8-2018

Generalizations of Permutation Statistics to Words and Labeled Forests

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Generalizations of Permutation Statistics to Words and Labeled Forests

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematics

by
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August 2018

Accepted by:
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Abstract

A classical result of MacMahon shows the equidistribution of the major index and inversion number over the symmetric groups. Since then, these statistics have been generalized in many ways, and many new permutation statistics have been defined, which are related to the major index and inversion number in may interesting ways. In this dissertation we study generalizations of some newer statistics over words and labeled forests.

Foata and Zeilberger defined the graphical major index, maj_U, and the graphical inversion index, inv_U, for words over the alphabet $\{1, \ldots, n\}$. In this dissertation we define a graphical sorting index, sor_U, which generalizes the sorting index of a permutation. We then characterize the graphs U for which sor_U is equidistributed with inv_U and maj_U on a single rearrangement class.

Björner and Wachs defined a major index for labeled plane forests, and showed that it has the same distribution as the number of inversions. We define and study the distributions of a few other natural statistics on labeled forests. Specifically, we introduce the notions of bottom-to-top maxima, cyclic bottom-to-top maxima, sorting index, and cycle minima. Then we show that the pairs $(\mathrm{inv}, \text{BT-max})$, $(\mathrm{sor}, \mathrm{Cyc})$, and $(\mathrm{maj}, \text{CBT-max})$ are equidistributed. Our results extend the result of Björner and Wachs and generalize results for permutations.

Lastly, we study the descent polynomial of labeled forests. The descent polynomial for permutations is known to be log-concave and unimodal. In this dissertation we discuss what properties are preserved in the descent polynomial of labeled forests.

Dedication

To my parents.

Acknowledgements

First I would like to thank my parents, my sister, and my Aunt Laurie for their constant love and support.

I would like to thank my advisor Dr. Svetlana Poznanović for all of her advice and support. I will certainly miss our weekly discussions. I would also like to thank the rest of my committee—Dr. Neil Calkin, Dr. Wayne Goddard, and especially Dr. Matthew Macauley—for all of their insightful ideas and suggestions.

I would also like to thank Dr. Gretchen Matthews and Dr. Lea Jenkins for their support of AWM and wonderful career advice.

In no particular order, I would like to thank my friends Rachel, Jennifer, Samantha, Stephanie, Tyler, Megan W., Megan L., the St. Mark's misfits, Shawna, Lydia, and Faith.

Last, but certainly not least, I would like to thank Will for all of his love and reassurance, as well as the rest of the Lassiter/Bowers clan for accepting me as one of their own.

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Chapter 1

Introduction

Given a set of combinatorial objects S, a combinatorial statistic associates with each element of S a nonnegative integer. Studying the distributions of combinatorial statistics over S allows us to gather more information from the set than simply counting the number of objects in S. Additionally, knowing the distributions of a certain statistic over S sometimes reveals similarities with other combinatorial objects, which in turn can be exploited to learn about their structure. This dissertation focuses on several statistics on words and labeled forests motivated by knowledge we have about permutations. In Chapter 2 we discuss the results for permutations that motivated this work.

In 1915, MacMahon [33] came to the surprising conclusion that two classical permutation statistics, inversion number (inv) and major index (maj), are equidistributed over the symmetric group S_n. Due to his discovery, the family of permutation statistics with this distribution is called Mahonian. Since his discovery, many more Mahonian statistics have been found, including Denert's statistic [12], the Rawlings major index [38], Kadell's weighted inversion number [29], the statistics introduced by Clarke [10], the maj-inv statistics of Kasraoui [30] and the sorting index [36]. Some of the aforementioned statistics have analogues over other combinatorial structures, among them Young tableaux, set partitions, and ordered partitions. In this thesis we discuss generalizations of Mahonian statistics over words and labeled forests.

It is natural to consider generalizing permutation statistics over words, otherwise known as multiset permutations. When MacMahon originally studied inv and maj, he showed that they have the same distribution over all permutations of a given multiset [33]. A natural question to consider is what happens to the distribution of these statistics if we consider a different ordering of the integers. Motivated by this question, Foata and Zeilberger defined a graphical inversion number (inv_U) and major index (maj_U) [21]. These new statistics compute the classical statistics with respect to a given relation, U, possibly different from the natural order of the integers. If we consider the natural ordering for the integers, then these statistics are equivalent to the classical statistics over permutations. Foata and Zeilberger showed that these statistics are equidistributed over all permutations of all multisets if and only if the relation U is bipartitional [21].

In Section 4.2, we define a graphical sorting index (sor_U) that reduces to the permutation statistic over S_n with the natural order of the integers. When the relation U is the natural ordering of the integers, sor_U defines a sorting index for words that is equidistributed with maj and inv over all permutations of a given multiset. Unfortunately, the statistics inv_U, maj_U, and sor_U are not equidistributed over permutations of a given multiset for every bipartitional relation U. However, in Theorem 3.1.2 we classify the relations U for which the three statistics are equidistributed over the set of permutations of a given multiset. In order to prove Theorem 3.1.2, we first strengthen the result of Foata and Zeilberger to show that the statistics inv_U and maj_U are equidistributed over all permutations of a single multiset, as opposed to all multisets, if and only if U is essentially bipartitional (Theorem 3.1.1).

In [11], Mallows and Riordan define an inversion number for rooted labeled forests, and in [4], Björner and Wachs define a descent set and major index for rooted labeled forests. A forest of size n is labeled by assigning to each vertex an integer label from the set $\{1, \ldots, n\}$, with each label used exactly once. Björner and Wachs showed that maj and inv are equidistributed over all possible labelings of a given forest. In Section 4.2, we define a sorting index for rooted labeled forests using a sorting algorithm, and show that it is equidistributed with inv and maj over all labelings of a given forest. These statistics are equivalent to the permutation statistics when they are considered on a straight tree.

The statistics right-to-left minima and number of cycles belong to a family of permutation statistics called Stirling statistics because their distribution is given by the unsigned Stirling numbers of the first kind. For permutations, it is known that the pairs (inv, RL-min), (maj, RL-min), and (sor, Cyc) are equidistributed over S_n [5, 36, 37]. These pairs of statistics are called Mahonian-Stirling statistics, and in Chapter 4 we discuss generalizations of these pairs of statistics to labeled forests. We define analogous statistics BT-max, Cyc, and CBT-max for labeled forests and, through a series of bijections, show that the pairs (inv, BT-max), (maj, CBT-max), and (sor, Cyc) are equidistributed over all labelings of a given forest F.

Many of these permutation statistics have been generalized over colored permutations, i.e. members of the generalized symmetric group S_n^k [1]. The group S_n^k is the wreath product, $C_k \wr S_n$, where C_k is the cyclic group on $\{0, 1, \ldots, k-1\}$. An element of S_n^k, a colored permutation, is a permutation $\sigma \in S_n$ with a "color" $c_i \in \{0, 1, \ldots, k-1\}$ assigned to each letter $\sigma(i)$. In Chapter 4, we generalize the statistics sor, right-to-left minima, and number of cycles for signed permutations, i.e. the case where $k = 2$, to signed labeled forests. This extends the work of Chen, Gao, and Guo [8], who defined an inversion number and major index for signed labeled forests. Unfortunately, the signed pair (maj_B, CBT-max_B) is not equidistributed with (inv_B, BT-max_B) and ($\mathrm{sor_B}$, Cyc_B) over all signed labelings of a forest F, but in Section 4.4, we conjecture that by using the sorting algorithm to define an inverse tree, we can consider the major index of the inverse tree (imaj) paired with BT-max to better generalize the results for permutation statistics.

Lastly, in Chapter 5 we discuss the descent polynomial for labeled trees. This polynomial is a generalization of the Eulerian polynomial, a polynomial known to have only real roots, which in turn implies log-concavity and unimodality of its coefficients. In Section 5.1, we prove the unimodality of the descent polynomial of a forest F. Then in Section 5.2 we discuss a conjecture about the log-concavity of these polynomials, and finally in Section 5.3 we show that the descent distribution for a tree T converges to the standard normal distribution under certain assumptions about the maximum degree of the tree.

Chapter 2

Background

In this chapter, we discuss the results for permutations that motivate the work in this dissertation. We begin by defining the Mahonian statistics inv, maj, and sor for permutations and discussing their distributions over S_n. Then in Section 2.2 we define the Stirling statistics RL-min and Cyc, and state the equidistribution results of the Mahonian-Stirling pairs $(\text{inv}, \text{RL-min})$, $(\text{maj}, \text{RL-min})$, and (sor, Cyc). In Section 2.3, we introduce the Eulerian polynomial and discuss some of its properties. Lastly, in Section 2.4, we discuss how some of the Mahonian and Stirling statistics we have defined for permutations have been generalized for signed permutations.

2.1 Mahonian Statistics

A permutation $\sigma \in S_n$ is a bijection of the set $S = \{1, \ldots, n\}$ to itself. We typically write the permutation σ in one-line notation, $\sigma(1)\sigma(2)\cdots\sigma(n)$. For example, the permutation in S_3 given by $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(3) = 1$ is written as 213. In general, the identity permutation of S_n is $123\cdots n$.

An inversion in a permutation σ is a pair $(\sigma(i), \sigma(j))$ where $i < j$ and $\sigma(i) > \sigma(j)$. In other words, it is a pair of numbers where the larger one appears to the left of the smaller one in one-line notation. For example, if $\sigma = 235416$, the pairs $(2, 1), (3, 1), (5, 4), (5, 4)$ and $(4, 1)$ are inversions, meaning that the inversion number of σ is 5. This is written as $\text{inv}(\sigma) = 5$.

In 1915, MacMahon defined a new statistic that he called the greater index, now commonly called the major index [35]. For a permutation $\sigma \in S_n$, the position i is a descent of σ if $\sigma(i) >$

$\sigma(i+1)$. The set of descents of σ is denoted by $\mathrm{Des}(\sigma)$. The major index of σ is the sum of its descents. For example, if $\sigma = 235416$ we have $\mathrm{Des}(\sigma) = \{3, 4\}$ and $\mathrm{maj}(\sigma) = 3 + 4 = 7$. MacMahon surprisingly showed that the major index and the inversion number have the same distribution over S_n. The distribution of these statistics over all permutations in S_n is given by the generating function

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}\,(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{maj}\,(\sigma)} = [n]!,$$

where $[n]$ denotes the polynomial $1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2]\cdots[n] = (1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$. In his honor, permutation statistics with this distribution are called Mahonian [3].

A newer Mahonian statistic of interest in this work is the sorting index, defined by Petersen in [36] and studied independently by Wilson in [45]. A permutation $\sigma \in S_n$ can be uniquely decomposed into a product of transpositions, $\sigma = (i_1, j_1)(i_2, j_2)\cdots(i_k, j_k)$, such that $j_1 < \cdots < j_k$ and $i_1 < j_1, \ldots, i_k < j_k$. The sorting index of σ is defined by $\mathrm{sor}(\sigma) = \sum_{r=1}^{k}(j_r - i_r)$. It can also be described as the total distance traveled by the elements of σ when σ is sorted using the Straight Selection Sort algorithm [31]. This algorithm first places n in the n^{th} position by applying a transposition, then places $n-1$ in the $(n-1)^{\text{st}}$ position by applying a transposition, etc. For example, consider the permutation $\sigma = 2413576$. We have

$$2413576 \overset{(67)}{\to} 2413567 \overset{(24)}{\to} 2314567 \overset{(23)}{\to} 2134567 \overset{(12)}{\to} 1234567.$$

Therefore, $\mathrm{sor}(\sigma) = (2-1) + (3-2) + (4-2) + (7-6) = 5$.

The aforementioned statistics have analogues over many combinatorial structures such as Young tableaux [25], set partitions [40], and ordered partitions [43]. In [34], MacMahon generalized the statistics inv and maj to words in the following ways. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a sequence of non-negative integers. We will denote by $\mathcal{R}(\boldsymbol{\alpha})$ the set of permutations of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n}\}$, i.e., $\mathcal{R}(\boldsymbol{\alpha})$ is the set of all words over the alphabet $\{1, \ldots, n\}$ containing α_i occurrences of the letter i for all $i = 1, 2, \ldots, n$. For $w = x_1 \ldots x_m \in \mathcal{R}(\boldsymbol{\alpha})$, the *inversion number* is defined as

$$\mathrm{inv}(w) = \sum_{1 \le i < j \le m} \mathcal{X}(x_i > x_j),$$

and the *major index* is defined as

$$\mathrm{maj}(w) = \sum_{i=1}^{m-1} i\mathcal{X}(x_i > x_{i+1}),$$

where $\mathcal{X}$ is the characteristic function defined as $\mathcal{X}(A) = 1$ when A is true and $\mathcal{X}(A) = 0$ when A is false. The set of all positions i such that $x_i > x_{i+1}$ is known as the descent set of w, $\mathrm{Des}(w)$, and its cardinality is denoted by $\mathrm{des}(w)$. So, $\mathrm{maj}(w) = \sum_{i\in \mathrm{Des}(w)} i$.

MacMahon showed that maj and inv are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$ [33, 35] . Namely,

$$\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{inv}(w)} = \sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{maj}(w)} = \begin{bmatrix} \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ \alpha_1, \alpha_2, \ldots, \alpha_n \end{bmatrix}$$

where

$$\begin{bmatrix} \alpha_1 + \alpha_2 + \ldots + \alpha_k \\ \alpha_1, \alpha_2, \ldots, \alpha_k \end{bmatrix} = \frac{[\alpha_1 + \alpha_2 + \cdots + \alpha_k]!}{[\alpha_1]![\alpha_2]! \ldots [\alpha_k]!}$$

is the q-multinomial coefficient.

The sorting index can also be naturally extended to words $w \in \mathcal{R}(\boldsymbol{\alpha})$ by using a stable variant of Straight Selection Sort which reorders the letters into a weakly increasing sequence. At each step transpositions are applied to place all the n's at the end, then all the $n-1$'s to the left of them, etc., so that for each $x \in X$, the α_x copies of x stay in the same relative order they were right before they were "processed". Then we define $\mathrm{sor}(w)$ to be the sum of the number of positions each element moved during the sorting. For example, applying this sorting algorithm to $w = 143123123$ yields

$$143123123 \to 133123124 \to 133122134 \to 131122334 \to 121123334 \to 111223334 \qquad (2.1)$$

and thus $\mathrm{sor}(w) = 7 + 2 + 4 + 4 + 2 = 19$.

In Chapter 3, we discuss a generalization of these statistics for words due to Foata and Zeilberger [21]. They defined graphical statistics (graphical inversions and graphical major index) parametrized by a general directed graph U and they described the graphs U for which these statistics are equidistributed on *all* rearrangement classes. We begin by strengthening their result to classify the graphs U such that the graphical major index and graphical inversions are equidistributed for a fixed rearrangement class. We then define a graphical sorting index and classify the graphs U that give equidistribution of all three statistics over a fixed rearrangement class.

The statistics inv and maj have a symmetric joint distribution over S_n [20]. This means that the pairs of statistics (inv, maj) and (maj, inv) are equidistributed over S_n. In [17], Foata defined a bijection commonly referred to as $\varphi : S_n \to S_n$ such that $\text{maj}(\sigma) = \text{inv}(\varphi(\sigma))$. Let $\sigma = \sigma(1) \cdots \sigma(n) \in S_n$ and construct words $w_1, \ldots, w_n$ such that w_k is a permutation of the set $\{\sigma(1), \ldots, \sigma(k)\}$ in the following way.

- Let $w_1 = \sigma(1)$.
- Assume that the word w_k has been defined.
- If the last letter in w_k (which is the same as $\sigma(k)$) is greater than $\sigma(k+1)$, split w_k after each letter greater than $\sigma(k+1)$.
- If the last letter in w_k is less than $\sigma(k+1)$, split w_k after each letter less than $\sigma(k+1)$.
- Cyclically shift each section of w_k one unit to the right.
- Place $\sigma(k+1)$ at the end and call this new word w_{k+1}.

Now set $\varphi(\sigma) = w_n$.

Example 2.1.1. *Let* $\sigma = 21435$.

$$
\begin{aligned}
w_1 &= 2 & \sigma(2) &= 1 \\
w_2 &= 2|1| & \sigma(3) &= 4 \\
w_3 &= 214| & \sigma(4) &= 3 \\
w_4 &= 4|2|1|3| & \sigma(5) &= 5
\end{aligned}
$$

Thus, $\varphi(\sigma) = 42135$ *and* $\text{maj}(\sigma) = 4 = \text{inv}(\varphi(\sigma))$.

This bijection also has the property that $\text{imaj}(\sigma) = \text{imaj}(\varphi(\sigma))$ [20], where the statistic imaj is defined as

$$\text{imaj}(\sigma) = \text{maj}(\sigma^{-1}).$$

The statistics (maj, imaj) have a symmetric joint distribution since taking the inverse of a permutation maps maj to imaj and vice versa. Applying the map φ takes maj to inv and preserves imaj, thus we deduce that (inv, imaj) has a symmetric joint distribution. Finally, taking the inverse of a permutation preserves inv and maps imaj to maj so we get that (inv, maj) has a symmetric joint distribution. In other words,

$$(\mathrm{inv}, \mathrm{maj}) \underset{i}{\sim} (\mathrm{inv}, \mathrm{imaj}) \underset{\varphi^{-1}}{\sim} (\mathrm{maj}, \mathrm{imaj}) \underset{i}{\sim} (\mathrm{imaj}, \mathrm{maj}) \underset{\varphi}{\sim} (\mathrm{imaj}, \mathrm{inv}) \underset{i}{\sim} (\mathrm{maj}, \mathrm{inv})$$

where $i : S_n \to S_n$ is the inverse map.

In Chapter 4 we discuss the generalization of the Mahonian statistics maj, inv, and sor to labeled forests. The statistics inv and maj do not have a symmetric joint distribution over all labelings of a forest F, but in Section 4.4 we discuss a generalization of imaj to labeled forests and discuss some properties that are preserved.

2.2 Sterling Statistics

Another family of permutation statistics are the Stirling statistics, i.e., permutation statistics whose distribution is governed by the unsigned Stirling numbers of the first kind. Two well-known Stirling statistics are the number of cycles, cyc, and the number of right-to-left minimum letters, rl-min. It is well known that

$$\sum_{\sigma \in S_n} t^{\mathrm{cyc}(\sigma)} = \sum_{\sigma \in S_n} t^{\text{rl-min}(\sigma)} = \prod_{k=0}^{n-1} (t+k).$$

In [5], Björner and Wachs showed that the pairs of statistics (inv, RL-min) and (maj, RL-min) are equidistributed over S_n, and they have the following generating function:

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}\,(\sigma)} \prod_{i \in \text{RL-min}} t_i = \sum_{\sigma \in S_n} q^{\mathrm{maj}\,(\sigma)} \prod_{i \in \text{RL-min}} t_i = t_1(t_2+q)\cdots(t_n+q+\cdots+q^{n-1}),$$

where RL-min(σ) is the set of right-to-left minimum letters in σ.

In [36], Peterson showed that the pair (sor, Cyc) is equidistributed with (inv, RL-min) and (maj, RL-min), where Cyc(σ) is the set of minimal elements in the cycles of σ. There is a canonical bijection that maps a permutation σ with k cycles to a permutation $\hat{\sigma}$ with k right-to-left minima by writing the permutation σ in cycle notation with the smallest element of each cycle written last, and then removing the parentheses to get the permutation $\hat{\sigma}$ in one-line notation. For example consider the permutation $\sigma = 541263 = (5631)(42)$. Removing the parentheses gives $\hat{\sigma} = 563142$, and $\text{cyc}(\sigma) = 2 = \text{rl-min}(\hat{\sigma})$. To reverse the map simply insert ")" to the right of each right-to-left minimum and "(" before the first element of $\hat{\sigma}$ and after each inserted ")". While this bijection gives the equidistribution of cyc and rl-min, it does not map sor to inv. In the example above, $\text{sor}(\sigma) = 9$ and $\text{inv}(\hat{\sigma}) = 11$.

In Chapter 4, we look at generalizations of these Mahonian-Stirling pairs over labeled forests. Mallows and Riordan [11] generalized inv to labeled trees, and Björner and Wachs [4] generalized maj, and showed the two statistics are equidistributed over all possible labelings of a fixed forest. We define the statistics sor, BT-max, Cyc, and CBT-max for labeled forests, and in Sections 4.1 through 4.3, we construct bijections to show that the pairs $(\text{inv}, \text{BT-max})$, (sor, Cyc), and $(\text{maj}, \text{CBT-max})$ are equidistributed over all labelings of a fixed forest, and we compute the generating function. In Section 4.4, we wrap up the Chapter with a discussion about a conjecture using the idea of an "inverse" tree to better generalize the pair of statistics $(\text{maj}, \text{RL-min})$ to labeled forests.

2.3 Eulerian Polynomial

The n^{th} Eulerian polynomial is

$$A_n(q) = \sum_{\sigma \in S_n} q^{\text{des}(\sigma)+1} = \sum_{k=1}^{n} A(n,k) q^k,$$

where $A(n,k)$ is the number of permutations $\sigma \in S_n$ with exactly $k-1$ descents, called an *Eulerian number*.

The Eulerian polynomials were first introduced by Euler in the form

$$\sum_{k=0}^{\infty} (k+1)^n q^n = \frac{A_n(q)}{(1-q)^{n+1}}$$

to evaluate values of the alternating zeta function at negative integers [15]. The combinatorial interpretation given above has led to many interesting combinatorial proofs of properties of these polynomials. In this thesis, we discuss two properties of the coefficients: unimodality and log-concavity.

