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Tight Representations of Specially-Structured 0-1 Linear, Quadratic, and Polynomial Programs

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Abstract

This research derives improved mathematical representations for various expressions of binary variables. There are three main efforts, consisting of: the Target Visitation Problem (TVP), the Quadratic Linear Ordering Problem (QLOP), and a special family of monomials of binary variables. For the first two efforts, which are known in the literature to be (mixed) 0-1 programs, we derive new families of valid inequalities that tighten existing forms and lead to superior performance. We give computational experience to show the advantages of our approaches. For the third effort, we provide an explicit convex hull representation for special forms of monomials of binary variables that arise in 0-1 polynomial programs; this convex hull form generalizes a classical linearization result.

Relative to the first effort, the TVP is a hybrid between the well-known Traveling Salesman Problem (TSP) and the Linear Ordering Problem (LOP). Both the TSP and LOP seek an "optimal" permutation of a collection of objects, but they differ in the manner in which the objective is defined. For each distinct pair of objects, the TSP incurs a cost when the first object immediately precedes the second, while the LOP realizes a reward when the first object anywhere precedes the second. The TVP incurs both the cost and reward. Our contribution is a new family of valid linear inequalities that are derived by exploiting the structure of the decision variables. This family is a strengthened version of the well-known Miller-Tucker-Zemlin inequalities for the TSP. The inequalities are derived via a two-step approach; the first step uses a conditional logic argument to compute valid quadratic inequalities, and the second step suitably surrogates these inequalities to eliminate the quadratic terms. Computational experience shows the advantage of using these inequalities within a commercial solver. While the emphasis is on the TVP, we show how this logic

is applicable to other families of problems.

For the second effort, the QLOP is a generalization of the LOP that, in addition to having linear rewards, also realizes rewards for products of pairwise orderings. Such products naturally lead to quadratic terms in the objective function. We exploit the problem structure to obtain three key results. First, we provide a lifting theorem which establishes that every facet-defining inequality (facet) for the convex hull of solutions on n objects is a facet for any collection of more than n objects. Such a result is known for the LOP but the complex structure of the QLOP necessitates a more involved argument. Second, we give an explicit listing of all the facets for the convex hull of the size $n = 3$ case; this listing allows for a mixed 0-1 linear form of the general size-n QLOP that uses only half the number of inequalities in the same variable space as recent work, while maintaining the same linear programming strength. Third, we give all the facets for the convex hull of the size $n = 4$ case. The facets are characterized into five general families. We provide computational experience that shows the merits of reduced numbers of constraints in a solution strategy, and we also examine the tightening effects of each of the five families.

For the third effort, we give the convex hull of any monomial in n binary variables $\mathbf{x}$ that has the characteristic that each variable is bounded above by an auxiliary binary variable y. Without y, the convex hull is known to be obtained by replacing the monomial with a continuous variable, and then enforcing $(n + 2)$ linear inequalities to ensure that the new variable equals the monomial value at all binary realizations. In the presence of y, we separately consider the cases having $n = 2$ and $n \geq 3$. We show that when $n = 2$, an implementation of a special-structure reformulation-linearization-technique (RLT) gives the convex hull, while for $n \geq 3$ we employ a different level-1 RLT implementation, in concert with the known convex hull result, to accomplish the same task. The argument for $n \geq 3$ allows extensions to convex hulls of various discrete and/or continuous sets including, for example, restricted forms of both special symmetric multilinear functions over box constraints and the Boolean Quadric Polytope.

Dedication

I dedicate this dissertation to my Heavenly Father who has been with me every step of this journey and has taught me to live by faith one day at a time. May this dissertation and the avenues it leads to, bring glory to God.

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Chapter 1

Introduction

The field of binary optimization is concerned with finding optimal solutions to decision problems in which the variables are restricted to realize values of 0 and 1. Such 0-1 problems are challenging to solve because of the exponential number of possible solutions. This challenge is commonly addressed through the construction of polyhedral outer-approximations of the associated discrete solution sets; in this manner, the difficult binary programs can be approximated by much simpler linear problems. The relaxations provide bounds that are typically used within enumerative strategies to implicitly eliminate from consideration inferior and/or infeasible binary realizations.

Considerable attention has historically been devoted to obtaining (mixed) integer linear representations whose continuous relaxations closely approximate the convex hull of feasible solutions. In this manner, tight bounds of the optimal integer solution are available. However, tight representations can be very challenging to construct. Various general techniques are available in the literature, including such approaches as reformulations, liftings, and cutting planes. Successful techniques are often problem-dependent, relying on the exploitation of special problem structures.

A key consideration in the derivation of attractive representations is the tradeoff between problem size and strength. In general, the more accurate the linear programming relaxation, the more effectively the bounds can be used to remove nonoptimal solutions. However, accuracy is often a function of problem size, with larger forms tending to produce stronger bounds. Unfortunately, the larger forms are more computationally expensive to solve. The challenge is to strike a balance between problem size and strength so that the overall effort of solving the binary program is minimized.

This dissertation contributes improved mixed 0-1 linear representations of three different 0-1 optimization problems: the Target Visitation Problem, the Quadratic Linear Ordering Problem, and special monomials of binary variables. All three problems are expressible in terms of binary variables, with the challenge being to obtain superior representations. The first two problems are naturally modelled as (mixed) 0-1 linear programs, and the third addresses monomials of 0-1 variables. For the first two problems, new families of valid inequalities are derived. For both, computational experience is presented that examines and evaluates the tradeoff between problem size and strength. For the third problem, a convex hull representation is given that generalizes known works for nonlinear expressions of binary variables.

Relative to the first contribution, we devise an improved 0-1 linear representation of the Target Visitation Problem (TVP). The key ingredient is a strengthened version of the Miller-Tucker-Zemlin (MTZ) subtour elimination inequalities that serve to tighten the linear programming relaxation and improve the efficiency of solving the integer program. The TVP is a hybrid between the Traveling Salesman Problem (TSP) and the Linear Ordering Problem (LOP). Common to both the TSP and LOP is the goal of constructing an optimal permutation of a collection of n objects, but they differ in the manner in which costs are incurred. For each distinct pair of objects i and j, the TSP incurs a cost, say c_{ij}, when object i *immediately* precedes object j, while the LOP incurs a reward, say r_{ij}, when object i *anywhere* precedes object j. The TVP incurs both the costs and rewards, simultaneously seeking the max-reward directed path and the least-cost directed cycle amongst n nodes. The objective function of the TVP is precisely that of the TSP joined with that of the LOP. The strengthened MTZ inequalities are obtained via a conditional logic approach that strategically computes nonnegative quadratic expressions so that surrogates eliminate the quadratic terms. The resulting inequalities, while designed for the TVP, are also valid for the TSP and LOP. Computational experience shows the merits of these linear inequalities for the TVP, with mixed success for the TSP and LOP. Further extensions of the conditional logic strategy on the MTZ inequalities give way to new strengthened versions for a quadratic variant of the TSP which allows for "quadratic" objective coefficients to record costs associated with the salesman traveling along *pairs* of consecutive arcs.

The second contribution deals with a quadratic variant of the LOP. The Quadratic Linear Ordering Problem (QLOP) is a generalization of the LOP that seeks an optimal permutation of n objects but also incurs quadratic rewards (or costs) based on products of pairwise orderings. Depending on the scenario, the LOP and QLOP can either be modeled to seek a max-reward permutation, as in the TVP, or a least-cost permutation, with cost coefficients as negative rewards. To be consistent with the literature, here we consider the QLOP as a min-cost problem. We obtain three results for the QLOP, all in a suitably-defined, linearized-variable space. First, we provide a "lifting theorem" that shows that every facet-defining inequality (facet) for the convex hull of solutions for n objects remains a facet for the convex hull for any collection of $s \geq n$ objects. Such a result is known for the LOP, but the complex structure of the QLOP necessitates a more involved argument. Second, we obtain the convex hull representation for $n = 3$ objects. This representation allows for a mixed 0-1 linear form of the general size-n QLOP that uses only half the number of inequalities in the same variable space as recent work, while maintaining the same linear programming relaxation strength. Third, we provide the convex hull representation for $n = 4$ objects, expressed in terms of five families of facets. To assess the computational merits of the latter two results, we provide experience to show the advantage of our reduced numbers of constraints in a solution strategy, and the tightening effects of each of the five families of facets.

All four problems—the TSP, LOP, QLOP, and TVP—are NP-hard and have applications in a variety of fields including military and civilian domains. Applications of the TSP include vehicle routing, computer wiring, scheduling, drilling operations, and the Very Large-Scale Integration placement problem [25, 27]. The QTSP has applications in biology, robotics, and bioinformatics [12, 24]. Applications of the LOP and QLOP include weighing of priorities, triangulation of input-output matrices, breaking sports ties, player rankings, aggregation of individual preferences, and scheduling problems [19, 26]. Popular applications of the TVP involve unmanned aerial vehicles tasked for surveillance, attack, search and rescue, or delivery of supplies [21]. An overarching research emphasis here is to devise new formulations of these important, popular optimization problems that provide superior computational results in terms of solution time and strength of the continuous relaxation.

The third contribution gives the convex hull representation of a monomial of binary variables where each variable is also upper-bounded by an auxiliary binary variable. Numerous strategies exist for linearizing polynomial expressions of binary variables. For a monomial of n such variables $\boldsymbol{x}$, a classical approach replaces the monomial with a continuous variable, and then enforces $(n+2)$ linear inequalities to ensure that the new variable equals the monomial value at all binary realizations. This approach was shown, together with the restrictions $\boldsymbol{x} \leq \mathbf{1}$, to give the convex hull of the corresponding set of 2^n points in $\mathbb{R}^{n+1}$ that have the new variable equal to the monomial value. We consider a simple, yet important, generalization where each variable is also upper-bounded by an auxiliary binary variable. When this auxiliary variable is fixed to 1, the classical approach results. For $n = 2$, a direct implementation of a specially-structured RLT gives the convex hull of this generalization. A small example tied to an application of a Channel Assignement Problem is explored to illustrate strength. For $n \geq 3$, a different level-1 RLT implementation is used. In fact, the approach used for $n \geq 3$ allows us to readily obtain the convex hulls of various nonlinear discrete and/or continuous sets including, for example, restricted forms of the Boolean Quadric Polytope and special multilinear functions over box constraints.

The contributions of this research are grouped into three main emphases, with one emphasis found in each of Chapters 2–4.

The focus of Chapter 2 is on the TVP with conditional logic based explorations and strengthenings of the MTZ inequalities, which also relate to the TSP, LOP, and can be extended for the Quadratic Traveling Salesman Problem (QTSP). The organization is as follows. Section 2.2 provides a literature review, presents a known, standard formulation of the TVP, and discusses select known TSP results, all of which serve as a foundation for the research contributions that follow. Section 2.3 illustrates a conditional logic strategy used to obtain tightened versions of the well-known Miller-Tucker-Zemlin subtour elimination constraints. Section 2.4 incorporates these new cuts into newly posed formulations of the TVP and presents computational experience. Section 2.5 closes the chapter with final remarks and insight for future research.

Chapter 3 focuses on the QLOP and is organized as follows. Section 3.2 states and proves the "lifting theorem" for the QLOP. Section 3.3 presents the convex hull representation of the QLOP for size $n = 3$ and explains the conciseness of our form compared to recent work. Section 3.4 presents the convex hull representation of the QLOP for size $n = 4$ and classifies the facets into five families. Section 3.5 provides computational experience, observations, and remarks. The final section, Section 3.6, closes the chapter with conclusions and comments on future research.

Chapter 4 focuses on monomials of binary variables where each variable is upper-bounded by an auxiliary binary variable. The organization is as follows. Section 4.2 gives the convex hull representation when $n = 2$, i.e. having quadratic product terms. The arguments are self-contained, relying only on the special-structure RLT (SSRLT). To clarify and show the usefulness, an application to a Channel Assignment Problem is given. Section 4.3 then considers the cases having $n \geq 3$. For these cases, the RLT is used in conjunction with [7]. Generalizations to symmetric multilinear functions and the Boolean Quadric Polytope are presented. Lastly, Section 4.4 gives conclusions and avenues for future research.

Chapters 2, 3, and 4 are written independently of each other so that the reader can choose to read any one work alone. Thus, each chapter contains its own introduction and conclusion. However, an encompassing bibliography is found at the end.

Chapter 2

Using Conditional Logic to Tighten Miller-Tucker-Zemlin Inequalities for the Target Visitation Problem

2.1 Introduction

The Target Visitation Problem (TVP) [21, 22, 23] is a combinatorial optimization problem that is a hybrid of the Traveling Salesman Problem (TSP) and the Linear Ordering Problem (LOP). Given a collection of n objects, all three problems desire an optimal permutation, with the difference being the manner in which the objectives are defined. For each pair of distinct objects i and j, the TSP incurs a cost, say c_{ij}, when object i *immediately* precedes object j whereas the LOP incurs a reward, say r_{ij}, when object i *anywhere* precedes object j. The TVP incurs both the costs and rewards, and arises in such instances as unmanned aerial vehicles tasked for surveillance, attack, search and rescue, or delivery of supplies in both military and civilian applications [21].

The TSP and LOP are known to be NP-hard, so that the more general TVP is also NP-hard. The challenge for each problem is the exponential number of permutations that must be considered. A distinguishing feature of the TVP is that, since the objective is a function of both the *immediate* and *anywhere* relative ordering of each pair of distinct objects i and j, the mathematical representation turns out to be larger than that of either the TSP or LOP. Specifically, for each such pair of objects, the TSP defines a binary variable, say x_{ij}, to represent whether object i immediately precedes object j, while the LOP defines a binary variable, say y_{ij}, to represent whether object i anywhere precedes object j. In order to record both the costs and rewards, the TVP uses both sets of variables. Consequently, the TVP can be considered as a generalization of both the TSP and LOP in the sense that it reduces to the first when the objective coefficients on the variables y_{ij} are all 0, and it reduces to the second when the objective coefficients on the variables x_{ij} are all 0.

The TSP, LOP, and TVP can be envisioned on a graph having n nodes, with each node corresponding to an object. The TSP and TVP seek optimal Hamiltonian circuits while the LOP seeks an optimal Hamiltonian path. In terms of this graph scenario, all three problems share the task, within a mathematical formulation, of eliminating subtours. Here, a subtour is a proper subset of two or more nodes that forms a Hamiltonian circuit. For the much-studied TSP, strategies have been devised for accomplishing this task [8, 32]. The "subtour elimination constraints" of [8] provide tight representations, but consist of an exponential number of inequalities. The MTZ inequalities [32], and their tightenings [9], are considerably fewer in number, being of order $O(n^2)$, but are known to yield weak relaxations. For the LOP, "3-dicycle" inequalities exist [19, 20, 30], whose number is of order $O(n^3)$. The 3-dicycle inequalities and tightenings thereof [43] have also been used [37] with mixed success for the TSP. These inequalities yield considerably tighter relaxations than the MTZ inequalities, but the relaxations are more expensive to solve. The challenge for all three problems is to develop formulations that have tight continuous relaxations which enable strong bounds to be obtained, but also so that the bounds are not prohibitively expensive to compute. The key is to balance the tradeoff between the strength of the bounds and the computational effort required to obtain them.

In this paper, we exploit the two sets of variables x_{ij} and y_{ij} found within the TVP to tighten the MTZ inequalities. Our approach consists of two steps. The first step uses conditional logic embedded within the reformulation-linearization technique [29, 41] to strategically compute pairwise products of linear restrictions that yield quadratic inequalities which are "conditionally" valid for the TVP. Simply put, the linear restrictions are not necessarily valid for the TVP, but the quadratic inequalities are. The second step suitably surrogates these quadratic inequalities to obtain the tightened linear MTZ inequalities. As a byproduct of this approach, we show that an application of the conditional logic arguments in terms of only the variables x_{ij} yields the tightenings of [9], and an application using only the variables y_{ij} yields a concise formulation for the LOP.

This paper is organized as follows. In the next section, we provide a brief literature review of formulations for each of the LOP, TSP, and TVP. This review serves as the basis for our motivating the improved MTZ inequalities. Section 3 then derives our strengthened cuts via the conditional logic and surrogation process. In Section 4, we use our cuts to construct new representations of the TVP, and provide computational experience. Of significance here is that our cuts tighten the continuous relaxations and expedite the solution process; this result is intriguing as the MTZ inequalities are known to be weak for the TSP. Section 5 closes with conclusions and avenues for future research.

2.2 Literature Review

In this section, we review known formulations for each of the LOP, TSP, and TVP. This review will form the basis of comparison for our contributions of Section 3. Throughout the remainder of the paper, we let n denote the number of objects and assume that all indices run from 1 to n unless otherwise stated.

2.2.1 Linear Ordering Problem

As described earlier, given a collection of n objects, the Linear Ordering Problem (LOP) seeks a permutation (ordering) of the objects that yields the maximum reward. Rewards are computed in terms of pairs of distinct objects i and j, with a reward r_{ij} realized when object i anywhere precedes object j in the permutation. The LOP is formulated [19, 20, 30] as follows.

LOP1: maximize $\sum_{i} \sum_{j \neq i} r_{ij} y_{ij}$

subject to

$$y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \forall\ (i,j,k) \text{ distinct} \tag{2.1}$$

$$y_{ij} + y_{ji} = 1 \qquad \forall\ (i,j),\ i < j \tag{2.2}$$

$$y_{ij} \text{ binary} \qquad \forall\ (i,j),\ i \neq j \tag{2.3}$$

The decision variables are the $2\binom{n}{2} = n(n-1)$ binary y_{ij} defined so that

$$y_{ij} = \begin{cases} 1 & \text{if object } i \textit{ anywhere} \text{ precedes object } j \text{ in the permutation} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ (i,j),\ i \neq j.$$

As mentioned above, for each $(i,j),\ i \neq j,\ r_{ij}$ denotes the reward for having object i anywhere precede object j. Inequalities (2.1) and equations (2.2) combine to enforce a valid permutation (eliminate subtours), with equations (2.2) stating that for each (i,j), either object i precedes object j or object j precedes object i. Inequalities (2.1) are known as “3-dicycle” inequalities and are of $O(n^3)$ in number. Of course, we have the option to remove half the variables found within LOP1 by substituting $y_{ji} = 1 - y_{ij}$ for all $(i,j),\ i < j$, throughout the objective function and inequalities (2.1), and by then removing equations (2.2).

2.2.2 Traveling Salesman Problem

As explained earlier, given n objects, the Traveling Salesman Problem (TSP) seeks a permutation of the objects that minimizes an overall cost. Given two distinct objects i and j, a cost is incurred when object i immediately precedes object j in the permutation. For any permutation, the last object is defined to immediately precede the first so that the associated cost is incurred. The TSP has been extensively studied, with a variety of mathematical representations found in the literature. A formulation of particular interest in this study is the following.

TSP1: minimize $\sum_{i} \sum_{j \neq i} c_{ij} x_{ij}$

subject to

$$\sum_{j \neq i} x_{ij} = 1 \qquad \forall\, i \tag{2.4}$$

$$\sum_{i \neq j} x_{ij} = 1 \qquad \forall\, j \tag{2.5}$$

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} \qquad \forall\, (i,j),\ i \neq j;\ i,j \geq 2 \tag{2.6}$$

$$x_{ij} \text{ binary} \qquad \forall\, (i,j),\ i \neq j \tag{2.7}$$

The decision variables are the $n(n-1)$ binary x_{ij} defined so that

$$x_{ij} = \begin{cases} 1 & \text{if object } i \textit{ immediately} \text{ precedes object } j \text{ in the permutation} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, (i,j),\ i \neq j,$$

together with $(n-1)$ "dummy variables" u_j, $j = 1, \ldots, n-1$,. For each (i,j),$i \neq j$, c_{ij} denotes the cost for having object i immediately precede object j. Equations (2.4) enforce that each object i has a single object following it, and equations (2.5) enforce that each object j has a single object preceding it. Inequalities (2.6) are the Miller-Tucker-Zemlin (MTZ) inequalities that serve to eliminate subtours. There exist $2\binom{(n-1)}{2} = (n-1)(n-2)$ such MTZ inequalities.

The TSP draws its name from the scenario of a salesman who must conduct a "tour" of n cities; the tour consists of the salesman beginning at some home city, visiting each city exactly once, and then returning to the home city. With this interpretation, each city takes the role of an object, and the cost of traveling from city i to city j is c_{ij}. The binary variables then have that $x_{ij} = 1$ if the salesman travels from city i to city j, and 0 otherwise. As the TSP forms a Hamiltonian cycle, it is usual to assume, without loss of generality, that city 1 is the first city in the tour (so that object 1 is listed first in the permutation).

The variables u_j in (2.6) have an interesting interpretation. Assuming that city 1 is the first city in the tour, for each $j \in \{2, \ldots, n\}$, the variable u_j can be thought of as the location of city j in the permutation, with $u_j = k$ indicating that city j is located in the k^{th} position after the first ($u_j = k$ indicates that city j is located in position $(k+1)$). Then we have

$$1 \leq u_j \leq n-1 \ \ \forall\, j \geq 2, \tag{2.8}$$

and these inequalities can be enforced as explicit restrictions.

The paper [9] strengthens inequalities (2.6) and (2.8). For the instances of the TSP in which $n \geq 4$, this paper strengthens (2.6) to

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} + (n-3)x_{ji} \ \ \forall\, (i,j),\ i \neq j;\ i,j \geq 2 \tag{2.9}$$

which is inequality (2.6) with the right side increased by the nonnegative quantity $(n-3)x_{ji}$. If $x_{ji} = 0$, then (2.9) reduces to (2.6) while if $x_{ji} = 1$, then $x_{ij} = 0$ and $u_j - u_i = -1$, verifying (2.9). The paper [9] strengthens (2.8) to

$$2 - x_{1j} + (n-3)x_{j1} \leq u_j \leq (n-2) + (3-n)x_{1j} + x_{j1} \ \ \forall\, j \geq 2, \tag{2.10}$$

which is inequality (2.8) with the left side increased by the nonnegative quantity $(1-x_{1j})+(n-3)x_{j1}$, and with the right side decreased by the nonnegative quantity $(n-3)x_{1j} + (1-x_{j1})$. There exist three possible feasible realizations of x_{1j} and x_{j1} : $x_{1j} = x_{j1} = 0$, $x_{1j} = (1-x_{j1}) = 0$, and $(1-x_{1j}) = x_{j1} = 0$. For these three possibilities, (2.10) gives us that $2 \leq u_j \leq n-2$, $u_j = (n-1)$, and $u_j = 1$, respectively, verifying the two inequalities. Therefore, we can replace (2.6) of TSP1 with (2.9) and (2.10) to obtain a tightened valid representation of the TSP.

Notably, the variables y_{ij} found within the LOP have also been used to eliminate subtours in the TSP [18, 37, 43]. For this modeling approach, the MTZ inequalities (2.6) of Problem TSP1 are replaced with those restrictions of (2.1) and (2.2) that have $i,j,k \geq 2$, together with the inequalities

$$x_{ij} \leq y_{ij} \ \forall\, (i,j),\ i \neq j;\ i,j \geq 2. \tag{2.11}$$

The resulting formulation is below.

TSP2: minimize $\sum_i \sum_{j \neq i} c_{ij} x_{ij}$

subject to

$$
\begin{aligned}
&(2.4), (2.5), (2.7), (2.11) && \\
&y_{ij} + y_{jk} + y_{ki} \leq 2 && \forall\ (i, j, k) \text{ distinct};\ i, j, k \geq 2 \qquad (2.12)\\
&y_{ij} + y_{ji} = 1 && \forall\ (i, j),\ 2 \leq i < j \qquad (2.13)
\end{aligned}
$$

In formulating TSP2, for each $j \in \{2, \ldots, n\}$, the variable y_{1j} can be envisioned as having been fixed to the value 1 and the value y_{j1} to the value 0 within (2.1) and (2.2) to reflect that city 1 is located first in the permutation. This fixing of variables allows for the simplification of these two families of restrictions to (2.12) and (2.13), respectively. In addition, the paper [37] explains that restrictions (2.3) are not needed, and that the elimination of subtours amongst cities 2 through n, as enforced by the constraints of TSP2, implies that no subtour can exist which includes city 1. (Note that (2.7), (2.11), and (2.13) combine to enforce that $0 \leq y_{ij} \leq 1\ \forall\ (i, j),\ i \neq j;\ i, j \geq 2$, so that these bounding restrictions are not included in TSP2, as they do not tighten the linear programming relaxation.)

Two properties of Problem TSP2 have been noted in the literature. First, computational experience of [37] showed that, while Problem TSP2 affords a much tighter lower bound on the optimal objective function value to the TSP than does TSP1, the additional $O(n^3)$ inequalities of (2.12), together with the additional variables y_{ij}, make the problem more expensive to solve. However, TSP2 performed well when the variables y_{ij} were used to model precedence constraints dictating that one object must be located prior to another in the permutation. Second, the paper [18] noted that the variables u_j of the MTZ inequalities (2.6), (2.8), (2.9), (2.10) can be related back to the variables y_{ij} as

$$u_j = \sum_{\substack{i \geq 2 \\ i \neq j}} y_{ij} + 1\ \forall\ j \geq 2. \qquad (2.14)$$

2.2.3 Target Visitation Problem

As discussed earlier, the Target Visitation Problem (TVP) is a combination of the LOP and TSP that incurs both a reward when an object i anywhere precedes a second object j, and a cost when it immediately precedes object j. As a result, the formulation uses both sets of variables y_{ij} and x_{ij}, and can be represented as a modification of Problem TSP2 [21, 22, 23] that adjusts the objective function to include the y_{ij} variables. The formulation follows.

$$\text{TVP0: maximize} \left\{ \sum_i \sum_{j \neq i} r_{ij} y_{ij} - \sum_i \sum_{j \neq i} c_{ij} x_{ij} : (2.4), (2.5), (2.7), (2.11)\text{–}(2.13) \right\}$$

The variables y_{ij} and x_{ij} retain the same definitions as in the LOP and TSP, respectively, as do the rewards r_{ij} and costs c_{ij}. The costs c_{ij} are negated in the objective function since the problem is a maximization. For the sake of consistency and simplicity, we carry through to the TVP the assumption from the TSP that object 1 is located first in the permutation. To accommodate this assumption, object 1 can be predefined as a "dummy" with all associated objective coefficients set to 0 so that $r_{1j} = r_{j1} = c_{1j} = c_{j1} = 0 \ \forall \ j \geq 2$.

