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Numerical study for non-Newtonian fluid-structure interaction problems

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Abstract

In this work, we consider non-Newtonian fluid structure problems, which have significant applications in biology and industry. Numerical approximation schemes are developed based on the Arbitrary Lagrangian-Eulerian (ALE) formulation of the flow equations. A spatial discretization is accomplished by the finite element method, and the time discretization is carried out by the implicit Euler method. We first consider a fluid-structure interaction problem that consists of a two-dimensional viscoelastic flow and a one-dimensional structure equation. We show how the system can be decoupled and how each subproblem can be solved using interface conditions. Numerical results of different algorithms are presented, showing the comparison between non-Newtonian and Newtonian fluids. We then extend the FSI problem into the 2D-2D case of a quasi-Newtonian fluid and a linear elastic structure. In this case, we present the stability and error estimation for both semi-discrete and fully discrete formulation. For the last part of this work, a 2D-2D viscoelastic FSI problem is considered with both monolithic algorithm and decoupled algorithm under Robin condition.
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Chapter 1

Introduction

1.1 Fluid-Structure Interaction Problem

Fluid-structure interaction (FSI) problems are multi-physics problems that consist of fluid flows and deformable structures. Such problems are widely used in engineering and biology applications where a surrounding or internal fluid interacts with a moveable structure. Some examples include a blood flow in vessels, airflow surround wings of micro-air vehicles and gas explosion in a pipeline, etc. For these interaction problems, a fluid follows structure motion, which implies that the domain of the fluid is determined by the structure displacement. In the meantime, the fluid stress deforms the structure through the interface conversely and the fluid velocity matches with the structure velocity on the interface. For most FSI problems, the analytical solution could be difficult or even impossible to obtain due to the equation complexity, and the laboratory experiments may also be unavailable due to the problem scale. Hence the numerical approximations could be helpful and necessary for simulating those cases.

Though various procedures have been developed to solve the FSI problems, they can be classified into two broad categories: the monolithic approach and the partitioned approach. Both of them have their own advantages and disadvantages in efficiency and stability.

As the name implies, the monolithic approach solves the entire problem with one complex system. The fluid and structure dynamics are treated in the same framework and the interface conditions are implicit. This approach is a popular research area due to the potential stability and accuracy of the numerical approximation, however the large size and complexity of the equations
pose difficulties in both analysis and computation aspects.

The partitioned approach, on the other hand, treats the fluid and structure dynamics separately. The whole interaction system is decoupled into two subproblems with their own equations and domains. The interface matching conditions are explicit as the bridge of the two subproblems. The computation complexity is relatively low by allowing smaller matrices for complex fluid and structure problems, but tracking the moving interface all the time may be difficult and error-prone. Various explicit/implicit algorithms are developed to communicate information between the fluid and structure subsystems. The explicit algorithm requires only one solver for each subproblem per time-step. In contrast, the implicit algorithms, which are more stable but expensive, cannot move to the next time step until the iterative solutions converge.

The treatment of meshes consists of the conforming mesh method and the non-conforming one. The immersed boundary method, which is a currently active research area, is known as a non-conforming mesh method. The two subsystems can be solved independently with a fixed mesh since the non-conforming mesh method does not require the mesh to be conformed to the interface. On the other hand, the conforming mesh method, which is applied in most partitioned approaches, requires conforming mesh based on the moving interface. This means the mesh needs to be updated at each time step.

Many numerical methods have been developed in the FSI field during the last several decades. Peskin proposed the immersed boundary method in 1972 which considers interface information as constraints imposed on model equations [43]. Many researchers, including Peskin and McQueen [44], Dillon and Fauci [13] and Zhu [52], published articles about this method after Peskin. A slightly different method called fictitious domain method was introduced in [13, 23], where a distributed Lagrange multiplier is used on the interface. The strong forms for the two methods are the same, but forces are imposed in a distributed manner with the multipliers in the weak form for the fictitious domain method. Reduced accuracy is observed near the interface due to the interpolation for these non-conforming mesh methods, although no change of mesh is required.

Opposed to the no-boundary-fitting method, the Arbitrary Lagrangian Eulerian method (ALE) concerns the position of the interface with the introduction of arbitrary mapping from a fixed reference domain to a current moving domain, which is also the method considered in this work. With an invertible and sufficiently regular ALE mapping, a conforming mesh that conforms to the interface movement at any arbitrary time could be obtained as the image of a fixed mesh.
which remains in the reference domain. Thus the time derivative terms in the equations can be rewritten in an ALE coordinate, while the spatial terms are left in the Eulerian coordinate. The ALE method was first proposed in the 1980’s [14, 28], and has been widely used thereafter. In 2001, Nobile employed the ALE method to simulate a FSI problem consisting of Navier-Stokes fluid and an elastic structure [41]. Related works for various boundary conditions [17, 18, 42] and stability investigation [9, 18] were published in following years. A similar technique was used for Stokes fluid-structure by Martin et al. [37].

Geometric Conservation Law (GCL) is a condition on time integration in the investigation of the ALE method. The inaccurate calculation of geometrical quantities of the control cells used in finite-volume computations may be the main reason for the instability and oscillations for some FSI problems, particularly for compressible fluid and aeroelasticity. GCL was developed to overcome this problem by governing the geometric parameters of given numerical schemes. In the finite volume case, the satisfying GCL condition was stated as a sufficient condition for a numerical scheme to be at least first order time-accurate on a moving mesh by Guillard [24]. Some related work were reported in [8, 16]. Unfortunately, complete analysis on the relationship between GCL condition and stability and accuracy for finite element case is still missing.

Other techniques including level set method [11], ALE formulation [8, 19] and space time approach [51, 38] can also be applied to simulate moving-domain problems.

1.2 Non-Newtonian Fluid

The reports of both numerical experiments and analysis for a non-Newtonian FSI are less common than the Newtonian case. A priori error estimates and numerical results for a quasi-Newtonian fluid with a known moving domain were investigated by Lee [33]. In simulations for blood flow in vessel, a Newtonian (Stokes or Navier-Stokes) fluid has been used for most cases [8, 19], however, it is not very accurate to model blood flow as Newtonian fluid due to the biology complexity. Several investigations have shown the significance of the non-Newtonian characteristics of blood flow [40, 36, 29, 6, 7]. This significance is the motivation for this investigation of both analytical and numerical aspects of FSI problems involving a quasi-Newtonian and viscoelastic fluid flow.

In general, the fluid could be classified as either Newtonian or non-Newtonian fluid based on
the relationship between the stress and the strain tensor. For a Newtonian fluid (like Navier-Stokes equation), the fluid stress is linear proportional to the strain tensor. However, the stress-deformation relation for a non-Newtonian fluid (like viscoelastic flow) is relatively more complicated.

A generalized Newtonian fluid model characterizes viscosity $\nu$ as a function of the deformation tensor $D(u)$ where $\sigma = \nu(\cdot)D(u)$. Unlike a Newtonian fluid where the viscosity is a constant, i.e. $\nu(\cdot) = \mu$, the $\nu(|D(u)|)$ for a quasi-Newtonian fluid gives a general viscosity function satisfying particular continuity and monotonicity properties. Typical models for such viscosity functions include the following:

- The power-law model

$$\nu_f(|D(u)|) := k |D(u)|^{r-1},$$

where $r$ is the dimensionless rate constant.

- The Carreau-Yasuda model

$$\nu_f(|D(u)|) := \nu_\infty + (\nu_0 - \nu_\infty)(1 + |\lambda D(u)|^2)^\frac{r}{2}, \quad \text{where} \quad r < 1,$$

$\nu_0, \nu_\infty \geq 0$ are the limiting viscosity values at a zero and infinite shear rate, respectively; $\lambda > 0$ is the relaxation time.

- The Cross model

$$\nu_f(|D(u)|) := \nu_\infty + \frac{(\nu_0 - \nu_\infty)}{1 + |\lambda D(u)|^{2-r}}, \quad \text{where} \quad 1 \leq r \leq 2, \quad 0 \leq \nu_\infty \leq \nu_0.$$

- The Ladyzhenskaya model

$$\nu_f(|D(u)|) := \nu_0 + \nu_1 |D(u)|_r^{r-2}, \quad \text{where} \quad r > 1, \quad \nu_0, \nu_1 \geq 0.$$

For a viscoelastic fluid, the relation between the fluid velocity $u$ and the extra stress tensor $\sigma$ can be given by an equation in the form of a constitutive law. The extra stress $\sigma$ can be decomposed into a Newtonian part and a viscoelastic part, i.e.

$$\sigma = \sigma_N + \sigma_V. \quad (1.2.1)$$
As mentioned above, the Newtonian part $\sigma_N$ has a linear relation with the tensor

$$\sigma_N = 2(1 - \alpha)D(u), \quad (1.2.2)$$

where $\alpha \in (0, 1)$ represents the portion of the total viscosity that is viscoelastic, therefore $(1 - \alpha)$ may be interpreted as the the part which is considered Newtonian.

With the deformation tensor

$$D(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) \quad (1.2.3)$$

and the vorticity tensor

$$W(u) = \frac{1}{2} (\nabla u - (\nabla u)^T), \quad (1.2.4)$$

we could introduce a function

$$g_\beta(\sigma, \nabla u) := \sigma W(u) - W(u) \sigma - \beta(D(u)\sigma + \sigma D(u))$$

$$= \frac{1 - \beta}{2} (\sigma \nabla u + \nabla u^T \sigma) - \frac{1 + \beta}{2} (\nabla u \sigma + \sigma \nabla u^T), \quad \beta \in [-1, 1]. \quad (1.2.5)$$

For the viscoelastic part $\sigma_V$, the constitutive law [48] is presented as

$$\sigma_V + \lambda \frac{\partial \sigma_V}{\partial t} - 2\alpha D(u) = 0, \quad (1.2.7)$$

where $\lambda$ is a dimensionless constant called Weissenberg number, which is the product of the relaxation time and a characteristic strain rate [5], and

$$\frac{\partial \sigma}{\partial t} := \frac{\partial \sigma}{\partial t} + u \cdot \nabla \sigma + g_\beta(\sigma, \nabla u) \quad (1.2.8)$$

is an objective derivative used to describe the Oldroyd model. Hence the constitutive law (1.2.7) can be written as

$$\sigma_V + \lambda \left( \frac{\partial \sigma_V}{\partial t} + u \cdot \nabla \sigma_V + g_\beta(\sigma_V, \nabla u) \right) - 2\alpha D(u) = 0. \quad (1.2.9)$$

If the density of the fluid is a constant, the fluid is called to be incompressible. The law of
conservation of mass for an incompressible fluid can be reduced to

\[ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega^f, \quad (1.2.10) \]

which is also called as the incompressibility condition (1.2.10). This condition implies that the flow of fluid into any subset of the fluid domain must be the same as the flow out of the subset.

By the law of conservation of momentum, the rate of change of momentum of a volume of flow must be equal to the sum of the forces acting on the flow. This momentum conservation law can be expressed by the equation

\[ Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \mathbf{f} + \nabla \cdot \mathbf{T}, \quad (1.2.11) \]

where \( Re \) is a constant called *Reynolds number* defined as

\[ Re = \frac{LV\rho}{\mu}. \quad (1.2.12) \]

Here \( L \) denotes characteristic length scale, \( V \) denotes characteristic velocity scale, and \( \rho, \mu \) denotes fluid density and viscosity, respectively. \( \mathbf{T} \) is a symmetric tensor that presents the force acting on the surface of the flow and it could be written as the sum of a pressure piece \( p \) and an extra stress tensor \( \sigma \):

\[ \mathbf{T} = -p\mathbf{I} + \sigma, \quad (1.2.13) \]

with which (1.2.11) can be rewritten as

\[ Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - \nabla \cdot \sigma = \mathbf{f}. \quad (1.2.14) \]

With the stress decomposition (1.2.1), the conservation of momentum (1.2.14) represents as

\[ Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - 2(1 - \alpha)\nabla \cdot D(\mathbf{u}) - \nabla \cdot \sigma_V = \mathbf{f}. \quad (1.2.15) \]

Together with the incompressibility condition (1.2.10) and the constitutive law (1.2.9), the
model equations for viscoelastic fluid reads as

\[
\sigma_V + \lambda \left( \frac{\partial \sigma_V}{\partial t} + u \cdot \nabla \sigma_V + g_\beta(\sigma_V, \nabla u) \right) - 2\alpha D(u) = 0 \quad \text{in } \Omega^f, \\
Re \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \nabla \cdot \sigma_V - 2(1 - \alpha) \nabla \cdot D(u) + \nabla p = f \quad \text{in } \Omega^f, \\
\nabla \cdot u = 0 \quad \text{in } \Omega^f.
\]

(1.2.16) (1.2.17) (1.2.18)

In following work throughout this thesis, the subscript V is dropped for notational simplicity. The difficulty of numerical simulation for viscoelastic fluid arise due mainly to (i) the large number of unknowns involved in computation and (ii) the hyperbolic, nonlinear term of the constitutive equation for the stress.

The FSI problems involving a non-Newtonian fluid flow are not as common as the Newtonian case in the literature. Chan, Ding and Tu presented numerical comparisons between FSI problems considering Carreau model and Power Law model in [10]. An energy estimate and numerical results using splitting method for a 3D generalized Newtonian shear-thinning FSI is presented by J.Janela et al. [29]. An extended work where several absorbing boundary conditions are discussed is also done by the same group [30]. Relevant numerical works considering a viscoelastic flow through a channel where part of the wall is flexible are done by Chen et al [12], and a mass-spring-dashpot prototype model is also studied by the same authors.

1.3 Thesis Organization

This work is inspired by the simulation of blood flow in vessels arisen in hemodynamics. The assumption that blood behaves as Newtonian flow is not accurate, particularly for small vessels. In this work, viscoelastic and quasi-Newtonian fluids are adopted for an incompressible non-Newtonian fluid modeling blood. In Chapter 2, we consider a 2D-1D fluid-structure interaction that consists of an incompressible fluid and an isotropic linear elastic structure. The fluid equation is given in Eulerian framework, thus the fluid domain is time-dependent due to the movement of the interface. The structure dynamics, on the other hand, is described in a Lagrangian frame of a reference, which means the structure domain is fixed. The system can be viewed as two subproblems for fluid and structure exchanging information through interface. Hence, with interface conditions
and appropriately chosen function spaces, we obtain the global formulation for the system, which accounts for the fluid and the structure at the same time. Two coupling algorithm (leap-frog and added mass implicit) are applied in the numerical experiments where the comparison of viscoelastic and Navier-Stokes fluid is shown.

Chapter 3 presents analytical and numerical results for a 2D-2D FSI system where the fluid is described as a quasi-Newtonian flow in Eulerian framework while the structure is considered as a 2D linear elastic in Lagrangian frame. The monolithic scheme is discussed in this chapter where a global weak formulation could be obtained by applying the interface matching conditions. The corresponding stability and error estimation analysis are shown for both semi-discrete and fully-discrete formulations. Numerical results for both the non-physical convergence test and physical simulation are presented in the latest part of this chapter.

In Chapter 4, numerical algorithms are discussed for a 2D-2D viscoelastic FSI system. We consider both monolithic and partitioned methods for this problem. In order to ensure the well-posedness of the viscoelastic fluid equation, an appropriate stress boundary condition is required on the inflow of the fluid subdomain. Besides the fixed inlet boundary, we also need to consider the inflow part on the interface which is time-dependent. The monolithic algorithm and a partitioned algorithm with Robin-Robin coupling condition are presented while the viscoelastic fluid is stabilized by using a streamline upwind Petrov-Galerkin (SUPG) approximation for the constitutive equation. The comparison of the simulation results with and without interface stress boundary conditions are presented in the numerical experiment part.
Chapter 2

2D-1D Viscoelastic Fluid-Structure Interaction Problem

There are only a few numerical studies found in literature for the non-Newtonian fluids coupled with elastic solids. Some simulation results for the interaction of non-Newtonian fluids with deformable bodies were reported in engineering journals [1, 49] for the purpose of model validation. Even though there are some reports [29, 36] on numerical methods for simulation of blood flows using non-Newtonian fluid models, detailed numerical analysis such as studies on stability of time-stepping schemes is, in general, lacking from the current literature.

Mathematical study for partial differential equations governing a viscoelastic fluid behavior is still far behind when compared to advances in computing. It is well-known that an analytical or numerical study of viscoelastic flows is very challenging due to complexity of governing equations. One of the difficulties in simulating viscoelastic flows arises from the hyperbolic nature of the constitutive equation for which one needs to use a stabilization technique such as the streamline upwinding Petrov-Galerkin (SUPG) [48] method or the discontinuous Galerkin method [4].

In this chapter, we consider an interaction problem consisting of a viscoelastic fluid and a deformable elastic tube through which the fluid flows. For analysis purposes, we introduced a modified Johnson-Segalman model, referred to as the Oseen-viscoelastic model, in which the velocity in the nonlinear pieces of the constitutive equation was taken to be a known function [15, 34]. For this model, we have been able to provide a rigorous mathematical analysis of the model equations.
and their approximation and provide some additional insights into the investigated problems (high Weissenberg number problem, domain decomposition, optimal control) in viscoelasticity. We use the Oseen model for analysis in this chapter, however, numerical tests will be for the standard Jonson-Segalman model. For the structure model, we use a one-dimensional string model introduced in [41, 46]. This dynamical interaction system can be widely used to model many physical and biological phenomenons including blood flows in vessels which was the focus in this work.

2.1 Model equations

We consider a viscoelastic flow problem, where flow equations are coupled with a one-dimensional elastic structure model. Let $\Omega^f_t$ be a bounded domain at time $t$ in $\mathbb{R}^2$ with the Lipschitz continuous boundary $\Gamma^f_t$. Suppose $\Gamma^f_t$ consists of three parts: $\Gamma^f_t := \Gamma^f_D \cup \Gamma^f_N \cup \Gamma^f_I$, where $\Gamma^f_D \cup \Gamma^f_N$ is a fixed boundary and $\Gamma^f_I$ a moving wall boundary.

Consider the Johnson-Segalman viscoelastic model equations:

\begin{align*}
\sigma + \lambda \left( \frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma + g_\beta(\sigma, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}) &= 0 \quad \text{in } \Omega^f_t, \quad (2.1.1) \\
Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \sigma - 2(1 - \alpha) \nabla \cdot D(\mathbf{u}) + \nabla p &= \mathbf{f} \quad \text{in } \Omega^f_t, \quad (2.1.2) \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega^f_t, \quad (2.1.3)
\end{align*}

where $\sigma$ denotes the extra stress tensor, $\mathbf{u}$ the velocity vector, $p$ the pressure of fluid, $Re$ the Reynolds number, and $\lambda$ is the Weissenberg number defined as the product of the relaxation time and a characteristic strain rate. In (2.1.1) and (2.1.2), $D(\mathbf{u}) := (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ is the rate of the strain tensor, $\alpha$ a number such that $0 < \alpha < 1$ which may be considered as the fraction of viscoelastic
viscosity, and \( f \) the body force. In (2.1.1), \( g_\beta(\sigma, \nabla u) \) is defined by

\[
g_\beta(\sigma, \nabla u) := \frac{1 - \beta}{2}(\sigma \nabla u + \nabla u^T \sigma) - \frac{1 + \beta}{2}(\nabla \sigma + \sigma \nabla u^T)
\]

for \( \beta \in [-1, 1] \). Note that (2.1.1) reduces to the Oldroyd-B model for the case \( \beta = 1 \).

For the viscoelastic fluid flow problem, the major difficulty in establishing existence of a solution to the continuous variation formulation is the constitutive equation (2.1.1). With this equation, the nonlinear operator associated with the model is neither coercive nor monotone. One way to overcome this difficulty is to consider a nearby problem where the \( g_\beta \) term is linearized with the given velocity \( b(x, t) \approx u(x, t) \). Consider the modified problem with the given velocity \( b \) in the \( g_\beta \) term:

\[
\sigma + \lambda \left( \frac{\partial \sigma}{\partial t} + u \cdot \nabla \sigma + g_\beta(\sigma, \nabla b) \right) - 2 \alpha D(u) = 0 \quad \text{in } \Omega^f_t,
\]

\[
\text{Re} \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \nabla \cdot \sigma - 2(1 - \alpha) \nabla \cdot D(u) + \nabla p = f \quad \text{in } \Omega^f_t,
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega^f_t,
\]

where \( b \) satisfies the following assumptions:

\[
b \in H^1(\Omega^f_t), \quad \nabla \cdot b = 0, \quad \|b\|_\infty \leq M, \quad \|\nabla b\|_\infty \leq M < \infty.
\]

To analyze the equation (2.1.5), we will need a small data condition on the Weissenberg number \( \lambda \) and/or on the bound \( M \) so that \( 1 - 4\lambda M > 0 \).

Initial and boundary conditions for \( u \) and \( \sigma \) are given as follows:

\[
u(x, 0) = u_0 \quad \text{in } \Omega_0^f,
\]

\[
\sigma(x, 0) = \sigma_0 \quad \text{in } \Omega_0^f,
\]

\[
u = 0 \quad \text{on } \Gamma_D^f,
\]

\[
(\sigma + 2(1 - \alpha) D(u) - pI) \cdot n = 0 \quad \text{on } \Gamma_N^f.
\]
where \( n \) is the outward unit normal vector to \( \Omega^f \).

As a model equation of elastic structure we consider the one-dimensional generalized string model \([46]\), which was developed to account for longitudinal action:

\[
\rho_w s \frac{\partial^2 \eta}{\partial t^2} - k G h \frac{\partial^2 \eta}{\partial z^2} - \gamma \frac{\partial^3 \eta}{\partial t \partial z^2} + \frac{E s}{(1 - \nu^2) R_0^2} \eta = \hat{\Phi},
\]

(2.1.13)

where \( \eta \) represents the radial displacement of the wall with respect to the rest configuration

\[
\Gamma_0 := \{(z, r) \in \mathbb{R}^2 : z \in (0, L), r \in (0, R_0)\}.
\]

(2.1.14)

In (2.1.13), \( \rho_w \) is the wall volumetric mass, \( s \) the wall thickness, \( k \) the Timoshenko shear correction factor, \( G \) the shear modulus, and \( E \) the Young modulus. The right hand side function \( \hat{\Phi} \) is the external force in the radial direction, \( \nu \) the Poisson ratio, \( \gamma \) a viscoelastic parameter and \( R_0 \) is the radius at rest. In order to simplify expressions, we rewrite (2.1.13) in the form of

\[
\rho_w \frac{\partial^2 \eta}{\partial t^2} - a \frac{\partial^2 \eta}{\partial t^2} - b \frac{\partial^3 \eta}{\partial t \partial z^2} + c \eta = \hat{\Phi},
\]

(2.1.15)

where \( a, b, c \) are positive constants related to the physical properties of the solid structure described above. The structure equation is accompanied with the conditions at \( z = 0, L \):

\[
\eta|_{z=0} = 0 \quad \text{for all } t, \quad \eta|_{z=L} = 0 \quad \text{for all } t.
\]

(2.1.16)

The interface conditions between the fluid and the structure are obtained by enforcing continuity of the velocity and the stress force:

\[
\frac{\partial \eta}{\partial t} e |_{t=0} = u_0 \quad \text{on } \Gamma_{t_0},
\]

(2.1.17)

\[
\frac{\partial \eta}{\partial t} e = u \quad \text{on } \Gamma_{t_1},
\]

(2.1.18)

\[
\Phi e = -(-p I + \sigma + 2(1 - \alpha) D(u)) \cdot n - p_{ext} n \quad \text{on } \Gamma_{t_1},
\]

(2.1.19)

where \( e \) is a unit vector in the radial direction and \( p_{ext} \) is the external pressure. Without loss of generality we let \( p_{ext} = 0 \) and \( \Gamma_0 = \Gamma_{t_0} \) in this chapter. In (2.1.19), \( \Phi \) is a representation of \( \hat{\Phi} \) on
It, which takes into account the change of configuration from the reference to the moving interface. A detailed form of $\Phi$ will be discussed in the next section. See (2.2.12).

**Remark 2.1.1** In general, the system of viscoelastic fluid equations (2.1.1)-(2.1.3) is completed with initial and boundary conditions (3.1.5)-(2.1.12) and a Dirichlet type boundary condition for the stress $\sigma$ along an inflow boundary of fluid domain, i.e., on a part of $\Gamma^f_t$ where $u \cdot n < 0$. In a fluid-structure system an inflow part on the moving boundary is changed from time to time due to the interface condition (2.1.18), which makes numerical studies for the system extremely challenging not only by the change of inflow boundaries but also by a lack of boundary information on the stress. Therefore, to simplify numerical analysis, we assume that the stress is unknown on the whole boundary. The analysis results in theorems are still valid if a stress condition is imposed. A possible way to implement a stress boundary condition is suggested in Chapter 4.

We use the Sobolev spaces $W^{m,p}(D)$ with norms $\| \cdot \|_{m,p,D}$ if $p < \infty$, $\| \cdot \|_{m,\infty,D}$ if $p = \infty$. We denote the Sobolev space $W^{m,2}$ by $H^m$ with the norm $\| \cdot \|_m$. The corresponding space of vector-valued or tensor-valued functions is denoted by $H^m$. If $D = \Omega^f_t$, $D$ is omitted, i.e., $(\cdot, \cdot) = (\cdot, \cdot)_{\Omega^f_t}$ and $\| \cdot \| = \| \cdot \|_{\Omega^f_t}$. For $\sigma$, $\tau$ tensors and $u$, $v$ vectors, we use $:$ and $\cdot$ to denote the tensor product $\sigma : \tau := \sum_{i,j=1}^2 \sigma_{ij} \tau_{ij}$ and the vector product $u \cdot v := \sum_{i=1}^2 u_i v_i$. For the structure equation, we will use $(\cdot, \cdot)$, $\| \cdot \|$ to denote $(\cdot, \cdot)_{\Gamma^0_t}$ and $\| \cdot \|_{\Gamma^0_t}$, respectively.

In the next theorem we show the stability of a solution satisfying the coupled problem (2.1.5)-(2.1.12), (2.1.15)-(2.1.19).

**Theorem 2.1.2** If $1 - 4\lambda M > 0$ and $u \cdot n \geq 0$ on $\Gamma^f_N$, a solution to the system (2.1.5)-(2.1.12), (2.1.15)-(2.1.19) satisfies the estimate

$$
\left[ \frac{\lambda}{2} \| \sigma \|_0^2 + \alpha Re \| u \|_0^2 + \rho_w \alpha \| \frac{\partial \eta}{\partial t} \|_0^2 + \rho \alpha \| \frac{\partial^2 \eta}{\partial z^2} \|_0^2 + c \alpha \| \eta \|_0^2 \right] \\
+ \int_0^t b \alpha \left[ \frac{\partial^2 \eta}{\partial t \partial z} \right]^2_0 + (1 - 4\lambda M) \| \sigma \|_0^2 + 2\alpha(1 - \alpha) \| D(u) \|_0^2 \, ds \\
\leq \ C,
$$

(2.1.20)

where $C$ is a constant depending the forcing term $f$ and initial data.

