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Dynamic Pricing for Competing Sellers

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Dynamic pricing for competing sellers

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Abstract

To optimize profit, pricing is of great importance for each company, especially when competitors exist. The optimal pricing strategy we are interested in is to achieve Nash equilibrium (NE) to prevent malignant competition. In this work, we study dynamic pricing for a duopoly with two competing sellers, each of which sells one product, using a simple linear model which incorporates the competition effect. Motivated from the modified pricing policy constructed in Liu and Cooper [12], we propose a policy, referred to as *randomized certainty equivalent pricing (RCEP)* policy, under which each seller applies *certainty equivalent pricing (CEP)* policy for most of the times and occasionally choose prices around the previous price according to uniform distribution. We use numerical experiments to investigate the convergence of the prices to NE under RCEP, and our results suggest that RCEP is optimal with probability 1. We also study the so-called *controlled variance pricing (CVP)* originally proposed by den Boer and Zwart [4] for the monopoly case. The essential idea of CVP is to apply CEP for most of the time, and during a time period, if the sample variance of the seller's prices is too small, the next price will be chosen to slightly deviate from the current price average to keep the sample variance large enough. The CVP policy is simple to apply, and is shown to be optimal in the monopoly case. However, it is still unknown whether the prices in the duopoly case converge to NE under CVP. Our numerical results show that CVP is actually not optimal with positive probability. We also conduct simulations under CEP and the policy proposed in [12], which support the theoretical results in [12].

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Chapter 1

Introduction

Competition plays an imperative role in modern market. However, in settings with multiple competing sellers, each seller typically uses models as if the seller is a monopolist to determine the selling price. Such models cannot explicitly take the competition effect from competitors into account. In this work, we study dynamic pricing for a duopoly with two competing sellers, each of which sells one product, using a simple linear model which incorporates the competition effect. More precisely, we model the demand of each product as a linear function of both products' prices with unknown parameters. We are interested in a pricing strategy which eventually achieves Nash equilibrium (NE). In practice, even if the NE exists and is unique, the sellers may not know what price the competitor takes or may doubt that the competitor will take the price in the NE. Thus both sellers' prices may deviate from the NE. Also in our model, the sellers initially have no information about the model parameters, and need to learn their values by experimenting the selling prices. Thus we consider the following two steps: Consider a sequence of time periods $\{1, 2, \dots, T\}$,

(i) at each time period, each seller uses the linear model, the observed demands, and the past prices of both sellers to derive the least square estimators of the model parameters;

(ii) since the two sellers don't cooperate, with the estimated parameters, each seller chooses price for the next time period by Cournot adjustment, namely, each seller chooses price that is the best response to the price chosen by the competing seller in the previous period.

Such intuitive pricing policy is called *certainty equivalent pricing (CEP)*. We are interested in the convergence of the price processes to the NE prices associated with the correct model. However, it is shown in Liu and Cooper [12] that, under certain initial conditions, such processes do not converge to the NE with positive probability. They also establish a sufficient condition for the convergence (see Theorem 2.1.1), which is given in terms of the sample covariance matrix of the two price processes, and then propose a modified policy, which allow each seller to apply CEP for most of the times and occasionally choose prices from a feasible price interval according to a probability distribution. Such occasional random price selection is independent of his/her past prices and the competitor's past and current prices. Under such modified pricing policy, it is shown that the estimated parameters converge to the true parameters in probability, and the average price for each seller converges to the price in the NE in probability. Another interesting pricing policy called *controlled variance pricing (CVP)* is proposed by den Boer and Zwart [4] for the monopoly case. The essential idea is to apply CEP for most of the time, however, during a time period, if the sample variance of the seller's prices is too small, then the next price will be chosen to slightly deviate from the current price average to keep the sample variance large enough. (The CEP in the monopoly setting is the same as that in the duopoly setting except at each time period it chooses the optimal price associated with the estimated parameters.) The CVP policy is simple to apply, and is shown to be optimal in the monopoly case. However, it is still unknown whether the prices in the duopoly case converge to NE under CVP.

Motivated from the modified pricing policy constructed in Liu and Cooper [12], we propose a policy, under which each seller applies CEP for most of the times and occasionally choose prices around the previous price according to uniform distribution, and such policy will be referred to as *randomized certainty equivalent pricing (RCEP)* policy. We believe such policy is more reasonable and applicable in practice. In this work, we use numerical experiments to investigate the convergence of the prices to NE under RCEP and CVP. For each policy, we randomly generate 30 sample paths and set up the total number of time periods $T = 100,000$. As the numerical results shows, for 30 sample paths, there are only 20% of them converging to NE price under CVP, and under RCEP, we observe very good parameter convergence for all sample paths and price convergence in 28 sample paths. Thus we conjecture that RCEP is optimal in the duopoly case with probability 1 and CVP is not optimal with positive probability. One of our future directions is to rigorously prove the optimality of RCEP using techniques from Markov processes. We also perform numerical experiments under CEP and the modified pricing policy proposed in Liu and Cooper [12], and the results support the theoretical results obtained in [12].