The paper [43] derived a tightening of the inequalities (2.12) for the TSP when both sets of variables x_{ij} and y_{ij} are present. (Within [43], the variables y_{ij} were used to enforce a precedence relationship and did not appear in the objective function.) This tightening replaces (2.12) with the inequalities

$$y_{ij} + y_{jk} + y_{ki} + x_{ji} \leq 2 \ \ \forall \ (i, j, k) \text{ distinct; } \ i, j, k \geq 2, \tag{2.15}$$

which, for each such (i, j, k), increases the left side of (2.12) by the nonnegative quantity x_{ji}. Since inequalities (2.15) are also valid for the TVP, we can replace (2.12) with (2.15) within TVP0 to obtain TVP1 below.

$$\text{TVP1: maximize} \left\{ \sum_i \sum_{j \neq i} r_{ij} y_{ij} - \sum_i \sum_{j \neq i} c_{ij} x_{ij} : (2.4), (2.5), (2.7), (2.11), (2.13), (2.15) \right\}$$

2.3 Strengthened MTZ Inequalities via Conditional Logic

We now present the conditional logic framework for computing strengthened MTZ inequalities for the TVP. We exploit the three sets of variables x_{ij}, y_{ij}, and u_j found within Problem TVP1 via a two-step approach. In the first step, we use conditional logic [29, 41] to compute nonnegative quadratic restrictions that are valid for the TVP. These restrictions are obtained by multiplying binary expressions of the variables x_{ij} and y_{ij} by linear restrictions in the variables u_j. The restrictions in u_j are only "conditionally" valid for the TVP in the sense that their validity is conditioned on the multiplying binary expressions. In the second step, we surrogate the quadratic restrictions to obtain linear inequalities.

This conditional logic framework also allows us to obtain other inequalities for the TVP (as well as for the TSP and LOP). Included here are the strengthened inequalities (2.9) and (2.10) of [9] for the TSP, and a variation of (2.6) that uses y_{ij} instead of x_{ij}. This section is divided so that the first subsection gives our main result of the strengthened MTZ inequalities, and the second provides alternate applications of the conditional logic approach to obtain (2.9) and (2.10) and the tightened (2.6) in terms of y_{ij}.

2.3.1 Strengthened MTZ Inequalities in x_{ij}, y_{ij}, and u_j.

To begin, recall the interpretation of the variables u_j which states that $u_j = k$ indicates that object j is located in the k^{th} position after the first within the permutation. Then, for each $(i, j),\ i \neq j;\ i, j \geq 2$, we can make the following conditional statements.

$$\begin{array}{ll} \text{If } x_{ij} = 1 & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -1. \\ \text{If } y_{ij} - x_{ij} = 1 & \text{then } u_j - u_i \geq 2. \\ \text{If } y_{ji} - x_{ji} = 1 & \text{then } u_j - u_i \geq 2 - n. \end{array}$$

Each of the expressions x_{ij}, x_{ji}, $y_{ij} - x_{ij}$, and $y_{ji} - x_{ji}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the TVP. Consequently, we can multiply each such expression by its associated consequence. For each such product, if the first expression is 0, then the product trivially holds. Otherwise, the first expression is 1 and the product holds by the associated conditional statement. We then obtain the following four quadratic restrictions that are valid for the TVP for every $(i, j),\ i \neq j;\ i, j \geq 2$.

$$x_{ij}(u_j - u_i = 1), x_{ji}(u_j - u_i = -1), (y_{ij} - x_{ij})(u_j - u_i \geq 2), (y_{ji} - x_{ji})(u_j - u_i \geq 2 - n)$$

The surrogation process to obtain a linear inequality consists of simply adding the four quadratic restrictions together. The following inequalities result upon using (2.13) to let $y_{ji} = 1 - y_{ij}$.

$$u_j - u_i \geq (2 - n) + ny_{ij} - x_{ij} + (n - 3)x_{ji} \ \ \forall\ (i, j), i \neq j; i, j \geq 2. \tag{2.16}$$

These inequalities are our desired tightened version of (2.9). For each $(i, j),\ i \neq j;\ i, j \geq 2$, the inequality of (2.16) can be obtained by adding the nonnegative expression $n(y_{ij} - x_{ij})$ to the right side of the corresponding inequality in (2.9). Inequalities (2.16) recognize that if object i *anywhere* precedes object j, but does not *immediately* precede it, then object j is located at least two positions after object i.

Remark 2.3.1. *The conditional logic statements are consistent with our earlier assumption that some object (object 1) is located first in the permutation. Otherwise, if we had not fixed the first position and instead let* $u_j = 0$ *indicate that object* j *is located in position 1 then, for given* (i, j)*, the first statement will fail if object* j *is in position 1, and the second statement will fail if object* j *is in position* n*. The last two statements will also fail since the antecedents are no longer binary, as* (2.11) *no longer holds true if object* j *is in position 1.*

2.3.2 Alternate Conditional Logic Applications

It is insightful to observe that the conditional logic approach affords intuitive derivations of the strengthened inequalities (2.9) and (2.10) of [9] for the TSP, as well as a modified version of (2.6) that uses y_{ij} instead of x_{ij}. As we will see, this last result will give rise to the most concise known formulation of the LOP.

1. Relative to (2.9), for each $(i, j),\ i \neq j;\ i, j \geq 2$, we have the following conditional statements.

$$\begin{array}{ll} \text{If } x_{ij} = 1 & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -1. \\ \text{If } 1 - x_{ij} - x_{ji} = 1 & \text{then } u_j - u_i \geq 2 - n. \end{array}$$

   Each of the expressions x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji}$ found in the antecedents of these statements is binary for all feasible solutions to the TSP. Then multiplying each such expression by its associated consequence and summing the resulting three quadratic expressions gives us (2.9).

2. Relative to (2.10), for each $j \geq 2$, we have the following conditional statements.

$$\begin{array}{ll} \text{If } x_{1j} = 1 & \text{then } u_j = 1. \\ \text{If } x_{j1} = 1 & \text{then } u_j = n - 1. \\ \text{If } 1 - x_{1j} - x_{j1} = 1 & \text{then } 2 \leq u_j \leq n - 2. \end{array}$$

   Each of the expressions x_{1j}, x_{j1}, and $1 - x_{1j} - x_{j1}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the TSP. Then, upon separately considering the two inequalities in the third statement, we can multiply each such expression by its associated consequence and sum to obtain (2.10). Here, $2 \leq u_j$ and then $u_j \leq n - 2$ of the third statement give rise to the left and right inequalities of (2.10), respectively.

3. Relative to expressing (2.6) in terms of y_{ij} instead of x_{ij}, suppose that we extend the definition of u_j to have $u_1 = k$ indicate that object 1 is located in the k^{th} position after the first, and we let $u_j = 0$ indicate that object j is located first in the permutation. (This more general definition does not restrict object 1 to be located first in the permutation.) Then (2.8) changes to

$$0 \leq u_j \leq n - 1 \ \ \forall\, j \geq 1.$$

As a result, for each $(i,j),\ i \neq j$, we have the following conditional statements.

$$\begin{array}{ll} \text{If } y_{ij} = 1 & \text{then } u_j - u_i \geq 1. \\ \text{If } 1 - y_{ij} = 1 & \text{then } u_j - u_i \geq 1 - n. \end{array}$$

Each of the expressions y_{ij} and $1-y_{ij}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the LOP. Then we can multiply each expression by its associated consequence and sum to obtain

$$u_j - u_i \geq (1-n) + n y_{ij} \ \ \forall\, (i,j),\ i \neq j. \tag{2.17}$$

Remark 2.3.2. *Unlike the conditional statements of Section 3.1, the third application above holds true regardless of whether an object is assumed to have been a priori located in position 1. Thus, inequalities* (2.17) *are valid for all* $(i,j), i \neq j$, *while inequalities* (2.16) *require that* $i, j \geq 2$.

Remark 2.3.2 leads to the most concise known formulation of the LOP. This formulation is obtained by replacing the $n(n-1)(n-2)$ inequalities (2.1) of LOP1 with the $n(n-1)$ inequalities of (2.17). The result is as follows.

$$\text{LOP2: maximize} \left\{ \sum_i \sum_{j \neq i} r_{ij} y_{ij} : (2.2), (2.3), (2.17) \right\} \tag{2.18}$$

Problem LOP2 has $2\binom{n}{2}$ variables y_{ij}, n variables u_j, and $\binom{n}{2}$ inequalities (2.2) and $2\binom{n}{2}$ inequalities (2.17), in addition to the binary restrictions on y_{ij}. As noted earlier for Problem LOP1, we can remove half the variables y_{ij} by substituting $y_{ji} = 1 - y_{ij}$ for all $(i,j), i < j$, throughout the problem, and then remove equations (2.2). The resulting formulation will have $\binom{n}{2}$ variables y_{ij}, n variables u_j, and the $2\binom{n}{2}$ inequalities (2.17), along with the binary restrictions on the remaining variables y_{ij}. The below proposition formally establishes Problem LOP2 as a valid formulation of the LOP.

Proposition 2.3.3. *Problem LOP2 is a valid formulation of the LOP in that the feasible solutions to* (2.18) *correspond to the* n*-factorial permutations of the set of* n *objects.*

Proof. It is sufficient to show that the constraints of Problem LOP2 imply (2.1). Toward this end, consider any distinct (i, j, k), and sum the three inequalities of (2.17) given by

$$u_j - u_i \geq (1 - n) + ny_{ij},\ u_k - u_j \geq (1 - n) + ny_{jk},\ u_i - u_k \geq (1 - n) + ny_{ki}$$

to obtain $0 \geq 3 + n(y_{ij} + y_{jk} + y_{ki} - 3)$. Then (2.3) gives us that $y_{ij} + y_{jk} + y_{ki} \leq 2$ must hold true. □

2.3.3 Strengthened MTZ Inequalities for the QTSP

If we allow for pairwise products, $x_{ik}x_{kj}$, as in the Quadratic Traveling Salesman Problem (QTSP) [11, 12], we can extend this conditional logic and surrogation approach to obtain further strengthenings of the MTZ inequalities. The QTSP is a generalization of the TSP that incurs an additional cost, for each distinct (i, j, k), if the salesman travels immediately from city i to city k and then immediately from city k to city j. Such costs require quadratic expressions in the TSP decision variables of the form $x_{ik}x_{kj}$. With such expressions, the paper [11] recognized the inequalities

$$x_{ij} + x_{ji} + \sum_{k \neq i,j} (x_{ik}x_{kj} + x_{jk}x_{ki}) \leq 1 \ \forall\ (i, j),\ i < j, \tag{2.19}$$

where $x_{ij} + x_{ji} = 1$ and $\sum\limits_{k \neq i,j} (x_{ik}x_{kj} + x_{jk}x_{ki}) = 1$ represent that cities i and j are exactly one step apart, and exactly two steps apart, respectively, in the salesman's tour. In fact, the left side of (2.19), as well as the sum of any subset of the expressions, must therefore be binary for all solutions to the QTSP. For $n \geq 4$, and treating city 1 as the city that occurs first in the permutation, we can make the following conditional statements for each $(i, j), i \neq j; i, j \geq 2$.

If $\sum_{k \neq 1,i,j} x_{ik}x_{kj} = 1$	then $u_j - u_i = 2$.
If $\sum_{k \neq 1,i,j} x_{jk}x_{ki} = 1$	then $u_j - u_i = -2$.
If $x_{ij} = 1$	then $u_j - u_i = 1$.
If $x_{ji} = 1$	then $u_j - u_i = -1$.
If $1 - x_{ij} - x_{ji} - \sum_{k \neq 1,i,j} x_{ik}x_{kj} - \sum_{k \neq 1,i,j} x_{jk}x_{ki} = 1$	then $u_j - u_i \geq 2 - n$.

Observe here that for any such (i, j), the equation $\sum_{k \neq 1,i,j} x_{ik}x_{kj} = 1$ represents that the salesman travels from city i to city j in exactly two steps, but not through city 1, and that the equation $\sum_{k \neq 1,i,j} x_{jk}x_{ki} = 1$ represents that the salesman travels from city j to city i in exactly two steps, but not through city 1. (We do not consider any expressions of the form $x_{i1}x_{1j}$ since this product equaling to 1 implies that $u_i = n - 1$ and $u_j = 1$.) From the above-mentioned binary property of the left side of (2.19), each of the expressions $\sum_{k \neq 1,i,j} x_{ik}x_{kj}$, $\sum_{k \neq 1,i,j} x_{jk}x_{ki}$, x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji} - \sum_{k \neq 1,i,j} x_{ik}x_{kj} - \sum_{k \neq 1,i,j} x_{jk}x_{ki}$ found in the antecedents is binary for all solutions to the QTSP. Then we can multiply each such expression by its associated consequence and surrogate to obtain the following quadratic inequalities.

$$u_j - u_i \geq (2 - n) + (n - 1)x_{ij} + (n - 3)x_{ji} + n \sum_{k \neq 1,i,j} x_{ik}x_{kj} + (n - 4) \sum_{k \neq 1,i,j} x_{jk}x_{ki} \quad \forall\, (i, j),\ i \neq j;\ i, j \geq 2 \tag{2.20}$$

For each (i, j), $i \neq j$; $i, j \geq 2$, the right side of the above inequality is greater than that of (2.9) by the nonnegative expression $n \sum_{k \neq 1,i,j} x_{ik}x_{kj} + (n - 4) \sum_{k \neq 1,i,j} x_{jk}x_{ki}$, as $n \geq 4$. (Note that while the inequality is valid for $n = 4$, the last antecedent always holds false in this case.) Inequalities (2.20) recognize that if the salesman travels from city i *immediately* to some city k and *immediately* thereafter to city j, then city j is exactly two steps *after* city i, and if the salesman travels from city j *immediately* to some city k and *immediately* thereafter to city i, then city j is exactly two steps *before* city i.

Inequalities (2.10) can also be tightened for the QTSP. For $n \geq 4$, we can make the following conditional statements for each $j \geq 2$.

If $x_{1j} = 1$	then $u_j = 1$.
If $\sum_{k \neq 1,j} x_{1k}x_{kj} = 1$	then $u_j = 2$.
If $x_{j1} = 1$	then $u_j = n - 1$.
If $\sum_{k \neq 1,j} x_{jk}x_{k1} = 1$	then $u_j = n - 2$.
If $1 - x_{1j} - x_{j1} - \sum_{k \neq 1,j} x_{1k}x_{kj} - \sum_{k \neq 1,j} x_{jk}x_{k1} = 1$.	then $3 \leq u_j \leq n - 3$.

We again use the binary property of the left side of (2.19), this time with $i = 1$, to note that the antecedents are all binary, so that we can multiply and surrogate to obtain the following two quadratic inequalities for each $j \geq 2$, upon separately considering the two possibilities in the last statement.

$$
\begin{aligned}
&3 - 2x_{1j} + (n-4)x_{j1} - \sum_{k \neq 1,j} x_{1k}x_{kj} + (n-5)\sum_{k \neq 1,j} x_{jk}x_{k1} \leq u_j, \\
&u_j \leq (n-3) + (4-n)x_{1j} + 2x_{j1} + (5-n)\sum_{k \neq 1,j} x_{1k}x_{kj} + \sum_{k \neq 1,j} x_{jk}x_{k1}
\end{aligned} \tag{2.21}
$$

Here, $3 \leq u_j$ and $u_j \leq n-3$ of the last statement give the first and second inequalities, respectively. (While these inequalities are valid for $n \in \{4, 5\}$, the last antecedent always holds false in these two cases.) These inequalities tighten (2.10) in the presence of (2.19). For each $j \geq 2$, the left side of the first inequality is greater than the left expression of (2.10) by

$$
1 - x_{1j} - x_{j1} - \sum_{k \neq 1,j} x_{1k}x_{kj} + (n-5)\sum_{k \neq 1,j} x_{jk}x_{k1}, \tag{2.22}
$$

and the right side of the second inequality is less than the right expression of (2.10) by this same amount. The expression (2.22) is nonnegative by (2.19) with $i = 1$ and the assumption that $n \geq 4$. Inequalities (2.21) recognize that if the salesman travels from city 1 to city j in two steps then $u_j = 2$, which signifies city j is in position 3 of the optimal permutation, and if the salesman travels from city j to city 1 in two steps then $u_j = n - 2$, which signifies city j is in position $n - 1$.

An additional tightening of (2.9) beyond (2.20) can be obtained for the QTSP when the variables y_{ij}, as defined above, are included within the formulation along with (2.13) and the inequalities

$$
x_{ij} + \sum_{k \neq 1,i,j} x_{ik}x_{kj} \leq y_{ij} \; \forall \, (i,j), \; i \neq j; \; i,j \geq 2. \tag{2.23}
$$

Inequalities (2.23) are a tightened version of (2.11) for the QTSP which recognize that if the salesman travels from city i to city j in one step or two steps, nowhere involving city 1, then city i must precede city j. As with (2.19), the left side of (2.23), as well as the sum of any subset of the expressions, must be binary for all solutions to the QTSP. For $n \geq 4$, we can make the following conditional statements for each $(i, j), i \neq j; i, j \geq 2$.

If $\sum_{k\neq 1,i,j} x_{ik}x_{kj} = 1$	then $u_j - u_i = 2$.
If $\sum_{k\neq 1,i,j} x_{jk}x_{ki} = 1$	then $u_j - u_i = -2$.
If $x_{ij} = 1$	then $u_j - u_i = 1$.
If $x_{ji} = 1$	then $u_j - u_i = -1$.
If $y_{ij} - x_{ij} - \sum_{k\neq 1,i,j} x_{ik}x_{kj} = 1$	then $u_j - u_i \geq 3$.
If $1 - y_{ij} - x_{ji} - \sum_{k\neq 1,i,j} x_{jk}x_{ki} = 1$	then $u_j - u_i \geq 2 - n$.

Using the binary property of the left side of (2.23), each of the expressions $\sum_{k\neq 1,i,j} x_{ik}x_{kj}$, $\sum_{k\neq 1,i,j} x_{jk}x_{ki}$, x_{ij}, x_{ji}, $y_{ij} - x_{ij} - \sum_{k\neq 1,i,j} x_{ik}x_{kj} = 1$, and $1 - y_{ij} - x_{ji} - \sum_{k\neq 1,i,j} x_{jk}x_{ki}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the QTSP. (The binary property of the last antecedent follows from (2.23) by interchanging the indices i and j, and by using (2.13).) We can then multiply each such expression by its associated consequence and surrogate to obtain the following quadratic inequalities.

$$u_j - u_i \geq (2-n) + (n+1)y_{ij} - 2x_{ij} + (n-3)x_{ji} - \sum_{k\neq 1,i,j} x_{ik}x_{kj} + (n-4)\sum_{k\neq 1,i,j} x_{jk}x_{ki} \quad \forall\ (i,j),\ i\neq j,\ i,j\geq 2 \tag{2.24}$$

Inequalities (2.24) are a tightened version of (2.20) in the presence of (2.23), and are thus a further tightening of (2.9). The right side of (2.24) is obtained by adding the nonnegative quantity $(n+1)(y_{ij} - x_{ij} - \sum_{k\neq 1,i,j} x_{ik}x_{kj})$ to the right side of (2.20). (Note that while (2.24) is valid for $n = 4$, the last two antecedents always hold false in this case.) Inequalities (2.24) recognize that if city i precedes city j, but not *immediately* before nor two steps before, then city j is located at least 3 positions beyond city i.

We chose to not create expressions of the form $x_{ik}x_{kj}$ to solve the TVP because of the resulting increased problem size due to the extra variables required for linearization. However, the QTSP naturally includes such product terms so that these variables are already present in the formulation.

As a final remark, we provide an additional family of inequalities in the variables $x_{ik}x_{kj}$ and y_{ij}. Recall our earlier statement that for any $(i,j), i \neq j; i,j \geq 2$, if $x_{i1}x_{1j} = 1$, then $u_i = n-1$ and $u_j = 1$. Hence, we can enforce the following inequalities.

$$x_{i1}x_{1j} \leq y_{ji} \;\; \forall \, (i,j), \; i \neq j; \; i,j \geq 2$$

2.4 Computational Experience

In this section, we examine the computational effectiveness of using MTZ inequalities within Problem TVP1 to solve the TVP, examining both the enforcement of (2.9) and (2.10) together, and then (2.10) with the tightened (2.16). We compare four different formulations. The first two are the base cases of TVP0 and TVP1 that use no MTZ inequalities. The third, referred to as Problem TVP2, includes the known MTZ inequalities (2.9) and (2.10) within TVP1. The fourth, Problem TVP3, replaces inequalities (2.9) within TVP2 with (2.16). Our intent is to analyze the usefulness of the inequalities (2.10) and (2.16) on the TVP, and to also learn the additional advantage of using (2.16) in lieu of (2.9). In addition, since the tightened version of (2.12) given by (2.15) has not previously been considered for the TVP, we also compare Problems TVP0 and TVP1.

$$\text{TVP2: maximize} \left\{ \sum_{i} \sum_{j \neq i} r_{ij} y_{ij} + \sum_{i} \sum_{j \neq i} c_{ij} x_{ij} : (2.4), (2.5), (2.7), (2.9) - (2.11), (2.13)\text{–}(2.15) \right\}$$

$$\text{TVP3: maximize} \left\{ \sum_{i} \sum_{j \neq i} r_{ij} y_{ij} + \sum_{i} \sum_{j \neq i} c_{ij} x_{ij} : (2.4), (2.5), (2.7), (2.10), (2.11), (2.13)\text{–}(2.16) \right\}$$

Since the MTZ inequalities serve to eliminate subtours, we have the option to either enforce, or not, the strengthened 3-dicycle inequalities (2.15) within TVP2 and TVP3. Based on preliminary computational experience, we chose to include these inequalities within both problems. We have a similar option relative to the inclusion of (2.14), and our preliminary results led us to also enforce these equations.

It is important to note that, while the MTZ inequalities (2.9), (2.10), and (2.16) are all implied by the restrictions of each of Problems TVP0 and TVP1 when the variables x_{ij} are enforced to be binary in (2.7), they are not implied in the continuous relaxations that are obtained by replacing these restrictions with $x_{ij} \geq 0$ for all $(i, j),\ i \neq j$. We denote the continuous relaxations to Problems TVP0, TVP1, TVP2, and TVP3 that are obtained by relaxing (2.7) in each as Problems $\overline{\text{TVP0}}$, $\overline{\text{TVP1}}$, $\overline{\text{TVP2}}$, and $\overline{\text{TVP3}}$, respectively.

All of our runs were made on Clemson University's high performance computer, Palmetto, with 2 CPU cores and 20gb RAM, using the CPLEX 12.8 solver in AMPL for Linux-Intel 64 with all pre-processing options turned off. Python 2.7.6 was used to generate data files with random integer objective coefficient values obtained via a uniform distribution for ranges specified below.

We ran a total of 440 problems of size $n = 20$, consisting of ten runs each for 11 families of problems, for each of four formulations. As the TVP reduces to the LOP when $c_{ij} = 0\ \forall\ (i, j)$, and to the TSP when $r_{ij} = 0\ \forall\ (i, j)$, our concern is with the usefulness of (2.9), (2.10), (2.15), and (2.16) over different coefficient ranges for c_{ij} and r_{ij}. We designed 11 families of various interval ranges and computed coefficients within each range via uniform distributions. Specifically, for c_{ij} in the interval $[0, 10]$, we let r_{ij} range in each of the six intervals $[0, 10]$, $[0, 20]$,[0,50], $[0, 100]$, $[0, 200]$, and $[0, 1000]$. Similarly for r_{ij} in the interval $[0, 10]$, we let c_{ij} range in the five intervals $[0, 20]$,[0,50], $[0, 100]$, $[0, 200]$, and $[0, 1000]$. Our output gives the averages of the ten instances within each family. Each instance was given a 3600 CPU second time limit.

Table 2.1 records, for each of the 11 families of 10 problems, the number of instances that were optimally solved in the 3600 CPU second time limit. The columns are arranged so that the first two give the ranges on the coefficients c_{ij} and r_{ij}, and the next four give the formulations used: TVP0, TVP1, TVP2, and TVP3. Each of the rows depicts a different family of coefficient ranges. The key observation of Table 2.1 is that Problems TVP2 and TVP3 were able to solve all instances, that TVP1 solve only 9 of the 10 instances when both c_{ij} and r_{ij} were in the interval $[0, 10]$, and that TVP0 was unable to solve a total of 18 out of the 110 instances.

Table 2.2 gives, for each of the 11 families of problems, the average times in seconds to solve the TVP and the average numbers of branch-and-bound nodes explored for each of the four formulations. To present meaningful results, both averages were computed only for those instances that were solved to optimality for all four formulations. As with Table 2.1, the first two columns give the ranges on the coefficients c_{ij} and r_{ij}, and the rows list the 11 families of problems. Columns 3 through 6 give the average times to solve the integer programs TVP0, TVP1, TVP2, and TVP3, respectively, while columns 7 through 10 give the average numbers of branch-and-bound nodes for these same problems.

Four notable observations of Table 2.2 are the following. First, for all cases, TVP1 outperformed TVP0 in terms of both the execution times and numbers of nodes explored. Thus, the tightening of (2.12) to (2.15) is advantageous. Second, for all cases except the final one having c_{ij} in the interval $[0, 1000]$ and r_{ij} in the interval $[0, 10]$, Problem TVP2 outperformed TVP1 in execution times and nodes explored. For the last case, TVP2 was better in terms of numbers of nodes but took a slightly longer execution time. Overall, the MTZ inequalities (2.9) and (2.10) tend to strengthen TVP1 to TVP2 and lead to more efficient solvings. Third, for 8 of the 11 cases, Problem TVP3 outperformed TVP2 in execution times and numbers of nodes explored. Thus, while the MTZ inequalities all serve to reduce the overall effort, the new cuts (2.16) tend to exhibit greater benefit than the weaker, known inequalities (2.9). Finally, the problems appear significantly more challenging when the c_{ij} and r_{ij} coefficients are of the same relative magnitudes. This difficulty is evident when both intervals are in one of the ranges $[0, 10]$, $[0, 20]$, or $[0, 50]$, in contrast with having one range as $[0, 10]$ and the other as $[0, 1000]$. Our conjecture here is that simplifications occur when one set of coefficients dominates the other because the TVP then reduces to take the form of either the TSP or LOP.

We are also interested in the additional strength afforded to the continuous relaxations: by (2.15) in TVP1 as opposed to (2.12) in TVP0, by (2.9) and (2.10) as found in TVP2, and by (2.10) and (2.16) as found in TVP3. Table 2.3 gives our results. For this table, the 11 rows again list the 11 families of problems, and the first two columns again give the coefficient ranges for c_{ij} and r_{ij}, respectively. Column 3 gives the optimal integer values, and columns 4 through 7 give the optimal objective values to the relaxations $\overline{\text{TVP0}}$, $\overline{\text{TVP1}}$, $\overline{\text{TVP2}}$, and $\overline{\text{TVP3}}$, respectively. Column

8 gives the percentage of the gap between the continuous relaxation values of $\overline{\text{TVP1}}$ and the optimal integer values that is reduced by $\overline{\text{TVP3}}$, computed as $(\frac{\overline{\text{TVP1}}-\overline{\text{TVP3}}}{\overline{\text{TVP1}}-\text{IP}}) \times 100$, and labeled G.R. for gap reduction, where IP is the optimal (integer programming) objective value to the TVP. While for this last column we could have instead computed the percentage gap relative to $\overline{\text{TVP0}}$ and $\overline{\text{TVP3}}$ instead of $\overline{\text{TVP1}}$ and $\overline{\text{TVP3}}$, our intent is to see the improved strength due to (2.10) and (2.16) on the tighter $\overline{\text{TVP1}}$.