**Proof:** Multiplying (2.1.15) by $\frac{\partial \eta}{\partial t}$ and integrating over $\Gamma^0_t$, we have that

$$
\left( \rho_w \frac{\partial^2 \eta}{\partial t^2} - a \frac{\sigma}{\partial z^2} - b \frac{\partial^3 \eta}{\partial t \partial z^2} + c \eta, \frac{\partial \eta}{\partial t} \right)_{\Gamma^0_t} = \left( \frac{\dot{\Phi}}{\eta}, \frac{\partial \eta}{\partial t} \right)_{\Gamma^0_t}.
$$

(2.1.21)
Using integration by parts and (2.1.16), (2.1.21) implies
\[
\rho_w \frac{d}{dt} \left( \frac{\partial \eta}{\partial t} \right)^2 + \frac{a}{2} \frac{d}{dt} \left( \frac{\partial \eta}{\partial z} \right)^2 + \frac{b}{2} \frac{\partial^2 \eta}{\partial t \partial z}^2 + \frac{c}{2} \frac{d}{dt} \| \eta \|_0^2 = \left( \frac{\phi}{\partial t} \right)_{\Gamma_0}.
\] (2.1.22)

On the other hand, multiplying (2.1.5), (2.1.6), (2.1.7) by \( \sigma \), \( 2\alpha \mathbf{u} \) and \( p \), respectively, integrating over \( \Omega_t \) and using the Green’s theorem, we have
\[
\lambda \int_{\Omega(t)} \frac{d}{dt} (\sigma : \sigma) \, d\Omega + \frac{\lambda}{2} \left( (\mathbf{u} \cdot \mathbf{n}) \sigma, \sigma \right)_{\Gamma_{fN}} + \mathbf{u} \cdot \mathbf{n} \sigma, \sigma \right)_{\Gamma_{fN}} + A((\sigma, \mathbf{u}), (\sigma, \mathbf{u}))
\]
\[
= 2\alpha (f, \mathbf{u}) + 2\alpha ((\sigma + 2(1 - \alpha) D(\mathbf{u}) - pI) \cdot \mathbf{n}, \mathbf{u})_{\Gamma_{tN}},
\] (2.1.23)

where \( A((\mathbf{u}, \sigma), (v, \tau)) \) is defined by
\[
A((\mathbf{u}, \sigma), (v, \tau)) := (\sigma, \tau) + \lambda (g_{\beta}(\sigma, \nabla \mathbf{b}), \tau) - 2\alpha (D(u), \tau)
\]
\[
+ 2\alpha (\sigma, D(v)) + 4\alpha (1 - \alpha) (D(u), D(v)).
\] (2.1.24)

Note that, since
\[
(g_{\beta}(\sigma, \nabla \mathbf{b}), \tau) \leq 4\|\nabla \mathbf{b}\|_{\infty} \sigma_0 \|\tau\|_0 \leq 4M \|\sigma\|_0 \|\tau\|_0,
\] (2.1.25)
if \( \lambda M \) is small so that \( 1 - 4\lambda M > 0 \), \( A \) satisfies
\[
A((\mathbf{u}, \sigma), (\mathbf{u}, \sigma)) \geq \|\sigma\|_0^2 - 4\lambda M \|\sigma\|_0^2 + 4\alpha (1 - \alpha) \|D(\mathbf{u})\|_0^2
\]
\[
= (1 - 4\lambda M) \|\sigma\|_0^2 + 4\alpha (1 - \alpha) \|D(\mathbf{u})\|_0^2.
\] (2.1.26)

Using (2.1.18) and the Reynolds transport formula
\[
\int_{\Omega(t)} \frac{\partial \psi}{\partial t} \, d\Omega = \frac{d}{dt} \int_{\Omega(t)} \psi \, d\Omega - \int_{\Gamma_{tN}} (\mathbf{w} \cdot \mathbf{n}) \psi \, d\Gamma,
\] (2.1.27)
where \( \mathbf{w} \) is the velocity of a moving boundary, (2.1.23) is reduced to
\[
\frac{d}{dt} \left( \frac{\lambda}{2} \|\sigma\|_0^2 + \alpha (\mathbf{u} \cdot \mathbf{n}) \sigma, \sigma \right)_{\Gamma_{fN}} + \frac{\lambda}{2} \left( (\mathbf{u} \cdot \mathbf{n}) \sigma, \sigma \right)_{\Gamma_{fN}} + \alpha (\mathbf{u} \cdot \mathbf{n}) \sigma, \sigma \right)_{\Gamma_{fN}} + A((\sigma, \mathbf{u}), (\sigma, \mathbf{u}))
\]
\[
= 2\alpha \left( \mathbf{f}, \mathbf{u} \right) + 2\alpha ((\sigma + 2(1 - \alpha) D(\mathbf{u}) - pI) \cdot \mathbf{n}, \mathbf{u})_{\Gamma_{tN}}.
\] (2.1.28)
Note that by the interface boundary conditions (2.1.18)-(2.1.19), the integral along the interface boundary in (2.1.28) is written as

\[
((\sigma + 2(1 - \alpha) D(u) - pI) \cdot n, u)_{\Gamma_{t_i}} = \left( -\Phi e, \frac{\partial \eta}{\partial t} \right)_{\Gamma_{t_i}} = \left( \hat{\Phi} e, \frac{\partial \eta}{\partial t} \right)_{\Gamma_0}.
\]  

(2.1.29)

If \( u \cdot n \geq 0 \) on \( \Gamma_{N} \), two boundary integrals along \( \Gamma_{N} \) in the left hand side of (2.1.28) are positive, therefore, using (2.1.26), (2.1.28) and (2.1.29), we have

\[
\frac{d}{dt} \left( \frac{\lambda}{2} \| \sigma \|_0^2 + \alpha Re \| u \|_0^2 \right) + (1 - 4\lambda M) \| \sigma \|_0^2 + 4\alpha(1 - \alpha) \| D(u) \|_0^2 \\
\leq 2\alpha (f, u) - 2\alpha \left( \hat{\Phi}, \frac{\partial \eta}{\partial t} \right)_{\Gamma_0}.
\]  

(2.1.30)

Now, multiplying (2.1.22) by \( 2\alpha \), adding to (2.1.30) and using the Young’s and Poincaré inequalities, we have that

\[
\frac{d}{dt} \left[ \frac{\lambda}{2} \| \sigma \|_0^2 + \alpha Re \| u \|_0^2 + \rho_w \alpha \| \frac{\partial \eta}{\partial t} \|_0^2 + a\alpha \| \frac{\partial \eta}{\partial z} \|_0^2 + c\alpha \| \eta \|_0^2 \right] \\
+ b\alpha \| \frac{\partial^2 \eta}{\partial t \partial z} \|_0^2 + (1 - 4\lambda M) \| \sigma \|_0^2 + 4\alpha(1 - \alpha) \| D(u) \|_0^2 \\
\leq 2\alpha \left( \frac{1}{4(1 - \alpha)} \| f \|_{-1}^2 + (1 - \alpha) \| D(u) \|_0^2 \right).
\]  

(2.1.31)

Simplifying (2.1.31), integrating over the time from 0 to \( t \) and using the Gronwall Lemma [47], the energy estimate (2.1.20) is obtained. □

## 2.2 ALE formulation

For the fluid-elastic system, the interface moves by the displacement of structure, therefore, the fluid subproblem is a moving boundary problem. Numerical simulation for the moving boundary problem can be performed using the Arbitrary Lagrangian Eulerian method [28, 45], where the Eulerian frame is used in the fluid domain while, in the solid domain, the Lagrangian formulation is used. In ALE formulation, an unknown coordinate transformation is usually introduced for the fluid domain and the fluid equations can be rewritten for a fixed reference domain.
In this work, let the initial domain configuration $\Omega_0^f$ be the reference domain. Then for any time $t \in (0,T]$, we define a bijective mapping $\Psi_t$ which maps the reference domain $\Omega_0^f$ to the physical domain $\Omega_t^f$,

$$\Psi_t : \Omega_0^f \rightarrow \Omega_t^f, \quad \Psi_t(y) = x(t,y),$$

where $x$ and $y$ are the spatial coordinates in $\Omega_t^f$ and $\Omega_0^f$, respectively. We refer to $x$ as the Eulerian coordinate and $y$ as the ALE coordinate. Assuming that $\Psi_t$ is invertible and $\Psi_t^{-1}$ is continuous, the ALE mapping introduces one-to-one coordinate transformations for the domains. For each time step, after determining the transformation function $\Psi_t$, the problem turns into a numerical simulation for a fluid defined on a fixed domain, which we are familiar with.

For a function $\phi : \Omega_t^f \times [0,T] \rightarrow \mathbb{R}$, posed on the Eulerian frame, we may define the corresponding function $\overline{\phi} = g \circ \Psi_t$ on the ALE frame as:

$$\overline{\phi} : \Omega_0^f \rightarrow \mathbb{R}, \quad \overline{\phi}(y,t) = \phi(\Psi_t(y),t).$$

Meanwhile, the corresponding time derivative in ALE frame is defined as

$$\frac{\partial \phi}{\partial t} |_{y} : \Omega_t^f \times [0,T] \rightarrow \mathbb{R}, \quad \frac{\partial \phi}{\partial t} |_{y} (y,t) = \frac{\partial \overline{\phi}}{\partial t} (y,t).$$

Using the above notation, the domain velocity can then be defined as $z := \frac{\partial x}{\partial t} |_{y}$, which is actually the time derivative of the Eulerian coordinate. Notice that $z$ gives the velocity of each mesh node when discretized, so it is also called the mesh velocity.

Applying the chain rule, the ALE derivative of $\phi$ can be computed as

$$\frac{\partial \overline{\phi}}{\partial t} |_{y} = \frac{\partial \phi}{\partial t} |_{x} + z \cdot \nabla_{x} \phi,$$

Hence, the time derivative term on the Eulerian frame can be replaced by the ALE derivative

$$\frac{\partial \phi}{\partial t} |_{x} = \frac{\partial \phi}{\partial t} |_{y} - z \cdot \nabla_{x} \phi.$$
the flow equations (2.1.5)-(2.1.7) can then be written in ALE formulation as follows:

\[
\sigma + \lambda \left( \frac{\partial \sigma}{\partial t} \bigg|_y + (u - z) \cdot \nabla \sigma + g_\beta(\sigma, \nabla b) \right) - 2 \alpha D(u) = 0 \quad \text{in } \Omega^f, \tag{2.2.6}
\]

\[
Re \left( \frac{\partial u}{\partial t} \bigg|_y + (u - z) \cdot \nabla u \right) - \nabla \cdot \sigma - 2(1 - \alpha) \nabla \cdot D(u) + \nabla p = f \quad \text{in } \Omega^f, \tag{2.2.7}
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega^f. \tag{2.2.8}
\]

In order to define the ALE mapping \(\Psi_t\), consider the boundary position function \(h\):

\[
h(y) = \begin{cases} 
y + \eta \mathbf{e} & \text{on } \Gamma_t, 
y & \text{on } \Gamma^f_D \cup \Gamma^f_N, \end{cases} \tag{2.2.9}
\]

where \(\eta\) is the displacement of the moving wall. The ALE mapping may then be determined by solving the equation

\[
\Delta \Psi_t = 0 \quad \text{in } \Omega^f_0, \quad \Psi_t = h \quad \text{on } \partial \Omega^f_0. \tag{2.2.10}
\]

This method is called the harmonic extension, where the boundary position function on the boundary is extended onto the whole domain [19, 20].

The forcing term \(\Phi\) in (2.1.19) given in the coordinate for \(\Gamma_t\), can be recast in the reference configuration \(\Gamma_0\) as

\[
\Phi(z, t) = \Phi \sqrt{1 + \left( \frac{\partial \eta}{\partial z} \right)^2} \bigg|_{y} \quad \text{on } \Gamma_0, \tag{2.2.11}
\]

where the expression \(\sqrt{1 + \left( \frac{\partial \eta}{\partial z} \right)^2}\) represents the change in the surface measure passing from \(\Gamma_t\) to \(\Gamma_0\). Therefore, (2.1.15) can be rewritten on \(\Gamma_0\) as

\[
\rho \omega \frac{\partial^2 \eta}{\partial t^2} - a \frac{\partial^2 \eta}{\partial z^2} - b \frac{\partial^3 \eta}{\partial t \partial z^2} + c \eta = \Phi \sqrt{1 + \left( \frac{\partial \eta}{\partial z} \right)^2}. \tag{2.2.12}
\]

Now the fluid in a moving boundary problem in ALE formulation is summarized as

\[\text{solve the fluid equations (2.2.6)-(2.2.8) and the structure equation (2.2.12) with the}\]
boundary and initial conditions (2.1.5)-(2.1.12), (2.1.16) and
\[ u = \frac{\partial \eta}{\partial t} e \quad \text{on } \Gamma_I, \quad (2.2.13) \]
\[ (-pI + \sigma + 2(1 - \alpha)D(u)) \cdot n = \Phi e \quad \text{on } \Gamma_I, \quad (2.2.14) \]

using the ALE mapping satisfying (2.2.10).

For the variational formulation of the flow equations (2.2.6)-(2.2.8) in ALE framework, define function spaces for the reference domain:
\[ U_0 := \{ v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_D \}, \]
\[ Q_0 := L^2(\Omega), \]
\[ \Sigma_0 := \{ \tau \in L^2(\Omega) : \tau_{ij} = \tau_{ji} \}. \]

The function spaces for \( \Omega_f^t \) are then defined as
\[ U_t := \{ v : \Omega_f^t \times [0, T] \rightarrow \mathbb{R}^2, \quad v = v \circ \Psi_t^{-1} \quad \text{for } v \in U_0 \}, \]
\[ Q_t := \{ q : \Omega_f^t \times [0, T] \rightarrow \mathbb{R}, \quad q = q \circ \Psi_t^{-1} \quad \text{for } p \in Q_0 \}, \]
\[ \Sigma_t := \{ \tau : \Omega_f^t \times [0, T] \rightarrow \mathbb{R}^{2 \times 2}, \quad \tau = \tau \circ \Psi_t^{-1} \quad \text{for } \tau \in \Sigma_0 \}. \]

The variational formulation of (2.2.6)-(2.2.8) in ALE framework is then to find \((u, p, \sigma) \in U_t \times Q_t \times \Sigma_t \) such that
\[ (\sigma, \tau) + \lambda \left( \frac{\partial \sigma}{\partial t} \big|_y + (u - z) \cdot \nabla \sigma + g_{ij}(\sigma, \nabla b), \tau \right) -2\alpha (D(u), \tau) = 0 \quad \forall \tau \in \Sigma_t, \quad (2.2.15) \]
\[ \text{Re} \left( \frac{\partial u}{\partial t} \big|_y + (u - z) \cdot \nabla u, v \right) + (\sigma, D(v)) + 2(1 - \alpha)(D(u), D(v)) \]
\[ - (p, \nabla \cdot v) = (f, v) + ((\sigma + 2(1 - \alpha)D(u) - pI) \cdot n, v)_{\Gamma_I} \quad \forall v \in U_t, \quad (2.2.16) \]
\[ (q, \nabla \cdot u) = 0 \quad \forall q \in Q_t. \quad (2.2.17) \]

In order to derive the conservative variational formulation [41], consider the Reynolds trans-
for any subdomain $V(t) \subset \Omega^f_t$ such that $V(t) = \Psi_t(V_0)$ with $V_0 \subset \Omega^f_0$. If $v$ is a function from $\Omega^f_t$ to $\mathbb{R}$ and $v = \tau \circ \Psi^{-1}_t$ for $\tau : \Omega^f_0 \to \mathbb{R}$, we have that

$$\frac{\partial v}{\partial t} |_{\gamma} = 0$$

and, therefore,

$$\frac{d}{dt} \int_{\Omega^f_t} v \, d\Omega = \int_{\Omega^f_t} v \nabla_x \cdot z \, d\Omega,$$

$$\frac{d}{dt} \int_{\Omega^f_t} \phi v \, d\Omega = \int_{\Omega^f_t} \left( \frac{\partial \phi}{\partial t} |_{\gamma} + \phi \nabla_x \cdot z \right) v \, d\Omega.$$

Using (2.2.15)-(2.2.17) and (2.2.21), we have the following weak formulation: find $(u, p, \sigma) \in U_t \times Q_t \times \Sigma_t$ such that

$$(\sigma, \tau) + \lambda \frac{d}{dt}(\sigma, \tau) + \lambda (-\sigma(\nabla \cdot z) + (u - z) \cdot \nabla)\sigma + g_\beta(\sigma, \nabla b), \tau)$$

$$-2\alpha (D(u), \tau) = 0 \quad \forall \tau \in \Sigma_t,$$

$$Re \frac{d}{dt}(u, v) + Re ((-u(\nabla \cdot z) + (u - z) \cdot \nabla u, v) + (\sigma, D(v))$$

$$+2(1 - \alpha)(D(u), D(v)) - (p, \nabla \cdot v)$$

$$= (f, v) + ((\sigma + 2(1 - \alpha)D(u) - pI) \cdot n, v)_{\Gamma_w(t)} \quad \forall v \in U_t,$$

$$\langle q, \nabla \cdot u \rangle = 0 \quad \forall q \in Q_t.$$
The variational formulation of (2.2.12) is then to find $\eta \in S$ such that
\begin{equation}
\rho_w \frac{\partial^2 \eta}{\partial t^2}, \xi) + a \left( \frac{\partial \eta}{\partial z}, \frac{\partial \xi}{\partial z} \right) + b \left( \frac{\partial^2 \eta}{\partial t \partial z}, \frac{\partial \xi}{\partial z} \right) + c(\eta, \xi) = (\Phi \sqrt{1 + \frac{\partial \eta}{\partial z}^2}, \xi) \quad \forall \xi \in S. \tag{2.2.25}
\end{equation}

For the coupled problem define the test function spaces for $u$, $\eta$ by
\begin{equation}
\tilde{U}_t \times \tilde{S} := \left\{ (v, \xi) \in U_t \times S : v \circ \Psi_t |_{\Gamma_0} = \xi e \right\}. \tag{2.2.26}
\end{equation}

Using (2.1.19), (2.2.11) and the ALE mapping, we have
\begin{equation}
\rho_w \left( \frac{\partial^2 \eta}{\partial t^2}, \xi \right) + a \left( \frac{\partial \eta}{\partial z}, \frac{\partial \xi}{\partial z} \right) + b \left( \frac{\partial^2 \eta}{\partial t \partial z}, \frac{\partial \xi}{\partial z} \right) + c(\eta, \xi) \\
= -((\sigma + 2(1 - \alpha) D(u) - pI) \cdot n, (\xi \circ \Psi_t^{-1}) e)_{\Gamma_0}. \tag{2.2.27}
\end{equation}

Thus, by (2.1.24), (2.2.13) and (2.2.14), the variational problem of the fluid-structure coupled problem in ALE frame is given by: find $(u, p, \sigma, \eta) \in U_t \times Q_t \times \Sigma_t \times S$ such that
\begin{align}
2\alpha \left[ \rho_w \left( \frac{\partial^2 \eta}{\partial t^2}, \xi \right) + a \left( \frac{\partial \eta}{\partial z}, \frac{\partial \xi}{\partial z} \right) + b \left( \frac{\partial^2 \eta}{\partial t \partial z}, \frac{\partial \xi}{\partial z} \right) + c(\eta, \xi) \right] \\
+ \lambda \frac{d}{dt}(\sigma, \tau) + \lambda (-\sigma(\nabla \cdot z), \tau) + \lambda ((u - z) \cdot \nabla) \sigma, \tau \\
+ 2\alpha Re \frac{d}{dt}(u, v) + 2\alpha Re (-u(\nabla \cdot z), v) + 2\alpha Re ((u - z) \cdot \nabla) u, v \\
+ A((\sigma, u), (\tau, v)) - 2\alpha(p, \nabla \cdot v) + 2\alpha(q, \nabla \cdot v) = 2\alpha (f, v) \quad \tag{2.2.28}
\end{align}

$\forall (v, q, \tau, \xi) \in \tilde{U}_t \times \tilde{S} \times Q_t \times \tilde{S}$, where $A((\sigma, u), (\tau, v))$ is defined as (2.1.24).

### 2.3 Discretization

We define finite element spaces for the approximation of $(u, p)$ in $\Omega_0^f$:
\begin{align*}
U_{h,0} &:= \left\{ v \in U_0 \cap (C^0(\overline{\Omega}))^2 : v|_K \in P_2(K)^2, \forall K \in T_{h,0} \right\}, \\
Q_{h,0} &:= \left\{ q \in Q_0 \cap C^0(\overline{\Omega}) : q|_K \in P_1(K), \forall K \in T_{h,0} \right\},
\end{align*}
where \( T_{h,0} \) is a triangularization of \( \Omega_0^f \). The stress \( \sigma \) is approximated in the discontinuous finite element space of piecewise linear polynomials:

\[
\Sigma_{h,0} := \{ \tau \in \Sigma_0 : \tau|_K \in P_1(K)^{2 \times 2}, \forall K \in T_{h,0} \}.
\]

Then the finite element spaces for \( \Omega_t^f \) are defined as

\[
U_{h,t} := \{ v_h : \Omega_t^f \times [0, T] \rightarrow \mathbb{R}^2, v_h = \nabla \Psi_{h,t}^{-1} \text{ for } v_h \in U_{h,0} \},
\]

\[
Q_{h,t} := \{ q_h : \Omega_t^f \times [0, T] \rightarrow \mathbb{R}, q_h = \Psi_{h,t}^{-1} \text{ for } q_h \in Q_{h,0} \},
\]

\[
\Sigma_{h,t} := \{ \sigma_h : \Omega_t^f \times [0, T] \rightarrow \mathbb{R}^{2 \times 2}, \sigma_h = \Psi_{h,t}^{-1} \text{ for } \sigma_h \in \Sigma_{h,0} \},
\]

where \( \Psi_{h,t} : \Omega_0^f \rightarrow \Omega_t^f \) is a discrete mapping approximated by \( P_1 \) Lagrangian finite elements such that \( \Psi_{h,t}(y) = x_h(y, t) \). For the discrete ALE mapping, define the set

\[
X_h := \{ x \in H^1(\Omega_0^f) : x|_K \in P_1(K)^2, \forall K \in T_{h,0} \}.
\]

(2.3.1)

For the displacement we define

\[
S_h := \{ \xi \in S \cap C^0([0, L]) : \xi|_E \in P_2(E), \forall E \in \mathcal{T}_h \},
\]

(2.3.2)

where \( \cup \mathcal{T}_h = [0, L] \) and \( \mathcal{T}_h \) has matching grid points with \( T_{h,0} \).

Introduce the operators \( \theta(\cdot, \cdot, \cdot) \) \( \kappa(\cdot, \cdot, \cdot) \) defined by

\[
\theta(u, w, v)_{\Omega_t^f} := \frac{1}{2} \left[ (u \cdot \nabla w, v)_{\Omega_t^f} - (u \cdot \nabla v, w)_{\Omega_t^f} \right],
\]

(2.3.3)

\[
\kappa(v - z, \sigma, \tau)_{\Omega_t^f} := \left( (v - z) \cdot \nabla \sigma, \tau \right)_{\Omega_t^f} + \frac{1}{2} (\nabla \cdot v \sigma, \tau)_{\Omega_t^f} + \left< \sigma^+ - \sigma^-, \tau^+ \right>_h, v - z,
\]

(2.3.4)

where the last term in (2.3.4) accounts for the jump in the discretized \( \sigma \) across an inflow edge of element associated \( v - z \).
Lemma 2.3.1 The trilinear operator $\theta(\cdot, \cdot, \cdot)$ has following properties \cite{25}:

\begin{align}
\text{(i):} & \quad \theta(u, v, w)_{\Omega'_t} \leq C\|u\|_{1,\Omega'_t}\|v\|_{1,\Omega'_t}\|w\|_{1,\Omega'_t}. \\
\text{(ii):} & \quad \theta(u, v, w)_{\Omega'_t} \leq C\|u\|^2_{0,\Omega'_t}\|v\|^2_{1,\Omega'_t}\|w\|_{1,\Omega'_t}.
\end{align}

\begin{align}
\kappa(u_h - z_h, \sigma_h, \sigma_h)_{\Omega'_t} & \geq \frac{1}{2} \left( (\nabla \cdot z_h)\sigma_h, \sigma_h \right)_{\Omega'_t} + (((u_h - z_h) \cdot n)\sigma_h, \sigma_h)_{\Gamma'_h \cup \Gamma_i}.
\end{align}

Proof: Applying integration by parts for $\kappa(\cdot, \cdot, \cdot)$, we have

\begin{align}
\kappa(u_h - z_h, \sigma_h, \tau_h)_{\Omega'_t} &= -(((u_h - z_h) \cdot \nabla)\tau_h, \sigma_h)_{\Omega'_t} - \frac{1}{2} (\nabla \cdot u_h \tau_h, \sigma_h)_{\Omega'_t} \\
&\quad - \langle \sigma_h^+, \tau_h^- \rangle_{h,u_h - z_h} + \langle \sigma_h^+, \sigma_h^- \rangle_{h,u_h - z_h} \\
&\quad + ((u_h - z_h) \cdot n)\sigma_h, \tau_h)_{\Gamma'_h \cup \Gamma_i},
\end{align}

therefore,

\begin{align}
\kappa(u_h - z_h, \sigma_h, \sigma_h)_{\Omega'_t} &= -(((u_h - z_h) \cdot \nabla)\sigma_h, \sigma_h)_{\Omega'_t} - \frac{1}{2} (\nabla \cdot u_h \sigma_h, \sigma_h)_{\Omega'_t} \\
&\quad - \langle \sigma_h^+, \sigma_h^- \rangle_{h,u_h - z_h} + \langle \sigma_h^+, -\sigma_h^- \rangle_{h,u_h - z_h} \\
&\quad + ((u_h - z_h) \cdot n)\sigma_h, \sigma_h)_{\Gamma'_h \cup \Gamma_i} \\
&\quad = -\kappa(u_h - z_h, \sigma_h, \sigma_h)_{\Omega'_t} + \langle \sigma_h^+, \sigma_h^- \rangle_{h,u_h - z_h} \\
&\quad + ((u_h - z_h) \cdot n)\sigma_h, \sigma_h)_{\Gamma'_h \cup \Gamma_i}.
\end{align}

That implies

\begin{align}
\kappa(u_h - z_h, \sigma_h, \sigma_h)_{\Omega'_t} &= \frac{1}{2} \left( (\nabla \cdot z_h)\sigma_h, \sigma_h \right)_{\Omega'_t} + (((u_h - z_h) \cdot n)\sigma_h, \sigma_h)_{\Gamma'_h \cup \Gamma_i} \\
&\quad + \langle \sigma_h^+, -\sigma_h^- \rangle_{h,u_h - z_h} \\
&\quad \geq \frac{1}{2} \left( (\nabla \cdot z_h)\sigma_h, \sigma_h \right)_{\Omega'_t} + (((u_h - z_h) \cdot n)\sigma_h, \sigma_h)_{\Gamma'_h \cup \Gamma_i}.
\end{align}
Using the Green’s theorem and $\nabla \cdot u = 0$,

\[
(u \cdot \nabla w, v)_{\Omega_t'} = \theta(u, w, v)_{\Omega_t'} + \frac{1}{2}((u \cdot n)w, v)_{\Gamma_{h,t}' \cup \Gamma_{I,t}}
\]

and

\[
\theta(u, v, v)_{\Omega_t'} = \frac{1}{2}((u \cdot n)v, v)_{\Gamma_{h,t}' \cup \Gamma_{I,t}} \quad \forall v \in U_{h,t}.
\]

The semi-discrete variational formulation of the coupled problem in ALE frame is then written as

\[
2\alpha \left[ \rho_w (\partial^2 \eta_h, \xi_h\gamma_0 + a (\partial \eta_h, \partial \xi_h))_{\gamma_0} + b (\partial^2 \eta_h, \partial \xi_h)_{\gamma_0} + c (\eta_h, \xi_h)_{\gamma_0} \right] \\
+ \lambda \left[ \frac{d}{dt} (\sigma_h, \tau_h)_{\Omega_t'} + \kappa (u_h - z_h, \sigma_h, \tau_h)_{\Omega_t'} - (\sigma_h (\nabla \cdot z_h), \tau_h)_{\Omega_t'} \right] \\
+ 2\alpha Re \left[ \frac{d}{dt} (u_h, v_h)_{\Omega_t'} - (u_h (\nabla \cdot z_h), v_h)_{\Omega_t'} \\
+ \theta(u_h, u_h, v_h)_{\Omega_t'} + \frac{1}{2}((u_h \cdot n)u_h, v_h)_{\Gamma_{h,t}' \cup \Gamma_{I,t}} - (z_h \cdot \nabla u_h, v_h)_{\Omega_t'} \right] \\
+ A((\sigma_h, u_h), (\tau_h, v_h)) - 2\alpha (p_h, \nabla \cdot v_h)_{\Omega_t'} + 2\alpha (q_h, \nabla \cdot u_h)_{\Omega_t'} \\
= 2\alpha (f, v)_{\Omega_t'} \quad \forall (v_h, \tau_h, q_h, \xi_h) \in \tilde{U}_h \times \tilde{S}_h \times Q_h \times \tilde{S}_h.
\]

In (2.3.13) $\tilde{U}_h \times \tilde{S}_h$ is a subspace of $U_h \times S_h$, where the matching condition in (2.2.26) is satisfied in a discrete sense.

Using the backward Euler for both the fluid and structure equations and applying the
geometric conservation law (GCL) [41], we have the fully discretized systems given as follows.

\[
2\alpha \left[ \rho_\omega \left( \eta_{h}^{n+1} - 2\eta_{h}^{n} + \eta_{h}^{n-1} \right) \right] + a \left( \frac{\partial \eta_{h}^{n+1}}{\partial z}, \right)_{\Gamma_{0}} + \frac{\partial \eta_{h}^{n+1}}{\partial z} , \xi_{h} \right)_{\Gamma_{0}} + b \left( \frac{\partial \eta_{h}^{n+1}}{\partial z} , \right)_{\Gamma_{0}} + c \left( \eta_{h}^{n+1} , \xi \right)_{\Gamma_{0}} + \lambda \frac{\Delta t}{2} \left( \sigma_{h}^{n+1} , \tau_{h} \right)_{\Omega_{i}^{f+1}} - \left( \sigma_{h}^{n} , \tau_{h} \right)_{\Omega_{i}^{f}} \\
+ \frac{\lambda}{\Delta t} \left[ \left( \sigma_{h}^{n+1} , \tau_{h} \right)_{\Omega_{i+1}^{f+1}} - \left( \sigma_{h}^{n} , \tau_{h} \right)_{\Omega_{i+1}^{f}} \right] + 2\alpha Re \left[ \left( u_{h}^{n+1} , v_{h} \right)_{\Omega_{i+1}^{f+1}} - \left( u_{h}^{n} , v_{h} \right)_{\Omega_{i+1}^{f}} \right] \\
+ \lambda \left[ \left( u_{h}^{n+1} - z_{h}^{n+1} , \sigma_{h}^{n+1} , \tau_{h} \right)_{\Omega_{i+1}^{f+1}} - \left( \sigma_{h}^{n+1} (\nabla \cdot z_{h}^{n+1} , \tau_{h} \right)_{\Omega_{i+1}^{f+1}} \right] \\
+ 2\alpha Re \left[ \theta \left(u_{h}^{n+1} , u_{h}^{n+1} , v_{h} \right)_{\Omega_{i+1}^{f+1}} + \frac{1}{2} \left( \left( u_{h}^{n+1} \cdot n \right) u_{h}^{n+1} , v_{h} \right)_{\Gamma_{i+1}^{f+1} \cup \Gamma_{i+1}^{f_+}} \\
- \left( u_{h}^{n+1} (\nabla \cdot z_{h}^{n+1} , v_{h} \right)_{\Omega_{i+1}^{f+1}} - \left( z_{h}^{n+1} \cdot \nabla u_{h}^{n+1} , v_{h} \right)_{\Omega_{i+1}^{f+1}} \right] \\
+ A \left( \left( \sigma_{h}^{n+1} , u_{h}^{n+1} \right), \left( \tau_{h} , v_{h} \right) \right)_{\Omega_{i+1}^{f+1}} - 2\alpha \left( p_{h}^{n+1} , \nabla \cdot v_{h} \right)_{\Omega_{i+1}^{f+1}} + 2\alpha \left( q_{h} , \nabla \cdot u_{h} \right) \\
= 2\alpha \left( f , v_{h} \right)_{\Omega_{i+1}^{f+1}} \\
\right) \tag{2.3.14}
\]

Stability of the fully discretized coupled system is proved in the next theorem. We omit the subscript $h$ in $(u_{h}^{n} , \sigma_{h}^{n} , p_{h}^{n} , \eta_{h}^{n})$ to simplify our notation.