There is a large collection of work on dynamic pricing. Lobo and Boyd [13] propose a convex approximation of the optimal price of monopolistic pricing over multiple time period. A linear demand function is assumed, and the accuracy of the model depend on the locality of the solution. Also, it requires the knowledge about the distribution on the demand parameters. In Carvalho and Puterman [5], they deal with the pricing problem with unknown demand distribution parameters. They applied Taylor series expansion to the future reward function to illustrate the trade-off between maximizing instant revenue and future information exploration, and suggest a pricing policy, which is referred to as one-step look ahead rule. Later on, Carvalho and Puterman [6] improve their work in 2004 by providing several methods that determine current price according to the sequence of prices of past prices. In Bertsimas and Perakis [2], they assume the parametric families of the demand function, in terms of price, are learned over time. First, they propose a dynamic programming algorithm to increase the computational intensity for jointly estimating the demand and setting price in the noncompetitive case. Then, a competitive oligopolistic case is considered, in which they introduce a more complicated model of demand learning, and methods of estimating other competitors' demand and price setting. Araman and Caldentey [1] model the uncertainty in the demand rate by a single factor θ, and in infinite many time periods, by Bayesian learning, the distribution of θ is updated after every price setting for maximizing the revenue. Farias and Van Roy [8] also consider the pricing problem in over an infinite time horizon, and they propose a new heuristic approaches, called Decay Balancing, which could achieve near-optimal performance on problems with high levels of uncertainty in market response, and compare it with other heuristic methods. In Broder and Rusmevichientong [3], they propose a forced-exploration policy (MLE-Cycle) based on maximum-likelihood estimation for general cases, which is shown to achieve the optimal $O(\sqrt{T})$ order of regret, and for a well-separated demand family case, they show that a myopic maximum-likelihood policy (MLE-GREEDY) achieve the optimal $O(\log T)$ order of regret. For duopoly case, Keller and Rady [9] characterize the so-called Markov Perfect Equilibria (MPE) which can be of two types: If the value of information is low, they charge the static duopoly price; otherwise, a mixed strategies will be applied to create price dispersion to increase the information content.

The rest of this paper is organized as follows. The linear model of demand is described in Chapter 2, followed by discussion about the sufficient conditions for parameter convergence in 2.1, then the optimal price, Nash equilibrium (NE) price, and the relation between parameter convergence and price convergence is presented in 2.2. In 2.3, we proposed Randomized certainty equivalent pricing policy (RCEP) which is a modification of Liu and Cooper [12]. Chapter 3 contains all the numerical experiments. First, in 3.1, we numerically verify that under RCEP both parameter and price estimates converge to NE price. Then we show that the Controlled variance pricing policy (CVP) can not provide convergence for duopoly case in 3.2. In 3.3, we present experiment results to prove the divergence of price and parameter. At the end of Chapter 3, we verify that the pricing policy proposed in [12] results in good convergence of price and parameter if we randomly select price in the continuous short time period, but it fails for discrete short time period because the convergence is too slow.

Chapter 2

Problem Formulation

We consider the dynamic pricing for a duopoly with two sellers, which are called seller 1 and seller -1. Each seller sells a product. Let $i = \pm 1$. We call the product of seller i product i. Denote by p_i and d_i the price and demand of product i, respectively. We study discrete time periods indexed by $k \in \mathbb{N}$. At the beginning of each time period k, the sellers need to choose a price $p_i^k \in [p_{i,l}, p_{i,h}]$, which yields an observation of the demand d_i^k at the end of time period k.The prices $p_{i,l}$ and $p_{i,h}$ are the lowest and highest acceptable prices for sell i. In this work, we suppose a linear model between demands and prices, which can be formulated by a multiple linear regression model

$$d_i^k = \beta_{i,0} + \beta_{i,i} p_i^k + \beta_{i,-i} p_{-i}^k + \epsilon_i^k, \ k \in \mathbb{N}, \tag{2.1}$$

where $\beta_{i,0}, \beta_{i,i}, \beta_{i,-i} \in \mathbb{R}$ are the model parameters, and $\{\epsilon_i^k\}$ are i.i.d normal random errors with mean 0 and variance ς_i^2. We assume $\beta_{i,0} > 0, \beta_{i,i} < 0, \beta_{i,-i} \geq 0$. For $k \in \mathbb{N}$, define the following matrices:

$$X_i^k = \begin{pmatrix} 1 & p_i^1 & p_{-i}^1 \\ 1 & p_i^2 & p_{-i}^2 \\ \vdots & \vdots & \vdots \\ 1 & p_i^k & p_{-i}^k \end{pmatrix}_{k\times 3}, \; D_i^k = \begin{pmatrix} d_i^1 \\ d_i^2 \\ \vdots \\ d_i^k \end{pmatrix}_{k\times 1}, \; E_i^k = \begin{pmatrix} \epsilon_i^1 \\ \epsilon_i^2 \\ \vdots \\ \epsilon_i^k \end{pmatrix}_{k\times 1}, \text{ and } \beta_i = \begin{pmatrix} \beta_{i0} \\ \beta_{ii} \\ \beta_{i,-i} \end{pmatrix}.$$