The table reaffirms the theoretical result that the objective values to $\overline{\text{TVP3}}$ are the tightest, followed by $\overline{\text{TVP2}}$ and $\overline{\text{TVP1}}$, with the majority of improvement occurring between $\overline{\text{TVP0}}$ and $\overline{\text{TVP1}}$. Interestingly, column eight shows that Problem $\overline{\text{TVP3}}$ reduced the gap between the optimal objective values to $\overline{\text{TVP1}}$ and the integer optimums between 2.75% and 31.14% on average. As evident in Table 2.2, these reductions had a marked effect on the overall solution times. This reduction runs counter to the generally-accepted belief that the MTZ inequalities are weak.

We believe that the success exhibited in these tables by using the tightened MTZ inequalities (2.16) is due both to the relatively few number of $(n-1)(n-2)$ inequalities in (2.16) and our not requiring additional, auxiliary variables. The paper [42] applied a partial level-1 RLT to a formulation of the TSP using MTZ inequalities by multiplying, for each i, the equation in (2.4) by u_i and, for each j, the equation in (2.5) by u_j to generate products of the form $u_i x_{ij}$ and $u_j x_{ij}$, which were in turn linearized through the use of additional variables. While the continuous relaxations were reported to have been strengthened, the overall solution times increased. In contrast, we strategically computed the quadratic inequalities so that they are readily surrogated to obtain linear restrictions, and thus eliminated the need for auxiliary variables.

2.5 Conclusions and Future Research

This paper presents a new family of valid linear inequalities for the Target Visitation Problem (TVP). The TVP is a hybrid between the Linear Ordering Problem (LOP) and the Traveling Salesman Problem (TSP) that seeks an optimal permutation of a collection of objects while incurring costs for both the *immediate* and *anywhere* relative location for each pair of distinct objects. The new inequalities exploit the variable structure of the TVP, and are a strengthened version

Table 2.1: Numbers of Instances Solved

Coefficient Ranges		**Number of Instances Solved**			
c_{ij}	r_{ij}	TVP0	TVP1	TVP2	TVP3
[0,10]	[0,10]	5	9	10	10
[0,10]	[0,20]	7	10	10	10
[0,10]	[0,50]	7	10	10	10
[0,10]	[0,100]	10	10	10	10
[0,10]	[0,200]	10	10	10	10
[0,10]	[0,1000]	10	10	10	10
[0,20]	[0,10]	7	10	10	10
[0,50]	[0,10]	6	10	10	10
[0,100]	[0,10]	10	10	10	10
[0,200]	[0,10]	10	10	10	10
[0,1000]	[0,10]	10	10	10	10

of the Miller-Tucker-Zemlin (MTZ) inequalities that are used to eliminate subtours in the TSP. Interestingly, while the MTZ restrictions are known to be weak for the TSP, they serve to add strength and reduce solution times for the TVP.

Our approach for generating the improved inequalities was to use the two steps of conditional logic and surrogation, specially tailored for the TVP. However, we believe that this approach will prove fruitful in more general scenarios than the MTZ inequalities discussed throughout Section 2.3. As an example, the paper [43] uses linearized products of the form $x_{ik}y_{kj}$ to tighten the continuous relaxation of the TSP. If we choose to allow such products, then we can form more complex conditional statements as follows for each (i, j, k) distinct; $i, j, k \geq 2$.

$$\begin{array}{ll} \text{If } x_{ji} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} \leq 1. \\ \text{If } x_{ik}y_{kj} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} = 1. \\ \text{If } 1 - x_{ji} - x_{ik}y_{kj} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} \leq 2. \end{array}$$

Using the same type logic as before, we have that each of the expressions x_{ji}, $x_{ik}y_{kj}$, and $1 - x_{ji} - x_{ik}y_{kj}$ found in the antecedents of these statements is binary for all feasible solutions to the TSP. Then multiplying each expression by its consequence and summing, we obtain the following

Table 2.2: IP Times & Branch-and-Bound Nodes

Coefficient Ranges		**IP Times (in Seconds)**				**Branch-and-Bound Nodes**			
c_{ij}	r_{ij}	TVP0	TVP1	TVP2	TVP3	TVP0	TVP1	TVP2	TVP3
[0,10]	[0,10]	769.41	196.56	121.21	83.93	66248.40	11017.20	3025.80	1919.00
[0,10]	[0,20]	1471.90	135.32	63.32	66.48	159489.00	8195.86	2017.86	2068.29
[0,10]	[0,50]	360.54	61.78	23.74	27.06	43860.86	4045.14	661.29	623.00
[0,10]	[0,100]	177.01	25.65	12.01	11.32	22824.50	1513.70	337.10	255.00
[0,10]	[0,200]	134.81	20.33	12.19	9.44	19481.10	1350.00	350.20	231.30
[0,10]	[0,1000]	5.63	4.29	1.71	1.30	485.00	189.90	20.10	5.40
[0,20]	[0,10]	1125.38	187.92	100.96	84.04	83289.86	8896.29	2899.00	2035.00
[0,50]	[0,10]	1386.31	252.36	123.43	108.43	95687.33	12794.00	3166.00	2711.17
[0,100]	[0,10]	119.12	33.17	20.93	17.32	6665.60	1323.90	337.40	236.70
[0,200]	[0,10]	38.26	20.13	12.87	11.20	1708.20	711.00	177.10	127.00
[0,1000]	[0,10]	2.80	2.70	3.13	3.24	50.70	36.20	14.20	14.30

inequalities that are valid for the TSP.

$$y_{ij} + y_{jk} + y_{ki} + x_{ji} + x_{ik}y_{kj} \leq 2 \ \ \forall\, (i,j,k) \text{ distinct; } \ i,j,k \geq 2$$

These inequalities are a tightened version of the expressions (2.15) of [43] since all the $x_{ik}y_{kj}$ products are nonnegative. For the TVP, we chose to not create such products because the additional linearized variables would have significantly increased the problem size.

Table 2.3: Continuous Relaxation Values

Coefficient Ranges		**IP**	**Continuous Relaxation Values**				**G.R.**
c_{ij}	r_{ij}	**Values**	$\overline{\text{TVP0}}$	$\overline{\text{TVP1}}$	$\overline{\text{TVP2}}$	$\overline{\text{TVP3}}$	**%**
[0,10]	[0,10]	972.80	1009.43	1001.41	1000.89	1000.53	3.08
[0,10]	[0,20]	1940.80	1984.22	1973.19	1972.75	1972.30	2.75
[0,10]	[0,50]	4890.71	4949.06	4921.03	4920.50	4919.93	3.63
[0,10]	[0,100]	10022.40	10070.49	10055.37	10054.92	10052.81	7.76
[0,10]	[0,200]	20025.50	20075.64	20057.23	20056.80	20054.52	8.54
[0,10]	[0,1000]	100528.00	100559.23	100543.77	100543.51	100538.86	31.14
[0,20]	[0,10]	934.70	979.36	968.31	967.14	966.63	5.00
[0,50]	[0,10]	832.50	911.68	894.80	890.90	889.42	8.64
[0,100]	[0,10]	667.90	753.07	734.14	729.61	726.84	11.02
[0,200]	[0,10]	572.10	662.88	646.34	634.68	630.89	20.81
[0,1000]	[0,10]	-683.70	-562.71	-583.11	-605.88	-611.49	28.21

Chapter 3

Quadratic Linear Ordering Problem

3.1 Introduction

Given a collection of n objects, the Quadratic Linear Ordering Problem (QLOP) seeks a permutation (ordering) of the objects that affords a minimal cost. There are two types of costs. First, for each distinct pair of objects i and j, a cost c_{ij} is incurred if object i is selected before j. Second, for each collection of two ordered-pairs of distinct objects (i, j) and (k, ℓ) with $(i, j) \neq (k, \ell)$, and $(i, j) \neq (\ell, k)$, a cost $C_{ijk\ell}$ is incurred if object i is selected before j *and* object k is selected before ℓ. To avoid duplication, and since $x_{ij}^2 = x_{ij}$ for all distinct i and j, we consider only such (i, j, k, ℓ) having $(i, j) \prec (k, \ell)$, where $(i, j) \prec (k, \ell)$ denotes that the ordered pair (i, j) is lexicographically less than (k, ℓ); that is, $i \leq k$ with $j < \ell$ if $i = k$.

Mathematically, the QLOP on n objects can be expressed as

$$\text{P1}(n): \quad \text{minimize} \quad \sum_{\substack{(i,j)\\ i\neq j}} c_{ij}x_{ij} + \sum_{\substack{(i,j,k,\ell)\\ i\neq j, k\neq \ell\\ (i,j)\neq(\ell,k)\\ (i,j)\prec(k,\ell)}} C_{ijk\ell}x_{ij}x_{k\ell}$$

$$\text{subject to} \quad x_{ij} + x_{jk} + x_{ki} \leq 2 \qquad \forall\ (i,j,k) \text{ distinct} \tag{3.1}$$

$$x_{ij} + x_{ji} = 1 \qquad \forall\ (i,j),\ i < j \tag{3.2}$$

$$x_{ij} \text{ binary} \qquad \forall\ (i,j),\ i \neq j$$

where the binary variable x_{ij} takes value 1 if object i is selected prior to object j, and takes value 0 otherwise. The first and second summations in the objective function record the linear and quadratic costs of selection, respectively. All indices are in the set $N \equiv \{1, \ldots, n\}$. For the special case in which every $C_{ijk\ell} = 0$, P1(n) reduces to the Linear Ordering Problem (LOP) [5, 20, 30, 36].

The problem constraints enforce a valid ordering. Given any solution to P1(n), inequalities (3.1) stipulate that for every distinct i, j, k, if object i is selected before j *and* object j is selected before k, then object i must be selected before k. Inequalities (3.2) state that, for any two objects i and j, either i is selected before j or j is selected before i.

Both the LOP and QLOP are similar to the traveling salesman problem (TSP) in that they seek a permutation of objects, but they differ in the manner in which the costs of the permutations are computed. The LOP and QLOP incur costs whenever an object i precedes an object j *anywhere* within a permutation, while the TSP incurs a cost only when object i *immediately* precedes object j. Due to the permutation structure, all three problems require some type of subtour elimination constraints. In fact, inequalities (3.1) and equations (3.2) have been used to eliminate subtours [18, 37] for the TSP.

Problem P1(n) can be trivially simplified by removing all variables $x_{ij}, i > j$, via the substitutions $x_{ij} = 1 - x_{ji}$ for all $i > j$, and then eliminating (3.2). For each quadratic expression in (i, j, k, ℓ), we then have $x_{ij}x_{k\ell} = x_{ij}x_{k\ell}$ if $i < j$ and $k < \ell$, $x_{ij}x_{k\ell} = x_{ij}(1 - x_{\ell k})$ if $i < j$ and $k > \ell$, $x_{ij}x_{k\ell} = (1 - x_{ji})x_{k\ell}$ if $i > j$ and $k < \ell$, and $x_{ij}x_{k\ell} = (1 - x_{ji})(1 - x_{\ell k})$ if $i > j$ and $k > \ell$. Upon making these substitutions, the below problem results, where we assume that the objective coefficients c_{ij} and $C_{ijk\ell}$ have been accordingly adjusted to b_{ij} and $B_{ijk\ell}$, respectively, and a constant possibly included. Also, repetitive inequalities have been removed to reduce, by two-thirds, the $n(n-1)(n-2)$ inequalities of (3.1) to the $2\binom{n}{3}$ inequalities of (3.3).

$$\text{P2}(n): \quad \text{minimize} \quad \sum_{(i,j)\in I_n} b_{ij}x_{ij} + \sum_{(i,j,k,\ell)\in J_n} B_{ijk\ell}x_{ij}x_{k\ell}$$

$$\text{subject to} \quad 0 \le x_{ij} + x_{jk} - x_{ik} \le 1 \qquad \forall\, (i,j,k) \in K_n \tag{3.3}$$

$$x_{ij} \text{ binary} \qquad \forall\, (i,j) \in I_n$$

Here, for convenience, we let $I_n \equiv \{(i,j) : 1 \le i < j \le n\}$, $J_n \equiv \{(i,j,k,\ell) : 1 \le i < j \le n, 1 \le k < \ell \le n, (i,j) \prec (k,\ell)\}$, and $K_n \equiv \{(i,j,k) : 1 \le i < j < k \le n\}$ so that $|I_n| = \binom{n}{2}$, $|J_n| = \binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$, and $|K_n| = \binom{n}{3}$. We explicitly write the subscript n on all sets for future reference. For the LOP having $B_{ijk\ell} = 0$ for all $(i,j,k,\ell) \in J_n$, inequalities (3.3) are known as *3-dicycle inequalities*.

Problem P2(n) is a 0-1 quadratic program that can be linearized using known techniques. A method variously ascribed to [13, 14, 15, 31] introduces a continuous variable $y_{ijk\ell}$ for every $(i,j,k,\ell) \in J_n$, and then removes the quadratic terms $x_{ij}x_{k\ell}$ via the substitutions

$$y_{ijk\ell} = x_{ij}x_{k\ell} \;\forall\, (i,j,k,\ell) \in J_n. \tag{3.4}$$

The sets

$$X_{ijk\ell} \equiv \{(x_{ij}, x_{k\ell}, y_{ijk\ell}) : 0 \le y_{ijk\ell},\; x_{ij} + x_{k\ell} - 1 \le y_{ijk\ell},$$
$$y_{ijk\ell} \le x_{ij},\; y_{ijk\ell} \le x_{k\ell}\} \;\forall\, (i,j,k,\ell) \in J_n \tag{3.5}$$

are used to enforce (3.4) for binary x_{ij} and $x_{k\ell}$ in such a manner that, for each $(i,j,k,\ell) \in J_n$, the equation of (3.4) is enforced by the four inequalities of the set $X_{ijk\ell}$.

Researchers [6, 28] have shown that the structure of the QLOP allows for additional restrictions that further relate the variables x_{ij} and $y_{ikj\ell}$ within P2(n), and that imply the 3-dicyle inequalities (3.3). Work of [28] replaces (3.3) with quadratic penalty functions as a means of prohibiting subtours. These functions were shown to yield a "valid infeasibility penalty" through the use of a logic table that demonstrates each function to be binary for binary x_{ij}, with all functions equal to 0 at a feasible solution, and with at least one function equal to 1 otherwise. The paper [6] expresses these quadratic functions as the equations of the sets

$$E_{ijk} \equiv \{(x_{ij}, x_{ik}, x_{jk}) : x_{ik}x_{jk} = x_{ik} - x_{ij}x_{ik} + x_{ij}x_{jk}\} \; \forall \; (i,j,k) \in K_n, \tag{3.6}$$

and then linearizes the quadratic terms within (3.6) using (3.4) to obtain

$$L_{ijk} \equiv \{(x_{ik}, y_{ijik}, y_{ijjk}, y_{ikjk}) : y_{ikjk} = x_{ik} - y_{ijik} + y_{ijjk}\} \; \forall \; (i,j,k) \in K_n. \tag{3.7}$$

The net result [6] is to reformulate Problem P2(n) as the mixed 0-1 linear program:

$$\text{LP2}(n) : \text{minimize} \left\{ \sum_{(i,j)\in I_n} b_{ij}x_{ij} + \sum_{(i,j,k,\ell)\in J_n} B_{ijk\ell}y_{ijk\ell} : (\boldsymbol{x}, \boldsymbol{y}) \in S_n, \; \boldsymbol{x} \text{ binary} \right\},$$

where

$$\begin{aligned} S_n \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{|I_n|} \times \mathbb{R}^{|J_n|} : (x_{ij}, x_{k\ell}, y_{ijk\ell}) \in X_{ijk\ell} \; \forall \; (i,j,k,\ell) \in J_n, \\ (x_{ik}, y_{ijik}, y_{ijjk}, y_{ikjk}) \in L_{ijk} \; \forall \; (i,j,k) \in K_n\}, \end{aligned} \tag{3.8}$$

and where $\boldsymbol{x}$ denotes the set of $x_{ij}, (i,j) \in I_n$, and $\boldsymbol{y}$ denotes the set of $y_{ijk\ell}, (i,j,k,\ell) \in J_n$.

Observe that Problem LP2(n) contains $\left(\binom{n}{2} + 3\binom{n+1}{4}\right)$ variables, $12\binom{n+1}{4}$ inequality restrictions, and $\binom{n}{3}$ equality restrictions. In fact, a count on the number of variables $\boldsymbol{y}$ is available in terms of the sets $X_{ijk\ell}$ as given in the equation

$$|J_n| = 3\binom{n+1}{4} = 3\binom{n}{3} + 3\binom{n}{4}, \tag{3.9}$$

where $\binom{n}{4} = 0$ for $n = 3$. To explain, Problem LP2(n) by definition has $|J_n| = 3\binom{n+1}{4}$ variables $\boldsymbol{y}$, and has a one-to-one correspondence between the variables $y_{ijk\ell}$ and the sets $X_{ijk\ell}$ for $(i,j,k,\ell) \in J_n$. There exist exactly $3\binom{n}{3}$ sets $X_{ijk\ell}$ with (i,j,k,ℓ) not distinct and $3\binom{n}{4}$ sets $X_{ijk\ell}$ with (i,j,k,ℓ) distinct, explaining (3.9).

We contribute to the study on the QLOP by providing different mixed 0-1 linear forms of

Problem P2(n) that are more concise in size and/or have tighter linear programming relaxations than LP2(n). In particular, letting $conv(\bullet)$ denote the convex hull of the set $\bullet$, we make three advances relative to the sets QLO(n) $:= conv(S_n \cap\ \boldsymbol{x}$ binary) for $n \geq 3$. These advances are as follows.

1. For $n \geq 3$, we present in Section 3.2 a "lifting theorem" that establishes that every facet-defining inequality (facet) for QLO(n) is also a facet for QLO(s) for $s \geq n$. While such a result is known for the LOP [20] and the Boolean Quadric Polytope (BQP) [35], the structure of the QLOP requires a more involved argument.

2. Section 3.3 derives the set of all facets for QLO(3), and shows how these facets motivate an equivalent mixed 0-1 linear formulation of Problem P2(n) that has half the inequalities of LP2(n). The facets are obtained by projecting a higher-dimensional convex hull representation, available from the reformulation-linearization-technique (RLT), onto the space of the variables $(\boldsymbol{x}, \boldsymbol{y})$. It will turn out that S_3 = QLO(3), but the inequalities of the sets X_{1213}, X_{1223}, and X_{1323} of (3.5) found within S_3 of (3.8) are not all facets. In fact, half of the four inequalities found within each set can be removed. Extending this result to general n, for each $(i, j, k) \in K_n$, half of the four inequalities found within each of the three sets X_{ijik}, X_{ijjk}, and X_{ikjk} of (3.8) can be removed. Thus, since all 4-tuples $(i, j, k, \ell) \in I_n$ with (i, j, k, ℓ) not distinct are of the form (i, j, i, k), (i, j, j, k), or (i, k, j, k) for some $(i, j, k) \in K_n$, we are able to halve the number of inequalities associated with all such sets $X_{ijk\ell}$. For the remaining sets $X_{ijk\ell}$: that is for those sets having $(i, j, k, \ell) \in J_n$ with (i, j, k, ℓ) distinct, we are again able to halve the number of inequalities, but this time using an observation of [1]. Notably, these reductions do not affect the strength of the continuous relaxations.

3. Section 3.4 gives an explicit description of QLO(4), characterized in terms of five families of facets. Analogous to the $n = 3$ case, these facets project from a higher-dimensional RLT space onto the variables $(\boldsymbol{x}, \boldsymbol{y})$. Due to the "lifting theorem," they are theoretically useful for providing tighter polyhedral relaxations than that afforded by S_n for $n \geq 4$.

As a matter of interest, we also provide computational experience in Section 3.5 to assess the merits of the latter two advances, emphasizing two points. First, we show the computational savings that are realized by solving reformulated instances of P2(n) using our reduced numbers of constraints. Second, we determine those facets defining QLO(4) that are most instrumental in tightening the linear programming relaxation of LP2(n). The paper concludes with a conclusion and avenues for future research in Section 3.6.

3.2 Lifting Theorem

In this section, we prove the lifting theorem. This theorem states that every facet for QLO(n) is also a facet for QLO(s) for all $s \geq n$. As mentioned earlier, similar results have been established for the LOP [20], but the proof for the quadratic linear ordering polytope requires a different approach. By induction, we need only show that every facet for QLO(n) is also a facet for QLO($n+1$).

Throughout this section, we use $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ to represent a point in the space of QLO(n), i.e., $\mathbb{R}^{|I_n|} \times \mathbb{R}^{|J_n|}$, and we use $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ to represent a point in the space of QLO($n+1$). We find it useful to partition a point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in$ QLO($n+1$) as $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}, \hat{\boldsymbol{x}})$ and $\bar{\boldsymbol{y}} = (\tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}})$, where $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in$ QLO(n) and $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in \mathbb{R}^n \times \mathbb{R}^{\frac{(n-1)n(n+1)}{2}}$, so that $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ represent the variables within QLO($n+1$) that are also found in QLO(n), and $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ represent the variables within QLO($n+1$) that are not found in QLO(n). We have the following lemma.

Lemma 3.2.1. *If $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an extreme point of QLO($n+1$), then $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is an extreme point of QLO(n).*

Proof. For $s \in \{n, n+1\}$, the set of extreme points of QLO(s) is the set of $s!$ points in the set ($S_s \cap \boldsymbol{x}$ binary). Since every $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in S_{n+1}$ has $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in S_n$, the result follows. □

An insightful explanation of Lemma 3.2.1 is as follows. Since for $s \in \{n, n+1\}$, the constraints of LP2(s) enforce a valid permutation of s objects, then $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an extreme point of QLO($n+1$) if and only if $\bar{\boldsymbol{x}}$ represents a permutation, say $\bar{\boldsymbol{\pi}}$, of $(n+1)$ objects with $\bar{y}_{ijk\ell} = \bar{x}_{ij}\, \bar{x}_{k\ell}$ for all $(i,j,k,\ell) \in J_{n+1}$. Likewise, $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is an extreme point of QLO(n) if and only if $\tilde{\boldsymbol{x}}$ represents a permutation, say $\tilde{\boldsymbol{\pi}}$, of n objects with $\tilde{y}_{ijk\ell} = \tilde{x}_{ij}\, \tilde{x}_{k\ell}$ for all $(i,j,k,\ell) \in J_n$. Relative to Lemma 3.2.1, given any permutation $\bar{\boldsymbol{\pi}}$ of $(n+1)$ objects, we can remove object $(n+1)$ to obtain a permutation $\tilde{\boldsymbol{\pi}}$ of n objects, which can be represented by the truncated vector $\tilde{\boldsymbol{x}}$. In this manner, every extreme point to QLO(n) can be associated with $(n+1)$ extreme points of QLO($n+1$) in the sense that there exist $(n+1)$ permutations of $(n+1)$ objects whose removal of object $(n+1)$ gives the same permutation over n objects.

A minimal equation system of a polyhedron contains the minimal number of equations that define the affine hull of the polyhedron. In [6], Buchheim et al characterized a minimal equation system for QLO(s). For completeness, we summarize their results in the following lemma.

Lemma 3.2.2 (Buchheim et al. [6])**.** *For any $s \geq 3$, the equations*

$$x_{ik} - y_{ijik} + y_{ijjk} - y_{ikjk} = 0 \quad \forall\ (i,j,k) \in K_s$$

that define the sets L_{ijk} of (3.7) *form a minimal equation system for QLO(s). As a result, the dimension of QLO(s) is $d_s = |I_s| + |J_s| - |K_s|$.*

We denote the minimal equation system in Lemma 3.2.2 by $A\boldsymbol{x} + B\boldsymbol{y} = \mathbf{0}$. Note that A is of size $|K_s| \times |I_s|$, B is of size $|K_s| \times |J_s|$, and the integrated matrix $\begin{bmatrix} A & B \end{bmatrix}$ is of full row rank $|K_s|$. Consistent with our earlier notation, for $s = n+1$, we let $\bar{A}\bar{\boldsymbol{x}} + \bar{B}\bar{\boldsymbol{y}} = \mathbf{0}$ denote the system of Lemma 3.2.2 while for $s = n$, we let $\widetilde{A}\tilde{\boldsymbol{x}} + \widetilde{B}\tilde{\boldsymbol{y}} = \mathbf{0}$ denote the system. Corresponding to the partition of $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}, \hat{\boldsymbol{x}})$ and $\bar{\boldsymbol{y}} = (\tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}})$, the structure of the system in Lemma 3.2.2 allows us to partition $\bar{A}$ and $\bar{B}$ into blocks

$$\bar{A} = \begin{bmatrix} \widetilde{A} & O \\ O & \widehat{A} \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} \widetilde{B} & O \\ O & \widehat{B} \end{bmatrix} \tag{3.10}$$

so that $\bar{A}\bar{\boldsymbol{x}} + \bar{B}\bar{\boldsymbol{y}} = \mathbf{0}$ can be equivalently written as

$$\begin{bmatrix} \widetilde{A} & O \\ O & \widehat{A} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \hat{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} \widetilde{B} & O \\ O & \widehat{B} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{y}} \\ \hat{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

with matrices $\widetilde{A}$ and $\widetilde{B}$ of sizes $\binom{n}{3} \times \binom{n}{2}$ and $\binom{n}{3} \times 3\binom{n+1}{4}$, respectively, and $\widehat{A}$ and $\widehat{B}$ of sizes $\binom{n}{2} \times n$

and $\binom{n}{2} \times \frac{(n-1)n(n+1)}{2}$, respectively. Alternately stated, matrices $\widetilde{A}$ and $\widetilde{B}$ are of sizes $|K_n| \times |I_n|$ and $|K_n| \times |J_n|$, respectively, and matrices $\widehat{A}$ and $\widehat{B}$ are of sizes $(|K_{n+1}| - |K_n|) \times (|I_{n+1}| - |I_n|)$ and $(|K_{n+1}| - |K_n|) \times (|J_{n+1}| - |J_n|)$, respectively.

The next lemma provides a tool to characterize facets of a polyhedron which is not full-dimensional. The proof of the lemma is standard and can be found in [33].