**Theorem 2.3.3** If $1 - 4\lambda M > 0$, a solution to the fully discretized system (2.3.14) satisfies the
\[
\begin{align*}
&\alpha c \|\eta^{n+1}\|^2 + \lambda \frac{\partial \eta^{n+1}}{\partial t} + \frac{\alpha \text{Re}}{\Delta t} \|u^{n+1}\|^2_{0,\Omega^f,\gamma^{n+1}} \\
&+\alpha \left[ \frac{\rho_w}{\Delta t^2} \|\eta^{n+1}\|^2 + a \|\eta^{n+1}\|_{\Omega}\right] \\
&+\sum_{i=0}^{n} \alpha \left[ \frac{\rho_w}{\Delta t^2} \|\eta^{i+1} - 2\eta^i + \eta^{i-1}\|^2 + a \|\frac{\partial \eta^{i+1} - \partial \eta^i}{\partial z}\|_{0}\right] \\
&+\sum_{i=0}^{n} \frac{2\alpha(1 - \alpha)}{\Delta t} \|D(u^{i+1})\|^2_{0,\Omega^f,\gamma^{n+1}} - (1 - 4\lambda M) \|\sigma^{i+1}\|^2_{0,\Omega^f,\gamma^{n+1}} \\
&\leq \alpha \left[ \frac{\rho_w}{\Delta t} \|\eta_0\|^2 + a \|\eta_0\|_{\Omega} + c \|\eta_0\|_{\Omega}^2\right] + \frac{\lambda}{2\Delta t} \|\sigma_0\|^2_{0,\Omega^f} + \frac{\alpha \text{Re}}{\Delta t} \|u_0\|^2_{0,\Omega^f} \\
&+ C \sum_{i=0}^{n} \|\eta^{i+1}\|^2_{0,\Omega^f,\gamma^{n+1}}. 
\end{align*}
\] (2.3.15)

**Proof:** Letting \( \tau = \sigma^{n+1}, \, \mathbf{v} = u^{n+1}, \, q = p^{n+1} \) and \( \xi = \eta^{n+1} - \eta^n \) in (2.3.14), we obtain

\[
\begin{align*}
2\alpha \left[ \rho_w \left( \frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\Delta t^2}, \eta^{n+1} - \eta^n \right) \right] \\
+ b \left( \frac{\partial \eta^{n+1}}{\partial z}, \frac{\partial \eta^{n+1} - \partial \eta^n}{\partial z} \right) + c \left( \eta^{n+1}, \eta^{n+1} - \eta^n \right) \right] \\
+ \frac{\lambda}{\Delta t} \|\sigma^{n+1}\|^2_{0,\Omega^f,\gamma^{n+1}} + \frac{2\alpha \text{Re}}{\Delta t} \|u^{n+1}\|^2_{0,\Omega^{n+1}} \\
+ \alpha \left[ \kappa(u^{n+1} - z^{n+1}, \sigma^{n+1})_{\Omega^{n+1}} - (\sigma^{n+1}(\nabla \cdot z^{n+1}), \sigma^{n+1})_{\Omega^{n+1}} \right] \\
+ 2\alpha \text{Re} \left[ \theta(u^{n+1}, u^{n+1}, u^{n+1})_{\Omega^{n+1}} + \frac{1}{2} (u^{n+1} \cdot n) u^{n+1} (u^{n+1})_{\Omega^{n+1}} \right] \\
- (u^{n+1}(\nabla \cdot z^{n+1}), u^{n+1})_{\Omega^{n+1}} - (z^{n+1} \cdot \nabla u^{n+1} + u^{n+1})_{\Omega^{n+1}} \right] \\
+ A((\sigma^{n+1}, u^{n+1}), (\sigma^{n+1}, u^{n+1}))_{\Omega^{n+1}} \right] \\
= \frac{\lambda}{\Delta t} \|\sigma^n - \sigma^{n+1}\|^2_{\Omega^n} + \frac{2\alpha \text{Re}}{\Delta t} \|u^n - u^{n+1}\|^2_{\Omega^{n+1}} + 2\alpha \|\mathbf{r}^{n+1} - u^{n+1}\|_{\Omega^{n+1}}. 
\end{align*}
\] (2.3.16)
Consider the left hand side of the equation first. The structure terms are turned to:

\[
2\alpha \left[ \rho_w (\frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\Delta t^2}, \eta^{n+1} - \eta^n)_{\Gamma_0} + a (\frac{\partial \eta^{n+1}}{\partial z}, \frac{\partial \eta^{n+1}}{\partial z})_{\Gamma_0} + b (\frac{\partial u^{n+1}}{\partial z} - \frac{\partial u^n}{\partial z}, \frac{\partial \eta^{n+1} - \partial \eta^n}{\Delta t})_{\Gamma_0} + c (\eta^{n+1}, \eta^{n+1} - \eta^n)_{\Gamma_0} \right]
\]

Thus, using (2.3.12) and (2.3.19),

\[
\alpha \left[ \frac{\rho_e}{\Delta t^2} \left( \| \eta^{n+1} - \eta^n \|^2_0 - \| \eta^n - \eta^{n-1} \|^2_0 + \| \eta^{n+1} - 2\eta^n + \eta^{n-1} \|^2_0 \right) + a \left( \| \frac{\partial \eta^{n+1}}{\partial z} \|^2_0 - \| \frac{\partial \eta^n}{\partial z} \|^2_0 + \| \frac{\partial \eta^{n+1} - \partial \eta^n}{\partial z} \|^2_0 \right) + \frac{2b}{\Delta t} \| \frac{\partial (\eta^{n+1} - \eta^n)}{\partial z} \|^2_0 \right] 
\]

By (2.3.10),

\[
\lambda \left[ \kappa (u^{n+1} - z^{n+1}, \sigma^{n+1}, \sigma^{n+1})_{\Omega_{r+\frac{1}{2}}} - (\sigma^{n+1} (\nabla \cdot z^{n+1}), \sigma^{n+1})_{\Omega_{r+\frac{1}{2}}} \right] 
\]

\[
\geq -\frac{\lambda}{2} \left[ (\sigma^{n+1} (\nabla \cdot z^{n+1}), \sigma^{n+1})_{\Omega_{r+\frac{1}{2}}} - ((u^{n+1} - z^{n+1}) \cdot n) (\sigma^{n+1}, \sigma^{n+1})_{\Gamma_N \cup \Gamma_{r+\frac{1}{2}}} \right]. 
\]

Using integration by parts and that \( z^{n+1} = 0 \) on the fixed boundary \( \Gamma_N \),

\[
(z^{n+1}, \nabla u^{n+1}, u^{n+1})_{\Omega_{r+\frac{1}{2}}} = -\frac{1}{2} ((\nabla \cdot z^{n+1})u^{n+1}, u^{n+1})_{\Omega_{r+\frac{1}{2}}} + \frac{1}{2} ((z^{n+1} \cdot n)u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{r+\frac{1}{2}}}. 
\]

Thus, using (2.3.12) and (2.3.19),

\[
2\alpha \text{ Re} \left[ \theta (u^{n+1}, u^{n+1}, u^{n+1})_{\Omega_{r+\frac{1}{2}}} + \frac{1}{2} ((u^{n+1} \cdot n)u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{r+\frac{1}{2}}} + \frac{2}{\Delta t} ((u^{n+1} \cdot n)u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{r+\frac{1}{2}}} \right] 
\]

\[
\geq \alpha \text{ Re} \left[ -(u^{n+1} (\nabla \cdot z^{n+1}), u^{n+1})_{\Omega_{r+\frac{1}{2}}} - (z^{n+1} \cdot \nabla u^{n+1}, u^{n+1})_{\Omega_{r+\frac{1}{2}}} + ((u^{n+1} - z^{n+1}) \cdot n) u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{r+\frac{1}{2}}} \right]. 
\]

(2.3.20)
and, by (2.1.26), (2.3.18) and (2.3.20),

\[
\text{Fluid terms at left} \\
\geq \alpha \text{Re} \left[ -(u^n+1(\nabla \cdot z^{n+1}), u^{n+1})_{\Omega}^{n+\frac{1}{2}} \\
+ ((u^{n+1} - z^{n+1}) \cdot n) u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{\Gamma_f}}^{n+\frac{1}{2}} \right] \\
+(1 - 4\lambda M) \| \sigma^{n+1} \|_{\partial \Omega_{\Gamma_f}}^{n+\frac{1}{2}} + 4\alpha (1 - \alpha) \| D(u^{n+1}) \|_{\partial \Omega_{\Gamma_f}}^{n+\frac{1}{2}} \\
- \frac{\lambda}{2} \left[ (\sigma^{n+1}(\nabla \cdot z^{n+1}), \sigma^{n+1})_{\Omega_f}^{n+\frac{1}{2}} \\
- (((u^{n+1} - z^{n+1}) \cdot n) \sigma^{n+1}, \sigma^{n+1})_{\Gamma_N \cup \Gamma_{\Gamma_f}}^{n+\frac{1}{2}} \right] \\
+ \frac{\lambda}{\Delta t} \| \sigma^{n+1} \|_{\partial \Omega_{\Gamma_f}}^{n+\frac{1}{2}} + \frac{2\alpha \text{Re}}{\Delta t} \| u^{n+1} \|_{\partial \Omega_{\Gamma_f}}^{n+\frac{1}{2}} .
\] (2.3.21)

On the other hand, using the Schwartz and Young’s inequalities, the right hand side of (2.3.16) is bounded as

\[
\text{RHS} \leq \frac{\lambda}{2\Delta t} \left( \| \sigma^n \|_{\partial \Omega_{\Gamma_f}}^2 + \| \sigma^{n+1} \|_{\partial \Omega_{\Gamma_f}}^2 \right) + \frac{\alpha \text{Re}}{\Delta t} \left( \| u^n \|_{\partial \Omega_{\Gamma_f}}^2 + \| u^{n+1} \|_{\partial \Omega_{\Gamma_f}}^2 \right) \\
+ C \| f^{n+1} \|_{\partial \Omega_{\Gamma_f}}^2 + \delta \| D(u^{n+1}) \|_{\partial \Omega_{\Gamma_f}}^2 .
\] (2.3.22)
By (2.3.17), (2.3.21) and (2.3.22), (2.3.16) implies

\[
\alpha \left[ \frac{\rho w}{\Delta t^2} \left( \|\eta^{n+1} - \eta^n\|_0^2 + \|\gamma^{n+1} - 2\eta^n + \gamma^{n-1}\|_0^2 \right) + a \left( \left\| \frac{\partial \eta^{n+1}}{\partial z} \right\|_0^2 + \left\| \frac{\partial \eta^{n+1} - \partial \eta^n}{\partial z} \right\|_0^2 \right) + \frac{2b}{\Delta t} \left( \left\| \frac{\partial (\eta^{n+1} - \eta^n)}{\partial z} \right\|_0^2 \right) + c \left( \|\eta^{n+1}\|_0^2 + \|\eta^{n+1} - \eta^n\|_0^2 \right) \right] \\
+ \frac{\lambda}{\Delta t} \left[ \|\sigma^{n+1}\|_{0, \Omega_{t_n}'}^2 - \frac{1}{2} \|\sigma^{n+1}\|_{0, \Omega_{t_n}}^2 \right] \\
+ \frac{2\alpha \text{Re}}{\Delta t} \left[ \|\mathbf{u}^{n+1}\|_{0, \Omega_{t_n}'}^2 - \frac{1}{2} \|\mathbf{u}^{n+1}\|_{0, \Omega_{t_n}}^2 \right] \\
+ (1 - 4\lambda M) \|\sigma^{n+1}\|_{0, \Omega_{t_n}'}^2 + (4\alpha(1 - \delta)) \|D(u^{n+1})\|_{0, \Omega_{t_n}'}^2 \\
- \frac{\lambda}{2} \left[ \|\sigma^{n+1}(\nabla \cdot \mathbf{z}^{n+1}), \sigma^{n+1}\|_{\Omega_{t_n}'}^2 \\
- ((u^{n+1} - \mathbf{z}^{n+1}) \cdot \mathbf{n}) \sigma^{n+1}, \sigma^{n+1}\|_{\Gamma_N \cup \Gamma_{\partial \Omega_{t_n}'}}^2 \right] \\
+ \alpha \text{Re} \left[ -(u^{n+1}(\nabla \cdot \mathbf{z}^{n+1}), u^{n+1})_{\Omega_{t_n}'}^2 \\
+ ((u^{n+1} - \mathbf{z}^{n+1}) \cdot \mathbf{n}) u^{n+1}, u^{n+1})_{\Gamma_N \cup \Gamma_{\partial \Omega_{t_n}'}}^2 \right] \leq \alpha \left[ \frac{\rho w}{\Delta t^2} \|\eta^{n+1} - \eta^n\|_0^2 + a \left\| \frac{\partial \eta^n}{\partial z} \right\|_0^2 + c \|\eta^n\|_0^2 \right] + \frac{\lambda}{2\Delta t} \|\sigma^{n+1}\|_{0, \Omega_{t_n}}^2 \\
+ \frac{\alpha \text{Re}}{\Delta t} \|D(u^n)\|_{0, \Omega_{t_n}}^2 + C \|f^{n+1}\|_{-1, \Omega_{t_n}'}^2 \cdot \tag{2.3.23}
\]

The time discretization scheme in (2.3.14) is based on the mid-point rule satisfying GCL [41]

\[
\int_{\Omega_{t_n}'} \mathbf{v}_h \, d\Omega - \int_{\Omega_{t_n}} \mathbf{v}_h \, d\Omega = \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \mathbf{v}_h \nabla \cdot \mathbf{z}_h \, d\Omega \, dt = \Delta t \int_{t_n}^{t_{n+1}} \mathbf{v}_h \nabla \cdot \mathbf{z}_h \, d\Omega, \tag{2.3.24}
\]

and we have

\[
\frac{\lambda}{2\Delta t} \left[ \|\sigma^{n+1}\|_{0, \Omega_{t_n}'}^2 - \|\sigma^{n+1}\|_{0, \Omega_{t_n}}^2 \right] = \frac{\lambda}{2} \int_{t_n}^{t_{n+1}} \|\sigma^{n+1}\|_{0, \Omega_{t_n}'}^2 \nabla \cdot \mathbf{z}^{n+1} \, d\Omega \\
= \frac{\lambda}{2} \left( \sigma^{n+1}(\nabla \cdot \mathbf{z}^{n+1}), \sigma^{n+1}\right)_{\Omega_{t_n}'}^2, \\
\frac{\alpha \text{Re}}{\Delta t} \left[ \|\mathbf{u}^{n+1}\|_{0, \Omega_{t_n}'}^2 - \|\mathbf{u}^{n+1}\|_{0, \Omega_{t_n}}^2 \right] = \alpha \text{Re} \int_{t_n}^{t_{n+1}} \|\mathbf{u}^{n+1}\|_{0, \Omega_{t_n}'}^2 \nabla \cdot \mathbf{z}^{n+1} \, d\Omega \\
= \alpha \text{Re} \left( \mathbf{u}^{n+1}(\nabla \cdot \mathbf{z}^{n+1}), \mathbf{u}^{n+1}\right)_{\Omega_{t_n}'}^2. \tag{2.3.25}
\]
Using (2.3.25) in (2.3.23) and letting $\delta = 2 \alpha (1 - \alpha)$, we obtain

\[
\alpha \left[ \frac{\rho w}{\Delta t^2} \left( \|\eta^{n+1} - \eta^n\|_0^2 + \|\eta^{n+1} - 2\eta^n + \eta^{n-1}\|_0^2 \right) + \left( \frac{\partial \eta^{n+1}}{\partial z} \right)_0^2 + \left( \frac{\partial \eta^{n+1} - \partial \eta^n}{\partial z} \right)_0^2 \right] + \frac{2b}{\Delta t} \left( \frac{\partial (\eta^{n+1} - \eta^n)}{\partial z} \right)_0^2 + c \left( \|\eta^{n+1}\|_0^2 + \|\eta^{n+1} - \eta^n\|_0^2 \right)
\]

\[
+ \frac{\lambda}{2\Delta t} \|\sigma^{n+1}\|_{0, \Omega^f_{i,n+\frac{1}{2}}}^2 + \frac{\alpha \Re}{\Delta t} \|u^{n+1}\|_{0, \Omega^f_{i,n+\frac{1}{2}}}^2 + (1 - 4\lambda M)\|\sigma^{n+1}\|_{0, \Omega^f_{i,n+\frac{1}{2}}}^2 + (2\alpha(1 - \alpha) - \delta)\|D(u^{n+1})\|_{0, \Omega^f_{i,n+\frac{1}{2}}}^2
\]

\[
+ \frac{\lambda}{2} \left( ((u^{n+1} - z^{n+1}) \cdot n)\sigma^{n+1}, \sigma^{n+1} \right)_{\Gamma_N \cup \Gamma_{i,n+\frac{1}{2}}} + \alpha \Re \left( ((u^{n+1} - z^{n+1}) \cdot n)u^{n+1}, u^{n+1} \right)_{\Gamma_N \cup \Gamma_{i,n+\frac{1}{2}}}
\]

\[
\leq \alpha \left[ \frac{\rho w}{\Delta t^2} \|\eta^n - \eta^{n-1}\|_0^2 + a\left( \frac{\partial \eta^n}{\partial z} \right)_0^2 + c\|\eta^n\|_0^2 \right] + \frac{\lambda}{2\Delta t} \|\sigma^n\|_{0, \Omega^f_{i,n}}^2 + \frac{\alpha \Re}{\Delta t} \|D(u^n)\|_{0, \Omega^f_{i,n}}^2 + C \|f^{n+1}\|_{-1, \Omega^f_{i,n+\frac{1}{2}}}^2 .
\]

(2.3.26)

Summing over $n$ in (2.3.26) and assuming that the fluid velocity matches with the domain velocity on $\Gamma_{i,n+\frac{1}{2}}$ and using $\mathbf{z}^{n+1} = \mathbf{0}$ on $\Gamma_{i,N}$, we obtain the estimate (2.3.15).

\[\square\]

### 2.4 Numerical results

In this section, we present results of numerical experiments for the viscoelastic fluid-structure system (2.1.1)-(2.1.3) and (2.1.15). The initial domain of the fluid is the rectangle of height $H = 1$ and length $L = 6$ whose upper bound is elastic. See Figure 2.2. Both the fluid and structure are initially at rest. We simulate the pressure pulse $P_{in} = 2000$ by imposing the following Neumann boundary conditions on the inflow and outflow sections:

\[
\begin{cases}
(\mathbf{\sigma} + 2(1 - \alpha)D(\mathbf{u}) - p\mathbf{I}) \cdot \mathbf{n} = -\frac{P_{in}}{2} \cos\left( \frac{\pi t}{0.025} \right) - 1) \mathbf{n} & \text{on } \Gamma_{in}, \\
(\mathbf{\sigma} + 2(1 - \alpha)D(\mathbf{u}) - p\mathbf{I}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_{out}.
\end{cases}
\]

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The parameters for the fluid equations are given as $\alpha = 0.9825$, $\beta = 0$, $\lambda = 0.9$, $1/Re = 0.035$, while for the structure equation, $a = 25000$, $c = 0.01$, $b = 400000$, $\rho_s = 1.1$. Note that for higher Weissenberg number $\lambda$, the numerical stability for the coupled system degrades due to the required condition $1 - 4\lambda M > 0$ of Theorem 2.3.3. One of the main goals of numerical experiments is to compare the viscoelastic case with the Newtonian case ($\lambda = 0$), and for this purpose, a choice of low Reynolds number (high viscosity) can be made to improve numerical stability when a larger $\lambda$ value is used for simulations.

A conforming space discretization is applied to the fluid-structure coupled system. For the fluid, the space discretization consists of the Taylor-Hood ($P_2, P_1$) finite elements for $u, p$ and $P_1$ discontinuous elements for $\sigma$. The structure is discretized by continuous $P_2$ finite elements, and the ALE mapping is approximated by $P_1$ elements.

The homogeneous Dirichlet boundary condition was applied for the structure equation in the previous analysis. However, in the numerical tests, we consider the first order absorbing boundary condition instead [17], which may be more realistic when applied to blood flow simulations:

$$
\begin{align*}
\frac{\partial \eta}{\partial t} - \sqrt{\frac{a}{\rho_w}} \frac{\partial \eta}{\partial z} &= 0 \quad \text{at } z = 0, \\
\frac{\partial \eta}{\partial t} + \sqrt{\frac{a}{\rho_w}} \frac{\partial \eta}{\partial z} &= 0 \quad \text{at } z = L.
\end{align*}
$$

### 2.4.1 Experiment 1

In this experiment, we approximate the system by the Leap-Frog algorithm. We set the mesh size to $h = 0.1$ and the time step to $\Delta t = 10^{-4}$. The wall displacements of the viscoelastic fluid at $t = 0.02s, 0.04s, 0.06s, 0.08s, 0.1s, 0.12s, 0.14s, 0.16s, 0.18s, 0.2s$ are presented in Figure 2.3. Observe that the bump of the wall, which is caused by fluid stress, moves from the inflow section to the outflow section repeatedly as time goes. The results meet our expectation based on the physical
Figure 2.3: Displacement of the wall for $P_{in} = 2000$
foundation. However, during numerical tests, we found out that the explicit Leap-Frog algorithm is not always stable. In fact, with a higher pressure input, the explicit algorithm can converge only up to the time $0.14s$.

### 2.4.2 Experiment 2

We implemented a more stable algorithm for the system by an implicit scheme with relaxation. Specifically, a sub-iteration involving both structure and fluid solvers is added in each time iteration [41]:

1. find the fluid subproblem solutions $u_k^{n+1}, p_k^{n+1}, \sigma_k^{n+1},$
2. find the structure subproblem solutions $\hat{\eta}_k^{n+1},$
3. relax the structure solver by
   $$\eta_k^{n+1} = \omega \hat{\eta}_k^{n+1} + (1 - \omega)\eta_k^{n+1}, \quad 0 \leq \omega \leq 1,$$
   and keep iterating until $|\eta_k^{n+1} - \eta_k^{n+1-1}| < \text{tolerance}.$

For this implicit algorithm, we used $\omega = 0.9$ and $p_{in} = 20000$, which is 10 times higher than the previous experiment. The structure displacement in this case is presented in Figure 2.4. Notice that the viscoelastic fluid is reduced to the Newtonian fluid in the case of $\lambda = 0$. Since the Newtonian flow is usually used to simulate the blood flow in many tests, we compared the wall displacements for both the viscoelastic and the Newtonian models with all the same parameters except $\lambda$. In Figure 2.4, displacements of the viscoelastic model are represented by blue curves while results for the Newtonian case are plotted with red curves. Clearly, similar patterns are observed from both models. A visible difference is observed as time goes, although the difference is not quite significant at initial times. During numerical tests, we also noticed that the viscoelastic case calls for more subiterations to converge as expected.

### 2.5 Conclusion

Viscoelastic flow model equations coupled with the String model was considered for stability analysis and numerical experiments. The numerical results indicate the approach based on the ALE method can be used to simulate the viscoelastic flow in an elastic medium. Non-negligible differences between the viscoelastic flows and the Newtonian flows were observed under a high
pressure input. Also, we noticed that the non-Newtonian property affects stability of decoupled algorithms significantly, i.e., larger Weissenberg numbers resulted in divergence of the algorithm at earlier times. This is obviously due to the small data assumption on $\lambda$ for numerical stability of the coupled system.
Chapter 3

2D-2D Quasi-Newtonian Fluid-Structure Interaction Problem

In this chapter, we consider a finite element approximation of the 2D-2D system of a quasi-Newtonian fluid and a linear elastic structure, and investigate both analysis and numerical experiments of that case. Compared to the analysis performed by Grandmont [2], which is based on a decoupled finite element approximation and a semi-implicit time-stepping strategy, our analysis and corresponding numerical tests are based on a monolithic scheme. The monolithic approach has been used widely, particularly for blood flow problems, where a stability issue caused by the added-mass effect exists in many partitioned algorithms [9, 21, 26, 27]. To our best knowledge, this is the first report that presents error estimation of an FSI problem in the monolithic framework. The fluid is quasi-Newtonian, where the fluid viscosity is a function of the magnitude of the deformation tensor, and the fluid does not have any memory or elastic properties. Examples of such fluids include blood, lubricants, and paints. Numerical studies on FSI involving this type of fluids are found in [7, 29, 36]. The fluid equation is given in an Eulerian framework; thus, the fluid domain is time-dependent due to the movement of the interface. The isotropic linear elastic structure, on the other hand, is described in a Lagrangian frame of reference, giving the structure a fixed domain. With interface conditions and appropriately chosen function spaces, we obtain the monolithic global formulation.
for the FSI problem that accounts for the fluid and the structure at the same time.

## 3.1 Model Description

![Figure 3.1: Fluid-structure interaction domain](image)

Let $\Omega_f^t$ be the moving fluid domain at $t$ in $\mathbb{R}^2$ with the boundary $\Gamma_f^t := \Gamma_{D,0}^f \cup \Gamma_D^f \cup \Gamma_I$, where $\Gamma_I$ is the moving boundary (interface). Let $\Omega^s$ be a fixed domain for the structure which is described in term of Lagrangian frame of reference. The boundary of structure is denoted as $\Gamma_s := \Gamma_N^s \cup \Gamma_D^s \cup \Gamma_I$. We considered the system with a Quasi-Newtonian flow and an isotropic linear elastic structure.

\[ \rho_f \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \nu_f(|D(\mathbf{u})|D(\mathbf{u}) + \nabla p) = \mathbf{f}_f \quad \text{in } \Omega_f^t, \tag{3.1.1} \]

\[ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_f^t, \tag{3.1.2} \]

\[ \rho_s \frac{\partial^2 \eta}{\partial t^2} - 2\nu_s \nabla \cdot D(\eta) - \bar{\lambda} \nabla (\nabla \cdot \eta) = \mathbf{f}_s \quad \text{in } \Omega^s, \tag{3.1.3} \]

where $\mathbf{u}$ denotes the velocity vector, $p$ the pressure of fluid, $\eta$ the displacement of structure, $\rho_f$ and $\rho_s$ are the densities of the fluid and the structure, respectively. In (3.1.1) and (3.1.3), $D(\mathbf{u}) := (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ is the rate of the strain tensor, and $\mathbf{f}_f$ and $\mathbf{f}_s$ are the body forces. $\nu$ and $\bar{\lambda}$ are the Lamé parameters defined as:

\[ \nu_s = \frac{E}{2(1+r)}, \quad \bar{\lambda} = \frac{rE}{(1-2r)(1+r)}, \tag{3.1.4} \]

where $E$ is the Young’s Modulus of the structure and $r$ is its Poisson ratio.
\[
\begin{align*}
\mathbf{u}(x, 0) &= \mathbf{u}_0 \quad \text{in } \Omega_f^t, \\
\eta(x, 0) &= \eta_0, \\
\mathbf{u} &= \mathbf{u}_{D} \quad \text{on } \Gamma_{D}^f, \\
\mathbf{u} &= 0 \quad \text{on } \Gamma_{D,0}^f, \\
2\nu_s D(\eta)\mathbf{n}_s + \bar{\lambda}(\nabla \cdot \eta)\mathbf{n}_s &= 0 \quad \text{on } \Gamma_N^s, \\
\eta &= 0 \quad \text{on } \Gamma_D^s,
\end{align*}
\]

where \(\mathbf{n}_f\) and \(\mathbf{n}_s\) are the outward unit normal vectors to \(\Omega_f^t\) and \(\Omega^s\), respectively. The moving interface \(\Gamma_I\) is determined by the displacement \(\eta\) at time \(t\) (Figure 3.1). To simplify numerical analysis, we use \(\mathbf{u}_D = 0\) on \(\Gamma_D^f\), but all our results hold for the case of \(\mathbf{u}_D \neq 0\) by the standard technique [22].