Definition 2.3.1. A sequence $a_0, a_1, \ldots, a_n$ is *unimodal* if for some j with $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \cdots \leq a_j \geq \cdots \geq a_n$ and is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i such that $1 \leq i \leq n-1$. We say a polynomial $a_0 + a_1 q + \cdots + a_n q^n$ is unimodal if the sequence of its coefficients is *unimodal*, and similarly the polynomial is *log-concave* if the sequence of its coefficients is log-concave.

The log-concavity of the Eulerian polynomial has been proven in many different ways [41, 7, 22]. For example, it is known that the Eulerian polynomial has only real roots which implies log-concavity [42, 41]. For a nonnegative sequence log-concavity implies unimodality, and thus the Eulerian polynomial is unimodal. In Chapter 5, we define and study the descent polynomial of labeled forests, a generalization of the Eulerian polynomial.

2.4 Signed Permutations

Let $\bar{i}$ denote $-i$. Signed permutations are permutations of $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ such that $\sigma(-i) = -\sigma(i)$. Note that the permutation σ is determined by the values $\sigma(1), \ldots, \sigma(n)$. As usual we will let B_n denote the group of signed permutations, also known as the hyperoctahedral group of rank n, or the Coxeter group of type B_n.

For a permutation $\sigma \in S_n$, its length $\ell(\sigma)$ as an element in a Coxeter group with the standard generators is equal to the number of inversions, i.e., $\ell(\sigma) = \mathrm{inv}(\sigma) = \#\{(i,j) : 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$. For an element $\sigma \in B_n$, the length function is given by

$$\ell_B(\sigma) = \mathrm{inv}(\sigma) + \mathrm{n}_1(\sigma) + \mathrm{n}_2(\sigma),$$

where

$$\mathrm{n}_1(\sigma) = \#\{i : 1 \leq i \leq n, \sigma(i) < 0\},$$

and

$$\mathrm{n}_2(\sigma) = \#\{(i,j) : 1 \leq i < j \leq n, \sigma(i) + \sigma(j) < 0\}.$$

The type B right-to-left minimum letters for $\sigma \in B_n$ are defined by

$$\text{RL-min}_B(\sigma) = \{\sigma(i) : 0 < \sigma(i) < |\sigma(j)| \text{ for all } j > i\}.$$

A signed permutation σ uniquley factors as a product of signed transpositions, $\sigma = (i_1, j_1) \cdots (i_k, j_k)$, where $i_s < j_s$ for $1 \leq s \leq k$ and $0 < j_1 < \cdots < j_k$. In [36], Peterson defined the type B sorting index by

$$\mathrm{sor_B}\,(\sigma) = \sum_{r=1}^{k} j_r - i_r - \mathcal{X}(i_r < 0)$$

where $\mathcal{X}$ is the characteristic function defined in Section 2.1. As before, this can be interpreted as the sum of distances traveled by the elements of σ as it is sorted.

For example, consider the permutation $\sigma = 4\bar{2}15\bar{3}$. We have

$$4\bar{2}15\bar{3} \overset{(45)}{\to} 4\bar{2}1\bar{3}5 \overset{(14)}{\to} \bar{3}\bar{2}145 \overset{(\bar{1}3)}{\to} \bar{1}\bar{2}345 \overset{(\bar{2}2)}{\to} \bar{1}2345 \overset{(\bar{1}1)}{\to} 12345.$$

Therefore $\sigma = (\bar{1}1)(\bar{2}2)(\bar{1}3)(14)(45)$, and thus $\mathrm{sor_B}(\sigma) = (1-(-1)-1)+(2-(-2)-1)+(3-(-1)-1)+(4-1)+(5-4) = 11$.

Signed permutations can be decomposed into two different types of cycles. Cycles of the form $(a_1, \ldots, a_m)$ which also takes $\bar{a}_1$ to $\bar{a}_2, \ldots, \bar{a}_{m-1}$ to $\bar{a}_m$, and $\bar{a}_m$ to $\bar{a}_1$ are called *balanced* and cycles of the form $(a_1, \ldots, a_m, \bar{a}_1, \ldots, \bar{a}_m)$ are called *unbalanced*.

For a signed permutation σ, we let $\mathrm{Cyc}_B\,(\sigma) = \{|k| : k$ is a minimal number in absolute value of a balanced cycle$\}$ [36, 37]. For example in the signed permutation $\sigma = 2\bar{4}513 = (12\bar{4}\bar{1}\bar{2}4)(35)$, the first cycle is unbalanced and the second is balanced. Thus $\mathrm{Cyc}_B\,(\sigma) = \{3\}$.

Signed permutation statistics that are equidistributed with the length function ℓ_B are called Mahonian. Two such statistics are the flag major index and the R-major index, generalizations of the major index for permutations. In [1], Adin and Roichman defined the statistic flag major index, fmaj, in terms of the Coxeter elements, and it can be expressed as

$$\mathrm{fmaj}(\sigma) = 2\,\mathrm{maj}(\sigma) + \mathrm{n}_1(\sigma)$$

for $\sigma \in B_n$ where $\mathrm{maj}(\sigma) = \sum_{\substack{1 \leq i \leq n-1 \\ \sigma(i) > \sigma(i+1)}} i$, analogous to the definition for unsigned permutations. In [18] Foata and Han give a bijective proof of the equidistribution of fmaj and ℓ_B.

Now we define a second Mahonian statistic generalizing the major index, called the R-major index, which appears implicitly in [39] and is defined explicitly in [8]. Let $\mathrm{Des}_B = \{1 \leq i \leq n :$

$\sigma(i) > \sigma(i+1)\}$ with $\sigma(n+1) = 0$, $\mathrm{maj}_B = \sum_{i \in \mathrm{Des}_B(\sigma)} i$, and $\mathrm{p}(\sigma)$ be the number of positive elements of σ, then

$$\mathrm{rmaj}(\sigma) = \mathrm{maj}_B(\sigma) + \mathrm{p}(\sigma).$$

In [8] Chen, Gao, and Guo generalized the statistics inv, fmaj, and rmaj to signed labeled forests. As mentioned in Section 2.2 in Chapter 4 we generalize Mahonian-Stirling pairs and whenever possible we work with signed labeled forests.

Chapter 3

Statistics on Words

A *directed graph* or a *binary relation* on $X = \{1, \ldots, n\}$ is defined by any subset U of the Cartesian product $X \times X$. For each such directed graph U, Foata and Zeilberger [21] defined the following statistics on each word $w = x_1 \cdots x_m$ over the alphabet X:

$$\begin{aligned}
\mathrm{inv}_U(w) &= \sum_{1 \le i < j \le m} \mathcal{X}((x_i, x_j) \in U), \\
\mathrm{Des}_U(w) &= \{i \colon 1 \le i \le m, (x_i, x_{i+1}) \in U\}, \\
\mathrm{des}_U(w) &= |\,\mathrm{Des}_U(w)|, \\
\mathrm{maj}_U(w) &= \sum_{i \in \mathrm{Des}_U(w)} i.
\end{aligned}$$

An *ordered bipartition* of X is a sequence $(B_1, \ldots, B_k)$ of nonempty disjoint subsets of X such that $B_1 \cup \cdots \cup B_k = X$, together with a sequence $(\beta_1, \ldots, \beta_k)$ of elements equal to 0 or 1. If $\beta_i = 0$ we say the subset B_i is *non-underlined*, and if $\beta_i = 1$ we say the subset B_i is *underlined*.

A relation U on $X \times X$ is said to be *bipartitional*, if there exists an ordered bipartition $((B_1, \ldots, B_k), (\beta_1, \ldots, \beta_k))$ such that $(x, y) \in U$ if and only if either $x \in B_i$, $y \in B_j$ and $i < j$, or x and y belong to the same underlined block B_i. Bipartitional relations were introduced in [21] as an answer to the question *"When are* inv_U *and* maj_U *equidistributed over all rearrangement classes?"*.

Theorem 3.0.1 ([21])**.** *The statistics* inv_U *and* maj_U *are equidistributed on each rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *is bipartitional.*

In particular, if U is bipartitional with blocks $((B_1, \ldots, B_k), (\beta_1, \ldots, \beta_k))$ then

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{inv}_U(w)} = \sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{maj}_U(w)} = \begin{bmatrix} |\boldsymbol{\alpha}| \\ m_1, \ldots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)} q^{\beta_j \binom{m_j}{2}}. \tag{3.1}$$

Here and later we use the notation

$$|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n,$$
$$m_i = |B_i|,$$
$$\boldsymbol{\alpha}(B) = (\alpha_{i_1}, \ldots, \alpha_{i_l}) \text{ if } B_i = \{i_1, \cdots, i_l\} \text{ with } i_1 < \cdots < i_l.$$

A similar result was proved in [19], where the definition of graphical inversions and major index is modified to allow different behavior of the letters at the end of the word. Hetyei and Krattenthaler [27] showed that the poset of bipartitional relations ordered by inclusion has nice combinatorial properties. Han [26] showed that bipartitional relations U can also be characterized as relations U for which both U and its complement are transitive. In particular, we will use Han's formulation of this characterization as stated in [26].

Theorem 3.0.2 ([26]). *U is bipartional if and only if the following two properties hold:*

(i) $(x, y) \in U, (y, z) \in U \implies (x, z) \in U$

(ii) $(x, y) \in U, (z, y) \notin U \implies (x, z) \in U$.

Here we do two different things. First, we strengthen Foata and Zeilberger's result by showing that the equidistribution of inv_U and maj_U on a *single* rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ implies that U is essentially bipartitional (Theorem 3.1.1). Second, we define a graphical sorting index on words, a statistic which generalizes the sorting index for permutations [36]. We then describe the directed graphs U for which sor_U is equidistributed with inv_U and maj_U on a fixed class $\mathcal{R}(\boldsymbol{\alpha})$ (Theorem 3.1.2).

In the next section we define the terminology we need and state the main results. Then we prove Theorem 3.1.1 and Theorem 3.1.2 in Section 3.2 and Section 4.2, respectively. The results in this Chapter are published in the Electronic Journal of Combinatorics [24].

3.1 Preliminaries and Main Results

It will be convenient to refer to $U \subseteq X \times X$ as a directed graph and a binary relation interchangeably and use language related to both terms. For example, in some places we will use the notation $x \geq_U y$ or $x \to y$ to represent the directed edge $(x, y) \in U$. Also, we will say x is related to y if $(x, y) \in U$ or $(y, x) \in U$.

We will be considering the distribution of inv_U and maj_U over a fixed rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$. Notice that if the multiplicity α_x of $x \in X$ is 1, then the pair (x, x) contributes neither to inv_U nor to maj_U. Therefore, omitting or adding such pairs to U doesn't change these two statistics over $\mathcal{R}(\boldsymbol{\alpha})$. For that purpose, we define U to be *essentially bipartitional relative to* $\boldsymbol{\alpha}$ if there are disjoint sets $I \subseteq X$ and $J \subseteq X$ such that

(1) $\alpha_x = 1$ for all $x \in I \cup J$ and

(2) $(U \setminus \{(x, x) \colon x \in I\}) \cup \{(x, x) \colon x \in J\}$ is bipartitional.

Theorem 3.1.1. *The statistics* inv_U *and* maj_U *are* equidistributed *over* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *is essentially bipartitional relative to* $\boldsymbol{\alpha}$*.*

In view of the comment preceding the theorem, the "if" part of Theorem 3.1.1 follows from Theorem 3.0.1. We prove the "only if" part in Section 3.2.

We define a graphical sorting index that depends on U using the same sorting algorithm but at each step, when sorting x, we only count how many elements y satisfying $(x, y) \in U$ it "jumps over". More formally, to compute $\text{sor}_U(w)$ for $w = x_1 x_2 \ldots x_m$:

- Begin with $i = m$, and $\text{sor}_U(w) = 0$.
- Consider the largest element in the first i letters of w with respect to the integer order. If there are multiple copies of the largest element, let x_j, $j \leq i$ be the rightmost one.
- For each $h = j + 1, j + 2, \ldots, i$, if $(x_j, x_h) \in U$ increase $\text{sor}_U(w)$ by 1.
- Interchange x_j with x_i and keep using the notation $w = x_1 x_2 \cdots x_m$.
- Repeat this process for $i = m - 1, \ldots, 1$.

For example, consider the sorting index of the word $w = 143123123$ under the relation $U = \{(4, 3), (3, 3), (3, 1), (2, 3), (1, 1)\}$. The sorting steps are the same as given in (2.1) and thus

$\text{sor}_U(w) = 3+1+2+2+0 = 8$. In particular, if U is the natural integer order $U = \{(x,y)\colon x > y\}$, then $\text{sor}(w) = \text{sor}_U(w)$. Our second main result follows.

Theorem 3.1.2. *The statistics* sor_U*,* inv_U *and* maj_U *are equidistributed on a fixed rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *has the following properties.*

1. *U is bipartitional with no underlined blocks.*

2. *If $(x,y) \in U$ then $x > y$.*

3. *All but the last block of U have size at most 2.*

4. *If U has blocks $B_1, \ldots, B_k$ and $|B_i| = 2$ for some $1 \le i \le k-1$, then $\alpha_{\max B_i} = 1$.*

We give the proof of Theorem 3.1.2 in Section 4.2.

3.2 The Proof of Theorem 3.1.1

The proof of Theorem 3.1.1 is based on a series of seven lemmas that we prove next. The first two describe how the distribution of maj_U and inv_U over $\mathcal{R}(\boldsymbol{\alpha})$ are related for general U. This will lead us to define special words in $\mathcal{R}(\boldsymbol{\alpha})$ which we call *maximal chain words*. Then we will show that when inv_U and maj_U are equidistributed over $\mathcal{R}(\boldsymbol{\alpha})$, the chains that are the building blocks of the same maximal chain word are nicely related to each other. We use this to show that U and U^c have to satisfy the properties of Theorem 3.0.2 modulo some relations (x,x) with $\alpha_x = 1$.

We begin with a simple but very useful observation.

Lemma 3.2.1. *The statistics* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if* maj_{U^c} *and* inv_{U^c} *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*.*

Proof. This follows from the fact that for every $w \in \mathcal{R}(\boldsymbol{\alpha})$,

$$\text{maj}_U(w) + \text{maj}_{U^c}(w) = \binom{|\boldsymbol{\alpha}|}{2} = \text{inv}_U(w) + \text{inv}_{U^c}(w).$$

□

Lemma 3.2.2. *For any* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *and any relation* U *on* $X = \{1, \ldots, n\}$*,*

$$\max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \text{maj}_U w \ge \max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \text{inv}_U w.$$

Proof. We will use induction on $|\boldsymbol{\alpha}|$. It is clear that the statement holds when $|\boldsymbol{\alpha}| = 1$. Assume that it holds for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq m$.

Consider a rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ such that $|\boldsymbol{\alpha}| = m+1$, and a relation U on $\{1, \ldots, n\}$. Let $(\boldsymbol{\alpha}, U)$ be a directed graph with vertex set $\{1^{\alpha_1}, \ldots, n^{\alpha_n}\}$ and a directed edge $x \to y$ whenever $(x, y) \in U$. Let $x_1 \to x_2 \to \cdots \to x_n$ be a directed path in $(\boldsymbol{\alpha}, U)$ of maximal possible length. This means we have a descending chain $x_1 \geq_U x_2 \geq_U \cdots \geq_U x_l$ of maximal possible length that uses at most α_i copies of each $i \in \{1, \ldots, n\}$. Set $\boldsymbol{\alpha}' = (\alpha_1', \ldots, \alpha_n')$ where

$$\alpha_i' = \alpha_i - \sum_{j=1}^{l} \mathcal{X}(x_j = i).$$

Let u' be a word that maximizes maj_U on the rearrangement class $\mathcal{R}(\boldsymbol{\alpha}')$. One can easily verify that for the word $u = u'x_1x_2\cdots x_l$ in $\mathcal{R}(\boldsymbol{\alpha})$ we have

$$\mathrm{maj}_U\, u = \mathrm{maj}_U\, u' + \frac{(l-1)(2m+2-l)}{2}. \tag{3.2}$$

To bound $\max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{inv}_U\, w$, first suppose there is an element $y \in (\boldsymbol{\alpha}', U)$ such that for all $i = 1, \ldots, l$ we have $(y, x_i) \in U$ or $(x_i, y) \in U$. If $(y, x_1) \in U$, then $y \to x_1 \to x_2 \to \cdots \to x_l$ is a longer directed path in $(\boldsymbol{\alpha}, U)$, therefore $(y, x_1) \notin U$ and $(x_1, y) \in U$. Similarly, if $(x_l, y) \in U$, we can form the longer directed path $x_1 \to x_2 \to \cdots \to x_l \to y$ in $(\boldsymbol{\alpha}, U)$; thus we must have $(x_l, y) \notin U$ and $(y, x_l) \in U$. However, this implies that there are elements x_i and x_{i+1} such that $(x_i, y), (y, x_{i+1}) \in U$, which yields a longer directed path $x_1 \to x_2 \to \cdots \to x_i \to y \to x_{i+1} \to \cdots \to x_l$. Therefore, every $y \in (\boldsymbol{\alpha}', U)$ is related to at most $l - 1$ elements in the chain $x_1 \to \cdots \to x_l$.

Now consider a word $v \in \mathcal{R}(\boldsymbol{\alpha})$ and the corresponding word $v' \in \mathcal{R}(\boldsymbol{\alpha}')$ obtained by deleting $x_1, \ldots, x_l$. By the argument in the previous paragraph, the $m + 1 - l$ letters in v' create at most $(m+1-l)(l-1)$ graphical inversions with $x_1, \ldots, x_l$. Therefore, by (3.2) and the induction hypothesis,

$$\max_{w\in\mathcal{R}(\boldsymbol{\alpha})} \operatorname{inv}_U w \le \max_{w'\in\mathcal{R}(\boldsymbol{\alpha}')} \operatorname{inv}_U w + (m+1-l)(l-1) + \binom{l}{2} \tag{3.3}$$

$$= \max_{w'\in\mathcal{R}(\boldsymbol{\alpha}')} \operatorname{inv}_U w + \frac{(l-1)(2m+2-l)}{2} \tag{3.4}$$

$$\le \max_{w'\in\mathcal{R}(\boldsymbol{\alpha}')} \operatorname{maj}_U w + \frac{(l-1)(2m+2-l)}{2} \tag{3.5}$$

$$\le \max_{w\in\mathcal{R}(\boldsymbol{\alpha})} \operatorname{maj}_U w. \tag{3.6}$$

□

The proof of Lemma 3.2.2 also shows that a word $w = w_k w_{k-1} \cdots w_1$ with the property $\operatorname{maj}_U(w) \ge \max_{v\in\mathcal{R}(\boldsymbol{\alpha})} \operatorname{inv}_U v$ can be constructed by greedily "peeling off" directed paths (i.e. descending chains) of maximal length from $(\boldsymbol{\alpha}, U)$ and ordering them from right to left, forming the subwords $w_1, \ldots, w_k$ in that order. These kind of words will be used in the proofs that follow and when the relation U is understood; we will call them *maximal chain words* in $\mathcal{R}(\boldsymbol{\alpha})$.

Moreover, if inv_U and maj_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$, equalities hold in (3.3), (3.5), and (3.6). Exploiting this, one can derive conclusions of how the elements from different chains in the maximal chain words are related to each other if maj_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. We list the properties that will be important later in a series of three lemmas.

Lemma 3.2.3. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*. Let* $w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ *be a maximal chain formed from the maximal chains* $w_1, w_2, \ldots, w_k$*. Then:*

(i) For each of the maximal descending chains $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$,

$$(x_r, x_s) \in U \ \text{ or } \ (x_s, x_r) \in U \ \text{ for all } \ i_{j-1}+1 \le r < s \le i_j. \tag{3.7}$$

(ii) Each letter y *in a maximal descending chain* w_i*,* $i > j$*, is related to exactly* $|w_j| - 1$ *elements from* $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$*, i.e., there is a unique* $r \in \{i_{j-1}+1, \ldots, i_j\}$ *such that* $(y, x_r) \notin U$ *and* $(x_r, y) \notin U$*. Moreover,* $(x_s, y) \in U$ *for* $i_{j-1}+1 \le s < r$ *and* $(y, x_s) \in U$ *for* $r < s \le i_j$*.*

Proof. Condition (i) is necessary for equality to hold in (3.3). The property (ii) also follows from the fact that equality holds in (3.3) and the definition of a maximal chain word which implies that the chain w_j is the longest one that can be formed among the letters in $w_k w_{k-1} \cdots w_j$. □

The following lemma shows that if maj_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$, the elements in the maximal chains can be reordered, if necessary, so that within each of them the following property holds: if x precedes y in the same chain of a maximal chain word, then $(x, y) \in U$.