We then write the linear regression (2.1) in the matrix form: For $i = \pm 1$,

$$D_i^k = X_i^k \beta_i + E_i^k, \; k \in \mathbb{N}.$$

2.1 Estimation of parameters

At each time period $k \in \mathbb{N}$, the least square estimate for β_i can be obtained as

$$\hat{\beta}_i^k = [(X_i^k)' X_i^k]^{-1} (X_i^k)' D_i^k.$$

Define the following sample moments of $\{p_i^j : i = \pm 1, 1 \le j \le k\}$:

$$\bar{p}_i^k = \frac{1}{k}\sum_{j=1}^k p_i^j, \;\; \bar{p}_{-i}^k = \frac{1}{k}\sum_{j=1}^k p_{-i}^j, \;\; \overline{p_i^2}^k = \frac{1}{k}\sum_{j=1}^k (p_i^j)^2, \;\; \overline{p_{-i}^2}^k = \frac{1}{k}\sum_{j=1}^k (p_{-i}^j)^2, \;\; \overline{p_i p_{-i}}^k = \frac{1}{k}\sum_{j=1}^k p_i^j p_{-i}^j.$$

By some elementary algebra calculations, we observe that

$$\Sigma_i^k := (X_i^k)' X_i^k = k \begin{pmatrix} 1 & \bar{p}_i^k & \bar{p}_{-i}^k \\ \bar{p}_i^k & \overline{p_i^2}^k & \overline{p_i p_{-i}}^k \\ \bar{p}_{-i}^k & \overline{p_i p_{-i}}^k & \overline{p_{-i}^2}^k \end{pmatrix}. \tag{2.2}$$

For an arbitrary square matrix M, denote by $\lambda_{max}(M)$ and $\lambda_{min}(M)$ the minimum and maximum eigenvalues of M. The following convergence result for $\hat{\beta}_i^k$ is an immediate consequence from Lai and Wei (1982).

Proposition 2.1.1 (Lai and Wei [11]). *Assume that as $k \to \infty$, $\lambda_{max}(\Sigma_i^k), \lambda_{min}(\Sigma_i^k) \to \infty$ a.s. and*

$$\frac{\log(\lambda_{max}(\Sigma_i^k))}{\lambda_{min}(\Sigma_i^k)} \to 0 \quad a.s. \tag{2.3}$$

Then the least square estimate $\hat{\beta}_i^k$ converges to β_i and in fact

$$\|\hat{\beta}_i^k - \beta_i\| = O\left(\sqrt{\frac{\log(\lambda_{max}(\Sigma_i^k))}{\lambda_{min}(\Sigma_i^k)}}\right) \; a.s.$$

Define the sample covariance matrix V_k of $\{(p_1^j, p_{-1}^j) : 1 \le j \le k\}$ as follows.

$$V_k = \begin{pmatrix} \widehat{\mathrm{Var}}^k(p_1) & \widehat{\mathrm{Cov}}^k(p_1, p_{-1}) \\ \widehat{\mathrm{Cov}}^k(p_1, p_{-1}) & \widehat{\mathrm{Var}}^k(p_{-1}) \end{pmatrix},$$

where

$$\widehat{\mathrm{Var}}^k(p_i) = \frac{1}{k}\sum_{j=1}^{k}(p_i^j - \bar{p}_i^k)^2 = \overline{p_i^2}^k - (\bar{p}_i^k)^2, \;\; i = \pm 1,$$

and

$$\widehat{\mathrm{Cov}}^k(p_1, p_{-1}) = \frac{1}{k}\sum_{j=1}^{k}(p_1^j - \bar{p}_1^k)(p_{-1}^j - \bar{p}_{-1}^k) = \overline{p_1 p_{-1}}^k - \bar{p}_1^k \bar{p}_{-1}^k.$$

Using Proposition 2.1.1, Liu and Cooper [12] establish the following sufficient condition for the convergence of $\hat{\beta}_i$.

Theorem 2.1.1 (Liu and Cooper [12]). *For any pricing policy, if*

$$\frac{\log(k)}{k\lambda_{min}(V_k)} \to 0, \tag{2.4}$$

then we have

$$\|\hat{\beta}_i^k - \beta_i\|^2 = O\left(\frac{\log(k)}{k\lambda_{min}(V_k)}\right).$$

Define

$$Z_k = \frac{\max\left\{\widehat{\mathrm{Var}}^k(p_1), \widehat{\mathrm{Var}}^k(p_{-1})\right\}}{\widehat{\mathrm{Var}}^k(p_1)\widehat{\mathrm{Var}}^k(p_{-1}) - (\widehat{\mathrm{Cov}}^k(p_1, p_{-1}))^2}.$$

and note that

$$Z_k \leq \frac{1}{\lambda_{min}(V_k)} \leq 2Z_k.$$

Then if

$$\frac{\log(k)}{k} Z_k \to 0, \tag{2.5}$$

we have

$$\|\hat{\beta}_i^k - \beta_i\|^2 = O\left(\frac{\log(k)}{k} Z_k\right).$$

2.2 Nash equilibrium (NE)

When the two sellers do not cooperate, a typical solution concept is a Nash equilibrium (NE). In a NE, each seller chooses a price that is the best response to its competitor's price. In our linear model, the best response of seller i to its competitor with price p_{-i} is given by

$$p_i = \arg\max_{p_i \geq 0} \left[p_i(\beta_{i,0} + \beta_{i,i} p_i + \beta_{i,-i} p_{-i})\right] = -\frac{\beta_{i,0} + \beta_{i,-i} p_{-i}}{2\beta_{i,i}}. \tag{2.6}$$

Solving (2.6) simultaneously for $i = \pm 1$, the unique NE prices are given as follows.

