1} \right\| \left\| \eta_v^{n+1} \right\| \left\| \phi_h^{n+1} \right\| \\
\leq C\nu^{-1} \left\| \omega_v^{n+1} \right\|^2 \left\| \eta_v^{n+1} \right\|^2 + \frac{\nu}{8} \left\| \nabla \phi_h^{n+1} \right\|^2.
\]

The next two terms will be bounded using Cauchy-Schwarz and Young’s inequalities to get
\[
\left(P_{n+1}, \nabla \cdot \phi_h^{n+1}\right) \leq \left\| P_{n+1} - r_h \right\| \left\| \nabla \phi_h^{n+1} \right\| \\
\leq C\nu^{-1} \left\| P_{n+1} - r_h \right\|^2 + \frac{\nu}{8} \left\| \nabla \phi_h^{n+1} \right\|^2,
\]
where \( r_h \in Q_h \) is chosen arbitrarily, and
\[
\nu(\nabla \eta_v^{n+1}, \nabla \phi_h^{n+1}) \leq C\nu \left\| \nabla \eta_v^{n+1} \right\|^2 + \frac{\nu}{8} \left\| \nabla \phi_h^{n+1} \right\|^2.
\]

The first interpolation term will be bounded with Cauchy-Schwarz inequality and
(2.2) to obtain

\[ \mu_1(I_H(\phi_{h,v}^{n+1}) - \phi_{h,v}^{n+1}, \phi_{h,v}^{n+1}) \leq \mu_1 \|I_H(\phi_{h,v}^{n+1}) - \phi_{h,v}^{n+1}\| \|\phi_{h,v}^{n+1}\| \]
\[ \leq \mu_1 C H \|\nabla \phi_{h,v}^{n+1}\| \|\phi_{h,v}^{n+1}\| \]
\[ \leq C \mu_1 H^2 \|\nabla \phi_{h,v}^{n+1}\|^2 + \frac{\mu_1}{32} \|\phi_{h,v}^{n+1}\|^2. \]

For the second term we apply inequality (2.3), yielding

\[ \mu_1 \|I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}\|^2 \leq \mu_1 H^2 \|\nabla \phi_{h,v}^{n+1}\|^2. \]

Finally, the last interpolation term will be bounded using the Cauchy-Schwarz and Young’s inequalities, as well as the bounds (2.2)-(2.3) to obtain

\[ \mu_1(I_H \eta_v^{n+1}, I_H \phi_{h,v}^{n+1}) = \mu_1(I_H \eta_v^{n+1}, I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}) + \mu_1(I_H \eta_v^{n+1}, \phi_{h,v}^{n+1}) \]
\[ \leq \mu_1 \|I_H \eta_v^{n+1}\| \left( \|I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}\| + \|\phi_{h,v}^{n+1}\| \right) \]
\[ \leq \mu_1 \|I_H \eta_v^{n+1}\| \left( C H \|\nabla \phi_{h,v}^{n+1}\| + \|\phi_{h,v}^{n+1}\|^2 \right) \]
\[ \leq C \mu_1 \|\eta_v^{n+1}\|^2 + \mu_1 H^2 \|\nabla \phi_{h,v}^{n+1}\|^2 + \frac{\mu_1}{32} \|\phi_{h,v}^{n+1}\|^2. \]

We can now move on to the vorticity equation. We decompose the velocity error in the first nonlinear term and add and subtract \(\omega^{n+1}\) in the second component, yielding

\[ b(\omega_v^{n+1}, \omega_w^{n+1}, \phi_{h,w}) = b(\omega_v^{n+1}, \omega_w^{n+1}, \phi_{h,w}) + b(\eta_v^{n+1}, \omega_w^{n+1}, \phi_{h,w}) + b(\phi_{h,v}^{n+1}, \omega^{n+1}, \phi_{h,w}^{n+1}) + b(\phi_{h,v}^{n+1}, \omega^{n+1}, \phi_{h,w}^{n+1}) + b(\eta_v^{n+1}, \omega^{n+1}, \phi_{h,w}^{n+1}). \]
For the first resulting term, we can apply Hölder’s and Young’s inequalities to get

\[
b(\phi^{n+1}_h, \eta^{n+1}_w, \phi^{n+1}_w) = (\phi^{n+1}_h \cdot \nabla \eta^{n+1}_w, \phi^{n+1}_w) + \frac{1}{2}((\nabla \cdot \phi^{n+1}_h) \eta^{n+1}_w, \phi^{n+1}_w) \\
\leq C||\phi^{n+1}_h||_\infty ||\nabla \eta^{n+1}_w||_\infty ||\phi^{n+1}_w||_\infty + ||\nabla \cdot \phi^{n+1}_h||_\infty ||\eta^{n+1}_w||_\infty ||\phi^{n+1}_w||_\infty \\
\leq C\nu^{-1}||\phi^{n+1}_h||_2^2 (||\nabla \eta^{n+1}_w||_\infty^2 + ||\eta^{n+1}_w||_\infty^2) + \frac{\nu}{16} ||\nabla \phi^{n+1}_h||_2^2. \quad (4.14)
\]

For the next term, we will apply Hölder’s and Young’s inequalities,

\[
b(\eta^{n+1}_v, e^{n+1}_w, \phi^{n+1}_w) = (\eta^{n+1}_v \cdot \nabla \eta^{n+1}_w, \phi^{n+1}_w) + \frac{1}{2}((\nabla \cdot \eta^{n+1}_v) \eta^{n+1}_w, \phi^{n+1}_w) \\
\leq C||\eta^{n+1}_v||_\infty ||\nabla \eta^{n+1}_w||_\infty ||\phi^{n+1}_w||_\infty + C||\nabla \cdot \eta^{n+1}_v||_\infty ||\eta^{n+1}_w||_\infty ||\phi^{n+1}_w||_\infty \\
\leq C\mu^{-1}_2 (||\eta^{n+1}_v||_L^2 ||\nabla \eta^{n+1}_w||_2^2 + ||\nabla \eta^{n+1}_w||_2^2 ||\eta^{n+1}_w||_L^2) + \frac{\mu_2}{16} ||\phi^{n+1}_w||_2^2. \quad (4.15)
\]

The third resulting term of (4.13) is bounded using the same inequalities,

\[
b(\phi^{n+1}_h, \omega^{n+1}_w, \phi^{n+1}_w) = \frac{1}{2} (\phi^{n+1}_h \cdot \nabla \omega^{n+1}_w, \phi^{n+1}_w) - \frac{1}{2} (\phi^{n+1}_h \cdot \nabla \phi^{n+1}_w, \omega^{n+1}_w) \\
\leq C||\phi^{n+1}_h||_L^6 ||\nabla \omega^{n+1}_w||_L^3 ||\phi^{n+1}_w||_L^3 + C||\phi^{n+1}_h||_L^6 ||\nabla \phi^{n+1}_w||_L^3 ||\omega^{n+1}_w||_L \\
\leq C\nu^{-1}||\phi^{n+1}_h||_2^2 ||\nabla \omega^{n+1}_w||_L^2 + \frac{\nu}{16} ||\nabla \phi^{n+1}_h||_2^2 + C\nu^{-1}||\phi^{n+1}_h||_2^2 ||\omega^{n+1}_w||_L^2. \quad (4.16)
\]

Finally, we apply similar arguments to the last resulting term to obtain

\[
b(\eta^{n+1}_v, \omega^{n+1}_w, \phi^{n+1}_w) = (\eta^{n+1}_v \cdot \nabla \omega^{n+1}_w, \phi^{n+1}_w) + \frac{1}{2}((\nabla \cdot \eta^{n+1}_v) \omega^{n+1}_w, \phi^{n+1}_w) \\
\leq C||\eta^{n+1}_v||_L^6 ||\nabla \omega^{n+1}_w||_L^3 ||\phi^{n+1}_w||_L^3 + C||\nabla \cdot \eta^{n+1}_v||_L^6 ||\omega^{n+1}_w||_L^3 ||\phi^{n+1}_w||_L \\
\leq C\mu^{-1}_2 (||\eta^{n+1}_v||_L^2 ||\nabla \omega^{n+1}_w||_L^2 + ||\nabla \eta^{n+1}_v||_L^2 ||\omega^{n+1}_w||_L^2) + \frac{\mu_2}{16} ||\phi^{n+1}_w||_2^2. \quad (4.17)
\]
Thus, (4.13) is bounded by

\begin{align*}
  b(e_v^{n+1}, w_h^{n+1}, \phi_{h,w}^{n+1}) &\leq \frac{\nu}{8} \|
  \nabla \phi_{h,v}^{n+1} \|^2 + \frac{\mu_2}{8} \|
  \phi_{h,w}^{n+1} \|^2 \\
  &+ C\nu^{-1} \|
  \phi_{h,w}^{n+1} \| \left( \|
  \nabla \eta_w^{n+1} \|_{L^\infty} + \|
  \eta_w^{n+1} \|_{L^\infty} \right) \\
  &+ C\mu_2^{-1} (\|
  \eta_v^{n+1} \|_{L^\infty} \|
  \nabla \eta_w^{n+1} \|^2 + \|
  \nabla \eta_v^{n+1} \| \|
  \eta_w^{n+1} \|_{L^\infty} \}
  \\
  &+ C\nu^{-1} \|
  \phi_{h,v}^{n+1} \| \|
  \nabla \omega^{n+1} \|^2 + C\nu^{-1} \|
  \phi_{h,v}^{n+1} \| \|
  \omega^{n+1} \|_{L^\infty} \\
  &+ C\mu_2^{-1} (\|
  \eta_v^{n+1} \| \|
  \nabla \omega^{n+1} \|^2_{L^\infty} + \|
  \nabla \eta_v^{n+1} \| \|
  \omega^{n+1} \|_{L^\infty} \right). \quad (4.18)
\end{align*}

For the next nonlinear term, we will apply Hölder’s and Young’s inequalities to get

\begin{align*}
  b(u_v^{n+1}, \eta_w^{n+1}, \phi_{h,w}^{n+1}) &\leq C \| u_v^{n+1} \|_{L^\infty} \|
  \nabla \eta_w^{n+1} \| \|
  \phi_{h,w}^{n+1} \| \\
  &\leq \frac{\mu_2}{16} \|
  \phi_{h,w}^{n+1} \| + C\mu_2^{-1} \|
  u_v^{n+1} \|_{L^\infty} \|
  \nabla \eta_w^{n+1} \|^2. \quad (4.19)
\end{align*}

Finally, the viscous and time derivative term on the left hand side, as well as interpolation terms are bounded using equivalent arguments as in the velocity equation. We can now replace the right hand sides with the computed bounds and add the two
equations together.

\[
\frac{1}{2\Delta t} \left[ \| \phi_{h,v}^{n+1} \|^2 + \| \phi_{h,w}^{n+1} \|^2 - \| \phi_{h,v}^n \|^2 - \| \phi_{h,w}^n \|^2 \right] + \frac{1}{2\Delta t} \| \phi_{h,v}^{n+1} - \phi_{h,v}^n \|^2 \\
+ \left( \frac{\nu}{2} - C\mu_1 \right) \| \nabla \phi_{h,v}^{n+1} \|^2 \\
+ \left( \frac{\nu}{2} - C\mu_2 \right) \| \nabla \phi_{h,w}^{n+1} \|^2 \\
+ \left( \frac{\mu_1}{2} - C\nu - 1 \right) \| \nabla \phi_{h,w}^{n+1} \|^2 \\
+ \left( \frac{\mu_2}{2} - C\nu - 1 \right) \| \phi_{h,w}^{n+1} \|^2 \\
\leq \frac{\nu}{\Delta t} \left( \| \phi_{h,v}^{n+1} \|^2 + \| \phi_{h,w}^{n+1} \|^2 \right) + \frac{1}{\Delta t} \| \phi_{h,v}^{n+1} - \phi_{h,v}^n \|^2 \\
+ \left( \frac{\mu_1}{2} - C\nu - 1 \right) \| \phi_{h,w}^{n+1} \|^2 \\
+ \left( \frac{\mu_2}{2} - C\nu - 1 \right) \| \phi_{h,w}^{n+1} \|^2.
\]

Note that we can drop the nonnegative terms on the left hand side with the \( \Delta t \) restriction of

\[
\frac{1}{2\Delta t} - C\nu - 1 \left( \| \phi_{h,v}^{n+1} \|^2 + \| \phi_{h,w}^{n+1} \|^2 \right) \geq 0.
\]

Provided \( \mu_1, \mu_2 \) are within the bounds stated in the theorem, we can set

\[
\lambda_1 := \left( \frac{\nu}{2} - C\mu_1 \right) \| \nabla \phi_{h,v}^{n+1} \|^2 + \frac{\mu_1}{2} - C\nu - 1 \| \phi_{h,w}^{n+1} \|^2,
\]
\[
\lambda_2 := \left( \frac{\nu}{2} - C\mu_2 \right) \| \nabla \phi_{h,w}^{n+1} \|^2 + \frac{\mu_2}{2} - C\nu - 1 \| \phi_{h,w}^{n+1} \|^2.
\]

and multiply through by \( 2\Delta t \) so that the above equation becomes

\[
(1 + 2\lambda_1 \Delta t) \| \phi_{h,v}^{n+1} \|^2 + (1 + 2\lambda_2 \Delta t) \| \phi_{h,w}^{n+1} \|^2 \leq C(\Delta t^3 + \Delta t h^{2k}) + \| \phi_{h,v}^n \|^2 + \| \phi_{h,w}^n \|^2.
\]
Take $\lambda = \min\{\lambda_1, \lambda_2\}$ to get the final result,

$$
\|\phi_{n,v}^{n+1}\|^2 + \|\phi_{h,w}^{n+1}\|^2 \leq (1 + \lambda \Delta t)^{-1} C (\Delta t^3 + \Delta t h^{2k}) + \|\phi_{h,v}^n\|^2 + \|\phi_{h,w}^n\|^2
$$

$$
\leq (\|\phi_{h,w}^n\|^2 + \|\phi_{h,v}^n\|^2) (1 + \lambda \Delta t)^{-n} + C (\Delta t^2 + h^{2k}).
$$

Triangle inequality completes the proof.

We will now show higher order convergence of Algorithm 4.1.1.

**Theorem 6.** Suppose $u, \in L^\infty(0, \infty; X)$, $\omega \in L^\infty(0, \infty; W)$ and $u_{tt}, \omega_{tt} \in L^\infty(0, \infty; L^2(\Omega))$ and that $H, \mu_1, \mu_2$ are chosen so that

$$
H^2 \leq \frac{C \nu}{\max\{\mu_1, \mu_2\}}.
$$

Then, provided $\Delta t$ is sufficiently small, for any time $t^n$, $n = 0, 1, 2, \ldots$, we have

$$
\|\nabla(v^n_h - u^n)\|^2 \leq \|\nabla(v^0_h - u^0)\|^2 (1 + \lambda_1 \Delta t)^{-n} + C (\Delta t^2 + h^{2k-2}),
$$

and

$$
\|\nabla(w^n_h - \text{rot } u^n)\|^2 \leq \|\nabla(w^0_h - \text{rot } u^0)\|^2 (1 + \lambda_2 \Delta t)^{-n} + C (\Delta t^2 + h^{2k}),
$$

with $C$ independent of $\Delta t$ and $h$, and

$$
\lambda_1 := C_P^{-2} (\nu - CH^2 \mu_1) + \mu_1 > 0,
$$

$$
\lambda_2 := C_P^{-2} (\nu - CH^2 \mu_2) + \mu_2 - C \nu^{-3} (\Delta t^2 + h^{2k}) - C \nu^{-1} \|u^{n+1}\|^2 > 0.
$$
Proof. Beginning the same way as Theorem 5, we will start with the difference equations and decompose the errors in an equivalent way. Now, we will choose 
\[ \chi_h = A_h \phi_{h,v}^{n+1} \] and \[ \psi_h = \Delta_h \phi_{h,w}^{n+1} \]. For the nudging terms, we will first add and subtract \( \phi_{h,v}^{n+1} \) (respectively, \( \phi_{h,w}^{n+1} \) in the vorticity equation) to the first component of the inner product, then \( A_h \phi_{h,v}^{n+1} \) (respectively \( \Delta_h \phi_{h,w}^{n+1} \)) to the second component. For the velocity equation, this yields

\[
(I_H \phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1}) = (\phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1}) + (I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1})
\]

\[= (\phi_{h,v}^{n+1}, A_h \phi_{h,v}^{n+1}) + (\phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1} - A_h \phi_{h,v}^{n+1})
\]

\[+ (I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1})
\]

\[= \| \nabla \phi_{h,v}^{n+1} \|^2 + (\phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1} - A_h \phi_{h,v}^{n+1})
\]

\[+ (I_H \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}, I_H A_h \phi_{h,v}^{n+1}),
\]

and similar arguments are used for the vorticity equation. The first term above will stay on the left hand side, while the other two are moved over to the right. Then the velocity equation becomes

\[
\frac{1}{2\Delta t} [\| \nabla \phi_{h,v}^{n+1} \|^2 - \| \nabla \phi_{h,v}^n \|^2 + \| \nabla (\phi_{h,v}^{n+1} - \phi_{h,v}^n) \|^2] + \nu \| A_h \phi_{h,v}^{n+1} \|^2 + \mu_1 \| \nabla \phi_{h,v}^{n+1} \|^2
\]

\[- \Delta t (u_{H}(t^r), A_h \phi_{h,v}^{n+1}) - \Delta t (\omega (t^r) \times u_{H}^{n+1}, A_h \phi_{h,v}^{n+1}) - (e_{w}^{n+1} \times v_{h}^{n+1}, A_h \phi_{h,v}^{n+1})
\]

\[- (\omega \phi_{h,v}^{n+1}, A_h \phi_{h,v}^{n+1}) - (\nabla P_{H}^{n+1}, A_h \phi_{h,v}^{n+1}) - \mu_1 (I_{H} \eta_{h}^{n+1}, I_{H}(A_h \phi_{h,v}^{n+1}))
\]

\[- \mu_1 (\phi_{h,v}^{n+1}, I_{H} A_h \phi_{h,v}^{n+1} - A_h \phi_{h,v}^{n+1}) - \mu_1 (I_{H} \phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}, I_{H} A_h \phi_{h,v}^{n+1})
\]

\[- \nu (A_h \eta_{h}^{n+1}, A_h \phi_{h,v}^{n+1}),
\]