Lemma 3.2.4. *If* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*, then there exists a maximal chain word* $w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ *with subwords* w_i *formed from descending chains such that for any* $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$ *we have*

$$(x_r, x_s) \in U \quad \text{for all} \quad i_{j-1} + 1 \le r < s \le i_j. \tag{3.8}$$

Proof. Since the equality in (3.3) holds, the elements $x_1, \ldots, x_l$ in the maximal chain can be arranged so that they form $\binom{l}{2}$ graphical inversions, which implies the lemma. □

Lemma 3.2.5. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*. Let* $w = w_k w_{k-1} \cdots w_1$ *be a maximal chain word in* $\mathcal{R}(\boldsymbol{\alpha})$ *for* U *with maximal chains* $w_1, \ldots, w_k$*. If* $(x, y) \in U$ *and* $(y, x) \in U$ *for some* $x \neq y$*, then the* α_x *copies of* x *and the* α_y *copies of* y *are all in the same chain* w_i*.*

Proof. Without loss of generality, suppose x that appears in chain w_{j_1} and y appears in chain w_{j_2}, $j_1 > j_2$. Consider the chain $w_{j_2} : b_1 \ge_U b_2 \ge_U \cdots \ge_U b_{l-1} \ge_U y \ge_U b_{l+1} \ge_U \cdots \ge_U b_m$. By Lemma 3.2.3, there is exactly one $i \in \{1, \ldots, m\}$ such that $(x, b_i), (b_i, x) \notin U$, $(b_1, x), \ldots, (b_{i-1}, x) \in U$, $(x, b_{i+1}), and \ldots, (x, b_m) \in U$. If $l < i$, then the chain $b_1 \ge_U \cdots > b_{l-1} \ge_U x \ge_U y \ge_U b_{l+1} \ge_U \cdots \ge_U b_m$ is longer than w_{j_2} and if $l > i$, then $b_1 \ge_U \cdots \ge_U b_{l-1} \ge_U y \ge_U x \ge_U b_{l+1} \ge_U \cdots \ge_U b_m$ is longer than w_{j_2}. This contradicts the definition of a maximal chain word. □

The remaining two lemmas are devoted to proving that the relations U and U^c are transitive modulo some relations (x, x) with $\alpha_x = 1$.

Lemma 3.2.6. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*. If* $(x, y), (y, x) \in U$ *and* $\alpha_x > 1$ *then* $(x, x) \in U$*.*

Proof. Since $(x, y), (y, x) \in U$, by Lemma 3.2.5, all the copies of x and y must be in the same maximal chain of a maximal chain word. In particular, since two x's are in the same chain, part (i) of Lemma 3.2.3 implies that $(x, x) \in U$. □

Lemma 3.2.7. *Suppose* maj_U *and* inv_U *are equidistributed over* $\mathcal{R}(\boldsymbol{\alpha})$ *and let* x *and* y *be distinct elements of* X *such that* $(x, y), (y, x) \in U$*. For every* $z \in \{1^{\alpha_1}, \ldots, n^{\alpha_n}\} \setminus \{x, y\}$*, we have*

$$(z, x) \in U \text{ if and only if } (z, y) \in U$$

and

$$(x, z) \in U \text{ if and only if } (y, z) \in U.$$

Proof. If $z = x$ then $\alpha_x > 1$ and the claim follows from Lemma 3.2.6. The same is true if $z = y$. So, suppose $z \neq x$, $z \neq y$. By symmetry, it suffices to prove

$$(z, x) \in U \implies (z, y) \in U \tag{3.9}$$

$$(x, z) \in U \implies (y, z) \in U. \tag{3.10}$$

To see (3.9), suppose that $(z, x) \in U$, but $(z, y) \notin U$. We consider two cases.

<u>Case 1:</u> $(y, z) \notin U$. Let $w = w_t w_{t-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word that satisfies (3.8). By Lemma 3.2.5, x and y are in the same chain w_i of w. By Lemma 3.2.3, z is in a different chain w_j and by Lemma 3.2.5, $(x, z) \notin U$. If $j > i$, notice that by Lemma 3.2.3, x cannot precede y in w_i, so w_i must be of the form $w_i = b_1 \cdots b_k y b_{k+1} \cdots b_l x b_{l+1} \cdots b_m$. Then $b_1 \cdots b_k z b_{k+1} \cdots b_l x y b_{l+1} \cdots b_m$ is a descending chain longer than w_i. If $j < i$, then $w_j = b_1 \cdots b_k z b_{k+1} \cdots b_l$. By Part (ii) of Lemma 3.2.3, $(b_k, x), (y, b_{k+1}) \in U$, which implies that $b_1 \cdots b_k x y b_{k+1} \cdots b_l$ is a descending chain longer than w_j.

<u>Case 2:</u> $(y, z) \in U$. By Lemma 3.2.1, maj_{U^c} and inv_{U^c} are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. Let $w = w_t w_{t-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word for U^c that satisfies (3.8). Suppose x, y, z are in the chains w_i, w_j, w_k, respectively. By Lemma 3.2.3, $i \neq j$ and $i \neq k$. If $i < j, k$ and $w_i = b_1 \cdots b_l x b_{l+1} \cdots b_m$, then a different maximal chain word w' could be constructed by taking the same chains $w_1, \ldots, w_{i-1}$ as in w and replacing w_i by $b_1 \cdots b_l y b_{l+1} \cdots b_m$. Since $(z, y) \in U^c$, it follows from Lemma 3.2.3 that z is not in relation U^c with some b_r, meaning there is some b_r such that $(z, b_r) \notin U^c$, $r \leq l$ and therefore $(z, x) \in U^c$, which contradicts $(z, x) \in U$. A similar argument holds if $j < i, k$. If $k < i, j$ and $w_k = b_1 \cdots b_l z b_{l+1} \cdots b_m$ then y is not in relation U^c with some b_r, $r > l$, and a different maximal chain word for U^c could be formed by replacing w_k

with $b_1 \cdots b_l z b_{l+1} \cdots b_{r-1} y b_{r+1} \cdots b_m$. Part (ii) of Lemma 3.2.3 now implies that $(z, x) \in U^c$, which contradicts $(z, x) \in U$. Finally, if $j = k < i$, then Lemma 3.2.3 implies that $(x, z) \in U^c$ and y precedes z in w_j since (z, x), (x, y), $(y, x) \notin U^c$. Therefore, $(y, z) \in U^c$, which contradicts $(y, z) \notin U$.

The implication (3.10) can be proven by considering completely analogous cases, so we omit it here. □

Proof of Theorem 3.1.1 . Assume inv_U and maj_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. Define the symmetric part of U to be

$$S(U) = \{(x, y) \in X \times X : (x, y), (y, x) \in U \text{ for some } y \neq x\}$$

and let

$$X_U = \{x \in X : (x, y) \in S(U) \text{ for some } y \in X\}.$$

Let

$$U' = (U \cup \{(x, x) \colon x \in X_U, \alpha_x = 1\}) \setminus \{(x, x) \colon x \notin X_U, \alpha_x = 1\}.$$

We will show that U' is bipartitional using the characterization given by Theorem 3.0.2, which will imply that U is essentially bipartitional relative to $\boldsymbol{\alpha}$.

To show that U' is transitive, suppose $(x, y), (y, z) \in U'$.

First consider the case when x, y, z are all different. If $(y, x) \in U$ or $(z, y) \in U$, then $(x, z) \in U$ by Lemma 3.2.7. Hence $(x, z) \in U'$. If $(y, x), (z, y), (x, z) \notin U$ then let $w \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word for U^c. If x, y, z all appear in the same chain w_i, by Lemma 3.2.4, the elements in w_i can be reordered to give a sequence $z_1, \ldots, z_l$ such that $(z_r, z_s) \notin U$ for all $1 \leq r < s \leq l$. This is possible only if z precedes x and $(z, x) \notin U$. Applying Lemma 3.2.7 to U^c, we get that $(x, y), (y, z) \notin U$ which contradicts the starting assumption. If not all x, y, z appear in the same maximal chain w_i, assume without loss of generality that x is the one that appears in the rightmost chain of the maximal chain word $w \in \mathcal{R}(\boldsymbol{\alpha})$ for U^c. Suppose y does not appear in w_i. Let t_y be the unique letter in w_i (guaranteed by Lemma 3.2.3) not related to y in U^c. Then another maximal chain word can be constructed in which t_y in the maximal chain w_i is replaced by y. Repeating this argument, we see that one can construct a maximal chain word for U^c in which x, y, z are all in the same maximal chain, which we saw is impossible.

If not all x, y, z are different, one only needs to consider the case $x = z \neq y$. Then $x \in X_U$.

If $\alpha_x = 1$, then $(x,x) \in U'$ by definition. Otherwise, $\alpha_x > 1$ and $(x,x) \in U$ by Lemma 3.2.6.

To show that U' has the second property from Theorem 3.0.2, assume that $(x,y) \in U'$ and $(z,y),(x,z) \notin U'$. If all x,y,z are different, then by the previous argument applied to U^c, we get $(x,y) \notin U$, which contradicts the assumption $(x,y) \in U$. The only case left to be considered is $x = y \neq z$. Then $(x,x) \in U'$ and $(x,z),(z,x) \notin U$. If $\alpha_x > 1$, then Lemma 3.2.6 applied to U^c yields $(x,x) \notin U$, which contradicts $(x,x) \in U'$. If $\alpha_x = 1$, then by the definition of U', $x \in X_U$. This means that $(x,w),(w,x) \in U$ for some $w \neq x$. Then x,z,w, are all different and by the preceding argument we get that $(w,x) \in U$ and $(z,x) \notin U$, which implies $(w,z) \in U$. But then Lemma 3.2.7 applied to U yields $(x,z) \in U$, a contradiction. □

3.3 Graphical Sorting Index

In this section we will prove Theorem 3.1.2. The "if" part follows from the following proposition and (3.1), while the "only if" part follows from Lemma 3.3.3 and Lemma 3.3.5.

Assume U satisfies the properties of Theorem 3.1.2 and has blocks $B_1,\ldots,B_k$. To each word $w \in \mathcal{R}(\boldsymbol{\alpha})$, we associate a pair of two sequences: a sequence of partitions and a sequence of nonnegative integers. This map is a generalization of the B-code defined for permutations [8, 37]. Precisely, we define a map $\phi : \mathcal{R}(\boldsymbol{\alpha}) \longrightarrow A$, where A is a set of pairs

$$((b_{1,1} \geq \ldots \geq b_{1,m_1}; b_{2,1} \geq \cdots \geq b_{2,m_2}; \ldots; b_{k,1} \geq \ldots \geq b_{k,m_k}),(p_1,\ldots,p_k))$$

satisfying

(1°) for $i < k$ in each partition $b_{i,1} \geq \ldots \geq b_{i,m_i} \geq 0$ each part has size $b_{i,j} \leq m_{i+1}+m_{i+2}+\cdots+m_k$, $1 \leq j \leq m_i$, while $b_{k,j} = 0$ for $1 \leq j \leq m_k$,

(2°) $p_i = 0$ if $|B_i| = 1$ and $1 \leq p_i \leq m_i$ if $|B_i| = 2$.

For $w = x_1 \cdots x_l \in \mathcal{R}(\boldsymbol{\alpha})$, $\phi(w)$ is computed as follows.

(1) Set $j = 1$.

(2) If $B_j = \{y_1, y_2\}$ has two integers $y_2 > y_1$ then let $p_j = i$ be the position of y_2 in the subword of w formed by the elements of B_j. Otherwise set $p_j = 0$.

(3) Sort the elements of the block B_j and form the partition $b_{j,1} \geq \ldots \geq b_{j,m_j} \geq 0$ from the contributions to $\operatorname{sor} w$ (listed in nonincreasing order) by the elements of B_j. Keep calling the partially sorted word w.

(4) If $j < k$ increase j by 1 and go to step (2). Otherwise stop.

Consider, for example, the bipartitional binary relation

$$U = \{(5,3),(5,2),(5,1),(4,3),(4,2),(4,1),(3,2),(3,1)\}$$

with blocks $B_1 = \{5,4\}, B_2 = \{3\}, B_3 = \{2,1\}$ and $\beta_1 = \beta_2 = \beta_3 = 0$. Take the word $w = 42345411 \in \mathcal{R}(2,1,1,3,1)$. Since the subword formed by the 4's and the 5 is 4454, we have $p_1 = 3$. The steps for sorting the 4's and the 5 are

$$4234541 \xrightarrow{+1} 42341415 \xrightarrow{+1} 42341145 \xrightarrow{+2} 42311445 \xrightarrow{+4} 12314445$$

and, therefore, the first partition in $\phi(w)$ is $4 \geq 2 \geq 1 \geq 1$. Then $p_2 = 0$ and sorting the 3 yields 12134445, therefore the second partition is 1. Finally, $p_3 = 2$ and

$$\phi(w) = ((4 \geq 2 \geq 1 \geq 1; 1; 0 \geq 0 \geq 0), (3,0,2)).$$

Since the parts of the partitions in the $\phi(w)$ represent contributions to the sorting index, the bound for their size $b_{i,j} \leq m_{i+1} + \cdots + m_k$ easily follows. Therefore, the $\phi(w)$ is clearly a map from $\mathcal{R}(\boldsymbol{\alpha})$ to the set of pairs of sequences of partitions and integers which satisfy (1°) and (2°), which we claim is a bijection. For describing the inverse, the crucial observation is that for blocks of size 2, $B_j = \{y_1 < y_2\}$, the contribution to the sorting index is given by b_{j,p_j}. Then given

$$((b_{1,1} \geq \cdots \geq b_{1,m_1}; b_{2,1} \geq \cdots \geq b_{2,m_2}; \ldots; b_{k,1} \geq \cdots \geq b_{k,m_k}), (p_1, \ldots, p_k))$$

which satisfies (1°) and (2°), the corresponding word $w \in \mathcal{R}(\boldsymbol{\alpha})$ is constructed as follows.

(1) Let $j = k$ and w be the empty word.

(2) Add to the end of w the elements of B_j with their multiplicities, listed in nondecreasing order $x_{j,1} x_{j,2} \cdots x_{j,m_j}$.

(3) If $|B_j| = 1$, then for $i = 1, \ldots, m_j$, swap $x_{j,i}$ with the element of w which is $b_{j,i}$ places to the left of $x_{j,i}$.

(4) If $B_j = \{y_1 < y_2\}$, then let $b'_{j,1} \geq \cdots \geq b'_{j,m_j-1}$ be the partition obtained from $b_{j,1} \geq \ldots \geq b_{j,m_j}$ by deleting the part b_{j,p_j}. Then for $i = 1, \ldots, m_j - 1$, swap $x_{j,i}$ with the element of w which is $b'_{j,i}$ places to the left of $x_{j,i}$. Finally, swap $x_{j,m_j} = y_2$ with the element in w which is $b_{j,p_j} + m_j - p_j$ positions to its left. (After this step there are b_{j,p_j} elements from $B_{j+1}, \ldots, B_k$ and $m_j - p_j$ elements from B_j to the right of y_2.)

(5) If $j > 1$ decrease j by 1 and go to step (2). Otherwise stop.

Proposition 3.3.1. *If U satisfies the properties of Theorem 3.1.2 and has blocks $B_1, \ldots, B_k$ then*

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}_U(w)} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)}.$$

Proof. The $\phi(w)$ is designed so that $\mathrm{sor}_U\, w = \sum_{i=1}^{k} \sum_{j=1}^{m_i} b_{i,j}$. The bijection described above then yields the generating function for sor_U. Let $p(j,k,n)$ denote the number of partitions of n into at most k parts, with largest part at most j. It is known that $\sum_{n \geq 0} p(j,k,n) q^n = \begin{bmatrix} j+k \\ j \end{bmatrix}$. The block B_j contributes

$$\binom{m_j}{\boldsymbol{\alpha}(B_j)} \sum_{n \geq 0} p(m_{j+1} + \cdots + m_n, m_j, n) q^n = \binom{m_j}{\boldsymbol{\alpha}(B_j)} \begin{bmatrix} m_j + m_{j+1} \cdots + m_n \\ m_j \end{bmatrix}$$

to $\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}_U(w)}$, where the leading binomial coefficient counts the number of possible values of p_j. Thus we have

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}_U(w)} = \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)} \begin{bmatrix} m_j + m_{j+1} \cdots + m_n \\ m_j \end{bmatrix} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)}.$$

□

In particular, we get the generating function for the standard sorting index for words.

Corollary 3.3.2.

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}(w)} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix}.$$

Finally, we prove the "only if" part of Theorem 3.1.2 via the following few lemmas.

Lemma 3.3.3. *If* sor_U*,* maj_U*, and* inv_U *are equidistributed over a fixed rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *then the relation* U *must be a subset of the integer order modulo relations* (x, x)*.*

Proof. Suppose the statistics sor_U, maj_U, and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. By Theorem 3.1.1, U must be essentially bipartitional relative to α. That means that there are subsets $I, J \subset \{x \colon \alpha_{x=1}\}$ such that $U' = (U \setminus \{(x,x) \colon x \in I\}) \cup \{(x,x) \colon x \in J\}$ is bipartitional. Without loss of generality we may assume that I, J are chosen so that U' does not have underlined blocks $\{x\}$ of size 1 such that $\alpha_x = 1$. We claim that U' is a subset of the natural order.

First we will show that there are no underlined blocks in U'. Suppose the contrary. Then there exist elements x and y such that $(x,y),(y,x) \in U'$ ($x \neq y$, or y is a second copy of the same element with $\alpha_x > 1$). Because we have both (x,y) and (y,x) in U', every word $w \in \mathcal{R}(\boldsymbol{\alpha})$ has at least one U'-inversion. Therefore the minimum inv_U over the rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ is 1. On the other hand, $\mathrm{sor}_U\, 11 \cdots 122 \cdots 2 \cdots nn \cdots n = 0$. This is a contradiction, and thus there are no underlined blocks in U'.

Now assume that U' is not a subset of the natural integer order. Then there exist at least two elements such that $(x,y) \in U'$, but $y > x$ with respect to the natural order. Let $B_1, B_2, \ldots, B_k$ be the blocks of U'. Now consider the words created by placing the elements of B_1 in some order followed by the elements of B_2 placed to the right of B_1 and continue the process until the elements of B_k in some order are the last elements of the word. The words of this type will have inv_U equal to the number of edges in the graph $(\boldsymbol{\alpha}, U')$ as defined in the proof of Lemma 3.2.2. Therefore, the maximum inv_U is bounded below by the number of edges in $(\boldsymbol{\alpha}, U')$ (it is in fact equal to the number of edges in $(\boldsymbol{\alpha}, U')$). In the sorting algorithm, however, elements are only sorted over elements that are smaller than them with respect to the natural order. Therefore x will never jump over y, and thus the relation (x,y) will never contribute to the sorting index. Since each edge of the graph $(\boldsymbol{\alpha}, U')$ contributes at most 1 to sor_U, we conclude that the maximum of sor_U on $\mathcal{R}(\boldsymbol{\alpha})$ is less than the maximum of inv_U. This is a contradiction, and so U' must be a subset of the natural order. □

The next inequality will be used to prove the remaining part of Theorem 3.1.2.

Lemma 3.3.4. *For* $a, b \in \mathbb{Z}_{\geq 1}$*,*

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} \leq \binom{a+b}{b}$$

and equality holds if and only if $b = 1$.

Proof. If $a \leq b$, then using the Vandermonde identity we have

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} = \sum_{i=0}^{a} \binom{a}{i} \leq \sum_{i=0}^{a} \binom{a}{i}\binom{b}{a-i} = \binom{a+b}{b}$$

and equality holds if and only if $a = b = 1$. Similarly, if $a > b$ then

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} = \sum_{i=0}^{b} \binom{a}{i} \leq \sum_{i=0}^{b} \binom{a}{i}\binom{b}{b-i} = \binom{a+b}{b}.$$

□

Lemma 3.3.5. *Suppose* U *is a bipartitional relation with blocks* $B_1, \ldots, B_k$*, none of which are underlined, such that* sor_U*,* maj_U*, and* inv_U *are equidistributed over* $\mathcal{R}(\boldsymbol{\alpha})$*. Then for every* $1 \leq i < k$*,* $|B_i| \leq 2$ *and if the equality* $|B_i| = 2$ *holds then* $\alpha_{\max B_i} = 1$*.*

Proof. By Lemma 3.3.3, the blocks $B_1, \ldots, B_k$ are consecutive intervals with $n \in B_1$ and $1 \in B_k$. If $k = 1$ there is nothing to prove, so suppose $k > 1$.

Let $\mathrm{i}(B_1, \ldots B_k)$ and $\mathrm{s}(B_1, \ldots, B_k)$ denote the number of words in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize inv_U and sor_U, respectively. Let $B_1 = \{s, s+1, \ldots, n\}$, $s \leq n-1$. The words in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize inv_U are exactly those formed by a permutation of the elements of B_1 (with their multiplicities) followed by a permutation of the elements from B_2, etc. So, $\mathrm{i}(B_1, \ldots, B_k) = \prod_{i=1}^{k} \binom{m_i}{\boldsymbol{\alpha}(B_i)}$.

On the other hand, if $w \in \mathcal{R}(\boldsymbol{\alpha})$ maximizes sor_U then after sorting the n's, one obtains a word $w' \in \mathcal{R}(\boldsymbol{\alpha}')$ that maximizes sor_U for $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1})$. The map $w \to w'$ is not one-to-one. One can write $w' = uv$ where u is the longest prefix of w' formed by elements of B_1. Then the number of words w that yield w' is at most $\sum_{i=0}^{\min\{|u|,\alpha_n\}} \binom{|u|}{i}$. Namely, such a w can be obtained by appending the α_n copies of n to w' and then swapping the leftmost i copies of n with i letters from u and the remaining $\alpha_n - i$ copies of n with the first $\alpha_n - i$ letters of v.

Since, by Lemma 3.3.4,

$$\sum_{i=0}^{\min\{|u|,\alpha_n\}} \binom{|u|}{i} \leq \binom{|u| + \alpha_n}{\alpha_n} \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n}$$

with equality when $\alpha_n = 1$, we have

$$\mathrm{s}(B_1, \ldots, B_k) \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n} \mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k),$$

where $\mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k)$ is the number of words in $\mathcal{R}(\boldsymbol{\alpha}')$ that maximize sor_U. Inductively, we get

$$\mathrm{s}(B_1, \ldots, B_k) \leq \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_s, \ldots, \alpha_{n-1}, \alpha_n} \mathrm{s}(B_2, \ldots, B_k) \leq \prod_{i=1}^{k} \binom{m_i}{\boldsymbol{\alpha}(B_i)} = \mathrm{i}(B_1, \ldots, B_k).$$

Since we have equalities everywhere, $\alpha_n = 1$. We also get that $\mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k) = \mathrm{i}(B_1 \setminus \{n\}, \ldots, B_k)$ and by the same argument, $\alpha_n = \alpha_{n-1} = \cdots = \alpha_{s+1} = 1$.