$$p_i^{NE} = \frac{\beta_{-i,0}\beta_{i,-i} - 2\beta_{i,0}\beta_{-i,-i}}{4\beta_{-i,-i}\beta_{i,i} - \beta_{-i,i}\beta_{i,-i}}.$$

We assume that $p_i^{NE} \in [p_{i,l}, p_{i,h}]$. In certainty equivalent pricing, for $k \geq 1$, using Cournot adjustment, define

$$\tilde{p}_i^{k+1} = \arg\max_{p_i \geq 0} \left[p_i(\hat{\beta}_{i0}^k + \hat{\beta}_{ii}^k p_i + \hat{\beta}_{i,-i}^k p_{-i}^k)\right] = -\frac{\hat{\beta}_{i0}^k + \hat{\beta}_{i,-i}^k p_{-i}^k}{2\hat{\beta}_{ii}^k}.$$

We then set

$$p_i^{k+1} = \min\{p_{ih}, \max\{p_{il}, \tilde{p}_i^{k+1}\}\}.$$

Proposition 2.2.1 (Cooper, Homem-de-Mello, and Kleywegt [7]). *Suppose* $\hat{\beta}_i^k \to \bar{\beta}_i$ *for some* $\bar{\beta}_i$

such that $\bar{\beta}_{1,-1} < -2\bar{\beta}_{1,1}$ *and* $\bar{\beta}_{-1,1} < -2\bar{\beta}_{-1,-1}$. *Then*

$$p_i^k \to \frac{\bar{\beta}_{-i,0}\bar{\beta}_{i,-i} - 2\beta_{i,0}\bar{\beta}_{-i,-i}}{4\bar{\beta}_{-i,-i}\bar{\beta}_{i,i} - \bar{\beta}_{-i,i}\bar{\beta}_{i,-i}}.$$

The following proposition says certainly equivalent pricing is asymptotically inconsistent with positive probability.

Proposition 2.2.2 (Liu and Cooper [12])**.** *Fix* $i = \pm 1$. *Let* p_i^{NE} *be the Nash equilibrium price and* $p_i^j, j = 1, 2, 3,$ *be the initial three prices. Assume that* (i) $p_i^{NE} \in (p_{i,l}, p_{i,h})$*;* (ii) $p_i^3 = p_{i,h}$*;* (iii) X_i^3 *and* Σ_i^3 *are nonsingular. Then the convergence of* p_i^k *to* p_i^{NE} *fails with positive probability.*

Example 2.2.1 (Liu and Cooper [12])**.** *Let* $p_i^k, k = 1, 2, 3$ *satisfy the conditions in Proposition 2.2.2 and let* $p_i^k = p_{i,h},\ k \geq 4$. *Then we have*

$$\widehat{\mathrm{Var}}^k(p_i) = \frac{1}{k}\sum_{j=1}^{3}(p_i^j - p_{ih})^2 - \frac{1}{k^2}\left(\sum_{j=1}^{3}(p_i^j - p_{ih})\right)^2,$$

$$\widehat{\mathrm{Var}}^k(p_{-i}) = \frac{1}{k}\sum_{j=1}^{3}(p_{-i}^j - p_{-ih})^2 - \frac{1}{k^2}\left(\sum_{j=1}^{3}(p_{-i}^j - p_{-ih})\right)^2,$$

$$\widehat{\mathrm{Cov}}^k(p_i, p_{-i}) = \frac{1}{k}\sum_{j=1}^{3}(p_i^j - p_{ih})(p_{-i}^j - p_{-ih}) - \frac{1}{k^2}\left(\sum_{j=1}^{3}(p_i^j - p_{ih})\right)\left(\sum_{j=1}^{3}(p_{-i}^j - p_{-ih})\right).$$

As $k \to \infty$,

$$\widehat{\mathrm{Var}}^k(p_i) \to 0, \widehat{\mathrm{Var}}^k(p_{-i}) \to 0, \widehat{\mathrm{Cov}}^k(p_i, p_{-i}) \to 0,$$

and

$$\begin{aligned}&\frac{1}{k}\frac{\max\left\{\widehat{\mathrm{Var}}^k(p_1), \widehat{\mathrm{Var}}^k(p_{-1})\right\}}{\widehat{\mathrm{Var}}^k(p_1)\widehat{\mathrm{Var}}^k(p_{-1}) - \widehat{\mathrm{Cov}}^k(p_1, p_{-1})^2} \\ &\to \frac{\max\left\{\sum_{j=1}^3(p_i^j - p_{i,h})^2, \sum_{j=1}^3(p_{-i}^j - p_{-i,h})^2\right\}}{\sum_{j=1}^3(p_i^j - p_{i,h})^2\sum_{j=1}^3(p_{-i}^j - p_{-i,h})^2 - (\sum_{j=1}^3(p_i^j - p_{i,h})(p_{-i}^j - p_{-i,h}))^2}.\end{aligned}$$

It's clear that sufficient condition in Theorem 2.1.1 doesn't hold.