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and the vorticity equation is

\[
\frac{1}{2\Delta t} \left[ \| \nabla \phi_{n+1} \|^2 - \| \nabla \phi_n \|^2 + \| \nabla (\phi_{n+1} - \phi_n) \|^2 \right] + \nu \left( \| \Delta_h \phi_{n+1} \|^2 + \mu_2 \| \nabla \phi_{n+1} \|^2 \right)
\]

\[
= -\Delta t \left( \omega_t(t^{**}) - \Delta_h \phi_{n+1} \right) - b(e_{n+1}^w, \Delta_h \phi_{n+1}^n) - b(u_{n+1}, \Delta_h \phi_{n+1}^n)
\]

\[
- \nu(\Delta_h \eta_{n+1}^w, \Delta_h \phi_{n+1}^n) - \mu_2(I_H \eta_{n+1}^w, I_H(\Delta_h \phi_{n+1}^n))
\]

\[
- \mu_2(\phi_{n+1}^w, I_H \Delta_h \phi_{n+1}^n - \Delta_h \phi_{n+1}^n) - \mu_2(I_H \phi_{n+1}^w - \phi_{n+1}^n, I_H \Delta_h \phi_{n+1}^n).
\]

We will now focus on the velocity result. We begin by bounding the terms on the right hand side. For the first term, we apply Cauchy-Schwarz and Young’s inequalities to get

\[
\Delta t(u_{tt}(t^{**}), A_h \phi_{n+1}^n) \leq C\nu^{-1}\Delta t^2 \| u_{tt} \|_{L^\infty(0,\infty;L^2(\Omega))}^2 + \frac{\nu}{8} \| A_h \phi_{n+1}^n \|^2.
\]

The next time derivative term will be bounded using Hölder’s and Young’s inequalities to get

\[
\Delta t(\omega_l(s^*) \times u_{n+1}, A_h \phi_{n+1}^n) \leq C\Delta t^2 \| \omega_l \|_{L^\infty(0,\infty;L^2(\Omega))} \| u_{n+1} \|_{L^\infty(0,\infty;L^2(\Omega))} \| A_h \phi_{n+1}^n \|
\]

\[
\leq C\Delta t^2 \nu^{-1} \| \omega_l \|_{L^\infty(0,\infty;L^2(\Omega))}^2 \| u_{n+1} \|_{L^\infty(0,\infty;L^2(\Omega))}^2 + \frac{\nu}{8} \| A_h \phi_{n+1}^n \|^2.
\]

For the next right hand side term, we will first add and subtract $u_{n+1}$ to $v_{n+1}^h$ to get $v_{n+1}^h = e_{n+1}^v + u_{n+1}$. Thus,

\[
(e_{n+1}^w \times v_{n+1}^h, A_h \phi_{n+1}^n)
\]

\[
= (e_{n+1}^w \times \eta_{n+1}^v, A_h \phi_{n+1}^n) + (e_{n+1}^w \times \phi_{n+1}^w, A_h \phi_{n+1}^n) + (e_{n+1}^w \times u_{n+1}, A_h \phi_{n+1}^n) \quad (4.20)
\]

We will now bound each term individually. For the first term, we will use Hölder’s
and Young’s inequalities to get

\[
(e_n^w \times \eta_{n+1}^v, A_h \phi_{h,v}^{n+1}) \leq \|e_n^w\| \|\eta_{n+1}^v\|_{L^\infty} \|A_h \phi_{h,v}^{n+1}\|
\]

\[
\leq C \nu^{-1} \|e_n^w\|^2 \|\eta_{n+1}^v\|^2_{L^\infty} + \frac{\nu}{32} \|A_h \phi_{h,v}^{n+1}\|^2.
\]

For the next term, we will also start by using Hölder’s inequality on the second right hand side term, then we will apply 2D Agmon’s inequality as well as generalized Young’s inequality:

\[
(e_n^w \times \phi_{h,v}^{n+1}, A_h \phi_{h,v}^{n+1}) \leq \|e_n^w\| \|\phi_{h,v}^{n+1}\|_{L^\infty} \|A_h \phi_{h,v}^{n+1}\|
\]

\[
\leq \|e_n^w\| \|\phi_{h,v}^{n+1}\|^{1/2} \|A_h \phi_{h,v}^{n+1}\|^{3/2}
\]

\[
\leq C \nu^{-4/3} \|e_n^w\|^4 \|\phi_{h,v}^{n+1}\|^2 + \frac{\nu}{32} \|A_h \phi_{h,v}^{n+1}\|^2.
\]

Finally, the last resulting term is bounded in a similar way,

\[
(e_n^w \times u_{n+1}^v, A_h \phi_{h,v}^{n+1}) \leq C \|e_n^w\| \|u_{n+1}^v\|_{L^\infty} \|A_h \phi_{h,v}^{n+1}\|
\]

\[
\leq C \nu^{-1} \|e_n^w\|^2 \|u_{n+1}^v\|^2_{L^\infty} + \frac{\nu}{16} \|A_h \phi_{h,v}^{n+1}\|.
\]

Thus, (4.20) becomes

\[
(e_n^w \times v_{h,v}^{n+1}, A_h \phi_{h,v}^{n+1}) \leq \frac{\nu}{8} \|\phi_{h,v}^{n+1}\|^2 + C \nu^{-1} \|e_n^w\|^2 \|\eta_{n+1}^v\|^2_{L^\infty} + C \nu^{-4/3} \|e_n^w\|^4 \|\phi_{h,v}^{n+1}\|^2
\]

\[
+ C \nu^{-1} \|e_n^w\|^2 \|u_{n+1}^v\|^2_{L^\infty}.
\]
For the next nonlinear term, we have

\[
(\omega^n \times e_v^{n+1}, A_h \phi_h^{n+1,v}) \leq \|\omega^n\|_{L^\infty} \|e_v^{n+1}\| \|A_h \phi_h^{n+1,v}\| \\
\leq C\nu^{-1}\|\omega^n\|_{L^\infty}^2 \|e_v^{n+1}\|^2 + \frac{\nu}{8} \|A_h \phi_h^{n+1,v}\|^2.
\]

The pressure term will be bounded with standard inequalities.

\[
(\nabla P^{n+1}, A_h \phi_h^{n+1,v}) \leq C\nu^{-1}\|\nabla (P^{n+1} - r_h)\|^2 + \frac{\nu}{16} \|A_h \phi_h^{n+1,v}\|^2,
\] (4.21)

where \(r_h \in Q_h\) is arbitrary. For the first interpolation term, applying Cauchy-Schwarz inequality, then use (2.3). Lastly, we apply Young’s inequality, yielding

\[
\mu_1(I_H \eta_v^{n+1}, I_H(A_h \phi_h^{n+1,v})) \leq \mu_1 \|I_H \eta_v^{n+1}\| \|I_H(A_h \phi_h^{n+1,v})\| \\
\leq \mu_1 \|I_H \eta_v^{n+1}\| \|A_h \phi_h^{n+1,v}\| \\
\leq C\nu^{-1} \mu_1^2 \|I_H \eta_v^{n+1}\|^2 + \frac{\nu}{16} \|A_h \phi_h^{n+1,v}\|^2.
\]

For the second nudging term, we will use Cauchy-Schwarz inequality, (2.2), and the inverse inequality to get

\[
\mu_1(\phi_h^{n+1,v}, I_H A_h \phi_h^{n+1,v} - A_h \phi_h^{n+1,v}) \leq \mu_1 \|\phi_h^{n+1,v}\| \|I_H A_h \phi_h^{n+1,v} - A_h \phi_h^{n+1,v}\| \\
\leq C\mu_1 \|\phi_h^{n+1,v}\| \|\nabla A_h \phi_h^{n+1,v}\| \\
\leq C\mu_1 \|\phi_h^{n+1,v}\| \|A_h \phi_h^{n+1,v}\| \\
\leq C\mu_1 \|\phi_h^{n+1,v}\|^2 + \frac{\nu}{16} \|A_h \phi_h^{n+1,v}\|^2.
\]
Finally, for the last nudging term, we have

$$\mu_1(I_H\phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}, I_H A_h\phi_{h,v}^{n+1}) \leq \mu_1 \|I_H\phi_{h,v}^{n+1} - \phi_{h,v}^{n+1}\| \|I_H A_h\phi_{h,v}^{n+1}\|$$

$$\leq CH\mu_1 \|
abla \phi_{h,v}^{n+1}\| \|I_H A_h\phi_{h,v}^{n+1}\|$$

$$\leq CH^2\mu_1 \|I_H A_h\phi_{h,v}^{n+1}\|^2 + \frac{\mu_1}{2} \|\nabla \phi_{h,v}^{n+1}\|^2.
$$

Now, replacing the right hand side of the velocity difference equation with the computed bounds, and simplifying, we obtain

$$\frac{1}{2\Delta t} \left[ \|\nabla \phi_{h,v}^{n+1}\|^2 - \|\nabla \phi_{h,v}^{n}\|^2 \right] + \left( \nu - CH^2\mu_1 \right) \|A_h\phi_{h,v}^{n+1}\|^2 + \frac{\mu_1}{2} \|\nabla \phi_{h,v}^{n+1}\|^2$$

$$\leq C\nu^{-1}\Delta t^2 \|\mathbf{u}_{tt}\|_{L^\infty(0,\infty;L^2(\Omega))} + C\nu^{-1} \|\mathbf{e}_{w}^{n+1}\|^2 \|\nu_h^{n+1}\|^2_{L^\infty} + C\nu^{-1} \|\omega^{n+1}\|_{L^3} \|\eta_v^{n+1}\|^2$$

$$+ C\nu^{-1} \|\nabla (P^{n+1} - r_h)\|^2 + CHh^{-1}\mu_1^2\nu^{-1} \|\phi_{h,v}^{n+1}\|^2 + C\nu^{-1} \|\mu_1^2 \|I_H\eta_v^{n+1}\|^2.$$

Provided $H$ is within the range stated in the theorem, applying the results of Theorem 5, and multiplying by $2\Delta t$ the above becomes

$$\|\nabla \phi_{h,v}^{n+1}\|^2 + \Delta t(\nu - CH^2\mu_1) \|A_h\phi_{h,v}^{n+1}\|^2 + \Delta t\mu_1 \|\nabla \phi_{h,v}^{n+1}\|^2 \leq C(\Delta t^3 + \Delta th^{2k-2}) + \|\nabla \phi_{h,v}^{n}\|^2.$$

Let $\lambda := C_P^{-2}(\nu - CH^2\mu_1) + \mu_1$ to obtain

$$(1 + \lambda \Delta t) \|\nabla \phi_{h,v}^{n+1}\|^2 \leq C(\Delta t^3 + \Delta th^{2k-2}) + \|\nabla \phi_{h,v}^{n}\|^2,$$

and at this point the result follows in an equivalent way as Theorem 5.

We will now move on to the vorticity equation. Note that we will use the $H^1$ velocity results in this part of the proof. Several of the right hand side terms of this equation will be bounded in a similar way as the velocity equation, so we will only discuss
the terms that differ. For the first nonlinear term, we will begin by adding and subtracting $\omega^{n+1}$ to $w^{n+1}_h$ to get

$$b(e^{n+1}_v, w^{n+1}_h, \Delta_h \phi^{n+1}_{h,w}) = b(e^{n+1}_v, e^{n+1}_w, \Delta_h \phi^{n+1}_{h,w}) + b(e^{n+1}_v, \omega^{n+1}_w, \Delta_h \phi^{n+1}_{h,w}). \quad (4.22)$$

We will now bound each term individually. For the first term, we will first write it as two terms then apply Hölder’s, 2D Agmon and Young’s inequalities. The second term will also require the inverse inequality.

$$b(e^{n+1}_v, e^{n+1}_w, \Delta_h \phi^{n+1}_{h,w}) = b(e^{n+1}_v, \phi^{n+1}_{h,w}, \Delta_h \phi^{n+1}_{h,w}) + b(e^{n+1}_v, \eta^{n+1}_w, \Delta_h \phi^{n+1}_{h,w})$$

$$\leq C \|e^{n+1}_v\|_{L^4} \|\nabla \phi^{n+1}_{h,w}\|_{L^4} \|\Delta_h \phi^{n+1}_{h,w}\| + C \|e^{n+1}_v\|_{L^4} \|\nabla \eta^{n+1}_w\|_{L^4} \|\Delta_h \phi^{n+1}_{h,w}\|$$

$$\leq C \|e^{n+1}_v\|^{1/2} \|\nabla e^{n+1}_v\|^{1/2} \|\nabla \phi^{n+1}_{h,w}\|^{1/2} \|\Delta_h \phi^{n+1}_{h,w}\|^{3/2}$$

$$+ C \|e^{n+1}_v\|^{1/2} \|\nabla e^{n+1}_v\|^{1/2} \|\nabla \eta^{n+1}_w\|^{1/2} \|\Delta_h \eta^{n+1}_w\|^{1/2} \|\Delta_h \phi^{n+1}_{h,w}\|$$

$$\leq C \nu^{-3} \|e^{n+1}_v\|^2 \|\nabla e^{n+1}_v\|^2 \|\nabla \phi^{n+1}_{h,w}\|^2 + \frac{\nu}{64} \|\Delta_h \phi^{n+1}_{h,w}\|^2$$

$$+ C \nu^{-1} \|e^{n+1}_v\| \|\nabla e^{n+1}_v\| \|\nabla \eta^{n+1}_w\| \|\Delta_h \eta^{n+1}_w\| + \frac{\nu}{64} \|\Delta_h \phi^{n+1}_{h,w}\|^2$$

$$\leq C \nu^{-3} (\Delta t^2 + h^{2k}) \|\nabla \phi^{n+1}_{h,w}\|^2$$

$$+ C \nu^{-1} (\Delta t^2 + h^{2k}) \|\nabla \eta^{n+1}_w\|^2 \|\Delta_h \eta^{n+1}_w\| + \frac{\nu}{32} \|\Delta_h \phi^{n+1}_{h,w}\|^2.$$
Therefore, (4.22) is bounded as

\[
b(e^{n+1}_v, w^{n+1}_h, \Delta_h\phi^{n+1}_{h,w}) \leq C\nu^{-3}(\Delta t^2 + h^{2k})\|\nabla\phi^{n+1}_{h,w}\|^2 + C\nu^{-1}(\Delta t^2 + h^{2k})\|\nabla\eta^{n+1}_w\|^2 \|\Delta_h\eta^{n+1}_w\|
\]
\[
+ C\nu^{-1}(\Delta t^2 + h^{2k})\|\nabla\omega^{n+1}\|^2 + \frac{\nu}{16} \|\Delta_h\phi^{n+1}_{h,w}\|^2.
\]

For the second nonlinear term, we have

\[
b(u^{n+1}, \eta^{n+1}_w, \Delta_h\phi^{n+1}_{h,w}) \leq \|u^{n+1}\|_{L^\infty} \|\nabla\eta^{n+1}_w\| \|\Delta_h\phi^{n+1}_{h,w}\|
\]
\[
\leq C\nu^{-1}\|u^{n+1}\|^2_{L^\infty} \|\nabla\eta^{n+1}_w\|^2 + \frac{\nu}{32} \|\Delta_h\phi^{n+1}_{h,w}\|^2.
\]

Finally, for the last nonlinear term, we will apply Hölder’s and Young’s inequality to

\[
b(u^{n+1}, \phi^{n+1}_{h,w}, \Delta_h\phi^{n+1}_{h,w}) \leq C\|u^{n+1}\|_{L^4} \|\nabla\phi^{n+1}_{h,w}\|_{L^4} \|\Delta_h\phi^{n+1}_{h,w}\|
\]
\[
\leq C\|u^{n+1}\|^2_{L^4} \|\nabla\phi^{n+1}_{h,w}\|_{L^4}^{1/2} \|\Delta_h\phi^{n+1}_{h,w}\|_{L^4}^{3/2}
\]
\[
\leq C\nu^{-3}\|u^{n+1}\|^2_{L^4} \|\nabla\phi^{n+1}_{h,w}\|^2 + \frac{\nu}{8} \|\Delta_h\phi^{n+1}_{h,w}\|^2.
\]

After replacing the right hand side of the vorticity difference equation with the computed bounds, we have

\[
\frac{1}{2\Delta t} \left[\|\nabla\phi^{n+1}_{h,w}\|^2 - \|\nabla\phi^n_{h,w}\|^2\right] + \left(\frac{\nu}{2} - CH^2\mu_2\right) \|\Delta_h\phi^{n+1}_{h,w}\|^2
\]
\[
+ \left(\frac{\mu_2}{2} - C\nu^{-3}(\Delta t^2 + h^{2k}) - C\nu^{-1}\|u^{n+1}\|^2_{L^4}\right) \|\nabla\phi^{n+1}_{h,w}\|^2
\]
\[
\leq C\nu^{-1}\Delta t^2\|\omega_t\|^2_{L^\infty(0,\infty;L^2(\Omega))} + C\nu^{-1}(\Delta t^2 + h^{2k})\|\nabla\eta^{n+1}_w\|^2 \|\Delta_h\eta^{n+1}_w\|^2
\]
\[
+ C\nu^{-1}(\Delta t^2 + h^{2k})\|\nabla\omega^{n+1}\|^2_{L^4} + C\nu^{-1}\|u^{n+1}\|^2_{L^\infty} \|\nabla\eta^{n+1}_w\|^2 + C\nu \|\Delta_h\eta^{n+1}_w\|^2
\]
\[
+ C\nu^{-1}\mu_2^2 \|I_{H\eta^{n+1}_w}\|^2 + C\nu^{-1}\mu_2 \|\phi^{n+1}_{h,w}\|^2.
\]
Again, using the results of Theorem 5 and provided $H$ is within the range stated in the theorem, we can multiply the above equation by $2\Delta t$ and lower bound the left had side the same way as the velocity equation to get

\[(1 + \lambda \Delta t) \| \nabla \phi_{h,w}^{n+1} \|^2 \leq C(\Delta t^3 + \Delta t h^{2k}) + \| \nabla \phi_{h,w}^n \|^2,\]

with the final result following immediately.