Based on the continuity of the velocity and the stress force, the matching conditions for the interface between the fluid and the structure domains are:

\[
\begin{align*}
\frac{\partial \eta}{\partial t} &= \mathbf{u} \quad \text{on } \Gamma_I, \\
\nu_f(|D(\mathbf{u})|)D(\mathbf{u}) - p\mathbf{n}_f &= -(2\nu_s D(\eta) + \bar{\lambda}(\nabla \cdot \eta))\mathbf{n}_s \quad \text{on } \Gamma_I.
\end{align*}
\]

For the nonlinear function \(\nu(|D(\mathbf{u})|)D(\mathbf{u})\), we make the following assumptions:

\[
\begin{align*}
(\nu(|\sigma|)|\sigma| - \nu(|\tau|)|\tau|) : (\sigma - \tau) &\geq K_1|\sigma - \tau|^2, \quad \forall \sigma, \tau \in \mathbb{R}^{2 \times 2}, \\
|\nu(|\sigma|)|\sigma| - \nu(|\tau|)|\tau| \leq K_2(|\sigma| + |\tau|)^{r-2}|\sigma - \tau|, \quad \forall \sigma, \tau \in \mathbb{R}^{2 \times 2}.
\end{align*}
\]

These properties imply that \(\nu(|\cdot|)\) is strongly monotone and Lipschitz continuous for bounded arguments [24]. In addition, the models also satisfy

\[
\|(\nu(|\sigma|)|\sigma| - \nu(|\tau|)|\tau|)\| \leq K_3\|\sigma - \tau\|_{0, \Omega_f^t}, \quad \forall \sigma, \tau \in L^r(\Omega_f^t),
\]

where \(1 < r \leq 2\). We consider the shear-thinning case \((1 < r < 2)\) for which the velocity is assumed to
be a $H^1$ function. The definition of inner product and norm notation are the same as the viscoelastic case presented in previous chapter.

### 3.2 The ALE Formulation

In most fluid-structure interaction problems, fluid equations and structure equations are posed from different perspectives in continuum mechanics: the Eulerian frame of reference is used for the fluid equations, and the Lagrangian frame of reference for elastic structures. The ALE [14] method allows the coupled problem to be posed in one framework, and therefore is widely used for simulating fluid flows in a moving domain.

With introduction of a family of time-dependent mappings from a fixed reference domain to a physical moving domain as presented in Chapter 2, the fluid equations can be rewritten in ALE formulation with respect to the reference domain. To compute the ALE mapping for our problem we solve the Laplace equation

\[
\begin{cases}
\Delta \psi_t(y) = 0 & \text{in } \Omega_f^t, \\
\psi_t(y) = h_t(y) & \text{on } \partial \Omega_f^0,
\end{cases}
\]

with the boundary position function $h_t : \partial \Omega_f^0 \to \partial \Omega_f^t$ defined by

\[
h_t(y) = \begin{cases} 
y + \eta & \text{on } \Gamma_t, \\
y & \text{on } \Gamma_D^f \cup \Gamma_D^f,0,
\end{cases}
\]  

where $\eta$ is the displacement of the moving interface. With the same notations as used in Chapter 2, we make the following assumptions for analysis throughout the rest of the chapter:

- $\Omega_f^t = \psi_t(\Omega_f^0)$ is a Lipschitz domain, 
  (3.2.2)
- $\psi_t \in W^{1,\infty}(\Omega_f^0)$ and $\psi_t^{-1} \in W^{1,\infty}(\Omega_f^0) \forall t \in [0, T]$, 
  (3.2.3)
- $z, \frac{\partial z}{\partial t} \in W^{1,\infty}(\Omega_f^t) \forall t \in [0, T]$. 
  (3.2.4)

These assumptions are reasonable for the movement and shape of the moving domain [19, 20].
We obtain the ALE formulation for the quasi-Newtonian flow equations (3.1.1)-(3.1.2) as

\[ \rho_f \left( \frac{\partial u}{\partial t} |_y + (u - z) \cdot \nabla_x u \right) - \nabla_x \cdot \nu_f(|D_x(u)|) D_x(u) + \nabla_x p = f_f \quad \text{in } \Omega_f^f, \quad (3.2.5) \]
\[ \nabla_x \cdot u = 0 \quad \text{in } \Omega_f^f. \quad (3.2.6) \]

For the variational formulation of the flow equations (3.1.1)-(3.1.2) in ALE framework, define function spaces for the reference domain:

\[ U_0 := \{ v \in H^1(\Omega_0^f) : v = 0 \text{ on } \Gamma_D^f \cup \Gamma_D^0 \}, \]
\[ Q_0 := L^2(\Omega_0^f). \]

The function spaces for physical domain \( \Omega_f^f \) is then defined as

\[ U_f := \{ v : \Omega_f^f \times [0, T] \to \mathbb{R}^d, \quad v = v \circ \Psi_t^{-1} \text{ for } v \in U_0 \}, \]
\[ Q_f := \{ q : \Omega_f^f \times [0, T] \to \mathbb{R}, \quad q = q \circ \Psi_t^{-1} \text{ for } q \in Q_0 \}. \]

For the structure equation, the function space is defined as

\[ S := \{ \xi \in H^1(\Omega^s) : \xi = 0 \text{ on } \Gamma_D^s \}. \]

The variational formulation of (3.2.5)-(3.2.6) and (3.1.3) in the ALE framework can then be written as

\[ \rho_f \left( \frac{\partial u}{\partial t} |_y + (u - z) \cdot \nabla_x u \right) \Omega_f^f + (\nu_f(|D(u)|) D(u), D(v))_{\Omega_f^f} - (p, \nabla \cdot v)_{\Omega_f^f} \]
\[ = (f_f, v)_{\Omega_f^f} + ((\nu_f(|D(u)|) D(u) - p) \cdot n, v)_{\Gamma_f^I}, \quad \forall v \in U_f, \quad (3.2.7) \]
\[ (q, \nabla \cdot u)_{\Omega_f^f} = 0 \quad \forall q \in Q_f, \quad (3.2.8) \]
\[ \rho_s \left( \frac{\partial^2 \eta}{\partial t^2} \right) \Omega_s + 2 \nu_s(D(\eta), D(\xi))_{\Omega_s} + \lambda(\nabla \cdot \eta_0, \nabla \cdot \xi_0)_{\Omega_s} \]
\[ = (f_s, \xi_0)_{\Omega_s} + ((2 \nu_s D(\eta) + \lambda(\nabla \cdot \eta_0)) n_s, \xi_0)_{\Gamma_f^I} \quad \forall \xi \in S. \quad (3.2.9) \]
Define the function space for the coupled problem as

\[ \tilde{U}_t \times \tilde{S} := \{(v, \xi) \in U_t \times S : v \big|_{\Gamma}\ = \ \left( \frac{\partial \xi}{\partial t} \circ \Psi_t^{-1} \right) \big|_{\Gamma}\}, \quad (3.2.10) \]

where the interface condition (3.1.11) is satisfied. Using (3.1.12), the boundary integral term in the right side of (3.2.9) can be substituted with \(- \left( (\nu_f(|D(u)|D(u) - p) \cdot n_f, \xi \circ \Psi_t^{-1} \right)_{\Gamma_t}. \) Hence, combining (3.2.7)-(3.2.8) with ((3.2.9), we obtain a monolithic formulation of the FSI system in the ALE framework: find \((u, p, \eta) \in \tilde{U}_t \times Q_t \times \tilde{S}\) such that

\[
\rho_s \frac{\partial^2 \eta}{\partial t^2}, \xi)_{\Omega_t} + 2\nu_s (D(\eta), D(\xi))_{\Omega_t} + \lambda (\nabla \cdot \eta, \nabla \cdot \xi)_{\Omega_t} + \rho_f \left( \left( \frac{\partial u}{\partial t} \big|_\gamma, v \right)_{\Omega_t} + ((u - z) \cdot \nabla u, v)_{\Omega_t} \right) + (\nu_f(|D(u)|D(u), D(v)) \right)_{\Omega_t} - (p, \nabla \cdot v)_{\Omega_t} + (q, \nabla \cdot u)_{\Omega_t}
\]

\[ = (f_f, v)_{\Omega_t} + (f_s, \xi)_{\Omega_t} \quad \forall (v, q, \xi) \in \tilde{U}_t \times Q_t \times \tilde{S}. \quad (3.2.11)\]

By Green’s theorem, we have

\[
(u \cdot \nabla u, v)_{\Omega_t} = -(u \cdot \nabla v, u)_{\Omega_t} - ((\nabla \cdot u)v, u)_{\Omega_t} + ((u \cdot n_f)v, u)_{\Gamma_t}, \quad (3.2.12)
\]

\[
(z \cdot \nabla u, v)_{\Omega_t} = -(z \cdot \nabla v, u)_{\Omega_t} - ((\nabla \cdot z)v, u)_{\Omega_t} + ((z \cdot n_f)v, u)_{\Gamma_t}, \quad (3.2.13)
\]

and, using \(\nabla \cdot u = 0\) and (3.2.12),

\[
(u \cdot \nabla u, v)_{\Omega_t} = \frac{1}{2}(u \cdot \nabla u, v)_{\Omega_t} + \frac{1}{2} \left[ -(u \cdot \nabla v, u)_{\Omega_t} - ((\nabla \cdot u)v, u)_{\Omega_t} \right]
\]

\[+ \frac{1}{2}((u \cdot n_f)v, u)_{\Gamma_t}
\]

\[= \theta(u, u, v)_{\Omega_t} + \frac{1}{2}((u \cdot n_f)v, u)_{\Gamma_t}. \quad (3.2.14)\]

Similarly,

\[
(z \cdot \nabla u, v)_{\Omega_t} = \theta(z, u, v)_{\Omega_t} - \frac{1}{2}((\nabla \cdot z)v, u)_{\Omega_t} + \frac{1}{2}((z \cdot n_f)v, u)_{\Gamma_t}. \quad (3.2.15)\]

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The interface condition (3.1.11) states that \( u = \frac{\partial n}{\partial t} = z \) on the interface, which implies
\[
\frac{1}{2}((u \cdot n_f)v, u)_{\Gamma_{I^d}} = \frac{1}{2}((z \cdot n_f)v, u)_{\Gamma_{I^d}}. \tag{3.2.16}
\]

Using (3.2.14)-(3.2.16), (3.2.11) can be rewritten as
\[
\rho_s(\frac{\partial^2 \eta}{\partial t^2}, \xi)_{\Omega^s} + 2\nu_s(D(\eta), D(\xi))_{\Omega^s} + \lambda(\nabla \cdot \eta, \nabla \cdot \xi)_{\Omega^s} \\
+ \rho_f \left( \frac{\partial u}{\partial t} \vert_y, v \right)_{\Omega^f_t} + \frac{1}{2}(u(\nabla \cdot z), v)_{\Omega^f_t} + \theta(u, u, v)_{\Omega^f_t} - \theta(z, u, v)_{\Omega^f_t} \\
+ (\nu_f(\|D(u)\|)D(u), D(v))_{\Omega^f_t} - (p, \nabla \cdot v)_{\Omega^f_t} + (q, \nabla \cdot u)_{\Omega^f_t} \\
= (f_f, v)_{\Omega^f_t} + (f_s, \xi)_{\Omega^s} \quad \forall (v, q, \xi) \in \bar{U}_t \times Q_t \times \bar{S}. \tag{3.2.17}
\]

### 3.3 Finite Element Discretization

Define finite element spaces for the approximation of \((u, p)\) in \(\Omega^f_0\) as
\[
U_{h,0} := \{ v \in U_0 \cap (C^0(\Omega^f_0))^2 : v\vert_K \in P_2(K)^2, \forall K \in T_{h,0} \}, \\
Q_{h,0} := \{ q \in Q_0 \cap C^0(\Omega^f_0) : q\vert_K \in P_1(K), \forall K \in T_{h,0} \},
\]
where \(T_{h,0}\) is a triangulation satisfying the quasi-uniform mesh condition. It is well known that the Taylor-Hood pair \((P_2, P_1)\) satisfies the LBB condition
\[
\inf_{0 \neq q_h \in Q_{h,0}} \sup_{0 \neq v_h \in U_{h,0}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_0} \geq C, \tag{3.3.1}
\]
where \(C\) is a positive constant independent of \(h\). The finite element spaces for \((u_h, p_h)\) in \(\Omega^f_t\) are then defined as
\[
U_{h,t} := \{ v_h : \Omega^f_t \times [0, T] \to \mathbb{R}^2, v_h = \overline{v}_h \circ \Psi_{h,t}^{-1} \text{ for } \overline{v}_h \in U_{h,0} \}, \\
Q_{h,t} := \{ q_h : \Omega^f_t \times [0, T] \to \mathbb{R}, q_h = \overline{q}_h \circ \Psi_{h,t}^{-1} \text{ for } \overline{q}_h \in Q_{h,0} \},
\]

40
where $\Psi_{h,t} : \Omega_0 \rightarrow \Omega_t$ is a discrete mapping approximated by $P_1$ Lagrangian finite elements such that $\Psi_{h,t}(y) = x_h(y, t)$. For the discrete ALE mapping, define the space

$$X_h := \{ x \in H^1(\Omega_0) : x|_K \in P_1(K)^2, \forall K \in T_{h,0}\}. \quad (3.3.2)$$

The corresponding discrete domain velocity is then defined as $z_h = \frac{\partial x_h}{\partial t}$ with the assumption that

$$\max\{\|z_h\|_{1,\infty,\Omega^f}, \|\frac{\partial}{\partial t} z_h\|_{1,\infty,\Omega}\} \leq M \quad (3.3.3)$$

based on the regularity of ALE mapping (3.2.4).

The finite element space for $\eta_h$ is defined as

$$S_h := \{ \xi_h \in S \cap (C^0(\overline{\Omega}))^2 : \xi_h|_K \in P_2(K)^2, \forall K \in \overline{T}_h\},$$

where $\overline{T}_h$ is a triangulation in the structure domain. Then with the discrete coupled function spaces

$$\tilde{U}_{h,t} \times \tilde{S}_h := \{(v_h, \xi_h) \in U_{h,t} \times S_h : v_h|_{\Gamma_{it}} = \left(\frac{\partial \xi_h}{\partial t} \circ \psi_{h,t}^{-1}\right)|_{\Gamma_{it}}\}, \quad (3.3.4)$$

the semi-discrete variational formulation of (3.2.17) is written as

$$\rho_s \left(\frac{\partial^2 \eta_h}{\partial t^2}, \xi_h\right)_{\Omega^t} + 2\nu_s (D(\eta_h), D(\xi_h))_{\Omega^t} + \lambda(\nabla \cdot \eta_h, \nabla \cdot \xi_h)_{\Omega^t}$$

$$+ \rho \left[\left(\frac{\partial u_h}{\partial t}\big|_{y}, v_h\right)_{\Omega^t} + \frac{1}{2}(u_h \cdot z_h, v_h)_{\Omega^t} + \theta(u_h, u_h, v_h)_{\Omega^t} - \theta(z_h, u_h, v_h)_{\Omega^t}\right]$$

$$+ (\nu_f(\|D(u_h)\|D(u_h), D(v_h))_{\Omega^f} - (p_h, \nabla \cdot v_h)_{\Omega^t} + (q_h, \nabla \cdot u_h)_{\Omega^t}$$

$$= (f_f, v_h)_{\Omega^t} + (f_s, \xi_h)_{\Omega^t} \quad \forall (v_h, q_h, \xi_h) \in \tilde{U}_{h,t} \times Q_{h,t} \times \tilde{S}_h. \quad (3.3.5)$$

We define a discrete divergence free space as

$$\tilde{V}^{f}_{h,t} := \{v_h \in \tilde{U}^{f}_{h,t} : (\nabla \cdot v_h, q_h)_{\Omega^t} = 0 \quad \forall q_h \in Q_{h,t}\}$$

which will be used in the proof of Theorem 3.3.2.
Theorem 3.3.1 A solution to the semi-discrete problem (3.3.5) satisfies the estimate

\[
\begin{aligned}
\frac{\rho_s}{2} \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0, \Omega'}^2 &+ \nu_s \left\| D(\eta_h) \right\|_{0, \Omega'}^2 + \frac{\lambda^2}{2} \left\| \nabla \cdot \eta_h \right\|_{0, \Omega'}^2 &+ \frac{\rho_f}{2} \left\| u_h \right\|_{0, \Omega'}^2 + \frac{K_1}{2} \int_0^t \left\| D(u_h) \right\|_{0, \Omega'}^2 \, dt \\
\leq \frac{\rho_s}{2} \left\| \eta_0 \right\|_{0, \Omega'}^2 &+ \nu_s \left\| D(\eta_0) \right\|_{0, \Omega'}^2 + \frac{\lambda^2}{2} \left\| \nabla \cdot \eta_0 \right\|_{0, \Omega'}^2 &+ \frac{\rho_f}{2} \left\| u_0 \right\|_{0, \Omega'}^2 \\
&+ C \int_0^t \left\| f_f \right\|_{0, \Omega'}^2 + \left\| f_s \right\|_{0, \Omega'}^2 \, dt.
\end{aligned}
\] (3.3.6)

Proof: Set \( v_h = u_h, q_h = p_h, \xi_h = \frac{\partial \eta_h}{\partial t} \) in (3.3.5). Using the Reynolds’ transportation formula [33, 41], we have

\[
\begin{aligned}
\left\langle \frac{\partial u_h}{\partial t} \left| y, u_h \right\rangle_{\Omega'_t} &= \frac{1}{2} \left\langle \frac{\partial^2 u_h}{\partial t^2} \left| y, 1 \right\rangle_{\Omega'_t} \\
&= \frac{1}{2} \left[ \frac{\partial}{\partial t} (u_h^2)_{|_{\Omega'_t}} - (u_h^2 \cdot z_h)_{|_{\Omega'_t}} \right] \\
&= \frac{1}{2} \frac{\partial}{\partial t} \left\| u_h \right\|_{0, \Omega'_t}^2 - \frac{1}{2} \left( u_h \cdot z_h, u_h \right)_{\Omega'_t}.
\end{aligned}
\] (3.3.7)

In the meantime,

\[
\begin{aligned}
\rho_s \left( \frac{\partial^2 \eta_h}{\partial t^2}, \frac{\partial \eta_h}{\partial t} \right)_{\Omega'} &+ 2\nu_s \left( D(\eta_h), D \left( \frac{\partial \eta_h}{\partial t} \right) \right)_{\Omega'} + \lambda \left( \nabla \cdot \eta_h, \nabla \cdot \frac{\partial \eta_h}{\partial t} \right)_{\Omega'} \\
&= \frac{\partial}{\partial t} \left( \frac{\rho_s}{2} \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0, \Omega'}^2 + \nu_s \left\| D(\eta_h) \right\|_{0, \Omega'}^2 + \frac{\lambda^2}{2} \left\| \nabla \cdot \eta_h \right\|_{0, \Omega'}^2 \right),
\end{aligned}
\] (3.3.8)

while

\[
\theta(u_h, u_h, u_h)_{\Omega'_t} - \theta(z_h, u_h, u_h)_{\Omega'_t} = 0.
\] (3.3.9)

By Poincaré inequality, the right hand side of (3.3.5) can be bounded by

\[
\begin{aligned}
\left\langle f_f, u_h \right\rangle_{\Omega'_t} + \left\langle f_s, \frac{\partial \eta_h}{\partial t} \right\rangle_{\Omega'} \\
\leq C'(1) \left\| f_f \right\|_{0, \Omega'_t}^2 + \epsilon_1 \left\| D(u_h) \right\|_{0, \Omega'_t}^2 + C'(\epsilon_2) \left\| f_s \right\|_{0, \Omega'}^2 + \epsilon_2 \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0, \Omega'}^2.
\end{aligned}
\] (3.3.10)
Applying (3.1.13), (3.3.7) - (3.3.10) in (3.3.5),

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{\rho_s}{2} \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0,\Omega}^2 + \nu_s \left\| \mathbf{D}(\eta_h) \right\|_{0,\Omega}^2 + \frac{\lambda}{2} \left\| \nabla \cdot \eta_h \right\|_{0,\Omega}^2 + \frac{\rho_f}{2} \left\| \mathbf{u}_h \right\|_{0,\Omega}^2 \right) \\
+ K_1 \left\| \mathbf{D}(\mathbf{u}_h) \right\|_{0,\Omega_t}^2 \\
\leq C(\epsilon_1) \left\| f_f \right\|_{0,\Omega_t}^2 + \epsilon_1 \left\| \mathbf{D}(\mathbf{u}_h) \right\|_{0,\Omega_t}^2 + C(\epsilon_2) \left\| f_s \right\|_{0,\Omega_t}^2 + \epsilon_2 \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0,\Omega_t}^2. 
\end{align*}
\]  

(3.3.11)

Setting \( \epsilon_1 = \frac{K_1}{2} \), \( \epsilon_2 = \frac{\rho_s}{2} \), (3.3.11) can be reduced to:

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{\rho_s}{2} \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0,\Omega}^2 + \nu_s \left\| \mathbf{D}(\eta_h) \right\|_{0,\Omega}^2 + \frac{\lambda}{2} \left\| \nabla \cdot \eta_h \right\|_{0,\Omega}^2 + \frac{\rho_f}{2} \left\| \mathbf{u}_h \right\|_{0,\Omega}^2 \right) \\
+ \frac{K_1}{2} \left\| \mathbf{D}(\mathbf{u}_h) \right\|_{0,\Omega_t}^2 \\
\leq \frac{\rho_s}{2} \left\| \frac{\partial \eta_h}{\partial t} \right\|_{0,\Omega}^2 + C(\epsilon_1) \left\| f_f \right\|_{0,\Omega_t}^2 + C(\epsilon_2) \left\| f_s \right\|_{0,\Omega_t}^2. 
\end{align*}
\]  

(3.3.12)

The estimate (3.3.6) can then be obtained by Gronwall’s Lemma. \( \square \)

We will present an a priori error estimate of the finite element solution in the following theorem.

**Theorem 3.3.2** Suppose \((\mathbf{u}, p, \eta)\) is a solution of (3.1.1)-(3.1.10) and

\[
\mathbf{u} \in L^4 \left( 0, T; \mathbf{H}^1(\Omega_f^t) \right).
\]
A solution to the semi-discrete problem (3.3.5) satisfies the error estimate

\[
\rho_s \left\| \frac{\partial}{\partial t} (\eta - \eta_h) \right\|_{0, \Omega_t^f}^2 + \nu_s \| D(\eta - \eta_h) \|_{0, \Omega_t^f}^2 + \frac{\lambda}{2} \| \nabla \cdot (\eta - \eta_h) \|_{0, \Omega_t^f}^2 \\
+ \rho_f \| u - u_h \|_{0, \Omega_t^f}^2 + K_1 \int_0^t \| D(u - u_h) \|_{0, \Omega_t^f}^2 \, dt
\leq C \left\{ \| u_0 - u_h(0) \|_{0, \Omega_t^f}^2 + \| D(\eta_0 - \eta_h(0)) \|_{0, \Omega_t^f}^2 + \| D(\tilde{\eta}_0 - \tilde{\eta}_h(0)) \|_{0, \Omega_t^f}^2 \right.
\left. + \inf_{u_h, \tilde{u}_h, \tilde{\eta}_h \in \tilde{V}_{h,t} \times Q_{h,t} \times \tilde{S}_h} \left[ \| u - \tilde{u}_h \|_{L^2(0,T; H^1)}^2 + \max_{0 \leq t \leq T} \| D(u - \tilde{u}_h) \|_{0, \Omega_t^f}^2 \\
+ \| D(\eta - \tilde{\eta}_h) \|_{0, \Omega_t^f}^2 + \int_0^t \| \frac{\partial^2}{\partial t^2} (\eta - \tilde{\eta}_h) \|_{0, \Omega_t}^2 + \| \frac{\partial}{\partial t} D(\eta - \tilde{\eta}_h) \|_{0, \Omega_t}^2 \right.ight.
\left. + \| \frac{\partial}{\partial t} (u - \tilde{u}_h) \|_{0, \Omega_t}^2 + \| \phi - \tilde{\phi} \|_{0, \Omega_t^f}^2 \, dt \right] \right\}. \tag{3.3.13}
\]

**Proof:** Let \( \tilde{u}_h, \tilde{p}_h, \tilde{\eta}_h \) be arbitrary functions in \( \tilde{V}_{h,t}, Q_{h,t}, \tilde{S}_h \), respectively. Then we have the relations:

\[
\eta - \eta_h = \phi - \psi \quad \text{where} \quad \phi = \tilde{\eta}_h - \eta_h, \psi = \tilde{\eta}_h - \eta.
\]

\[
u = u - u_h = I - g \quad \text{where} \quad I = \tilde{u}_h - u_h, g = \tilde{u}_h - u.
\]

To simplify the analysis, we slightly modify (3.2.17) by taking the discrete domain velocity (i.e., replacing \( z \) by \( z_h \)). The error estimate for \( \| z - z_h \|_{1, \Omega_t^f} \) is shown in [20]. Assuming \( z \) is uniformly bounded and using (3.3.3), it is easily seen that the same error estimate result will hold if (3.2.17) is used without the replacement. Subtracting the semi-discretized weak formulation (3.3.5) from the continuous weak formulation (3.2.17) gives

\[
\rho_s \left( \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \psi}{\partial t^2}, \xi_h \right)_{\Omega_t^f} + 2 \nu_s \left( D(\phi) - D(\psi), D(\xi_h) \right)_{\Omega_t^f} + \lambda \left( \nabla \cdot \phi - \nabla \cdot \psi, \nabla \cdot \xi_h \right)_{\Omega_t^f} \\
+ \rho_f \left( \frac{\partial}{\partial t} (I - g), v_h \right)_{\Omega_t^f} + \rho_f \left( \theta(u, u, v_h)_{\Omega_t^f} - \theta(u_h, u_h, v_h)_{\Omega_t^f} \right) \\
+ \rho_f \left( \frac{1}{2} \left[ (I - g) \nabla \cdot z_h, v_h \right]_{\Omega_t^f} - \theta(z_h, I - g, v_h)_{\Omega_t^f} \right) \\
+ \nu_f \left( \| D(u) \|_{0, \Omega_t^f} \| D(v_h) \|_{0, \Omega_t^f} - \| D(u_h) \|_{0, \Omega_t^f} \right) \\
- \left( p, \nabla \cdot v_h \right)_{\Omega_t^f} + \left( p_h, \nabla \cdot v_h \right)_{\Omega_t^f} - \left( q_h, \nabla \cdot u_h \right)_{\Omega_t^f} + \left( q_h, \nabla \cdot u \right)_{\Omega_t^f} = 0. \tag{3.3.14}
\]
Note that

\[
\theta(u, u, v_h)_{\Omega^t} - \theta(u_h, u_h, v_h)_{\Omega^t} = \theta(u, u, v_h)_{\Omega^t} - \theta(u_h, u, v_h)_{\Omega^t} + \theta(u_h, u, v_h)_{\Omega^t} - \theta(u_h, u_h, v_h)_{\Omega^t} \\
= \theta(I - g, u, v_h)_{\Omega^t} + \theta(u_h, I - g, v_h)_{\Omega^t} \\
= \theta(I, u, v_h)_{\Omega^t} - \theta(g, u, v_h)_{\Omega^t} + \theta(u_h, I, v_h)_{\Omega^t} - \theta(u_h, g, v_h)_{\Omega^t}.
\]

(3.3.15)

Using (3.3.15) and the fact that \( \nabla \cdot u = 0 \) for the strong solution, moving some terms to the right hand side of the equation and adding \( (\nu f)(D(\tilde{u}_h), D(v_h))_{\Omega^t} - (\tilde{p}_h, \nabla \cdot v_h)_{\Omega^t} \) to both sides, we obtain

\[
\rho_s \left( \frac{\partial^2 \phi}{\partial t^2}, \xi_h \right)_{\Omega^t} + 2\nu_s (D(\phi), D(\xi_h))_{\Omega^t} + \bar{\lambda}(\nabla \cdot \phi, \nabla \cdot \xi_h)_{\Omega^t} \\
+ \rho_f \left( \frac{\partial I}{\partial t}, |y, v_h| \right)_{\Omega^t} + \rho_f \theta(u_h, I, v_h)_{\Omega^t} + \rho_f \left[ \frac{1}{2}(I\nabla \cdot z_h, v_h)_{\Omega^t} - \theta(z_h, I, v_h)_{\Omega^t} \right] \\
+ (\nu f(|D(\tilde{u}_h)|)D(\tilde{u}_h), D(v_h))_{\Omega^t} - (\nu f(|D(u_h)|)D(u_h), D(v_h))_{\Omega^t} \\
= - (\tilde{p}_h - p_h, \nabla \cdot v_h)_{\Omega^t}.
\]

(3.3.16)

In (3.3.16) set \( \xi_h = \frac{\partial \phi}{\partial t} \), then the structure terms on the left hand side satisfies

\[
\rho_s \left( \frac{\partial^2 \phi}{\partial t^2}, \frac{\partial \phi}{\partial t} \right)_{\Omega^t} + 2\nu_s (D(\phi), D(\frac{\partial \phi}{\partial t}))_{\Omega^t} + \bar{\lambda} \left( \nabla \cdot \phi, \nabla \cdot \frac{\partial \phi}{\partial t} \right)_{\Omega^t} \\
= \frac{\partial}{\partial t} \left[ \rho_s \left\| \frac{\partial \phi}{\partial t} \right\|^2_{0, \Omega^t} + \nu_s \left\| D(\phi) \right\|^2_{0, \Omega^t} + \frac{\bar{\lambda}}{2} \left\| \nabla \cdot \phi \right\|^2_{0, \Omega^t} \right].
\]

(3.3.17)

By Cauchy-Schwartz and Young’s inequalities, the structure part on the right hand side of (3.3.16)
is bounded as

$$
\rho_s \left( \frac{\partial^2 \psi}{\partial t^2}, \frac{\partial \phi}{\partial t} \right)_{\Omega^s} + 2\nu_s \left( D(\psi), D(\frac{\partial \phi}{\partial t}) \right)_{\Omega^s} + \bar{\lambda} \left( \nabla \cdot \psi, \nabla \cdot \frac{\partial \phi}{\partial t} \right)_{\Omega^s} 
\leq \frac{\rho_s}{2} \left( \left\| \frac{\partial^2 \psi}{\partial t^2} \right\|_{(\Omega^s)^2}^2 + \left\| \frac{\partial \phi}{\partial t} \right\|_{(\Omega^s)^2}^2 \right) + 2\nu_s \left( D(\psi), D(\frac{\partial \phi}{\partial t}) \right)_{\Omega^s} + \bar{\lambda} \left( \nabla \cdot \psi, \nabla \cdot \frac{\partial \phi}{\partial t} \right)_{\Omega^s} 
$$