2.3 Randomized certainty equivalent pricing

In this section, we introduce the modified pricing policy proposed in Liu and Cooper [12]. For $k \in \mathbb{N}$, let $\mathcal{K} = \{1, 2, \ldots, k\}$ and $\mathcal{K}_i \subset \mathcal{K}$. Denote by $|\mathcal{K}_i|$ the cardinality of $\mathcal{K}_i$, and assume as $k \to \infty$, there exist $\kappa_i \in (0, \infty)$ such that

$$\frac{|\mathcal{K}_i|}{k^{\alpha_i}} \to \kappa_i, \tag{2.7}$$

where $\alpha_i \in (0, 1)$. Under the policy proposed in [12], seller i uses certainty equivalent pricing policy for time periods in $\mathcal{K} - \mathcal{K}_i$, and chooses prices from interval $[p_{i,l}, p_{i,h}]$ according to a probability distribution in time periods T_i, which has mean $\mu_i \in [p_{i,l}, p_{i,h}]$ and finite variance $\sigma_i^2 \in (0, \infty)$. The main results from [12] are as follows.

Theorem 2.3.1 (Liu and Cooper [12]). *Let* $\alpha = \max\{\alpha_1, \alpha_{-1}\}$. *Then*

$$\|\hat{\beta}_i^k - \beta_i\|^2 = O(k^{-\alpha} \log(k)).$$

Furthermore, suppose $p_i^{NE} \in [p_{i,l}, p_{i,h}]$, $\beta_{1,-1} < -2\beta_{1,1}$, *and* $\beta_{-1,1} < -2\beta_{-1,-1}$. *Then*

$$\bar{p}_i^k \to p_i^{NE}, \text{ with probability } 1, \tag{2.8}$$

and

$$k^{\theta} \left| \bar{p}_i^k - p_i^{NE} \right| \to 0, \text{ with probability } 1, \tag{2.9}$$

where $\theta \in (0, \alpha/2)$ *and* $\alpha + \theta < 1$.

We propose a policy, referred to as randomized certainty equivalent pricing (RCEP), which is the same as the above policy except that for each time period t in $\mathcal{K}_i$, seller i will select price from a neighborhood of the previous price, i.e., $[p_i^{t-1} - \tau, p_i^{t-1} + \tau]$, according to uniform distribution, where ϵ is a small positive constant. In next chapter, we use numerical experiments to investigate the convergence of prices under RCEP.

Chapter 3

Numerical Experiments

We numerically investigate the performances of randomized certainty equivalent pricing policy, controlled variance pricing policy, certainty equivalent pricing policy, and the policy proposed in Liu and Cooper [12] by computing the difference between the simulated prices and Nash price. For each policy, we randomly generate 30 different sample paths and set up the initial prices for first three time periods and the total number of time periods as $T = 100,000$. The lowest and highest possible prices for seller 1 are set to be $p_{1,l} = 1$,$p_{1,h} = 15$, and for seller -1 are $p_{-1,l} = 1$, $p_{-1,h} = 10$. The sufficient conditions in Theorem 2.1.1 for the convergence of parameters will be checked out for each policy. More precisely, we compute the left hand side of (2.5) for each sample path, and observe whether these values are close to 0. For all the policies, the true values of parameters are

$$\beta_{1,0} = 15,\ \beta_{1,1} = -1,\ \beta_{1,-1} = 0.5,\ \beta_{-1,0} = 20,\ \beta_{-1,-1} = -2,\ \beta_{-1,1} = 0.5,$$

and the prices in Nash equilibrium are

$$p_1^{NE} = 9.0323,\ p_{-1}^{NE} = 6.1290.$$

Furthermore, the price errors are computed as follows.

$$\text{Price error } = \frac{\text{Simulated value - True value}}{\text{True value}}.$$

The parameter errors are defined in the same way. We decide that if the relative price error is less than 0.01, we say that the price converges to NE price, and the prices that diverge from NE prices are shaded in the tables for your convenience.

3.1 Randomized certainty equivalent pricing

We choose $\tau = 0.01$. The simulation results are summarized in the following two tables. We can see that both parameters and prices have very good approximations (only two sample paths have price errors slightly greater than 0.01).

Table 3.1: Simulated Prices of Seller 1 using RCEP

Sample	Prices	Price Err.	Parameter Err. ($\beta_{-1,0}$,$\beta_{-1,-1}$,$\beta_{-1,1}$)		
1	9.0371	0.0005	-0.0005	-0.0019	-0.0029
2	9.0735	0.0046	-0.0050	-0.0098	-0.0048
3	9.0961	0.0071	-0.0066	-0.0128	-0.0058
4	9.0889	0.0063	-0.0059	-0.0116	-0.0053
5	9.0211	-0.0012	0.0019	0.0035	0.0010
6	9.0514	0.0021	-0.0023	-0.0034	0.0010
7	9.0133	-0.0021	0.0008	0.0035	0.0064
8	9.0256	-0.0007	-0.0005	-0.0011	-0.0008
9	9.0249	-0.0008	0.0017	0.0032	0.0012
10	9.0527	0.0023	-0.0005	-0.0008	0.0001
11	9.0753	0.0048	-0.0021	-0.0079	-0.0130
12	9.0546	0.0025	0.0004	-0.0042	-0.0143
13	8.9501	-0.0091	0.0105	0.0182	0.0016
14	9.1133	0.0090	-0.0087	-0.0166	-0.0068
15	9.0173	-0.0017	-0.0002	0.0018	0.0062
16	8.9734	-0.0065	0.0057	0.0111	0.0044
17	9.0058	-0.0029	0.0033	0.0019	-0.0108
18	9.1367	0.0116	-0.0095	-0.0205	-0.0146
19	9.1378	0.0117	-0.0110	-0.0228	-0.0143
20	9.0203	-0.0013	0.0004	0.0018	0.0035
21	9.0720	0.0044	-0.0045	-0.0086	-0.0036
22	9.0988	0.0074	-0.0080	-0.0142	-0.0029
23	9.0122	-0.0022	0.0022	0.0043	0.0023
24	9.0602	0.0031	-0.0021	-0.0058	-0.0069
25	9.0898	0.0064	-0.0073	-0.0133	-0.0035
26	9.0245	-0.0009	0.0006	-0.0004	-0.0043
27	9.0153	-0.0019	0.0000	0.0006	0.0018
28	9.0873	0.0061	-0.0072	-0.0121	-0.0010
29	9.0377	0.0006	-0.0025	-0.0018	0.0071
30	9.1155	0.0092	-0.0109	-0.0212	-0.0098