-\[
\text{Remark 7.} \quad \text{In this case, we lose a power of } h \text{ due to the pressure term (4.21). If divergence-free elements, like Scott-Vogelius, are used instead, this term is 0 and we gain that power of } h \text{ back.}
\]

-\[
\text{Remark 8.} \quad \text{Thanks to the } G\text{-norm, stability and convergence results for an analogous BDF2 second order scheme can be achieved using similar argument as above. In this case, we expect the difference between the NSE solution and the second order algorithm solution to converge up to a } \Delta t^2 + h^k \text{ dependent constant, independent of the initial condition, provided some assumptions on data hold.}
\]
Chapter 5

Continuous DA applied to reduced order models of fluid flow

In this chapter we are analyzing and testing a continuous DA reduced order model (DA-ROM) for incompressible flows. ROMs have been shown to be successful on certain problems with recurring dominant structures, and they are very attractive algorithms to use because of the very short computational time needed to generate results. However, they do not perform as well on complicated problems or longer time intervals. In order to address some shortcomings of ROMs, we incorporate continuous DA into the algorithm. We prove that with a properly chosen nudging parameter, the proposed DA-ROM algorithm converges exponentially fast in time to the true solution, up to discretization and ROM truncation errors.

The numerical tests included at the end of the chapter confirm the analytical results and illustrate the improvement in the ROM when DA is added. We also propose a strategy for nudging adaptively in time by adjusting the nudging parameter so that the dissipation better matches the true solution energy. The numerical experiments
show that this adaptive nudging algorithm out-performs all other ROMs considered.

5.1 Analysis of a first order DA-ROM algorithm

In this section, we provide error estimates for a DA-ROM algorithm, after presenting necessary preliminaries for the algorithm.

5.1.1 ROM preliminaries

Let \( \{u_h^1, \ldots, u_h^M\} \) be snapshots of finite element solutions at \( M \) different time instances. The proper orthogonal decomposition seeks a low-dimensional basis that approximates these snapshots optimally with respect to a certain norm, which we choose to be the \( L^2 \) norm. This minimization can be set up as an eigenvalue problem \( YY^T M_h \varphi_j = \lambda_j \varphi_j, \ j = 1, \ldots, N_h \). where \( Y \) is the matrix of snapshots, whose columns correspond to the finite element coefficients, \( M_h \) is the finite element mass matrix, and \( N_h \) is the dimension of the finite element space. The eigenvalues are real and non-negative, so they can be ordered as \( \lambda_1 \geq \ldots \geq \lambda_d \geq \lambda_{d+1} = \ldots = \lambda_{N_h} = 0 \), where \( d \) is the rank of the snapshot matrix. We take the ROM space to be \( X_r := \text{span}\{\varphi_i\}_{i=1}^r \), and note that \( X_r \subset V_h \). The ROM approximation of the velocity is defined as

\[
u_r(x,t) = \sum_{j=1}^r a_j(t) \varphi_j(x),
\]

where the coefficients \( a_j(t) \) are determined by solving the Galerkin ROM:

\[
(u_{r,t}, \varphi_i) + \nu (\nabla u_r, \nabla \varphi_i) + b(u_r, u_r, \varphi_i) = (f, \varphi_i).
\]
We define the $L^2$ ROM projection $P_r : L^2 \to X_r$ by: for all $v \in L^2(\Omega)$, $P_r(v)$ is the unique element of $X_r$ such that

$$(P_r(v), v_r) = (v, v_r) \quad \forall \, v_r \in X_r. \tag{5.1}$$

In addition, the following inverse inequality holds for our ROM basis [60].

**Lemma 5** (POD inverse estimate).

$$\|\nabla \varphi\| \leq ||| S_R |||^{1/2} \|\varphi\| \quad \forall \varphi \in X_r, \tag{5.2}$$

where $||| S_R |||_2$ is the matrix 2-norm of the ROM stiffness matrix, as in Lemma 3.1 of [50].

In order to establish an error estimate for the ROM projection, we first make the following assumption on the finite element error:

**Assumption 9.** Let $C(\nu, p)$ denote a constant which is dependent upon the viscosity and pressure. We assume that the finite element error $u_h$ satisfies the following error estimate

$$\|u^M - u_h^M\|^2 + \nu h^2 \Delta t \sum_{n=1}^M \|\nabla (u^n - u_h^n)\|^2 \leq C(\nu, p)(h^{2k+2} + \Delta t^4). \tag{5.3}$$

**Remark 10.** Error estimates of this form have been proven for varying amounts of regularity on the continuous solution $u$ and $p$. Some examples include the scheme used in the numerical experiments in Section 4.

Using Assumption 9 the following error estimates for the ROM projection can be proven [50]:

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**Lemma 6.** The $L^2$ ROM projection of $u^n$ satisfies the following error estimates:

$$
\sum_{n=1}^{M} \|u^n - P_r(u^n)\|^2 \leq C(\nu, p) \left( h^{2k+2} + \Delta t^4 + \sum_{j=r+1}^{d} \lambda_j \right), \\
\sum_{n=1}^{M} \|\nabla(u^n - P_r(u^n))\|^2 \leq C(\nu, p) \left( h^{2k} + |||S_R|||_2 h^{2k+2} + (1 + |||S_R|||_2) \Delta t^4 \right. \\
\left. + \sum_{j=r+1}^{d} \|\nabla \varphi_j\|^2 \lambda_j \right).
$$

(5.4) (5.5)

We then make the following assumption similar to that made in [50]:

**Assumption 11.** The $L^2$ ROM projection of $u^n$ satisfies the following error estimates:

$$
\max_n \|u^n - P_r(u^n)\|^2 \leq C(\nu, p) \left( h^{2k+2} + \Delta t^4 + \sum_{j=r+1}^{d} \lambda_j \right), \\
\max_n \|\nabla(u^n - P_r(u^n))\|^2 \leq C(\nu, p) \left( h^{2k} + |||S_R|||_2 h^{2k+2} + (1 + |||S_R|||_2) \Delta t^4 \right. \\
\left. + \sum_{j=r+1}^{d} \|\nabla \varphi_j\|^2 \lambda_j \right).
$$

(5.6) (5.7)

**Remark 12.** If we assumed in Assumption 9 that the finite element error satisfies

$$
\|u^M - u_h^M\|^2 + h^2 \|\nabla(u^M - u_h^M)\|^2 \leq C(\nu, p) (h^{2k+2} + \Delta t^4),
$$

then the bound in Assumption 11 would hold. Error estimates of this form have been proven for varying amounts of regularity on the continuous solution $u$ and $p$. Some examples include the incremental pressure correction schemes in [38] and chapter 7 of [83].
5.1.2 Stability and convergence analysis

For simplicity of exposition, our analysis considers a first order DA-ROM algorithm, which takes the following form:

**Algorithm 5.1.1.** Given an external forcing \( f \in L^\infty(0, \infty; L^2(\Omega)) \) and (true solution) measurements \( u(t^n) \), \( n = 1, 2, \ldots \), find \( u^{n+1}_r \in X_r \) such that for all \( v_r \in X_r \),

\[
\frac{1}{\Delta t} (u^{n+1}_r - u^n_r, v_r) + b(u^{n+1}_r, u^{n+1}_r, v_r) + \nu(\nabla u^{n+1}_r, \nabla v_r) + \mu(I_H(u^{n+1}_r - u(t^{n+1})), I_H v_r) = (f^{n+1}, v_r),
\]

for \( n = 1, 2, \ldots, M \), with \( v_0 = P_r(u_0) \), and where \( \mu \geq 0 \) is the nudging parameter, and \( I_H \) is an interpolation operator satisfying (2.2)-(2.3).

**Remark 13.** Extension to other time stepping methods is possible, and, for example, extension to BDF2 can be done following the usual techniques [64]. All of our numerical tests use the analogous BDF2 algorithm.

We first prove a stability estimate for the DA ROM algorithm.

**Lemma 7.** The solutions to (5.8) satisfy for all \( M > 1 \),

\[
\|u^M_r\|^2 \leq \|u^0_r\|^2 \left( \frac{1}{1 + \lambda \Delta t} \right)^M + C \lambda^{-1}(\nu^{-1} F^2 + \mu U^2) := C_{data},
\]

where \( F := \|f\|_{L^\infty(0,\infty; H^{-1})} \), \( U := \|u\|_{L^\infty(0,\infty; L^2)} \), and \( \lambda = \nu C_p^{-2} \).

**Proof.** This result follows as in [64] by letting \( v_r = u^{n+1}_r \) in (5.8) and using Cauchy-Schwarz and Young’s inequalities. Additionally, the non-negative DA term \( \|I_H u^{n+1}_r\|^2 \) can be dropped from the left after bounding the right hand side.
To analyze rates of convergence of the approximation we make the following regularity assumptions on the NSE [66]:

**Assumption 14.** We assume that the solution of the NSE satisfies

\[ u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \cap H^2(0,T;H^1(\Omega)), \]

\[ p \in L^2(0,T;H^{k+1}(\Omega)), \]

\[ f \in L^2(0,T;L^2(\Omega)). \]

We next prove that solutions to (5.8) converge to the true solution exponentially fast, up to discretization and ROM projection error.

**Theorem 15.** Define

\[ \alpha_1 := \nu - 2\mu(\beta_2 - 1)C_i^2 H^2, \]

\[ \alpha_2 := 2\mu - \frac{\mu C_i^2}{2\beta_1} - \frac{\mu}{2\beta_2} - 6\nu^{-1}M^2 |||S_R|||_2 \|\nabla u^{n+1}\|_2, \]

which have parameters \( \mu, H, \beta_i > 0, i = 1, 2 \) that are chosen so that \( \alpha_i > 0, i = 1, 2. \)

Then under the regularity assumptions of Assumption 14, we have that

\[
\begin{align*}
\|u^{n+1} - u_r^{n+1}\|^2 &\leq \|u^0 - u_r^0\|^2 \left( \frac{1}{1 + 2\lambda \Delta t} \right)^{n+1} \\
&+ C\lambda^{-1}\left\{ \Delta t^2 + \nu^{-1}h^{2k} + \beta_1 C_i^2 \mu \left( h^{2k+2} + \Delta t^4 + \sum_{j=r+1}^d \lambda_j \right) \right. \\
&\left. + (\nu^{-1}M^2 + \nu^{-1}M^2 |||S_R|||_2) \left( h^{2k} + \Delta t^4 + \sum_{j=r+1}^d \|\nabla \varphi_j\|^2 \lambda_j \right) \right\},
\end{align*}
\]

where \( \lambda = \min\{\alpha_1 C_F^{-2}, \alpha_2\} \).

**Remark 16.** The \( \Delta t^2 \) term that shows up on the right hand side of (5.9) is a result of the first order time stepping in Algorithm 5.8. If we instead used a second order
approximation, like BDF2, then this term would be replaced by $\Delta t^4$.

**Remark 17.** If $\beta_1, \beta_2$ are chosen to be $1/2$, the condition $\alpha_1 > 0$ reduces to $\nu - C\mu H^2 > 0$, which is the same condition found in [64] and references therein, for a relationship between the nudging parameter, viscosity, and coarse mesh width. Choosing $\beta_1, \beta_2$ larger can allow one to choose the coarse mesh width $H$ larger (and thus require less observational data) while still satisfying $\alpha_i > 0$, $i = 1, 2$. However, there is a trade-off because $\beta_1$ appears on the right hand side of equation (5.9): As $\beta_1$ increases, the bound on the DA-ROM error grows.

**Proof.** The NSE (true) solution satisfies

$$
\frac{1}{\Delta t}(u^{n+1} - u^n, v_r) + b(u^{n+1}, u^n, v_r) + \nu(\nabla u^{n+1}, \nabla v_r) + (p^{n+1}, \nabla \cdot v_r)
= (f^{n+1}, v_r) + \left(\frac{1}{\Delta t}(u^{n+1} - u^n) - u_t^{n+1}, v_r\right).
$$

(5.10)

Note that we can write the time derivative term above as $C\Delta t u_{tt}(t^*)$ for some $t^* \in (t^n, t^{n+1})$ [64]. Subtracting (5.8) from (5.10) and letting $e^n := u^n_r - u^n$, we obtain

$$
\frac{1}{\Delta t}(e^{n+1} - e^n, v_r) + \nu(\nabla e^{n+1}, \nabla v_r) + \mu(I_{H}e^{n+1}, I_{H}v_r)
\leq C\Delta t (u_{tt}(t^*), v_r) + b(u^{n+1}_r, e^{n+1}, v_r) + b(e^{n+1}, u^{n+1}, v_r) + (p^{n+1}, \nabla \cdot v_r).
$$

(5.11)

Decompose the error as a part inside the ROM space and one outside by adding and subtracting the $L^2$ projection of $u^n$ into the ROM space, $P_r(u^n)$ (see (5.1)):

$$
e^n = (u^n_r - P_r(u^n)) + (P_r(u^n) - u^n) =: \phi^n_r + \eta^n.
$$
Letting $v_r = \phi_r^{n+1}$ in (5.11), we note that since $\phi_r^{n+1} \in X_r \subset V_h$, for any $q_h \in Q_h$,

$$(p^{n+1}, \nabla \cdot \phi_r^{n+1}) = (p^{n+1} - q_h, \nabla \cdot \phi_r^{n+1}).$$  \hspace{1cm} (5.12)

Adding and subtracting $\phi_r^{n+1}$ to both components of the nudging term we have

$$(I_H\phi_r^{n+1} + I_H\eta^{n+1} + \phi_r^{n+1} - \phi_r^{n+1}, I_H\phi_r^{n+1} + \phi_r^{n+1} - \phi_r^{n+1})$$

$$= \|\phi_r^{n+1}\|^2 + (\phi_r^{n+1}, I_H\phi_r^{n+1} - \phi_r^{n+1}) + (I_H\phi_r^{n+1} + I_H\eta^{n+1} - \phi_r^{n+1}, I_H\phi_r^{n+1} + \phi_r^{n+1} - \phi_r^{n+1})$$

$$= \|\phi_r^{n+1}\|^2 + (\phi_r^{n+1}, I_H\phi_r^{n+1} - \phi_r^{n+1}) + (I_H\eta^{n+1}, I_H\phi_r^{n+1} + \phi_r^{n+1} - \phi_r^{n+1})$$

$$+ (I_H\phi_r^{n+1} - \phi_r^{n+1}, I_H\phi_r^{n+1} - \phi_r^{n+1} + \phi_r^{n+1})$$

$$= \|\phi_r^{n+1}\|^2 + 2(\phi_r^{n+1}, I_H\phi_r^{n+1} - \phi_r^{n+1}) + (I_H\eta^{n+1}, I_H\phi_r^{n+1}) + \|I_H\phi_r^{n+1} - \phi_r^{n+1}\|^2.$$  \hspace{1cm} (5.13)