Now consider a permutation p of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, (s-1)^{\alpha_{s-1}}\}$ which maximizes sor_U. By appending α_s copies of s to p and then swapping them with the first α_s letters of p we get the word

$$\underbrace{ss\cdots s}_{\alpha_s} p'.$$

One can readily see that the word

$$w' = (n-1)\underbrace{ss\cdots s}_{\alpha_s - 1} p' s(s+1)(s+2)\cdots(n-2) \in \mathcal{R}(\boldsymbol{\alpha}')$$

maximizes sor_U over $\mathcal{R}(\boldsymbol{\alpha}')$. Also, there are exactly $\alpha_s + 1$ words w in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize sor_U which can be obtained from w', namely,

$$\begin{aligned}
&n\underbrace{ss\cdots s}_{\alpha_s-1} p' s(s+1)(s+2)\cdots(n-2)(n-1),\\
&(n-1)n\underbrace{ss\cdots s}_{\alpha_s-2} p' s(s+1)(s+2)\cdots(n-2)s,\\
&(n-1)sn\underbrace{ss\cdots s}_{\alpha_s-3} p' s(s+1)(s+2)\cdots(n-2)s,\\
&\vdots\\
&(n-1)\underbrace{ss\cdots s}_{\alpha_s-2} np' s(s+1)(s+2)\cdots(n-2)s,\\
&(n-1)\underbrace{ss\cdots s}_{\alpha_s-1} np'' s(s+1)(s+2)\cdots(n-2)a,
\end{aligned}$$

where a is the first letter of p'. However, as we saw above, if sor_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$, each word w' corresponds to exactly $\binom{\alpha_n+\alpha_{n-1}+\cdots+\alpha_s}{\alpha_n}$ words w. So,

$$\binom{\alpha_n+\alpha_{n-1}+\cdots+\alpha_s}{\alpha_n} = \alpha_s + 1$$

and therefore $s = n-1$.

This proves that either $B_1 = \{n-1, n\}$ with $\alpha_n = 1$ or $B_1 = \{n\}$. Since the block is of this form, reasoning as in the proof of Proposition 3.3.1 one can see that

$$\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}_U(w)} = \binom{m_1}{\boldsymbol{\alpha}(B_1)} \begin{bmatrix} m_1+\cdots+m_n \\ m_j \end{bmatrix} \sum_{w\in\mathcal{R}(\boldsymbol{\alpha}'')} q^{\mathrm{sor}_U(w)},$$

where $\mathcal{R}(\boldsymbol{\alpha}'')$ is the set of all permutations of the elements of $B_2, \ldots, B_k$ with the multiplicities given by $\boldsymbol{\alpha}$. Since

$$\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{inv}_U(w)} = \binom{m_1}{\boldsymbol{\alpha}(B_1)} \begin{bmatrix} m_1+\cdots+m_n \\ m_j \end{bmatrix} \sum_{w\in\mathcal{R}(\boldsymbol{\alpha}'')} q^{\mathrm{inv}_U(w)},$$

we conclude that sor_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha}'')$ and inductively, we get that each of the remaining blocks $B_2, \ldots, B_{k-1}$ has either size 1 or size 2 with the multiplicity of the largest element being 1. □

This completes the proof of Theorem 3.1.2.

Chapter 4

Mahonian Stirling Pairs for Labeled Forests

As we saw in the Introduction, the Mahonian-Stirling paris $(\mathrm{maj}, \mathrm{RL\text{-}min})$, $(\mathrm{inv}, \mathrm{RL\text{-}min})$, and $(\mathrm{sor}, \mathrm{Cyc})$ are equidistributed over S_n. In this chapter, we generalize these results over labeled forests. We begin with some notation and background.

A forest is an acyclic graph, and throughout this dissertation we will be considering planer rooted forests. Let F be a plane forest with vertex set $V(F) = \{v_1, \ldots, v_n\}$. We will draw F with the roots on top and think of it as a Hasse diagram of the poset $(V(F), <_F)$ with the root as the maximal element. Throughout this chapter, we assume that the vertices of F are naturally indexed. That is, if $v_i <_F v_j$, then $i < j$.

A labeling w of F is a bijection

$$w : V(F) \to \{1, \ldots, n\}.$$

Let $\mathcal{W}(F)$ be the set of all labelings of a forest F. For each vertex $v \in V(F)$ the *hook length of* x, denoted by h_v, is the number of vertices of the subtree of F rooted at v. In other words, h_v is the size of the principal order ideal generated by v.

Mallows and Riordan [11] generalized inversions to labeled forests in the following way:

$$\mathrm{inv}(F, w) = \#\{(u, v) : u <_F v, w(u) > w(v)\}.$$

If the forest F is a linear tree, this is simply the inversion index of the corresponding permutation obtained by reading the labels of F from bottom to top. Mallows and Riordan [11] studied the inversions of unordered labeled trees, but we will be discussing the distribution of inversions over all labelings of a fixed ordered forest F.

Björner and Wachs [4] extended the major index to labeled forests. Namely, they defined the descent set of a labeled forest as

$$\mathrm{Des}(F, w) = \{v \in V(F) : w(v) > w(u), u \text{ is the parent of } v\},$$

the major index as

$$\mathrm{maj}(F, w) = \sum_{v \in \mathrm{Des}(F,w)} h_v,$$

and they showed that the major index has the same distribution as the inversion index on labeled forests of fixed shape (see [32] for a bijective proof):

$$\sum_{w \in \mathcal{W}(F)} q^{\mathrm{maj}(F,w)} = \sum_{w \in \mathcal{W}(F)} q^{\mathrm{inv}(F,w)} = \frac{n!}{\prod_{v \in V(F)} h_v} \prod_{v \in V(F)} [h_v].$$

(a) A tree F with naturally indexed vertices (b) A signed labeling of F

Figure 4.1: A forest F with a signed labeling

A signed labeling of the forest F of size n is a one-to-one map

$$w : V(F) \to \{\pm 1, \ldots, \pm n\}$$

such that if $i \in w(V(F))$ then $-i \notin w(V(F))$. We denote $-i$ with $\bar{i}$, see Figure 4.1 for an example. The set of all signed labelings of F will be denoted by $\mathcal{W}_B(F)$. Chen et al. [8] extended the notion of inversions and major index to signed labeled forests, the latter one in two different ways. The inversion number inv_B for signed labelings is motivated by the length function for signed permutations, while the major indices fmaj and rmaj are based on the major indices of signed permutations introduced by Adin and Roichman [1] and Reiner [39], respectively, which we defined in Section 2.4. The authors in [8] showed that

$$\sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{fmaj}(F,w)} = \sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{rmaj}(F,w)} = \sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{inv}_B(F,w)} = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} [2h_v].$$

Precise definitions of the statistics BT-max, CBT-max (cyclic bottom-to-top maximum positions), sor (sorting index), and Cyc (minimal elements in cycles) will be given below. Our main result is that these three pairs of statistics are equidistributed over all (signed) labelings of a forest F. We give a bijective proof of this fact by mapping the labelings to certain integer sequences in three different ways. This also gives us an explicit formula for the generating function of each of the three pairs. Explicitly, we prove that

$$\begin{aligned}\sum_{w\in\mathcal{W}(F)} q^{\mathrm{inv}(F,w)} \prod_{v\in\text{BT-max}(F,w)} t_v = \sum_{w\in\mathcal{W}(F)} q^{\mathrm{sor}(F,w)} \prod_{v\in\mathrm{Cyc}(F,w)} t_v = \sum_{w\in\mathcal{W}(F)} q^{\mathrm{maj}(F,w)} \prod_{v\in\text{CBT-max}(F,w)} t_v \\ = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([h_v] - 1 + t_v\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{inv}_B(F,w)} \prod_{v\in\text{BT-max}_B(F,w)} t_v = \sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{sor}_\mathrm{B}(F,w)} \prod_{v\in\mathrm{Cyc}_B(F,w)} t_v \\ = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([2h_v] - 1 + t_v\right).\end{aligned}$$

When the forest is a linear tree, we show how these statistics specialize to known permutation statistics, and we discuss how our results are a generalization of some results for (signed) permutations.

Sections 4.1, 4.2, and 4.3 deal with each of the pairs (inv, BT-max), (sor, Cyc), and (maj, CBT-max) separately. Whenever possible, we work with signed labeled forests and in our results we keep track of the negative signs in the labeling, so that the results for unsigned labeled forests are a corollary. At places, we also discuss the case of even signed labeled forests, which is related to the case of even signed permutations.

As one can see above the pair (maj, CBT-max) has not been generalized over signed labeled forests. In Section 4.4, we define a new statistic imaj using the sorting algorithm given in Section 4.2

and discuss a conjecture that may better generalize the result for permutations some of the results in this Chapter are published in Advances of Applied Mathematics [23].

4.1 Inversions and Bottom-To-Top Maxima

Recall that a signed labeling of a forest F is a one-to-one map $w : V(F) \to \{\pm 1, \ldots, \pm n\}$ such that if $i \in w(V(F))$ then $-i \notin w(V(F))$. As usual, we denote $-i$ by $\bar{i}$. A labeling is even-signed if the number of negative labels used is even. We will use $\mathcal{W}_B(F)$ and $\mathcal{W}_D(F)$ to denote the set of all signed labelings and the set of all even-signed labelings of a forest F, respectively. The type B and type D analogues of the inversion number of a labeled forest, introduced by Björner and Wachs [4], was proposed by Chen et al. [8]. The definition follows. Let $\mathrm{n}_1(F, w)$ be the number of negative labels in w, and define

$$\mathrm{n}_2(F, w) = \#\{(x, y) : x <_F y, w(x) + w(y) < 0\}.$$

The *inversion number* of a signed labeled forest is given by

$$\mathrm{inv}_B(F, w) = \mathrm{inv}(F, w) + \mathrm{n}_1(F, w) + \mathrm{n}_2(F, w),$$

while for $w \in W_D(F)$, the *type D inversion number* is defined by

$$\mathrm{inv}_D(F, w) = \mathrm{inv}(F, w) + \mathrm{n}_2(F, w).$$

For example, for the even signed labeled forest (F, w) from Figure 4.1, $\mathrm{inv}_B(F, w) = 2 + 2 + 3 = 7$ and $\mathrm{inv}_D(F, w) = 2 + 3 = 5$. Note that if a signed labeling w is in fact in $\mathcal{W}(F)$, then $\mathrm{inv}_B(F, w) = \mathrm{inv}(F, w)$. Chen et al. [8] showed that for a forest F with n vertices,

$$\sum_{w \in \mathcal{W}_B(F)} p^{\mathrm{n}_1(F,w)} q^{\mathrm{inv}_B(F,w)} = \frac{n!}{\prod_{v \in V(F)} h_v} \prod_{v \in V(F)} (1 + pq^{h_v})[h_v].$$

As a corollary, they derived

$$\sum_{w \in \mathcal{W}_B(F)} q^{\mathrm{inv}_B(F,w)} = \frac{n!}{\prod_{v \in V(F)} h_v} \prod_{v \in V(F)} [2h_v]$$

and

$$\sum_{w\in\mathcal{W}_D(F)} q^{\mathrm{inv}_D(F,w)} = \frac{n!}{2\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} (1+q^{h_v-1})[h_v].$$

In this section, we refine these results by looking at the joint distribution of inversions and bottom-to-top maxima.

Definition 4.1.1. Let F be a forest. For $w \in \mathcal{W}(F)$, we define the *bottom-to-top maximum positions* to be

$$\text{BT-max}(F,w) = \{v : w(v) > w(u) \text{ for all } u <_F v\}.$$

For $w \in \mathcal{W}_B(F)$, we define the *signed bottom-to-top maximum positions* to be

$$\text{BT-max}_B(F,w) = \{v : w(v) > 0 \text{ and } w(v) > |w(u)| \text{ for all } u <_F v\}.$$

Finally, for $w \in \mathcal{W}_D(F)$, we define the even *signed bottom-to-top maximum positions* to be

$$\text{BT-max}_D(F,w) = \{v : v \text{ is not a leaf}, w(v) > 0 \text{ and } w(v) > |w(u)| \text{ for all } u <_F v\}.$$

For example, for the signed labeled forest (F,w) from Figure 4.1, $\text{BT-max}_B(F,w) = \{v_1\}$ and $\text{BT-max}_D(F,w) = \emptyset$. In this and following sections we make use of maps between labeled forests and certain sequences that in the case of the symmetric group reduce to inversion tables. We will use SE_F and SE_F^B to denote the type A and type B subexcedent sequences that correspond to a forest F, respectively:

$$\mathrm{SE}_F = \{(a_1,\dots,a_n) : a_i \in \mathbb{Z}, 0 \le a_i \le h_{v_i} - 1\},$$

$$\mathrm{SE}_F^B = \{(a_1,\dots,a_n) : a_i \in \mathbb{Z}, 0 \le a_i \le 2h_{v_i} - 1\}.$$

For a forest F and $w \in \mathcal{W}_B(F)$, we define its A-code(F,w) to be the sequence $(a_1,\dots,a_n) \in \mathrm{SE}_F^B$ given by

$$a_i = \#\{u : u <_F v_i \text{ and } w(u) > w(v_i)\} + \#\{u : u <_F v_i \text{ and } w(u) + w(v_i) < 0\} + \chi(w(v_i) < 0),$$

where χ is the truth indicator function. For example, for the tree F with labeling w in Figure 4.1, A-code$(F, w) = (0, 1, 2, 1, 3)$.

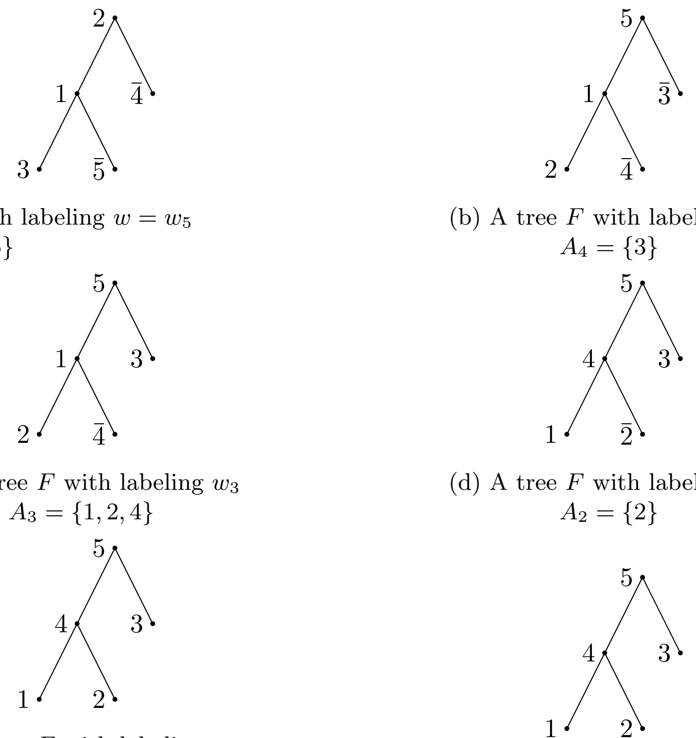
Lemma 4.1.2. *Let* $w \in \mathcal{W}_B(F)$ *and suppose* A-code$(F, w) = (a_1, a_2, \ldots, a_n)$. *Then*

1. $\mathrm{inv}_B(F, w) = \sum_{i=1}^{n} a_i$
2. $\text{BT-max}_B(F, w) = \{v_i : a_i = 0\}$
3. $w(v_i) < 0$ *if and only if* $h_{v_i} \le a_i \le 2h_{v_i} - 1$ *and therefore* $\mathrm{n}_1(F, w) = \#\{i : h_{v_i} \le a_i \le 2h_{v_i} - 1\}$.

Proof. It is clear from the definition of the map A-code that $\mathrm{inv}_B(F, w) = \sum_{i=1}^{n} a_i$ because each a_i is a sum of the amounts that the vertex v_i contributes to $\mathrm{inv}(F, w)$, $\mathrm{n}_1(F, w)$, and $\mathrm{n}_2(F, w)$. Now, by definition, v_i is a signed bottom-to-top maximum position if and only if $w(v_i) > 0$ and $w(v_i) > |w(u)|$ for all $u <_F v_i$. Furthermore, $w(v_i) > |w(u)|$ for all $u <_F v_i$ if and only if v_i does not create any inversions with vertices below it and $w(v_i) + w(u) > 0$ for all $u <_F v_i$. Therefore, v_i is a signed bottom-to-top maximum position if and only if $a_i = 0$. This proves the second part of the lemma. For the third part, note that if $w(v_i) > 0$, each vertex u such that $u <_F v_i$ belongs to at most one of the sets $\{v_j : v_j <_F v_i \text{ and } w(v_j) > w(v_i)\}$ and $\{v_j : v_j <_F v_i \text{ and } w(v_j) + w(v_i) < 0\}$. Therefore, in this case $a_i < h_{v_i}$. On the other hand, if $w(v_i) < 0$, each vertex u such that $u <_F v_i$ belongs to at least one of these two sets and therefore $a_i \ge h_{v_i}$. □

A labeling $w \in \mathcal{W}(F)$ is said to be natural if it preserves the order $<_F$. The map A-code is not a bijection between $\mathcal{W}_B(F)$ and SE_F^B, but it can be used to define the following bijection ϕ from $\mathcal{W}_B(F)$ to the set $\{(w', (a_1, \ldots, a_n)) : w' \in \mathcal{W}(F) \text{ is a natural labeling and } (a_1, \ldots, a_n) \in \mathrm{SE}_F^B\}$. First we set A-code$(F, w) = (a_1, \ldots, a_n)$. The natural positive labeling w' is obtained by a sequence of n modifications applied to w in the following way. Start with $w_n = w$. If the labeling w_i has been defined for $i > 0$, construct w_{i-1} as follows. Set $A_i = \{|w_i(u)| : u \le_F v_i\}$. Find the largest element in A_i, say it is $|w_i(v_j)|$, and define the new labeling w_{i-1} of F so that

1. $w_{i-1}(v_i) = |w_i(v_j)|$
2. for all $u \not<_F v_i$, $w_{i-1}(u) = w_i(u)$
3. in w_{i-1}, the absolute values of the labels of the vertices below v_i are given by $A_i \backslash \{|w_i(v_j)|\}$ so that for all $u, u' <_F v_i$, $|w_{i-1}(u)| < |w_{i-1}(u')|$ if and only $|w_i(u)| < |w_i(u')|$ and $\operatorname{sgn} w_{i-1}(u) = \operatorname{sgn} w_i(u)$ for all $u <_F v_i$.



(a) A tree F with labeling $w = w_5$
$A_5 = \{1, 2, 3, 4, 5\}$

(b) A tree F with labeling w_4
$A_4 = \{3\}$

(c) A tree F with labeling w_3
$A_3 = \{1, 2, 4\}$

(d) A tree F with labeling w_2
$A_2 = \{2\}$

(e) A tree F with labeling w_1
$A_1 = \{1\}$

(f) A tree F with labeling w_0

Figure 4.2: Example of the algorithm used to find the natural positive labeling of the A-code

Finally, we set $w' = w_0$. See Figure 4.2 for an example.

Lemma 4.1.3. *If* A-code$(F, w) = (a_1, \ldots, a_n)$*, then* A-code$(F, w_k)$*, defined in the previous paragraph, is* $(a_1, \ldots, a_k, 0, \ldots, 0)$*,* $1 \leq k \leq n$*. Thus* w' *has no inversions and is a natural positive labeling.*

Proof. Assume that A-code$(F, w_i) = (a_1, \ldots, a_i, 0, \ldots, 0)$. In both w_i and w_{i-1}, all vertices u with $u \not<_F v_i$ form the same number of inversions with vertices below them. So, the corresponding entries in A-code(F, w_i) and A-code(F, w_{i-1}) are equal. Furthermore, the choice of the label $w_{i-1}(v_i)$ is such that it is clear that the i-th entry of A-code(F, w_{i-1}) is 0. What remains is to show that the entries in A-code(F, w_i) and A-code(F, w_{i-1}) corresponding to the vertices v_j below v_i are the same. The third property of w_{i-1} directly implies $\#\{u : u <_F v_j \text{ and } w_{i-1}(u) > w_{i-1}(v_j)\} = \#\{u : u <_F v_j \text{ and } w_i(u) > w_i(v_j)\}$ and $\chi(w_{i-1}(v_j) < 0) = \chi(w_i(v_j) < 0)$. So, we only need to check that

$$\#\{u : u <_F v_j \text{ and } w_{i-1}(u) + w_{i-1}(v_j) < 0\} = \#\{u : u <_F v_j \text{ and } w_i(u) + w_i(v_j) < 0\}.$$

Note that from w_i to w_{i-1}, the labels below v_i can stay the same, can increase by 1 (in which case they were negative), or can decrease by 1 (in which case they were positive). Therefore, $|(w_i(u)+w_i(v_j))-(w_{i-1}(u)+w_{i-1}(v_j))| \leq 2$. Thus, a change in the sign of the sum of two labels can possibly occur only when $w_i(u)+w_i(v_j) \in \{-2,-1,1,2\}$. In case when $w_i(u)+w_i(v_j) \in \{-2,-1\}$, we have that one of $w_i(u), w_i(v_j)$ is positive while the other one is negative. So $w_{i-1}(u)+w_{i-1}(v_j) \leq w_i(u)+w_i(v_j)+1 \leq 0$. Since all labels are different in absolute value, $w_{i-1}(u)+w_{i-1}(v_j) < 0$. Similarly, if $w_i(u)+w_i(v_j) \in \{1,2\}$, then $w_{i-1}(u)+w_{i-1}(v_j)$ must be positive as well. □

Theorem 4.1.4. *Let F be a forest with n vertices. The map*

$$\phi : \mathcal{W}_B(F) \to \{(w', (a_1, \ldots, a_n)) : w' \in \mathcal{W}(F) \text{ is a natural labeling and } (a_1, \ldots, a_n) \in \mathrm{SE}_F^B\}$$

is a bijection.