Table 3.2: Simulated Prices of Seller -1 using RCEP

Sample	Prices	Price Err.	Parameter Err. ($\beta_{-1,0}$,$\beta_{-1,-1}$,$\beta_{-1,1}$)			Sufficient Conditions
1	6.112581854	-0.0027	0.0019	-0.0079	0.0003	0.1876
2	6.117432482	-0.0019	-0.0037	0.0197	0.0012	0.1973
3	6.148655641	0.0032	0.0013	-0.0046	0.0003	0.4332
4	6.131923121	0.0005	0.0091	-0.0468	-0.0024	0.1694
5	6.141509738	0.0020	0.0044	-0.0189	0.0002	0.1270
6	6.139035335	0.0016	-0.0013	0.0025	-0.0012	0.2253
7	6.114348422	-0.0024	-0.0044	0.0206	0.0003	0.1881
8	6.089862769	-0.0064	0.0012	-0.0061	-0.0003	0.1297
9	6.085672896	-0.0071	-0.0080	0.0414	0.0023	0.1655
10	6.112406614	-0.0027	-0.0059	0.0304	0.0016	0.1399
11	6.154529285	0.0042	0.0037	-0.0242	-0.0029	0.1229
12	6.14115035	0.0020	-0.0006	-0.0014	-0.0015	0.1554
13	6.119115822	-0.0016	0.0018	-0.0186	-0.0038	0.2271
14	6.154822121	0.0042	0.0065	-0.0394	-0.0041	0.4571
15	6.12122952	-0.0013	-0.0025	0.0141	0.0012	0.1210
16	6.096085797	-0.0054	-0.0055	0.0271	0.0009	0.4001
17	6.11940302	-0.0016	-0.0032	0.0253	0.0041	0.1607
18	6.1700492	0.0067	-0.0179	0.0936	0.0056	0.3886
19	6.136163538	0.0012	0.0004	-0.0048	-0.0011	0.2021
20	6.130527474	0.0002	0.0015	-0.0047	0.0008	0.2041
21	6.126455933	-0.0004	0.0008	-0.0039	-0.0001	0.1874
22	6.140928427	0.0019	0.0003	-0.0050	-0.0014	0.2696
23	6.119384441	-0.0016	-0.0035	0.0200	0.0015	0.2527
24	6.143942366	0.0024	0.0047	-0.0246	-0.0014	0.1503
25	6.11193037	-0.0028	0.0006	-0.0016	0.0004	0.3144
26	6.101687866	-0.0045	-0.0038	0.0230	0.0021	0.3279
27	6.117735016	-0.0018	-0.0131	0.0719	0.0051	0.3945
28	6.135871042	0.0011	0.0074	-0.0419	-0.0035	0.2368
29	6.121223493	-0.0013	0.0015	-0.0018	0.0018	0.1477
30	6.113826592	-0.0025	-0.0036	0.0204	0.0016	0.2299

3.2 Controlled variance pricing policy (CVP)

The lower bound of variance is controlled to be no less than $k^{-1/2}, k = 1, 2, \ldots, T$ by taking $c = 1$ and $\alpha = 0.5$, which is the same with the experiments in [4]. We still simulate 30 different sample paths, and the results are listed in the following two tables.