Using the polarization identity, the fact that $(\eta^{n+1} - \eta^n, \phi_r^{n+1}) = 0$ (by the definition of the $L^2$ projection), and dropping the nonnegative term $\frac{1}{2\Delta t}\|\phi_r^{n+1} - \phi_r^n\|^2$ on the left hand side, we have

$$\frac{1}{2\Delta t}\left[\|\phi_r^{n+1}\|^2 - \|\phi_r^n\|^2\right] + \nu\left|\nabla \phi_r^{n+1}\right|^2 + \mu\|\phi_r^{n+1}\|^2 + \mu\left|I_H\phi_r^{n+1} - \phi_r^{n+1}\right|^2$$

$$\leq \nu\left|\nabla \eta^{n+1}, \nabla \phi_r^{n+1}\right| + \left|C \Delta t \left(u_{ht}(t^*), \phi_r^{n+1}\right)\right| + \left|b(u_r^{n+1}, \eta^{n+1}, \phi_r^{n+1})\right|$$

$$+ \left|b(\eta^{n+1}, u^{n+1}, \phi_r^{n+1})\right| + \left|b(\phi_r^{n+1}, u^{n+1}, \phi_r^{n+1})\right| + \left|(p^{n+1} - q_h, \nabla \cdot \phi_r^{n+1})\right|$$

$$+ \mu\left|(I_H\eta^{n+1}, I_H\phi_r^{n+1})\right| + 2\mu\left|(\phi_r^{n+1}, I_H\phi_r^{n+1} - \phi_r^{n+1})\right|.  \hspace{1cm} (5.14)$$

By Poincaré, Cauchy Schwarz, and Young’s inequalities, we bound the first two terms
on the right hand side and the pressure term,

$$
\nu(\nabla \eta^{n+1}, \nabla \phi_r^{n+1}) \leq \frac{\nu}{4c_1} \|
abla \eta^{n+1}\|^2 + C_1 \nu \|
abla \phi_r^{n+1}\|^2, \\
$$

$$
C \Delta t \left( u_{tt}(t^*), \phi_r^{n+1} \right) \leq \frac{C \Delta t^2 \nu^{-1}}{4c_2} \| u_{tt}(t^*) \|^2 + C_2 \nu \|
abla \phi_r^{n+1}\|^2, \\
$$

$$
(p^{n+1} - q_h, \nabla \cdot \phi_r^{n+1}) \leq \frac{\nu^{-1}}{4c_3} \| p^{n+1} - q_h^{n+1} \|^2 + C_3 \nu \|
abla \phi_r^{n+1}\|^2. \\
$$

The first two nonlinear terms are now bounded similarly to those in [76] using Cauchy-Schwarz and Young’s inequalities, and the first inequality from Lemma 1:

$$
b(\eta^{n+1}, u^{n+1}, \phi_r^{n+1}) \leq \frac{\nu^{-1} M^2}{4c_4} \| \nabla u^{n+1}\|^2 \|
abla \eta^{n+1}\|^2 + C_4 \nu \|
abla \phi_r^{n+1}\|^2, \\
$$

$$
b(u_r^{n+1}, \eta^{n+1}, \phi_r^{n+1}) \leq \frac{\nu^{-1} M^2}{4c_5} \| \nabla u_r^{n+1}\|^2 \|
abla \eta^{n+1}\|^2 + C_5 \nu \|
abla \phi_r^{n+1}\|^2. \\
$$

How we treat the third nonlinear term is the key difference in the proof from standard schemes (see chapter 9 of [67]). Due to the added dissipation from the DA term on the left-hand side of (5.14), we will be able to hide the term containing $\phi_r^{n+1}$, rather than invoking a discrete Gronwall’s inequality. Thus, for this term we use the second inequality from Lemma 1 and the ROM inverse inequality (5.2) to obtain

$$
b(\phi_r^{n+1}, u^{n+1}, \phi_r^{n+1}) \leq M \| \phi_r^{n+1}\|^{1/2} \|
abla \phi_r^{n+1}\|^{1/2} \|
abla u^{n+1}\| \|
abla \phi_r^{n+1}\| \\
\leq \frac{\nu^{-1} M^2 \| \mathcal{S}_R \|^2}{4c_6} \| \nabla u^{n+1}\|^2 \| \phi_r^{n+1}\|^2 + c_6 \nu \|
abla \phi_r^{n+1}\|^2. \\
$$

The first nudging terms on the right hand side of (5.14) are bounded using (2.3), Cauchy Schwarz, and Young’s inequality,

$$
\mu(I_H \eta^{n+1}, I_H \phi_r^{n+1}) \leq \frac{\mu}{4\beta_1} \| I_H \phi_r^{n+1}\|^2 + \mu \beta_1 \| I_H \eta^{n+1}\|^2 \\
\leq \frac{\mu C_I^2}{2\beta_1} \| \phi_r^{n+1}\|^2 + 2\mu \beta_1 C_I^2 \| \eta^{n+1}\|^2. \\
$$

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The second nudging term is bounded using Cauchy Schwarz and Young’s inequality, and (2.2), yielding

\[
2\mu (\phi_r^{n+1}, I_H \phi_r^{n+1} - \phi_r^{n+1}) \leq \frac{\mu}{4\beta_2} \|\phi_r^{n+1}\|^2 + \mu \beta_2 \|I_H \phi_r^{n+1} - \phi_r^{n+1}\|^2
\]

\[
= \frac{\mu}{4\beta_2} \|\phi_r^{n+1}\|^2 + \mu(\beta_2 - 1) \|I_H \phi_r^{n+1} - \phi_r^{n+1}\|^2 + \mu \|I_H \phi_r^{n+1} - \phi_r^{n+1}\|^2 \quad (5.20)
\]

\[
\leq \frac{\mu}{4\beta_2} \|\phi_r^{n+1}\|^2 + C_2^2 H^2 \mu(\beta_2 - 1) \|\nabla \phi_r^{n+1}\|^2 + \mu \|I_H \phi_r^{n+1} - \phi_r^{n+1}\|^2.
\]

Now letting \( c_i = \frac{1}{12}, i = 1, 2, \ldots, 6 \), combining terms, and recalling our definition of \( \alpha_1 \) and \( \alpha_2 \) given in the statement of the theorem, (5.14) becomes

\[
\|\phi_r^{n+1}\|^2 + \alpha_1 \Delta t \|\nabla \phi_r^{n+1}\|^2 + \alpha_2 \Delta t \|\phi_r^{n+1}\|^2
\]

\[
\leq \|\phi_r^n\|^2 + C \Delta t^3 \nu^{-1} \|u_{tt}(t^*)\|^2 + C \Delta t \nu^{-1} M^2 \|\nabla \eta^{n+1}\|^2 \|\nabla u^{n+1}\|^2
\]

\[
+ C \Delta t \nu^{-1} M^2 \|\nabla \eta^{n+1}\|^2 \|\nabla u_r^{n+1}\|^2 + C \nu \Delta t \|\nabla \eta^{n+1}\|^2 + C \nu^{-1} \Delta t \|p^{n+1} - q_h\|^2
\]

\[
+ 2C_1^2 \beta_1 \Delta t \mu \|\eta^{n+1}\|^2,
\]

(5.21)

where \( C \) is a generic constant which is independent of \( \nu, p, u, T, H, C_1 \). Next, we bound the fourth term on the right hand side further using the ROM inverse inequality (5.2), and the stability result from Lemma 7

\[
C \Delta t \nu^{-1} M^2 \|\nabla \eta^{n+1}\|^2 \|\nabla u^{n+1}\|^2 \leq CC_{\text{data}} \Delta t \nu^{-1} M^2 \|||S_z||\|2 \|\nabla \eta^{n+1}\|^2.
\]

(5.22)

Now applying Lemma 6, using our regularity assumptions, and taking \( \lambda := \min\{C^{-2} \alpha_1, \alpha_2\} \)
in (5.21), it then follows that

\[(1 + 2\lambda \Delta t)\|\phi_{r}^{n+1}\|^2 \leq \|\phi_{r}^n\|^2 + C \Delta t^3 + C \nu^{-1} \Delta t h^{2k} + 2C^2 I \beta_1 \Delta t \mu \left( h^{2k+2} + \Delta t^4 + \sum_{j=r+1}^{d} \lambda_j \right) + C \Delta t (\nu^{-1} M^2 + C_{data} \nu^{-1} M^2 \|||S_R|||_2 + \nu) \left( h^{2k} + \Delta t^4 + \sum_{j=r+1}^{d} \|\nabla \varphi_j\|^2 \lambda_j \right).\]

Finally, by Lemma 2, we obtain

\[\|\phi_{r}^{n+1}\|^2 \leq \|\phi_{r}^0\|^2 \left( \frac{1}{1 + 2\lambda \Delta t} \right)^{n+1} + 2\lambda^{-1} \Delta t^{-1} \left\{ C \Delta t^3 + C \nu^{-1} \Delta t h^{2k} + 2\beta_1 C^2 I \Delta t \mu \left( h^{2k+2} + \Delta t^4 + \sum_{j=r+1}^{d} \lambda_j \right) + C \Delta t (\nu^{-1} M^2 + C_{data} \nu^{-1} M^2 \|||S_R|||_2 + \nu) \left( h^{2k} + \Delta t^4 + \sum_{j=r+1}^{d} \|\nabla \varphi_j\|^2 \lambda_j \right) \right\}.\]

The triangle inequality completes the proof.

5.2 Numerical Experiments

In this section, we perform a numerical investigation of the new DA-ROM. In Section 5.2.1, we illustrate the theoretical scalings proved in Section 5.1. In Section 5.2.2, we investigate the numerical accuracy of the new DA-ROM. In Section 5.2.3, we investigate the new DA-ROM when inaccurate snapshots are used in its construction. Finally, in Section 5.2.4, we propose and investigate an adaptive nudging procedure.

We consider Algorithm 5.8 (except here with BDF2) applied to 2D channel flow past a cylinder [90], with Reynolds number $Re=500$ and $\nu = 0.0002$. We enforce the zero-
traction boundary condition with the usual ‘do-nothing’ condition at the outflow.

The DNS is run to $t=15$ with the usual BDF2-FEM discretization [66] using $(P_2, P_1^{disc})$
Scott-Vogelius elements on a barycenter refined Delaunay mesh that provided 103K
velocity dof, a time step of $\Delta t = 0.002$, and with the simulations starting from rest
($v^0_h = v^1_h = 0$). Lift and drag calculations were performed for the computed solution
and compared to the literature [90, 95], which verified the accuracy of the DNS. We
used the snapshots from $t=5$ to $t=6$ to generate the ROM modes.

The coarse mesh for DA is constructed using the intersection of a uniform rectangular
mesh with the domain. We take $H$ to be the width of each rectangle, and use $H = \frac{2.2}{20}$
(400 measurement locations) in our tests. Figure 6.6 shows in red a 35K dof mesh
and associated $H = \frac{2.2}{8}$ coarse mesh in black.

Figure 5.1: Shown above is a FE mesh (in red) and the $H = \frac{2.2}{8}$ coarse mesh and
nodes (in black).

For the DA-ROM computations, we start from zero initial conditions $v^1_h = v^0_h = 0$,
use the same spatial and temporal discretization parameters as the DNS, and start
assimilation with the $t = 5$ DNS solution (i.e., time 0 for DA-ROM corresponds to
t = 5 for the DNS).

5.2.1 Convergence rate test

In this section, we illustrate numerically the rates of convergence in Section 5.1.
Theorem 15 gave a DA-ROM error estimate that depends on the ROM eigenvalues
and eigenvectors, for sufficiently large $n$ and assumptions on $\mu$ and $H$:

$$\|u^{n+1} - u_r^{n+1}\| \leq C(\nu) \left( \Delta t^2 + h^{k+1} + \left( \sum_{j=r+1}^{d} \lambda_j (1 + \|\nabla \varphi_j\|^2) \right)^{1/2} \right),$$

where $(\lambda_j, \varphi_j)$ are the eigenpairs of the ROM eigenvalue problem described in Section 2.1. Table 5.1 illustrates the dependence of the error bound on the dimension of the DA-ROM space, $r$. Taking $T=1$, $\mu = 100$, $H = \frac{22}{20}$, Re=500, we run the ROM with varying $r$ and calculate the $L^2$ spatial error at the last time step. We also calculate the quantity in the error estimate corresponding to the eigenvalues and eigenvectors (i.e., $\sum_{j=r+1}^{d} \lambda_j (1 + \|\nabla \varphi_j\|^2)^{1/2}$), and use this and the error to calculate the corresponding convergence rate with respect to increasing $r$. From the theorem, we expect a rate of 1, and our results are consistent with this rate.

<table>
<thead>
<tr>
<th>No. modes</th>
<th>$\sum_{j=r+1}^{d} \lambda_j (1 + |\nabla \varphi_j|^2)^{1/2}$</th>
<th>Error</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.218e+2</td>
<td>4.980e-2</td>
<td>–</td>
</tr>
<tr>
<td>10</td>
<td>1.077e+2</td>
<td>4.850e-2</td>
<td>1.74</td>
</tr>
<tr>
<td>12</td>
<td>9.246e+1</td>
<td>3.046e-2</td>
<td>2.51</td>
</tr>
<tr>
<td>14</td>
<td>7.680e+1</td>
<td>1.793e-2</td>
<td>1.70</td>
</tr>
<tr>
<td>16</td>
<td>4.590e+1</td>
<td>1.360e-2</td>
<td>1.36</td>
</tr>
<tr>
<td>18</td>
<td>3.334e+1</td>
<td>9.498e-3</td>
<td>1.12</td>
</tr>
<tr>
<td>20</td>
<td>2.601e+1</td>
<td>6.974e-3</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 5.1: DA-ROM rates of convergence with respect to the ROM truncation.

### 5.2.2 Numerical Accuracy

In this section, we investigate the numerical accuracy of the new DA-ROM. Specifically, we compare the performance of the DA-ROM to that of the standard ROM ($\mu = 0$) and the DNS solution in predicting energy and drag (lift is accurate in all of our tests, so we omit it here). We run to $t=10$, and run tests with both $N = 8$
and $N = 16$ modes, and with varying $\mu = 0, 10, 100$ (we also ran $\mu = 1$, results were omitted because they are very close to those for $\mu = 0$). Results are shown in figure 5.2 for energy and drag prediction, and we observe a big improvement from using DA. For $N = 16$ and $\mu = 100$, very good accuracy is achieved from the DA-ROM. For $N = 8$, $\mu = 10$ is somewhat more accurate than for $\mu = 100$, but both are better than no DA.

![Figure 5.2: Shown above are the energy and drag coefficient versus time for $Re = 500$ DA-ROM with different choices of $\mu$, $H = \frac{22}{20}$, and with 8 modes (top) and 16 modes (bottom).](image)

### 5.2.3 Inaccurate Snapshots

In this section, we investigate the DA-ROM performance when the snapshots are inaccurate. Specifically, we consider the same test as in Section 5.2.2, but now with only a small amount of data being used to build the ROM basis. This is an important
Figure 5.3: Pictured above are the first 5 basis functions generated by the ROM for first the full basis, then inaccurate bases 1 and 2, which both use less than one period of data to generate the basis.

aspect of the ROM to investigate, because in practical applications complete data is generally not available, or the amount of data needed to sufficiently capture the behavior of the true solution is unknown.

We generated these inaccurate snapshots for $Re=500$ using less than one period of data: basis 1 used 64% of one period of data while basis 2 used 84%. See figure 5.3 for the first five basis functions generated by the ROM; the basis functions for the full ROM are also included for comparison.

In figure 5.4, we show the results of the DA-ROM using only 8 modes, with basis 1 and 2 defined above, and $\mu$ ranging from 100 to 500. DA significantly improves the accuracy of the ROM, and basis 2 does better at predicting the drag coefficient than basis 1.

Figure 5.5 shows energy and drag coefficient plots versus time using $N=12$ modes, and the nudging parameter $\mu$ is varied from 100 to 500. We see similar results as
Figure 5.4: Energy and drag coefficient versus time plots with different values of $\mu$ for $Re = 500$ using 8 modes and $H = \frac{2.2}{20}$.

Figure 5.5: Energy and drag coefficient versus time plots with different values of $\mu$ for $Re = 500$ using 12 modes and $H = \frac{2.2}{20}$.

the case of using 8 modes; for both bases, DA significantly improves the accuracy of the ROM, compared to the ROM without DA ($\mu = 0$), which becomes more and
more inaccurate as time goes on. Basis 2 is able to very accurately predict the drag coefficient.

The results in this section suggest that DA can dramatically improve the accuracy of a ROM when insufficient data is available to build the ROM, which is the general case in practical applications. We also emphasize that the improvement in the DA-ROM accuracy over the standard ROM accuracy is significantly larger in the realistic case of inaccurate snapshot construction. Indeed, comparing figures 5.4 and 5.5 with figure 5.2, we notice that the absolute improvement in the DA-ROM is much larger in the former than in the latter (this could be clearly seen from the magnitude of the $y$-axis).

### 5.2.4 Adaptive Nudging

To further improve the accuracy of the DA-ROM solution, we also consider nudging that is adaptive in time. While the error estimate we prove guarantees convergence up to discretization error and ROM truncation error exponentially fast in time, it may not be sufficient to expect good numerical results. In practice, the ROM truncation error is often quite large, and can make the error bounds be too large to guarantee accurate predictions, especially over long time intervals. We propose below an adaptive nudging technique that will help produce better results by forcing the DA-ROM predicted energy to be more accurate.

#### 5.2.4.1 Algorithm

In this section, we propose to change $\mu$ adaptively in time, based on the accuracy of the energy prediction of the ROM as well as the sign of the contribution of the data
assimilation term to the energy balance. The semi-discrete algorithm reads: Find \( u_r \in X_r \) such that for all \( v_r \in X_r \),

\[
((u_r)_t, v_r) + b(u_r, u_r, v_r) + \nu (\nabla u_r, \nabla v_r) + \mu (I_H(u_r - u), I_H v_r) = (f, v_r),
\]

(5.23)

with \( v_0 = P_r(u_0) \), and \( \mu \) is the adaptive nudging parameter.