Proof. First note that by Lemma 4.1.3 the map ϕ is well defined. We now describe the inverse of ϕ. Given a pair $(w', (a_1, \ldots, a_n))$ where $w' \in \mathcal{W}(F)$ is a natural labeling and $(a_1, \ldots, a_n) \in \mathrm{SE}_F^B$, the corresponding labelings w_i from the definition of ϕ can be obtained in the following way. First, $w_0 = w'$. If w_{i-1} has been constructed for $i \leq n$, let $A_i = \{|w_{i-1}(u)| : u \leq v_i\}$. If $a_i < h_{v_i}$, find the (a_i+1)-st largest element in A_i, say it is $|w_{i-1}(v_j)|$, and set $w_i(v_i) = |w_{i-1}(v_j)|$. If $h_{v_i} \leq a_i \leq 2h_{v_i}-1$, find the $(a_i - h_{v_i}+1)$-st smallest element of A_i, say it is $|w_{i-1}(v_j)|$, and set $w_i(v_i) = -|w_{i-1}(v_j)|$. In either case, relabel the vertices below v_i with the elements from $A_i \backslash \{|w_{i-1}(v_j)|\}|$ while preserving the order of the original labels in absolute values as well as the signs at the vertices in w_{i-1} (similar to the third property in the definition of ϕ above), and call this new labeling w_i. The desired labeling w that corresponds to $(w', (a_1, \ldots, a_n))$ is simply w_n constructed in this process. Note that like in Lemma 4.1.3, one can show that the A-code$(F, w_i) = (a_1, \ldots, a_i, 0, \ldots, 0)$ and therefore A-code$(F, w) = (a_1, \ldots, a_n)$. □

Corollary 4.1.5. *Given a forest F of size n and a sequence $(a_1, \ldots, a_n) \in \mathrm{SE}_F^B$, there are $\frac{n!}{\prod_{v\in V(F)} h_v}$ signed labelings w of F such that* A-code$(F, w) = (a_1, \ldots, a_n)$.

Proof. This follows from Theorem 4.1.4 and the well-known fact that there are $\frac{n!}{\prod_{v\in V(F)} h_v}$ natural labelings of the forest F. □

Theorem 4.1.6. *Let F be a forest of size n. Then*

$$\sum_{w\in\mathcal{W}_B(F)} p^{\mathrm{n}_1(F,w)} q^{\mathrm{inv}_B(F,w)} \prod_{v\in\text{BT-max}_B(F,w)} t_v = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \Big((1+pq^{h_v})[h_v]-1+t_v\Big). \quad (4.1)$$

Proof. This is a direct consequence of Lemma 4.1.2, Theorem 4.1.4, and Corollary 4.1.5. □

As a corollary, we obtain a generalization of the results of Björner and Wachs [4] and Chen, Gao, and Guo [8].

Corollary 4.1.7. *Let F be a forest of size n. Then*

$$\sum_{w\in\mathcal{W}(F)} q^{\mathrm{inv}(F,w)} \prod_{v\in\text{BT-max}(F,w)} t_v = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([h_v]-1+t_v\right), \quad (4.2)$$

$$\sum_{w\in\mathcal{W}_B(F)} q^{\mathrm{inv}_B(F,w)} \prod_{v\in\text{BT-max}_B(F,w)} t_v = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([2h_v]-1+t_v\right), \quad (4.3)$$

$$\sum_{w\in\mathcal{W}_D(F)} q^{\mathrm{inv}_D(F,w)} \prod_{v\in\text{BT-max}_D(F,w)} t_v = \frac{n!\times 2^{\#\textit{leaves in } F-1}}{\prod_{v\in V(F)} h_v} \prod_{\substack{v\in V(F)\\ v \textit{ is not a leaf}}} \Big((1+q^{h_v-1})[h_v]-1+t_v\Big). \quad (4.4)$$

Proof. For $w\in\mathcal{W}_B(F)$ which is actually in $\mathcal{W}(F)$, $\mathrm{inv}(F,w)=\mathrm{inv}_B(F,w)$ and $\text{BT-max}(F,w)=\text{BT-max}_B(F,w)$. Therefore, (4.2) follows from (4.1) by setting $p=0$. The equation (4.3) is obtained by setting $p=1$ in (4.1). To get (4.4), let

$$D_n(p,q,t) = \sum_{w\in\mathcal{W}_B(F)} p^{\mathrm{n}_1(F,w)} q^{\mathrm{inv}_D(F,w)} \prod_{v\in\text{BT-max}_D(F,w)} t_v.$$

Then

$$\begin{aligned}
D_n(p,q,t) &= \sum_{w\in\mathcal{W}_B(F)} p^{\mathrm{n}_1(F,w)} q^{\mathrm{inv}(F,w)+\mathrm{n}_2(F,w)} \prod_{v\in \text{BT-max}_B(F,w)} t_v \Big|_{\substack{t_v=1\\ v \text{ is a leaf}}} \\
&= \sum_{w\in\mathcal{W}_B(F)} \left(\frac{p}{q}\right)^{\mathrm{n}_1(F,w)} q^{\mathrm{inv}_B(F,w)} \prod_{v\in \text{BT-max}_B(F,w)} t_v \Big|_{\substack{t_v=1\\ v \text{ is a leaf}}} \\
&\overset{(4.1)}{=} \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left((1+pq^{h_v-1})[h_v]-1+t_v\right) \Big|_{\substack{t_v=1\\ v \text{ is a leaf}}} \\
&= \frac{n!}{\prod_{v\in V(F)} h_v}(1+p)^{\#\text{leaves in } F} \prod_{\substack{v\in V(F)\\ v \text{ is not a leaf}}} \left((1+pq^{h_v-1})[h_v]-1+t_v\right).
\end{aligned}$$

Since F has at least one leaf, $D_n(-1,q,t)=0$, which implies

$$\sum_{i \text{ is even}} [p^i]D_n(p,q,t) = \sum_{i \text{ is odd}} [p^i]D_n(p,q,t),$$

where $[p^i]D_n(p,q,t)$ denotes the coefficient in $D_n(p,q,t)$ in front of $[p^i]$. Therefore,

$$\begin{aligned}
\sum_{w\in\mathcal{W}_D(F)} q^{\mathrm{inv}_D(F,w)} \prod_{v\in \text{BT-max}_D(F,w)} t_v &= \sum_{i \text{ is even}} [p^i]D_n(p,q,t) \\
&= \frac{D_n(1,q,t)}{2} \\
&= \frac{n!\times 2^{\#\text{leaves in } F-1}}{\prod_{v\in V(F)} h_v} \prod_{\substack{v\in V(F)\\ v \text{ is not a leaf}}} \left((1+q^{h_v-1})[h_v]-1+t_v\right).
\end{aligned}$$

□

Now, we show how Corollary 4.1.7 generalizes results for permutations. Let F be a tree of size n with one leaf whose vertices are naturally indexed and $w\in\mathcal{W}_B(F)$. Let σ be the signed permutation obtained by reading the labeling w of F from bottom to top, i.e., $\sigma = w(v_1)\cdots w(v_n)$. Then clearly $\mathrm{inv}(F,w)=\mathrm{inv}(\sigma)$, $\mathrm{n}_1(F,w)=\mathrm{n}_1(\sigma)$, and $\mathrm{n}_2(F,w)=\mathrm{n}_2(\sigma)$. Therefore, if $\sigma\in S_n$, then $\ell(\sigma)=\mathrm{inv}(F,w)$; if $\sigma\in B_n$, then $\ell_B(\sigma)=\mathrm{inv}_B(F,w)$; and if $\sigma\in D_n$, then $\ell_D(\sigma)=\mathrm{inv}_D(F,w)$.

The statistic RL-min is related to BT-max in the following way.

Lemma 4.1.8. *Let F be a linear tree with n vertices. Let $w\in\mathcal{W}_B(F)$ and let $\sigma = w(v_1)\cdots w(v_n)$ be the corresponding signed permutation. Then* $\text{BT-max}_B(F,w) = \text{RL-min}_B(\sigma^{-1})$. *Moreover, if $w\in\mathcal{W}(F)$ then* $\text{BT-max}(F,w)=\text{RL-min}(\sigma^{-1})$ *and if $w\in\mathcal{W}_D(F)$ then* $\text{BT-max}_D(F,w) =$

RL-$\min_D(\sigma^{-1})$.

Proof. Assume that the first statement holds for all linear trees of size at most n. Let F be a linear tree of size $n+1$ and let $w \in \mathcal{W}_B(F)$. Now let F' be the tree of size n obtained by removing the root v_{n+1} of F, and let $w' \in \mathcal{W}_B(F')$ be the corresponding standardized labeling obtained from w by decreasing the absolute values of all labels larger than $|w(v_{n+1})|$ by 1 and preserving the signs. Note that $v_{n+1} \in \text{BT-max}_B(F)$ if and only if $w(v_{n+1}) = n+1$. Therefore, if $w(v_{n+1)} \neq n+1$ then $\text{BT-max}_B(F, w) = \text{BT-max}_B(F'', w')$, and if $w(v_{n+1}) = n+1$ then $\text{BT-max}_B(F, w) = \text{BT-max}_B(F', w') \cup \{v_{n+1}\}$. Let $\sigma = w(v_1)\cdots w(v_{n+1})$, and $\sigma' = w'(v_1)\cdots w'(v_n)$. The permutation σ' can be obtained from σ by deleting the last letter $w(v_{n+1})$ and standardizing, and therefore, σ'^{-1} is obtained by removing $n+1$ or $\overline{(n+1)}$ from σ^{-1}. Since $w(v_{n+1})$ determines the position of $n+1$ in σ^{-1}, $n+1 \in \text{RL-min}_B(\sigma^{-1})$ if and only if $w(v_{n+1}) = n+1$. Applying this and the induction hypothesis, we have that if $w(v_{n+1}) \neq n+1$ then $\text{RL-min}_B(\sigma^{-1}) = \text{RL-min}_B(\sigma'^{-1}) = \text{BT-max}_B(F', w') = \text{BT-max}_B(F, w)$, and if $w(v_{n+1}) = n+1$ then $\text{RL-min}_B(\sigma^{-1}) = \text{RL-min}_B(\sigma'^{-1}) \cup \{n+1\} = \text{BT-max}_B(F', w') \cup \{n+1\} = \text{BT-max}_B(F, w)$. Therefore $\text{BT-max}_B(F, w) = \text{RL-min}_B(\sigma^{-1})$. If there are no signed letters, the same argument shows that $\text{BT-max}(F, w) = \text{RL-min}(\sigma^{-1})$. □

Since $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ and the corresponding statement is also true for signed and even signed permutations, as a direct consequence of Lemma 4.1.8 and Corollary 4.1.7 we get the following results for permutations.

Corollary 4.1.9.

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} \prod_{i \in \text{RL-min}(\sigma)} t_i = \prod_{i=1}^{n} ([i] - 1 + t_i), \tag{4.5}$$

$$\sum_{\sigma \in B_n} q^{\text{inv}_B(\sigma)} \prod_{i \in \text{RL-min}_B(\sigma)} t_i = \prod_{i=1}^{n} \left([2i] - 1 + t_i\right), \tag{4.6}$$

$$\sum_{\sigma \in D_n} q^{\text{inv}_D(\sigma)} \prod_{i \in \text{RL-min}_D(\sigma)} t_i = \prod_{i=2}^{n} \left((1 + q^{i-1})[i] - 1 + t_i\right). \tag{4.7}$$

Equation (4.5) was first shown in [5], while (4.6) and (4.7) can be found in [37], where a more general case of restricted permutations was also studied.

4.2 Sorting Index and Cycles

In this section, we introduce two new statistics for labeled forests, sorting index and cycle minima, and we study their joint distribution. They are motivated by corresponding permutation statistics $\mathrm{sor_B}$ and Cyc_B, or in the unsigned case, sor and Cyc, which we discussed in the introduction.

We begin by introducing the sorting index for signed labeled forests, which is computed via a sorting algorithm related to Straight Selection Sort. To describe it we introduce the following notation. For a signed forest (F, w), and a vertex v, let w_v denote the labeling of the subtree of F rooted at v which is induced by w. The algorithm for computing the sorting index of type B, $\mathrm{sor_B}(F, w)$, is as follows:

- Begin with $i = n$ and $\mathrm{sor_B}(F, w) = 0$.
- Let v be the vertex with $|w(v)| = i$ and let u be the largest vertex such that $u \geq_F v$ and $|w(u)| \leq i$. If $w(u) > 0$, then let $\mathrm{sor_B}(F, w) = \mathrm{sor_B}(F, w) + |w_u(v)| - w_u(u)$. Otherwise let $\mathrm{sor_B}(F, w) = \mathrm{sor_B}(F, w) + |w_u(v)| - w_u(u) - 1$
- If $w(v) > 0$, interchange the labels on the vertices u and v, and if not, first multiply $w(u)$ and $w(v)$ by -1, and then interchange the labels on the vertices u and v.
- Call this new labeling w, and repeat this process for $i = n - 1, \ldots, 1$.

For $w \in \mathcal{W}(F)$, we will define the type A sorting index $\mathrm{sor}(F, w)$ to be the same as $\mathrm{sor_B}(F, w)$. Since in this case there are no negative labels, the sorting algorithm can be simplified and we present it here for convenience.

- Begin with $i = n$ and $\mathrm{sor}(F, w) = 0$.
- Let v be the vertex with $|w(v)| = i$, and let u be the largest vertex such that $u \geq_F v$ and $w(u) \leq i$, then $\mathrm{sor}(F, w) = \mathrm{sor}(F, w) + w_u(v)$ - $w_u(u)$.
- Interchange the labels on the vertices u and v, and call the new labeling w.
- Repeat this process for $i = n - 1, \ldots, 1$.

An example of sorting a signed labeled tree is given in Figure 4.6. For this tree we have

$$\mathrm{sor}(F, w) = (5 - 2) + (1 + 1 - 1) + (3 - 1) + (1 + 1 - 1) + (1 - 1) = 7.$$

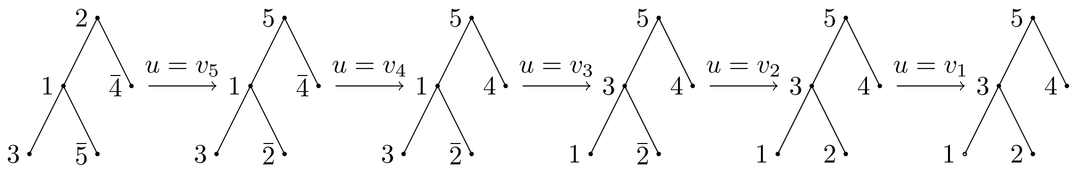


Figure 4.3: Sorting of the signed labeled tree from Figure 4.1.

The sorting algorithm applied to a labeling w produces a positive natural labeling w' of F. Though for signed labelings w, the map $w \circ (w')^{-1}$ is technically a map $\{1, \ldots, n\} \to \{\pm 1, \ldots, \pm n\}$, it can be uniquely extended to a signed permutation in B_n.

Definition 4.2.1. For $w \in \mathcal{W}(F)$, we define the *minimal cycle vertices* of (F, w) to be

$$\mathrm{Cyc}(F, w) = \{v : w'(v) \in \mathrm{Cyc}(w \circ (w')^{-1})\}.$$

For $w \in \mathcal{W}_B(F)$, we define the *type B minimal cycle vertices* of (F, w) to be

$$\mathrm{Cyc}_B(F, w) = \{v : w'(v) \in \mathrm{Cyc}_B(w \circ (w')^{-1})\}.$$

For example, the signed permutation that corresponds to the signed labeled tree from Figure 4.1 is $w \circ (w')^{-1} = 3\bar{5}1\bar{4}2 = (13)(2\bar{5}\bar{2}5)(4\bar{4})$. Therefore, $\mathrm{Cyc}_B(F, w) = \{v_1\}$.

Note that each vertex in F plays the role of u in the sorting algorithm exactly once. We define the map B-code$(F, w) = (b_1, \ldots, b_n) \in \mathrm{SE}_F^B$ where b_i is equal to the amount added to the sorting index in the step of the algorithm when $u = v_i$, (i.e. $b_i = |w_u(v)| - w_u(u) - 1$ or $b_i = |w_u(v)| - w_u(u)$). One can think of b_i as the amount contributed to the sorting index by the vertex v_i. For example, for the tree in Figure 4.1, B-code$(F, w) = (0, 1, 2, 1, 3)$.

Lemma 4.2.2. *Let $w \in \mathcal{W}_B(F)$ and suppose* B-code$(F, w) = (b_1, \ldots, b_n)$. *Then*

1. $\mathrm{sor_B}(F, w) = \sum_{i=1}^{n} b_i$
2. $\mathrm{Cyc}_B(F, w) = \{v_i : b_i = 0\}$
3. *$w \in \mathcal{W}(F)$ if and only if $b_i < h_{v_i}$ for all i.*

Proof. The first part follows from the way b_i is defined.

For the second part, we will use induction on the size of F. Suppose that the statement is true for all forests of size less than n. First, assume F is a forest with trees $T_1, \ldots, T_k$ for some $k > 1$. The sequence B-code(F, w) is a concatenation of the sequences B-code$(T_1, w_1), \ldots,$ B-code(T_k, w_k) (with possible rearrangements depending on the indexing of the vertices), where w_j is w restricted to T_j. The vertex v_i is in $\mathrm{Cyc}_B(F, w)$ if and only if for some $j \leq k$, $v_i \in \mathrm{Cyc}_B(T_j, w_j)$. By the induction hypothesis, $v_i \in V(T_j)$ is an element of $\mathrm{Cyc}_B(T_j, w_j)$ if and only if $b_i = 0$. Therefore i is an element of $\mathrm{Cyc}_B(F, w)$ if and only if $b_i = 0$.

Now assume that $k = 1$, i.e., F is a tree. Let F_1 be the forest obtained by removing the root v_n from F, and let w_1 be the labeling obtained by restricting w to F_1 and replacing the label n with $w(v_n)$. Now let w' and w'_1 be the labelings of F and F_1, respectively, obtained by sorting w and w_1, respectively. The permutation $w_1 \circ w_1'^{-1}$ can be obtained from $w \circ w'^{-1}$ by deleting the elements n and $\overline{n}$ in the cycle notation of $w \circ w'^{-1}$. Thus for $i = 1, \ldots, n-1$, $v_i \in \mathrm{Cyc}_B(F_1, w_1)$ if and only if $v_i \in \mathrm{Cyc}_B(F, w)$. Applying the induction hypothesis we see $v_i \in \mathrm{Cyc}_B(F, w)$ if and only if $b_i = 0$ for all $i = 1, \ldots, n-1$. The value n is a minimal element of a balanced cycle in $w \circ w'^{-1}$ if and only if it is in a cycle by itself, and thus $w(v_n) = n$, which happens exactly when $b_n = 0$. Therefore, $\mathrm{Cyc}_B(F, w) = \{v_i : b_i = 0\}$.

For the third part, note that the value $|w_u(v)|$ that appears in the sorting algorithm is equal to h_u. Therefore, the contribution of u to $\mathrm{sor_B}(F, w)$ is less than h_u if and only if the current label of the vertex u is positive. Because of the rule of interchanging the signs of the labels during the sorting process, if the starting labeling w has at least one negative sign, there will be a step in the process in which u has a negative label. On the other hand, if $w \in \mathcal{W}(F)$, then all the labels remain positive throughout the sorting. $\square$

Similarly to the A-code, the B-code also induces a map ψ from $\mathcal{W}_B(F)$ onto the set of pairs $(w', (b_1, \ldots, b_n))$ of a natural positive labeling w' of F and a sequence $(b_1, \ldots, b_n) \in \mathrm{SE}_F^B$. The natural labeling w' is the one obtained by sorting w, while $(b_1, \ldots, b_n) =$ B-code(F, w).

Theorem 4.2.3. *Let F be a forest with n naturally indexed vertices $v_1, \ldots, v_n$. The map*

$$\psi : \mathcal{W}_B(F) \to \{(w', (b_1, \ldots, b_n)) : w' \in \mathcal{W}(F) \text{ is a natural labeling and } (b_1, \ldots, b_n) \in \mathrm{SE}_F^B\}$$

is a bijection. Restricted to positive labelings, ψ is a bijection from $\mathcal{W}(F)$ to the set of pairs

$(w', (b_1, \ldots, b_n))$ *where* $w' \in \mathcal{W}(F)$ *is a natural labeling and* $(b_1, \ldots, b_n) \in \mathrm{SE}_F$.