Lemma 3.2.3. *Let $D\boldsymbol{z} = \boldsymbol{d}$ be a minimal equation system of a polyhedron $P \subseteq \mathbb{R}^d$, where D is an $m \times d$ matrix and $\boldsymbol{d} \in \mathbb{R}^m$. Let $F = \{\boldsymbol{z} \in P : \boldsymbol{\pi}^T\boldsymbol{z} = \boldsymbol{\pi}_0\}$ be a proper face of P. Then the following two statements are equivalent:*

1. *F is a facet of P.*
2. *If $\boldsymbol{\lambda}^T\boldsymbol{z} = \boldsymbol{\lambda}_0$ for all $\boldsymbol{z} \in F$, then*

$$\begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\lambda}_0 \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi}_0 \end{bmatrix} + \begin{bmatrix} D^T \\ \boldsymbol{d}^T \end{bmatrix} \boldsymbol{\beta}$$

for some $\alpha \in \mathbb{R}$ and $\boldsymbol{\beta} \in \mathbb{R}^m$.

Our first result is the following proposition that provides a necessary condition for a valid inequality for QLO(s) to be a facet.

Proposition 3.2.4. *For $s \geq 3$, let $\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{y} \leq c$ be a facet of QLO(s), and let $A\boldsymbol{x} + B\boldsymbol{y} = \boldsymbol{0}$ be the minimal equation system in Lemma 3.2.2. Then $\boldsymbol{b} \notin \text{Range}(B^T)$.*

Proof. We first claim that $\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{y} \leq c$ is not a facet if $\boldsymbol{b} = \boldsymbol{0}$. To prove the claim, suppose that

$$\boldsymbol{a}^T\boldsymbol{x} = \sum_{(i,j) \in I_s} a_{ij} x_{ij} \leq c$$

is a valid inequality that defines a proper face $G := \{(\boldsymbol{x}, \boldsymbol{y}) \in \text{QLO}(s) : \boldsymbol{a}^T\boldsymbol{x} = c\}$ of QLO(s). Without loss of generality, we assume that $a_{12} \neq 0$. Multiplying both sides of $\boldsymbol{a}^T\boldsymbol{x} = c$ by $x_{s-1,s}$, we have

$$\sum_{(i,j) \in I_s} a_{ij}\, x_{ij}\, x_{s-1,s} = c\, x_{s-1,s}. \tag{3.11}$$

For each extreme point $(\boldsymbol{x}, \boldsymbol{y})$ of G, $(\boldsymbol{x}, \boldsymbol{y})$ is automatically an extreme point of $\text{QLO}(s)$, since G is a face of $\text{QLO}(s)$. Therefore, $(x_{ij})^2 = x_{ij}$ for all $(i, j) \in I_s$, and $y_{ijk\ell} = x_{ij}\, x_{k\ell}$ for all $(i, j, k, \ell) \in J_s$. As a result of the above equalities and (3.11),

$$(a_{s-1,s} - c)\, x_{s-1,s} + \sum_{(i,j) \in I_s \setminus \{(s-1,s)\}} a_{ij}\, y_{i,j,s-1,s} = 0 \tag{3.12}$$

for all extreme points of G. Therefore, (3.12) holds for all points of G. Denote (3.12) by $\boldsymbol{p}^T \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{y} = 0$. By Lemma 3.2.3, G is a facet of $\text{QLO}(s)$ if and only if there exist $\alpha \in \mathbb{R}$ and $\boldsymbol{\beta} \in \mathbb{R}^{|K_s|}$ such that

$$\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{0} \\ c \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \\ \boldsymbol{0}^T \end{bmatrix} \boldsymbol{\beta}. \tag{3.13}$$

Consider the following two cases: $s \geq 4$ and $s = 3$.

For $s \geq 4$, observe that $y_{1,2,s-1,s}$ does not appear in any equation of the minimal equation system

$$A\boldsymbol{x} + B\boldsymbol{y} = \boldsymbol{0}.$$

As such, for any α and $\boldsymbol{\beta}$ in (3.13),$q_{1,2,s-1,s} = 0$. However, $q_{1,2,s-1,s} = a_{12} \neq 0$, which is a contradiction. Hence there do not exist parameters α and $\boldsymbol{\beta}$ such that $\boldsymbol{q} = \alpha\boldsymbol{0} + B^T\boldsymbol{\beta}$.

For $s = 3$, the system simplifies to

$$\begin{bmatrix} 0 \\ 0 \\ a_{23} - c \\ 0 \\ a_{12} \\ a_{13} \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} a_{12} \\ a_{13} \\ a_{23} \\ 0 \\ 0 \\ 0 \\ c \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

It is easy to check that no such $(\alpha, \boldsymbol{\beta})$ satisfies (3.13) given that $a_{12} \neq 0$. Therefore, in both cases, G is not a facet of QLO(s) by Lemma 3.2.3.

Next we suppose $\boldsymbol{b} \neq \boldsymbol{0}$ and prove by contradiction that $\boldsymbol{b} \notin \text{Range}(B^T)$. Suppose that $\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{y} \leq c$ is a facet and $B^T\boldsymbol{g} = \boldsymbol{b}$. Then, by Lemma 3.2.3, $(\boldsymbol{a} - A^T\boldsymbol{g})^T\boldsymbol{x} = (\boldsymbol{a} - A^T\boldsymbol{g})^T\boldsymbol{x} + (\boldsymbol{b} - B^T\boldsymbol{g})^T\boldsymbol{y} \leq c$ is also a facet for QLO(s), which contradicts to the fact that $\boldsymbol{b} \neq \boldsymbol{0}$. Therefore, $\boldsymbol{b} \notin \text{Range}(B^T)$. □

As an application, the following corollary is immediate.

Corollary 3.2.5. *No facet of the linear ordering polytope is a facet of the quadratic linear ordering polytope.*

Before proceeding to the lifting theorem, we first prove a technical lemma.

Lemma 3.2.6. *Suppose that $(p, q) \in I_s$. Let $\boldsymbol{\pi}^+_{pq}$ be a permutation of s objects such that object q is placed right after object p. Let $\boldsymbol{\pi}^-_{pq}$ be the same permutation as $\boldsymbol{\pi}^+_{pq}$ but with objects p and q switched. Let $(\boldsymbol{x}^+, \boldsymbol{y}^+)$ and $(\boldsymbol{x}^-, \boldsymbol{y}^-)$ be the extreme points of QLO(s) corresponding to $\boldsymbol{\pi}^+_{pq}$ and $\boldsymbol{\pi}^-_{pq}$, respectively. If both $(\boldsymbol{x}^+, \boldsymbol{y}^+)$ and $(\boldsymbol{x}^-, \boldsymbol{y}^-)$ are in $\{(\boldsymbol{x}, \boldsymbol{y}) \in QLO(s) : \boldsymbol{u}^T\boldsymbol{x} + \boldsymbol{v}^T\boldsymbol{y} = w\}$, then*

$$u_{pq} + \sum_{\substack{(i,j)\in I_s \\ (i,j)\prec(p,q)}} v_{ijpq}\, x^{\pm}_{ij} + \sum_{\substack{(i,j)\in I_s \\ (i,j)\succ(p,q)}} v_{pqij}\, x^{\pm}_{ij} = 0. \tag{3.14}$$

Proof. By definition, the difference between $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$ is:

$$x^+_{ij} - x^-_{ij} = \begin{cases} 0, & \text{if } (i,j) \neq (p,q), \\ 1, & \text{if } (i,j) = (p,q), \end{cases} \tag{3.15}$$

for $(i,j) \in I_s$. Since $(\boldsymbol{x}^\pm, \boldsymbol{y}^\pm)$ are extreme points of QLO(s), $y^\pm_{ijk\ell} = x^\pm_{ij}\, x^\pm_{k\ell}$ for all $(i,j,k,\ell) \in J_s$. Then (3.15) indicates that

$$\begin{cases} y^+_{ijk\ell} = y^-_{ijk\ell}, & \text{if } (i,j) \neq (p,q), (k,\ell) \neq (p,q) \text{ and } (i,j) \prec (k,\ell), \\ y^+_{ijpq} - y^-_{ijpq} = x^\pm_{ij}, & \text{if } (i,j) \prec (p,q), \\ y^+_{pqij} - y^-_{pqij} = x^\pm_{ij}, & \text{if } (i,j) \succ (p,q). \end{cases}$$

As a result,

$$\sum_{(i,j)\in I_s} u_{ij}\, x^+_{ij} + \sum_{(i,j,k,\ell)\in J_s} v_{ijk\ell}\, y^+_{ijk\ell} = w = \sum_{(i,j)\in I_s} u_{ij}\, x^-_{ij} + \sum_{(i,j,k,\ell)\in J_s} v_{ijk\ell}\, y^-_{ijk\ell}$$

simplifies to (3.14). ☐

Now we are ready to prove the lifting theorem.

Theorem 3.2.7. *Let* $\tilde{\boldsymbol{a}} \in \mathbb{R}^{|I_n|}, \tilde{\boldsymbol{b}} \in \mathbb{R}^{|J_n|}, \bar{\boldsymbol{a}} = (\tilde{\boldsymbol{a}}, \boldsymbol{0}) \in \mathbb{R}^{|I_{n+1}|}$, *and* $\bar{\boldsymbol{b}} = (\tilde{\boldsymbol{b}}, \boldsymbol{0}) \in \mathbb{R}^{|J_{n+1}|}$. *If* $\tilde{\boldsymbol{a}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{b}}^T\tilde{\boldsymbol{y}} \leq c$ *is a facet of QLO*(n)*, then* $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} \leq c$ *is a facet of QLO*$(n+1)$*.*

Proof. First, we show that $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} \leq c$ is valid for QLO$(n+1)$. For each extreme point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ of QLO$(n+1)$,

$$\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} = \tilde{\boldsymbol{a}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{b}}^T\tilde{\boldsymbol{y}}. \tag{3.16}$$

By Lemma 3.2.1, we have that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in$ QLO(n), and therefore $\tilde{\boldsymbol{a}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{b}}^T\tilde{\boldsymbol{y}} \leq c$. As a result, (3.16) indicates that $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} \leq c$ holds for all extreme points of QLO$(n+1)$, and thus is valid for QLO$(n+1)$.

Now we show that $G_{n+1} := \{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \text{QLO}(n+1) : \bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} = c\}$ is a facet of QLO$(n+1)$. By Proposition 3.2.4, $\tilde{\boldsymbol{b}} \notin \text{Range}(\widetilde{B}^T)$, which implies that $\bar{\boldsymbol{b}} \notin \text{Range}(\bar{B}^T)$. As a result, $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} =$

c is not a linear combination of the equations in the minimal equation system $\bar{A}\bar{\boldsymbol{x}} + \bar{B}\bar{\boldsymbol{y}} = \boldsymbol{0}$, and therefore, $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} = c$ does not hold for all $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \text{QLO}(n+1)$. That is, G_{n+1} is a *proper* face of $\text{QLO}(n+1)$. Suppose that for coefficients $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, w) \in \mathbb{R}^{|I_{n+1}|} \times \mathbb{R}^{|J_{n+1}|} \times \mathbb{R}$,

$$\bar{\boldsymbol{u}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{v}}^T\bar{\boldsymbol{y}} = w \tag{3.17}$$

holds for all $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in G_{n+1}$. By Lemma 3.2.3, it suffices to show that there exist $\alpha \in \mathbb{R}$ and $\boldsymbol{\beta} \in \mathbb{R}^{|K_{n+1}|}$ such that

$$\begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \\ w \end{bmatrix} = \alpha \begin{bmatrix} \bar{\boldsymbol{a}} \\ \bar{\boldsymbol{b}} \\ c \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ \boldsymbol{0}^T \end{bmatrix} \boldsymbol{\beta},$$

or equivalently,

$$\begin{bmatrix} \tilde{\boldsymbol{u}} \\ \hat{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}} \\ \hat{\boldsymbol{v}} \\ w \end{bmatrix} = \alpha \begin{bmatrix} \tilde{\boldsymbol{a}} \\ \boldsymbol{0} \\ \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \\ c \end{bmatrix} + \begin{bmatrix} \widetilde{A}^T & O \\ O & \widehat{A}^T \\ \widetilde{B}^T & O \\ O & \widehat{B}^T \\ \boldsymbol{0}^T & \boldsymbol{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\gamma} \end{bmatrix}$$

for some $\alpha \in \mathbb{R}$, $\boldsymbol{\theta} \in \mathbb{R}^{|K_n|}$, and $\boldsymbol{\gamma} \in \mathbb{R}^{|K_{n+1}|-|K_n|}$, where $\bar{\boldsymbol{u}} = (\tilde{\boldsymbol{u}}, \hat{\boldsymbol{u}})$ and $\bar{\boldsymbol{v}} = (\tilde{\boldsymbol{v}}, \hat{\boldsymbol{v}})$ follow the same pattern of partitioning as $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}, \hat{\boldsymbol{x}})$ and $\bar{\boldsymbol{y}} = (\tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}})$.

Given an extreme point $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ of $G_n := \{(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \text{QLO}(n) : \tilde{\boldsymbol{a}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{b}}^T\tilde{\boldsymbol{y}} = c\}$, there is a permutation $\tilde{\boldsymbol{\pi}}$ of n objects corresponding to $\tilde{\boldsymbol{x}}$. For each object $r = 1, \ldots, n$, we construct two points $(\bar{\boldsymbol{x}}^{r+}, \bar{\boldsymbol{y}}^{r+})$ and $(\bar{\boldsymbol{x}}^{r-}, \bar{\boldsymbol{y}}^{r-})$ in $\mathbb{R}^{|I_{n+1}|} \times \mathbb{R}^{|J_{n+1}|}$ as follows.

$$\bar{x}^{r+}_{ij} = \begin{cases} x_{ij}, & \text{if } (i,j) \in I_n \\ x_{ir}, & \text{if } 1 \le i < r,\ j = n+1 \\ 1 - x_{ri}, & \text{if } r < i \le n,\ j = n+1 \\ 1, & \text{if } i = r,\ j = n+1 \end{cases}, \qquad \bar{y}^{r+}_{ijkl} = \bar{x}^{r+}_{ij}\bar{x}^{r+}_{kl}$$

and

$$\bar{x}_{ij}^{r-} = \begin{cases} x_{ij}, & \text{if } (i,j) \in I_n \\ x_{ir}, & \text{if } 1 \le i < r,\ j = n+1 \\ 1 - x_{ri}, & \text{if } r < i \le n,\ j = n+1 \\ 0, & \text{if } i = r,\ j = n+1 \end{cases}, \qquad \bar{y}_{ijkl}^{r-} = \bar{x}_{ij}^{r-}\bar{x}_{kl}^{r-}.$$

By construction, $\bar{\boldsymbol{x}}^{r+}/\bar{\boldsymbol{x}}^{r-}$ represents the permutation of $n+1$ objects, in which objects $1,\dots,n$ keep the order as in $\tilde{\boldsymbol{\pi}}$ and object $n+1$ is placed right after/before object r. As a result, $(\bar{\boldsymbol{x}}^{r+}, \bar{\boldsymbol{y}}^{r+})$ and $(\bar{\boldsymbol{x}}^{r-}, \bar{\boldsymbol{y}}^{r-})$ are extreme points of QLO$(n+1)$. Let $P := \{(\bar{\boldsymbol{x}}^{r\pm}, \bar{\boldsymbol{y}}^{r\pm}) : r = 1,\dots,n\}$ be the set of points defined as above. Then $P \subseteq G_{n+1}$ since $\bar{\boldsymbol{a}}^T\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}^T\bar{\boldsymbol{y}} = c$ for all $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in P$.

Among the points in P, we use $(\bar{\boldsymbol{x}}^0, \bar{\boldsymbol{y}}^0)$ to denote the one such that

$$\bar{x}_{ij}^0 = \begin{cases} x_{ij}, & \text{if } (i,j) \in I_n \\ 0, & \text{otherwise} \end{cases}, \qquad \bar{y}_{ijk\ell}^0 = \bar{x}_{ij}^0\bar{x}_{k\ell}^0 = \begin{cases} y_{ijk\ell}, & \text{if } (i,j,k,\ell) \in J_n \\ 0, & \text{otherwise} \end{cases}.$$

Note that $(\bar{\boldsymbol{x}}^0, \bar{\boldsymbol{y}}^0) \in P$ because $\bar{\boldsymbol{x}}^0$ represents the permutation of $n+1$ objects, in which objects $1,\dots,n$ keep the order as in $\tilde{\boldsymbol{\pi}}$ and object $n+1$ is placed at the end of the queue. Since (3.17) holds for all $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in G_{n+1}$, for each extreme point $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ of G_n,

$$\tilde{\boldsymbol{u}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{y}} = \bar{\boldsymbol{u}}^T\bar{\boldsymbol{x}}^0 + \bar{\boldsymbol{v}}^T\bar{\boldsymbol{y}}^0 = w. \tag{3.18}$$

Consequently,

$$\tilde{\boldsymbol{u}}^T\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{y}} = w$$

for every point $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in G_n$. By Lemma 3.2.3, we know that there exist $\alpha \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbb{R}^{|K_n|}$ such that

$$\begin{bmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}} \\ w \end{bmatrix} = \alpha \begin{bmatrix} \tilde{\boldsymbol{a}} \\ \tilde{\boldsymbol{b}} \\ c \end{bmatrix} + \begin{bmatrix} \widetilde{A}^T \\ \widetilde{B}^T \\ \boldsymbol{0}^T \end{bmatrix} \boldsymbol{\theta}.$$

We complete the proof by showing that there exists $\boldsymbol{\gamma} \in \mathbb{R}^{|K_{n+1}|-|K_n|}$ such that

$$\begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \widehat{A}^T \\ \widehat{B}^T \end{bmatrix} \boldsymbol{\gamma}.$$

Applying Lemma 3.2.6 to $(\bar{\boldsymbol{x}}^{r\pm}, \bar{\boldsymbol{y}}^{r\pm})$ with $p = r$ and $q = n+1$, we have

$$u_{r,n+1} + \left(\sum_{\substack{1 \le i \le r \\ i < j \le n}} + \sum_{\substack{1 \le i < r \\ j = n+1}} \right) v_{i,j,r,n+1}\, \bar{x}_{ij}^{r+} + \sum_{r < i < j \le n+1} v_{r,n+1,i,j}\, \bar{x}_{ij}^{r+} = 0.$$

After substituting $\bar{x}_{ij}^{r+}$ and rearranging similar terms, we get

$$\begin{aligned} \sum_{r<i<j\le n} v_{r,n+1,i,j}\, x_{ij} + \sum_{\substack{1 \le i < r \\ i < j \le n \\ j \ne r}} v_{i,j,r,n+1}\, x_{ij} + \sum_{1 \le i < r} \left(v_{i,r,r,n+1} + v_{i,n+1,r,n+1}\right) x_{ir} \\ + \sum_{r < i \le n} \left(v_{r,i,r,n+1} - v_{r,n+1,i,n+1}\right) x_{ri} = -\left(u_{r,n+1} + \sum_{r<i\le n} v_{r,n+1,i,n+1} \right). \end{aligned} \tag{3.19}$$

Let (3.19) be denoted by $(\tilde{\boldsymbol{u}}')^T \tilde{\boldsymbol{x}} = w'$. Since $(\tilde{\boldsymbol{u}}')^T \tilde{\boldsymbol{x}} = w'$ holds for all extreme points of G_n, it is satisfied by all points of G_n. By Lemma 3.2.3, there exist $\lambda \in \mathbb{R}$ and $\boldsymbol{\mu} \in \mathbb{R}^{|K_n|}$ such that

$$\begin{bmatrix} \tilde{\boldsymbol{u}}' \\ \boldsymbol{0} \\ w' \end{bmatrix} = \lambda \begin{bmatrix} \tilde{\boldsymbol{a}} \\ \tilde{\boldsymbol{b}} \\ c \end{bmatrix} + \begin{bmatrix} \widetilde{A}^T \\ \widetilde{B}^T \\ 0 \end{bmatrix} \boldsymbol{\mu}.$$

Since $\tilde{\boldsymbol{b}} \notin \mathrm{Range}(\widetilde{B}^T)$ by Proposition 3.2.4, λ must be zero. Moreover, $\boldsymbol{\mu} = \boldsymbol{0}$ because the columns of $\widetilde{B}^T$ are linearly independent. Therefore, $\tilde{\boldsymbol{u}}' = \boldsymbol{0}$ and $w' = 0$. For each $r = 1, \ldots, n$, the following equations hold:

$$v_{r,n+1,i,j} = 0, \qquad \forall\ r < i < j \le n \tag{3.20}$$

$$v_{i,j,r,n+1} = 0, \qquad \forall\ 1 \le i < r,\ i < j \le n,\ j \ne r \tag{3.21}$$

$$v_{i,r,r,n+1} + v_{i,n+1,r,n+1} = 0, \qquad \forall\ 1 \le i < r \tag{3.22}$$

$$v_{r,i,r,n+1} - v_{r,n+1,i,n+1} = 0, \qquad \forall\ r < i \le n \tag{3.23}$$

$$u_{r,n+1} + \sum_{r<i\le n} v_{r,n+1,i,n+1} = 0. \tag{3.24}$$

Let $\mathcal{L}$ be the system of equations (3.20–3.24) of $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$ for all $r = 1, \dots, n$. Since each of the variables in (3.20–3.21) and each of the first variables in (3.22–3.24) appear only once in $\mathcal{L}$, $\mathcal{L}$ is a linearly independent system. Moreover, the total number of equations in $\mathcal{L}$ is

$$|\mathcal{L}| := n\left[\binom{n-r}{2} + \left(\binom{r-1}{2} + (r-1)(n-1)\right) + (r-1) + (n-r) + 1\right] = n\left[\binom{n}{2} + 1\right].$$

Let $S_{\mathcal{L}}$ be the solution set of $\mathcal{L}$. Then $S_{\mathcal{L}}$ is a subspace of $\mathbb{R}^{(|I_{n+1}|-|I_n|)} \times \mathbb{R}^{(|J_{n+1}|-|J_n|)}$ with dimension

$$\dim S_{\mathcal{L}} = (|I_{n+1}| - |I_n|) + (|J_{n+1}| - |J_n|) - |\mathcal{L}| = \binom{n}{2}.$$

Noticing the structure of the equations in Lemma 3.2.2, it is not difficult to check that the coefficients of each equation in $\begin{bmatrix} \widehat{A} & \widehat{B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{bmatrix} = \mathbf{0}$ satisfy the system $\mathcal{L}$. Therefore,

$$\text{Range}\left(\begin{bmatrix} \widehat{A}^T \\ \widehat{B}^T \end{bmatrix}\right) \subseteq S_{\mathcal{L}}.$$

Since the equations in $\begin{bmatrix} \widehat{A} & \widehat{B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{bmatrix} = \mathbf{0}$ are linearly independent,

$$\dim \operatorname{Range}\left(\begin{bmatrix} \widehat{A}^T \\ \widehat{B}^T \end{bmatrix}\right) = |K_{n+1}| - |K_n| = \binom{n}{2} = \dim S_{\mathcal{L}}.$$

As a result,

$$\begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \end{bmatrix} \in S_{\mathcal{L}} = \operatorname{Range}\left(\begin{bmatrix} \widehat{A}^T \\ \widehat{B}^T \end{bmatrix}\right),$$

which completes the proof. □

3.3 RLT implementation using $n = 3$ objects

In this section, we apply the reformulation-linearization technique (RLT) described in [3, 38, 40] to Problem P2(3) in order to obtain, upon applying a suitable projection, an explicit description of QLO(3). For this simple case, we number the objects as i, j, k, with $i < j < k$ so that, relative to the descriptions of Problems P2(3) and LP2(3), we have $I_3 = \{(i,j),(i,k),(j,k)\}$, $J_3 = \{(i,j,i,k),(i,j,j,k),(i,k,j,k)\}$, and $K_3 = \{(i,j,k)\}$. As we will demonstrate, it turns out that $S_3 = \text{QLO}(3)$, but we require only six inequalities, as opposed to the 12 found within the sets X_{ijik}, X_{ijjk}, and X_{ikjk} of (3.8) for $n = 3$. We will then show how the six missing inequalities are implied by the remaining restrictions.*

The RLT process performs the two steps of *Reformulation* and *Linearization*. There is considerable flexibility in the implementation, due primarily to the definition of the "product factors" that multiply the problem constraints to compute nonlinear inequalities. In general, these factors can consist of any products of the original problem constraints (or implications thereof). Here, we choose to compute all possible 3-way products of the three binary variables x_{ij}, x_{ik}, x_{jk} and their complements so as to obtain the cubic functions $F_1 = (1 - x_{ij})(1 - x_{ik})(1 - x_{jk})$, $F_2 = x_{ij}(1-x_{ik})(1-x_{jk})$, $F_3 = (1-x_{ij})x_{ik}(1-x_{jk})$, $F_4 = (1-x_{ij})(1-x_{ik})x_{jk}$, $F_5 = x_{ij}x_{ik}(1-x_{jk})$,

*In fact, $(i,j,k) = (1,2,3)$. We use generic (i,j,k) for easier reference.

$F_6 = x_{ij}(1-x_{ik})x_{jk}$, $F_7 = (1-x_{ij})x_{ik}x_{jk}$, and $F_8 = x_{ij}x_{ik}x_{jk}$. These functions are clearly nonnegative for all $0 \leq x_{ij}, x_{ik}, x_{jk} \leq 1$. We then use these factors as follows.

Reformulation

Multiply the two inequalities of (3.3) by each of the eight nonnegative functions F_j, $j = 1, \ldots, 8$, enforcing the idempotent property that each binary variable squared is itself, and maintaining the inequality restrictions. Include the binary restrictions on x_{ij}, x_{ik}, x_{jk}. Notably, $F_3(x_{ij}+x_{jk}-x_{ik}) \geq 0$ and $F_3(1-x_{ij}-x_{jk}+x_{ik}) \geq 0$ combine to give

$$(1-x_{ij})x_{ik}(1-x_{jk}) = 0, \tag{3.25}$$

while $F_6(x_{ij}+x_{jk}-x_{ik}) \geq 0$ and $F_6(1-x_{ij}-x_{jk}+x_{ik}) \geq 0$ combine to give

$$x_{ij}(1-x_{ik})x_{jk} = 0, \tag{3.26}$$

which together imply

$$x_{ik}x_{jk} = x_{ik} - x_{ij}x_{ik} + x_{ij}x_{jk}, \tag{3.27}$$

where (3.27) is (3.6), attributed to [6].

Linearization

Linearize the resulting problem by substituting $y_{ijik} = x_{ij}x_{ik}$, $y_{ijjk} = x_{ij}x_{jk}$, $y_{ikjk} = x_{ik}x_{jk}$, and $y_{ijikjk} = x_{ij}x_{ik}x_{jk}$ throughout the objective function and constraints.

The feasible region to the resulting problem consists of the binary restrictions on x_{ij}, x_{ik}, x_{jk}, the two equations

$$x_{ik} - y_{ijik} - y_{ikjk} + y_{ijikjk} = 0 \tag{3.28}$$

and

$$y_{ijjk} - y_{ijikjk} = 0, \tag{3.29}$$

which are the linearized versions of (3.25) and (3.26), respectively, and the six inequalities expressed in matrix notation below.