(3.3.18)

Setting $v_h = I$ in (3.3.16),

$$
\left( \frac{\partial I}{\partial t}, \mathbf{y} \cdot I \right)_{\Omega^f} = \frac{1}{2} \frac{\partial}{\partial t} \left\| I \right\|_{0, \Omega^f}^2 - \frac{1}{2} \left( \mathbf{I} \nabla \cdot \mathbf{z}_h, I \right)_{\Omega^f} 
$$

(3.3.19)

as shown in (3.3.7). Using (3.1.13), (3.3.19), Poincare inequality and the fact that $(\hat{p}_h - p_h, \nabla \cdot I) = 0$, $	heta(u_h, I)_{\Omega^f} = \theta(z_h, I)_{\Omega^f} = 0$, we obtain a lower bound for the fluid terms on the left hand side of (3.3.16),

$$
\rho_f \left( \frac{\partial I}{\partial t}, \mathbf{y} \cdot I \right)_{\Omega^f} + \rho_f \theta(u_h, I, I)_{\Omega^f} + \rho_f \left[ \frac{1}{2} \left( \mathbf{I} \nabla \cdot \mathbf{z}_h, I \right)_{\Omega^f} - \theta(z_h, I, I)_{\Omega^f} \right] + (\nu_f([D(\tilde{u}_h)]))D(\tilde{u}_h), D(I))_{\Omega^f} - (\nu_f([D(u_h)]))D(u_h), D(I))_{\Omega^f} - (\hat{p}_h - p_h, \nabla \cdot I)_{\Omega^f} 
\geq \frac{\rho_f}{2} \frac{\partial}{\partial t} \left\| I \right\|_{0, \Omega^f}^2 + K_1 \left\| D(I) \right\|_{0, \Omega^f}^2. 
$$

(3.3.20)

Poincaré inequality and Young’s inequality imply

$$
\rho_f \left( \frac{\partial g}{\partial t}, \mathbf{y} \cdot I \right)_{\Omega^f} - (\hat{p}_h - p, \nabla \cdot I)_{\Omega^f} 
\leq C(\epsilon_1) \left( \left\| \frac{\partial g}{\partial t}, \mathbf{y} \right\|_{0, \Omega^f}^2 + \left\| p - \hat{p}_h \right\|_{0, \Omega^f}^2 \right) + \epsilon_1 \left\| D(I) \right\|_{0, \Omega^f}^2. 
$$

(3.3.21)

The estimate (2.3.6), Young’s inequality and the stability result (3.3.6) imply

$$
\rho_f \theta(u_h, g, I)_{\Omega^f} \leq C \left\| u_h \right\|_{0, \Omega^f}^{1/2} \left\| D(u_h) \right\|_{0, \Omega^f}^{1/2} \left\| D(g) \right\|_{0, \Omega^f} \left\| D(I) \right\|_{0, \Omega^f} 
\leq C(\epsilon_1) \left\| D(u_h) \right\|_{0, \Omega^f} \left\| D(g) \right\|_{0, \Omega^f}^2 + \epsilon_1 \left\| D(I) \right\|_{0, \Omega^f}^2. 
$$

(3.3.22)
The inequality (2.3.5) and Young’s inequality imply that

\[
\rho_f \theta(g, u, I)_{\Omega'_t} \leq C \|D(u)\|_{0, \Omega'_t} \|D(g)\|_{0, \Omega'_t} \|D(I)\|_{0, \Omega'_t} \\
\leq C(\epsilon_1) \|D(u)\|_{0, \Omega'_t}^2 \|D(g)\|_{0, \Omega'_t}^2 + \epsilon_1 \|D(I)\|_{0, \Omega'_t}^2. 
\] (3.3.23)

Also, (2.3.6) and the inequality \(ab \leq c a^\frac{2}{3} + C(\epsilon)b^4\) imply

\[
-\rho_f \theta(I, u, I)_{\Omega'_t} \leq C |I|^\frac{1}{2} \|D(I)\|_{0, \Omega'_t}^2 \|D(u)\|_{0, \Omega'_t}^2 \\
\leq C(\epsilon_1) |I|_{0, \Omega'_t}^2 \|D(u)\|_{0, \Omega'_t}^4 + \epsilon_1 \|D(I)\|_{0, \Omega'_t}^2. 
\] (3.3.24)

By (2.3.5) and (3.3.3) we have

\[
\frac{1}{2} (g \nabla \cdot z_h, I)_{\Omega'_t} - \theta(z_h, g, I)_{\Omega'_t} \leq C \|z_h\|_{1, \infty, \Omega_t} \|D(I)\|_{0, \Omega_t} \|D(g)\|_{0, \Omega_t} \\
\leq C(\epsilon_1) \|D(g)\|_{0, \Omega'_t}^2 + \epsilon_1 \|D(I)\|_{0, \Omega'_t}^2. 
\] (3.3.25)

The bound of the viscosity term is determined by (3.1.15) as

\[
(\nu_f(D(\tilde{u}_h)), D(I))_{\Omega'_t} - (\nu_f(D(u)), D(I))_{\Omega'_t} \\
\leq K_3 \|D(g)\|_{0, \Omega'_t} \|D(I)\|_{0, \Omega'_t} \leq C(\epsilon_1) \|D(g)\|_{0, \Omega'_t}^2 + \epsilon_1 \|D(I)\|_{0, \Omega'_t}^2. 
\] (3.3.26)

Using the estimates (3.3.21) - (3.3.26), a bound of the fluid part in the right hand side of (3.3.16) can be obtained as

\[
\rho_f \left( \frac{\partial g}{\partial t} |_{\Omega'_t} \right) + \rho_f \left[ \theta(u_h, g, I)_{\Omega'_t} + \theta(g, u, I)_{\Omega'_t} - \theta(I, u, I)_{\Omega'_t} \right] \\
+ \rho_f \left[ \frac{1}{2} (g \nabla \cdot z_h, I)_{\Omega'_t} - \theta(z_h, g, I)_{\Omega'_t} \right] \\
+ (\nu_f(D(\tilde{u}_h)), D(I))_{\Omega'_t} - (\nu_f(D(u)), D(I))_{\Omega'_t} - (\tilde{p}_h - p, \nabla \cdot I)_{\Omega'_t} \\
\leq C(\epsilon_1) \left[ \|D(u)_h\|_{0, \Omega'_t}^2 + \|D(u)\|_{0, \Omega'_t}^2 \|D(g)\|_{0, \Omega'_t}^2 + \|D(u)\|_{0, \Omega'_t}^2 \|D(g)\|_{0, \Omega'_t}^2 \\
+ \|D(u)\|_{0, \Omega'_t}^4 + \|D(g)\|_{0, \Omega'_t}^2 + \|\tilde{p}_h - p\|_{0, \Omega'_t}^2 \right] + 6\epsilon_1 \|D(I)\|_{0, \Omega'_t}^2. 
\] (3.3.27)
Setting $\epsilon_1 = \frac{K_4}{4}$, using (3.3.17), (3.3.18), (3.3.20), (3.3.27) and multiplying both sides by 2, (3.3.16) implies

\[
\frac{\partial}{\partial t} \left[ \rho_s \left\| \frac{\partial \phi}{\partial t} \right\|_{0,\Omega_t}^2 + 2\nu_s \|D(\phi)\|^2_{0,\Omega_t^s} + \lambda \|\nabla \cdot \phi\|^2_{0,\Omega_t^s} + \rho_f \|I\|^2_{0,\Omega_t^s} \right] + K_1 \|D(I)\|^2_{0,\Omega_t^s}
\]

\[
\leq \rho_s \left( \frac{\partial^2 \psi}{\partial t^2} \right)_{0,\Omega_t^s} + \rho_s \left( \frac{\partial \phi}{\partial t} \right)_{0,\Omega_t^s}^2 + 4\nu_s \left( D(\psi), D \left( \frac{\partial \phi}{\partial t} \right) \right)_{0,\Omega_t^s} + 2\lambda \left( \nabla \cdot \psi, \nabla \cdot \frac{\partial \phi}{\partial t} \right)_{0,\Omega_t^s}
\]

\[
+ C \left( \left\| \frac{\partial \psi}{\partial t} \right\|_{0,\Omega_t^s}^2 + \|D(u)\|_{0,\Omega_t^s} \|D(g)\|_{0,\Omega_t^s} + \|D(u)\|^2_{0,\Omega_t^s} + \|D(g)\|^2_{0,\Omega_t^s} \right) \cdot \tag{3.3.28}
\]

Assuming $u \in L^4(0,T;H^1(\Omega_t^s))$, and using the same technique shown in [32] (p.157), which is equivalent to the Gronwall's inequality, we have

\[
\rho_s \left( \frac{\partial \phi}{\partial t} \right)_{0,\Omega_t^s}^2 + 2\nu_s \|D(\phi)\|^2_{0,\Omega_t^s} + \lambda \|\nabla \cdot \phi\|^2_{0,\Omega_t^s} + \rho_f \|I\|^2_{0,\Omega_t^s} + K_1 \int_0^t \|D(I)\|^2_{0,\Omega_t^s} \, dt
\]

\[
\leq C \left( \rho_s \left( \frac{\partial \phi(0)}{\partial t} \right)_{0,\Omega_t^s}^2 + 2\nu_s \|D(\phi(0))\|^2_{0,\Omega_t^s} + \lambda \|\nabla \cdot \phi(0)\|^2_{0,\Omega_t^s} + \rho_f \|I(0)\|^2_{0,\Omega_t^s} \right)
\]

\[
+ \frac{\rho_s}{2} \int_0^t \|\frac{\partial \phi}{\partial t}\|^2_{0,\Omega_t^s} \, dt + \int_0^t 4\nu_s \left( D(\psi), D \left( \frac{\partial \phi}{\partial t} \right) \right)_{0,\Omega_t^s} \, dt + \int_0^t 2\lambda \left( \nabla \cdot \psi, \nabla \cdot \frac{\partial \phi}{\partial t} \right)_{0,\Omega_t^s} \, dt
\]

\[
+ C \int_0^t \left( \left\| \frac{\partial \phi}{\partial t} \right\|_{0,\Omega_t^s}^2 + \|D(u)\|_{0,\Omega_t^s} \|D(g)\|_{0,\Omega_t^s} \right) + \|D(u)\|^2_{0,\Omega_t^s} + \|D(g)\|^2_{0,\Omega_t^s} + \|\tilde{p}_h - p\|^2_{0,\Omega_t^s} \right) \, dt. \tag{3.3.29}
\]

Using integration by parts,

\[
\int_0^t 4\nu_s \left( D(\psi), D \left( \frac{\partial \phi}{\partial t} \right) \right)_{0,\Omega_t^s} \, dt
\]

\[
= 4\nu_s \int_{\Omega_t^s} \int_0^t D(\psi)D \left( \frac{\partial \phi}{\partial t} \right) \, d\Omega \, dt
\]

\[
= 4\nu_s \int_{\Omega_t^s} \left[ D(\psi(t))D(\phi(t)) - D(\psi(0))D(\phi(0)) - \int_0^t D \left( \frac{\partial \psi}{\partial t} \right) D(\phi) \, dt \right] \, d\Omega, \tag{3.3.30}
\]
and Cauchy-Schwartz and Young’s inequalities yield that

\[
\int_0^t 4\nu_s \left( D(\psi), D\left( \frac{\partial \phi}{\partial \tilde{t}} \right) \right) d\tilde{t} \leq 4C(\epsilon_2)\nu_s \|D(\psi)\|^2_{0,\Omega^*} + 4\epsilon_2\nu_s \|D(\phi)\|^2_{0,\Omega^*} \\
+ 2\nu_s \left( \|D(\psi(0))\|^2_{0,\Omega^*} + \|D(\phi(0))\|^2_{0,\Omega^*} \right) \\
+ \int_0^t \left( 4C(\epsilon_2)\nu_s \|D(\partial_{\tilde{t}} \phi)\|^2_{0,\Omega^*} + 4\epsilon_2\nu_s \|D(\phi)\|^2_{0,\Omega^*} \right) d\tilde{t}. \tag{3.3.31}
\]

Similarly, we have

\[
\int_0^t 2\bar{\lambda} \left( \nabla \cdot \psi, \nabla \cdot \partial_{\tilde{t}} \phi \right) d\tilde{t} \leq 2C(\epsilon_3)\bar{\lambda} \|\nabla \cdot \psi\|^2_{0,\Omega^*} + 2\epsilon_3\bar{\lambda} \|\nabla \cdot \phi\|^2_{0,\Omega^*} \\
+ \bar{\lambda} \left( \|\nabla \cdot \psi(0)\|^2_{0,\Omega^*} + \|\nabla \cdot \phi(0)\|^2_{0,\Omega^*} \right) \\
+ \int_0^t \left( 2C(\epsilon_3)\bar{\lambda} \|\nabla \cdot \partial_{\tilde{t}} \phi\|^2_{0,\Omega^*} + 2\epsilon_3\bar{\lambda} \|\nabla \cdot \phi\|^2_{0,\Omega^*} \right) d\tilde{t}. \tag{3.3.32}
\]

Hölder inequality implies

\[
\int_0^t \|D(u_h)\|_{0,\Omega^*} \|D(g)\|_{0,\Omega^*} d\tilde{t} \leq C \|D(u_h)\|_{L^2(0,T;L^2)} \|D(g)\|_{L^4(0,T;L^4)}, \tag{3.3.33}
\]

where \(\|D(u_h)\|^2_{L^2(0,T;L^2)}\) is bounded by the stability result (3.3.6). Also,

\[
\int_0^t \|D(u)\|_{0,\Omega^*} \|D(g)\|_{0,\Omega^*} d\tilde{t} \leq C \|D(u)\|_{L^4(0,T;L^4)} \|D(g)\|_{L^4(0,T;L^4)} \leq C \|D(g)\|_{L^4(0,T;L^4)}, \tag{3.3.34}
\]

since \(u \in L^4(0, T; H^1(\Omega^*))\).

Setting \(\epsilon_2 = \epsilon_3 = \frac{1}{4}\), applying (3.3.31) - (3.3.34) and using the fact \(\|\nabla \cdot \xi\|_{0,\Omega^*} \leq C \|\nabla \xi\|_{0,\Omega^*}\).
for $\xi \in H^1(\Omega^*)$, (3.3.29) implies that

$$
\begin{align*}
\rho_s \left\| \frac{\partial \phi}{\partial t} \right\|_{0,\Omega^*}^2 + \nu_s \| D(\phi) \|_{0,\Omega^*}^2 + \frac{\lambda}{2} \| \nabla \cdot \phi \|_{0,\Omega^*}^2 + \rho_f \| I \|_{0,\Omega_f}^2 + K_1 \int_0^t \| D(I) \|_{0,\Omega_f}^2 \, dt \\
\leq C \left( \left\| \frac{\partial \phi(0)}{\partial t} \right\|_{0,\Omega^*}^2 + \| D(\phi(0)) \|_{0,\Omega^*}^2 + \| \nabla \cdot \phi(0) \|_{0,\Omega^*}^2 + \| I(0) \|_{0,\Omega_f}^2 + \| D(\psi(0)) \|_{0,\Omega^*}^2 \right) \\
+ \int_0^t \left( \frac{\rho_s}{2} \left\| \frac{\partial \phi}{\partial t} \right\|_{0,\Omega^*}^2 + \nu_s \| D(\phi) \|_{0,\Omega^*}^2 + \frac{\lambda}{2} \| \nabla \cdot \phi \|_{0,\Omega^*}^2 \right) \, dt \\
+ C \int_0^t \| \frac{\partial^2 \psi}{\partial t^2} \|_{0,\Omega^*}^2 + \| \frac{\partial g}{\partial t} \|_{0,\Omega_f}^2 + \| D(g) \|_{0,\Omega_f}^2 + \| \bar{p} - p \|_{0,\Omega_f}^2 + \| D \left( \frac{\partial \psi}{\partial t} \right) \|_{0,\Omega^*}^2 \, dt \\
+ C \left( \| D(g) \|_{L^4(0,T;L^2)}^4 + \| D(\psi) \|_{0,\Omega^*}^4 \right).
\end{align*}
$$

(3.3.35)

The stated error estimate can be obtained applying Gronwall’s lemma, the triangular inequality and the inf-sup condition (3.3.1).

\[\square\]

### 3.4 Time Discretization

In order to discretize the time-derivative term in time, we introduce the Reynolds transport formula

$$
\left( \frac{\partial \phi}{\partial t} \big|_{y,v} \right)_{\Omega_f} = \frac{\partial}{\partial t} (\phi, v)_{\Omega_f} - (\phi \nabla_x \cdot z, v)_{\Omega_f},
$$

(3.4.1)

which implies

$$
\left( \frac{\partial u_h}{\partial t} \big|_{y,v} \right)_{\Omega_f} + \frac{1}{2} (u_h \nabla \cdot z_h, v_h)_{\Omega_f} = \frac{\partial}{\partial t} (u_h, v_h)_{\Omega_f} - \frac{1}{2} (u_h \nabla \cdot z_h, v_h)_{\Omega_f}.
$$
The semi-discrete variational formulation considering the time-derivative term based on (3.4.1) is then obtained as

\[
\rho_s \left( \frac{\partial^2 \eta_h}{\partial t^2} \xi_h \right)_{\Gamma_I} + 2\nu_s (D(\eta_h), D(\xi_h))_{\Omega_f} + \bar{\lambda}(\nabla \cdot \eta_h, \nabla \cdot \xi_h)_{\Omega_f} + \rho_f \left[ \frac{\partial}{\partial t} (u_h, v_h)_{\Omega_f} - \frac{1}{2} (u_h \nabla \cdot z_h, v_h)_{\Omega_f} + \theta(u_h, u_h, v_h)_{\Omega_f} - \theta(z_h, u_h, v_h)_{\Omega_f} \right] + (\nu_f (|D(u_h)|) D(u_h), D(v_h))_{\Omega_f} - (p_h, \nabla \cdot v_h)_{\Omega_f} + (q_h, \nabla \cdot u_h)_{\Omega_f} = (f_f, v_h)_{\Omega_f} + (f_s, \xi_h)_{\Omega_f} \quad \forall (v_h, q_h, \xi_h) \in \tilde{U}_{h,t} \times Q_{h,t} \times \tilde{S}_h.
\]

In order to define the time-discretized ALE mapping, let us first define

\[
\Psi_{h, t} (y, t) = \frac{t - t^{n-1}}{\Delta t} \Psi_{h, t^n} (y) + \frac{t^n - t}{\Delta t} \Psi_{h, t^{n-1}} (y), \quad \forall t \in [t^{n-1}, t^n],
\]

where \(\Psi_{h, t^{n-1}}\) and \(\Psi_{h, t^n}\) are the harmonic extensions onto \(\Omega_0^f\) of \(\eta^{n-1}|_{\Gamma_0}\) and \(\eta^n|_{\Gamma_0}\), respectively. Here, \(\eta^{n-1}\) and \(\eta^n\) are the time-discretized displacement solutions to (3.4.10), described later. The corresponding discrete domain velocity \(z_h\) can then be defined as

\[
z_h(x, t) = \frac{\partial \Psi_{h, t} (y, t)}{\partial t} \circ \Psi_{h, t}^{-1} (x, t) = \frac{\Psi_{h, t^n} (y) - \Psi_{h, t^{n-1}} (y)}{\Delta t} \circ \Psi_{h, t}^{-1} (x, t), \quad \forall t \in [t^{n-1}, t^n].
\]

In other words, \(z_h^n = z_h(t^n)\) is the mesh velocity at time step \(t^n\) and for all times \(t \in [t^{n-1}, t^n]\). Let \(J_t\) denote the Jacobian matrix of the ALE mapping with its determinant given by

\[
J_t := \det(J_t) = \det \left( \frac{\partial \Psi_t (y)}{\partial y} \right).
\]

Under the assumptions (3.2.2), (3.2.3), proposition 2.1 of [19] gives

\[
\exists \kappa_{\min}, \kappa_{\max} \in \mathbb{R}^+ \text{ such that } 0 < \kappa_{\min} \leq J_t \leq \kappa_{\max} < \infty \quad \forall t \in [0, T].
\]

It has been further shown in [8] that

\[
|J_t - J_{t+1}| \leq C J \Delta t,
\]
where $C_J = \tilde{C} \| \nabla_y (z_h^{n+1} \circ \Psi_{t_n+1}) \|_{\infty, \Omega_t^n} \| \nabla \Psi_{t_n+1} \|_{\infty, \Omega_t^n}$ for $\tilde{C}$ independent of $h$, $\Delta t$ and the mapping.

To deal with test functions on different time domains, define $\Psi_{t_1, t_2}$ by

$$\Psi_{t_1, t_2} := \Psi_{h, t_2} \circ \Psi_{h, t_1}^{-1}. \quad (3.4.8)$$

Since any test function $v_h$ on $\Omega_{t_{n+1}}$ is given by $v_h = v_h^0 \circ \Psi_{t_n+1}^{-1}$ for some $v_h^0 \in \Omega_0$, the corresponding test function on $\Omega_{t_n}^+$ can be obtained by $v_h \circ \Psi_{t_n, t_{n+1}}$. Define the function space satisfying the continuity of velocities on the interface in the discrete sense:

$$\hat{U}_{h, t_{n+1}} \times \hat{S}_h := \{(v_h, \xi_h) \in U_{h, t_{n+1}} \times S_h : v_h |_{\Gamma_{t_{n+1}}} = \left( \frac{\xi_h^{n+1} - \eta_h^n}{\Delta t} \circ \Psi_{h, t_{n+1}}^{-1} \right) |_{\Gamma_{t_{n+1}}} \}, \quad (3.4.9)$$

where $\eta_h^n$ is the solution of the previous time step. We now consider the fully-discrete system for the stability and error estimate:

$$\frac{\rho_s}{2} \left( \eta_h^{n+1} - 2 \eta_h^n + \eta_h^{n-1} \right)_{\Delta t^2} \Omega_t^n + 2 \nu_s \left( D(\eta_h^{n+1}), D(\xi_h) \right)_{\Omega_t^n} + \lambda (\nabla \cdot \eta_h^{n+1}, \nabla \cdot \xi_h)_{\Omega_t^n} + \rho_f [ (u_h^{n+1}, v_h)_{\Omega_{t_{n+1}}} - (u_h^n, v_h \circ \Psi_{t_n, t_{n+1}})_{\Omega_{t_{n+1}}} ]$$

$$\quad + \theta(u_h^{n+1}, u_h^{n+1}, v_h)_{\Omega_{t_{n+1}}} - \theta(z_h^{n+1}, u_h^{n+1}, v_h)_{\Omega_{t_{n+1}}} \right]$$

$$\quad + \nu_f (D(u_h^{n+1}) \lvert D(u_h^{n+1}), D(v_h) \rvert_{\Omega_{t_{n+1}}} - (p_h^{n+1}, \nabla \cdot v_h)_{\Omega_{t_{n+1}}} + (q_h, \nabla \cdot u_h^{n+1})_{\Omega_{t_{n+1}}}$$

$$= (f_f^{n+1}, v_h)_{\Omega_{t_{n+1}}} + (f_s^{n+1}, \xi_h)_{\Omega_{t_{n+1}}} \forall (v_h, q_h, \xi_h) \in \hat{U}_{h, t_{n+1}} \times Q_{h, t_{n+1}} \times \hat{S}_h. \quad (3.4.10)$$

**Theorem 3.4.1** The solution of (3.4.10) satisfies the following estimate if $\Delta t$ satisfies $C_J M \Delta t \leq 1$.

$$\frac{\rho_f}{2} \| u_h^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 + \frac{\rho_s}{2} \left\| \eta_h^{n+1} - \eta_h^n \right\|_{0, \Omega_t^n}^2 + \nu_s \| D(\eta_h^{n+1}) \|_{0, \Omega_t^n}^2 + \frac{\lambda}{2} \| \nabla \cdot \eta_h^{n+1} \|_{0, \Omega_t^n}^2$$

$$+ \sum_{i=0}^n \left[ \rho_s \left( \frac{\eta_h^{i+1} - 2 \eta_h^i + \eta_h^{i-1}}{\Delta t} \right)_{\Omega_t^n}^2 + \nu_s \| D(\eta_h^{i+1}) \|_{0, \Omega_t^n}^2 \right] + \frac{\lambda}{2} \| \nabla \cdot (\eta_h^{i+1} - \eta_h^i) \|_{0, \Omega_t^n}^2 + \Delta t \frac{K_1}{2} \| D(u_h^{i+1}) \|_{0, \Omega_{t_{n+1}}}^2$$

$$\leq \frac{\rho_f}{2} \| u_0 \|_{0, \Omega_0}^2 + \frac{\rho_s}{2} \left( \frac{\eta_h^1 - \eta_h^0}{\Delta t} \right)_{\Omega_t^n}^2 + \nu_s \| D(\eta_0) \|_{0, \Omega_t^n}^2 + \frac{\lambda}{2} \| \nabla \cdot \eta_0 \|_{0, \Omega_t^n}^2$$

$$+ C \Delta t \sum_{i=0}^n \left[ \| f_f^{i+1} \|_{-1, \Omega_{t_{n+1}}}^2 + \| f_s^{i+1} \|_{0, \Omega_t^n}^2 \right]. \quad (3.4.11)$$

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Proof:

Set $v_h = u_h^{n+1}$, $\xi_h = \frac{\eta^{n+1} - \eta^n}{\Delta t}$, $q_h = p_h^{n+1}$ in (3.4.10). Using the identity $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$, the structure terms on the left hand side of (3.4.10) satisfy

$$\rho_s \left( \frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\Delta t}, \frac{\eta^{n+1} - \eta^n}{\Delta t} \right)_{\Omega^*} + 2\nu_s \left( \frac{D(\eta^{n+1}), D(\eta^n)}{\Delta t} \right)_{\Omega^*} + \lambda \left( \nabla \cdot \eta_h^{n+1}, \nabla \cdot \eta_h^{n+1} \right)_{\Omega^*}$$

$$= \frac{\rho_s}{2\Delta t} \left( \left\| \frac{\eta^{n+1} - \eta^n}{\Delta t} \right\|_{0,\Omega^*}^2 - \left\| \frac{\eta^n - \eta^{n-1}}{\Delta t} \right\|_{0,\Omega^*}^2 + \left\| \frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\Delta t} \right\|_{0,\Omega^*}^2 \right)$$

$$+ \frac{\nu_s}{\Delta t} \left( ||D(\eta^{n+1})||^2_{0,\Omega^*} - ||D(\eta^n)||^2_{0,\Omega^*} + ||D(\eta^{n+1} - \eta^n)||^2_{0,\Omega^*} \right)$$

$$+ \frac{\lambda}{2\Delta t} \left( ||\nabla \cdot \eta_h^{n+1}||^2_{0,\Omega^*} - ||\nabla \cdot \eta_h^n||^2_{0,\Omega^*} + ||\nabla \cdot (\eta_h^{n+1} - \eta_h^n)||^2_{0,\Omega^*} \right). \quad (3.4.12)$$

The structure terms on the right hand side of (3.4.10) is bounded as

$$\left( \frac{\rho_s}{\Delta t}, \frac{\eta^{n+1} - \eta^n}{\Delta t} \right)_{\Omega^*}$$

$$\leq ||f_n^{n+1}||_{0,\Omega^*} \cdot \left\| \frac{\eta^{n+1} - \eta^n}{\Delta t} \right\|_{0,\Omega^*} \leq C(\epsilon_1)||f_n^{n+1}||^2_{0,\Omega^*} + \epsilon_1 \left\| \frac{\eta^{n+1} - \eta^n}{\Delta t} \right\|_{0,\Omega^*}^2. \quad (3.4.13)$$

For the fluid terms,

$$-\frac{1}{2}(\nabla \cdot z_h^{n+1}, |u_h^{n+1}|^2)_{\Omega_{t_{n+1}}} + \theta(u_h^{n+1}, u_h^{n+1}, u_h^{n+1})_{\Omega_{t_{n+1}}} - \theta(z_h^{n+1}, u_h^{n+1}, u_h^{n+1})_{\Omega_{t_{n+1}}}$$

$$= -\frac{1}{2}(\nabla \cdot z_h^{n+1}, |u_h^{n+1}|^2)_{\Omega_{t_{n+1}}}, \quad (3.4.14)$$

and, using (3.1.13),

$$(\nu_f ||D(u_h^{n+1})||D(u_h^{n+1}), D(u_h^{n+1}))_{\Omega_{t_{n+1}}} \geq K_1 ||D(u_h^{n+1})||^2_{0,\Omega_{t_{n+1}}}. \quad (3.4.15)$$