Proof. We describe the inverse of ψ. Given a pair $(w', (b_1, \ldots, b_n))$ of a natural labeling $w' \in \mathcal{W}(F)$ and $(b_1, \ldots, b_n) \in \mathrm{SE}_F^B$, the original labeling w can be recovered in the following way. Begin with $i = 1$, and let j be such that $|w'(v_j)| = i$. Let $B_i = \{|w'(v)| : v \leq_F v_j\}$. If $b_j < h_{v_j}$, let u be the vertex so that $|w'(u)|$ is the $(b_i + 1)$-st largest element in B_i. Then if $w'(u) > 0$, simply interchange the labels of u and v_j. Otherwise, first change the signs of the labels of u and v_j and then interchange them. Otherwise, if $h_{v_j} \leq b_j < 2h_{v_j}$, let u be the vertex so that $|w'(u)|$ is the $(b_j - h_{v_j} + 1)$-st smallest element of B_i. Then if $w'(u) < 0$, simply interchange the labels of u and v_j. Otherwise, first change the signs of the labels of u and v_j and then interchange them. Keep calling the new labeling w'. Repeat for $i = 2, \ldots, n$. The final labeling is the desired $w \in \mathcal{W}_B(F)$.
The second part of the theorem follows from the third part of Lemma 4.2.2. □

Corollary 4.2.4. *Let F be a forest of size n, then*

$$\sum_{w \in \mathcal{W}(F)} q^{\mathrm{sor}(F,w)} \prod_{v \in \mathrm{Cyc}(F,w)} t_v = \frac{n!}{\prod_{v \in V(F)} h_v} \prod_{v \in V(F)} \left([h_v] - 1 + t_v\right), \tag{4.8}$$

$$\sum_{w \in \mathcal{W}_B(F)} q^{\mathrm{sor_B}\,(F,w)} \prod_{v \in \mathrm{Cyc}_B\,(F,w)} t_v = \frac{n!}{\prod_{v \in V(F)} h_v} \prod_{v \in V(F)} \left([2h_v] - 1 + t_v\right). \tag{4.9}$$

Proof. This is a direct consequence of Lemma 4.2.2 and Theorem 4.2.3. The products on the right-hand side of (4.8) and (4.9) are the generating functions for the sequences in SE_F and SE_F^B, respectively, according to total sum of elements and positions of zeros. The factor $n!/\prod_{v \in V(F)} h_v$ is due to the fact that the B-code is a $(n!/\prod_{v \in V(F)} h_v)$-to-1 map. □

Our definition of sor and Cyc for labeled forests was motivated by corresponding permutation statistics. Petersen [36] showed that

$$\sum_{\sigma \in B_n} q^{\mathrm{sor_B}(\sigma)} t^{\mathrm{cyc}_B(\sigma)} = \sum_{\sigma \in B_n} q^{\mathrm{inv}_B(\sigma)} t^{\mathrm{rl\text{-}min}_B(\sigma)}.$$

This equidistribution was later generalized to include r-colored permutations and, instead of just sor and cyc, the result was refined in terms of set-valued statistics RL-min and Cyc as well as additional statistics that allow us to deduce results for restricted permutations [37, 9, 14].

The following two theorems reveal the relation of the statistics sor and Cyc for labeled forests with the corresponding permutation statistics.

Theorem 4.2.5. *Let F be a linear tree of size n and $w \in \mathcal{W}_B(F)$. Let $\sigma = w(v_1)\cdots w(v_n)$ be the corresponding signed permutation, then* $\mathrm{sor}_{\mathrm{B}}(F,w) = \mathrm{sor}_{\mathrm{B}}(\sigma^{-1})$. *Consequently, if $w \in \mathcal{W}(F)$, then* $\mathrm{sor}(F,w) = \mathrm{sor}(\sigma^{-1})$.

Proof. Assume that the theorem holds for all linear trees of size at most n. Let F be a linear tree of size $n+1$ and let F_1 be the tree of size n obtained by removing the root v_{n+1} of F.

Consider first the case when $n+1$ appears as a label in w. Let w_1 be the labeling of F_1 obtained by restricting w to F_1 and replacing the label $n+1$ with $w(v_{n+1})$. If $w(v_{n+1}) < 0$, then $\mathrm{sor}_{\mathrm{B}}(F,w) = \mathrm{sor}_{\mathrm{B}}(F_1,w_1) + (n+1) - w(v_{n+1}) - 1$, and if $w(v_{n+1}) > 0$ then $\mathrm{sor}_{\mathrm{B}}(F,w) = \mathrm{sor}_{\mathrm{B}}(F_1,w_1) + (n+1) - w(v_{n+1})$. Now let $\sigma = w(v_1)\cdots w(v_{n+1})$, and $\sigma_1 = w_1(v_1)\cdots w_1(v_n)$. Note that σ_1^{-1} is the permutation obtained from σ^{-1} by performing the first step of the Straight Selection Sort Algorithm and deleting $n+1$. Moreover, $w(v_{n+1})$ is the position of $n+1$ in σ^{-1}. Therefore, if $w(v_{n+1}) < 0$ then $\mathrm{sor}_{\mathrm{B}}(\sigma^{-1}) = \mathrm{sor}_{\mathrm{B}}(\sigma_1^{-1}) + (n+1) - w(v_{n+1}) - 1$, and if $w(v_{n+1}) > 0$ then $\mathrm{sor}_{\mathrm{B}}(\sigma^{-1}) = \mathrm{sor}_{\mathrm{B}}(\sigma_1^{-1}) + (n+1) - w(v_{n+1})$. The claim follows by applying the induction hypothesis.

The proof in the case when $\overline{n+1}$ appears as a label in w is similar. We set w_1 to be the labeling of F_1 obtained by restricting w to F_1 and replacing the label $\overline{n+1}$ with $\overline{w(v_{n+1})}$. If $w(v_{n+1}) < 0$ then $\mathrm{sor}_{\mathrm{B}}(F,w) = \mathrm{sor}_{\mathrm{B}}(F_1,w_1) + (n+1) - w(v_{n+1}) - 1$, and if $w(v_{n+1}) > 0$ then $\mathrm{sor}_{\mathrm{B}}(F,w) = \mathrm{sor}_{\mathrm{B}}(F_1,w_1) + (n+1) - w(v_{n+1})$. So, we can continue as above.

Finally, note that if a labeling or a permutation is in fact positive, then the type B and type A sorting indices coincide. □

Theorem 4.2.6. *Let F be a linear tree with naturally indexed vertices $v_1,\ldots,v_n$. Let $w \in \mathcal{W}_B(F)$ and let $\sigma = w(v_1)\cdots w(v_n) \in B_n$. Then $v_i \in \mathrm{Cyc}_B(F,w)$ if and only if $i \in \mathrm{Cyc}_B(\sigma^{-1})$. Consequently, if $w \in \mathcal{W}(F)$, then $v_i \in \mathrm{Cyc}(F,w)$ if and only if $i \in \mathrm{Cyc}(\sigma^{-1})$.*

Proof. By definition, $\mathrm{Cyc}_B(F,w) = \{v_i : w'(v_i) \in \mathrm{Cyc}_B(w \circ w'^{-1})\}$. For a linear tree the sorted labeling w' is given by $w'(v_i) = i$. Thus for every $w \in \mathcal{W}_B(F)$, $w \circ w'^{-1} = \sigma$, and hence $v_i \in \mathrm{Cyc}_B(F,w)$ if and only if $i \in \mathrm{Cyc}_B(\sigma)$. The claim now follows from the fact that $\mathrm{Cyc}_B(\sigma) = \mathrm{Cyc}_B(\sigma^{-1})$. □

As a corollary to Corollary 4.2.4, Theorem 4.2.5, and Theorem 4.2.6, we get the following result.

Corollary 4.2.7 ([9, 37])**.**

$$\sum_{\sigma\in S_n} q^{\mathrm{sor}\,(\sigma)} \prod_{i\in \mathrm{Cyc}\,(\sigma)} t_i = \prod_{i=1}^{n}([i]-1+t_i)$$

$$\sum_{\sigma\in B_n} q^{\mathrm{sor_B}\,(\sigma)} \prod_{i\in \mathrm{Cyc}_B\,(\sigma)} t_i = \prod_{i=1}^{n}([2i]-1+t_i).$$

4.3 Major Index and Cyclic Bottom-To-Top Maxima

While for permutations it is true that

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} t^{\text{rl-min}(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)} t^{\text{rl-min}(\sigma)},$$

for a general forest F, $(\mathrm{inv}, \#\,\text{BT-max})$ and $(\mathrm{maj}, \#\,\text{BT-max})$ are not equidistributed over $\mathcal{W}(F)$. In this section, we find a suitable Stirling partner for maj for labeled forests and then discuss the case of signed labelings.

Definition 4.3.1. Let (F, w) be a labeled forest. A vertex v is a *cyclic bottom-to-top maximum* if its label is a bottom-to-top maximum with respect to the cyclic shift of the natural ordering of the integers $1, \ldots, n$ beginning with the label of the parent of v, p. Precisely, if $w(v) < w(p)$, then v is a cyclic bottom-to-top maximum if

$$\{u : u <_F v, w(u) \in [w(v), w(p)]\} = \emptyset.$$

If $w(p) < w(v)$, then v is a cyclic bottom-to-top maximum if

$$\{u : u <_F v, w(u) \notin [w(p), w(v)]\} = \emptyset.$$

Let CBT-max(F, w) denote the set of all cyclic bottom-to-top maxima of the labeled forest (F, w).

Let F be a forest of size n with naturally indexed vertices $\{v_1, \ldots, v_n\}$. We will denote by p_i be the parent of v_i, and while for a root v_j, p_j is not defined, we will use the convention

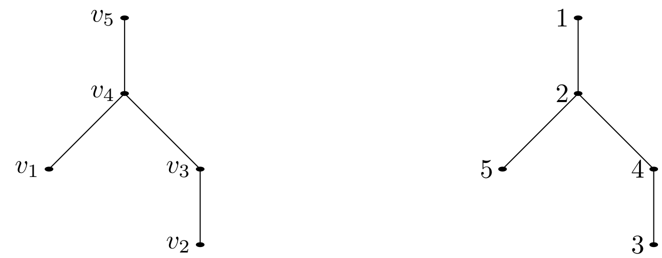


Figure 4.4: For this tree F with labeling w, we have $\mathrm{Cyc}(F,w) = \{v_1, v_2, v_3\}$ and $\text{M-code}(F,w) = (0,0,0,3,4)$. Also, $\mathrm{Des}(F,w) = \{v_1, v_3, v_4\}$ and therefore $\mathrm{maj}(F,w) = 1+2+4 = 7$.

$w(p_j) = n+1$. Define $\text{M-code}(F,w) = (m_1, \ldots, m_n)$ as follows (see Figure 4.4):

$$m_i = \#\{u : u <_F v_i, w(u) \in [w(v_i), w(p_i)]\} \text{ if } w(v_i) < w(p_i)$$

$$m_i = \#\{u : u <_F v_i, w(u) \notin [w(p_i), w(v_i)]\} \text{ otherwise.}$$

The special case of this code for permutations was used under the name "McMahon code" in [44], where its relationship to the Lehmer code was explained.

Theorem 4.3.2. *Let* $\text{M-code}(F,w) = (m_1, \ldots, m_n)$*, then* $\sum_{i=1}^n m_i = \mathrm{maj}\,(F,w)$*, and* $m_i = 0$ *if and only if* $v_i \in \text{CBT-max}\,(F,w)$*.*

Proof. The second part follows directly from the definitions.

For the first part, assume that the statement holds for all forests of size less than n. Suppose first that F is a forest of size n with trees $T_1, \ldots, T_k$ for $k > 1$. It is clear that $\mathrm{maj}\,(F,w) = \sum_{i=1}^n \mathrm{maj}\,(T_i, w_i)$, where w_i is the labeling of T_i induced by w. Also, the sequence $\text{M-code}(F,w)$ is just a concatenation of the sequences $\text{M-code}(T_1,w_1), \ldots, \text{M-code}(T_k,w_k)$, with reordering as necessary. Therefore $\sum_{i=1}^n m_i = \mathrm{maj}\,(F,w)$.

Now suppose $k = 1$. Note that v_{n-1} is a child of v_n. Let F' be the forest obtained by deleting the edge (v_{n-1}, v_n) from F. Let $\text{M-code}(F',w) = (m'_1, \ldots, m'_n)$, $A = \{u : u <_F v_{n-1} \text{ and } w(u) < w(v_{n-1})\}$, $B = \{u : u <_F v_{n-1} \text{ and } w(v_{n-1}) < w(u) < w(v_n)\}$, and $C = \{u : u <_F v_{n-1} \text{ and } w(v_n) < w(u)\}$. We will consider two cases.

Case 1. $w(v_{n-1}) < w(v_n)$

In this case, $\mathrm{Des}(F,w) = \mathrm{Des}(F',w)$ and hence $\mathrm{maj}(F,w) = \mathrm{maj}(F',w)$. Comparing the two M-codes, we have $m'_{n-1} = m_{n-1} + \#C$, $m'_n = m_n - \#C$, and $m'_i = m_i$ for all $i \neq n-1, n$. Therefore,

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m'_i = \text{maj}(F', w) = \text{maj}(F, w).$$

Case 2. $w(v_{n-1}) > w(v_n)$

In this case, $\text{Des}(F, w) = \text{Des}(F', w) \cup \{v_{n-1}\}$ and hence $\text{maj}(F, w) = \text{maj}(F', w) + h_{v_{n-1}}$. Comparing the two MacMahon codes, we have $m'_{n-1} = h_{v_{n-1}} - 1 - \#A$, $m'_n = m_n - 1 - \#C$, and $m'_i = m_i$ for all $i \neq n-1, n$. For the code of (F, w), we notice that

$$\begin{aligned} m_{n-1} &= \#\{u : u <_F v_{n-1} \text{ and } w(u) > w(v_{n-1})\} + \#\{u : u <_F v_{n-1} \text{ and } w(u) < w(v_n)\} \\ &= (h_{v_{n-1}} - 1 - \#A) + (h_{v_{n-1}} - 1 - \#C). \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m'_i + h_{v_{n-1}} = \text{maj}(F', w) + h_{v_{n-1}} = \text{maj}(F, w).$$

□

The M-code induces a map θ from $\mathcal{W}(F)$ to the set of pairs $(w', (m_1, \ldots, m_n))$, where w' is a natural labeling of F and $(m_1, \ldots, m_n) \in \text{SE}_F$ is defined in the following way. For $w \in \mathcal{W}(F)$, the corresponding subexcedent sequence $(m_1, \ldots, m_n)$ is M-code(F, w). The natural labeling w' is obtained in n steps by sorting w as follows. Start with $i = n$. Let $\ell_1 < \cdots < \ell_{h_{v_i}}$ be the labels of the subtree rooted at v_i. Replace each label ℓ_j by ℓ_{j+m_i}, where the addition is performed modulo h_{v_i}. Note that after this, the vertex v_i is a cyclic bottom-to-top maximum in the new labeling, while the other cyclic bottom-to-top maxima remain unchanged. It is not difficult to see that the M-code of the new labeling is $(m_1, \ldots, m_{i-1}, 0 \ldots, 0)$. Decrease i by 1 and repeat until $i = 0$. This will produce a labeling w' with an M-code of $(0, \ldots, 0)$ so it is natural. Because of this discussion, it is not difficult to see that the steps are reversible and therefore θ is a bijection. We summarize this in the following theorem.

Theorem 4.3.3. *The map*

$$\theta : \mathcal{W}(F) \to \{(w', (m_1, \ldots, m_n)) : w' \in \mathcal{W}(F) \text{ is a natural labeling and } (m_1, \ldots, m_n) \in \text{SE}_F\}$$

described above is a bijection.

As a corollary to Theorem 4.3.3 and Theorem 4.3.2 we get the following result.

Theorem 4.3.4.

$$\sum_{w\in\mathcal{W}(F)} q^{\mathrm{maj}(F,w)} \prod_{v\in\text{CBT-max}(F,w)} t_v = \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([h_v]-1+t_v\right), \tag{4.10}$$

Liang and Wachs [32] constructed a bijection on labeled forests to prove that the enumerator for the inversion index on labeled forests is identical to the enumerator for the major index on labeled forests. For the symmetric group, their bijection reduces to a map similar to Foata's second fundamental transformation. Note that as a consequence of the properties of the A-code and B-code for labeled forests, the map $\theta^{-1}\circ\phi : \mathcal{W}(F) \to \mathcal{W}(F)$ has the stronger property: it takes $(\mathrm{inv}, \text{BT-max})$ to $(\mathrm{maj}, \text{CBT-max})$. This map is different from the one in [32].

In [8], Chen, Gao, and Guo defined two major indices for signed labeled forests and showed that they are equidistributed with inv_B. The first one is based on the flag major index for signed permutations introduced by Adin and Roichman [1]. The second one is based on a Mahonian statistic for signed permutations that implicitly appears in [39].

Definition 4.3.5 ([8])**.** For a signed labeled forest (F, w),

$$\mathrm{fmaj}(F,w) = 2\,\mathrm{maj}(F,w) + n_1(F,w).$$

For a signed forest (F, w) let

$$\mathrm{Des}_B(F,w) = \mathrm{Des}(F,w) \cup \{u \in F : u \text{ is a root of } F \text{ with a positive label}\}$$

and

$$\mathrm{maj}_B\,(F,w) = \sum_{u\in \mathrm{Des}_B(F,w)} h_u.$$

Let $\mathrm{p}(F, w)$ be the number of positive labels of w.

Definition 4.3.6 ([8])**.** For a signed labeled forest (F, w),

$$\mathrm{rmaj}(F,w) = 2\,\mathrm{maj}_B(F,w) - \mathrm{p}(F,w).$$

As observed in [8], there is a simple map that sends fmaj to rmaj, so here we will discuss only finding a Stirling partner for fmaj. One could define a generalization of CBT-max for signed labelings as follows.

Definition 4.3.7. Let (F, w) be a signed labeled forest of size n. A vertex v is a *cyclic bottom-to-top maximum* if its label is positive and is a bottom-to-top maximum with respect to the cyclic shift of the natural ordering of the integers $-n, \ldots, -1, 1, \ldots, n$ beginning with the label of the parent of v, p. Precisely, for a vertex v with a positive label, if $w(v) < w(p)$, then v is a cyclic bottom-to-top maximum if

$$\{u : u <_F v, w(u) \in [w(v), w(p)]\} = \emptyset.$$

If $w(p) < w(v)$, then v is a cyclic bottom-to-top maximum if

$$\{u : u <_F v, w(u) \notin [w(p), w(u)]\} = \emptyset.$$

Let $\text{CBT-max}_B\, F, w$ denote the set of all cyclic bottom-to-top maxima of the signed labeled forest (F, w).

Unfortunately, the pairs $(\text{fmaj}, \#\,\text{CBT-max})$ and $(\text{inv}_B, \#\,\text{BT-max})$ are not equidistributed over $\mathcal{W}_B(F)$. It would be interesting to see if there is a better definition of $\text{CBT-max}(F, w)$ or if there is another natural Stirling partner for fmaj. Here we will only show that there is an analogue of Theorem 4.3.2 for signed forests.

For a signed labeled forest (F, w) with naturally indexed vertices $\{v_1, \ldots, v_n\}$, we define $\text{M-code}(F, w)$ to be the sequence $(m_1, \ldots, m_n)$ given by

$$\begin{aligned} m_i &= 2\#\{u : u <_F v_i, w(u) \in [w(v_i), w(p_i)]\} + \chi(w(v_i) < 0) \quad \text{if } w(v_i) < w(p_i) \\ m_i &= 2\#\{u : u <_F v_i, w(u) \notin [w(p_i), w(v_i)]\} + \chi(w(v_i) < 0) \quad \text{otherwise.} \end{aligned}$$

Here we use the same convention that p_i is the parent of v_i and if v_i is a root of F, then $w(p_i) = n+1$.

Theorem 4.3.8. *For a forest F with signed labeling w and* $\text{M-code}(F, w) = (m_1, \ldots, m_n)$*,* $\sum_{i=1}^{n} m_i = \text{fmaj}\,(F, w)$*, and $m_i = 0$ if and only if $v_i \in \text{CBT-max}_B(F, w)$.*

Proof. It is clear from the definitions that $m_i = 0$ if and only if $v_i \in \text{CBT-max}_B\,(F, w)$.

For the first part, we use induction on n, the number of vertices of F. If F is a forest, the claim follows the same way as in the unsigned case (Theorem 4.3.2). Therefore, suppose that F is a tree. Then v_n is the root of F. Consider the child of v_n, v_{n-1}, and let F' be the forest obtained from deleting the edge (v_{n-1}, v_n) from F. Let $(m'_1, \ldots, m'_n)$ be the signed MacMahon code of F'. We will use the sets $A = \{u \in F : u <_F v_{n-1} \text{ and } w(u) < w(v_{n-1})\}$, $B = \{u \in F : u <_F v_{n-1} \text{ and } w(v_{n-1}) < w(u) < w(v_n)\}$, and $C = \{u \in F : u <_F v_{n-1} \text{ and } w(v_n) < w(u)\}$.

Case 1. $w(v_{n-1}) < w(v_n)$.

In this case, $v_{n-1} \notin \mathrm{Des}(F, w)$ and therefore $\mathrm{maj}(F, w) = \mathrm{maj}(F', w)$. So, $\mathrm{fmaj}(F, w) = \mathrm{fmaj}(F', w)$. Note that $m'_{n-1} = m_{n-1} + 2\#C$, $m'_n = m_n - 2\#C$, and $m'_j = m_j$ for all $j \neq n-1, n$. Thus

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m'_i = \mathrm{maj}(F', w) = \mathrm{maj}(F, w).$$

Case 2. $w(v_{n-1}) > w(v_n)$.