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_{ij} \\ x_{ik} \\ x_{jk} \\ y_{ijik} \\ y_{ijjk} \\ y_{ikjk} \\ y_{ijikjk} \end{pmatrix} \geq \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.30}$$

Restrictions (3.28)–(3.30) can be simplified. Use (3.29) to substitute the variable y_{ijikjk} from (3.28) and (3.30), and remove (3.29). This substitution effectively projects the region defined by (3.28)–(3.30) onto the space of the variables $(x_{ij}, x_{ik}, x_{jk}, y_{ijik}, y_{ijjk}, y_{ikjk})$. Equation (3.28) becomes

$$x_{ik} - y_{ijik} + y_{ijjk} - y_{ikjk} = 0, \tag{3.31}$$

which is the linearized form of (3.27). Then add equation (3.31) to each of the first, fourth, and fifth (revised) inequalities of (3.30). The result is the polytope Z_{ijk} below, together with the binary restrictions on x_{ij}, x_{ik}, x_{jk}.

$$\begin{aligned} Z_{ijk} \equiv \{(x_{ij}, x_{ik}, x_{jk}, y_{ijik}, y_{ijjk}, y_{ikjk}) : (3.31),\ & y_{ijik} \leq x_{ij},\ y_{ijik} \leq x_{ik}, \\ & y_{ikjk} \leq x_{ik},\ y_{ikjk} \leq x_{jk},\ 0 \leq y_{ijjk},\ x_{ij} + x_{jk} - 1 \leq y_{ijjk}\} \end{aligned} \tag{3.32}$$

The polytope Z_{ijk} has special structure. By construction, the RLT theory [3, 38, 40] gives us that it has exactly six extreme points, and that each such point corresponds to a realization of (x_{ij}, x_{ik}, x_{jk}) that identifies a valid ordering, and has $y_{ijik} = x_{ij}x_{ik}$, $y_{ijjk} = x_{ij}x_{jk}$, and $y_{ikjk} = x_{ik}x_{jk}$. Then we have $Z_{ijk} = \text{QLO}(3)$.

Note the relationship between the set Z_{ijk} of (3.32) and the set S_3 of (3.8). Equation (3.31) of (3.32) is as found in L_{ijk} of (3.8). Since all six inequalities of Z_{ijk} are found in S_3, we can use the fact that $Z_{ijk} = \text{QLO}(3)$ to conclude that $S_3 = \text{QLO}(3)$. The additional six inequalities found within S_3 but not within Z_{ijk} must then be implied by the restrictions of Z_{ijk}. Each of these additional inequalities is listed in the table below, together with two restrictions in Z_{ijk} that imply it in the presence of equation (3.31). The first column gives the inequality, and the second column gives the associated restrictions in Z_{ijk}.

Table 3.1: Inequalities in S_3 but not in Z_{ijk} that are implied by the restrictions in Z_{ijk}.

Inequality in S_3 but not in Z_{ijk}	Restrictions in Z_{ijk} that imply the inequality
$y_{ijik} \geq 0$	$y_{ikjk} \leq x_{ik},\ \ y_{ijjk} \geq 0$
$y_{ijik} \geq x_{ij} + x_{ik} - 1$	$y_{ikjk} \leq x_{jk},\ \ y_{ijjk} \geq x_{ij} + x_{jk} - 1$
$y_{ijjk} \leq x_{ij}$	$y_{ikjk} \leq x_{ik},\ \ y_{ijik} \leq x_{ij}$
$y_{ijjk} \leq x_{jk}$	$y_{ijik} \leq x_{ik},\ \ y_{ikjk} \leq x_{jk}$
$y_{ikjk} \geq 0$	$y_{ijik} \leq x_{ik},\ \ y_{ijjk} \geq 0$
$y_{ikjk} \geq x_{ik} + x_{jk} - 1$	$y_{ijik} \leq x_{ij},\ \ y_{ijjk} \geq x_{ij} + x_{jk} - 1$

Problem LP2(n) can now be made more concise in size using two reductions. The first reduction follows from the above discussion.

1. For each $(i, j, k, \ell) \in J_n$ with (i, j, k, ℓ) not distinct; that is, for each 4-tuple of the form (i, j, i, k), (i, j, j, k), or (i, k, j, k) for some $(i, j, k) \in K_n$, half the inequalities defining $X_{ijk\ell}$ of (3.5) are redundant in LP2(n) and can be removed. The above discussion and Table 3.1 explain that, for each $(i, j, k) \in K_n$, the three sets X_{ijik}, X_{ijjk}, and X_{ikjk} of (3.8) can be replaced with the single set Z_{ijk} of (3.32). Since there are $3|K_n|$ such sets $X_{ijk\ell}$, the net savings is $6|K_n|$ inequalities.

2. For each $(i, j, k, \ell) \in J_n$ with (i, j, k, ℓ) distinct, half the inequalities defining $X_{ijk\ell}$ of (3.5) are redundant at optimality and can be removed. Specifically, we can apply an observation of [1] to Problem LP2(n) to recognize that, for each such (i, j, k, ℓ) having $B_{ijk\ell} \geq 0$, the two upper-bounding inequalities on $y_{ijk\ell}$ of (3.5) are redundant at optimality, while for each such (i, j, k, ℓ) having $B_{ijk\ell} \leq 0$, the two lower-bounding inequalities on $y_{ijk\ell}$ of (3.5) are redundant at optimality. Consequently, each such set $X_{ijk\ell}$ can be represented using at most two inequalities. Since there are $|J_n| - 3|K_n|$ such sets $X_{ijk\ell}$, the net savings is $2\left(|J_n| - 3|K_n|\right)$ inequalities. (In fact, as noted in [16] for the case in which $B_{ijk\ell} < 0$, a variable substitution $w_{ijk\ell} = x_{ij} - y_{ijk\ell}$, or similarly $w_{ijk\ell} = x_{k\ell} - y_{ijk\ell}$, can reduce $X_{ijk\ell}$ to having a single structural

inequality.) For each such (i,j,k,ℓ), the removal of the two inequalities from the set $X_{ijk\ell}$ gives rise to the sets below, where $X'_{ijk\ell}$ is obtained from $X_{ijk\ell}$. Here, if $B_{ijk\ell}=0$, then the set $X_{ijk\ell}$ can be removed.

$$X'_{ijk\ell} \equiv \{(x_{ij}, x_{k\ell}, y_{ijk\ell}) : 0 \le y_{ijk\ell},\ x_{ij} + x_{k\ell} - 1 \le y_{ijk\ell}\}$$
$$\forall\ (i,j,k,\ell) \in J_n,\ (i,j,k,\ell) \text{ distinct},\ B_{ijk\ell} > 0 \tag{3.33}$$

$$X'_{ijk\ell} \equiv \{(x_{ij}, x_{k\ell}, y_{ijk\ell}) : y_{ijk\ell} \le x_{ij},\ y_{ijk\ell} \le x_{k\ell}\}$$
$$\forall\ (i,j,k,\ell) \in J_n,\ (i,j,k,\ell) \text{ distinct},\ B_{ijk\ell} < 0 \tag{3.34}$$

Upon applying these reductions to Problem LP2(n), the following problem emerges.

$$\text{LP2}'(n): \quad \text{minimize} \quad \sum_{(i,j)\in I_n} b_{ij}x_{ij} + \sum_{(i,j,k,\ell)\in J_n} B_{ijk\ell}y_{ijk\ell}$$

subject to $\boldsymbol{x}$ binary,

$(x_{ij}, x_{k\ell}, y_{ijk\ell}) \in X'_{ijk\ell}\ \forall\ (i,j,k,\ell) \in J_n,\ (i,j,k,\ell)$ distinct, $B_{ijk\ell} \neq 0$

$(x_{ij}, x_{ik}, x_{jk}, y_{ijik}, y_{ijjk}, y_{ikjk}) \in Z_{ijk}\ \forall\ (i,j,k) \in K_n$

In summary, the size of Problem LP2(n) compares to LP2$'(n)$ as follows. As mentioned in Section 3.1, Problem LP2(n) contains $\left(\binom{n}{2} + 3\binom{n+1}{4}\right)$ variables, $12\binom{n+1}{4}$ inequality restrictions, and $\binom{n}{3}$ equality restrictions. Problem LP2$'(n)$ contains $6\binom{n+1}{4}$ fewer inequality restrictions due to reductions 1 and 2 above.

We conclude this section with two remarks.

Remark 3.3.1. *We can reduce the size of Problem LP2$'(n)$ by $\binom{n}{3}$ variables $\boldsymbol{y}$ and $\binom{n}{3}$ equality restrictions via substitution. Specifically, for each $(i,j,k) \in K_n$, we can use equation* (3.7) *of L_{ikj} found in LP2$'(n)$ to set $y_{ikjk} = x_{ik} - y_{ijik} + y_{ijjk}$, and then remove the variable y_{ikjk} and this* *equation from the formulation. Each set Z_{ijk} with $(i,j,k) \in K_n$ would then become*

$$Z'_{ijk} \equiv \{(x_{ij}, x_{ik}, x_{jk}, y_{ijik}, y_{ijjk}) : y_{ijik} \leq x_{ij},\ y_{ijik} \leq x_{ik},\ y_{ijjk} \leq y_{ijik},$$
$$x_{ik} + y_{ijjk} \leq x_{jk} + y_{ijik},\ 0 \leq y_{ijjk},\ x_{ij} + x_{jk} - 1 \leq y_{ijjk}\}. \quad (3.35)$$

While theoretically appealing, these substitutions yield a more dense coefficient matrix which, as later mentioned in Section 3.5, do not necessarily reduce computation times.

Remark 3.3.2. *For each $(i,j,k) \in K_n$, the equation (3.27) can be alternately derived using a different RLT implementation. For each $(i,j,k) \in K_n$, the expression $(x_{ij} + x_{jk} - x_{ik})$ of (3.3) is itself binary. Then we can conclude that $(x_{ij} + x_{jk} - x_{ik}) = (x_{ij} + x_{jk} - x_{ik})^2$ which, upon applying the binary identities $x_{ij} = x_{ij}^2$, $x_{jk} = x_{jk}^2$, and $x_{ik} = x_{ik}^2$, and dividing by 2, gives (3.27).*

3.4 RLT implementation using $n = 4$ objects

This section gives an explicit description of QLO(4). Section 3.3 showed, for the case having $n = 3$ objects $i < j < k$, that $S_3 = \text{QLO}(3)$, and that the set Z_{ijk} of (3.32) can be obtained from S_3 by removing 6 redundant inequalities. However, for $n \geq 4$, the restrictions of (3.8) are not sufficient to describe QLO(n). Our description of QLO(4) consists of 126 inequalities, characterized in terms of five families, together with the four equations:

$$y_{1323} = x_{13} - y_{1213} + y_{1223},\ y_{1424} = x_{14} - y_{1214} + y_{1224},$$
$$y_{1434} = x_{14} - y_{1314} + y_{1334},\ y_{2434} = x_{24} - y_{2324} + y_{2334}, \quad (3.36)$$

that are of the form (3.31). The inequalities are expressed in terms of the six variables x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, and the eleven variables y_{1213}, y_{1214}, y_{1223}, y_{1224}, y_{1234}, y_{1314}, y_{1324}, y_{1334}, y_{1423}, y_{2324}, and y_{2334}. The four equations (3.36) serve to link the variables y_{1323}, y_{1424}, y_{1434}, and y_{2434} to these seventeen variables.

We use the following notation. As earlier defined, we have $I_4 = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$, $J_4 = \{(1,2,1,3),\ (1,2,1,4),\ (1,2,2,3),\ (1,2,2,4),\ (1,2,3,4),\ (1,3,1,4),\ (1,3,2,3),$

$(1,3,2,4)$, $(1,3,3,4)$, $(1,4,2,3)$, $(1,4,2,4)$, $(1,4,3,4)$, $(2,3,2,4)$, $(2,3,3,4)$, $(2,4,3,4)\}$, and $K_4 = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$. We let

$$\mathcal{P} \equiv \{(i,j,k,\ell) : \ i,j,k,\ell \text{ distinct}\}$$

be the set of all 24 permutations of the numbers 1 through 4, and

$$\mathcal{P}' \equiv \{(i,j,k,\ell) : \ i,j,k,\ell \text{ distinct}, \ i < \ell\}$$

be the set of all 12 such permutations having the first index less than the fourth. We also let

$$D_{ijk} = (x_{ij} + x_{jk} - x_{ik}) \ \forall \ (i,j,k) \in K_4 \tag{3.37}$$

to represent the expressions found within (3.3) for $n = 4$.

We again use the RLT to obtain the convex hull in a higher-variable space, and then project onto the desired $(\boldsymbol{x}, \boldsymbol{y})$ space. Begin by computing $2^6 = 64$ distinct 6^{th} degree polynomials, say $F_1, \ldots, F_{64}$, to serve as the RLT product factor multipliers in the following way. For each of the six binary variables x_{ij}, choose either the binary variable itself or its complement $(1 - x_{ij})$, and then compute the product of the six selections. Define each function F_j to be one such 6^{th} degree polynomial, and observe that every F_j is nonnegative for all $0 \leq x_{ij} \leq 1$. Perform the following operations, analogous to those found in Section 3.3 for the case of $n = 3$. (Note that the functions F_j defined here differ from the functions F_j of Section 3.3 due to the value of n; here we take products of the six above-mentioned variables x_{ij} and their complements since $n = 4$, while Section 3.3 used only the three variables x_{12}, x_{13}, x_{23} and their complements since $n = 3$.)

Reformulation

Multiply the eight inequalities of (3.3) by each of the 64 functions F_j, $j = 1, \ldots, 64$, enforcing the idempotent property that each binary variable squared is itself, and maintaining the inequality restrictions. Include the binary restrictions on x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}.

Linearization

Linearize the resulting problem by substituting a continuous variable for each of the 57 distinct products of two or more binary variables throughout the objective function and constraints. Included within these new variables are the $15 = |J_4|$ linearized quadratic variables $y_{ijk\ell}$ for all $(i, j, k, \ell) \in J_4$.

Upon simplification, the RLT yields 63 variables in 64 restrictions, with 24 restrictions being inequalities and the remaining 40 being equalities. In fact, one inequality exists for each of the $4! = 24$ feasible arrangements of the objects. The 40 equalities are linearly independent, and can be expressed so that the columns corresponding to 36 of the linearized cubic and higher-degree variables are linearly independent, and so that the remaining four equations are (3.36). We use this independence to remove, via substitution, these 36 cubic and higher-degree variables, and also the four variables $y_{1323}, y_{1424}, y_{1434}, y_{2434}$, resulting in a system of 24 linear inequalities in 6 variables x_{ij}, 11 linearized quadratic variables, and 6 linearized higher-degree variables. The task now is to compute the extreme directions of the projection cone that is defined in terms of the 24×6 matrix that consists of the columns of the remaining linearized cubic and higher-degree variables. Then the set QLO(4) will be given by the projected inequalities, together with the four equations of (3.36).

We enumerated all the extreme directions using PORTA. The resulting 126 inequalities are categorized into five families below, with each family explained in terms of the RLT. Since the restrictions at any given level of the RLT (other than the highest level) are known [3, 38, 40] to be implied by the restrictions from the level above it, we simplify our presentation by explaining these inequalities in terms of lower-level RLT restrictions, wherever possible. For further simplification, we present the inequalities as quadratic expressions, computed prior to the linearization operation. Quadratic expressions $x_{ij}x_{k\ell}$ within the tables are written so that $(i, j, k, \ell) \in J_4$. Consistent with the above discussion, for each $(i, j, k) \in K_4$, we used the substitutions of (3.27) to remove all occurrences of the products $x_{13}x_{23}$, $x_{14}x_{24}$, $x_{14}x_{34}$, and $x_{24}x_{34}$ from Families 2 through 5. (Such

$y_{ijk\ell} = x_{ij}x_{k\ell}$ expressions were already removed from the sets Z'_{ijk} found in Family 1.) Immediately following each family are remarks for clarification and insight.

Each of the 126 linearized inequalities is indeed a facet of QLO(4). We computationally verified this result as follows; since these inequalities collectively define QLO(4), for each such inequality expressed in generic form as $\boldsymbol{ax} \geq b$, we minimized the expression $\boldsymbol{ax}$ over the remaining 125 inequalities to obtain a value $b' < b$.

Five Families of Inequalities for QLO(4)

1. For each $(i, j, k) \in K_4$, include the six inequalities from Z'_{ijk} of (3.35). For each $(i, j, k, \ell) \in J_n$ with (i, j, k, ℓ) distinct, include the four inequalities from (3.5). Since $n = 4$, there exist $6\binom{4}{3} + 4 \times 3\binom{4}{4} = 36$ such inequalities. The quadratic forms of these inequalities are found in Table 3.2. The first column gives either the 3-tuple $(i, j, k) \in K_4$ or the 4-tuple (i, j, k, ℓ) with (i, j, k, ℓ) distinct, and the second column gives the associated inequalities.

Table 3.2: $(i, j, k) \in K_4$ or $(i, j, k, \ell) \in J_4, (i, j, k, \ell)$ distinct, and Family 1 Inequalities

$(i, j, k) \in K_4$	Quadratic Inequalities
$(1, 2, 3)$	$x_{12} + x_{23} - 1 \leq x_{12}x_{23} \leq x_{12}x_{13} \leq x_{12},\ x_{13} - x_{23} + x_{12}x_{23} \leq x_{12}x_{13} \leq x_{13},\ 0 \leq x_{12}x_{23}$
$(1, 2, 4)$	$x_{12} + x_{24} - 1 \leq x_{12}x_{24} \leq x_{12}x_{14} \leq x_{12},\ x_{14} - x_{24} + x_{12}x_{24} \leq x_{12}x_{14} \leq x_{14},\ 0 \leq x_{12}x_{24}$
$(1, 3, 4)$	$x_{13} + x_{34} - 1 \leq x_{13}x_{34} \leq x_{13}x_{14} \leq x_{13},\ x_{14} - x_{34} + x_{13}x_{34} \leq x_{13}x_{14} \leq x_{14},\ 0 \leq x_{13}x_{34}$
$(2, 3, 4)$	$x_{23} + x_{34} - 1 \leq x_{23}x_{34} \leq x_{23}x_{24} \leq x_{23},\ x_{24} - x_{34} + x_{23}x_{34} \leq x_{23}x_{24} \leq x_{24},\ 0 \leq x_{23}x_{34}$
$(i, j, k, \ell) \in J_4$, (i, j, k, ℓ) distinct	
$(1, 2, 3, 4)$	$0 \leq x_{12}x_{34} \leq x_{12}, \quad x_{12} + x_{34} - 1 \leq x_{12}x_{34} \leq x_{34}$
$(1, 3, 2, 4)$	$0 \leq x_{13}x_{24} \leq x_{13}, \quad x_{13} + x_{24} - 1 \leq x_{13}x_{24} \leq x_{24}$
$(1, 4, 2, 3)$	$0 \leq x_{14}x_{23} \leq x_{14}, \quad x_{14} + x_{23} - 1 \leq x_{14}x_{23} \leq x_{23}$

Remark 3.4.1. *Reduction 2 of the previous section is not applicable in this setting because, in the presence of the remaining families of inequalities, the deleted inequalities are no longer redundant. Thus, for each $(i, j, k, \ell) \in J_4$ with (i, j, k, ℓ) distinct, we use all four inequalities of the set $X_{ijk\ell}$ of* (3.5)*, as opposed to the two inequalities of the more concise set $X'_{ijk\ell}$ of* (3.33) *or* (3.34)*.*

2. For each $(i, j, k) \in K_4$, separately multiply the two inequalities in (3.3) by each x_{rs} and

$(1 - x_{rs})$, where $r < s$ and $\{r, s\} \not\subset \{i, j, k\}$; that is, compute

$$(0 \leq D_{ijk} \leq 1)\, x_{rs} \;\forall\; (i, j, k, r, s) \in \Omega$$

and

$$(0 \leq D_{ijk} \leq 1)\, (1 - x_{rs}) \;\;\forall\; (i, j, k, r, s) \in \Omega$$

to obtain, respectively, the two families of inequalities

$$0 \leq x_{ij}x_{rs} + x_{jk}x_{rs} - x_{ik}x_{rs} \leq x_{rs} \;\forall\; (i, j, k, r, s) \in \Omega \tag{3.38}$$

and

$$0 \leq x_{ij} + x_{jk} - x_{ik} - x_{ij}x_{rs} - x_{jk}x_{rs} + x_{ik}x_{rs} \leq 1 - x_{rs} \;\forall\; (i, j, k, r, s) \in \Omega, \tag{3.39}$$

where $\Omega \equiv \big\{(i, j, k, r, s) : (i, j, k) \in K_4, r < s, \{r, s\} \not\subset \{i, j, k\}\big\}$. We have $|\Omega| = 3\binom{4}{3} = 12$, so that (3.38) and (3.39) consist of 48 inequalities. These inequalities are found in Table 3.3. The first column gives the 5-tuple $(i, j, k, r, s) \in \Omega$, and the second column gives the two corresponding inequalities (3.38) followed by the two inequalities (3.39). Inequalities from distinct 5-tuples are delineated by horizontal lines so that, for example, the first four inequalities correspond to (3.38) and (3.39) for $(1, 2, 3, 1, 4) \in \Omega$.

Remark 3.4.2. *We defined the set Ω to have $\{r, s\} \not\subset \{i, j, k\}$ for each $(i, j, k) \in K_4$ since otherwise the inequalities* (3.38) *and* (3.39) *would be valid for QLO*(3) *in the variables x_{ij}, x_{ik}, x_{jk}, and thus would be implied by the Family 1 inequalities.*

3. For each distinct (i, j) pair, let $\alpha_{ij} = x_{ij}$ if $i < j$ and let $\alpha_{ij} = 1 - x_{ji}$ if $i > j$. Then, for each $(i, j, k, \ell) \in \mathcal{P}'$, sum the two RLT inequalities

$$(1 - \alpha_{ij})(1 - \alpha_{jk})(1 - \alpha_{k\ell}) \geq 0 \text{ and } \alpha_{ij}\alpha_{jk}\alpha_{k\ell} \geq 0 \tag{3.40}$$

Table 3.3: 5-tuple $(i,j,k,r,s) \in \Omega$, and Family 2 Inequalities

$(i,j,k,r,s) \in \Omega$	Quadratic Inequalities
$(1,2,3,1,4)$	$0 \leq x_{12}x_{14} - x_{13}x_{14} + x_{14}x_{23} \leq x_{14}$ $0 \leq x_{12} - x_{13} + x_{23} - x_{12}x_{14} + x_{13}x_{14} - x_{14}x_{23} \leq 1 - x_{14}$
$(1,2,3,2,4)$	$0 \leq x_{12}x_{24} - x_{13}x_{24} + x_{23}x_{24} \leq x_{24}$ $0 \leq x_{12} - x_{13} + x_{23} - x_{12}x_{24} + x_{13}x_{24} - x_{23}x_{24} \leq 1 - x_{24}$
$(1,2,3,3,4)$	$0 \leq x_{12}x_{34} - x_{13}x_{34} + x_{23}x_{34} \leq x_{34}$ $0 \leq x_{12} - x_{13} + x_{23} - x_{12}x_{34} + x_{13}x_{34} - x_{23}x_{34} \leq 1 - x_{34}$
$(1,2,4,1,3)$	$0 \leq x_{12}x_{13} - x_{13}x_{14} + x_{13}x_{24} \leq x_{13}$ $0 \leq x_{12} - x_{14} + x_{24} - x_{12}x_{13} + x_{13}x_{14} - x_{13}x_{24} \leq 1 - x_{13}$
$(1,2,4,2,3)$	$0 \leq x_{12}x_{23} - x_{14}x_{23} + x_{23}x_{24} \leq x_{23}$ $0 \leq x_{12} - x_{14} + x_{24} - x_{12}x_{23} + x_{14}x_{23} - x_{23}x_{24} \leq 1 - x_{23}$
$(1,2,4,3,4)$	$0 \leq -x_{14} + x_{24} + x_{12}x_{34} + x_{13}x_{14} - x_{13}x_{34} - x_{23}x_{24} + x_{23}x_{34} \leq x_{34}$ $0 \leq x_{12} - x_{12}x_{34} - x_{13}x_{14} + x_{13}x_{34} + x_{23}x_{24} - x_{23}x_{34} \leq 1 - x_{34}$
$(1,3,4,1,2)$	$0 \leq x_{12}x_{13} - x_{12}x_{14} + x_{12}x_{34} \leq x_{12}$ $0 \leq x_{13} - x_{14} + x_{34} - x_{12}x_{13} + x_{12}x_{14} - x_{12}x_{34} \leq 1 - x_{12}$
$(1,3,4,2,3)$	$0 \leq x_{13} - x_{12}x_{13} + x_{12}x_{23} - x_{14}x_{23} + x_{23}x_{34} \leq x_{23}$ $0 \leq -x_{14} + x_{34} + x_{12}x_{13} - x_{12}x_{23} + x_{14}x_{23} - x_{23}x_{34} \leq 1 - x_{23}$
$(1,3,4,2,4)$	$0 \leq -x_{14} + x_{24} + x_{12}x_{14} - x_{12}x_{24} + x_{13}x_{24} - x_{23}x_{24} + x_{23}x_{34} \leq x_{24}$ $0 \leq x_{13} - x_{24} + x_{34} - x_{12}x_{14} + x_{12}x_{24} - x_{13}x_{24} + x_{23}x_{24} - x_{23}x_{34} \leq 1 - x_{24}$
$(2,3,4,1,2)$	$0 \leq x_{12}x_{23} - x_{12}x_{24} + x_{12}x_{34} \leq x_{12}$ $0 \leq x_{23} - x_{24} + x_{34} - x_{12}x_{23} + x_{12}x_{24} - x_{12}x_{34} \leq 1 - x_{12}$
$(2,3,4,1,3)$	$0 \leq x_{13} - x_{12}x_{13} + x_{12}x_{23} - x_{13}x_{24} + x_{13}x_{34} \leq x_{13}$ $0 \leq -x_{13} + x_{23} - x_{24} + x_{34} + x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{24} - x_{13}x_{34} \leq 1 - x_{13}$
$(2,3,4,1,4)$	$0 \leq x_{12}x_{14} - x_{12}x_{24} - x_{13}x_{14} + x_{13}x_{34} + x_{14}x_{23} \leq x_{14}$ $0 \leq x_{23} - x_{24} + x_{34} - x_{12}x_{14} + x_{12}x_{24} + x_{13}x_{14} - x_{13}x_{34} - x_{14}x_{23} \leq 1 - x_{14}$

to obtain

$$1 - \alpha_{ij} - \alpha_{jk} - \alpha_{k\ell} + \alpha_{ij}\alpha_{jk} + \alpha_{ij}\alpha_{k\ell} + \alpha_{jk}\alpha_{k\ell} \geq 0.$$

We then have the 12 inequalities given by

$$1 - \alpha_{ij} - \alpha_{jk} - \alpha_{k\ell} + \alpha_{ij}\alpha_{jk} + \alpha_{ij}\alpha_{k\ell} + \alpha_{jk}\alpha_{k\ell} \geq 0 \;\forall\; (i,j,k,\ell) \in \mathcal{P}'. \tag{3.41}$$

These inequalities take the form of the triangle inequalities for the Boolean Quadric Polytope [35], and are listed in Table 3.4. The first column gives the 4-tuple $(i,j,k,\ell) \in \mathcal{P}'$, and the second column gives the corresponding inequality (3.41).