We begin the discussion with an energy estimate. Choosing \( v_r = u_r \) vanishes the nonlinear term, and after bounding the forcing term in the usual way we obtain the energy estimate

\[
\frac{d}{dt} \| u_r \|^2 + \nu \| \nabla u_r \|^2 + \mu \left( \| I_H(u_r) \|^2 - \| I_H(u) \|^2 + \| I_H(u_r - u) \|^2 \right) \leq \nu^{-1} \| f \|^2_{-1}.
\]

We assume this estimate is sharp in the following analysis, and that we know \( \| u(t^n) \| \) in addition to \( I_H(u(t^n)) \).

The adaptive strategy is to adjust \( \mu \) so the contribution of the data assimilation term removes dissipation if the ROM-DA energy is too small, and adds dissipation if the energy is too large. We use the term dissipation loosely, since here we refer to dissipation from the DA term only meaning that it adds positivity to the left hand side of the energy estimate. Now after step \( n \) we can calculate (1) the DA-ROM energy \( \frac{1}{2} \| u_r^n \|^2 \) and the true energy \( \frac{1}{2} \| u(t^n) \|^2 \); and (2) the sign of the contribution of the data assimilation term (DAT):

\[
DAT := \| I_H(u_r^n) \|^2 - \| I_H(u(t^n)) \|^2 + \| I_H(u_r - u)(t^n) \|^2.
\]

With this information, we check the energy error to see if it is too high (or too low), and if so, then add dissipation by increasing \( \mu \) if \( DAT > 0 \) and decreasing \( \mu \) otherwise;
or do the opposite to decrease dissipation.

How often to adjust $\mu$, and by how much each time, are interesting questions. In our numerical tests below, we checked the value of $DAT$ every 10 time steps, since there is some calculation cost involved, and changed $\mu$ by $\pm 1$ each time, as large sudden changes in $\mu$ gave bad results.

### 5.2.4.2 Numerical Results

We follow the same problem set up outlined in Section 4.2 (again using the full ROM basis), but now choosing $\mu$ adaptively in time. We note that, in addition to the Reynolds number we considered in the previous numerical experiments (i.e., $Re = 500$), we also consider $Re = 1000$. Figures 5.6 and 5.7 show the energy and drag plots for the DA-ROM algorithm with the adaptive nudging described above, and for constant $\mu$, for no DA. For both $Re$, the adaptive DA-ROM yields the most accurate results, outperforming the ROM without DA, and the DA-ROM with a constant $\mu$. Also included are plots of the $\mu$ values chosen by the algorithm at each timestep. We observe that the behavior of the values of $\mu$ is similar to that of the plots of $DAT$ in the figures.
Figure 5.6: Energy and drag coefficient versus time for $Re = 500$ DA-ROM with different choices of $\mu$, with $N = 8$ modes and $H = \frac{22}{20}$. Also included are the optimal choices of $\mu$ and the energy terms versus time, for the adaptive $\mu$ simulation.
Figure 5.7: Energy and drag coefficients versus time for $Re = 1000$ DA-ROM with different choices of $\mu$, with $N = 8$ modes and $H = \frac{22}{20}$. Shown at the bottom is $\mu$ and the contribution of the DA term versus time, for the adaptive $\mu$ simulation.
Chapter 6

Simple and efficient continuous data assimilation of evolution equations via algebraic nudging

In this chapter we are introducing a new interpolation operator to use with continuous DA algorithms of time-dependent PDE that are discretized in space with the finite element method. This operator can be thought of as an approximation to the $L^2$ projection onto piecewise constant functions defined on the coarse mesh used for the observational data. It allows the DA nudging to be done entirely at the linear algebraic level that results in only adding a diagonal matrix to the existing FEM algorithm. This means that we have completely eliminated the need to construct the coarse mesh.

We prove stability and accuracy properties of the operator, and then apply it to both fluid transport and NSE. In each situation, we are able to show that the DA solutions converge to the true solution up to discretization error, independent of the initial
conditions. Results of several numerical tests are also given, which both illustrate the
theory and demonstrate its usefulness on practical problems.

6.1 A new interpolation operator for efficient continuous data assimilation

Let \( X_h = P_k(\tau_h), \ k \geq 1, \) be a FE space consisting of globally continuous piecewise
degree \( k \geq 1 \) polynomials on a regular triangular mesh \( \tau_h \), and \( X_H = P_0(\tau_H) \) be a
FE space consisting of piecewise constant functions over a coarser mesh \( \tau_H \); denote
the maximum element diameters of these meshes by \( h \) and \( H \), respectively (and we
assume that \( 0 < h \leq H \leq 1 \)). See figure 6.1 for an example of two such meshes.

We make the assumption that every element of \( \tau_H \) contain at least one node from \( \tau_h \),
which is expected to be true since \( \tau_H \) is typically much coarser than \( \tau_h \) in practice. For
our purposes, one can assume that the nodes of \( \tau_H \) are the points where observations
of the true solution are made. Additionally, we assume that the mesh \( \tau_h \) is sufficiently
regular so that the inverse inequality holds for functions in \( X_h \).

Denote by \( \{x_j\}_{j=1}^M \) the set of nodes of \( X_h \), and \( \{x_{k_j}\}_{j=1}^N \) the set of nodes of \( X_H \),
noting that each coarse mesh node is also a fine mesh node. Further, the coarse mesh
nodes also satisfy the property that for each coarse mesh element \( E_j^H \), the node \( x_{k_j} \)
is contained in element \( E_j^H \) and is closest to its center. The assumed relationship
between the fine and coarse meshes guarantees the existence of such a node for each
element.
Denote the basis functions of these two FE spaces by

\[ X_h : \{ \psi_1, \psi_2, \psi_3, \ldots, \psi_M \}, \]
\[ X_H : \{ \phi_1, \phi_2, \phi_3, \ldots, \phi_N \}. \]

We assume the usual property of FE basis functions that \( \psi_i(x_j) = 1 \) if \( i = j \) and 0 otherwise, and \( \phi_i(x_{k_j}) = 1 \) if \( j = i \) and 0 otherwise. Note that this implies that \( \phi_i = 1 \) on all of \( E_h^i \) since basis functions of \( X_H \) are piecewise constant.

To help define our new interpolant, we first consider the usual \( L^2 \) projection of a function \( u \in L^2(\Omega) \) onto \( X_H \), which is defined by: Find \( P_{L^2}^H(u) \in X_H \) satisfying

\[ (P_{L^2}^H(u), v_H) = (u, v_H) \quad \forall v_H \in X_H, \]

which is equivalent to

\[ (P_{L^2}^H(u), \phi_j) = (u, \phi_j) \]

holding for all \( 1 \leq j \leq N \). Since \( P_{L^2}^H(u) \in X_H \), we can write \( P_{L^2}^H(u) = \sum_{m=1}^{N} \beta_m \phi_m \), and thus

\[ \sum_{m=1}^{N} \beta_m (\phi_m, \phi_j) = (u, \phi_j) \]

for all \( 1 \leq j \leq N \). Since \( X_H \) consists of piecewise constant basis functions with non-overlapping support, each of these equations reduces, yielding for \( j = 1, 2, \ldots, N \),

\[ \beta_j (\phi_j, \phi_j) = (u, \phi_j) \implies \beta_j = \frac{1}{\text{meas}(E_j^h)} \int_{E_j^h} u \, dx. \]  (6.1)
We now define our new interpolation operator, denoted $\tilde{P}_{L^2}^H$, by

$$\tilde{P}_{L^2}^H(u) = \sum_{j=1}^{N} u(x_{k_j}) \phi_j. \quad (6.2)$$

This operator can be considered an approximation of $P_{L^2}^H$, as it differs only in that the last integral in (6.1) is approximated with a quadrature rule that is exact on constant functions. Indeed, if on each coarse mesh element $E_j^H$ we make the quadrature approximation in (6.1) by

$$\int_{E_j^H} u \, dx \approx u(x_{k_j}) \text{meas}(E_j^H),$$

then the interpolation operator $\tilde{P}_{L^2}^H$ is recovered from $P_{L^2}^H$.

### 6.1.1 Implementation of the nudging term with interpolation operator $\tilde{P}_{L^2}^H$

A key property of $\tilde{P}_{L^2}^H$ is how it acts on the basis functions of $X_h$. For each basis function $\psi_i$ of $X_h$, we calculate using (6.2) that

$$\tilde{P}_{L^2}^H(\psi_i) = \sum_{j=1}^{N} \psi_i(x_{k_j}) \phi_j = \begin{cases} \phi_j & \text{if } i = k_j, \\ 0 & \text{else.} \end{cases}$$

Thus for each coarse mesh (piecewise constant) basis function, there is exactly one fine mesh basis function that $\tilde{P}_{L^2}^H$ maps to it (the $k_j$th basis function); all other fine mesh basis functions get mapped to zero by $\tilde{P}_{L^2}^H$. Hence for the $M$ finite element basis functions $\psi_i, i = 1, 2, ..., M$, the new operator $\tilde{P}_{L^2}^H$ maps $N$ of them one to one and onto the $X_H$ basis functions, and maps the other $M - N$ of them to 0.
Consider now the FE implementation of the nudging term using this new interpolation operator \( \tilde{P}_{L^2}^H \). It will be written in DA algorithms as (see sections 3 and 4) \(^1\)

\[
\mu(\tilde{P}_{L^2}^H(u_h), \tilde{P}_{L^2}^H(\chi_h)),
\]

and so creates a matrix contribution to the linear system of the form

\[
\mu D_{mn} = \mu(\tilde{P}_{L^2}^H(\psi_m), \tilde{P}_{L^2}^H(\psi_n)) = \mu \left( \sum_{i=1}^{N} \psi_m(x_{k_i})\phi_i, \sum_{j=1}^{N} \psi_n(x_{k_j})\phi_j \right) = \mu \sum_{j=1}^{N} \left( \psi_m(x_{k_j})\phi_j, \psi_n(x_{k_j})\phi_j \right),
\]

with the last step holding since the \( \phi_i \)'s are non-overlapping piecewise constants. But since \( \psi_m(x_{k_j}) \) is only nonzero if \( m = k_j \), we have shown that

\[
D_{mn} = \begin{cases} 
\text{meas}(E_j^H) & \text{if } n = m = k_j, \\
0 & \text{else}.
\end{cases}
\]

This reveals that \( D \) is diagonal, and is nonzero only at entries \((k_j, k_j)\).

The right hand side nudging term takes the form \( \mu(\tilde{P}_{L^2}^H(u_{true}), \tilde{P}_{L^2}^H(\chi_h)) \), and we can

\(^1\)While it is typical for the weak formulation of DA nudging terms to take the form \( \mu(I_H(u_h), v_h) \), additionally applying the interpolation operator to the test function is necessary for a simple and efficient implementation, and as we show in later sections this does not adversely affect stability or convergence results of the associated DA algorithms.
Similarly derive
\[ \mu(\tilde{P}_{L^2}^H(u_{\text{true}}), \tilde{P}_{L^2}^H(\psi_m)) = \mu \left( \sum_{i=1}^{N} u_{\text{true}}(x_{k_i}) \phi_i, \sum_{j=1}^{N} \psi_m(x_{k_j}) \phi_j \right) \]
\[ = \mu \sum_{j=1}^{N} \left( u_{\text{true}}(x_{k_j}) \phi_j, \psi_m(x_{k_j}) \phi_j \right). \]

Hence if \( m \neq k_j \) for all \( 1 \leq j \leq N \), then the term is zero, but otherwise
\[ \mu(\tilde{P}_{L^2}^H(u_{\text{true}}), \tilde{P}_{L^2}^H(\psi_m)) = \mu u_{\text{true}}(x_{k_j}) \text{ meas}(E_H^j). \tag{6.3} \]

Denoting the vector \( \hat{u}_{\text{true}} \) by \( \hat{u}_{\text{true}j} = u_{\text{true}}(x_j) \), we can write this right hand side nudging contribution as \( \mu D \hat{u}_{\text{true}} \).

Remark 18. While it would complicate the convergence analysis in the following sections (but not affect the main result), algebraic nudging could be implemented with
\[ \tilde{\mu} D_{mn} = \begin{cases} \tilde{\mu} & \text{if } n = m = k_j, \\ 0 & \text{else}. \end{cases} \]
In this case, one could still consider \( D \) to be the matrix arising from nudging, but with \( \mu \) chosen locally to produce \( \tilde{\mu} \). On quasi-uniform meshes, this could be a reasonable approach, if an even simpler implementation is desired.

6.1.2 Properties of \( \tilde{P}_{L^2}^H \)

We now prove the fundamental stability and accuracy properties for the new interpolation operator. This result assumes the definitions and assumptions above for the meshes \( \tau_h \) and \( \tau_H \), and finite element spaces \( X_h \) and \( X_H \).
Lemma 8. Let \( \epsilon > 0 \) and suppose \( w \in L^\infty(\Omega) \) and \( \nabla w \in L^{d+\epsilon}(\Omega) \), \( d = 2 \) or \( 3 \). Then

\[
\| \tilde{P}_{L^2}^H(w) - w \| \leq C H \| \nabla w \|_{L^{d+\epsilon}(\Omega)}. \tag{6.4}
\]

Proof. We begin the proof by using the triangle inequality and Proposition 1.135 in [21] for approximation error in the \( L^2 \) projection \( P_{L^2}^H \), which provides

\[
\| \tilde{P}_{L^2}^H(w) - w \| \leq \| \tilde{P}_{L^2}^H(w) - P_{L^2}^H(w) \| + \| P_{L^2}^H(w) - w \| \\
\leq \| \tilde{P}_{L^2}^H(w) - P_{L^2}^H(w) \| + C H \| \nabla w \|. \tag{6.5}
\]

Hence it remains to bound \( \| \tilde{P}_{L^2}^H(w) - P_{L^2}^H(w) \| \). Expanding \( \tilde{P}_{L^2}^H(w) \) and \( P_{L^2}^H(w) \) in the \( X_H \) basis, and using their definitions yields

\[
P_{L^2}^H(w) = \sum_{i=1}^{N} \alpha_j \phi_i, \quad \left( \alpha_j = \frac{1}{\text{meas}(E_j^H)} \int_{E_j^H} w \, dx \right),
\]

\[
\tilde{P}_{L^2}^H(w) = \sum_{i=1}^{N} \beta_j \phi_i, \quad \left( \beta_j = w(x_k) \right).
\]

Now subtracting their difference inside of the \( L^2(\Omega) \) norm yields, thanks to these basis functions being non-overlapping piecewise constant functions,

\[
\| \tilde{P}_{L^2}^H(w) - P_{L^2}^H(w) \| = \left\| \sum_{i=1}^{N} (\alpha_i - \beta_i) \phi_i \right\| = \sum_{i=1}^{N} \left| \alpha_i - \beta_i \right| \| \phi_i \| = \sum_{i=1}^{N} \text{meas}(E_i^H) \cdot \left| \alpha_i - \beta_i \right|.
\tag{6.6}
\]

Consider now \( |\alpha_i - \beta_i| \). Using the definitions of \( \alpha_i \) and \( \beta_i \), we expand the difference by

\[
|\alpha_i - \beta_i| = \left| \frac{1}{\text{meas}(E_i^H)} \int_{E_i^H} w \, dx - w(x_k) \right| = \left| \frac{1}{\text{meas}(E_i^H)} \int_{E_i^H} w \, dx - w(x_k) \cdot \text{meas}(E_i^H) \right|.
\]
Combining with (6.6), this gives
\[ \| \tilde{P}_L^H(w) - P_L^H(w) \| = \sum_{i=1}^N \left| \int_{E_i^H} w \, dx - w(x_k) \, \text{meas}(E_i^H) \right|. \]

The differences in the sum are precisely the errors in a quadrature rule over \( E_i^H \) that is exact on constants. Applying Lemma 8.4 from [21] with \( s = 1, p = 2 + \epsilon, p' = \frac{2+\epsilon}{1+\epsilon} \), \( k_q = 0 \), for \( d = 2 \), we have \( sp > d \) and thus
\[ \left| \int_{E_i^H} w \, dx - w(x_k) \, \text{meas}(E_i^H) \right| \leq C \text{meas}(E_i^H)^{\frac{1}{p'}} \| \nabla w \|_{L^{2+\epsilon}(E_i^H)}. \]

Combining the last two equations and then following the proof of Theorem 8.5 in [21] by using Hölder’s inequality we obtain
\[ \| \tilde{P}_L^H(w) - P_L^H(w) \| \leq C H \sum_{i=1}^N \text{meas}(E_i^H)^{\frac{1}{p'}} \| \nabla w \|_{L^{2+\epsilon}(E_i^H)} \leq C H \| \nabla w \|_{L^{2+\epsilon}(\Omega)}. \] (6.7)

Combining this with (6.5) finishes the proof for (6.4). For the 3D case, we proceed similarly but with \( p = 3 + \epsilon \). \( \blacksquare \)

We next prove a result similar to Lemma 8, but for functions in \( X_h \).