In this case, $\mathrm{Des}(F, w) = \mathrm{Des}(F', w) \cup \{v_{n-1}\}$ and therefore $\mathrm{maj}(F, w) = \mathrm{maj}(F', w) + h_{v_{n-1}}$. This implies $\mathrm{fmaj}(F, w) = \mathrm{fmaj}(F', w) + 2h_{v_{n-1}}$. Note that $m'_{n-1} = m_{n-1} - 2(h_{v_{n-1}} - 1 - \#C)$, $m'_n = m_n - 2 - 2\#C$, and $m'_j = m_j$ for all $j \neq n-1, n$. Thus

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m'_i + 2h_{v_{n-1}} = \mathrm{maj}(F', w) + 2h_{v_{n-1}} = \mathrm{maj}(F, w).$$

□

The difference between the signed and the unsigned case is that the map from $\mathcal{W}_B(F) \to \mathrm{SE}_F^B$ given by the M-code is not onto.

4.4 Inverse Major Index and Bottom-To-Top Maxima

As we discussed in the previous section, the statistics maj and CBT-max do not generalize over signed forests in a straightforward way. Additionally, they do not generalize the equidistribution result $(\mathrm{inv}, \text{rl-min}) \sim (\mathrm{maj}, \text{rl-min})$ for permutations. As we mentioned in Section 2.2, the pairs of statistics $(\mathrm{inv}, \text{RL-min})$ and $(\mathrm{maj}, \text{RL-min})$ are equidistributed over S_n. Recall that in Section 2.1, we discussed that inv and maj have symmetric joint distributions over S_n, but this is not true for

(a) Sorted labeling w' (b) Inverse labeling w^{inv}

Figure 4.5: Sorted and Inverse labelings of (F, w) from Figure 4.4

labeled forests. For example, for the tree F in Figure 4.1,

$$\sum_{w\in\mathcal{W}(F)} q^{\text{inv}(F,w)}t^{\text{maj}(F,w)} - \sum_{w\in\mathcal{W}(F)} q^{\text{maj}(F,w)}t^{\text{inv}(F,w)} =$$

$$4q^5t^4 - 4q^4t^5 - 2q^5t^3 + 2q^3t^5 - 4q^4t^3 + 4q^3t^4 + 4q^3t^2 - 4q^2t^3 + 2q^3t - 2qt^3 - 4q^2t + 4qt^2 \neq 0.$$

Foata's bijection also preserves RL-min, and therefore the statistics (inv, RL-min), (maj, RL-min) and (imaj, RL-min) are all equidistributed over S_n. In this section, we define iDes and imaj for labeled forests and discuss the properties that hold for trees, and state a conjecture that would generalize parts of this result to trees.

Recall that in Definition 4.2.1 we used the permutation that arises from $w \circ (w')^{-1}$ where w' is the positive decreasing labeling obtained from sorting the labeling w.

Definition 4.4.1. For $w \in \mathcal{W}(F)$ with sorted labeling w', we define the inverse labeling, w^{inv}, to be the labeling $(w \circ (w')^{-1})^{-1}$.

For example, consider the forest F with labeling w in Figure 4.4. Figure 4.5a gives the sorted labeling w', and thus $w \circ (w')^{-1} = 53421$. Therefore, the inverse labeling w^{inv} shown in Figure 4.5b is given by the permutation $(w \circ (w')^{-1})^{-1} = 54231$.

Proposition 4.4.2. *For a labeling $w \in \mathcal{W}(F)$, $(w^{inv})^{inv} = w$.*

In other words, taking the inverse labeling is an involution.

Proof. Let F be a forest of size n with a labeling w. First assume that w and w^{inv} sort to the same labeling w' and let $\sigma = w \circ (w')^{-1}$. Then w^{inv} is the labeling given by σ^{-1} and thus $(w^{\text{inv}})^{\text{inv}}$ is given by $(\sigma^{-1})^{-1} = \sigma$. Therefore $(w^{\text{inv}})^{\text{inv}} = w$. So, it suffices to show that w and w^{inv} do in fact sort to the same tree. We do this by induction on the size of the tree.

For the base case, when T is a tree with only one vertex, the statement is clearly true. Thus assume that for all trees of size less than n a labeling and its inverse sort to the same labeling.

Consider a forest F of size n with labeling w_F and sorted labeling w'_F. Perform the first step of the sorting algorithm to get the new labeling $\tilde{w}_F$. Now remove the root labeled n and adjacent edges from F to form a forest F_1 of size $n-1$ with the labeling $\tilde{w}_{F_1}$ defined by $\tilde{w}_{F_1}(v) = \tilde{w}_F(v)$ for all $v \in V(F_1)$. Let $\tilde{w}'_{F_1}$ be the labeling that $\tilde{w}_{F_1}$ sorts to, and by the induction hypothesis $\tilde{w}^{\text{inv}}_{F_1}$ sorts to $\tilde{w}'_{F_1}$ as well. Note that the forest F_1 with the labeling $\tilde{w}'_{F_1}$ is the same labeled forest found from removing the root from the original forest F with the sorted labeling w'_F. Let $\sigma = \tilde{w}_F \circ (\tilde{w}'_F)^{-1}$, $\tau = w_F \circ (w'_F)^{-1}$, and $i = w_F(v_n)$ where v_n is the root of F we removed to get F_1. To get the labeling $\tilde{w}_F$ from w_F the labels i and n are interchanged. Therefore $\tau = (i\ n)\sigma$. Now let j be the label such that $\sigma(j) = i$. In other words, if you let v be the vertex in F such that $w_F(v) = n$ then $j = \tilde{w}_F(v)$. Consider the forest F_1 with labeling $\tilde{w}^{\text{inv}}_{F_1}$ and add the root with a label of n to get the original forest F with the labeling $\tilde{w}^{\text{inv}}_F$. Now interchange the labels j and n to get a new labeling $l_F \in \mathcal{W}(F)$. By construction the labelings w_F and l_F both sort to w'_F, and we need to show that $l_F = w^{\text{inv}}_F$. Since $\sigma = \tilde{w}_F \circ (\tilde{w}'_F)^{-1}$ we know $\sigma^{-1} = \tilde{w}^{\text{inv}}_F \circ (\tilde{w}_F)^{-1}$. To get the labeling l_F, we started with the labeling $\tilde{w}^{\text{inv}}_F$ and interchanged j and n. Therefore the permutation $l_F \circ (w'_F)^{-1} = (j\ n)\sigma^{-1}$. Now we show that the permutations τ and $l_F \circ (w'_F)^{-1}$ are inverses. Since $\sigma(j) = i$ and $\sigma(n) = n$ we get

$$\tau \circ (l_T \circ (w'_T)^{-1}) = (i\ n)\sigma(j\ n)\sigma^{-1} = (1)\cdots(n).$$

Therefore $\tau^{-1} = l_F \circ (w'_F)^{-1}$ and thus $l_F = w^{\text{inv}}_F$. □

Definition 4.4.3. For a forest F and labeling $w \in \mathcal{W}(F)$, the *inverse descent set* is defined as $\text{iDes}(F, w) = \text{Des}(F, w^{\text{inv}})$ and the *inverse major index* is $\text{imaj}(F, w) = \text{maj}(F, w^{\text{inv}})$.

For example, consider (F, w) in Figure 4.4 and its inverse given in Figure 4.5b. Then $\text{iDes}(F, w) = \text{Des}(F, w^{\text{inv}}) = \{v_4, v_1, v_2\}$, and $\text{imaj}(F, w) = \text{maj}(F, w^{\text{inv}}) = 6$.

Conjecture 4.4.4. *For a forest F of size n,*

$$\sum_{w\in\mathcal{W}(F)} q^{\mathrm{inv}(F,w)} \prod_{v\in \text{BT-max}(F,w)} t_v = \sum_{w\in\mathcal{W}(F)} q^{\mathrm{imaj}(F,w)} \prod_{v\in \text{BT-max}(F,w)} t_v$$
$$= \frac{n!}{\prod_{v\in V(F)} h_v} \prod_{v\in V(F)} \left([h_v]-1+t_v\right).$$

If Conjecture 4.4.4 is true, then it generalizes the result for permutations in the following way. Let F be a linear tree with n vertices. Let $w \in \mathcal{W}(F)$ and let $\sigma = w(v_1)\cdots w(v_n)$ be the corresponding permutation. As we noted before in Lemma 4.1.8 $\mathrm{inv}(w,F) = \mathrm{inv}(\sigma^{-1})$ and $\text{BT-max}(F,w) = \text{RL-min}(\sigma^{-1})$. By definition of imaj we have $\mathrm{imaj}(w,F) = \mathrm{imaj}(\sigma) = \mathrm{maj}(\sigma^{-1})$.

We have tested this Conjecture on trees with up to 8 verticies. In the examples that we have run, there is evidence suggesting the following inductive argument. For a forest F of size $n-1$ with labeling w_F, add a root labeled n to get a tree T with a labeling w_1. Let w_1' be the labeling that w_1 sorts to. Now there is a permutation σ_w with $\sigma_{w_F}(1) = n$ and a way to manipulate the labeling w_1 to create $n-1$ new labelings $w_2,\ldots,w_n$ such that the root of w_i is $\sigma_w(i)$, $\mathrm{imaj}(T,w_i) = \mathrm{imaj}(F,w_F)+i$, w_i sorts to w_1' for all $i = 1,\ldots,n$, and $\text{BT-max}(T,w_i) = \text{BT-max}(T,w_1)\setminus\{v_n\}$ where v_n is the root of T. In other words, we can write the generating function in the following way:

$$\sum_{w\in\mathcal{W}(T)} q^{\mathrm{imaj}(T,w)} y_{w(v_n)} \prod_{v\in \text{BT-max}(T,w)} t_v =$$
$$\sum_{\tilde{w}\in\mathcal{W}(F)} \left(q^{\mathrm{imaj}(F,w)}(t_n y_{\sigma_{\tilde{w}}(1)} + q y_{\sigma_{\tilde{w}}(2)} + \cdots + q^{n-1} y_{\sigma_{\tilde{w}}(n)}) \prod_{v\in \text{BT-max}(F,\tilde{w})} t_v \right).$$

We have found a way to construct σ_w and the desired labelings $w_2,\ldots w_n$ when we begin with a decreasing labeling for the forest F which we explain next. However, we have not been able to extend this construction for nondecreasing labelings and complete the proof of Conjecture 4.4.4.

Consider a forest F of size $n-1$ with a decreasing labeling and add a root labeled n to the forest by connecting this new root to the root of each tree in the forest, creating a tree T of size n with a decreasing labeling that we call w. Notice that the tree T with labeling w sorts to itself, and therefore $w^{\mathrm{inv}} = w$. Also note that $\mathrm{imaj}(w,T) = \mathrm{maj}(w,T) = 0$, and every vertex is a bottom-to-top maximum. Now we will create $n-1$ new labelings for the tree T such that $\mathrm{imaj}(w_i) = i$, and every

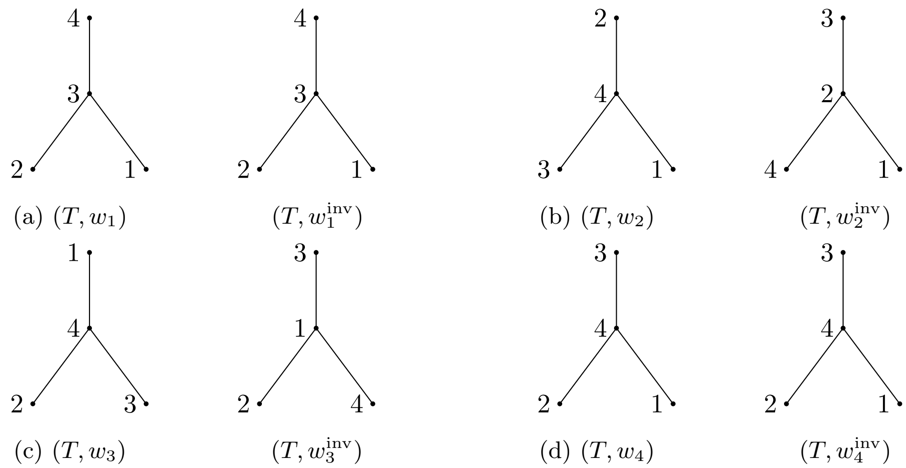


Figure 4.6: Example of the inductive argument for the Conjecture 4.4.4 beginning with a decreasing labeling.

vertex of w_i is a bottom-to-top maximum except the root.

- Begin with $i = 1$ and let S be the set of all vertices in T.
- Start from the root and move down the tree using only elements of S and always choosing the child with the largest label.
- When a vertex v that is a leaf or has no children in the set S is reached, stop. Let the label of v be r_i, remove v from S, and let S_i be the set $V(T) \setminus S$.
- Move the label r_i to the root of T by interchanging r_i with the label of its parent and repeating this process until r_i "slides" all the way to the root. Call this new labeling w_i.
- Continue this process for $i = 2, \ldots, n$.

Note that each new labeling w_i will be decreasing upon removing the root so it is clear that every vertex except the root will be a bottom-to-top maximum. An example of this construction is given in Figure 4.6.

Next, we will show that $\text{imaj}(T, w_i) = i$ by proving the following slightly stronger statement: the vertices in the set S_i are exactly the vertices counted when calculating $\text{imaj}(T, w_i)$.

To begin, consider a tree T of size n with decreasing labeling w found by adding a root labeled n to a forest F with a decreasing labeling as before. Now consider the new labeling w_1 found using the algorithm described above. Let $v_{i_1} \cdots v_{i_k}$, for $n = i_1 > \cdots > i_k$, with labelings

$n = w(v_{i_1}) > \cdots > w(v_{i_k}) = r_1$ be the path traversed in the tree T with labeling w to find the label r_1. Note that in the labeling w_1 this path is now labeled $r_1 < n > w(v_{i_2}) > \cdots > w(v_{i_{k-1}})$ from the root to the leaf v_{i_k}. Notice that the labeling w_1 will sort to the labeling w and that $\sigma = w_1 \circ (w^{-1}) = (r_1\ w(v_{i_{k-1}}) \cdots w(v_{i_3})\ w(v_{i_2})\ n)$. Therefore, $\sigma^{-1} = (n\ w(v_{i_2}) \cdots w(v_{i_{k-1}})\ r_1)$ and we see that the path traversed in the algorithm will be labeled $w(v_{i_2}) > \cdots > w(v_{i_{k-1}}) > r_1 < n$ (from root to leaf) in the labeling w_1^{inv} and the labels on the vertices not on this path do not change. Thus $\text{imaj}(T, w_1) = 1$, since the pair r_1 and n create the only descent and the vertex v_k is a leaf.

Now, assume that for all $i < j$ the vertices in S_i are the ones counted when computing $\text{maj}(T, w_i^{\text{inv}})$ and thus $\text{maj}(T, w_i^{\text{inv}}) = \text{imaj}(T, w_i) = i$. Use the algorithm to find r_j and the new labeling w_j. Let $v_{j_l} \in V(T)$ be the vertex such that $w(v_{j_m}) = r_j$. Suppose that r_j appears on the path $v_{j_1} \cdots v_{j_{l-1}} v_{j_l} v_{j_{l+1}} \cdots v_{j_m}$ for $n = j_1 > \cdots > j_{l-1} > j_l > j_{l+1} > \cdots > j_m$ with the labels $n = w(v_{j_1}) > \cdots > w(v_{j_{l-1}}) > w(v_{j_l}) > w(v_{j_{l+1}}) > \cdots > w(v_{j_m})$ where v_{j_m} is a leaf in T. Note that from the way the algorithm is defined we get the following properties:

1. every descendent of the vertex v_{j_l} must be in the set S_j,

2. if a vertex v_{j_s} for $s = 1, \ldots, j_{l-1}$ has a child $y \in S_j$, then $w(y) > w(v_{j_{s+1}})$ and all of the descendants of y are in S_j, and

3. the vertices $v_{j_1}, \ldots, v_{j_{l-1}}$ are not in S_j.

Sorting the tree T with labeling w_j gives $\sigma = w_j \circ (w^{-1}) = (r_j\ \ w(v_{j_{l-1}}) \cdots w(v_{j_2})\ \ n)$. Therefore $\sigma^{-1} = (n\ w(v_{j_2}) \cdots w(v_{j_{l-1}})\ r_j)$ and thus the same branch in T with the labeling w_j^{inv} is now labeled $w(v_{j_2}) > \cdots > w(v_{j_{l-1}}) > r_j < n > w(v_{j_{l+1}}) > \cdots > w(v_{j_m})$ from root to leaf.

The pair r_j and n create a descent, and when we add the hook length to the $\text{imaj}(T, w_j)$, the vertex v_{j_l} and all of its descendants are counted. Then for any vertex v_{j_s} for $s = 1, \ldots, l-1$ with a child $y \in S_j$, $w_j^{\text{inv}}(v_{j_s}) = w(v_{j_{s+1}}) < w(y) = w_j^{\text{inv}}(y)$ by property 2 above. Therefore, the pair $w_j^{\text{inv}}(v_{j_s})$ and $w_j^{\text{inv}}(y)$ create a descent in w_j^{inv} and adding this to $\text{maj}(T, w_j^{\text{inv}})$ counts the vertex y and all of its descendants. Thus we have counted all of the vertices in S_j.

Chapter 5

Descents

In this chapter we will generalize the Eulerian polynomials by considering the descent polynomials of labeled rooted forests.

Definition 5.0.1. For a rooted forest F, the *descent polynomial* is $A_F(q) = \sum_{l \in \mathcal{W}(F)} q^{\text{des}(F,l)}$.

In Section 5.1 we prove that the descent polynomial of a forest F is unimodal and in Section 5.2 we discuss log-concavity. Then in Section 5.3 we prove that the distribution of descents converges to the normal distribution as the size of the tree grows to infinity if we place a restriction on the maximum degree of the tree.

5.1 Unimodality

In this section, we prove that the descent polynomial of a rooted forest is unimodal. The following result on the product of symmetric unimodal polynomials is used in the proof. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is symmetric, or palindromic, if $a_i = a_{n-i}$ for $i = 0, 1, \ldots, n$.

Proposition 5.1.1 ([42])**.** *If $A(q)$ and $B(q)$ are symmetric unimodal polynomials with nonnegative coefficients, then so is $A(q)B(q)$.*

First we show that $A_F(q)$ is symmetric.

Lemma 5.1.2. *For a forest F, the descent polynomial $A_F(q)$ is symmetric.*

Proof. Let F be a forest and consider a labeling $w \in \mathcal{W}(F)$. Let $w' \in \mathcal{W}(F)$ be defined as $w'(x) = n + 1 - w(x)$. Then for a vertex x with a child y, clearly $w(x) < w(y)$ if and only if $w'(x) > w'(y)$.

Therefore every descent in (F, w) corresponds to an ascent in (F, w') and vice versa. Therefore, the number of labelings with k descents is equal to the number of labelings with $n - k$ descents. $\square$

Now we are ready for the main result of this section.

Theorem 5.1.3. *For a forest F, the descent polynomial $A_F(q)$ is unimodal.*

Proof. Suppose $A_F(q)$ is unimodal for all forests F of size less than or equal to n. Consider a forest F of size $n+1$ that consists of trees $T_1, \ldots, T_m$. We will consider the cases when $m > 1$ and $m = 1$ seperatly.

First suppose that $m > 1$, or in other words, F is not a tree. Then

$$A_F(q) = \binom{n}{k_1, \ldots, k_m} A_{T_1}(q) \cdots A_{T_m}(q)$$

for $k_i = |V(t_i)|$. $A_{T_i}(q)$ is unimodal for all $i = 1, \ldots, m$ by the inductive hypothesis, and thus by Proposition 5.1.1 $A_F(q)$ is unimodal.

Now suppose that $m = 1$, or in other words F, is a tree with $n+1$ vertices. For a vertex v, let $F_v = F - v$, or in other words, F_v is the tree F with vertex v and incident edges removed. For $l \in \mathcal{W}(F)$, let v be the vertex such that $l(v) = 1$. Consider the map defined by removing the vertex v to get the forest F_v and the labeling $l' \in \mathcal{W}(F_v)$ defined by $l'(y) = l(y) - 1$ for all vertices y in F_v . This defines a bijection from $\mathcal{W}(F)$ to the set of pairs of a vertex v and a labeling $l' \in \mathcal{W}(F_v)$. Notice that v creates a descent with all of its children since $l(v) = 1$ so we have

$$A_F(q) = \sum_{l \in \mathcal{W}(F)} q^{\operatorname{des}(l,F)} = \sum_{v \in F} \sum_{l' \in \mathcal{W}(F_v)} q^{\operatorname{des}(l',F_v)+d_v}$$

where d_v is the down-degree of v, i.e. the number of children of v.