Table 3.4: 4-tuple $(i,j,k,\ell) \in \mathcal{P}'$, and Family 3 Inequalities

$(i,j,k,\ell) \in \mathcal{P}'$	Quadratic Inequality
$(1,2,3,4)$	$1 - x_{12} - x_{23} - x_{34} + x_{12}x_{23} + x_{12}x_{34} + x_{23}x_{34} \geq 0$
$(1,2,4,3)$	$-x_{24} + x_{34} + x_{12}x_{24} - x_{12}x_{34} + x_{23}x_{24} - x_{23}x_{34} \geq 0$
$(1,3,2,4)$	$-x_{13} + x_{23} + x_{12}x_{13} - x_{12}x_{23} + x_{13}x_{24} - x_{23}x_{24} \geq 0$
$(1,3,4,2)$	$-x_{13}x_{24} + x_{13}x_{34} + x_{23}x_{24} - x_{23}x_{34} \geq 0$
$(1,4,2,3)$	$-x_{14} + x_{24} + x_{12}x_{14} - x_{12}x_{24} + x_{14}x_{23} - x_{23}x_{24} \geq 0$
$(1,4,3,2)$	$x_{13}x_{14} - x_{13}x_{34} - x_{14}x_{23} + x_{23}x_{34} \geq 0$
$(2,1,3,4)$	$x_{12} - x_{12}x_{13} - x_{12}x_{34} + x_{13}x_{34} \geq 0$
$(2,1,4,3)$	$-x_{12}x_{14} + x_{12}x_{34} + x_{13}x_{14} - x_{13}x_{34} \geq 0$
$(2,3,1,4)$	$x_{12}x_{13} - x_{12}x_{23} - x_{13}x_{14} + x_{14}x_{23} \geq 0$
$(2,4,1,3)$	$x_{12}x_{14} - x_{12}x_{24} - x_{13}x_{14} + x_{13}x_{24} \geq 0$
$(3,1,2,4)$	$x_{13} - x_{12}x_{13} + x_{12}x_{24} - x_{13}x_{24} \geq 0$
$(3,2,1,4)$	$x_{14} - x_{12}x_{14} + x_{12}x_{23} - x_{14}x_{23} \geq 0$

Remark 3.4.3. *We defined the set $\mathcal{P}'$ to have $i < \ell$ so that there is not a repetition of inequalities in* (3.41)*. Here, for every $(i,j,k,\ell) \in \mathcal{P}'$, the inequality* (3.41) *would be identical for the pair of "symmetric" permutations (i,j,k,ℓ) and (ℓ,k,j,i). In addition, the structure of* (3.41) *indicates why no other triangle inequalities can possibly be facet-defining for QLO*(4)*. Given an $(i,j,k,\ell) \in \mathcal{P}'$, the inequality* (3.41) *is satisfied with equality at each of the* $4! = 24$ *extreme points of QLO*(4)*, less the two points corresponding to the permutations (i,j,k,ℓ) and (ℓ,k,j,i). Suppose now, that for any three pairs of distinct indices (i_1,j_1), (i_2,j_2), and (i_3,j_3), we compute a triangle inequality by modifying* (3.40) *to*

$$(1-\alpha_{i_1j_1})(1-\alpha_{i_2j_2})(1-\alpha_{i_3j_3}) \geq 0 \text{ and } \alpha_{i_1j_1}\alpha_{i_2j_2}\alpha_{i_3j_3} \geq 0,$$

and then summing these inequalities to obtain

$$1 - \alpha_{i_1j_1} - \alpha_{i_2j_2} - \alpha_{i_3j_3} + \alpha_{i_1j_1}\alpha_{i_2j_2} + \alpha_{i_1j_1}\alpha_{i_3j_3} + \alpha_{i_2j_2}\alpha_{i_3j_3} \geq 0. \tag{3.42}$$

By construction, provided that (3.42) *cannot be expressed in the form of* (3.41)*, the extreme points of QLO*(4) *corresponding to at least two pairs of symmetric permutations must satisfy* (3.42) *with strict inequality. Then* (3.42) *would be satisfied with equality at a strict subset of* *the extreme points satisfying at least two different inequalities of* (3.41) *with equality.*

4. Compute, and set nonnegative, six specially-selected pairwise products of the 3-dicycle expressions found within (3.3), as given in the six rows of Table 3.5 below, with one product per row. The first column states the product computed in terms of the D_{ijk} values of (3.37), and the second column gives the quadratic inequality.

Table 3.5: Product, and Family 4 Inequalities

Product	Quadratic Inequality
$D_{123}(1-D_{124})$	$-x_{13}+x_{23}+x_{12}x_{13}+x_{12}x_{14}-x_{12}x_{23}-x_{12}x_{24}-x_{13}x_{14}+x_{13}x_{24}+x_{14}x_{23}-x_{23}x_{24}\geq 0$
$(1-D_{123})D_{124}$	$-x_{14}+x_{24}+x_{12}x_{13}+x_{12}x_{14}-x_{12}x_{23}-x_{12}x_{24}-x_{13}x_{14}+x_{13}x_{24}+x_{14}x_{23}-x_{23}x_{24}\geq 0$
$D_{123}D_{134}$	$-x_{12}x_{14}+x_{12}x_{23}+x_{12}x_{34}+x_{13}x_{14}-x_{13}x_{34}-x_{14}x_{23}+x_{23}x_{34}\geq 0$
$(1-D_{123})(1-D_{134})$	$-x_{12}+x_{14}-x_{23}-x_{34}-x_{12}x_{14}+x_{12}x_{23}+x_{12}x_{34}+x_{13}x_{14}-x_{13}x_{34}-x_{14}x_{23}+x_{23}x_{34}\geq -1$
$D_{124}(1-D_{134})$	$x_{12}-x_{12}x_{13}+x_{12}x_{24}-x_{12}x_{34}-x_{13}x_{24}+x_{13}x_{34}+x_{23}x_{24}-x_{23}x_{34}\geq 0$
$(1-D_{124})D_{134}$	$x_{13}-x_{24}+x_{34}-x_{12}x_{13}+x_{12}x_{24}-x_{12}x_{34}-x_{13}x_{24}+x_{13}x_{34}+x_{23}x_{24}-x_{23}x_{34}\geq 0$

Remark 3.4.4. *There exist 24 different possible pairwise products of distinct 3-dicycle expressions, but we list only six in Table 3.5 because nonnegativity of the remaining 18 are implied by nonnegativity of the given six. To explain, first recall that the final remark of Section 3.3 gives us*

$$(D_{ijk})^2 = D_{ijk}\ \forall\ (i,j,k) \in K_4. \tag{3.43}$$

Now consider the following lemma.

Lemma 3.4.5. *Given that* (3.43) *is enforced, we have*

$$D_{123} = D_{123}(D_{124} + D_{234} - D_{134}), \tag{3.44}$$

$$D_{124} = D_{124}(D_{123} + D_{134} - D_{234}), \tag{3.45}$$

$$D_{134} = D_{134}(D_{124} + D_{234} - D_{123}), \tag{3.46}$$

and

$$D_{234} = D_{234}(D_{123} + D_{134} - D_{124}). \tag{3.47}$$

Proof. From (3.37), the definition of the variables $D_{ijk}\ \forall\ (i,j,k) \in K_4$ gives us that

$$D_{123} + D_{134} = D_{124} + D_{234}. \tag{3.48}$$

Separately multiply equation (3.48) by each of the variables D_{123}, D_{124}, D_{134}, and D_{234}, and apply (3.43) to obtain (3.44) – (3.47), respectively. □

As a result of Lemma 4.1, provided that equations (3.43) *are enforced, we have* (3.44)–(3.47) *holding true. The utility of these equations is that* (3.44)–(3.48) *allow us to readily express each of the 18 possible nonnegative pairwise products of distinct 3-dicycle expressions that do not appear in Table 3.5 as the sum of at most two such products that do appear. Table 3.6 summarizes these expressions; it contains 24 rows and 3 columns in such a manner that each row corresponds to a distinct product. The first column states the product of concern, the second column gives the desired representation, and the third column provides an ordered 5-tuple of surrogate multipliers for* (3.44) – (3.48) *that verifies the representation; that is, column 3 shows the equivalence of columns 1 and 2 through surrogates of* (3.44)–(3.48)*. The six products that already appear in Table 3.5 are labeled "Table 3.5" in the second and third columns. Here,* (3.43) *is implicitly enforced via the substitution of* (3.27) *for* $(i,j,k) \in K_4$ *that removes all occurrences of the variables* y_{1323}, y_{1424}, y_{1434}, *and* y_{2434} *from all pairwise products of the 3-dicycle expressions.*

5. As with Family 3 above, for each distinct (i,j) pair, let $\alpha_{ij} = x_{ij}$ if $i < j$ and let $\alpha_{ij} = 1 - x_{ji}$ if $i > j$. Then, for each $(i,j,k,\ell) \in \mathcal{P}$, consider the following three RLT inequalities:

$$(1-\alpha_{ij})\alpha_{i\ell}\alpha_{jk}(1-\alpha_{j\ell})(1-\alpha_{k\ell})(\alpha_{ij}+\alpha_{j\ell}-\alpha_{i\ell}) \geq 0, \tag{3.49}$$

$$(1-\alpha_{ij})(1-\alpha_{ik})\alpha_{i\ell}\alpha_{jk}\alpha_{j\ell}(1-\alpha_{k\ell})(\alpha_{ik}+\alpha_{k\ell}-\alpha_{i\ell}) \geq 0, \tag{3.50}$$

Table 3.6: Representations of 3-dicycle Products for Family 4 Inequalities

Product	Representation	Multipliers on (3.44)–(3.48)
$D_{123}D_{124}$	$D_{123}D_{134} + D_{124}(1 - D_{134})$	$(-.5, -.5, -.5, .5, .5)$
$D_{123}(1 - D_{124})$	Table 3.5	Table 3.5
$(1 - D_{123})D_{124}$	Table 3.5	Table 3.5
$(1 - D_{123})(1 - D_{124})$	$(1 - D_{123})(1 - D_{134}) + (1 - D_{124})D_{134}$	$(-.5, -.5, -.5, .5, .5)$
$D_{123}D_{134}$	Table 3.5	Table 3.5
$D_{123}(1 - D_{134})$	$D_{123}(1 - D_{124}) + D_{124}(1 - D_{134})$	$(-.5, -.5, -.5, .5, .5)$
$(1 - D_{123})D_{134}$	$(1 - D_{123})D_{124} + (1 - D_{124})D_{134}$	$(-.5, -.5, -.5, .5, .5)$
$(1 - D_{123})(1 - D_{134})$	Table 3.5	Table 3.5
$D_{123}D_{234}$	$D_{123}D_{134} + D_{123}(1 - D_{124})$	$(-1, 0, 0, 0, 0)$
$D_{123}(1 - D_{234})$	$D_{124}(1 - D_{134})$	$(.5, -.5, -.5, .5, .5)$
$(1 - D_{123})D_{234}$	$(1 - D_{124})D_{134}$	$(.5, -.5, -.5, .5, -.5)$
$(1 - D_{123})(1 - D_{234})$	$(1 - D_{123})(1 - D_{134}) + (1 - D_{123})D_{124}$	$(-1, 0, 0, 0, 1)$
$D_{124}D_{134}$	$D_{123}D_{134} + (1 - D_{123})D_{124}$	$(-.5, -.5, -.5, .5, .5)$
$D_{124}(1 - D_{134})$	Table 3.5	Table 3.5
$(1 - D_{124})D_{134}$	Table 3.5	Table 3.5
$(1 - D_{124})(1 - D_{134})$	$D_{123}(1 - D_{124}) + (1 - D_{123})(1 - D_{134})$	$(-.5, -.5, -.5, .5, .5)$
$D_{124}D_{234}$	$D_{123}D_{134}$	$(-.5, .5, -.5, .5, .5)$
$D_{124}(1 - D_{234})$	$(1 - D_{123})D_{124} + D_{124}(1 - D_{134})$	$(0, -1, 0, 0, 0)$
$(1 - D_{124})D_{234}$	$D_{123}(1 - D_{124}) + (1 - D_{124})D_{134}$	$(0, -1, 0, 0, -1)$
$(1 - D_{124})(1 - D_{234})$	$(1 - D_{123})(1 - D_{134})$	$(-.5, .5, -.5, .5, -.5)$
$D_{134}D_{234}$	$D_{123}D_{134} + (1 - D_{124})D_{134}$	$(0, 0, -1, 0, 0)$
$D_{134}(1 - D_{234})$	$(1 - D_{123})D_{124}$	$(-.5, -.5, .5, .5, .5)$
$(1 - D_{134})D_{234}$	$D_{123}(1 - D_{124})$	$(-.5, -.5, .5, .5, -.5)$
$(1 - D_{134})(1 - D_{234})$	$(1 - D_{123})(1 - D_{134}) + D_{124}(1 - D_{134})$	$(0, 0, -1, 0, 1)$

and

$$(1 - \alpha_{ij})\alpha_{ik}\alpha_{i\ell}\alpha_{jk}\alpha_{j\ell}\alpha_{k\ell} \geq 0, \tag{3.51}$$

where (3.49) and (3.50) result from computing products of binary variables and their complements with nonnegative 3-dicycle expressions, and where (3.51) uses products of binary variables and their complements. Then we have

$$\begin{aligned}
0 &\leq (1-\alpha_{ij})\alpha_{i\ell}\alpha_{jk}\left[-(1-\alpha_{j\ell})(1-\alpha_{k\ell})+\alpha_{j\ell}(\alpha_{ik}+\alpha_{k\ell}-1)\right] \\
&= (1-\alpha_{ij})\alpha_{i\ell}\alpha_{jk}(-1+\alpha_{k\ell}+\alpha_{ik}\alpha_{j\ell}) \\
&= \alpha_{i\ell}\alpha_{jk}[-1+\alpha_{ij}+(1-\alpha_{ij})\alpha_{k\ell}+(1-\alpha_{ij})\alpha_{ik}\alpha_{j\ell}] \\
&\leq -\alpha_{i\ell}\alpha_{jk}+\alpha_{ij}\alpha_{jk}+\alpha_{jk}\alpha_{k\ell}+\alpha_{ik}\alpha_{j\ell}.
\end{aligned}$$

The first inequality is the sum of (3.49)–(3.51), upon using the RLT strengthening step that the square of a binary variable is itself. Here, the first term results from (3.49), and the second term results from the sum of (3.50) and (3.51). The two equalities are algebra. The final inequality follows from the RLT-type restrictions $\alpha_{ij}\alpha_{jk}(1-\alpha_{i\ell}) \geq 0$, $\alpha_{jk}\alpha_{k\ell}[1-(1-\alpha_{ij})\alpha_{i\ell}] \geq 0$, and $\alpha_{ik}\alpha_{j\ell}[1-(1-\alpha_{ij})\alpha_{i\ell}\alpha_{jk}] \geq 0$. We thus have 24 inequalities given by

$$-\alpha_{i\ell}\alpha_{jk}+\alpha_{ij}\alpha_{jk}+\alpha_{jk}\alpha_{k\ell}+\alpha_{ik}\alpha_{j\ell} \geq 0 \;\forall\; (i,j,k,\ell) \in \mathcal{P}. \tag{3.52}$$

These inequalities are listed in Table 3.7. The first column gives the 4-tuple $(i,j,k,\ell) \in \mathcal{P}$, and the second column gives the quadratic inequality.

Remark 3.4.6. *Unlike the first four families, the Family 5 inequalities are computed using products of linear functions taken more than 3 at a time, as shown in* (3.49)–(3.51).

3.5 Computational Experience

In this section, we provide computational experience to assess the merits of the contributions of Sections 3.3 and (4) 3.4. This experience consists of two studies: the first determines the savings realized by using the smaller Problem LP2′(n) in lieu of LP2(n) of [6] to solve Problem P1(n), equivalently P2(n), focusing on each of the two reduction strategies of Section 3.3. The second individually examines each family of facets found within Section 3.4 to determine those that tend to provide increased relaxation strength. The two studies are found in the two subsections below, with one study per subsection.

Table 3.7: 4-tuple $(i, j, k, \ell) \in \mathcal{P}$, and Family 5 Inequalities

$(i, j, k, \ell) \in \mathcal{P}$	Quadratic Inequality
(1,2,3,4)	$x_{12}x_{23} + x_{13}x_{24} - x_{14}x_{23} + x_{23}x_{34} \geq 0$
(1,2,4,3)	$x_{12}x_{24} - x_{13}x_{24} + x_{14}x_{23} + x_{23}x_{24} - x_{23}x_{34} \geq 0$
(1,3,2,4)	$-x_{14} + x_{24} + x_{12}x_{13} - x_{12}x_{23} + x_{12}x_{34} + x_{14}x_{23} - x_{23}x_{24} \geq 0$
(1,3,4,2)	$x_{14} - x_{24} + x_{34} - x_{12}x_{34} + x_{13}x_{34} - x_{14}x_{23} + x_{23}x_{24} - x_{23}x_{34} \geq 0$
(1,4,2,3)	$x_{12} - x_{13} + x_{23} + x_{12}x_{14} - x_{12}x_{24} - x_{12}x_{34} + x_{13}x_{24} - x_{23}x_{24} \geq 0$
(1,4,3,2)	$-x_{12} + x_{13} - x_{23} - x_{34} + x_{12}x_{34} + x_{13}x_{14} - x_{13}x_{24} - x_{13}x_{34} + x_{23}x_{34} \geq -1$
(2,1,3,4)	$x_{13} - x_{12}x_{13} - x_{13}x_{24} + x_{13}x_{34} + x_{14}x_{23} \geq 0$
(2,1,4,3)	$x_{14} - x_{12}x_{14} + x_{13}x_{14} + x_{13}x_{24} - x_{13}x_{34} - x_{14}x_{23} \geq 0$
(2,3,1,4)	$-x_{13} + x_{14} + x_{23} - x_{24} + x_{34} + x_{12}x_{13} - x_{12}x_{23} - x_{12}x_{34} - x_{13}x_{14} + x_{13}x_{24} \geq 0$
(2,3,4,1)	$-x_{14} + x_{24} + x_{12}x_{34} + x_{13}x_{14} - x_{13}x_{24} - x_{13}x_{34} + x_{23}x_{34} \geq 0$
(2,4,1,3)	$-x_{12} + x_{13} - x_{14} - x_{23} + x_{24} - x_{34} + x_{12}x_{14} - x_{12}x_{24} + x_{12}x_{34} - x_{13}x_{14} + x_{14}x_{23} \geq -1$
(2,4,3,1)	$x_{12} - x_{13} + x_{23} - x_{12}x_{34} + x_{13}x_{34} - x_{14}x_{23} + x_{23}x_{24} - x_{23}x_{34} \geq 0$
(3,1,2,4)	$x_{12} + x_{14} - x_{12}x_{13} + x_{12}x_{24} - x_{12}x_{34} - x_{14}x_{23} \geq 0$
(3,1,4,2)	$x_{12}x_{14} - x_{12}x_{24} + x_{12}x_{34} - x_{13}x_{14} + x_{14}x_{23} \geq 0$
(3,2,1,4)	$-x_{12} + x_{14} - x_{23} + x_{24} - x_{34} - x_{12}x_{14} + x_{12}x_{23} + x_{12}x_{34} - x_{13}x_{24} \geq -1$
(3,2,4,1)	$-x_{14} + x_{24} + x_{34} + x_{12}x_{14} - x_{12}x_{24} - x_{12}x_{34} + x_{13}x_{24} - x_{23}x_{24} \geq 0$
(3,4,1,2)	$x_{12} - x_{13} + x_{23} - x_{24} + x_{34} - x_{12}x_{14} + x_{13}x_{14} + x_{13}x_{24} - x_{13}x_{34} - x_{14}x_{23} \geq 0$
(3,4,2,1)	$-x_{12} + x_{13} - x_{14} - x_{23} - x_{24} + x_{34} + x_{12}x_{24} - x_{13}x_{24} + x_{14}x_{23} + x_{23}x_{24} - x_{23}x_{34} \geq -1$
(4,1,2,3)	$x_{13} - x_{12}x_{14} + x_{12}x_{23} + x_{12}x_{34} - x_{13}x_{24} \geq 0$
(4,1,3,2)	$x_{12} + x_{12}x_{13} - x_{12}x_{23} - x_{12}x_{34} - x_{13}x_{14} + x_{13}x_{24} \geq 0$
(4,2,1,3)	$x_{13} + x_{23} - x_{24} + x_{34} - x_{12}x_{13} + x_{12}x_{24} - x_{12}x_{34} - x_{14}x_{23} \geq 0$
(4,2,3,1)	$-x_{12} - x_{13} + x_{23} - x_{34} + x_{12}x_{13} - x_{12}x_{23} + x_{12}x_{34} + x_{14}x_{23} - x_{23}x_{24} \geq -1$
(4,3,1,2)	$x_{12} - x_{14} - x_{23} + x_{24} - x_{34} - x_{12}x_{13} - x_{13}x_{24} + x_{13}x_{34} + x_{14}x_{23} \geq -1$
(4,3,2,1)	$-x_{12} - x_{13} + x_{14} - x_{23} - x_{24} - x_{34} + x_{12}x_{23} + x_{13}x_{24} - x_{14}x_{23} + x_{23}x_{34} \geq -2$

For both studies, all computations were performed on Clemson University's high performance computer, Palmetto, with 4 CPU cores and 10gb RAM, using the CPLEX 12.8 solver in AMPL for Linux-Intel 64 with all pre-processing options turned off.

3.5.1 Reductions Strategies for Problem LP2(n)

Recall that the two reduction strategies of Section 3.3 that were used to obtain the more concise formulation LP2$'(n)$ from the formulation LP2(n) are as follows. The first strategy, for each $(i, j, k) \in K_n$, replaced the twelve inequalities collectively found within the three sets X_{ijik}, X_{ijjk}, and X_{ikjk} of (3.5) with the six inequalities of the set Z_{ijk} of (3.32). The second strategy, for each $(i, j, k, \ell) \in J_n$ with (i, j, k, ℓ) distinct, removed half the inequalities of $X_{ijk\ell}$ of (3.5), depending on the objective function coefficient $B_{ijk\ell}$, to obtain either (3.33) or (3.34).

We examine the usefulness of these strategies by solving various instances of Problem P1(n) using each of four different forms: Problem LP2(n), Problem LP2(n) with reduction strategy 1 only, Problem LP2(n) with reduction strategy 2 only, and Problem LP2$'$(n). All four forms are equivalent mixed 0-1 linear representations of P1(n), and they share the same optimal continuous relaxation value. The notation LP2(n), R1(n), R2(n), and LP2$'$(n) found within our results of Tables 3.8 and 3.9 is used to denote these four forms, respectively.

We ran instances of sizes $n = 10, \ldots, 15$ for different densities $d = 10, 20, \ldots, 90,$. We report the "LP Solution Times in Second" in Table 3.8 and both the "IP Solution Times in Seconds" and the numbers of "Branch-and-Bound Nodes" explored in Table 3.9. The first column of each table gives the problem size n and the second column gives the density d as a percentage of the number of nonzero objective coefficients. All such nonzero coefficients were randomly generated in Python as integers following a uniform distribution in the interval $[-100, 100]$. In Table 3.9, columns 3 through 6 give the "IP Solution Times in Seconds" for each of the four forms, and columns 7 through 10 give the numbers of "Branch-and-Bound Nodes" explored. Because the problems become increasingly more difficult for higher densities, only lesser densities were considered for sizes $n = 13, 14, 15$. Each entry in the table records the average of ten instances. Every instance was restricted to have a one-hour CPU limit, and a hyphen within the table represents that the time to solve at least one of the ten instances exceeded the one-hour limit.

Problem LP2$'$(n) shows an advantage, for all cases, over LP2(n) in terms of IP execution time. Insofar as the individual reduction strategies are concerned, both exhibit improvement over LP2(n), but the impact of strategy 2 is more profound. This relative improvement is to be expected, as strategy 2 removes more inequalities from LP2(n) than does strategy 1. As for the numbers of Branch-and-Bound Nodes explored, the conclusions are not as obvious, although it appears that LP2$'$(n) tends to encounter the fewest nodes, followed by R2(n), LP2(n), and R1(n).