**Lemma 9.** Let \( \epsilon > 0 \). For any \( w \in X_h \), there exists a \( C \) independent of \( H \) and \( \epsilon \) satisfying
\[ \| \tilde{P}_L^H(w) - w \| \leq C H h^{-\frac{3+\epsilon}{1+\epsilon}} \| \nabla w \| \, \text{in 3D}, \] (6.8)
\[ \| \tilde{P}_L^H(w) - w \| \leq C H h^{-\frac{3}{1+\epsilon}} \| \nabla w \| \, \text{in 2D}. \] (6.9)

**Proof.** We begin by noting that since \( w \) is a finite element function in \( X_h \), we have that \( w \in H^1(\Omega) \). Agmon’s inequality and the inverse inequality then imply that
\( w \in L^\infty(\Omega) \) and \( \nabla w \in L^{d+\epsilon}(\Omega) \). Since \( w \) is also continuous by the definition of \( X_h \), \( \tilde{P}^H_{L^2}(w) \) is well defined. Hence we have that Lemma 2.5 is applicable to \( w \in X_h \) and so

\[
\| \tilde{P}^H_{L^2}(w) - w \| \leq C H \| \nabla w \|_{L^{d+\epsilon}(\Omega)}.
\]

The results (6.8) and (6.9) now follow immediately from applying the inverse inequality (see [12] page 112).

\[ \blacksquare \]

**Assumption 19.** We make the assumption that for any \( w \in X_h \), or \( w \in L^\infty(\Omega) \) and \( \nabla w \in L^{d+\epsilon}(\Omega) \), there exists a \( C \) independent of \( H \) and \( h \) satisfying

\[
\| \tilde{P}^H_{L^2}(w) - w \| \leq C H \| \nabla w \|_{L^{d+\epsilon}(\Omega)}.
\]

The results of Lemmas 9 and 8 hold for any \( \epsilon > 0 \). Analytically, having \( \epsilon \) be positive has important regularity consequences. However, numerically, \( \epsilon \) can be taken sufficiently small so that its effect in the estimates of these lemmas is negligible, which is what we assume from here forward. Note that in 2D, we have that

\[
\| \tilde{P}^H_{L^2}(w) - w \| \leq C H \| \nabla w \|,
\]

which is the same bound as is given by the \( L^2 \) projection. In 3D, one half power of \( h \) is lost, and we point out in the analysis that follows the differences that arise between the 2D and 3D cases.
6.2 Application: Data assimilation in fluid transport equations

As a first application, we consider applying DA with the new interpolation operator to the fluid transport equation, given by

\[ c_t + U \cdot \nabla c - \epsilon \Delta c = f, \]
\[ c(0) = c_0, \]

where \( U \in L^\infty(\Omega) \) is constant with respect to time, and with boundary conditions \( c|_{\Gamma_1} = 0 \) and \( \nabla c \cdot n|_{\Gamma_2} = 0 \), where \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) and \( \text{meas}(\Gamma_1 \cap \Gamma_2) = 0 \). The DA algorithm we consider is given as follows, with a regular, conforming finite element mesh \( \tau_h \), function space

\[ X_h = \{ v \in H^1(\Omega), v|_{\Gamma_1} = 0 \} \cap P_k(\tau_h), \]

and appropriately chosen coarse mesh \( \tau_H \) and \( X_H = P_0(\tau_H) \) (constructed as discussed in section 2). For simplicity of analysis, we will assume a smooth boundary and that \( \partial \Omega = \Gamma_1 \), however in the numerical tests we do use mixed boundary conditions.

The DA algorithm we consider reads as follows, with a BDF2 temporal discretization and FE spatial discretization.

**Algorithm 6.2.1.** Given any initial conditions \( c_h^0, c_h^1 \in X_h \), divergence free velocity field \( U \in (L^\infty(\Omega))^d \), forcing \( f \in L^\infty(0, \infty; L^2(\Omega)) \), true solution \( c \in L^\infty(0, \infty; H^2(\Omega)) \), and nudging parameter \( \mu \geq 0 \), find \( c_h^{n+1} \in X_h \) for \( n = 1, 2, \ldots \), satisfying for all
\( \chi_h \in X_h, \)

\[
\frac{1}{2\Delta t} \left( 3c_h^{n+1} - 4c_h^n + c_h^{n-1}, \chi_h \right) + (U \cdot \nabla c_h^{n+1}, \chi_h) \\
+ \epsilon (\nabla c_h^{n+1}, \nabla \chi_h) + \mu (\tilde{P}^H L_2(c_h^{n+1} - c(t^{n+1})), \tilde{P}^H L_2 \chi_h) = (f^{n+1}, \chi_h). \quad (6.13)
\]

The implementation of Algorithm 6.2.1 is rather straightforward. Standard finite element packages can construct the matrices \( M, S, \) and \( N \) arising from \((c_h^{n+1}, \chi_h), (\nabla c_h^{n+1}, \nabla \chi_h),\) and \((U \cdot \nabla c_h^{n+1}, \chi_h),\) respectively. Once the observation points \( \{x_{k_j}\} \) are defined, a coarse mesh can be constructed so that element \( E_j^H \) contains \( x_{k_j}. \) Exactly how to construct the coarse mesh is somewhat arbitrary, so long as the elements are convex, one can calculate the area of each element, and a minimum angle condition is enforced. Once this is done, the diagonal nudging matrix \( D \) can be constructed, as defined above. This gives, at each time step, the linear algebraic system for the unknown coefficient vector

\[
\left( \frac{1.5}{\Delta t} M + N + \epsilon S + \mu D \right) \hat{c}^{n+1} = \frac{1}{\Delta t} M \left( 2\hat{c}^n - \frac{1}{2}\hat{c}^{n-1} \right) + \hat{f}^{n+1} + \mu D \hat{c}_{true}^{n+1}.
\]

In this way, the method can easily be adapted to enable DA to work with existing and/or legacy codes.

### 6.2.1 Analysis of the DA algorithm

We prove in this section the long-time stability, well-posedness and accuracy of Algorithm 6.2.1. We begin with stability and well-posedness. In all of our analysis, we invoke the \( G \)-norm and \( G \)-stability theory often used with BDF2 analysis, see e.g. [45, 19].
Lemma 10. For any $\Delta t > 0$, $\epsilon > 0$, and $\mu \geq 0$, Algorithm 6.2.1 is well-posed globally in time, and solutions are long time stable: for any $n > 1$,

\[
\left( C^{-2}_u (\|c^n_h\|^2 + \|c^n_h\|^2) + \frac{\epsilon \Delta t}{4} \|\nabla c^{n+1}_h\|^2 \right) \\
\leq \left( C^{-2}_u (\|c^n_h\|^2 + \|c^n_h\|^2) + \frac{\epsilon \Delta t}{4} \|\nabla c^n_h\|^2 \right) \left( \frac{1}{1 + \lambda \Delta t} \right)^n \\
+ C\epsilon^{-1} \lambda^{-1} \|f\|_{L^2(0,\infty;H^{-1})} + C\mu \lambda^{-1} \left( H^2 \|\tilde{c}\|_{L^2(0,\infty;H^2)} + \|\tilde{c}\|_{L^2(0,\infty;L^2)} \right),
\]

where $\lambda = \min\{2\Delta t^{-1}, \frac{C \epsilon^{-2} c^2}{2}\}$.

Proof. Choose $\chi_h = c^{n+1}_h$, which vanishes the convective term, and, after dropping the non-negative term $\frac{1}{4\Delta t} \|c^{n+1}_h - 2c^n_h + c^{n-1}_h\|^2$ on the left hand side, yields

\[
\frac{1}{2\Delta t} \|[c^{n+1}_h;c^n_h]\|^2 + \epsilon \|\nabla c^{n+1}_h\|^2 + \mu \|\tilde{P}_L^H c^{n+1}_h\|^2 \\
\leq \frac{1}{2\Delta t} \|[c^n_h;c^{n-1}_h]\|^2 + \mu \|\tilde{P}_L^H c^{n+1}_h\| + |(f^{n+1}, c^{n+1}_h)|. \tag{6.14}
\]

The nudging term on the right hand side is bounded using Cauchy-Schwarz and Young’s inequalities to obtain

\[
\mu \|\tilde{P}_L^H c(t^{n+1}), \tilde{P}_L^H c^{n+1}_h\| \leq C\mu \|\tilde{P}_L^H c(t^{n+1})\|^2 + \frac{H}{2} \|\tilde{P}_L^H c^{n+1}_h\|^2.
\]

The forcing term is bounded using the $H^{-1}(\Omega)$ norm as well as Young’s inequality, via

\[
|(f^{n+1}, c^{n+1}_h)| \leq \frac{\epsilon^{-1}}{2} \|f^{n+1}\|^2 + \frac{\epsilon}{2} \|\nabla c^{n+1}_h\|^2.
\]

Replacing the right hand side of (6.14) with these bounds, using the assumed uniform
bounds in time of $f$, and multiplying by $2\Delta t$, we obtain

$$
\|[cn^+; c^n_h]\|_G^2 + \epsilon \Delta t \|\nabla c_h^{n+1}\|^2 + \mu \Delta t \|\tilde{P}L_2^H c_h^{n+1}\|^2 \\
\leq \|[cn^n; c_{n-1}^h]\|_G^2 + C \mu \Delta t \|\tilde{P}L_2^H c(t^{n+1})\|^2 + \epsilon^{-1} \Delta t \|f\|^2_{L^\infty(0,\infty;H^{-1})}.
$$

Next, drop the positive nudging term on the left hand side and add $\frac{\epsilon \Delta t}{4} \|\nabla c^n_h\|^2$ to both sides of the equation to obtain

$$
\left(\|[cn^+; c^n_h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c_h^{n+1}\|^2\right) + \frac{\epsilon \Delta t}{4} \left(\|\nabla c_h^{n+1}\|^2 + \|\nabla c^n_h\|^2\right) + \frac{\epsilon \Delta t}{2} \|\nabla c^n_{n+1}\|^2 \\
\leq \|[cn^n; c_{n-1}^h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c^n_h\|^2 + C \mu \Delta t \|\tilde{P}L_2^H c(t^{n+1})\|^2 + \epsilon^{-1} \Delta t \|f\|^2_{L^\infty(0,\infty;H^{-1})},
$$

which then reduces using $G$-norm equivalence and Poincaré’s inequality,

$$
\left(\|[cn^+; c^n_h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c_h^{n+1}\|^2\right) + \frac{\epsilon \Delta t}{4} \left(\|[cn^+; c^n_h]\|_G^2 + \frac{\epsilon \Delta t}{2} \|\nabla c^n_{n+1}\|^2\right) \\
\leq \|[cn^n; c_{n-1}^h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c^n_h\|^2 + C \mu \Delta t \|\tilde{P}L_2^H c(t^{n+1})\|^2 + \epsilon^{-1} \Delta t \|f\|^2_{L^\infty(0,\infty;H^{-1})}.
$$

(6.15)

Using $\lambda = \min \left\{2\Delta t^{-1}, \frac{\epsilon C_P^{-2}C^2_l}{4}\right\}$, equation (6.15) can be written as

$$
(1 + \lambda \Delta t) \left(\|[cn^+; c^n_h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c_h^{n+1}\|^2\right) \\
\leq \left(\|[cn^n; c_{n-1}^h]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla c^n_h\|^2\right) + \Delta t(C \mu \|\tilde{P}L_2^H c(t^{n+1})\|^2 + \epsilon^{-1} \|f\|^2_{L^\infty(0,\infty;H^{-1})}).
$$

(6.16)
We next use Lemma 2 to write
\[
\|c_n^{n+1}; c_n^h\|_G^2 + \frac{\epsilon \Delta t}{4} \| \nabla c_n^{n+1}\|^2 \\
\leq \left( \frac{1}{1 + \lambda \Delta t} \right)^n \left( \|c_1^h; c_0^h\|_G^2 + \frac{\epsilon \Delta t}{4} \| \nabla c_1^h\|^2 \right) + \lambda^{-1}(C\mu \| \tilde{P}_{L^2} c(t^{n+1})\|^2 + \epsilon^{-1} \|f\|_{L^\infty(0,\infty;H^{-1})}).
\]

Lastly, we bound the \( \tilde{P}_{L^2} c(t^{n+1}) \) term using the triangle inequality, Lemma 8, and a Sobolev inequality to get
\[
C\mu \| \tilde{P}_{L^2} c(t^{n+1})\|^2 \leq C\mu \| \tilde{P}_{L^2} c(t^{n+1}) - c(t^{n+1})\|^2 + C\mu \|c(t^{n+1})\|^2 \\
\leq C\mu H^2 \|c(t^{n+1})\|_{H^2}^2 + C\mu \|c(t^{n+1})\|^2.
\]

The stability result is completed using the \( G \)-norm equivalence and assumed regularity. At each time step, the scheme is linear and finite dimensional, and thus this stability result immediately implies existence and uniqueness of the solutions, and thus well-posedness of the algorithm.

Next, we prove that solutions to Algorithm 6.2.1 converge to the true solution, exponentially fast in time, up to discretization error, provided restrictions on \( H \) and \( \mu \) are satisfied. Our analysis will use the \( H^1_0 \) projection onto \( X_h \), denoted by \( \pi_h \) and defined by: Given \( \phi \in H^1(\Omega) \), \( \pi_h \phi \in X_h \) satisfies
\[
(\nabla \pi_h \phi, \nabla v_h) = (\nabla \phi, \nabla v_h)
\]
for all \( v_h \in X_h \). For \( \phi \in H^1_0(\Omega) \), we have the following estimate [53],
\[
\| \pi_h \phi - \phi \| + h \| \nabla (\pi_h \phi - \phi)\| \leq C h^{k+1} |\phi|_{k+1}, \quad (6.17)
\]
where $| \cdot |_{k+1}$ denotes the $(k+1)$st seminorm.

**Theorem 20.** Let $c \in L^\infty(0, \infty; H^{k+1}(\Omega))$ denote the true solution to the fluid transport equation with given $f \in L^\infty(0, \infty; L^2(\Omega))$, initial condition $c^0, c^1 \in H^1(\Omega)$, and $c_t, c_{tt}, c_{ttt} \in L^\infty(0, \infty; L^2(\Omega))$. Then for any $\mu \geq 0$ and $\Delta t > 0$, the difference between the DA solution and the true solution satisfies, for all $n$,

$$
\frac{1}{\lambda \Delta t} \left( \left\| c(t^n) - c^n_h \right\|_2^2 + \frac{\epsilon \Delta t}{4} \left\| \nabla (c^1 - c^1_h) \right\|_2^2 \right) + R \leq C \left( 1 + \frac{1}{\lambda \Delta t} \right)^n \left( \left\| c^0 - c^0_h \right\|_2^2 + \left\| c^1 - c^1_h \right\|_2^2 \right) + \frac{R}{\lambda}, \quad (6.18)
$$

where $R = C\epsilon^{-1}h^{2k+2} + C\epsilon^{-1}\Delta t^4 + C\mu(h^{2k+2} + H^2h^{2k}) + C\mu^{-1}h^{2k+2}$ and

$$
\lambda = \min \left\{ 2\Delta t^{-1}, \frac{\epsilon C^{-2}C^2}{4} \right\}.
$$

Furthermore, if $\mu \leq \frac{\epsilon}{CH^2h^{2+\frac{d}{2}}}$, then we can take

$$
\lambda = \min \left\{ 2\Delta t^{-1}, \frac{(\epsilon-C\mu H^2h^{2-\frac{d}{2}})C^{-2}+\mu)C^2}{4} \right\}
$$

and get the same bound (6.18) but with

$$
R = C\epsilon^{-1}h^{2k+2} + C\epsilon^{-1}\Delta t^4 + C\mu(h^{2k+2} + H^2h^{2k}h^{\frac{d-2}{2}}) + C\mu^{-1}h^{2k+2}
$$

and

$$
\lambda = \min \left\{ 2\Delta t^{-1}, \frac{\epsilon C^{-2}C^2}{4} \right\}.
$$

**Remark 21.** It is no surprise with fluid transport that the DA algorithm will converge to the true solution (up to discretization error), even when $\mu = 0$. This is because the initial condition will eventually diffuse away, leaving the forcing (and boundary conditions) to drive the system. Hence if the algorithm has the correct forcing and boundary conditions, then it must converge to the true solution even without nudging, as we prove below. However, if $H$ and $\mu$ satisfy the stated restriction, we have proven that the DA nudging can significantly speed up the convergence to the true solution, and we observe exactly this phenomena in our numerical tests.

**Remark 22.** If we consider $H$ to be related to $h$ in the sense of $H = ch$, then in 2D
the above estimate is optimal and the restriction of $\mu$ and the coarse mesh width $H$ is the same as what is found in recent literature, e.g. [64, 4, 5]. In 3D, however, the estimate is suboptimal by $\frac{1}{2}$ power of $h$ in the $L^2$ norm, and the restriction on $\mu$ and $H$ requires the coarse mesh width to be finer than in the 2D case.