Table 3.3: Simulated Prices of Seller 1 using CVP policy

Sample	Prices	Price Err.	Parameter Err. ($\beta_{1,0}$,$\beta_{1,1}$,$\beta_{1,-1}$)		
1	8.6413	-0.0433	-0.0340	0.0947	0.4329
2	9.0349	0.0003	-0.0087	-0.0049	0.0270
3	8.8648	-0.0185	0.0282	0.0137	-0.0994
4	9.0117	-0.0023	-0.0239	-0.0539	-0.0463
5	9.1208	0.0098	0.0517	0.0867	-0.0105
6	8.9710	-0.0068	0.0109	-0.0178	-0.1023
7	8.6952	-0.0373	-0.0093	0.0750	0.2622
8	9.1837	0.0168	-0.0089	-0.1497	-0.4371
9	9.1001	0.0075	0.0015	-0.0237	-0.0772
10	9.0385	0.0007	-0.0362	-0.0198	0.1168
11	9.1668	0.0149	-0.0032	0.0412	0.1362
12	9.0613	0.0032	-0.0664	-0.0942	0.0323
13	8.8694	-0.0180	0.0063	-0.0168	-0.0792
14	9.1356	0.0114	-0.0251	-0.1335	-0.2908
15	8.9246	-0.0119	-0.0394	-0.0117	0.1480
16	9.1940	0.0179	0.0354	-0.0175	-0.2170
17	8.7273	-0.0338	0.0091	-0.0062	-0.0641
18	9.0319	0.0000	0.0016	-0.0258	-0.0844
19	9.0232	-0.0010	-0.0131	0.0071	0.0840
20	9.2935	0.0289	0.0809	-0.0819	-0.7117
21	9.0799	0.0053	-0.0689	-0.1401	-0.1058
22	8.9163	-0.0128	-0.0430	-0.0440	0.0780
23	9.2877	0.0283	0.0316	0.0365	-0.0503
24	8.8910	-0.0156	-0.0143	-0.0042	0.0569
25	9.0216	-0.0012	-0.0207	-0.0012	0.0965
26	8.8611	-0.0189	-0.0005	0.0111	0.0366
27	8.8389	-0.0214	-0.0414	-0.0841	-0.0569
28	8.9095	-0.0136	0.0238	0.0228	-0.0518
29	8.8477	-0.0204	0.0064	0.0106	-0.0017
30	9.0693	0.0041	0.0652	0.1675	0.1383

Table 3.4: Simulated Prices of Seller -1 using CVP policy

Sample	Prices	Price Err.	Parameter Err. ($\beta_{1,0}$,$\beta_{1,1}$,$\beta_{1,-1}$)			Correlations
1	6.1840	0.0090	-0.0116	-0.0122	-0.0234	0.6251
2	6.0140	-0.0188	0.0012	-0.0070	-0.0004	0.9212
3	5.9545	-0.0285	0.0045	0.0533	0.0270	0.7229
4	6.1755	0.0076	-0.0071	0.2262	0.0755	0.7890
5	6.1233	-0.0009	-0.0035	0.2255	0.0769	0.6797
6	6.0745	-0.0089	-0.0320	-0.0655	-0.0734	0.9404
7	6.0070	-0.0199	-0.0095	0.0846	0.0144	0.7438
8	6.3719	0.0396	-0.0597	0.3365	0.0365	0.8367
9	6.1967	0.0110	-0.0113	0.0410	-0.0028	0.8290
10	6.2285	0.0162	0.0023	-0.0853	-0.0277	0.3244
11	6.1019	-0.0044	-0.0285	0.2202	0.0338	0.8280
12	6.0936	-0.0058	0.0033	-0.1641	-0.0566	0.8765
13	6.1516	0.0037	-0.0035	-0.0072	-0.0084	0.7420
14	6.2857	0.0256	0.0493	-0.3127	-0.0424	0.8778
15	6.2129	0.0137	-0.0025	-0.2074	-0.0778	0.7467
16	6.1009	-0.0046	-0.0545	0.0608	-0.0643	0.5414
17	6.1657	0.0060	-0.0054	0.0370	0.0050	-0.5078
18	6.1667	0.0061	0.0060	-0.0567	-0.0110	0.8648
19	5.9337	-0.0319	-0.0273	0.0621	-0.0211	-0.4435
20	6.1587	0.0048	0.0678	0.2294	0.2187	0.9588
21	6.0957	-0.0054	0.0115	-0.1088	-0.0235	0.9520
22	6.1202	-0.0014	-0.0007	0.0124	0.0030	-0.2544
23	6.2014	0.0118	-0.0003	-0.0017	-0.0012	0.6326
24	6.1982	0.0113	-0.0048	-0.0675	-0.0319	0.7776
25	6.2698	0.0230	-0.0127	-0.0484	-0.0381	0.5774
26	6.0686	-0.0099	-0.0135	0.2159	0.0584	0.7407
27	6.1220	-0.0011	0.0123	-0.0590	-0.0027	0.8514
28	6.0284	-0.0164	-0.0197	0.0754	-0.0047	0.2996
29	6.1719	0.0070	-0.0115	-0.0398	-0.0325	0.8513
30	5.9869	-0.0232	0.0138	-0.0985	-0.0106	0.6660

Sample paths 4-6, 12, 18, 21 converge to NE price for both sellers, which suggests $P(\lim_{t\to\infty} p_i^t = p_i^{NE}) \approx 20\%$. With the variances controlled by a lower bound, the sufficient condition in (2.5) is equivalent to that the sample correlations is not close to 1 or -1. Thus the bad convergence for CVP also can be explained by the large correlations between these two sellers (27 sample paths' correlations greater than 0.4). Therefore, the CVP doesn't work for 2 sellers case. Figure 3.1 plots the simulated price of sample path 1. It's obvious that the simulated prices of both sellers diverge from NE prices. And Figure 3.2 shows our control on variances, that is we keep variance no less than the variance bound $k^{-1/2}, k = 1, 2, \ldots, T$.