To provide a graphical depiction of the results, we construct a performance profile as described in [10]. For each problem size n and density d found in Tables 3.8 and 3.9 for which no

Table 3.8: LP Solution Times in Seconds

n	d	LP2(n)	R1(n)	R2(n)	LP2′(n)
10	10	0.231	0.204	0.108	0.016
10	20	0.262	0.203	0.130	0.097
10	30	0.264	0.222	0.131	0.095
10	40	0.244	0.150	0.089	0.068
10	50	0.245	0.167	0.101	0.065
10	60	0.238	0.208	0.131	0.105
10	70	0.194	0.206	0.116	0.085
10	80	0.241	0.196	0.187	0.113
10	90	0.279	0.227	0.199	0.120
11	10	0.501	0.278	0.095	0.078
11	20	0.405	0.242	0.081	0.069
11	30	0.415	0.290	0.111	0.080
11	40	0.375	0.328	0.123	0.087
11	50	0.394	0.283	0.170	0.132
11	60	0.394	0.318	0.205	0.130
11	70	0.339	0.291	0.221	0.139
11	80	0.302	0.303	0.184	0.144
11	90	0.319	0.238	0.288	0.174
12	10	0.894	0.570	0.200	0.179
12	20	0.781	0.462	0.135	0.105
12	30	0.814	0.400	0.170	0.103
12	40	0.851	0.470	0.207	0.158
12	50	0.795	0.387	0.232	0.201
12	60	0.793	0.447	0.222	0.172
12	70	0.834	0.448	0.293	0.234
12	80	0.761	0.396	0.302	0.223
12	90	0.659	0.398	0.321	0.209
13	10	0.726	0.614	0.191	0.108
13	20	0.698	0.600	0.191	0.115
13	30	0.843	0.649	0.224	0.140
13	40	0.831	0.546	0.227	0.116
13	50	0.919	0.624	0.333	0.184
13	60	0.970	0.626	0.341	0.211
13	70	0.992	0.683	0.397	0.227
13	80	0.980	0.675	0.430	0.269
13	90	0.935	0.694	0.564	0.257
14	10	0.711	0.716	0.230	0.108
14	20	0.919	0.785	0.246	0.147
14	30	1.115	0.897	0.311	0.190
14	40	1.193	0.946	0.401	0.212
14	50	1.370	0.993	0.525	0.262
14	60	1.296	1.085	0.760	0.409
14	70	1.553	1.122	0.692	0.370
14	80	1.596	1.119	0.716	0.358
14	90	1.297	1.014	1.115	0.488
15	10	0.960	0.922	0.303	0.165
15	20	1.191	1.052	0.376	0.228
15	30	1.455	1.254	0.726	0.335
15	40	1.815	1.348	0.663	0.317
15	50	1.988	1.477	0.769	0.391
15	60	2.223	1.614	0.853	0.456
15	70	1.913	1.478	1.320	0.590
15	80	2.448	1.805	1.115	0.589
15	90	2.563	1.962	1.194	0.776
16	10	1.214	1.109	0.423	0.229
16	20	1.503	1.472	0.489	0.302
16	30	2.130	1.447	0.688	0.376
16	40	2.652	1.787	0.997	0.495
16	50	3.028	2.132	1.286	0.605
16	60	2.147	1.764	1.773	0.849
16	70	3.387	2.507	1.535	0.894
16	80	3.484	2.639	1.584	1.052
16	90	3.755	2.863	1.743	1.261
17	10	1.685	1.526	0.553	0.294
17	20	2.308	1.642	0.684	0.421
17	30	2.797	2.053	1.099	0.577
17	40	3.017	2.231	1.771	0.724
17	50	4.244	2.705	1.760	0.939
17	60	1.277	1.033	0.847	1.428
17	70	5.044	3.511	2.161	1.411
17	80	2.838	2.226	1.760	1.437
17	90	3.062	2.288	1.660	1.368
18	10	1.783	1.647	0.584	0.362
18	20	2.016	1.808	1.030	0.483
18	30	4.175	2.625	1.690	0.842
18	40	5.568	3.236	2.211	1.044
18	50	2.544	1.916	1.399	1.215
18	60	6.091	4.415	2.702	1.600
18	70	6.531	4.903	3.136	1.960
18	80	7.096	5.146	3.526	2.433
18	90	7.523	5.700	4.059	2.713
19	10	2.233	2.092	0.696	0.593
19	20	2.497	2.466	1.467	0.714
19	30	3.430	2.577	1.723	1.099
19	40	7.643	4.169	2.886	1.490
19	50	7.092	4.369	2.573	2.071
19	60	4.868	3.324	2.249	1.713
19	70	3.845	2.874	2.108	1.728
19	80	10.126	7.118	5.149	3.511
19	90	10.285	7.816	5.650	4.207
20	10	2.675	2.779	0.979	0.796
20	20	6.219	4.029	2.144	1.279
20	30	28.513	16.095	5.733	4.363
20	40	6.615	3.438	2.127	1.769
20	50	5.050	3.094	2.159	1.796
20	60	11.755	7.828	5.236	3.419
20	70	7.314	5.129	3.516	2.503
20	80	13.275	9.855	6.716	5.136
20	90	8.535	6.443	4.498	3.313

Table 3.9: IP Solution Times in Seconds and Numbers of Branch-and-Bound Nodes

		Solution Time in Seconds				**Branch-and-Bound Nodes**			
n	d	LP2(n)	R1(n)	R2(n)	LP2′(n)	LP2(n)	R1(n)	R2(n)	LP2′(n)
10	10	3.5	3.1	2.0	1.9	57	43	63	48
10	20	5.5	4.8	2.7	2.3	361	286	237	213
10	30	6.3	5.9	3.1	2.7	563	650	431	414
10	40	13.1	12.3	5.4	4.3	776	1004	709	688
10	50	13.6	11.9	6.3	4.6	1201	1159	1183	1017
10	60	14.6	13.3	7.5	6.2	1617	1774	1553	1595
10	70	10.3	9.7	5.8	4.8	1775	1912	1804	1740
10	80	14.4	13.9	8.9	7.5	2027	2175	1991	1888
10	90	17.9	16.4	11.3	9.9	2866	2717	2640	2719
11	10	11.2	10.4	3.1	2.4	264	262	214	189
11	20	22.4	19.5	5.4	3.8	1133	1214	861	869
11	30	33.2	30.4	8.6	6.3	2286	2577	1886	1896
11	40	42.4	36.4	12.0	9.1	3418	3524	3176	2989
11	50	49.7	41.0	15.7	11.2	5231	5180	4991	4395
11	60	70.3	66.5	27.0	21.9	6056	6996	6451	5990
11	70	76.6	64.1	37.3	27.0	9861	9186	9749	8445
11	80	57.6	50.3	28.5	24.5	7820	7971	7340	7963
11	90	76.4	72.1	49.0	38.0	11679	12351	12313	11742
12	10	20.3	20.8	5.7	4.1	812	1333	789	660
12	20	96.0	85.4	17.6	10.9	4088	4894	3601	3015
12	30	165.1	131.7	26.7	20.0	8946	8933	6316	6257
12	40	150.2	149.2	41.0	25.4	9034	9958	8977	7263
12	50	289.0	252.4	71.7	59.6	23517	22402	18628	20266
12	60	336.8	273.2	111.4	75.6	30926	28877	27709	23502
12	70	361.6	319.5	149.9	104.8	39645	35993	37382	33566
12	80	525.2	450.1	548.7	338.4	59685	60012	138709	103210
12	90	483.5	415.7	735.6	481.3	56577	55418	177509	138598
13	10	179.1	122.4	23.3	14.8	3013	3552	2786	2501
13	20	602.7	439.7	76.8	38.8	16053	14787	14893	10553
13	30	1722.8	1449.5	231.8	152.8	42691	49359	40290	34760
13	40	2101.5	1655.9	317.4	196.7	71106	75637	72364	59287
13	50	–	–	1558.9	1098.6	–	–	364462	321507
14	10	1381.7	877.8	92.4	59.4	20004	16834	13481	12842
14	20	–	–	458.8	295.7	–	–	78004	76063
15	10	–	–	385.9	247.1	–	–	44869	44828

dash appears, we compute the value

$$\mu_{i,n,d} = \frac{t_{i,n,d} - t_{\min,n,d}}{t_{\max,n,d} - t_{\min,n,d}} \text{ for all } i \in \Lambda \equiv \{\text{LP2}(n), \text{R1}(n), \text{R2}(n), \text{LP2}'(n)\},$$

where $t_{i,n,d}$ is the data measurement (either LP Solution Time, IP Solution Time, or Branch-and-Bound Nodes) of Tables 3.8 and 3.9, and

$$t_{\min,n,d} = \min_{i\in\Lambda}\{t_{i,n,d}\} \quad \text{and} \quad t_{\max,n,d} = \max_{i\in\Lambda}\{t_{i,n,d}\}.$$

In this manner, for each n and d, the value $\mu_{i,n,d} \in [0, 1]$ has $100\mu_{i,n,d}$ denoting the percentage of the interval $(t_{\max,n,d} - t_{\min,n,d})$ that is less than or equal to the value $t_{i,n,d}$, for every $i \in \Lambda$. Using these values of $\mu_{i,n,d}$, once each for the LP Solution Time, IP Solution Time, and Branch-and-Bound Nodes enumerated, the graphs of Figure 3.1 are constructed as follows. For each graph, the horizontal axis denotes the "Profile Statistic" defined as a variable ranging from 0 to 1 inclusive and the vertical axis denotes percentage so that, for each $i \in \Lambda$, the plot is the percentage of test runs for which $\mu_{i,n,d}$ evaluated over all n and d is less than or equal to the Profile Statistic. Thus, for example, the IP Time Performance Profile having percentage 90 for the Profile Statistic equal to 0.2 and $i = \text{R2}(n)$ indicates that 90% of the test runs using reduction strategy R2(n) have the associated $\mu_{i,n,d}$ value less than or equal to 0.2. Clearly then, the higher the curve, the better the associated form performs. The left graph reinforces our earlier observation that LP2′(n) performed the best in terms of IP Time, followed by R2(n), R1(n), and LP2(n). The right graph suggests a similar performance for the Branch-and-Bound Nodes in that LP2′(n) and R2(n) performed the best although, in this case, LP2(n) outperformed R1(n).

3.5.2 Five Families of Facets describing QLO(4)

The five families of facets in Section 3.4 that describe QLO(4), as a consequence of Section 3.2, are valid (and also facets) for QLO(n) for all $n \geq 4$. The question arises as to the relative strength afforded by each family towards tightening the continuous relaxation of LP2(n), and the overall effect on solving the integer program P2(n) itself. Notably, while Problems LP2(n), R1(n),

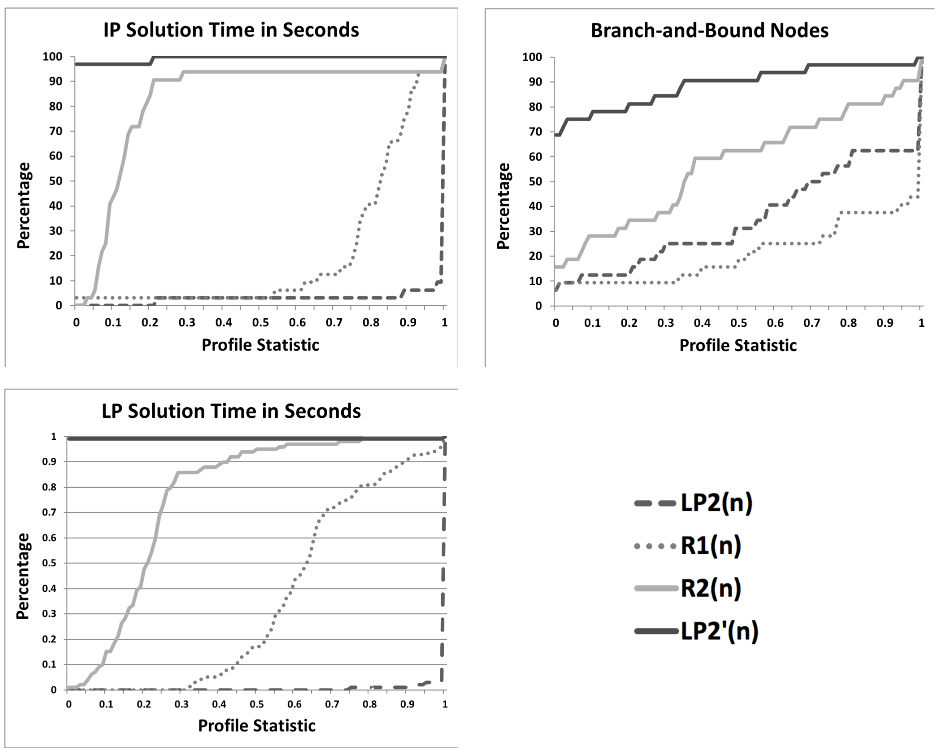

Figure 3.1: Performance Profiles for Problems LP2(n), R1(n), R2(n), and LP2′(n)

R2(n), and LP2$'$(n) share the same optimal objective value to the continuous relaxations, we conduct our study using Problem R1(n). The reason is that, consistent with Remark 3.4.1 in Section 3.4, the inequalities deleted from LP2(n) via reduction strategy 2 are not necessarily redundant when additional inequalities are present, and thus their removal can weaken the relaxation value in the presence of additional facets.

Our study reconsiders the problems of Table 3.9 having $n = 10$ and $d = 10, 20, \ldots, 90$. We are interested in the improved relaxation strengths and the times to compute these values, as well as the times to solve LP2(n) and the numbers of nodes explored. For each family, we computed the facets of Section 3.4 for all $\binom{10}{4}$ selections of 4 indices from amongst the $n = 10$ possibilities. (Duplications of facets within family 1 due to the definition of the sets K_4 are removed.)

Our results are summarized in the four charts of Table 3.10. These charts are labeled "Objective Gap in Percentage," "LP Solution Time in Seconds," "IP Solution Time in Seconds," and "Branch-and-Bound Nodes." Each chart consists of six columns. The first column gives the density of nonzero objective coefficients d, and the remaining five columns give Problem R1 strengthened by each of families 1 through 5, denoted by R1, F2, F3, F4, and F5. Since the first family of facets comprises the constraints of Problem R1, we use the column title R1(n) instead of F1 to denote that no additional inequalities were included in R1. (Unlike Table 3.9, we suppress the problem size, since $n = 10$ for all instances.) For the first chart, the "Objective Gap in Percentage" is computed as:

$$\frac{(\text{IP value}) - (\text{LP value})}{|\text{IP value}|} * 100\%.$$

Three observations from the charts of Table 3.10 are as follows. First, relative to the "Objective Gap Percentage" and number of "Branch-and-Bound Nodes" explored, F3 always performed the best, followed by F2, F5, F4, and R1. Thus, for the given instances, the third family of facets added the most strength. Second, relative to "LP Solution Time in Seconds," R1 always performed the best, followed by F4. This performance is to be expected since R1 has the fewest number of inequalities, followed by F4. Third, for "IP Solution Time in Seconds," R1 performed best in 7 of 9 cases tested, even though this form reported the most nodes enumerated. For the remaining 2 cases, F3 performed the best. It appears then, that the additional strength gained by enforcing any single family of facets does not tend to reduce computation time, with the possible exception of the third family.

We again give a graphical depiction of our results by computing a performance profile for each chart in Table 3.10. The graphs are found in Figure 3.2. Observe that the graphs "Objective Gap in Percentage" and number of "Branch-and-Bound Nodes" are to be expected from Table 3.10, with F3, F2, F5, F4, R1 appearing from highest to lowest. The superior results of F3 for all test cases resulted in the horizontal plots at value 100 for F3 in both graphs. Similarly, for "Branch-and-Bound Nodes," since R1 always performed worse, the plot is a horizontal line at 0, making the jump to 1 when the Profile Statistic is 1. Notably, the graphs provide additional insight beyond Table 3.10 relative to "LP Solution Time in Seconds" and "IP Solution Time in Seconds." For "LP Solution Time in Seconds," since R1 always performed the best, followed by F4, the associated two plots were the highest. However, the graph further suggests that the subsequent order of performance is F5, F3, and F2. For "IP Solution Time in Seconds," as noted above relative to Table 3.10, R1 and F3 performed the best, and thus have the highest plots. According to the graph, their performance is then followed by F4, F2, and F5.

Table 3.10: Comparison of the Five Families of Facets Describing QLO(4) for $n = 10$.

Objective Gap in Percentage					
d	R1	F2	F3	F4	F5
10	24.7	13.9	2.6	23.2	22.3
20	64.0	36.5	3.8	62.5	61.3
30	90.4	44.1	4.7	82.7	79.8
40	112.3	52.8	7.4	96.1	91.2
50	142.7	70.7	2.6	117.1	109.1
60	176.0	89.3	15.4	134.0	126.0
70	175.4	93.0	13.2	130.0	117.2
80	179.9	98.1	13.0	127.1	115.8
90	222.7	129.0	19.5	154.3	139.5

LP Solution Time in Seconds					
d	R1	F2	F3	F4	F5
10	0.2	5.7	1.9	0.9	2.2
20	0.2	5.3	5.1	0.4	1.0
30	0.2	5.3	5.5	0.5	1.3
40	0.2	5.4	4.6	0.7	1.9
50	0.2	5.1	1.5	0.8	2.0
60	0.2	4.9	5.8	1.2	2.4
70	0.2	6.5	6.2	1.2	2.7
80	0.2	5.2	6.3	1.6	3.2
90	0.2	4.5	5.8	1.6	3.4

IP Solution Time in Seconds					
d	R1	F2	F3	F4	F5
10	3.1	28.5	4.7	17.0	55.4
20	4.8	44.1	10.7	22.6	99.7
30	5.9	57.5	12.5	24.7	111.3
40	12.3	66.0	11.6	31.2	134.3
50	11.9	90.1	4.0	44.3	136.9
60	13.3	112.1	25.0	51.2	183.5
70	9.7	345.8	24.0	55.2	158.5
80	13.9	335.6	22.3	51.9	188.5
90	16.4	414.7	28.8	50.2	218.5

Branch-and-Bound Nodes					
d	R1	F2	F3	F4	F5
10	43.3	8.8	2.5	40.5	23.2
20	285.5	23.4	3.5	108.3	85.7
30	650.2	31.5	8.7	139.2	134.3
40	1003.7	42.8	12.2	207.7	161.2
50	1158.5	82.9	2.5	293.1	192.0
60	1773.8	122.5	28.0	333.8	264.0
70	1912.0	132.9	22.6	366.4	270.8
80	2174.6	137.8	23.3	278.1	216.9
90	2717.0	189.4	24.7	344.4	281.3

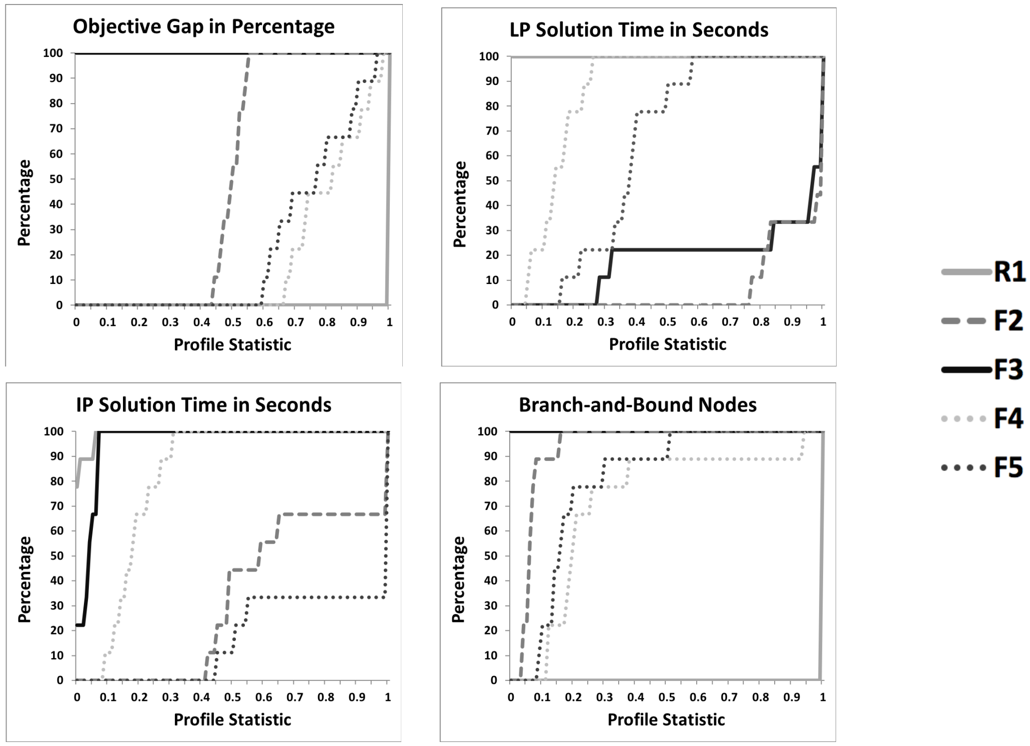

Figure 3.2: Comparison of the Families of Facets Describing QLO(4)

3.6 Conclusions and Future Research

The theoretical advances of this paper are a "lifting theorem" that establishes that every facet for the size n problem is also a facet for the size N problem for all $N > n$, and a complete listing of all facets for QLO(3) and QLO(4). The lifting theorem and facets for QLO(3), together with a constraint reduction strategy, gave rise to a more concise mixed 0-1 linear formulation of the QLOP than that available in the literature. The smaller form led to improved computational experience on all problems tested.

Additional research is needed to determine effective strategies for using the newly-derived facets of QLO(4) to solve general instances of the QLOP. The five identified families of facets gave mixed results when naively inserted into a branch-and-bound framework, with none showing superior performance. In fact, only family 3 served to improve the overall solution time for any case, and this improvement occurred in only 2 of 9 instances. However, the reduction in "Objective Gap in Percentage" and number of "Branch-and-Bound Nodes" for each family shows that these facets provide strength - the challenge is handling the large numbers of additional inequalities. A direction for future work is the development of separation algorithms that sequentially compute violated inequalities as needed, in order to capture the additional strength without being encumbered by the increased sizes.

Chapter 4

Convex Hull Representations of Special Monomials of Binary Variables

4.1 Introduction

A classical approach for optimizing a 0-1 polynomial program is to first linearize the polynomial expressions, and then use linear programming techniques to compute bounds, typically within some type of enumerative strategy. Such linearizations can be accomplished as follows. Given a monomial expression $\prod_{j \in N} x_j$ in n binary variables $\mathbf{x} = (x_1, \ldots, x_n)$ with $N \equiv \{1, \ldots, n\}$, a new variable, say w, is used to replace $\prod_{j \in N} x_j$, and linear restrictions are then defined to enforce that $w = \prod_{j \in N} x_j$ at all binary realizations of $\mathbf{x}$. In this manner, the original 0-1 polynomial program is transformed into an equivalent mixed 0-1 linear problem.

Arguably, the most commonly-used linearization for such monomial expressions is due to [17]. This approach enforces that $\begin{pmatrix} \mathbf{x} \\ w \end{pmatrix} \in A$, where

$$A \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \end{pmatrix} \in \mathbb{R}^{n+1} : \; 0 \leq w, \; \sum_{j \in N} x_j - (n-1) \leq w, \; w \leq x_j \leq 1 \; \forall \, j \in N \right\}.$$

(For the case of $n = 2$, the restrictions $x_1 \leq 1$ and $x_2 \leq 1$ are implied by the remaining 4 inequalities.) The popularity of A is due to the fact that w is continuous and the linear programming relaxation is tight; the paper [7] showed that the set A is a "best" linearization of $\prod_{j \in N} x_j$ in the sense that $A = conv(B)$, where

$$B \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \end{pmatrix} \in \mathbb{R}^{n+1} : \; w = \prod_{j \in N} x_j, \; \mathbf{x} \text{ binary} \right\},$$

and where $conv(\bullet)$ denotes the convex hull of the set $\bullet$.

In this paper, we use reformulation-linearization technique (RLT) arguments of [39] and [41] to show that the result of [7] can be extended to a generalization of B wherein each binary variable x_j is further constrained to be upper-bounded by an auxiliary binary variable y, so that $x_j \leq y$ for all $j \in N$. Then the generalization of B takes the form

$$C \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \\ y \end{pmatrix} \in \mathbb{R}^{n+2} : \; w = \prod_{j \in N} x_j, \; x_j \leq y \; \forall \, j \in N, \; \mathbf{x} \text{ binary}, \; y \text{ binary} \right\}.$$

While the set C is a simple generalization of B, we are not aware of any explicit description of $conv(C)$. This lack of description is curious because the set C arises in practice, as noted in Subsection 4.2.2 with the Channel Assignment Problem, and as discussed in Section 4.4 with variants of the Chromatic Number Problem and Uncapacitated Facility Location Problem.

Our RLT acquisition of the convex hull consists of three steps: a *reformulation* of the problem that is obtained by computing special pairwise-products of the constraints, a *linearization* that substitutes a continuous variable for each resulting quadratic term to express the problem in an extended-variable space, and a *projection* that recasts the extended form in terms of the original variables. The applications for the $n = 2$ and $n \geq 3$ cases, however, employ different such implementations, with the first using the special-structure RLT (SSRLT) of [41] and the second using the level-1 RLT of [39]. For the latter, a key observation is a trivial projection that consists of variable substitutions. Using this second implementation, we are able to obtain interpretations for known convex hull forms of certain symmetric multilinear functions, and also for the Boolean Quadric Polytope.

The paper is organized as follows. The next section shows that, when $n = 2$, an implementation of the SSRLT that uses a family of six inequalities which are valid for C gives the convex hull form. To clarify and show the usefulness, an application to a Channel Assignment Problem is given. Section 4.3 then considers the cases having $n \geq 3$. For these cases, the RLT is used in conjunction with [7]. Generalizations to symmetric multilinear functions and the Boolean Quadric Polytope are presented. The final section gives conclusions and avenues for future research.

4.2 Convex Hull Representation for the Case having $n = 2$

We derive the convex hull representation of C for the case having $n = 2$, and then show an application to a Channel Assignment Problem.

4.2.1 Convex Hull Derivation

The special case of C having $n = 2$ is given by

$$C' = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 : \ w = x_1 x_2, \ x_1 \leq y, \ x_2 \leq y, \ x_1, x_2, y \text{ binary} \right\}.$$

The following six inequalities are valid for C' :

$$0 \leq x_1 \leq y \leq 1 \text{ and } 0 \leq x_2 \leq y \leq 1. \tag{4.1}$$

An implementation of the SSRLT of [41] applied to C' consists of the following three steps, with the first two illustrated in Table 4.1.

1. **Reformulation.** Compute 9 nonnegative, quadratic expressions by multiplying each of the first 3 nonnegative expressions $(1-y)$, $(y-x_1)$, and x_1 of (4.1) by each of the last 3 nonnegative expressions $(1-y)$, $(y-x_2)$, and x_2. Enforce the binary identity that $y^2 = y$ to remove the resulting four occurrences of y^2.

2. **Linearization.** Substitute a continuous variable for each of the 3 remaining quadratic terms, say $w = x_1x_2$, $z_1 = x_1y$, and $z_2 = x_2y$, and omit the binary restrictions on all variables.

3. **Projection.** Project the resulting variable-space $(x_1, x_2, w, y, z_1, z_2)$ onto the space of the variables (x_1, x_2, w, y).

Table 4.1: Illustration of the Special-Structure RLT Applied to Inequalities (4.1)

Nonnegative Product	Linearized Inequality
$(1-y)(1-y) \geq 0$	$1-y \geq 0$
$(1-y)(y-x_2) \geq 0$	$-x_2 + z_2 \geq 0$
$(1-y)x_2 \geq 0$	$x_2 - z_2 \geq 0$
$(y-x_1)(1-y) \geq 0$	$-x_1 + z_1 \geq 0$
$(y-x_1)(y-x_2) \geq 0$	$y - z_1 - z_2 + w \geq 0$
$(y-x_1)x_2 \geq 0$	$z_2 - w \geq 0$
$x_1(1-y) \geq 0$	$x_1 - z_1 \geq 0$
$x_1(y-x_2) \geq 0$	$z_1 - w \geq 0$
$x_1x_2 \geq 0$	$w \geq 0$

The left column of Table 4.1 gives the nonnegative products that are computed in the *reformulation* step, and the right column gives the inequalities that are obtained from enforcing $y^2 = y$ and applying the subsequent *linearization* step. The polytope D below is formed from the 9 inequalities of the right column.

$$D = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \\ z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^6 : \ y \leq 1, \ 0 \leq w \leq z_1, \ z_1 + z_2 - y \leq w \leq z_2, \ x_i \leq z_i \leq x_i, \ i = 1, 2 \right\}$$

The Projection onto the (x_1, x_2, w, y) space fixes $z_1 = x_1$ and $z_2 = x_2$. Upon applying this projection to D and eliminating the variables z_1 and z_2, we obtain the set

$$E \equiv \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 : \ y \le 1,\ 0 \le w \le x_1,\ x_1 + x_2 - y \le w \le x_2 \right\}. \tag{4.2}$$

It is straightforward to show that $E = conv(C')$; we accomplish this task by establishing a nonsingular linear transformation between the set E and that simplex in $\mathbb{R}^4$ whose extreme points are the 5 (affinely-independent) feasible points to C'. This argument is a variation of the line of thought used in [4, Lemma 1]; here we specifically tailor the arguments to the sets C' and E, as opposed to nonnegative Lagrange interpolating polynomials and Vandermonde matrices as found in [4]. We have that

$$\begin{aligned} E &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 : \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ w \\ y \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 : \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ w \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} \text{ for some } \lambda_1, \ldots, \lambda_5 \ge 0 \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ w \\ y \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} \text{ for some } \lambda_1, \ldots, \lambda_5 \ge 0 \right\}, \end{aligned}$$

as desired since each column in the 5×5 matrix of the final description of E, less the first entry, is a distinct feasible point to C'. The first equality is (4.2) in matrix form, the second equality is trivial, and the third equality is due to the matrices $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ being inverses.