**Proof.** After applying Taylor’s theorem, the true solution $c(t^n) =: c^n$ satisfies, for all $\chi_h \in X_h$,

$$
\frac{1}{2\Delta t} (3c^{n+1} - 4c^n + c^{n-1}, \chi_h) + (U \cdot \nabla c^{n+1}, \chi_h) + \epsilon(\nabla c^{n+1}, \nabla \chi_h)
= (f^{n+1}, \chi_h) + \frac{\Delta t^2}{3} (c_{ttt}(t^*), \chi_h),
$$

where $t^* \in [t^{n-1}, t^{n+1}]$. Subtracting (6.13) and (6.19) and letting $e^n = c_h^n - c^n$ yields the difference equation

$$
\frac{1}{2\Delta t} (3e^{n+1} - 4e^n + e^{n-1}, \chi_h) + \epsilon(\nabla e^{n+1}, \nabla \chi_h) + \mu(\tilde{P}^H_L e^{n+1}, \tilde{P}^H_L \chi_h)
= -(U \cdot \nabla e^{n+1}, \chi_h) - \frac{\Delta t^2}{3} (c_{ttt}(t^*), \chi_h).
$$

(6.20)

Now decompose the error by adding and subtracting $\pi_h(c^n)$ to $e^n$, denote $\eta^n = \pi_h(c^n) - c^n$ and $\phi_h^n = c_h^n - \pi_h(c^n)$ so that $e^n = \eta^n + \phi_h^n$, with $\phi_h^n \in X_h$. Choose $\chi_h = \phi_h^{n+1}$ and use the $G$-stability framework to write the difference equation as

$$
\frac{1}{2\Delta t} \left( \|\phi_h^{n+1}; \phi_h^n\|_G^2 - \|\phi_h^n; \phi_h^{n-1}\|_G^2 \right) + \epsilon \|\nabla \phi_h^{n+1}\|^2 + \mu \|\tilde{P}^H_L \phi_h^{n+1}\|^2
\leq |(U \cdot \nabla \eta^{n+1}, \phi_h^{n+1})| + \frac{\Delta t^2}{3} |(c_{ttt}(t^*), \phi_h^{n+1})| + \mu |(\tilde{P}^H_L \eta^{n+1}, \tilde{P}^H_L \phi_h^{n+1})|
+ \left| \left( \frac{\eta^{n+1} - 4\eta^n + \eta^{n-1}}{2\Delta t}, \phi_h^{n+1} \right) \right|.
$$

(6.21)

Except for the nudging term, all right hand side terms above are bounded using
standard inequalities, as in [35, 66, 93, 22]:

\[
\|(U \cdot \nabla \eta^{n+1}, \phi_h^{n+1})\| \leq C \epsilon^{-1} \|\eta^{n+1}\|^2 + \frac{\epsilon}{8} \|\nabla \phi_h^{n+1}\|^2,
\]

\[
\frac{\Delta t^2}{3} |(c_{ttt}(t^*), \phi_h^{n+1})| \leq C \epsilon^{-1} \Delta t^4 \|c_{ttt}\|_{L^\infty(0,\infty;L^2)} + \frac{\epsilon}{8} \|\nabla \phi_h^{n+1}\|^2,
\]

\[
\frac{1}{2\Delta t} |(3\eta^{n+1} - 4\eta^n + \eta^{n-1}, \phi_h^{n+1})| \leq C \epsilon^{-1} \left( \|\eta_t\|_{L^\infty(0,\infty;L^2)}^2 + \int_{t_{n-1}}^{t_n} \|\eta_{tt}\|^2 dt \right) + \frac{\epsilon}{4} \|\nabla \phi_h^{n+1}\|^2.
\]

For the nudging term on the right hand side, we first apply Cauchy-Schwarz and Young’s inequalities, and then (6.10), which yields

\[
\mu |(\bar{P}_{L^2}^H \eta^{n+1}, \bar{P}_{L^2}^H \phi_h^{n+1})| \leq \mu \|\bar{P}_{L^2}^H \eta^{n+1}\|^2 + \frac{\mu}{4} \|\bar{P}_{L^2}^H \phi_h^{n+1}\|^2
\]

\[
\leq C \mu (\|\eta^{n+1}\|^2 + \|\bar{P}_{L^2}^H (\eta^{n+1} - \eta^{n+1})\|^2) + \frac{\mu}{4} \|\bar{P}_{L^2}^H \phi_h^{n+1}\|^2
\]

\[
\leq C \mu \left( \|\eta^{n+1}\|^2 + H^2 \|\nabla \eta^{n+1}\|^2 \right) + \frac{\mu}{4} \|\bar{P}_{L^2}^H \phi_h^{n+1}\|^2. \tag{6.22}
\]

Applying these bounds in (6.21), we next use inequality (6.17) and the assumed regularity on the true solution \(c\), drop the nudging term on the left hand side, and multiply by \(2\Delta t\) to get

\[
\|\phi_h^{n+1} - \phi_h^n\|_G^2 + \epsilon \Delta t \|\nabla \phi_h^{n+1}\|^2 \\
\leq \|\phi_h^n - \phi_h^{n-1}\|_G^2 + C \epsilon^{-1} \Delta t \|\eta^{n+1}\|^2 + C \epsilon^{-1} \Delta t^5 \|c_{ttt}\|_{L^\infty(0,\infty;L^2)} \\
+ C \mu \left( \|\eta^{n+1}\|^2 + H^2 \|\nabla \eta^{n+1}\|^2 + \frac{\mu}{4} \|\bar{P}_{L^2}^H \phi_h^{n+1}\|^2 \right) \\
+ C \epsilon^{-1} \Delta t \left( \|\eta_t\|_{L^\infty(0,\infty;L^2)}^2 + \int_{t_{n-1}}^{t_n} \|\eta_{tt}\|^2 dt \right) + C \epsilon^{-1} \Delta th^{2k+2}
\]

\[
\leq \|\phi_h^n - \phi_h^{n-1}\|_G^2 + C \Delta t \epsilon^{-1} h^{2k+2} + C \Delta t^5 + C \mu \Delta t \left( h^{2k+2} + H^2 h^{2k} \right). \tag{6.23}
\]
Setting $R := C\epsilon^{-1}h^{2k+2} + C\Delta t^4 + C\mu(h^{2k+2} + H^2h^{2k}) + C\epsilon^{-1}h^{2k+2}$ reveals
\[
\|[\phi_h^{n+1}; \phi_h^n]\|_G + \epsilon \Delta t\|\nabla \phi_h^{n+1}\|^2 \leq \|[\phi_h^n; \phi_h^{n-1}]\|_G + \Delta tR.
\]

We can now proceed as in the long time stability proof above to get
\[
\|[\phi_h^{n+1}; \phi_h^n]\|_G + \frac{\epsilon \Delta t}{4} \|\nabla \phi_h^{n+1}\|^2 \leq \left(\|[\phi_h^{n}; \phi_h^0]\|_G^2 + \frac{\epsilon \Delta t}{4} \|\nabla \phi_h^1\|^2\right) \left(\frac{1}{1 + \lambda \Delta t}\right)^n + R.
\]

G-norm equivalence and triangle inequality complete the first part of proof, without any restriction on $H$ or $\mu$.

We will now show that with an added restriction on $\mu$, we obtain a faster convergence rate to the true solution. Starting back at (6.20), add and subtract $\phi_h^{n+1}$ in both components of the nudging inner product on the left hand side to obtain
\[
\frac{1}{2\Delta t} \left(\|[\phi_h^{n+1}; \phi_h^n]\|_G^2 - \|[\phi_h^n; \phi_h^{n-1}]\|_G^2\right) + \epsilon \|\nabla \phi_h^{n+1}\|^2 + \mu \|\phi_h^{n+1}\|^2
\]
\[
\leq \left|(U \cdot \nabla \eta^{n+1}, \phi_h^{n+1})\right| + \frac{\Delta t^2}{3} \left|(ctu(t^*), \phi_h^{n+1})\right| + \mu \left|(\tilde{P}_{L^2}H\eta^{n+1}, \tilde{P}_{L^2}H\phi_h^{n+1})\right|
\]
\[
+ \mu \left|(\tilde{P}_{L^2}H\phi_h^{n+1} - \phi_h^{n+1})^2 + 2\mu \left|(\tilde{P}_{L^2}H\phi_h^{n+1} - \phi_h^{n+1}, \phi_h^{n+1})\right|\right.
\]
\[
\left. + \frac{1}{2\Delta t} \left|(3\eta^{n+1} - 4\eta^n + \eta^{n-1}; \phi_h^{n+1})\right|\right). \quad (6.24)
\]

This then leads to two additional right hand side terms to bound, and we have to adjust the bound on the original right hand side nudging term. We upper bound the first one with inequality (6.10), yielding
\[
\mu \|\tilde{P}_{L^2}H\phi_h^{n+1} - \phi_h^{n+1}\|^2 \leq C\mu H^2h^{\frac{d+2}{2}} \|\nabla \phi_h^{n+1}\|^2.
\]

In a similar manner, we start with Cauchy-Schwarz on the last nudging term, then
apply inequality (6.10) to obtain

\[ 2\mu |(\tilde{P}_{L^2}^H \phi_{h}^{n+1} - \phi_{h}^{n+1}, \phi_{h}^{n+1})| \leq C\mu \| \tilde{P}_{L^2}^H \phi_{h}^{n+1} - \phi_{h}^{n+1} \| \| \phi_{h}^{n+1} \| \]

\[ \leq C\mu H^2 h^{\frac{d-2}{2}} \| \nabla \phi_{h}^{n+1} \|^2 + \frac{\mu}{8} \| \phi_{h}^{n+1} \|^2. \]

Lastly, we need to adjust the way we bound the following DA term. To do so, we start by adding and subtracting \( \phi_{h}^{n+1} \) to the second term of the inner product then apply Cauchy-Schwarz and Young's inequalities to each term to get

\[ \mu(\tilde{P}_{L^2}^H \eta^{n+1}, \tilde{P}_{L^2}^H \phi_{h}^{n+1}) = \mu(\tilde{P}_{L^2}^H \eta^{n+1}, \tilde{P}_{L^2}^H \phi_{h}^{n+1} - \phi_{h}^{n+1}) + \mu(\tilde{P}_{L^2}^H \eta^{n+1}, \phi_{h}^{n+1}) \]

\[ \leq \mu \| \tilde{P}_{L^2}^H \eta^{n+1} \| \| \tilde{P}_{L^2}^H \phi_{h}^{n+1} - \phi_{h}^{n+1} \| + \mu \| \tilde{P}_{L^2}^H \eta^{n+1} \| \| \phi_{h}^{n+1} \| \]

\[ \leq C\mu \| \tilde{P}_{L^2}^H \eta^{n+1} \|^2 + C\mu \| \tilde{P}_{L^2}^H \phi_{h}^{n+1} - \phi_{h}^{n+1} \|^2 + \frac{\mu}{8} \| \phi_{h}^{n+1} \|^2 \]

\[ \leq C\mu \left( \| \eta^{n+1} \|^2 + H^2 h^{\frac{d-2}{2}} \| \nabla \eta^{n+1} \|^2 + H^2 h^{\frac{d-2}{2}} \| \nabla \phi_{h}^{n+1} \|^2 + \frac{1}{8} \| \phi_{h}^{n+1} \|^2, \right) \]

where in the last step we followed equivalent arguments as in (6.22) to further bound the first term, and applied inequality (6.10) to the second term. Replace the right hand side of (6.24) with the bounds above, reduce, and multiply by \( 2\Delta t \) to obtain

\[ \| [\phi_{h}^{n+1}; \phi_{h}^{n}] \|_{G}^2 + \Delta t (\epsilon - C\mu H^2 h^{\frac{d-2}{2}}) \| \nabla \phi_{h}^{n+1} \|^2 + \mu \Delta t \| \phi_{h}^{n+1} \|^2 \]

\[ \leq \| [\phi_{h}^{n}; \phi_{h}^{n-1}] \|_{G}^2 + C\epsilon^{-1} \Delta t \| \eta^{n+1} \|^2 + C\epsilon^{-1} \Delta t^5 \| c_{ut} \|^2_{L^\infty(0,\infty;L^2)} \]

\[ + C\mu \Delta t(\| \eta^{n+1} \|^2 + H^2 h^{\frac{d-2}{2}} \| \nabla \eta^{n+1} \|^2) + C\mu^{-1} \Delta t \| \eta^t \|^2_{L^\infty(0,\infty,L^2)} \]

\[ + C\mu^{-1} \Delta t \int_{t_{n-1}}^{t_{n-1}} \| \eta_t \|^2 dt \]

\[ \leq \| [\phi_{h}^{n}; \phi_{h}^{n-1}] \|_{G}^2 + C\Delta t \epsilon^{-1} h^{2k+2} + C\epsilon^{-1} \Delta t^5 + C\mu \Delta t (h^{2k+2} + H^2 h^{2k} h^{\frac{d-2}{2}}) \]

\[ + C\mu^{-1} \Delta t h^{2k+2}. \]
This using the assumed restriction on \( \mu \) and \( H \), and now with \( R := C\epsilon^{-1}h^{2k+2} + C\epsilon^{-1}\Delta t^4 + C\mu(h^{2k+2} + H^2h^{2k}h^{\frac{d-2}{2}}) + C\mu^{-1}h^{2k+2} \), we can complete the proof using similar arguments as above. ■

### 6.2.2 Numerical Tests

We now give results for numerical tests of Algorithm 6.2.1. We illustrate the predicted convergence rates with respect to the discretization parameters \( h \), \( H \) and \( \Delta t \), the convergence in time for changing \( \mu \) and \( H \), and also show the method works very well on a more practical test problem.

#### 6.2.2.1 Convergence rates

![Figure 6.1: Shown above is an example of a fine mesh and associated coarse mesh used in the convergence rate test.](image)

To test the convergence rates with respect to the discretization parameters, we select the true solution \( c = \sin(x + y + t) \) on \( \Omega = (0, 1)^2 \) with time domain \([0, 5] \), transport velocity \( U = (1, 0)^T \), and \( \epsilon = 1 \). The forcing \( f \) is calculated from this chosen solution and (6.11). We compute approximate solutions using Algorithm 6.2.1 with the
calculated \( f \), \( P_2 \) finite elements, Dirichlet boundary conditions enforced nodally to be equal to the true solution, zero initial conditions \( c_h^0 = c_h^1 = 0 \), \( \mu = 1 \), a uniform triangular mesh \( \tau_h \), and a uniform square mesh \( \tau_H \), see figure 6.1. Computations are done with varying \( h \), \( H \), and \( \Delta t \), but we tie the discretization parameters together via \( h = H/4 \) and \( \Delta t = Ch^{3/2} \) with \( C=0.9051 \). We then successively refine \( h \) (and thus \( H \) and \( \Delta t \) as well), and calculate the \( L^2 \) norm of the difference to the true solution at \( t=5 \). Due to the way the parameters are tied together, we expect from our above theory that \( \| c_h^n - c(t^n) \|_{L^2} = O(h^3) \), which is exactly what we observe in table 6.1.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( | c_h - c |_{L^2} )</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>9.1235e-05</td>
<td>-</td>
</tr>
<tr>
<td>1/16</td>
<td>1.1249e-05</td>
<td>3.02</td>
</tr>
<tr>
<td>1/32</td>
<td>1.4136e-06</td>
<td>2.99</td>
</tr>
<tr>
<td>1/64</td>
<td>1.7687e-07</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Table 6.1: Convergence rates of Algorithm 6.2.1 to the true solution at \( t=5 \), for varying \( h \), \( H = 4h \) and \( \Delta t = 0.9051h^{3/2} \). Third order convergence is observed, as predicted by the theory.

### 6.2.2.2 Effect of \( \mu \) and \( H \) on convergence to the true solution as \( t \to \infty \)

![Figure 6.2](image)

Figure 6.2: Shown above is the \( L^2 \) difference to the true solution versus time, for DA simulations of fluid transport with varying \( \mu \) and \( H \).

Our theory predicts that even with \( \mu = 0 \), the DA solution will converge to the true
solution (up to discretization error), exponentially fast in time. However, we also prove that under restrictions that \( \mu H^2 \) is sufficiently small, the speed of convergence will be increased as \( \mu \) increases (until it becomes so large that \( \mu H^2 \) is no longer sufficiently small). We test this theory now.

Repeating the test above for convergence rates, but now with \( \epsilon = 0.01 \), the fine mesh fixed to be a uniform triangular mesh with \( h = 1/32 \), and time step size \( \Delta t = 0.01 \) fixed. We then run Algorithm 6.2.1 with varying \( H \) and \( \mu \), calculating for each run the \( L^2 \) error versus time. Results of these tests are shown in figure 6.2. We observe that for the largest \( H = 1/4 \), convergence is only slightly increased with \( \mu = 1 \) over \( \mu = 0 \), and even very large \( \mu \) does not produce significant speed up in the convergence. However, as \( H \) is decreased, we observe that large \( \mu \) has a much greater impact; in particular, for \( H = 1/32 \), and \( \mu \geq 1,000 \), convergence to the true solution is almost immediate.