The estimates (3.4.14), (3.4.15) and the identity $-ab = \frac{(a-b)^2-a^2-b^2}{2}$ provide a lower bound of the
fluid terms on the left hand side of (3.4.10) as

\[ \rho_f \left[ \left\| u_h^{n+1} \right\|_{0, \Omega'_n}^2 - \left\langle u_h^n, u_h^{n+1} \circ \Psi_{n,n+1} \right\rangle_{\Omega'_n} - \frac{1}{2} \left( \nabla \cdot z_h^{n+1}, |u_h^{n+1}|^2 \right)_{\Omega'_n} + \frac{1}{2} \right] \]

\[ + \theta(u_h^{n+1}, u_h^{n+1}, u_h^{n+1})_{\Omega'_n} - \theta(z_h^{n+1}, u_h^{n+1}, u_h^{n+1})_{\Omega'_n} + \left( \nu_f \left( D(u_h^{n+1}) | D(u_h^{n+1}), D(u_h^{n+1}) \right)_{\Omega'_n} \right. \]

\[ \geq \frac{\rho_f}{\Delta t} \left( \frac{\left\| u_h^{n+1} \right\|_{0, \Omega'_n}^2}{2} - \frac{\left\| u_h^n \right\|_{0, \Omega'_n}^2}{2} + \frac{\left\| u_h^{n+1} \circ \Psi_{n,n+1} - u_h^n \right\|_{0, \Omega'_n}^2}{2} \right) \]

\[ + \frac{\rho_f}{\Delta t} \left( \frac{\left\| u_h^{n+1} \right\|_{0, \Omega'_n}^2}{2} - \frac{\left\| u_h^n \right\|_{0, \Omega'_n}^2}{2} \right) + \frac{\rho_f}{\Delta t} \left( \frac{\left\| u_h^{n+1} \circ \Psi_{n,n+1} \right\|_{0, \Omega'_n}^2}{2} + K_1 \left( D(u_h^{n+1}) \right)_{\Omega'_n}^2 \right) \]

\[ + \frac{\rho_f}{\Delta t} \left( \frac{\left\| u_h^{n+1} \circ \Psi_{n,n+1} \right\|_{0, \Omega'_n}^2}{2} - \frac{\left\| u_h^n \circ \Psi_{n,n+1} \right\|_{0, \Omega'_n}^2}{2} - \frac{\Delta t}{2} \left( \nabla \cdot z_h^{n+1}, |u_h^{n+1}|^2 \right)_{\Omega'_n} \right). \tag{3.4.16} \]

It is noteworthy that with the Reynolds transport formula (3.4.1),

\[ \left( \nabla \cdot z_h, |u_h^{n+1} \circ \Psi_{t,t+1}|^2 \right)_{\Omega'_f} \]

\[ = \frac{\partial}{\partial t} \left( u_h^{n+1} \circ \Psi_{t,t+1}, u_h^{n+1} \circ \Psi_{t,t+1} \right)_{\Omega'_f} - \left( \frac{\partial(u_h^{n+1} \circ \Psi_{t,t+1})}{\partial t} |y, u_h^{n+1} \circ \Psi_{t,t+1+1} \right)_{\Omega'_f} \]

\[ = \frac{\partial}{\partial t} \left| u_h^{n+1} \circ \Psi_{t,t+1} \right|_{0, \Omega'_{t+1}}^2, \tag{3.4.17} \]

since \( u_h^{n+1} \) is time-independent. Integrating (3.4.17) from \( t^n \) to \( t^{n+1} \), and using (3.3.3), (3.4.7) and
(3.4.17),

\[
\frac{\|\mathbf{u}_h^{n+1}\|_{0,\Omega_{t_{n+1}}}^2}{2} - \frac{\|\mathbf{u}_h^{n+1} \circ \Psi_{t_{n+1}}\|_{0,\Omega_{t_n}}^2}{2} - \frac{\Delta t}{2} (\nabla \cdot \mathbf{z}_h^{n+1}, \|\mathbf{u}_h^{n+1}\|_{\Omega_{t_{n+1}}}^2)
\]

\[
= \frac{1}{2} \int_{t_n}^{t_{n+1}} \left( (\nabla \cdot \mathbf{z}_h^{n+1}, \mathbf{u}_h^{n+1} \circ \Psi_{t_{n+1}}) - (\nabla \cdot \mathbf{z}_h^{n+1}, \|\mathbf{u}_h^{n+1}\|_{\Omega_{t_{n+1}}}^2) \right) dt
\]

\[
= \frac{1}{2} \int_{t_n}^{t_{n+1}} (\nabla \cdot \mathbf{z}_h^{n+1}, \mathbf{u}_h^{n+1} \circ \Psi_{t_{n+1}}^2 (J_t - J_{t_{n+1}})) dt
\]

\[
\geq -\frac{1}{2} C \Delta t^2 (\|\nabla \cdot \mathbf{z}_h^{n+1}\|, \|\mathbf{u}_h^{n+1}\|_{\Omega_{t_{n+1}}}^2)
\]

\[
\geq -\frac{1}{2} C M \Delta t^2 \|\mathbf{u}_h^{n+1}\|_{0,\Omega_{t_{n+1}}}^2.
\]

(3.4.18)

Substituting (3.4.18) into (3.4.16), we have

\[
\rho_f \left[ \frac{\|\mathbf{u}_h^{n+1}\|_{\Omega_{t_{n+1}}}^2}{2} - (\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1} \circ \Psi_{t_{n+1}}) \right] - \frac{\Delta t}{2} (\nabla \cdot \mathbf{z}_h^{n+1}, \|\mathbf{u}_h^{n+1}\|_{\Omega_{t_{n+1}}}^2)
\]

\[
+ \theta (\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1})_{\Omega_{t_{n+1}}} - \theta (\mathbf{z}_h^{n+1}, \mathbf{u}_h^{n+1})_{\Omega_{t_{n+1}}}
\]

\[
+ (\nu_f (D(\mathbf{u}_h^{n+1}) \|D(\mathbf{u}_h^{n+1}) \|D(\mathbf{u}_h^{n+1}) \|D(\mathbf{u}_h^{n+1})_{\Omega_{t_{n+1}}}^2)
\]

\[
\geq -\frac{\rho_f}{\Delta t} \left( \frac{\|\mathbf{u}_h^{n+1}\|_{0,\Omega_{t_{n+1}}}^2}{2} \right) + K_1 \|D(\mathbf{u}_h^{n+1})\|_{0,\Omega_{t_{n+1}}}^2
\]

\[
-\frac{1}{2} \rho_f C M \Delta t \|\mathbf{u}_h^{n+1}\|_{0,\Omega_{t_{n+1}}}^2.
\]

(3.4.19)

A bound of right side fluid term is obtained by Poincaré and Young’s inequalities,

\[
(F_{t_{n+1}}^{n+1}, \mathbf{u}_h^{n+1})_{\Omega_{t_{n+1}}} \leq C(\epsilon) \|F_{t_{n+1}}^{n+1}\|_{-1,\Omega_{t_{n+1}}}^2 + \epsilon_2 \|D(\mathbf{u}_h^{n+1})\|_{0,\Omega_{t_{n+1}}}^2.
\]

(3.4.20)

We now substitute (3.4.12), (3.4.13), (3.4.19) and (3.4.20) in (3.4.10) and move the negative term
Multiplying (3.4.22) by \( \Delta \) over time steps gives
to the right to get
\[
\begin{aligned}
\frac{\rho_f}{2\Delta t} & \left( \| u_h^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 - \| u_h^n \|_{0, \Omega_{t_n}}^2 \right) + (K_1 - \epsilon_2) \| D(u_h^{n+1}) \|_{0, \Omega_{t_{n+1}}}^2 \\
+ \frac{\rho_s}{2\Delta t} & \left( \| \eta_h^{n+1} - \eta_h^n \|_{\Delta t}^2 \right) - \| \eta_h^n - \eta_h^{n-1} \|_{\Delta t}^2 \\
+ \frac{\nu_s}{\Delta t} & \left( \| D(\eta_h^{n+1}) \|_{0, \Omega_{t_n}}^2 - \| D(\eta_h^n) \|_{0, \Omega_{t_{n+1}}}^2 \right) + \frac{\lambda}{2\Delta t} \left( \| \nabla \cdot \eta_h^{n+1} \|_{0, \Omega_{t_n}}^2 - \| \nabla \cdot \eta_h^n \|_{0, \Omega_{t_{n+1}}}^2 \right) \\
+ \frac{\nu_s}{\Delta t} & \| D(\eta_h^{n+1} - \eta_h^n) \|_{0, \Omega_{t_n}}^2 + \frac{\lambda}{2\Delta t} \| \nabla \cdot (\eta_h^{n+1} - \eta_h^n) \|_{0, \Omega_{t_{n+1}}}^2 \\
\leq & \ C(\epsilon_2) \| f_j^{n+1} \|_{-1, \Omega_{t_{n+1}}}^2 - C(\epsilon_1) \| f_s^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 + \frac{\epsilon_1}{\Delta t} \| \eta_h^{n+1} - \eta_h^n \|_{0, \Omega_{t_{n+1}}}^2 \\
+ & \frac{1}{2} \rho_f C_J M \Delta t \| u_h^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 \\
\leq & \ C(\epsilon_2) \| f_j^{n+1} \|_{-1, \Omega_{t_{n+1}}}^2 - C(\epsilon_1) \| f_s^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 + \frac{\epsilon_1}{\Delta t} \| \eta_h^{n+1} - \eta_h^n \|_{0, \Omega_{t_{n+1}}}^2 \\
+ & \frac{\rho_f}{2} \| u_h^{n+1} \|_{0, \Omega_{t_{n+1}}}^2,
\end{aligned}
\] (3.4.21)

with a small enough \( \Delta t \) such that \( C_J M \Delta t \leq 1 \). Setting \( \epsilon_1 = \frac{\nu_s}{2} \), \( \epsilon_2 = \frac{K_1}{2} \), and summing (3.4.21) over time steps gives
\[
\begin{aligned}
\frac{\rho_f}{2\Delta t} & \| u_h^{n+1} \|_{0, \Omega_{t_{n+1}}}^2 + \frac{\rho_s}{2\Delta t} \left( \| \eta_h^{n+1} - \eta_h^n \|_{\Delta t}^2 \right) - \| \eta_h^n - \eta_h^{n-1} \|_{\Delta t}^2 \\
+ & \sum_{i=0}^n \left[ \frac{\rho_s}{2\Delta t} \| \eta_h^{i+1} - 2\eta_h^i + \eta_h^{i-1} \|_{\Delta t}^2 \right] + \frac{\nu_s}{\Delta t} \| D(\eta_h^{n+1} - \eta_h^n) \|_{0, \Omega_{t_n}}^2 \\
+ \frac{\lambda}{2\Delta t} \| \nabla \cdot (\eta_h^{n+1} - \eta_h^n) \|_{0, \Omega_{t_{n+1}}}^2 + \frac{K_1}{2} \| D(u^{i+1}) \|_{0, \Omega_{t_{i+1}}}^2 \\
\leq & \sum_{i=0}^n \left[ \frac{\rho_f}{2} \| u_h^{i+1} \|_{0, \Omega_{t_{i+1}}}^2 + \frac{\rho_s}{2} \left( \| \eta_h^{i+1} - \eta_h^i \|_{\Delta t}^2 \right) + \frac{\rho_f}{2\Delta t} \| u^0 \|_{0, \Omega_0}^2 + \frac{\rho_s}{2\Delta t} \| \eta^0 \|_{\Delta t}^2 \right] \\
+ & \frac{\nu_s}{\Delta t} \| D(\eta^0) \|_{0, \Omega_{t_n}}^2 + \frac{\lambda}{2\Delta t} \| \nabla \cdot \eta^0 \|_{0, \Omega_{t_{n+1}}}^2 + C \sum_{i=0}^n \left[ \| f_j^{i+1} \|_{-1, \Omega_{t_{i+1}}}^2 + \| f_s^{i+1} \|_{0, \Omega_{t_{n+1}}}^2 \right].
\end{aligned}
\] (3.4.22)

Multiplying (3.4.22) by \( \Delta t \) and applying discrete Gronwall’s Lemma, we obtain the estimate (3.4.11).

\[ \square \]
In the remainder of this section, we prove a convergence estimate for the time-discretization scheme. Error estimation for (3.4.10) is based on the assumption that \((u_h, p_h, \eta_h)\) exists in \(\bar{U}_{h,t} \times Q_{h,t} \times \bar{S}_h\) for \(t \in (0, T]\) and \(u_h \in L^\infty(0, T; H^1(\Omega_f))\). We begin with introducing the following identity for domain velocities \(z_h(t^n+1), z_h^{n+1}\):

\[
\begin{align*}
  z_h^{n+1} &= \frac{\partial \Psi_{h,t}(t^n+1)}{\partial t} = \frac{\Psi_{h,t,n+1} - \Psi_{h,t,n}}{\Delta t} + \frac{1}{\Delta t} \int_{t^n}^{t^n+1} (\tilde{t} - t^n) \frac{\partial^2 \Psi_{h,t}(\tilde{t})}{\partial \tilde{t}^2} \left( \Psi_{h,t,n+1}(x), \tilde{t} \right) d\tilde{t} \\
  &= z_h^n + \frac{1}{\Delta t} \int_{t^n}^{t^n+1} (\tilde{t} - t^n) \frac{\partial^2 \Psi_{h,t}(\tilde{t})}{\partial \tilde{t}^2} \left( \Psi_{h,t,n+1}(x), \tilde{t} \right) d\tilde{t},
\end{align*}
\]

which is obtained by the Taylor expansion of \(\Psi_{h,t}\) around \(t = t^n+1\).

**Theorem 3.4.2** A solution to the fully discretized equation (3.4.10) satisfies the error estimate for sufficiently small \(\Delta t\):

\[
\begin{align*}
  \rho_s &\| \frac{\partial \eta_{n+1}}{\partial t} - \frac{\partial \eta_n}{\partial t} \|_{0,\Omega_s}^2 + \nu_s \| D(\mathbf{e}_{\eta}^{n+1}) \|_{0,\Omega_s}^2 + \frac{\lambda}{2} \| \nabla \cdot \mathbf{e}_{\eta}^{n+1} \|_{0,\Omega_s}^2 \\
  &\quad + \frac{\rho_f}{2} \| \mathbf{e}_{u}^{n+1} \|_{0,\Omega_{f,t}^{n+1}}^2 + \sum_{i=0}^{n} \| D(\mathbf{e}_{u}^{t+i}) \|_{0,\Omega_{f,t}^{t+i}}^2 \\
  \leq \quad &C \Delta t^3 \int_0^T \| \frac{\partial^2 \eta_h(t)}{\partial t^2} \|^2_{0,\Omega_s} dt + C \Delta t^2 \int_0^T K^2(t) dt \\
  &\quad + C \Delta t^2 \| D(\mathbf{u}_h) \|_{L^\infty(0, T; L^2(\Omega_f))}^2,
\end{align*}
\]

where \(\mathbf{e}_{\eta}^k = \eta_h^k - \eta_h(t^k)\), \(\mathbf{e}_{u}^k = \mathbf{u}_h^k - \mathbf{u}_h(t^k)\) and

\[
K(t) = C \left[ \left\| \frac{\partial^2 \mathbf{u}_h(t)}{\partial t^2} \right\|_{0,\Omega_f} + \left\| \frac{\partial \mathbf{u}_h(t)}{\partial t} \right\|_{0,\Omega_f} + \left\| \mathbf{u}_h(t) \right\|_{0,\Omega_f} \right].
\]
Proof: Letting $t = t^{n+1}$ in (3.4.2) and adding same terms on both sides, we have

\[
\begin{align*}
\rho_s & \left( \eta_h(t^{n+1}) - 2\eta_h(t^n) + \eta_h(t^{n-1}) \right) \frac{\Delta t^2}{\Omega_s} \\
+ 2\nu_s \left( D(\eta_h(t^{n+1})) - D(\eta_h(t^n)) + \lambda (\nabla \cdot \eta_h(t^{n+1}), \nabla \cdot \xi_h) \right)_{\Omega_s} \\
+ \rho_f & \left[ \left( u_h(t^{n+1}), v_h \right)_{\Omega_{f_{n+1}}} - \left( u_h(t^n), v_h \circ \Psi_{t^n,t^{n+1}} \right)_{\Omega_{f_n}} \right] \frac{\Delta t}{\Delta t} \\
- & \frac{1}{2} \left( u_h(t^{n+1}) \nabla \cdot z_h(t^{n+1}), v_h \right)_{\Omega_{f_{n+1}}} \\
\end{align*}
\]

Let $\eta_h^k - \eta_h(t^k) = e_h^k$, $u_h^k - u_h(t^k) = e_u^k$, and subtract (3.4.25) from the fully discretized formulation (3.4.10). We have:

\[
\sum_{i=1}^{3} L_i = \sum_{i=1}^{6} R_i. \quad (3.4.26)
\]

where

\[
\begin{align*}
L_1 & := \rho_s \left( \frac{e_h^{n+1} - 2e_h^n + e_h^{n-1}}{\Delta t^2}, \xi_h \right)_{\Omega_s} + 2\nu_s \left( D(e_h^{n+1}), D(\xi_h) \right)_{\Omega_s} + \lambda (\nabla \cdot e_h^{n+1}, \nabla \cdot \xi_h)_{\Omega_s}, \\
L_2 & := \rho_f \left[ \left( e_u^{n+1}, v_h \right)_{\Omega_{f_{n+1}}} - \left( e_u^n, v_h \circ \Psi_{t^n,t^{n+1}} \right)_{\Omega_{f_{n}}} \right] , \\
L_3 & := (\nu_f (|D(u_h^{n+1})|))_n D(u_h(t^{n+1})), D(v_h)_{\Omega_{f_{n+1}}} \\
& \quad - (\nu_f (|D(u_h(t^{n+1})|))_n D(u_h(t^{n+1})), D(v_h)_{\Omega_{f_{n+1}}}. \quad (3.4.27)
\end{align*}
\]
With the test functions chosen above, we could do the following estimation

\[ \xi_{h} = \frac{\eta_{h}^{n+1} - \eta_{h}^{n}}{\Delta t} - \frac{\eta_{h}(t^{n}) - 2\eta_{h}(t^{n}) + \eta_{h}(t^{n-1})}{\Delta t}, \]

Step 1. Estimate the lower bounds for the left hand side terms in (3.4.26)

\[ L_{1} = \rho_{s} \left( \frac{\partial^{2} \eta_{h}(t^{n+1})}{\partial t^{2}} - \frac{\eta_{h}(t^{n+1}) - 2\eta_{h}(t^{n}) + \eta_{h}(t^{n-1})}{\Delta t} \right) \Omega_{s}, \]

\[ L_{2} := \rho_{f} \left[ \frac{\partial}{\partial t} \left( u_{h}(t), v_{h} \right)_{\Omega_{t}^{n+1}} - \frac{\left( u_{h}(t^{n+1}), v_{h} \right)_{\Omega_{t}^{n+1}} - \left( u_{h}(t^{n}), v_{h} \circ \Psi_{t_{n}, t^{n+1}} \right)_{\Omega_{t}^{n+1}}}{\Delta t} \right], \]

\[ L_{3} := \frac{\rho_{f}}{2} \left[ (u_{h}^{n+1} \nabla \cdot z_{h}^{n+1}, v_{h})_{\Omega_{t}^{n+1}} - (u_{h}(t^{n+1}) \nabla \cdot z_{h}(t^{n+1}), v_{h})_{\Omega_{t}^{n+1}} \right], \]

\[ L_{4} := \rho_{f} \left[ \theta \left( u_{h}(t^{n+1}), u_{h}(t^{n+1}), v_{h} \right)_{\Omega_{t}^{n+1}} - \theta \left( u_{h}^{n+1}, u_{h}^{n+1}, v_{h} \right)_{\Omega_{t}^{n+1}} \right], \]

\[ L_{5} := \rho_{f} \left[ \theta \left( z_{h}^{n+1}, u_{h}^{n+1}, v_{h} \right)_{\Omega_{t}^{n+1}} - \theta \left( z_{h}(t^{n+1}), u_{h}(t^{n+1}), v_{h} \right)_{\Omega_{t}^{n+1}} \right], \]

\[ L_{6} := (p_{h}^{n+1} - p_{h}(t^{n+1}), \nabla \cdot v_{h})_{\Omega_{t}^{n+1}} - (q_{h}, \nabla \cdot e_{u}^{n+1})_{\Omega_{t}^{n+1}}. \]

(3.4.28)

Set the test functions for the fluid as

\[ v_{h} = e_{u}^{n+1} := u_{h}^{n+1} - u_{h}(t^{n+1}), \quad q_{h} = p_{h}^{n+1} - p_{h}(t^{n+1}). \]

The test function for the structure, \( \xi_{h} \), is chosen based on the velocity continuity on the interface as

\[ \xi_{h} = \frac{\eta_{h}^{n+1} - \eta_{h}^{n}}{\Delta t} - \frac{\eta_{h}(t^{n}) - \eta_{h}(t^{n-1})}{\Delta t} = \frac{e_{\eta}^{n+1} - e_{\eta}^{n}}{\Delta t}. \]

(3.4.29)

With the test functions chosen above, we could do the following estimation

Step 1. Estimate the lower bounds for the left hand side terms in (3.4.26)

\[ L_{1} = \rho_{s} \left( \frac{e_{\eta}^{n+1} - 2e_{\eta}^{n} + e_{\eta}^{n-1}}{\Delta t^{2}} \right) \Omega_{s} + 2\nu_{s} \left( D \left( e_{\eta}^{n+1} \right), D \left( e_{\eta}^{n+1} - e_{\eta}^{n} \right) \right) \Omega_{s}, \]

\[ + \lambda \left( \nabla \cdot e_{\eta}^{n+1}, \nabla \cdot \left( \frac{e_{\eta}^{n+1} - e_{\eta}^{n}}{\Delta t} \right) \right) \Omega_{s} \]

\[ = \frac{\rho_{s}}{\Delta t} \left( \frac{e_{\eta}^{n+1} - 2e_{\eta}^{n} + e_{\eta}^{n-1}}{\Delta t} \right) \Omega_{s} + 2\nu_{s} \left( D \left( e_{\eta}^{n+1} \right), D \left( e_{\eta}^{n+1} - e_{\eta}^{n} \right) \right) \Omega_{s}, \]

\[ + \frac{\lambda}{\Delta t} \left( \nabla \cdot e_{\eta}^{n+1}, \nabla \cdot \left( e_{\eta}^{n+1} - e_{\eta}^{n} \right) \right) \Omega_{s}. \]

(3.4.30)
By the identity that \((a - b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2\),

\[
L_1 = \frac{\rho_s}{2\Delta t} \left( \left\| \frac{\partial}{\partial t} \eta^n - \frac{\partial}{\partial t} \eta^n \right\|_{0,\Omega_s}^2 - \left\| \frac{\partial}{\partial t} \eta^n - \frac{\partial}{\partial t} \eta^n \right\|_{0,\Omega_s}^2 + \left\| \frac{\partial}{\partial t} \eta^n + \frac{\partial}{\partial t} \eta^n \right\|_{0,\Omega_s}^2 \right)
\]

\[+ \frac{\nu_s}{\Delta t} \left( \left\| D(e^{n+1}) \right\|_{0,\Omega_s}^2 - \left\| D(e^n) \right\|_{0,\Omega_s}^2 + \left\| D(e^{n+1}) - D(e^n) \right\|_{0,\Omega_s}^2 \right)
\]

\[+ \lambda \frac{1}{\Delta t} \left( \left\| \nabla \cdot e^{n+1} \right\|_{0,\Omega_s}^2 - \left\| \nabla \cdot e^n \right\|_{0,\Omega_s}^2 + \left\| \nabla \cdot (e^{n+1} - e^n) \right\|_{0,\Omega_s}^2 \right)
\]

\[= \eta^{n+1} - \eta^n + \frac{\rho_s}{2\Delta t} \left\| \frac{\partial}{\partial t} \eta^n - \frac{\partial}{\partial t} \eta^n \right\|_{0,\Omega_s}^2 + \frac{\nu_s}{\Delta t} \left\| D(e^{n+1}) \right\|_{0,\Omega_s}^2 + \lambda \frac{1}{\Delta t} \left\| \nabla \cdot e^{n+1} \right\|_{0,\Omega_s}^2.
\] (3.4.31)

where \(E^{n+1}_\eta := \frac{\rho_s}{2\Delta t} \left\| \frac{\partial}{\partial t} \eta^n - \frac{\partial}{\partial t} \eta^n \right\|_{0,\Omega_s}^2 + \frac{\nu_s}{\Delta t} \left\| D(e^{n+1}) \right\|_{0,\Omega_s}^2 + \lambda \frac{1}{\Delta t} \left\| \nabla \cdot e^{n+1} \right\|_{0,\Omega_s}^2.

For the terms in \(L_2\), notice that by (3.4.1),

\[
\frac{\partial}{\partial t} (e^{n+1}_u + \Psi_{t,n+1}, e^{n+1}_u + \Psi_{t,n+1})_{\Omega_f} = (e^{n+1}_u + \Psi_{t,n+1} (\nabla \cdot z(t)), e^{n+1}_u + \Psi_{t,n+1})_{\Omega_f},
\] (3.4.32)

which further implies

\[
\left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 - (e^{n+1}_u + \Psi_{t,n+1}, e^{n+1}_u + \Psi_{t,n+1})_{\Omega_{t,n+1}} = \int_{t_n}^{t_{n+1}} ((e^{n+1}_u + \Psi_{t,t+1})^2, \nabla \cdot z(t))_{\Omega_f} dt.
\] (3.4.33)

Therefore

\[
(e^{n+1}_u, e^{n+1}_u + \Psi_{t,n+1})_{\Omega_{t,n+1}} \leq \frac{1}{2} \left[ \left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 + \left\| e^{n+1}_u + \Psi_{t,n+1} \right\|_{0,\Omega_{t,n+1}}^2 \right]
\]

\[
\leq \frac{1}{2} \left[ \left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 + \left\| e^{n+1}_u + \Psi_{t,n+1} \right\|_{0,\Omega_{t,n+1}}^2 - \int_{t_n}^{t_{n+1}} ((e^{n+1}_u + \Psi_{t,t+1})^2, \nabla \cdot z(t))_{\Omega_f} dt \right].
\] (3.4.34)

\(L_2\) is then bounded by

\[
L_2 = \frac{\rho_f}{\Delta t} \left[ \left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 - (e^{n}_u, e^{n+1}_u + \Psi_{t,n+1})_{\Omega_{t,n+1}} \right]
\]

\[
\geq \frac{\rho_f}{2\Delta t} \left[ \left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 - \left\| e^{n+1}_u \right\|_{0,\Omega_{t,n+1}}^2 + \int_{t_n}^{t_{n+1}} ((e^{n+1}_u + \Psi_{t,t+1})^2, \nabla \cdot z(t))_{\Omega_f} dt \right].
\] (3.4.35)
where the integration part satisfies
\[
\int_{t_n}^{t_{n+1}} (\mathbf{e}_u^{n+1} \circ \Psi_{t',t_{n+1}+1})^2, \nabla \cdot \mathbf{z}_h(\tilde{t})}_{\Omega_{t'}} \, d\tilde{t}
\]
\[
\leq \sup_{\tilde{t} \in (t_n, t_{n+1})} \| J_{\tilde{t}} \nabla \cdot \mathbf{z}_h(\tilde{t}) \|_{\infty, \Omega_{t'}} \| J_{\tilde{t}}^{-1} \|_{\infty, \Omega_{t'}} \int_{t_n}^{t_{n+1}} \| \mathbf{e}_u^{k+1} \|^2_{0, \Omega_{t'}} \, d\tilde{t}
\]
\[
= \Delta t \sup_{\tilde{t} \in (t_n, t_{n+1})} \| J_{\tilde{t}} \nabla \cdot \mathbf{z}_h(\tilde{t}) \|_{\infty, \Omega_{t'}} \| J_{\tilde{t}}^{-1} \|_{\infty, \Omega_{t'}} \| \mathbf{e}_u^{k+1} \|^2_{0, \Omega_{t'}} .
\] (3.4.36)

Thus, we obtain a lower bound of \( L_2 \) as
\[
L_2 \geq \frac{\rho f}{2\Delta t} \left[ \| \mathbf{e}_u^{n+1} \|_{0, \Omega_{t_n}}^2 - \| \mathbf{e}_u^n \|_{0, \Omega_{t_n+1}}^2 \right]
\]
\[
- \frac{\rho f}{2} \sup_{\tilde{t} \in (t_n, t_{n+1})} \| J_{\tilde{t}} \nabla \cdot \mathbf{z}_h(\tilde{t}) \|_{\infty, \Omega_{t'}} \| J_{\tilde{t}}^{-1} \|_{\infty, \Omega_{t'}} \| \mathbf{e}_u^{k+1} \|^2_{0, \Omega_{t'}} .
\] (3.4.37)

The lower bounds of \( L_3 \) can be easily found using the strong monotonicity of \( \nu_f(\cdot) \) as we present in (3.1.13),
\[
L_3 = (\nu_f(|D(\mathbf{u}_h^{n+1})|)D(\mathbf{u}_h^{n+1}) - \nu_f(|D(\mathbf{u}_h(t^{n+1}))|)D(\mathbf{u}_h(t^{n+1})), D(\mathbf{u}_u^{n+1}))_{\Omega_{t_n}}
\]
\[
\geq K_1 \| D(\mathbf{e}_u^{n+1}) \|^2_{0, \Omega_{t_n+1}} .
\] (3.4.38)