The set A for $n = 2$ has arisen in various contexts [3, 13, 14, 31], but the set E of (4.2) has not previously been identified as the convex hull of C'. Given a 0-1 quadratic expression, say x_1x_2, that is present within some larger optimization problem, the usual linearization approach is to substitute $w = x_1x_2$ and then enforce the four non-implied inequalities of A within the larger problem. This approach applied to C' results in E_R below, where the inequalities $x_1 \leq y,\ x_2 \leq y,\ y$ binary, are the "additional" restrictions of the larger problem.

$$E_R \equiv \left\{ \begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} \in \mathbb{R}^4 :\ y \leq 1,\ 0 \leq w \leq x_1 \leq y,\ x_1 + x_2 - 1 \leq w \leq x_2 \leq y \right\}$$

The notation E_R is used to recognize that E_R is a relaxation of E of (4.2), with $E_R \supseteq E = conv(C')$. The containment follows since every point feasible to E can be shown feasible to E_R, and the equality follows from the above arguments. However, $E_R \neq conv(C')$ as, for example,

$$\begin{pmatrix} x_1 \\ x_2 \\ w \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \in (E_R - E).$$

4.2.2 Application to the Channel Assignment Problem

We demonstrate the improved strength of the polytope E of (4.2) over E_R on a formulation of the Channel Assignment Problem studied in [34]. Within this subsection only, we double-subscript the variables $\mathbf{x}$ to obtain x_{ij} and single-subscript the variable y to obtain y_i to reflect the given application, and to reflect the upper bounding of the variables x_{ij} by y_i for each (i, j) pair. Given p electromagnetic channels and q units, the problem is to assign a single channel to each unit so as to use the minimum number of channels. Allocating the same channel i to different units j and k with $j < k$ creates co-channel interference α_{ijk}, and the overall assignment of channels to units cannot violate predesignated maximum allowable interferences β_{ij} associated with each channel-unit pair (i, j). Letting y_i for all $i \in P \equiv \{1, \ldots, p\}$ be a binary variable equaling 1 if channel i is used and 0 otherwise, and x_{ij} for all $i \in P$ and $j \in Q \equiv \{1, \ldots, q\}$ be a binary variable equaling 1 if channel i is assigned to unit j and 0 otherwise, the problem can be formulated as the 0-1 quadratic program below. Here, the index i is assumed within P and the indices j and k within Q.

CAP: minimize $\sum_i y_i$

subject to

$$\sum_i x_{ij} = 1 \qquad \forall\ j \tag{4.3}$$

$$x_{ij} \leq y_i \qquad \forall\ (i,j) \tag{4.4}$$

$$\sum_{k<j} \alpha_{ikj} x_{ik} x_{ij} + \sum_{k>j} \alpha_{ijk} x_{ij} x_{ik} \leq \beta_{ij} x_{ij} \qquad \forall\ (i,j) \tag{4.5}$$

$$(\mathbf{x}, \mathbf{y}) \text{ binary} \tag{4.6}$$

Equations (4.3) restrict that every unit j must have exactly one channel assigned to it, inequalities (4.4) enforce that each channel i which is assigned to at least one unit is counted in the objective function, and inequalities (4.5) enforce the maximum interference restrictions. For this last set of inequalities, the co-channel interference α_{ijk} for $j < k$ is recorded if and only if channel i is assigned to both units j and k.

The paper [34] computes a mixed 0-1 linear representation of Problem CAP by applying, for each $(i, j, k), j < k$, the approach of E_R to replace the quadratic term $x_{ij}x_{ik}$ with the continuous variable w_{ijk}. (The paper does not include the x_{ij} expressions found in the right side of (4.5), nor in the right side of the upcoming (4.7), but we include them in both instances to strengthen the continuous relaxations.) The below mixed 0-1 linear program results:

LP: minimize $\sum_i y_i$

subject to

$$
\begin{aligned}
& (4.3), (4.4), (4.6), && \\
& \sum_{k<j} \alpha_{ikj} w_{ikj} + \sum_{k>j} \alpha_{ijk} w_{ijk} \leq \beta_{ij} x_{ij} && \forall\, (i,j), \qquad (4.7)\\
& (x_{ij}, x_{ik}, w_{ijk}) \in T_{ijk} && \forall\, (i,j,k),\ j<k,
\end{aligned}
$$

where

$$T_{ijk} \equiv \{(x_{ij}, x_{ik}, w_{ijk}) :\ 0 \leq w_{ijk} \leq x_{ij},\ x_{ij} + x_{ik} - 1 \leq w_{ijk} \leq x_{ik}\} \ \forall\, (i,j,k),\ j<k.$$

Here, the restrictions $x_{ij} \leq y_i$ and $x_{ik} \leq y_i$ are not found in the sets T_{ijk} because they are present in (4.4), and the $y_i \leq 1$ restrictions will be later enforced in the relaxation of LP.

Our approach replaces each set T_{ijk} with the tighter set T'_{ijk} defined as

$$T'_{ijk} \equiv \{(x_{ij}, x_{ik}, w_{ijk}, y_i) :\ 0 \leq w_{ijk} \leq x_{ij},\ x_{ij} + x_{ik} - y_i \leq w_{ijk} \leq x_{ik}\} \ \forall\, (i,j,k),\ j<k,$$

and removes (4.4) due to it being implied by these sets T'_{ijk} to obtain the below linear reformulation of Problem CAP.

$$LP' : \text{minimize} \left\{ \sum_i y_i :\ (4.3), (4.6), (4.7),\ (x_{ij}, x_{ik}, w_{ijk}, y_i) \in T'_{ijk} \ \forall\, (i,j,k),\ j<k \right\}$$

Problem LP' is superior to Problem LP in two respects. First, while the two problems contain the same number of variables, Problem LP' contains pq fewer constraints, since (4.4) are implied. Second, Problem LP' has a continuous relaxation that is at least as tight as that of LP, as every inequality in LP is implied by the inequalities of LP'. Specifically, denote the continuous relaxations of Problems LP and LP' obtained by replacing (4.6) with the restrictions $y_i \leq 1 \ \forall\, i \in P$ within each, by Problems $\overline{LP}$ and $\overline{LP'}$, respectively. Letting $\nu(\bullet)$ denote the optimal objective function value of Problem $\bullet$, we then have that $\nu(CAP) = \nu(LP) = \nu(LP') \geq \nu(\overline{LP'}) \geq \nu(\overline{LP})$.

The following numeric example shows the improved relaxation strength of Problem $\overline{LP'}$

over Problem $\overline{LP}$.

Example

Suppose that $p = q = 2$ with $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 1$ and $\alpha_{112} = \alpha_{212} = 4$ in Problem CAP. Then Problem LP takes the form

LP: minimize $\quad y_1 + y_2$

subject to

$$x_{11} + x_{21} = 1, x_{12} + x_{22} = 1,$$

$$x_{11} \leq y_1,\ x_{12} \leq y_1, x_{21} \leq y_2,\ x_{22} \leq y_2,$$

$$4w_{112} \leq x_{11},\ 4w_{112} \leq x_{12}, 4w_{212} \leq x_{21},\ 4w_{212} \leq x_{22},$$

$$0 \leq w_{112} \leq x_{11}, x_{11} + x_{12} - 1 \leq w_{112} \leq x_{12},$$

$$0 \leq w_{212} \leq x_{21}, x_{21} + x_{22} - 1 \leq w_{212} \leq x_{22},$$

$$x_{11}, x_{12}, x_{21}, x_{22}, y_1, y_2 \text{ binary},$$

and Problem LP' takes the form

LP': minimize $\quad y_1 + y_2$

subject to

$$x_{11} + x_{21} = 1, x_{12} + x_{22} = 1,$$

$$4w_{112} \leq x_{11},\ 4w_{112} \leq x_{12}, 4w_{212} \leq x_{21},\ 4w_{212} \leq x_{22},$$

$$0 \leq w_{112} \leq x_{11}, x_{11} + x_{12} - y_1 \leq w_{112} \leq x_{12},$$

$$0 \leq w_{212} \leq x_{21}, x_{21} + x_{22} - y_2 \leq w_{212} \leq x_{22},$$

$$x_{11}, x_{12}, x_{21}, x_{22}, y_1, y_2 \text{ binary}.$$

We have $\nu(LP) = \nu(LP') = 2$, with a solution of

$$(x_{11}, x_{12}, x_{21}, x_{22}, w_{112}, w_{212}, y_1, y_2) = (1, 0, 0, 1, 0, 0, 1, 1).$$

$\nu(\overline{LP'}) = \frac{7}{4}$ with a solution of

$$(x_{11}, x_{12}, x_{21}, x_{22}, w_{112}, w_{212}, y_1, y_2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8}\right).$$

$\nu(\overline{LP}) = 1$ with a solution of

$$(x_{11}, x_{12}, x_{21}, x_{22}, w_{112}, w_{212}, y_1, y_2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}\right).$$

The above example, presented for illustrative purposes, is a small instance of the Channel Assignment Problem for which the given values of α_{ijk} and β_{ij} enforce that two channels must be used. To provide insight that Problem $\overline{LP'}$ can yield strength beyond that of $\overline{LP}$ for nontrivial problems, we present Table 4.2 below that reports on instances having $p = 4$ channels and all $\alpha_{ijk} = 1$. Within this table, each row records a different instance. The first column gives the number of q units as being either 4 or 8, the second column gives the common value to which all parameters β_{ij} are set, and the last three columns give the optimal objective function values of the integer program CAP, and the linear programs $\overline{LP'}$ and $\overline{LP}$, respectively. As noted theoretically above, we have $\nu(CAP) \geq \nu(\overline{LP'}) \geq \nu(\overline{LP})$. Table 4.2 shows that, depending on the parameters p and β_{ij}, we can satisfy this inequality system with a mixture of equalities and strict inequalities.

4.3 Convex Hull Representations for the Cases having $n \geq 3$

We derive the convex hull representation of C for the cases having $n \geq 3$, and then show how the RLT arguments allow for extensions to other discrete and/or continuous sets.

4.3.1 Convex Hull Derivation

For the cases of C having $n \geq 3$, we introduce a set F that is obtained by including the binary variable y and the restrictions $x_j \leq y$ for all $j \in N$ of C within A. Specifically, let

$$F \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \\ y \end{pmatrix} \in \mathbb{R}^{n+2} : \ 0 \leq w, \sum_{j \in N} x_j - (n-1) \leq w, \ w \leq x_j \leq y \ \forall \, j \in N, \ y \text{ binary} \right\}.$$

Table 4.2: Optimal Objective Function Values for $p = 4$ Channels when all $\alpha_{ijk} = 1$

q units	β_{ij}	$\nu(CAP)$	$\nu(\overline{LP'})$	$\nu(\overline{LP})$
4	1	2	$\frac{5}{3}$	1
4	2	2	$\frac{4}{3}$	1
4	3	1	1	1
8	1	4	$\frac{13}{7}$	1
8	2	3	$\frac{12}{7}$	1
8	3	2	$\frac{11}{7}$	1
8	4	2	$\frac{10}{7}$	1
8	5	2	$\frac{9}{7}$	1
8	6	2	$\frac{8}{7}$	1
8	7	1	1	1

We next apply the three level-1 RLT steps [39] of Reformulation, Linearization, and Projection described below to the set F to obtain $conv(F)$. The first two steps are illustrated in Table 4.3.

1. **Reformulation.** Multiply each of the $(2n + 2)$ inequalities of F by each of the nonnegative "product factors" y and $(1 - y)$. Enforce the binary identity that $y^2 = y$ to remove all occurrences of y^2.

2. **Linearization.** Substitute a continuous variable, say z_j, for each of the resulting quadratic terms $x_j y$ so that $z_j = x_j y$ for all $j \in N$. Substitute a continuous variable, say u, for each resulting quadratic term wy so that $u = wy$. Omit the binary restriction on y.

3. **Projection.** Project the resulting variable-space $(\mathbf{x}, w, y, \mathbf{z}, u)$ onto the space of the variables $(\mathbf{x}, w, y)$.

The left column of Table 4.3 gives the nonnegative products that are computed in the *reformulation* step, and the right column gives the inequalities obtained from enforcing $y^2 = y$ and applying the subsequent *linearization* step. The polytope G below is formed from the $(4 + 4n)$ inequalities of the right column.

Table 4.3: Illustration of the Level-1 RLT Applied to the Set F

Nonnegative Product	Linearized Inequality
$y(w \geq 0)$	$u \geq 0$
$y(w - \sum_{j\in N} x_j + (n-1) \geq 0)$	$u - \sum_{j\in N} z_j + (n-1)y \geq 0$
$y(x_j - w \geq 0)\ \forall\, j \in N$	$z_j - u \geq 0\ \forall\, j \in N$
$y(y - x_j \geq 0)\ \forall\, j \in N$	$y - z_j \geq 0\ \forall\, j \in N$
$(1-y)(w \geq 0)$	$w - u \geq 0$
$(1-y)(w - \sum_{j\in N} x_j + (n-1) \geq 0)$	$w - u - \sum_{j\in N}(x_j - z_j) + (n-1)(1-y) \geq 0$
$(1-y)(x_j - w \geq 0)\ \forall\, j \in N$	$x_j - z_j - w + u \geq 0\ \forall\, j \in N$
$(1-y)(y - x_j \geq 0)\ \forall\, j \in N$	$-x_j + z_j \geq 0\ \forall\, j \in N$

$$G \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \\ y \\ \mathbf{z} \\ u \end{pmatrix} \in \mathbb{R}^{2n+3} : 0 \leq u,\ \sum_{j\in N} z_j - (n-1)y \leq u,\ u \leq z_j \leq y\ \forall\, j \in N, \right.$$

$$0 \leq (w-u),\ \sum_{j\in N}(x_j - z_j) - (n-1)(1-y) \leq (w-u),$$

$$\left. (w-u) \leq (x_j - z_j) \leq 0\ \forall\, j \in N \right\}$$

The Projection then simply substitutes $z_j = x_j$ for all $j \in N$ and $u = w$ within G, and eliminates the variables $\mathbf{z}$ and u to obtain the set H below.

$$H \equiv \left\{ \begin{pmatrix} \mathbf{x} \\ w \\ y \end{pmatrix} \in \mathbb{R}^{n+2} :\ y \leq 1,\ 0 \leq w,\ \sum_{j\in N} x_j - (n-1)y \leq w,\ w \leq x_j \leq y\ \forall\, j \in N \right\}$$

Then we have the following result.

Theorem 4.3.1. *$conv(F) = H$.*

Theorem 4.3.1 follows directly from the RLT theory [39, Theorem 3.5] for a mixed 0-1 set having a single binary variable y. In fact, for this special case, the RLT arguments simplify to the following two statements: (i) $conv(F) \subseteq H$ because $F \subseteq H$ with H a convex set, and (ii) $conv(F) \supseteq H$ because every point $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{w} \\ \hat{y} \end{pmatrix} \in H$ with $\hat{y}$ binary is also in F, and every point $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{w} \\ \hat{y} \end{pmatrix} \in H$ with $\hat{y} \in (0,1)$ can be expressed as $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{w} \\ \hat{y} \end{pmatrix} = (1-\hat{y}) \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} + \hat{y} \begin{pmatrix} \frac{\hat{\mathbf{x}}}{\hat{y}} \\ \frac{\hat{w}}{\hat{y}} \\ 1 \end{pmatrix}$ with $\begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \in F$ and $\begin{pmatrix} \frac{\hat{\mathbf{x}}}{\hat{y}} \\ \frac{\hat{w}}{\hat{y}} \\ 1 \end{pmatrix} \in F$.

We can now get the desired result.

Corollary 4.3.2. $conv(C) = H$.

Proof. By Theorem 4.3.1, we need to show that $conv(C) = conv(F)$. We consider the two containments $conv(C) \subseteq conv(F)$ and $conv(C) \supseteq conv(F)$.

- $\mathbf{conv(C) \subseteq conv(F)}$

  Follows from $C \subseteq F$ with $conv(F)$ a convex set.

- $\mathbf{conv(C) \supseteq conv(F)}$

  It is sufficient to show that every extreme point of $conv(F)$ is in C. Clearly, $conv(F)$ has y binary at all extreme points. For $y = 0$, $conv(F)$ enforces $\mathbf{x} = \mathbf{0}$ and $w = 0$, and we have $\begin{pmatrix} \mathbf{x} \\ w \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \in C$. For $y = 1$, every extreme point $\begin{pmatrix} \mathbf{x} \\ w \\ 1 \end{pmatrix}$ of $conv(F)$ has $\begin{pmatrix} \mathbf{x} \\ w \end{pmatrix}$ an extreme point of A. But since $A = conv(B)$ by [7], we have that every extreme point of A is feasible to B, and thus that $\begin{pmatrix} \mathbf{x} \\ w \\ 1 \end{pmatrix} \in C$.

□

Three comments are warranted. First, upon looking back at the structure of the sets A and H, we see that H is a homogenization [45] of the polytope A, with the homogenization following from the RLT. Using this type application of the RLT, we are able to construct convex hull forms of related sets. Such forms are discussed in the following subsection. Second, while the set F was used to motivate the polytope H via the level-1 RLT operations, the proof of Corollary 4.3.2 does not need to explicitly rely on F. To explain, we can modify the two RLT arguments following the statement of Theorem 4.3.1 to obtain: (i) $conv(C) \subseteq H$ because $C \subseteq H$ with H a convex set, and (ii) every extreme point $\begin{pmatrix} \mathbf{x} \\ w \\ y \end{pmatrix}$ of H has y binary because, given $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{w} \\ \hat{y} \end{pmatrix} \in H$ with $\hat{y} \in (0,1)$, we have $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{w} \\ \hat{y} \end{pmatrix} = (1-\hat{y}) \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} + \hat{y} \begin{pmatrix} \frac{\hat{\mathbf{x}}}{\hat{y}} \\ \frac{\hat{w}}{\hat{y}} \\ 1 \end{pmatrix}$ with $\begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \in H$ and $\begin{pmatrix} \frac{\hat{\mathbf{x}}}{\hat{y}} \\ \frac{\hat{w}}{\hat{y}} \\ 1 \end{pmatrix} \in H$. Then we can invoke the latter part (last three sentences) of the proof of Corollary 4.3.2 to argue that every extreme point of H is in C by replacing "$conv(F)$" with "H" in each of the two occurrences, obtaining $conv(C) \supseteq H$. Third, we note that the set E of (4.2) is the special case of H when $n = 2$. For this case, the inequalities $x_1 \leq y$ and $x_2 \leq y$ of H are redundant in E, similar to the manner in which the restrictions $x_1 \leq 1$ and $x_2 \leq 1$ are redundant in A when $n = 2$, and consistent with the removal of (4.4) from Problem LP'.

4.3.2 Extensions to Other Discrete and/or Continuous Sets

Theorem 4.3.1 and Corollary 4.3.2 continue to hold true for any generalization of the sets A and B satisfying $A = conv(B)$, with the sets C and H suitably defined in terms of B and A, respectively. Thus, provided the convex hull of a given set in discrete and/or continuous variables is known, the convex hull of the resulting form obtained by bounding each variable in terms of a binary variable y is also known. Various scenarios come to mind, with three discussed below.

As a first scenario, we can replace the monomial expression $\prod_{j \in N} x_j$ of B and C with any supermodular function of variables $\mathbf{x}$, and allow $\ell y \leq x_j \leq uy$ for all $j \in N$ for variable lower and upper bounds ℓ and u, respectively, within C. This replacement is possible because the convex hull form of B is known [44] for such functions. In this manner, we have the convex hulls of such functions when the variables are bounded in terms of the binary y.

A second scenario is to replace the $\mathbf{x}$ binary restrictions of B with the inequalities $-1 \leq x_j \leq 1$ for all $j \in N$. The paper [2] shows the convex hull for this set to be defined by the following two families of inequalities where, for simplicity, $x_{n+1} = w$ and $N' \equiv \{1, \ldots, n+1\}$.

$$-1 \leq x_j \leq 1 \quad \forall\, j \in N'$$
$$(n-1) - \left(\sum_{j \in J} x_j - \sum_{j \in (N'-J)} x_j \right) \geq 0 \quad \forall\, J \subseteq N' \text{ with } (n+1-|J|) \text{ odd}$$

In this case, these two families of inequalities are as A. Then the convex hull of C with the $\mathbf{x}$ binary restrictions replaced with these same $-1 \leq x_j \leq 1$ restrictions is given as follows, together with $y \leq 1$.

$$-y \leq x_j \leq y \quad \forall\, j \in N'$$
$$(n-1)y - \left(\sum_{j \in J} x_j - \sum_{j \in (N'-J)} x_j \right) \geq 0 \quad \forall\, J \subseteq N' \text{ with } (n+1-|J|) \text{ odd}$$

These inequalities become H.

A third scenario deals with the Boolean Quadric Polytope [35]. This problem on t variables $\mathbf{x}$ is defined as follows.

$$BQP(t) = conv\left(\left\{ (\mathbf{x}, \mathbf{w}) \in \mathbb{R}^{\frac{t(t+1)}{2}} : \mathbf{x} \text{ binary}, \; w_{ij} = x_i x_j \; \forall\, (i,j),\; 1 \leq i < j \leq t \right\}\right)$$

Here, $BQP(t)$ plays the role of A. For every t for which $BQP(t)$ can be computed, our study allows us to obtain

$$BQP'(t) = conv\left(\left\{ (\mathbf{x}, \mathbf{w}, y) \in \mathbb{R}^{\frac{t(t+1)}{2}+1} : \mathbf{x} \text{ binary},\; y \text{ binary}, \; x_j \leq y \; \forall\, j = 1, \ldots, t, \; w_{ij} = x_i x_j \; \forall\, (i,j),\; 1 \leq i < j \leq t \right\}\right).$$

To demonstrate, the paper [35] shows that $BQP(3)$ has

$$
\begin{aligned}
BQP(3) = \Big\{ & (x_1, x_2, x_3, w_{12}, w_{13}, w_{23}) : \\
& 0 \le w_{12} \le x_1,\ x_1 + x_2 - 1 \le w_{12} \le x_2, \\
& 0 \le w_{13} \le x_1,\ x_1 + x_3 - 1 \le w_{13} \le x_3, \\
& 0 \le w_{23} \le x_2,\ x_2 + x_3 - 1 \le w_{23} \le x_3, \\
& - x_1 + w_{12} + w_{13} - w_{23} \le 0, \\
& - x_2 + w_{12} - w_{13} + w_{23} \le 0, \\
& - x_3 - w_{12} + w_{13} + w_{23} \le 0, \\
& x_1 + x_2 + x_3 - w_{12} - w_{13} - w_{23} \le 1 \Big\}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
BQP'(3) = \Big\{ & (x_1, x_2, x_3, w_{12}, w_{13}, w_{23}, y) : y \le 1, \\
& 0 \le w_{12} \le x_1,\ x_1 + x_2 - y \le w_{12} \le x_2, \\
& 0 \le w_{13} \le x_1,\ x_1 + x_3 - y \le w_{13} \le x_3, \\
& 0 \le w_{23} \le x_2,\ x_2 + x_3 - y \le w_{23} \le x_3, \\
& - x_1 + w_{12} + w_{13} - w_{23} \le 0, \\
& - x_2 + w_{12} - w_{13} + w_{23} \le 0, \\
& - x_3 - w_{12} + w_{13} + w_{23} \le 0, \\
& x_1 + x_2 + x_3 - w_{12} - w_{13} - w_{23} \le y \Big\}.
\end{aligned}
$$

Then $BQP(3)$ and $BQP'(3)$ are of the forms A and H, respectively. These convex hull representations give flexibility in solving 0-1 quadratic programs such as Problem CAP. For Problem CAP, we have the option to group subsets of the variables relating to the same channel i into different sets of various sizes t to obtain tighter representations.

4.4 Conclusions and Future Research

This paper generalizes convex hull results for a classical linearization of monomial expressions of binary variables; the generalization allows each variable to be upper-bounded by an auxiliary binary variable, as expressed in the set C. For monomials having $n = 2$ variables, the convex hull is obtained by applying a special-structure RLT that computes products of constraints taken two at a time while, for cases having $n \geq 3$, a level-1 RLT application is used on a suitably-defined polytope. As a consequence, the level-1 RLT is shown to explain the homogenization of a polytope. The approach is generally applicable to various discrete and/or continuous sets that include special symmetric multilinear functions over box constraints, and the Boolean Quadric Polytope.

For monomials having $n = 2$ variables, the set C can arise in various scenarios. As explained in the paper, one scenario is the Channel Assignment Problem where, for a given channel that can service multiple units, each variable x_j denotes the assignment of that channel to unit j, the quadratic terms are used to model co-channel interference between two units assigned the same channel, and the variable y indicates whether the channel is used. The set C can also model variants of known problems. For example, consider the Chromatic Number Problem (see [33, page 582] for example), modified so that adjacent nodes are permitted to realize the same color, subject to pairwise penalties. Here, given a color, each variable x_j denotes the assignment of that color to node j, the quadratic terms are used to record the penalty of assigning the color to adjacent nodes, and the variable y indicates whether the color is used. Another example is the Uncapacitated Facility Location Problem [33, page 8], modified so that costs are also incurred based on the pairwise assignments of a facility to multiple clients. Given a facility, each variable x_j denotes the assignment of that facility to client j, the quadratic terms are used to record the pairwise costs of assigning the facility to different clients, and the binary variable y indicates whether the facility is opened.

Future work is to determine the computational merits of the posed forms, as well as to assess effective implementations. We showed, in a small numeric example, theoretical improvement in relaxation strength relative to the Channel Assignment Problem when computing convex hull forms for quadratic expressions, and we also showed similar strengthenings in the somewhat larger instances of Table 4.2. However, more detailed studies are needed. Relative to implementation, the theory allows for different treatments of the variables. As an example, for 0-1 quadratic programs, we can either consider each quadratic term individually, or we can strategically group terms to obtain convex hull forms over larger subsets of variables. The second option produces tighter linear programming relaxations, but increases problem size. Optimal tradeoffs between strength and size need to be determined.

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