Let $A_F(q) = a_0 + a_1 q + \cdots + a_n q^n$ and consider a_k and a_{k+1} for some $k < \lfloor \frac{n}{2} \rfloor$. Let $v_1, \ldots, v_{n+1}$ be the vertices of F, and let $A_{F_{v_i}}(q) = a_{i,0} + a_{i,1}q + a_{i,2}q^2 + \cdots + a_{i,e_i}q^{e_i}$, where e_i is the number of edges in F_{v_i}. The coefficient $a_{i,j}$ is the number of labelings of the forest F_{v_i} with j descents. Using the bijection described above, we can count the number of labelings of F with k descents by counting the number of labelings of F_{v_i} with $k - d_{v_i}$ descents for all $i = 1, \ldots, n$. This means that

$$a_k = a_{1,k-d_{v_1}} + \cdots + a_{n,k-d_{v_n}},$$

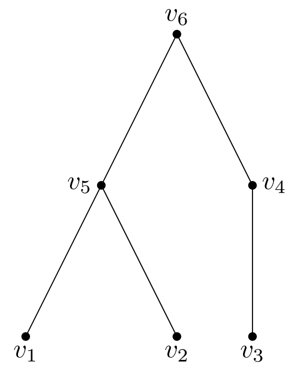


Figure 5.1: A tree T with a descent polynomial that does not have all real roots.

and similarly,

$$a_{k+1} = a_{1,k-d_{v_1}+1} + \cdots + a_{n,k-d_{v_n}+1}.$$

Let us consider a fixed i. If F_{v_i} has less than $k - d_{v_i}$ edges, then $a_{i,k-d_{v_i}} = a_{i,k-d_{v_i}+1} = 0$. If instead F_{v_i} has exactly $k - d_{v_i}$ edges, then $a_{i,k-d_{v_i}} > 0 = a_{i,k-d_{v_i}+1}$. We will consider the case where F_{v_i} has more than $k - d_{v_i}$ edges. Since we have $k < \lfloor \frac{n}{2} \rfloor$ we get $k + 1 \leq \lfloor \frac{n}{2} \rfloor$, and thus $2k \leq n - 2 = e_i + d_{v_i} + 1 - 2 = e_i + d_{v_i} + 1$. We have $2k - 2d_{v_i} < 2k - d_{v_i} \leq e_i - 1 < e_i$ and thus $k - d_{v_i} < \frac{e_i}{2}$. By the induction hypothesis, we have $a_{i,k-d_{v_i}} \leq a_{i,k-d_{v_i}+1}$. This holds for all i and thus we get $a_k \leq a_{k+1}$. By symmetry we get that $A_F(q)$ is unimodal. □

5.2 Log-Concavity Conjecture

As we mentioned in the introduction, there are several different types of proofs of the fact that the classical Eulerian polynomial is log-concave [42, 41]. It is known that the Eulerian polynomial has all real roots which implies that it is log-concave. The descent polynomials $A_F(q)$ do not in general have all real roots. For example, for the tree T in Figure 5.1, $A_T(q) = 20q^5 + 90q^4 + 250q^3 + 250q^2 + 90q + 20$ which has only one real root, $q = -1$.

The log-concavity of the Eulerian polynomial has also been proven combinatorially. Two examples are by Gasharov in [22] and by Bóna and Ehrenborg in [7]. These proofs use bijections between permutations of size n with k descents and the set $\mathcal{P}(n,k)$ of labeled northeastern lattice paths with n edges, exactly k of which are vertical. They construct a map from $\mathcal{P}(n,k-1) \times \mathcal{P}(n,k+1)$ to $\mathcal{P}(n,k) \times \mathcal{P}(n,k)$ that is injective but not surjective. Therefore we get $A(n,k)A(n,k+2) =$

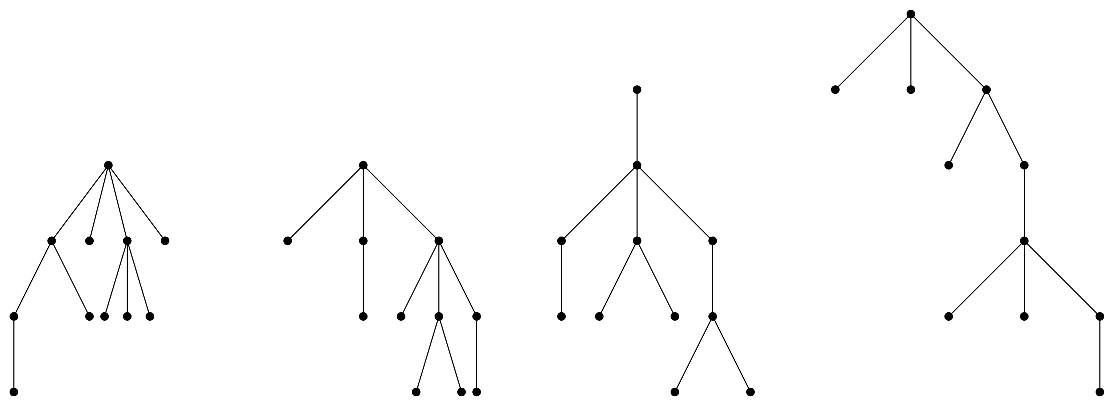


Figure 5.2: Some of the trees tested for log-concave descent polynomials

$|\mathcal{P}(n,k-1)||\mathcal{P}(n,k+1)| \leq |\mathcal{P}(n,k)|^2 = A(n,k+1)^2$.

Another method used to prove log concavity of a sequence is by interpreting the values in the sequence as mixed volumes of convex bodies. For convex bodies K and L in $\mathbb{R}^n$, the *Minkowski sum* is $K + L = \{x + y : x \in K, y \in L\}$ and for a real number γ, the *dilation* of K by γ is the set $\gamma \cdot K = \{\gamma x : x \in K\}$. Let V denote the n-dimensional volume. Minkowski showed that there are real numbers $V_i(K,L) \geq 0$ satisfying $V(\gamma K + \mu L) = \sum_{i=0}^{n} \binom{n}{i} V_i(K,L)\gamma^{n-i}\mu^i$ [42]. The number $V_i(K,L)$ is called the i^{th} *mixed volume* of K and L, and it has been proven by Aleksandrov [2] and Fenchel [16] independently that the sequence $V_0(K,L),\ldots,V_n(K,L)$ is log-concave. Therefore, any sequence of numbers that can be interpreted as mixed volumes of some convex bodies is log-concave. Ehrenborg, Readdy, and Steingrímsson use this technique in [13] to prove that a refinement of the Eulerian numbers $A(n,k,i)$ is log-concave, where $A(n,k,i)$ is the number of permutations in S_n ending with i an having k descents.

The following conjecture is based on running examples on randomly generated trees up to size 11.

Conjecture 5.2.1. *For a forest F the descent polynomial $A_F(q)$ is log-concave.*

Some of the trees we have checked are shown in Figure 5.2, and all of them had a log-concave descent polynomial. We have looked at generalizing the techniques mentioned above, but have not yet found a proof for this Conjecture.

5.3 Central Limit Theorem

We begin with some definitions and notation. For a random variable Z, let $\bar{Z} = Z - \mathrm{E}(Z)$, $\tilde{Z} = \frac{\bar{Z}}{\sqrt{\mathrm{Var}(Z)}}$, and we write $Z_n \to N(0,1)$ to mean that Z_n converges in distribution to the standard normal distribution as n grows.

Consider the random variable X_n which is counting the number of descents in a randomly generated labeling of a tree of size n. In this section, we show in the main Theorem given below that under some assumptions on the maximum degree of a tree, $\tilde{X_n} \to N(0,1)$ as $n \to \infty$.

Theorem 5.3.1. *Let $\{T_n\}$ be a sequence of trees of size n and X_n be the random variable that counts the number of descents in a random labeling of T_n. If $D_n \leq Cn^{\frac{1}{2}-\epsilon}$ for some constant C and some $0 < \epsilon < \frac{1}{2}$ where D_n is the maximum down degree in the tree T_n, then $\tilde{X_n} \to N(0,1)$.*

The main result used in the proof is Janson's dependency theorem, a central limit theorem. This method is related to the method used by Bóna to prove that the distribution of certain generalized descents on S_n also converges to the normal distribution [6].

Definition 5.3.2. Let $\{Y_{n,k} \mid k = 1, 2, \ldots\}$ be an array of random variables. Then a graph G is a dependency graph for $\{Y_{n,k} \mid k = 1, 2, \ldots\}$ if the following conditions are satisfied:

1. There exists a bijection between the random variables and the vertices of G, and
2. if V_1 and V_2 are disjoint sets of vertices of G such that no edge of G has one endpoint in V_1 and the other in V_2, then the corresponding sets of random variables are independent.

Note that the dependency graph for an array of random variables is not unique because if the graph is not complete one can add another edge to obtain a new dependency graph. We can now state Janson's dependency criterion.

Theorem 5.3.3 ([28])**.** *Let $Y_{n,k}$ be an array of random variables such that for all n, and for all $k = 1, \ldots, N_n$, the inequality $|Y_{n,k}| \leq A_n$ holds for some real number A_n, and that the maximum degree of a dependency graph of $\{Y_{n,k} | k = 1, \ldots, N_n\}$ is Δ_n. Set $Y_n = \sum_{k=1}^{N_n} Y_{n,k}$ and $\sigma_n^2 = \mathrm{Var}(Y_n)$. If there is a natural number m so that*

$$N_n \Delta_n^{m-1} \left(\frac{A_n}{\sigma_n} \right)^m \to 0, \tag{5.1}$$

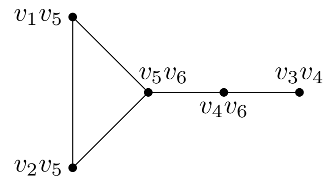


Figure 5.3: Dependency graph G of the tree T in Figure 5.1

then

$$\tilde{Y}_n \to N(0,1).$$

Fix an ordering of the edges of a tree T_n of size n. To apply Janson's criterion, let $Y_{n,k}$ be the indicator random variables $X_{n,k}$ of the event that the edge k is a descent in a randomly selected labeling of T_n. Thus $N_n = n-1$, the number of edges in a tree of size n. By the definition of $Y_{n,k}$, we have $|Y_{n,k}| \leq 1$ so we will set $A_n = 1$.

Now we will look at a dependency graph G for the random variables $X_{n,k}$ to get a bound on Δ_n. Certainly, the variables X_{n,k_1} and X_{n,k_2} are independent if and only if the edges k_1 and k_2 do not share a vertex. Therefore we can let the vertices of the dependency graph G be the edges of the tree T_n, and connect the edges that share an endpoint. Each vertex in G corresponds to an edge from the tree T_n and two vertices of G are adjacent if the corresponding edges of T_n share an endpoint. In other words, G is the line graph of the tree T_n. For example, Figure 5.3 shows this dependency graph for the tree in Figure 5.1. Let D_n denote the largest down-degree of a vertex in the tree T, then $\Delta_n \leq 2D_n$.

Lemma 5.3.4. *For a tree T with size n,*

$$\mathrm{Var}(X_n) = \frac{2d_r + \sum_{v\in T} d_v^2}{12}$$

where d_v is the down-degree of vertex v, and d_r is the down-degree of the root.

Proof.

$$\begin{aligned}
\mathrm{Var}(X_n) &= \mathrm{E}(X_n^2) - (\mathrm{E}(X_n)^2) && (5.2)\\
&= \mathrm{E}\left(\left(\sum_{k=1}^{n-1} X_{n,k}\right)^2\right) - \left(\mathrm{E}\left(\sum_{k=1}^{n-1} X_{n,k}\right)\right)^2 && (5.3)\\
&= \mathrm{E}\left(\left(\sum_{k=1}^{n-1} X_{n,k}\right)^2\right) - \left(\sum_{k=1}^{n-1} \mathrm{E}(X_{n,k})\right)^2 && (5.4)\\
&= \sum_{k_1,k_2} \mathrm{E}(X_{n,k_1}, X_{n,k_2}) - \sum_{k_1,k_2} \mathrm{E}(X_{n,k_1})\,\mathrm{E}(X_{n,k_2}), && (5.5)
\end{aligned}$$

where the sums run over all ordered pairs $(k_1, k_2) \in \{1, \ldots, n-1\} \times \{1, \ldots, n-1\}$.

Now, $\mathrm{E}(X_{n,k}) = \frac{1}{2}$ because as we saw in the proof of Corollary 5.1.2, an edge is a descent in half of the labelings of a given tree T, and therefore the $\mathrm{E}(X_{n,k_1})\,\mathrm{E}(X_{n,k_2})$ terms appearing in (5.5) are all equal to $\frac{1}{4}$. We will now calculate the values for the $\mathrm{E}(X_{n,k_1}, X_{n,k_2})$ terms in (5.5). If the edges k_1 and k_2 do not share a vertex, then they are independent and we get $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \mathrm{E}(X_{n,k_1})\,\mathrm{E}(X_{n,k_2}) = \frac{1}{4}$, and if $k_1 = k_2$ then $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \mathrm{E}(X_{n,k_1}^2) = \mathrm{E}(X_{n,k_1}) = \frac{1}{2}$. Let k_1 be the edge v_iv_j and k_2 be the edge v_rv_s. There are three cases left to consider: if $i = r$ we have the case shown in Figure 5.4a, if $i = s$ we have the case in Figure 5.4b, and if $j = r$ we have the case shown in Figure 5.4c. If $i = r$, then $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{1}{3}$ since $X_{n,k_1} = X_{n,k_2} = 1$ if and only if $w(v_i) < w(v_j)$ and $w(v_i) < w(v_s)$. There are 6 ways to order those labels and two of them satisfy that requirement, so $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{2}{6} = \frac{1}{3}$. Similarly, if $i = s$ or $j = r$ we see $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{1}{6}$.

Next, we will count how many of the terms $\mathrm{E}(X_{n,k_1}, X_{n,k_2})$ are $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$.

- We know that $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{1}{2}$ only when $k_1 = k_2$ and therefore this occurs $n-1$ times, once for each edge.

- Next, if $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{1}{3}$, we have the case from Figure 5.4a. For each vertex v in the tree T this will occur $d_v(d_v - 1)$ times since we have d_v choices for the first child and then $d_v - 1$ choices for the second. This occurs a total of $\sum_{v\in T} d_v(d_v - 1)$ times in the tree T.

- Lastly, if $\mathrm{E}(X_{n,k_1}, X_{n,k_2}) = \frac{1}{6}$, we have the cases shown in Figures 5.4b and 5.4c. If r is the root of T, then for each vertex $v_i \neq r$ in T this occurs $2d_v$ times. There are d_{v_i} choices for the child v_r and one choice for v_j and the edges could appear in either order. Therefore this case appears $\sum_{v\in T} 2d_v$ times throughout the tree T.

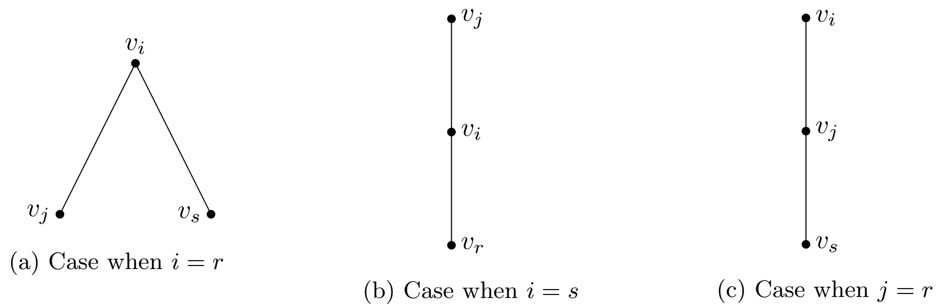


Figure 5.4: The cases where the edges $k_1 = v_iv_j$ and $k_2 = v_rv_s$ are adjacent

Plugging this information into (5.5), we get

$$\begin{aligned}
\operatorname{Var}(X_n) &= \left(\frac{1}{2}-\frac{1}{4}\right)(n-1) + \left(\frac{1}{2}-\frac{1}{6}\right)\sum_{v\in T} d_v(d_v-1) + \left(\frac{1}{2}-\frac{1}{3}\right)\sum_{\substack{v\in T\\ v\neq r}} 2d_v &(5.6)\\
&= \frac{1}{4}(n-1) + \frac{1}{12}\sum_{v\in T} d_v(d_v-1) - \frac{1}{12}\sum_{\substack{v\in T\\ v\neq r}} 2d_v &(5.7)\\
&= \frac{3(n-1) + 2d_r + \sum_{v\in T} d_v^2 - \sum_{v\in T} d_v - \sum_{v\in T} 2d_v}{12} &(5.8)\\
&= \frac{3(n-1) + 2d_r + \sum_{v\in T} d_v^2 - 3\sum_{v\in T} d_v}{12} &(5.9)\\
&= \frac{3(n-1) - 3(n-1) + 2d_r + \sum_{v\in T} d_v^2}{12} &(5.10)\\
&= \frac{2d_r + \sum_{v\in T} d_v^2}{12}. &(5.11)
\end{aligned}$$

□

Using the variance calculated above and the values we found for N_n, Δ_n, and A_n we can now apply Janson's criterion to prove the following theorem.

Proof of Theorem 5.3.1. In Lemma 5.3.4, we showed that $\operatorname{Var}(X_n) = \frac{2d_r + \sum_{v\in T} d_v^2}{12}$. At least one of the vertices in the tree must have down-degree D_n. If v^* is one such vertex, then we get

$$\begin{aligned}
\operatorname{Var}(X_n) &= \frac{2d_r + \sum_{v\in T} d_v^2}{12} = \frac{2d_r + D_n^2 + \sum_{v\neq v^*} d_v^2}{12} \geq \frac{2d_r + D_n^2 + \sum_{v\neq v^*} d_v^2}{12}\\
&= \frac{2d_r + D_n^2 + (n-1-D_n)}{12} \geq \frac{D_n^2 + n - 1 - D_n + 2}{12} = \frac{n + D_n^2 - D_n + 1}{12}.
\end{aligned} \tag{5.12}$$

Note that this bound is tight when the maximum degree does not appear at the root, and the rest of the vertices have down-degree one.

To apply Janson's criterion with $N_n = n-1$, $\Delta_n \le 2D_n$, $A_n = 1$, and the estimate (5.12), we need to show there is a natural number m such that

$$(n-1)(2D_n)^{m-1}\left(\frac{12}{n+D_n^2-D_n+1}\right)^{\frac{m}{2}} \to 0.$$

It suffices to show that there is a natural number m such that

$$\frac{nD_n^{m-1}}{(n+D_n^2-D_n+1)^{\frac{m}{2}}} = \frac{\frac{nD_n^{m-1}}{n^{\frac{m}{2}}}}{(1+\frac{D_n^2}{n}-\frac{D_n}{n}+\frac{1}{n})^{\frac{m}{2}}} \to 0. \tag{5.13}$$

Clearly $\lim_{n\to\infty}\dfrac{1}{n} = 0$, and so we have

$$\frac{D_n}{n} \le \frac{D_n^2}{n} \le \frac{C^2(n^{\frac{1}{2}-\epsilon})^2}{n} \to 0$$

for $0 < \epsilon < \frac{1}{2}$. This means that

$$\frac{nD_n^{m-1}}{n^{\frac{m}{2}}} = \frac{D_n^{m-1}}{n^{\frac{m}{2}-1}} \le \frac{C^{m-1}n^{(\frac{1}{2}-\epsilon)(m-1)}}{n^{\frac{m}{2}-1}} \to 0 \tag{5.14}$$

if $\frac{m}{2}-1 > (\frac{1}{2}-\epsilon)(m-1) = \frac{1}{2}m - \frac{1}{2} - \epsilon(m-1)$. In other words, (5.14) holds if $\epsilon > \frac{1}{2(m-1)}$.

Note that $\frac{1}{2(m-1)}$ approaches 0 as $m\to\infty$ and thus for any $0<\epsilon<\frac{1}{2}$, there will be some $m \ge 3$ such that (5.13) holds. Therefore, by Janson's criterion, we get that $\tilde{X}_n \to N(0,1)$.

□

Chapter 6

Conclusion and Future Directions

In this thesis we studied many different generalizations of permutation statistics.

In Chapter 3, motivated by Foata and Zeilberger's graphical major index and inversions [21], we defined a graphical sorting index. We began by strengthening Foata and Zeilberger's result and showing that the graphical major index and inversion number are equidistributed for a fixed rearrangement class if and only if the relation U is conditionaly bipartitional, and the main result of the chapter is the classification of the types of relation U that give rise to the equidistribution of all three statistics. There are still many open questions with regards to the graphical statistics. One future direction is generalizing these statistics to signed or colored permutations of multisets. Another question to investigate, is how we could define generalizations of the Stirling statistics, and study the joint distributions of the graphical Stirling statistics with the graphical Mahonian statistics.

We defined the statistics sor, BT-max, Cyc, and CBT-max for labeled forests in Chapter 4. Björner and Wachs [4] defined maj for labeled forests, and showed that it is equidistributed with inv, defined by Mallows and Riordan [11], over all labelings of a fixed forest. Our main results show that the pairs $(\mathrm{inv}, \mathrm{BT\text{-}max})$, $(\mathrm{sor}, \mathrm{Cyc})$, and $(\mathrm{maj}, \mathrm{CBT\text{-}max})$ are equidistributed over all labelings of a forest, and that the pairs $(\mathrm{inv}, \mathrm{BT\text{-}max})$ and $(\mathrm{sor}, \mathrm{Cyc})$ are equidistridued over all signed labelings of a forest. In Section 4.4 we discussed the conjecture that the pair $(\mathrm{imaj}, \mathrm{BT\text{-}max})$ is equdistributed with $(\mathrm{inv}, \mathrm{BT\text{-}max})$ and $(\mathrm{sor}, \mathrm{Cyc})$ over all labelings of a forest. There are many other open questions such as generalizing other Mahonian statistics, or considering statistics for labelings of more general posets.

Lastly, in Chapter 5 we discussed the properties of the Eulerian polynomial that are preserved in the descent polynomial of labeled forests. We proved that the descent polynomial of a forest is unimodal, and discussed a conjecture that it is log-concave. In Section 5.3 we proved for a sequence of random trees with a bound on the maximum degree of each tree that as the size of the tree approaches infinity, the descent distribution converges to the normal distribution.

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