6.2.2.3 DA prediction of contaminant transport

Our last test for Algorithm 6.2.1 is on the following test problem, which is intended to simulate contaminant transport in a river. The domain is constructed from the curves \( y = \sin(x) \) and \( y = 1 + \sin(x) \) as lower and upper boundaries, with \( x = 0 \) and \( x = 4\pi \) as left and right boundaries. Using the mesh \( \tau_h \) shown in figure 6.3, we use \((P_2^2, P_1)\) Taylor-Hood elements (which gives a total of 13,928 dof) to compute a solution to the Stokes equations with viscosity 0.01 and zero forcing, using no-slip boundary conditions on the top and bottom boundaries, a plug inflow of \( u_{\text{in}} = 3 \), and a zero-traction outflow enforced with the do-nothing condition (see e.g. [66] for more details on FE implementation of Stokes equations). We take our transport velocity \( U \) to be the velocity solution of this discrete Stokes problem.
Fine mesh $\tau_h$

Coarse mesh $\tau_H$

Figure 6.3: Shown above are the fine and coarse meshes used in the fluid transport numerical test. The coarse mesh is created by intersecting the rectangular grid with the domain, and for each coarse mesh element $E_j^H$, we also plot the fine mesh node $x_{k_j}$ that is closest to the center of $E_j^H$.

Figure 6.4: Shown above are plots of convergence of the DA solutions to the true solution, for varying $\mu$.

We next solve (6.11)-(6.12) equipped with a boundary condition of 0 contaminant at the inflow ($c_{in} = 0$), the transport velocity $U$ from the discrete Stokes problem above, $\epsilon = 0.01$, and zero Neumann conditions $\nabla c \cdot n = 0$ at all other boundaries. For the initial condition, there is zero contaminant except for two ‘blobs’, which are represented with $c = 3$ inside the circles centered at $(1, 1.5)$ and $(5, -0.5)$, with radius 0.1 (see figure 6.5, top right plot). We compute a direct numerical simulation (DNS) for the concentration $c$ using Algorithm 6.2.1 with no data assimilation ($\mu = 0$), $P_2$.
on the same fine mesh as for the Stokes FE problem, and $\Delta t = 0.02$. Plots of the DNS solution are shown in figure 6.5 on the right side.

![Figure 6.5: Contour plots of DA and DNS velocity magnitudes at times 0, 0.5, 1, 2.5, and 4, with nudging parameter $\mu = 100$.](image)

Finally, we compute the DA solution, using Algorithm 6.2.1 with the same parameters as the DNS except zero initial conditions, taking the DNS solution $c$ as the true solution, and testing the algorithm using several choices of $\mu = \{0.01, 0.1, 1, 10, 100, 1000\}$. For the coarse mesh, we (purposely) choose a crude and simple mesh to show the robustness of the DA scheme, making a rectangular grid of $[0, 4\pi] \times [-1, 2]$, and in-
intersecting it with the domain (see figure 6.3 at bottom). The nodes $x_{kj}$ for the coarse mesh are the nodes from the fine mesh that are in element $E_j^H$ and closest to its center, and so for some elements, this node is on the boundary.

Convergence of the DA solution to the true (DNS) solution is shown in figure 6.4, for varying $\mu$, in relative $L^2$ norms of the difference (relative norms are used since the true solution decays significantly over this time period). We observe almost identical convergence for $\mu=1, 10, 100$, and 1000. The DA solution for $\mu = 0.1$ seems to also converge, but at a slower rate. Convergence for $\mu = 0.01$ is even slower, but does still appear to be converging.

Note that if $\mu = 0$, the relative error will always be one since the DA solution will always be 0 (0 initial condition, no forcing, homogenous Dirichlet and Neumann boundary conditions). The absolute error will go to zero for large enough $t$, but this corresponds to the case of all contaminant leaving the river through the outflow or finally diffusing away. Hence waiting for the solution to assimilate with no nudging is not useful.

For the case $\mu = 100$ (which is very closely resembled by the solutions to $\mu =1, 10,$ and 1000), we show contour plots of the DA and true (DNS) solution at $t=0, 0.5, 1,$ 2.5 and 4 in figure 6.5. We observe agreement between the solutions increasing, until finally by $t=4$ the solutions are visually indistinguishable.
6.3 Application: Data assimilation in incompressible Navier-Stokes equations

We consider now application of DA with the new interpolant for the incompressible NSE. The associated DA scheme we consider uses an IMEX BDF2 temporal discretization, finite element spatial discretization, and uses our new proposed interpolant. We again use the velocity-pressure finite element spaces \((X_h, Q_h) = ((P_k)^d, P_{k-1}) \cap (X, Q)\), where \(X\) and \(Q\) are the natural velocity and pressure spaces, respectively, \(k \geq 2\).

The scheme reads as follows.

\begin{algorithm}
Given any initial conditions \(v_h^0, v_h^1 \in V_h\), forcing \(f \in L^\infty(0, \infty; L^2(\Omega))\), true solution \(u \in L^\infty(0, \infty; H^2(\Omega))\), and nudging parameter \(\mu > 0\), find \((v_h^{n+1}, q_h^{n+1}) \in (X_h, Q_h)\) for \(n = 1, 2, \ldots\), satisfying

\[
\frac{1}{2\Delta t} \left( 3v_h^{n+1} - 4v_h^n + v_h^{n-1}, \chi_h \right) + b(2v_h^n - v_h^{n-1}, v_h^{n+1}, \chi_h) - (q_h^{n+1}, \nabla \cdot \chi_h) \nonumber
\]
\[
+ \nu(\nabla v_h^{n+1}, \nabla \chi_h) + \mu(\tilde{P}_L^H(v_h^{n+1} - u^{n+1}), \tilde{P}_L^H \chi_h) = (f^{n+1}, \chi_h), \tag{6.25}
\]
\[
(\nabla \cdot v_h^{n+1}, r_h) = 0, \tag{6.26}
\]

for all \((\chi_h, r_h) \in X_h \times Q_h\).
\end{algorithm}

6.3.1 Analysis of the DA algorithm for NSE

We now state a lemma for long-time stability and well-posedness. In our analysis, we will use the parameter \(\alpha := \nu - C \mu H^2 h^{d/2 - 2}\), where \(C\) depends on the size of the true solution. We will require that \(\alpha > 0\), which can be thought of as the coarse mesh \(H\) being sufficiently fine.
Lemma 11. Assume $\alpha > 0$. Then for any $\Delta t > 0$, Algorithm 6.3.1 is well-posed globally in time, and solutions are nonlinearly long-time stable: for any $n > 1$,

$$C_u^{-2}(v_{n+1}^h + v_n^h) \leq \left( C_t^{-2}(v_1^h + v_0^h) + \frac{\alpha \Delta t}{4} \|\nabla v_1^h\|^2 + \frac{\mu \Delta t}{4} \|v_1^h\|^2 \right) \left( \frac{1}{1 + \lambda \Delta t} \right)^n + C \lambda^{-1}(\nu^{-1} F^2 + \mu U^2).$$

where $\lambda = \min\{\frac{\mu C_2^2}{4}, \frac{\alpha C_1^{-2} C_3^2}{4}, 2\Delta t^{-1}\}$, and $F := \|f\|_{L^\infty(0,\infty;H^{-1})}$, $U := C\|u\|_{L^\infty(0,\infty;H^2)}$.

Proof. Well-posedness and long-time stability were proven in chapter 3 for a similar algorithm, except with a different treatment of the nudging term. This proof can be adapted with just minor modifications, using the analysis of the nudging term in the proof of Lemma 9 above, to immediately provide the long time stability result. With this established, well-posedness follows directly. ■

We now prove that solutions to Algorithm 6.3.1 converge to the true NSE solution (up to discretization error) in the $L^2$ norm, globally in time, provided restrictions on $\Delta t$ and $\mu$ are satisfied. The time derivative term will again be handled with the G-stability theory in a manner similar to the stability proof.

Theorem 23. Let $u, p$ solve the NSE (1.1)-(1.2) with given $f \in L^\infty(0, \infty; L^2(\Omega))$ and $u_0, u_1 \in H^1(\Omega)$, with $u \in L^\infty(0, \infty; H^{k+1}(\Omega))$, $p \in L^\infty(0, \infty; H^k(\Omega))$ ($k \geq 1$), $u_{tt} \in L^\infty(0, \infty; L^2(\Omega))$, and $u_{ttt} \in L^\infty(0, \infty; H^1(\Omega))$. Denote $U := \|u\|_{L^\infty(0,\infty;H^{k+1})}$ and $P := \|p\|_{L^\infty(0,\infty;H^k)}$. Assume that the time step size satisfies

$$\Delta t < CM^2 \nu^{-1} \left( h^{2k-3} U^2 + \|\nabla u^{n+1}\|_{L^3}^2 + \|u^{n+1}\|_{L^\infty}^2 \right)^{-1},$$

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and the parameter $\mu$ satisfies

$$CM^2\nu^{-1}\left(h^{2k-3}U^2 + \|\nabla u^{n+1}\|_{L^3}^2 + \|u^{n+1}\|_{L^\infty}^2\right) < \mu < \frac{2\nu}{CH^2h^{\frac{d-2}{2}}}.$$ 

Then if the boundary is sufficiently smooth so that the discrete Stokes projection has optimal approximation properties, the error in solutions to Algorithm 6.3.1 satisfy, for any $n$,

$$\|u^n - v_h^n\|^2 \leq C\left(\frac{1}{1 + \lambda \Delta t}\right)^n \left(\|u_0 - v_h^0\|^2 + \|u_1 - v_h^1\|^2 + \nu \Delta t \|\nabla (u_1 - v_h^1)\|^2\right) + \frac{R}{\lambda},$$

where $R = C(\Delta t^4 + H^2h^{\frac{d-2}{2}}h^{2k})$ and $\lambda = 2\alpha C_l^2 C_p^{-2}$.

**Remark 24.** Although the true solution regularity given in the statement of this theorem is not known to be satisfied for a general forcing term $f$ and initial conditions (particularly for $d = 3$), these assumptions are necessary for the result to hold.

**Remark 25.** If we consider $H$ to be related to $h$ in the sense of $H = ch$, then in 2D the above estimate is optimal and the restriction of $\mu$ and $H$ is same as in recent literature, e.g. [64, 4, 5]. In 3D, however, the estimate is suboptimal by $\frac{1}{2}$ power of $h$, and the restriction on $\mu$ and $H$ requires $H$ to be finer than in the 2D case.

**Proof.** The proof of this theorem follows similar to Theorem 3.7 in [64], but with some minor modifications, in particular using the treatment of the nudging term from the previous section, and (as pointed out in [33]) using the discrete Stokes projection in the definition of the interpolation error term $\eta$ instead of the $L^2$ projection into $V_h$. ■
6.3.2 Numerical experiments for incompressible NSE

To test the DA algorithm for incompressible NSE, we consider Algorithm 6.3.1 applied to 2D channel flow past a cylinder [90], introduction in section 2.2.2. We will consider Reynolds numbers $Re = 100$ and $Re = 500$. There is no external forcing ($f = 0$), and for $Re=100$, we take $\nu = 0.001$, for $Re=500$, we take $\nu = 0.0002$.

We prescribe different outflow boundary conditions for the two cases. For $Re = 100$, we enforce the Dirichlet condition that the outflow be the same as the inflow, and for $Re = 500$, we use the zero-traction boundary condition and enforce it with the usual ‘do-nothing’ condition. The nonlinear term is also treated differently, as no skew symmetry is used. Thus the $Re = 500$ test does not fit the assumptions of our analysis above (which assumes full Dirichlet boundary conditions), and the difference is important since the nonlinear terms that vanish in our analysis will no longer vanish (additional boundary integrals will arise, even if divergence-free elements are used). Still, channel flow with no stress / no traction outflow conditions is important in practice since Dirichlet outflow is not physical for higher Reynolds numbers, and thus this is an important practical test for DA algorithms.

Since we do not have access to a true solution for this problem, we instead use computed solutions. They are obtained using Algorithm 6.3.1 but without nudging (i.e. $\mu = 0$), using $(P_2, P_1^{\text{disc}})$ Scott-Vogelius elements on barycenter refined Delaunay meshes that provide 35K velocity dof for $Re=100$, and 103K velocity dof for $Re=500$, a time step of $\Delta t = 0.002$, and with the simulations starting from rest ($u_h^0 = u_h^1 = 0$).

We will refer to these solutions as the DNS solutions. Lift and drag calculations were performed for the computed solution and compared to the literature [90, 95], which verified the accuracy of the DNS solutions.
The coarse meshes for DA are constructed using the intersection of uniform rectangular meshes with the domain. We take $H$ to be the width of each rectangle, and use several choices of $H$ in our tests. Figure 6.6 shows in red the 35K dof mesh and associated $H = \frac{2.2}{8}$ coarse mesh in black.

Figure 6.6: Shown above is the FE mesh (in red) and the $H = \frac{2.2}{8}$ coarse mesh and nodes (in black).

For the DA computations, we start from zero initial conditions $v^1_h = v^0_h = 0$, use the same spatial and temporal discretization parameters as the DNS for that Reynolds number, and start assimilation with the $t=5$ DNS solution (i.e., time 0 for DA corresponds to $t=5$ for the DNS). The simulations are run on $[5,10]$.

6.3.2.1 $Re=100$

Figure 6.7: Shown above is the $L^2$ difference between the DA and DNS solutions versus time, for varying $H$.

Results are shown for the $Re=100$ tests in figures 6.7-6.9. For all $Re = 100$ tests, we
Figure 6.8: Shown above are the lift and drag coefficients for Re=100 simulations for DA with varying $H$, and for the DNS.

Figure 6.9: Speed contour plots of DA solutions with $\mu = 10$ with $H = 0.55$ (left) and $H = 0.1375$ (center), and DNS solutions, at times 5, 5.5, 6 and 10.

use $\mu = 10$, as this was sufficiently large for the DA to be effective ($\mu = 1$ was not large enough). Figure 6.7 shows convergence in time of the DA schemes to the DNS. We observe that the DA solutions from the two finest $H$'s converge to the DNS, and
quickly. For $H = 0.275$, the DA solution does appear to be converging, although slowly, and agrees with the DNS to $10^{-5}$ by $t=10$. We do not observe convergence for $H = 0.55$ in this time interval. Plots of lift and drag coefficients versus time are shown in figure 6.8, and all DA solution except for $H = 0.55$ give good lift and drag predictions by $t=7$ (the solution from $H = 0.55$ never gives good drag coefficient prediction).

Figure 6.9 shows speed contour plots of DA and DNS solutions at $t=5$ (the start time for DA), 5.5, 6, and 10. The DA scheme with $H = 0.1375$ (middle column) is already close to the DNS by $t=5.5$, and we observe no difference from the DNS by $t=6$. The DA solution from $H = 0.55$, on the other hand, does not converge by $t=10$. At $t=5.5$ and $t=6$, it is clearly quite far from the DNS solution. By $t=10$, it looks closer, but still shows significant differences from the DNS.

Overall, we observe good convergence of the DA solution to the DNS solution, provided the coarse mesh is fine enough. However, ‘fine enough’ is still quite coarse, as we observe good convergence even when only 64 measurement points ($H=0.275$) are used.

6.3.2.2 $Re=500$

We now give results for $Re = 500$ numerical tests. We remark again that due to the outflow boundary condition, the analysis in this section is not applicable, since the nonlinear terms behave in a different way.

Results for varying $H$ with $\mu = 10$ are shown in figure 6.10, as $L^2$ error, and lift and drag coefficients. An interesting phenomena is that the error appears to be bounded below, which does not happen in the $Re = 100$ tests. However, as we see in figures
Figure 6.10: Shown above are the $L^2$ error (left), drag (center) and lift (right) coefficient predictions for $Re = 500$ simulations for DA with $\mu = 10$ and varying $H$, and for the DNS.

DA, $H=0.275$ (t=6.0)  
DA, $H=0.0688$ (t=6.0)  
DNS (t=6.0)

DA, $H=0.275$ (t=10)  
DA, $H=0.0688$ (t=10)  
DNS (t=10)

Figure 6.11: Speed contour plots of DA solutions for $Re = 500$ with $\mu = 10$, $H = 0.275$ (left) and $H = 0.0688$ (center), and DNS solutions, at times 6 and 10.

6.10 and 6.11, this level of accuracy of $L^2$ error around $10^{-3}$ is enough so that the lift and drag coefficients are accurately predicted. Moreover, the contour plots from figure 6.11 match the DNS very well by t=10, both for $H = 0.275$ and $H = 0.0688$, although at t=6 only the solution with $H = 0.0688$ matches the DNS well.

To consider further the seeming lower bound on the error in the $Re = 500$ tests so far, we consider additional runs with varying $H$ and $\mu$. We show the $L^2$ errors for these tests in figure 6.12, and observe that the error seems to be bounded below by $O(\mu^{-1})$,  

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seemingly independent of \( H \) (even though the DNS solution is in the finite element space and thus 0 error is possible, just as in the \( Re = 100 \) case). Up to this lower bound, the DA solutions converge quickly, in particular for \( H = \frac{2.2}{500} \) and \( \mu = 1000 \) the convergence is rapid.

Overall we conclude that results for \( Re = 500 \) are quite good. While it appears that the error depends on \( O(\mu^{-1}) \), the only reason why we see this error in these tests is that the DNS was done on the same discretization as the DA. In practice, there will also be spatial and temporal errors present, and in particular we would expect spatial error to dominate any \( O(\mu^{-1}) \) errors when \( \mu = 100 \) or 1000.

![Figure 6.12](image)

Figure 6.12: Shown above are the \( L^2 \) errors versus time for \( Re = 500 \) simulations with varying \( \mu \) and \( H = \frac{2.2}{48} \) (left) and \( H = \frac{2.2}{500} \) (right).
References