**Step 2.** Estimate the upper bounds for the right hand side terms in (3.4.26)

From Young’s inequality and Taylor expansion
\[
R_1 = \rho_s \left( \frac{\partial^2 \eta_h(t^{n+1})}{\partial t^2} - \frac{\partial^2 \eta_h(t^n) - 2\eta_h(t^n) + \eta_h(t^{n-1})}{\Delta t^2} \right)_{\Omega_\eta}
\]
\[
\leq C(\epsilon_1) \rho_s \left( \frac{\partial^2 \eta_h(t^{n+1})}{\partial t^2} - \frac{\partial^2 \eta_h(t^n) - 2\eta_h(t^n) + \eta_h(t^{n-1})}{\Delta t^2} \right)_{\Omega_\eta}^2
\]
\[
+ \epsilon_1 \rho_s \| \mathbf{e}_\eta^{n+1} - \mathbf{e}_\eta^n \|^2_{0, \Omega_\eta} \Delta t
\]
\[
\leq C(\epsilon_1) \Delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \eta_h(t)}{\partial t^2}(\tilde{t}) \right\|_{0, \Omega_\eta}^2 \, d\tilde{t} + \epsilon_1 \rho_s \| \mathbf{e}_\eta^{n+1} - \mathbf{e}_\eta^n \|^2_{0, \Omega_\eta} \Delta t .
\] (3.4.39)
To estimate $R_2$, first consider the Taylor expansion of \( \frac{\partial}{\partial t} (u_h(t), e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}})_{\Omega} \) at $t^{n+1}$

\[
\frac{\partial}{\partial t} (u_h(t), e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}})_{\Omega} = \frac{1}{\Delta t} \left[ (u_h(t^{n+1}), e_{u_i}^{n+1})_{\Omega_{t^{n+1}}} - (u_h(t^n), e_{u_i}^{n+1} \circ \Psi_{t^n,t^{n+1}})_{\Omega_{t^n}} + \int_{t^n}^{t^{n+1}} (t - t^n) \frac{\partial^2}{\partial t^2} (u_h(t), e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}})_{\Omega} \right] .
\]

(3.4.40)

Applying (3.4.1) twice, we have

\[
\frac{\partial^2}{\partial t^2} (u_h(t), e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}})_{\Omega_{t^n}}
\]

\[
= \left( \frac{\partial^2 u_h(t)}{\partial t^2} \bigg|_y e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}} \right)_{\Omega_{t^n}} + \left( \frac{\partial u_h(t)}{\partial t} \bigg|_y \nabla \cdot z_h + u_h(t) \frac{\partial (\nabla \cdot z_h) \bigg|_y, e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}} \right)_{\Omega_{t^n}}
\]

\[
+ \left( u_h(t) (\nabla \cdot z_h), e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}} \right)_{\Omega_{t^n}}
\]

\[
\leq \left[ \left\| \frac{\partial^2 u_h}{\partial t^2} \bigg|_y \right\|_{\Omega_{t^n}} + 2 \left\| \nabla \cdot z_h \right\|_{\infty, \Omega_{t^n}} \left\| \frac{\partial u_h}{\partial t} \bigg|_y \right\|_{\Omega_{t^n}} + \left\| u_h \right\|_{\Omega_{t^n}} \left\| \frac{\partial (\nabla \cdot z_h)}{\partial t} \bigg|_{\Omega_{t^n}} \right\|_{\Omega_{t^n}}
\]

\[
+ \left\| \nabla \cdot z_h \right\|_{\infty, \Omega_{t^n}} \left\| u_h \right\|_{\Omega_{t^n}} \right] \left\| e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}} \right\|_{\Omega_{t^n}}
\]

:= K(t) \left\| e_{u_i}^{n+1} \circ \Psi_{t,t^{n+1}} \right\|_{\Omega_{t^n}},
\]

(3.4.41)

where

\[
K(t) \quad = \left[ \left\| \frac{\partial^2 u_h}{\partial t^2} \bigg|_y \right\|_{\Omega_{t^n}} + 2 \left\| \nabla \cdot z_h \right\|_{\infty, \Omega_{t^n}} \left\| \frac{\partial u_h}{\partial t} \bigg|_y \right\|_{\Omega_{t^n}} + \left\| u_h \right\|_{\Omega_{t^n}} \left\| \frac{\partial (\nabla \cdot z_h)}{\partial t} \right\|_{\Omega_{t^n}}
\]

\[
+ \left\| \nabla \cdot z_h \right\|_{\infty, \Omega_{t^n}} \left\| u_h \right\|_{\Omega_{t^n}} \right] \left\| u_h \right\|_{\Omega_{t^n}} \right]
\]

\[
\leq C \left[ \left\| \frac{\partial^2 u_h}{\partial t^2} \bigg|_y \right\|_{\Omega_{t^n}} + \left\| \frac{\partial u_h}{\partial t} \bigg|_y \right\|_{\Omega_{t^n}} + \left\| u_h \right\|_{\Omega_{t^n}} \right]
\]

(3.4.42)
by (3.3.3). Exploiting (3.4.40) and (3.4.41), we obtain

\[
R_2 = \frac{\partial f}{\partial t} \int_{t_n}^{t_{n+1}} (\tilde{t} - t^n) \frac{\partial^2}{\partial t^2} (u_h(\tilde{t}), e_u^{n+1} \circ \Psi_{t, t_{n+1}})_{\Omega_f^t} d\tilde{t}
\]

\[
\leq \frac{\partial f}{\partial t} \int_{t_n}^{t_{n+1}} (\tilde{t} - t^n) K(\tilde{t}) \| e_u^{n+1} \circ \Psi_{t, t_{n+1}} \|_{0, \Omega_f^t} d\tilde{t}
\]

\[
\leq \frac{\partial f}{\partial t} \int_{t_n}^{t_{n+1}} \| J_\tilde{t} \|_{\infty, \Omega_f^t}^{1/2} \| J_{t_{n+1}}^{-1} \|_{\infty, \Omega_f^t}^{1/2} (\tilde{t} - t^n) K(\tilde{t}) \| e_u^{n+1} \|_{0, \Omega_f^t} d\tilde{t}
\]

\[
\leq \frac{\partial f}{\partial t} \left( \int_{t_n}^{t_{n+1}} \| J_\tilde{t} \|_{\infty, \Omega_f^t} \| J_{t_{n+1}}^{-1} \|_{\infty, \Omega_f^t} \left( \int_{t_n}^{t_{n+1}} K(\tilde{t})^2 d\tilde{t} \right)^{1/2} \right)^{1/2} \| e_u^{n+1} \|_{0, \Omega_f^t} d\tilde{t}
\]

\[
\leq \Delta t C(\epsilon_2) \sup_{\tilde{t} \in (t_n, t_{n+1})} \| J_\tilde{t} \|_{\infty, \Omega_f^t}^{1/2} \| J_{t_{n+1}}^{-1} \|_{\infty, \Omega_f^t}^{1/2} \int_{t_n}^{t_{n+1}} K(\tilde{t}) d\tilde{t}
\]

\[
+ \rho_f D(e_u^{n+1})^2_{0, \Omega_f^t_{n+1}}.
\]

Next, we use (3.4.23) to estimate \( R_3 \).

\[
R_3 = \frac{\partial f}{2} \left[ (u_h^{n+1} \nabla \cdot z_h^{n+1}, e_u^{n+1})_{\Omega_f^t_{n+1}} - (u_h(t^{n+1}) \nabla \cdot z_h(t_n^{n+1}), e_u^{n+1})_{\Omega_f^t_{n+1}} \right]
\]

\[
= \frac{\partial f}{2} \left[ (u_h^{n+1} \nabla \cdot z_h^{n+1}, e_u^{n+1})_{\Omega_f^t_{n+1}} - (u_h(t_n^{n+1}) \nabla \cdot z_h^{n+1}, e_u^{n+1})_{\Omega_f^t_{n+1}} \right]
\]

\[
- \left( u_h(t_n^{n+1}) \nabla \cdot \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (\tilde{t} - t^n) \frac{\partial^2}{\partial t^2} \left( \Psi_{h, t, t_n}(x), \tilde{t} \right) d\tilde{t}, e_u^{n+1} \right)_{\Omega_f^t_{n+1}}
\]

\[
= \frac{\partial f}{2} \left[ (e_u^{n+1} \nabla \cdot z_h^{n+1}, e_u^{n+1})_{\Omega_f^t_{n+1}}
\]

\[
- \frac{1}{\Delta t} \left( J_{t_n^{n+1}} (u_h \circ \Psi_{h, t, t_n}^{-1}) (t_n^{n+1}) \nabla \cdot \int_{t_n}^{t_{n+1}} (\tilde{t} - t^n) \frac{\partial^2}{\partial t^2} \left( \Psi_{h, t, t_n}^{-1}(x), \tilde{t} \right) d\tilde{t}, e_u^{n+1} \circ \Psi_{h, t, t_n}^{-1} \right)_{\Omega_f^t_{n+1}}
\]

where the last term is written as the inner product in the reference domain \( \Omega_f^0 \) so that the order of space-time integration is changeable in the following step. The first term is then bounded by

\[
(e_u^{n+1} \nabla \cdot z_h^{n+1}, e_u^{n+1})_{\Omega_f^t_{n+1}} \leq C(\epsilon_3) \| e_u^{n+1} \|_{0, \Omega_f^t_{n+1}}^2 \| \nabla \cdot z_h^{n+1} \|_{\infty, \Omega_f^t_{n+1}}^2 + \epsilon_3 \| e_u^{n+1} \|_{0, \Omega_f^t_{n+1}}^2,
\]
and the next term is estimated as

\[
\frac{1}{\Delta t} \left( J_{t+1} \left( u_h \circ \Psi^{-1}_{t+1} \right) (t^{n+1}) \nabla \cdot \int_{t^n}^{t^{n+1}} (\tilde{t} - t^n) \frac{\partial^2 \Psi_h}{\partial t^2} \, d\tilde{t}, e_{n+1}^u \circ \Psi^{-1}_{t+1} \right)_{\Omega_0^n} \\
= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left( J_{t+1} \left( u_h \circ \Psi^{-1}_{t+1} \right) (t^{n+1}) (\tilde{t} - t^n) \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2}, e_{n+1}^u \circ \Psi^{-1}_{t+1} \right)_{\Omega_0^n} \, d\tilde{t} \\
\leq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|_{\infty, \Omega_0^n} | \tilde{t} - t^n | (|u_h(t^{n+1})|, |e_{n+1}^u|)_{\Omega_{t+1}^n} \, d\tilde{t} \\
\leq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|_{\infty, \Omega_0^n} | \tilde{t} - t^n | \|u_h(t^{n+1})\|_{0, \Omega_{t+1}^n} \|e_{n+1}^u\|_{0, \Omega_{t+1}^n} \, d\tilde{t} \\
\leq \frac{1}{\Delta t} \left( \int_{t^n}^{t^{n+1}} (\tilde{t} - t^n)^2 \|e_{n+1}^u\|_{0, \Omega_{t+1}^n} \right)^{\frac{1}{2}} \left( \int_{t^n}^{t^{n+1}} \|u_h(t^{n+1})\|^2_{0, \Omega_{t+1}^n} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|^2_{\infty, \Omega_0^n} \right)^{\frac{1}{2}} \\
\leq \sqrt{\frac{\Delta t}{3}} \|e_{n+1}^u\|_{0, \Omega_{t+1}^n} \left( \int_{t^n}^{t^{n+1}} \|u_h(t^{n+1})\|^2_{0, \Omega_{t+1}^n} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|^2_{\infty, \Omega_0^n} \right)^{\frac{1}{2}} \\
\leq \Delta t^2 C(\varepsilon_3) \|D(u_h(t^{n+1}))\|^2_{0, \Omega_{t+1}^n} \sup_{\tilde{t} \in (t^n, t^{n+1})} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|^2_{\infty, \Omega_0^n} + \varepsilon_3 \|D(e_{n+1}^u)\|^2_{0, \Omega_{t+1}^n}.
\]

Thus, combining the above estimates together gives

\[
R_3 \leq C(\varepsilon_3) \frac{\rho_f}{2} \|e_{n+1}^u\|^2_{0, \Omega_{t+1}^n} \|\nabla \cdot z_{n+1}^f\|^2_{0, \Omega_{t+1}^n} + \varepsilon_3 \rho_f \|D(e_{n+1}^u)\|^2_{0, \Omega_{t+1}^n} \\
+ \Delta t^2 C(\varepsilon_3) \frac{\rho_f}{2} \|D(u_h(t^{n+1}))\|^2_{0, \Omega_{t+1}^n} \sup_{\tilde{t} \in (t^n, t^{n+1})} \left\| \nabla \cdot \frac{\partial^2 \Psi_h}{\partial t^2} \right\|^2_{\infty, \Omega_0^n}.
\tag{3.444}
\]

For the trilinear terms, we have

\[
R_4 = \rho_f \left[ \theta(u_h(t^{n+1}), u_h(t^{n+1}), e_{n+1}^u)_{\Omega_{t+1}^n} - \theta(u_h^{n+1}, u_h^{n+1}, e_{n+1}^u)_{\Omega_{t+1}^n} \right] \\
= \rho_f \left[ \theta(u_h(t^{n+1}), u_h(t^{n+1}), e_{n+1}^u)_{\Omega_{t+1}^n} - \theta(u_h^{n+1}, u_h(t^{n+1}), e_{n+1}^u)_{\Omega_{t+1}^n} \right. \\
\left. + \theta(u_h^{n+1}, u_h(t^{n+1}), e_{n+1}^u)_{\Omega_{t+1}^n} - \theta(u_h^{n+1}, u_h^{n+1}, e_{n+1}^u)_{\Omega_{t+1}^n} \right] \\
= \rho_f \left[ \theta(-e_{n+1}^u, u_h(t^{n+1}), e_{n+1}^u)_{\Omega_{t+1}^n} + \theta(u_h^{n+1}, -e_{n+1}^u, e_{n+1}^u)_{\Omega_{t+1}^n} \right].
\]

Using (2.3.6) and the inequality \(ab \leq \varepsilon a^p + C(\varepsilon, p) b^q\) for any \(\varepsilon > 0, 1 \leq p \leq \infty\), satisfying \(\frac{1}{p} + \frac{1}{q} = 1\),
we have

\[ R_4 = \rho_f \theta(-e_{u_{t+1}}, u_{h_{t+1}})_{t_{n+1}} \]
\[ \leq C\rho_f \| e_{u_{t+1}} \|_{0, \Omega_{t_{n+1}}}^\frac{1}{2} \| D(e_{u_{t+1}}) \|_{0, \Omega_{t_{n+1}}} \| D(u_{h_{t+1}}) \|_{0, \Omega_{t_{n+1}}} \]
\[ \leq \epsilon_4 \rho_f \| D(e_{u_{t+1}}) \|_{0, \Omega_{t_{n+1}}}^2 + C(\epsilon_4) \rho_f \| e_{u_{t+1}} \|_{0, \Omega_{t_{n+1}}}^2 \| D(u_{h_{t+1}}) \|_{0, \Omega_{t_{n+1}}}^4 \]  \hspace{1cm} (3.4.45)

By (3.4.23) and the same technique as we did to estimate \( R_3 \),

\[ R_5 = \rho_f \left[ \theta(z_{h_{t+1}}, u_{h_{t+1}}, e_{u_{t+1}})_{t_{n+1}} - \theta(z_{h_{t+1}}, u_{h_{t+1}}, e_{u_{t+1}})_{t_{n+1}} \right] \]
\[ = \rho_f \left[ \theta(z_{h_{t+1}}, u_{h_{t+1}}, e_{u_{t+1}})_{t_{n+1}} - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (i - t^n) \frac{\partial^2 \Psi h_{i}}{\partial t^2} \left( \Psi h_{i+1}^{-1}(x), \hat{t} \right) \hat{d}, \left( u_{h_{t+1}}, e_{u_{t+1}} \right)_{t_{n+1}} \right] \]
\[ = -\frac{\rho_f}{\Delta t} \theta \left( \int_{t_n}^{t_{n+1}} (i - t^n) \frac{\partial^2 \Psi h_{i}}{\partial t^2} \left( \Psi h_{i+1}^{-1}(x), \hat{t} \right) \hat{d}, \left( u_{h_{t+1}}, e_{u_{t+1}} \right)_{t_{n+1}} \right) \]
\[ \leq \frac{\rho_f}{\Delta t} \theta \left( \sup_{t \in [t_n, t_{n+1}]} \left\| \frac{\partial^2 \Psi h_{i}}{\partial t^2} \right\|_{0, \Omega_0}^\frac{\Delta t^2}{2} \cdot \left( u_{h_{t+1}}, e_{u_{t+1}} \right)_{t_{n+1}} \right) \]
\[ \leq \Delta t^2 C(\epsilon_5) \| D(u_{h_{t+1}}) \|_{0, \Omega_{t_{n+1}}}^2 \sup_{i \in [t_n, t_{n+1}]} \left\| \frac{\partial^2 \Psi h_{i}}{\partial t^2} \right\|_{0, \Omega_0}^2 + \epsilon_5 \rho_f \| D(e_{u_{t+1}}) \|_{0, \Omega_{t_{n+1}}}^2 \] \hspace{1cm} (3.4.46)

and, finally

\[ R_6 = (p_{h_{t+1}} - p_{h_{t+1}}), \nabla \cdot e_{u_{t+1}} \|_{\Omega_{t_{n+1}}} - (p_{h_{t+1}} - p_{h_{t+1}}), \nabla \cdot e_{u_{t+1}} \|_{\Omega_{t_{n+1}}} = 0 \] \hspace{1cm} (3.4.47)

**Step 3.** Combine the bounds for each sides and estimate \( E_n \).
Combining (3.4.31)-(3.4.47), we obtain the following inequality

\[
E^{n+1}_\eta - E^n_\eta + \frac{\rho_s}{2\Delta t} \left\| e^{n+1}_\eta - 2e^n_\eta + e^{n-1}_\eta \right\|^2_{\Omega,\Delta t} + \frac{\nu_s}{\Delta t} \left\| D(e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + \frac{\lambda}{2\Delta t} \left\| \nabla \cdot (e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + K_1 \left\| D(e^{n+1}_u) \right\|^2_{\Omega,\Delta t} + \frac{\rho_l}{2\Delta t} \left( \| e^{n+1}_u \|_{0,\Omega_{n+1}}^2 - \| e^n_u \|_{0,\Omega_{n+1}}^2 \right) \\
\leq C(\epsilon_1) \Delta t^2 \int_{t^n}^{t^{n+1}} \left\| \frac{\partial^2 \eta}{\partial \tau^2} (\tilde{t}) \right\|^2_0 \; d\tilde{t} + \epsilon_1 \rho_s \left\| \frac{e^{n+1}_\eta - e^n_\eta}{\Delta t} \right\|^2_{\Omega,\Delta t} + C(\epsilon_2) \Delta t \sup_{\tilde{t} \in (t^n, t^{n+1})} \left\| J_{\tilde{t}} \right\|_{1/2, \Omega_{t^n+1}} \left\| J_{t^{n+1}} \right\|_{1/2, \Omega_{t^n+1}} \int_{t^n}^{t^{n+1}} K^2(\tilde{t}) \; d\tilde{t} + (\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) \rho_l \left\| D(e^{n+1}_u) \right\|^2_{0,\Omega_{t^n+1}} + C(\epsilon_3, \epsilon_4) \left\| e^{n+1}_u \right\|^2_{0,\Omega_{t^n+1}} \max \left\{ \left\| D(u_h(t^{n+1})) \right\|^4_{0,\Omega_{t^n+1}}, \left\| \nabla \cdot z_h^{n+1} \right\|^2_{0,\Omega_{t^n+1}} \right\} + C(\epsilon_3, \epsilon_5) \Delta t^2 \left\| D(u_h(t^{n+1})) \right\|^2_{0,\Omega_{t^n+1}} \sup_{\tilde{t} \in (t^n, t^{n+1})} \max \left\{ \left\| \frac{\partial^2 \Psi_{h,\tilde{t}}}{\partial \tau^2} \right\|^2_{\Omega_{t^n+1}}, \left\| \nabla \cdot \frac{\partial^2 \Psi_{h,\tilde{t}}}{\partial \tau^2} \right\|^2_{\Omega_{t^n+1}} \right\} + \frac{\rho_l}{2} \sup_{\tilde{t} \in (t^n, t^{n+1})} \left\| J_{\tilde{t}} \nabla \cdot z_h(\tilde{t}) \right\|_{\Omega_{t^n+1}} \left\| J_{t^{n+1}} \right\|_{\Omega_{t^n+1}} \left\| e^{n+1}_u \right\|^2_{0,\Omega_{t^n+1}}. \tag{3.4.48}
\]

Let \( E^{k+1} := E^{k+1}_\eta + \frac{\rho_l}{2\Delta t} \left\| e^{k+1}_u \right\|^2_{0,\Omega_{k+1}} \). If we let \( \epsilon_1 = \frac{1}{2}, \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \frac{K_1}{8} \), (3.4.48) becomes

\[
E^{n+1} + \frac{\rho_s}{2\Delta t} \left\| e^{n+1}_\eta - 2e^n_\eta + e^{n-1}_\eta \right\|^2_{\Omega,\Delta t} + \frac{\nu_s}{\Delta t} \left\| D(e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + \frac{\lambda}{2\Delta t} \left\| \nabla \cdot (e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + \frac{\rho_l}{2} \left\| e^{n+1}_\eta - e^n_\eta \right\|^2_{\Omega,\Delta t} \\
\leq E^n + \frac{\rho_s}{2\Delta t} \left\| e^{n+1}_\eta - 2e^n_\eta + e^{n-1}_\eta \right\|^2_{\Omega,\Delta t} + \frac{\nu_s}{\Delta t} \left\| D(e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + \frac{\lambda}{2\Delta t} \left\| \nabla \cdot (e^{n+1}_\eta - e^n_\eta) \right\|^2_{\Omega,\Delta t} + \frac{\rho_l}{2} \left\| e^{n+1}_\eta - e^n_\eta \right\|^2_{\Omega,\Delta t} \\
\leq C \Delta t^2 \int_{t^n}^{t^{n+1}} \left\| \frac{\partial^2 \eta}{\partial \tau^2} (\tilde{t}) \right\|^2_0 \; d\tilde{t} + C \Delta t \sup_{\tilde{t} \in (t^n, t^{n+1})} \left\| J_{\tilde{t}} \right\|_{1/2, \Omega_{t^n+1}} \left\| J_{t^{n+1}} \right\|_{1/2, \Omega_{t^n+1}} \int_{t^n}^{t^{n+1}} K^2(\tilde{t}) \; d\tilde{t} + C \left\| e^{n+1}_u \right\|^2_{0,\Omega_{t^n+1}} \max \left\{ \left\| D(u_h(t^{n+1})) \right\|^4_{0,\Omega_{t^n+1}}, \left\| \nabla \cdot z_h^{n+1} \right\|^2_{0,\Omega_{t^n+1}} \right\} + C \Delta t^2 \left\| D(u_h(t^{n+1})) \right\|^2_{0,\Omega_{t^n+1}} \sup_{\tilde{t} \in (t^n, t^{n+1})} \max \left\{ \left\| \frac{\partial^2 \Psi_{h,\tilde{t}}}{\partial \tau^2} \right\|^2_{\Omega_{t^n+1}}, \left\| \nabla \cdot \frac{\partial^2 \Psi_{h,\tilde{t}}}{\partial \tau^2} \right\|^2_{\Omega_{t^n+1}} \right\}. \tag{3.4.49}
\]
Summing over time steps gives

\[
E^{n+1} + \sum_{i=0}^{n} \left( \frac{\rho_s}{2\Delta t} \| e^{i+1}_\eta - 2e^i_\eta + e^{i-1}_\eta \|_{0,\Omega_s}^2 + \frac{\nu_s}{\Delta t} \| D(e^{i+1}_\eta - e^i_\eta) \|_{0,\Omega_s}^2 + \frac{\lambda}{2\Delta t} \| \nabla \cdot (e^{i+1}_\eta - e^i_\eta) \|_{0,\Omega_s}^2 + \frac{K_1}{2} \| D(e^{i+1}_u) \|_{0,\Omega_{t+1}}^2 \right)
\]

\[
+ \sum_{i=0}^{n} \left( \frac{\rho_f}{2} \| e^{i+1}_\eta - e^i_\eta \|_{0,\Omega_s}^2 + C\| e^{i+1}_\eta \|_{0,\Omega_{t+1}}^2 \right)
\]

\[
\leq E^0 + \sum_{i=0}^{n} \left[ \frac{\rho_s}{2} \| e^{i+1}_\eta - e^i_\eta \|_{0,\Omega_s}^2 + \| e^{i+1}_\eta \|_{0,\Omega_{t+1}}^2 \right] + C \Delta t^2 N_2 + C \Delta t N_3 + C \Delta t T \| D(u_h) \|_{L^\infty(0,T;L^2(\Omega_t))}^2 N_4,
\]

where

\[
N_1 := \max \left\{ \| D(u_h) \|_{L^\infty(0,T;L^2(\Omega_t))}^2, \| \nabla \cdot z_h \|_{L^\infty(0,T;L^2(\Omega_t))}^2 \right\},
\]

\[
N_2 := \int_0^T \| \frac{\partial^2 z_h(\tilde{t})}{\partial \tilde{t}^2} \|_{0,\Omega_s}^2 d\tilde{t},
\]

\[
N_3 := \| J_i \|_{L^\infty(0,T;L^\infty(\Omega_t))} \| J_{i-1} \|_{L^\infty(0,T;L^\infty(\Omega_t))} \int_0^T K^2(\tilde{t}) d\tilde{t},
\]

\[
N_4 := \sup \left\{ \max \left\{ \left\| \frac{\partial^2 \Psi_{h,i}}{\partial \tilde{t}^2} \right\|_{0,\Omega_s}^2, \left\| \nabla \cdot \frac{\partial^2 \Psi_{h,i}}{\partial \tilde{t}^2} \right\|_{0,\Omega_s}^2 \right\} \right\}.
\]

Note that there exist \( C > 0 \) such that \( N_1 \leq C, N_4 \leq C \) and \( N_3 \leq C \int_0^T K^2(\tilde{t}) d\tilde{t} \) by (3.3.3) and (3.4.6). Recall the definition of \( E^n \). Multiplying (3.4.50) by \( \Delta t \) and using the fact that \( E^0 = 0 \),

\[
\sum_{i=0}^{n} \left( \frac{\rho_s}{4} \| e^{i+1}_\eta - 2e^i_\eta + e^{i-1}_\eta \|_{0,\Omega_s}^2 + \frac{\nu_s}{\Delta t} \| D(e^{i+1}_\eta - e^i_\eta) \|_{0,\Omega_s}^2 + \frac{\lambda}{2} \| \nabla \cdot (e^{i+1}_\eta - e^i_\eta) \|_{0,\Omega_s}^2 + \frac{K_1}{2} \| D(e^{i+1}_u) \|_{0,\Omega_{t+1}}^2 \right)
\]

\[
\leq \Delta t \sum_{i=0}^{n} \left[ \frac{\rho_s}{2} \| e^{i+1}_\eta - e^i_\eta \|_{0,\Omega_s}^2 + C\| e^{i+1}_\eta \|_{0,\Omega_{t+1}}^2 \right] + C \Delta t^2 N_2 + C \Delta t N_3 + C \Delta t T \| D(u_h) \|_{L^\infty(0,T;L^2(\Omega_t))}^2 N_4.
\]

Finally, if we drop the first three positive terms under the summation in the left side of
(3.4.56) and $\Delta t$ is sufficiently small, (3.4.24) follows by Gronwall’s Lemma.

\[ \Box \]

3.5 Numerical experiments

3.5.1 Convergence Test

We started with several numerical tests carried out on a non-physical problem to check the convergence rates. Although the previous analysis is based on the general quasi-Newtonian fluid involving a general viscosity function $\nu(|\mathbf{D}(\mathbf{u})|)$, we consider a specific case, the Cross model, as the quasi-Newtonian flow for numerical tests. The viscosity function of the Cross model is given by

\[ \nu_f(|\mathbf{D}(\mathbf{u})|) := \nu_\infty + \frac{(\nu_0 - \nu_\infty)}{1 + (\kappa |\mathbf{D}(\mathbf{u})|)^{2-r}}, \]

where $\kappa > 0$ is a time constant, $1 \leq r \leq 2$ is a dimensionless rate constant, $\nu_\infty$ and $\nu_0$ denote limiting viscosity values at an infinite and zero shear rate, respectively, assumed to satisfy $0 \leq \nu_\infty \leq \nu_0$. Throughout this section, we will restrict our focus to the case where $\kappa = 1$, $\nu_0 = 1$, $\nu_\infty = 0$. Note that with the choice of $r = 2$, the fluid becomes Newtonian.

Since we are interested in convergence results of the FSI problem, we make the same assumption as in [2]: the system have infinitesimal displacements of the fluid domain and the structure, but with non-negligible velocity of the interface. Parameters chosen for the simulations are: $\rho_f = 1.0$, $\rho_s = 1.9$; $\nu_s = 3$ and $\lambda = 4.5$. Initial conditions, body forces, and boundary conditions are appropriately given so that the exact solutions on the computational domain $\Omega_f = [0,1] \times [0,1]$ and...
\[ \Omega^s = [0, 1] \times [0, 1] \] are

\[ u_1 = \cos(x + t)\sin(y + t) + \sin(x + t)\cos(y + t), \]
\[ u_2 = -\sin(x + t)\cos(y + t) - \cos(x + t)\sin(y + t), \]
\[ p = \left( 0.5 + \frac{0.5}{1 + (\sin(x + t)\sin(y + t) - \cos(x + t)\cos(y + t))^{4-2r}} \right) \]
\[ \cdot (\sin(x + t)\sin(y + t) - \cos(x + t)\cos(y + t)) \]
\[ + 2\nu_s\cos(x + t)\sin(y + t), \]
\[ \eta_1 = \sin(x + t)\sin(y + t), \]
\[ \eta_2 = \cos(x + t)\cos(y + t). \]

We performed simulations over one time step to check convergence rates. The finite element pair \((Q_2, Q_1)\) was used to solve the fluid equations, while \(Q_2, Q_1\) finite elements were used for the structure displacement and the discrete ALE mapping, respectively.

3.5.1.1 Test I

We performed simulations over one time step to check convergence rates with \(\Delta t = 1e-5s\) and uniform meshes.

We first check the case where \(r = 2\) in the Cross model, which is equivalent to a linear Newtonian Fluid-structure system. The FSI problem was solved with a sequence of decreasing mesh size, and the results are presented in Table 3.1 - 3.2. We obtained the theoretical spatial convergence rate at which the computed solution converges upon the true solution.

<table>
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<th>(h)</th>
<th>(|u^n - u^{true}|_{L^2})</th>
<th>Rate</th>
<th>(|u^n - u^{true}|_{H^1})</th>
<th>Rate</th>
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Table 3.1: Fluid convergence result for linear FSI

In the second spatial convergence test we considered a nonlinear quasi-Newtonian case by setting \(r = 1.5\). Similar simulations have been done as the first test, and the results are presented in Table 3.3 - 3.4. Again, we obtained the theoretical convergence rate upon the true solution, and observed no significant difference between the Newtonian and non-Newtonian cases.
Table 3.2: Structure convergence result for linear FSI

<table>
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<th>h</th>
<th>$|\eta^n - \eta^{true}|_{L^2}$</th>
<th>Rate</th>
<th>$|\eta^n - \eta^{true}|_{H^1}$</th>
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Table 3.3: Fluid convergence result for non-linear FSI

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Table 3.4: Structure convergence result for non-linear FSI

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Table 3.5: Convergence result

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<td>2.96</td>
<td>6.559e-06</td>
<td>3.02</td>
</tr>
<tr>
<td>1/64</td>
<td>1.504e-06</td>
<td>3.13</td>
<td>8.142e-07</td>
<td>3.01</td>
</tr>
</tbody>
</table>

Table 3.5: Convergence result

3.5.1.2 Test II

In this test we considered a nonlinear quasi-Newtonian case with $r = 1.5$. The FSI problem was solved for the final time $T = 0.5$ with a sequence of decreasing mesh size, and the time-steps were decreased accordingly so that the error would not be dominated by time-steps. The results are presented in Table 3.5, which shows the theoretical spatial convergence rate at which the computed solution converges upon the true solution.

<table>
<thead>
<tr>
<th>h</th>
<th>$\Delta t$</th>
<th>$|\eta^n - \eta^{true}|_{L^\infty(L^2)}$</th>
<th>Rate</th>
<th>$|\eta^n - \eta^{true}|_{L^\infty(L^2)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>1/100</td>
<td>7.539e-04</td>
<td>-</td>
<td>4.083e-04</td>
<td>-</td>
</tr>
<tr>
<td>1/16</td>
<td>1/1000</td>
<td>1.025e-04</td>
<td>2.88</td>
<td>5.321e-05</td>
<td>2.94</td>
</tr>
<tr>
<td>1/32</td>
<td>1/10000</td>
<td>1.317e-05</td>
<td>2.96</td>
<td>6.559e-06</td>
<td>3.02</td>
</tr>
<tr>
<td>1/64</td>
<td>1/100000</td>
<td>1.504e-06</td>
<td>3.13</td>
<td>8.142e-07</td>
<td>3.01</td>
</tr>
</tbody>
</table>

Table 3.5: Convergence result

The next test focused more on the time-discretized error by using a sequence of decreasing time steps and a fixed mesh of $h = 1/20$. Errors and convergence rates are presented in Table 3.6.
3.5.2 Blood flow simulation

We considered a blood flow problem reported in [31, 39], where modeling parameters in the structure equation are consistent with blood flow in a human body. The reference domain for the fluid subsystem has height 1 cm and length 6 cm. The structure domain has height 0.1 cm and length 6 cm. The density of the structure, \( \rho_s \), is 1.1 g/cm\(^3\). The Young’s Modulus of the structure, \( E \), is \( 3 \times 10^6 \) dyne/cm\(^2\) and its Poisson ratio, \( \nu \), is 0.3. The Lamé parameters \( \bar{\lambda} \) and \( \nu_s \) are defined as in (3.1.4). All fluid parameters except \( r \) are the same as in Test I.

A force \( b(t) \) is applied to the left fluid boundary (Fig. 3.2) at \( t \) sec, where

\[
b(t) = \begin{cases} 
-10^3(1 - \cos \frac{2\pi t}{0.025}),0) & \text{dyne/cm}^2, \quad t \leq 0.025 \\
(0,0), & \quad 0.025 < t < T.
\end{cases}
\]

The function \( b(t) \) defines the stress on the inlet denoted by \( u_N \). For numerical tests, we impose the Neumann condition on both the inflow and outflow boundaries as in the literature. The volume force for the fluid and structure are \( f(t) = (0,0) \) dyne/cm\(^2\). The other boundary conditions on the domain configuration are homogeneous Dirichlet or Neumann (Fig. 3.2), and the simulation begins at rest.

In Figure 3.3 we present the vertical displacement of the structure at three locations on the interface. Comparison is made between the solutions of different fluid types: \( r = 1.5 \) for the shear-thinning case, \( r = 2 \) for the Newtonian case and \( r = 3 \) for the shear-thickening case. Figure 3.5

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( | u^n - u^{true} |_{L^2} )</th>
<th>Rate</th>
<th>( | \eta^n - \eta^{true} |_{L^2} )</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
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<td>-</td>
<td>1.952e-03</td>
<td>-</td>
</tr>
<tr>
<td>1/40</td>
<td>1.936e-03</td>
<td>0.96</td>
<td>9.062e-04</td>
<td>1.11</td>
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<tr>
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<td>9.390e-04</td>
<td>1.04</td>
<td>4.309e-04</td>
<td>1.07</td>
</tr>
<tr>
<td>1/160</td>
<td>4.314e-04</td>
<td>1.12</td>
<td>2.044e-04</td>
<td>1.08</td>
</tr>
</tbody>
</table>

Table 3.6: Convergence result

![Figure 3.2: Domain and boundary conditions for numerical experiment](image-url)
shows the vertical displacement at a sequence of time $t = 0.02s$, $t = 0.05s$, $t = 0.08s$. The structure displacement is most significant for the shear-thinning case as expected. The pressure profiles at $t = 0.01s$, $t = 0.025s$, $t = 0.035s$ are also presented in Figure 3.4 for $r = 1.5$.

![Figure 3.3: Vertical interface displacement at different points](image)

![Figure 3.4: Fluid pressure profile](image)

### 3.6 Conclusion

We considered a fully discretized monolithic system for a quasi-Newtonian fluid-structure interaction problem. An Arbitrary Lagrangian Eulerian mapping was introduced to deal with the time derivative term in the fluid equation, and the two dynamic equations were combined into one formulation using interface conditions. After defining finite element spaces, the finite element formulation was obtained for which we proved the stability and error estimate. Numerical tests
were performed on both Newtonian and non-Newtoninan cases, and for both cases we obtained the theoretical convergence rates in $L_2$ and $H_1$ norms. The fully discrete system was analyzed for stability and time-discretization error, and numerically tested. We obtained the theoretical convergence rate in numerical experiments where a known analytical solution is given and, in the blood flow example, both Newtonian and non-Newtonian fluids were considered, then the results were compared. In the next chapter, we extend this work to viscoelastic fluid-structure interaction problems.
Chapter 4

2D-2D Viscoelastic Fluid-Structure Interaction Problem

For a Newtonian fluid, a separate constitutive equation is not necessary in the model equations since the extra stress is assumed to be linear proportional to the deformation tensor. However, for viscoelastic fluids such as Oldroyd-B, FENE-P and Owens models, a separate hyperbolic differential constitutive equation is required to describe the complicated stress-deformation relation. Extra difficulty arises from both analytical and computational aspects due to the hyperbolic character and the lack of stabilizing term for the stress.

A viscoelastic fluid, unlike a Newtonian fluid, usually develops not only the shear stresses, but also the normal stresses. Polymers in the fluid tend to align with the streamlines under the action of the local shear while the entropic forces acting to its historical position cause an extra tension in the direction of the flow [50]. The viscosity effect is important for a small scale problem such as a biological simulation where deformation may be caused by the extra polymeric stresses acting on cells.

In both computational and physical perspectives, we concern the stress for a viscoelastic fluid-structure interaction problem. It is well known that for a viscoelastic fluid, the stress boundary condition on the inflow must be imposed to ensure the well-posedness of the equation. In this chapter, we simulated the viscoelastic FSI problem by both monolithic and partitioned algorithms, and investigate how the stress boundary condition affects the system.
4.1 Models Equations and Framework

The FSI system we consider in this chapter consists of a 2D viscoelastic fluid and an isotropic linear elastic structure as shown in Figure 4.1. The fluid equations are given by incompressible viscoelastic fluid equations

\[
\sigma + \lambda \left( \frac{D\sigma}{Dt} + \mathbf{u} \cdot \nabla \sigma + g_\beta(\sigma, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}) = 0 \quad \text{in } \Omega_f, \quad (4.1.1)
\]

\[
\rho_f \left( \frac{D\mathbf{u}}{Dt} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \sigma - 2(1 - \alpha) \nabla \cdot D(\mathbf{u}) + \nabla p = f_f \quad \text{in } \Omega_f, \quad (4.1.2)
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_f. \quad (4.1.3)
\]

And the wall is described by an isotropic linear elastic structure as

\[
\rho_s \frac{D^2 \eta}{Dt^2} - 2\nu_s \nabla \cdot D(\eta) - \lambda \nabla (\nabla \cdot \eta) = f_s \quad \text{in } \Omega_s, \quad (4.1.4)
\]

where the notations are exactly the same as what we used in Chapters 2 and 3.
Initial and boundary conditions for \( u \) and \( \sigma \) are given as follows:

\[
\begin{align*}
    u(x,0) &= u_0, \quad \sigma(x,0) = \sigma_0 \quad \text{in } \Omega_f^t, \\
    \eta(x,0) &= \eta_0, \quad \eta_t(x,0) = \dot{\eta}_0 \quad \text{in } \Omega_s^t, \\
    u &= u_D \quad \text{on } \Gamma_{D}^f, \\
    \sigma &= \sigma_D \quad \text{on } \Gamma_{inlet}^f, \\
    2\nu_s D(\eta) \mathbf{n}_s + \bar{\lambda} (\nabla \cdot \eta) \mathbf{n}_s &= 0 \quad \text{on } \Gamma_N^s, \\
    \eta &= 0 \quad \text{on } \Gamma_D^s.
\end{align*}
\] (4.1.5-4.1.11)

As presented in the previous chapter, the matching conditions satisfying velocity and the stress continuity on the interface are:

\[
\begin{align*}
    \frac{\partial \eta}{\partial t} &= u \quad \text{on } \Gamma_I, \\
    (\sigma + 2(1-\alpha)D(u) - pI)\mathbf{n}_f &= -(2\nu_s D(\eta) + \bar{\lambda} (\nabla \cdot \eta)) \mathbf{n}_s \quad \text{on } \Gamma_I, \\
    \rho_s \frac{\partial^2 \eta}{\partial t^2} - 2\nu_s \nabla \cdot D(\eta) - \bar{\lambda} \nabla \cdot (\nabla \cdot \eta) &= f_s \quad \text{in } \Omega_s.
\end{align*}
\] (4.1.12-4.1.17)

and the ALE formulation for (4.1.1)-(4.1.4) are

\[
\begin{align*}
    \sigma + \lambda \left( \left. \frac{\partial \sigma}{\partial t} \right|_y + (u - z) \cdot \nabla_x \sigma + g_c(\sigma, \nabla_x u) \right) - 2\alpha D_x(u) &= 0 \quad \text{in } \Omega_f^t, \\
    Re \left( \left. \frac{\partial u}{\partial t} \right|_y + (u - z) \cdot \nabla_x u \right) - \nabla_x \cdot \sigma - 2(1-\alpha) \nabla_x \cdot D_x(u) + \nabla_x p &= f_f \quad \text{in } \Omega_f^t, \\
    \nabla_x \cdot u &= 0 \quad \text{in } \Omega_f^t, \\
    \rho_s \frac{\partial^2 \eta}{\partial t^2} - 2\nu_s \nabla \cdot D(\eta) - \bar{\lambda} \nabla \cdot (\nabla \cdot \eta) &= f_s \quad \text{in } \Omega_s.
\end{align*}
\]
With the chosen function spaces as

$$U_0 := \{ v \in H^1(\Omega_f^0) : \nabla v = 0 \text{ on } \Gamma^f_{D} \cup \Gamma^f_{D,0} \},$$

$$Q_0 := L^2(\Omega_f^0),$$

$$\Sigma_0 := \{ \tau \in L^2(\Omega) : \tau_{ij} = \tau_{ji} \},$$

$$U_t := \{ v : \Omega_f^t \times [0,T] \rightarrow \mathbb{R}^2, v = \nabla \circ \Psi^{-1}_t \text{ for } v \in U_0 \},$$

$$Q_t := \{ q : \Omega_f^t \times [0,T] \rightarrow \mathbb{R}, q = \beta \circ \Psi^{-1}_t \text{ for } p \in Q_0 \},$$

$$\Sigma_t := \{ \tau : \Omega(t) \times [0,T] \rightarrow \mathbb{R}^{2 \times 2}, \tau = \tau \circ \Psi^{-1}_t \text{ for } \tau \in \Sigma_0 \},$$

$$S := \{ \xi \in H^1(\Omega_s) : \xi = 0 \text{ on } \Gamma_s^D \},$$

and Reynolds transportation formula (3.4.1) we obtain the conservative variational formulation

$$(\sigma, \tau)_{\Omega^t_f} + \lambda \frac{d}{dt} (\sigma, \tau)_{\Omega^t_f} + \lambda (-\sigma \cdot \nabla \cdot \nabla + ((u - z) \cdot \nabla)\sigma + \beta\sigma, \nabla u)_{\Omega^t_f}$$

$$-2\alpha (D(u), \tau)_{\Omega^t_f} = 0 \quad \forall \tau \in \Sigma_t,$$  \hspace{1cm} (4.1.18)

$$Re \frac{d}{dt} (u, v)_{\Omega^t_f} + Re (u(\nabla \cdot z) + (u - z) \cdot \nabla u, v)_{\Omega^t_f} + (\sigma, D(v))_{\Omega^t_f} + 2(1-\alpha)(D(u), D(v))_{\Omega^t_f}$$

$$-(p, \nabla \cdot v)_{\Omega^t_f} = (f, v)_{\Omega^t_f} + ((\sigma + 2(1-\alpha)D(u) - pI) \cdot n, v)_{\Gamma^t_f} \quad \forall v \in U_t,$$  \hspace{1cm} (4.1.19)

$$(q, \nabla \cdot u)_{\Omega^t_f} = 0 \quad \forall q \in Q_t,$$  \hspace{1cm} (4.1.20)

$$\rho_s (\frac{\partial^2 \eta}{\partial t^2}, \xi) + 2\nu_s (D(\eta), D(\xi)) + \tilde{\lambda}(\nabla \cdot \eta, \nabla \cdot \xi)$$

$$= (f_s, \xi) - ((\sigma + 2(1-\alpha)D(u) - p) \cdot n, \xi \circ \Psi^{-1}_t)_{\Gamma^t_f} \quad \forall \xi \in S.$$  \hspace{1cm} (4.1.21)

Using the test function space

$$\bar{U}_t \times \bar{S} := \{ (v, \xi) \in U_t \times S : v |_{\Gamma^t_f} = \left(\frac{\partial \xi}{\partial t} \circ \Psi^{-1}_t\right) |_{\Gamma^t_f} \},$$  \hspace{1cm} (4.1.22)
the monolithic scheme of the weak formulation is written as

\[
\rho_s \frac{\partial^2 \eta}{\partial t^2} \xi_{\Omega^s} + 2\nu_s (D(\eta), D(\xi))_{\Omega^s} + \lambda (\nabla \cdot \eta, \nabla \cdot \xi)_{\Omega^s} \\
+ \lambda \left[ \frac{d}{dt} (\sigma, \tau)_{\Omega^f} - (\sigma(\nabla \cdot z), \tau)_{\Omega^f} + \left( (\nabla \cdot z) \cdot \nabla \sigma, \tau \right)_{\Omega^f} \right] \\
+ Re \left[ \frac{d}{dt} (u, v)_{\Omega^f} - (u(\nabla \cdot z), v)_{\Omega^f} + ((u - z) \cdot \nabla u, v)_{\Omega^f} \right] \\
+ A((\sigma, u), (\tau, v))_{\Omega^f} - (p, \nabla \cdot v)_{\Omega^f} + (q, \nabla \cdot v)_{\Omega^f} \\
= (f_f, v)_{\Omega^f} + (f_s, \xi)_{\Omega^s} \quad \forall (v, q, \tau, \xi) \in \tilde{U}_t \times Q_t \times \Sigma_t \times \tilde{S}, \quad (4.1.23)
\]

where \(A((u, \sigma), (v, \tau)) := (\sigma, \tau) + \lambda (g_\beta (\sigma, \nabla u), \tau) - 2\alpha (D(u), \tau) + (\sigma, D(v)) + 2(1-\alpha) (D(u), D(v)). \)

We notice that there is no issue on a stress boundary condition on the interface in the monolithic scheme. However, we will need a boundary condition on the inflow part of the interface when a partitioned scheme for (4.1.18)-(4.1.21) is considered. The semidiscrete formulation for the structure is given in Chapter 3, while a streamline upwind Petrov-Galerkin (SUPG) method is applied to stabilize the constitutive equation for the fluid. The fully discretized formulation is then obtained by the backward Euler method.

### 4.2 Partitioned Scheme

In order to investigate the stress boundary condition on interface, we applied a partitioned algorithm where the FSI problem could be split into two separate sub-problems. The two sub-problems are coupled through the conditions on the interface. The most commonly used partitioned transmission condition is the Dirichlet-Neumann algorithm, where the fluid subproblem is solved with the Dirichlet boundary condition

\[
u = \frac{\partial \eta}{\partial t} \quad \text{on} \quad \Gamma_{I_t},
\]

and the structure subproblem is solved with a Neumann boundary condition

\[
(2\nu_s D(\eta) + \lambda (\nabla \cdot \eta)) n_s = -(\sigma + 2(1-\alpha)D(u) - p) n_f \quad \text{on} \quad \Gamma_{I_t}.
\]
These are also the transmission conditions we applied in the previous chapter which comes directly from the velocity and stress continuity matching conditions. This scheme is a standard implementation for FSI partitioned procedures, but it may require a large number of iterations to converge when fluid and structure densities are comparable (added mass effect).

In the case where the fluid and the structure have the same spatial dimension, a linear combination of Dirichlet and Neumann conditions (Robin condition) could be used as the transmission condition, and the good convergence properties of Robin condition were presented in [3].

**Implicit Algorithm**

For \( n=0,1,... \) do until arrive at the final time step

Initial guess of \( \eta_0^{n+1} \)

for \( k=0,1,... \) do until convergence

1. Solve the fluid subproblem with Robin boundary condition

\[
\sigma^{n+1}_{k+1} + \lambda \left( \frac{\partial \sigma^{n+1}_{k+1}}{\partial t} |_y + (u^{n+1}_{k+1} - z^{n+1}_{k}) \cdot \nabla \sigma^{n+1}_{k+1} + g_o (\sigma^{n+1}_{k+1}, \nabla u^{n+1}_{k+1}) \right) \\
-2\alpha D_x (u^{n+1}_{k+1}) = 0 \quad \text{in } \Omega^n_{f_{k+1}} \\
Re \left( \frac{\partial u^{n+1}_{k+1}}{\partial t} |_y + (u^{n+1}_{k+1} - z^{n+1}_{k}) \cdot \nabla u^{n+1}_{k+1} \right) \\
- \nabla_x \cdot \sigma^{n+1}_{k+1} - 2(1-\alpha) \nabla_x \cdot D_x (u^{n+1}_{k+1}) + \nabla_x p^{n+1}_{k+1} = f_f \quad \text{in } \Omega^n_{f_{k+1}} \\
\nabla_x \cdot u^{n+1}_{k+1} = 0 \quad \text{in } \Omega^n_{f_{k+1}} \\
w_f u^{n+1}_{k+1} + (\sigma^{n+1}_{k+1} + 2(1-\alpha)D(u^{n+1}_{k+1}) - p^{n+1}_{k+1}) \cdot n_f \\
= w_f \frac{\partial \eta^{n+1}_{k+1}}{\partial t} - (2\nu_s D(\eta^{n+1}_{k+1}) + \lambda(\nabla \cdot \eta^{n+1}_{k+1})) \cdot n_s \quad \text{on } \Gamma^n_{I_{k+1}}.
\]

2. Solve the structure subproblem with Robin boundary condition

\[
\rho_s \frac{\partial^2 \eta^{n+1}_{k+1}}{\partial t^2} - 2\nu_s \nabla \cdot D(\eta^{n+1}_{k+1}) - \lambda \nabla (\nabla \cdot \eta^{n+1}_{k+1}) = f_s \quad \text{in } \Omega^n \\
w_s \frac{\eta^{n+1}_{k+1}}{\Delta t} + (2\nu_s D(\eta^{n+1}_{k+1}) + \lambda(\nabla \cdot \eta^{n+1}_{k+1})) \cdot n_s \\
= w_s \frac{\partial \eta^{n+1}_{k+1}}{\partial t} + w_s u^{n+1}_{k+1} - (\sigma^{n+1}_{k+1} + 2(1-\alpha)D(u^{n+1}_{k+1}) - p^{n+1}_{k+1}) \cdot n_s \quad \text{on } \Gamma^n_{I_{k+1}}.
\]

3. Update the \( z^{n+1}_{k+1}, \Omega^n_{f_{k+1}}, \Gamma^n_{I_{k+1}} \) using \( \eta^{n+1}_{k+1} \).
end for

End for

The algorithm presented above applies the general Robin-Robin boundary condition, where $w_f, w_s$ are the combination parameters for the transmission conditions. With an appropriate choice of $w_f$ and $w_s$, we could obtain all type of mixed schemes like Dirichlet-Neumann, Dirichlet-Robin, Robin-Neumann etc.

4.3 Numerical Test

4.3.1 Code validation

As shown above, solving the viscoelastic fluid sub-problem is part of the algorithm which requires an appropriate stress boundary condition for the inflow part of boundary. For our setting, the inflow part consists of both the inlet $\Gamma_{inlet}$ and the inflow part (when $\mathbf{u} \cdot \mathbf{n} < 0$) on the deformable interface (Figure 4.2).

![Figure 4.2: Viscoelastic fluid boundary](image)

Three different strategies regarding stress boundary condition are considered for numerical approximations

1. Monolithic Scheme: No stress boundary condition is required on the inflow part of the interface since the interface terms are canceled in the monolithic formulation;

2. Decoupled Scheme with do-nothing stress boundary condition: No stress boundary condition is given on the inflow part of the interface;

3. Decoupled Scheme with Dirichlet-type stress boundary condition: At $(k+1)$-th subiteration, the inflow stress boundary condition is approximated by $\mathbf{u}^k$. 

80
For any fixed time, the constitutive equation (4.1.1) could be reduced to
\[
\sigma + \lambda (u \cdot \nabla \sigma + g_\beta(\sigma, \nabla u)) - 2\alpha D(u) = 0 \quad \text{in } \Omega_t^f \tag{4.3.1}
\]

The boundary condition for stress could be approximated by the steady-state constitutive equation (4.3.1) and the velocity conditions
\[
\sigma_{11} = -\alpha \beta u_{1,y} \quad \text{(4.3.2)}
\]
\[
\sigma_{12} = -\alpha u_{1,y} \quad \text{(4.3.3)}
\]
\[
\sigma_{22} = -\alpha (\beta - 1) u_{1,y} \quad \text{(4.3.4)}
\]
where \(u_{1,y} = \frac{\partial u_1}{\partial y}\).

For numerical experiments, we start with the convergence test for both monolithic and partitioned algorithms. These non-physical convergence tests are performed with the same parameters as we presented in previous chapters.

<table>
<thead>
<tr>
<th>h</th>
<th>|u^n - u^{true}|_{L^2}</th>
<th>Rate</th>
<th>|u^n - u^{true}|_{H^1}</th>
<th>Rate</th>
<th>|\sigma^n - \sigma^{true}|_{L^2}</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
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<td>4.739e-004</td>
<td>-</td>
<td>5.016e-003</td>
<td>-</td>
<td>4.957e-003</td>
<td>-</td>
</tr>
<tr>
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<tr>
<td>1/64</td>
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<td>3.06</td>
<td>9.852e-005</td>
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<td>1.040e-004</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Table 4.1: Fluid convergence result for monolithic viscoelastic FSI

<table>
<thead>
<tr>
<th>h</th>
<th>|u^n - u^{true}|_{H^1}</th>
<th>Rate</th>
<th>|\sigma^n - \sigma^{true}|_{L^2}</th>
<th>Rate</th>
</tr>
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<td>-</td>
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<td>1.684e-003</td>
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</tr>
<tr>
<td>1/32</td>
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<td>1.92</td>
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</tr>
<tr>
<td>1/64</td>
<td>1.378e-004</td>
<td>1.51</td>
<td>1.341e-004</td>
<td>1.64</td>
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</tbody>
</table>

Table 4.2: Convergence result for partitioned scheme without stress boundary condition

<table>
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<tr>
<th>h</th>
<th>|u^n - u^{true}|_{H^1}</th>
<th>Rate</th>
<th>|\sigma^n - \sigma^{true}|_{L^2}</th>
<th>Rate</th>
</tr>
</thead>
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<td>5.037e-003</td>
<td>-</td>
</tr>
<tr>
<td>1/16</td>
<td>1.057e-003</td>
<td>2.38</td>
<td>1.143e-003</td>
<td>2.14</td>
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<tr>
<td>1/32</td>
<td>2.698e-004</td>
<td>1.97</td>
<td>3.063e-004</td>
<td>1.90</td>
</tr>
<tr>
<td>1/64</td>
<td>6.561e-005</td>
<td>2.04</td>
<td>7.397e-005</td>
<td>2.05</td>
</tr>
</tbody>
</table>

Table 4.3: Convergence result for partitioned scheme with Dirichlet stress boundary condition
In this test, we obtained the expected convergence rates by both the monolithic and decoupled schemes. However, we observed that the optimal convergence rates are lost if no stress boundary condition is imposed for the decoupling scheme. These results motivate us to investigate the effect of stress boundary conditions in a physical setting.

4.3.2 Hemodynamic simulation

The experiments presented in this part are aimed at simulating the problem in Section 3.5.2 for a viscoelastic fluid and comparing results for different stress boundary conditions. The first test is performed with a relatively larger Weissenburg number $\lambda = 0.9$, where the viscoelastic behavior is more significant. Figure 4.3 presents the pressure profile (with Dirichlet stress boundary condition on inflow) at a sequence of increasing times $t=0.02, 0.04, 0.06, 0.08$ s.
Figure 4.3: Pressure profile on the fluid domain with $\lambda = 0.9$

The corresponding vertical structure displacement (scaled by 10) at different times are compared in Figure 4.4- Figure 4.7. Since the monolithic scheme does not have the stress boundary issue on the interface, we compared the partition scheme results (with or without Dirichlet stress boundary conditions) with the one obtained from the monolithic scheme. The difference among the three cases is obvious from the graphs, and we observed that the partition scheme result is improved with Dirichlet stress boundary condition.
Figure 4.4: Structure displacement at $t=0.02$

Figure 4.5: Structure displacement at $t=0.04$

Figure 4.6: Structure displacement at $t=0.06$
In order to investigate the effect of the stress boundary condition for a fluid close to being Newtonian, we do similar experiments for a smaller Weissenberg number $\lambda = 0.06$ which is used to simulate blood flow in [40, 35]. The corresponding pressure profile is presented in Figure 4.8, where a similar pattern to Figure 4.3 is observed. We also notice that the pressure decreases with the smaller Weissenberg number $\lambda$. The vertical structure deformations with $\lambda_1 = 0.9$ and $\lambda_2 = 0.06$ are compared in Figure (4.9).
Figure 4.8: Pressure profile on the fluid domain with $\lambda = 0.06$

The comparison for structure deformations under different schemes when $\lambda = 0.06$ is presented in Figure 4.10. Although the deformation difference is still noticeable, the error is not significant due to the small Weissenberg number.
Figure 4.9: Structure displacement at t=0.02,0.04,0.06,0.08s with $\lambda_1 = 0.9$, $\lambda_2 = 0.06$
Figure 4.10: Structure displacement at t=0.02,0.04,0.06,0.08s with $\lambda = 0.06$
4.4 Conclusion

In this chapter, we considered both monolithic and partitioned algorithms for a viscoelastic fluid-structure interaction problem. For the partition algorithm, a Robin-Robin transmission condition is applied for coupling two subproblems, where two different approaches for stress boundary conditions are considered on the inflow part of the moving fluid boundary. Numerical tests were performed for both monolithic and partitioned schemes to investigate the effect of the stress boundary condition for the viscoelastic fluid-structure interaction problem, and the partition scheme without stress boundary failed to obtain the optimal convergence rate in the convergence test. In the blood flow simulation, the partition algorithm with the stress boundary condition yielded a more accurate numerical solution, especially when the viscoelastic property of the fluid was significant.
Bibliography