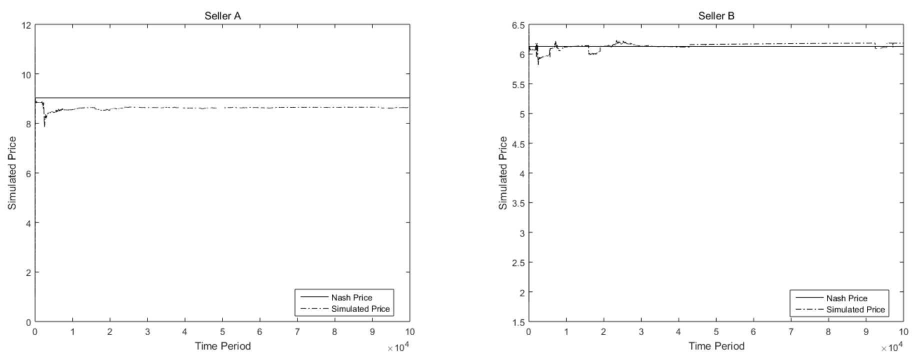

Figure 3.1: Simulated Prices (Sample Path 1)

3.3 Certainty equivalent pricing policy (CEP)

Table 3.5 and 3.6 suggest that sample paths 1-8, 11-15, 20-22, 24,25,27 and 28 converge for both sellers, that is, $P(\lim_{t\to\infty} p_i^t = p_i^{NE}) \approx 66.7\%$, and we plot the simulated price of sample path 30, because its divergence can be obviously shown in the graph. It's noticeable that for sample path 30, the value of sufficient condition is 11.6727, which violates (2.5) in Theorem 2.1.1. Also when the value of "Sufficient Condition" quantity is small, e.g. sample path 25 in Table 3.5, and 12,13,25 in Table 3.6, the parameters tend to have good convergence.

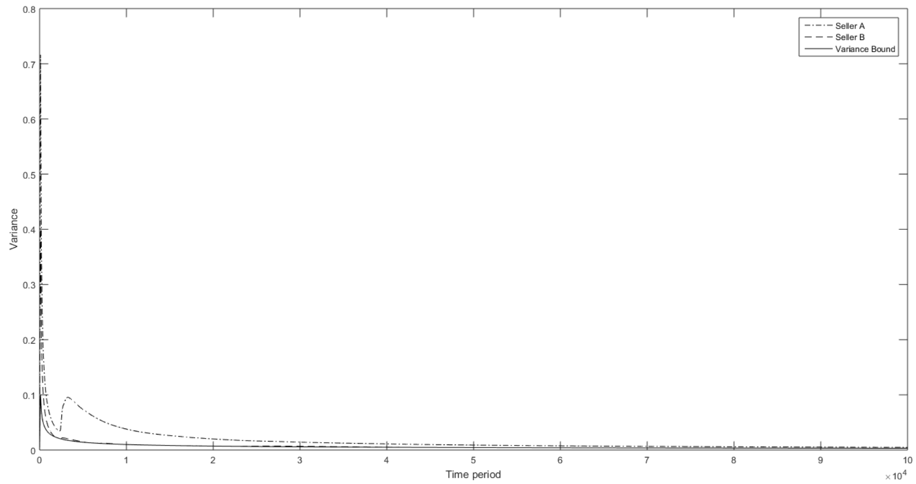

Figure 3.2: Variance (Sample Path 1)

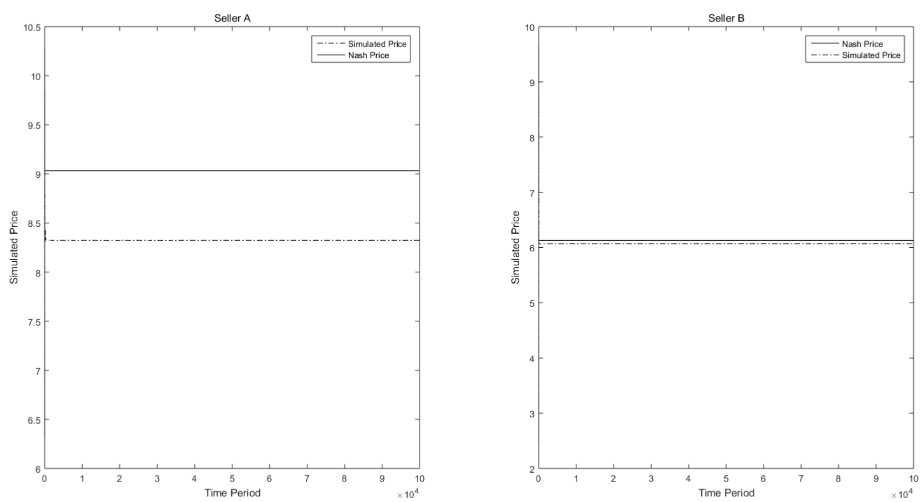

Figure 3.3: Simulated Prices (Sample Path 30)

Table 3.5: Simulated Prices of Seller 1 using Certainty Equivalent Pricing Policy

Sample	Prices	Price Err.	Parameter Err. ($\beta_{1,0}$,$\beta_{1,1}$,$\beta_{1,-1}$)		
1	9.0197	-0.0014	-0.0053	0.0025	0.0336
2	9.0887	0.0063	-0.0026	-0.0053	-0.0032
3	9.0500	0.0020	0.0071	0.0140	0.0063
4	9.0818	0.0055	0.0140	-0.0163	-0.1174
5	9.0794	0.0052	-0.0016	0.0088	0.0339
6	9.0491	0.0019	-0.0114	-0.0279	-0.0272
7	9.0099	-0.0025	0.0054	0.0151	0.0176
8	8.9984	-0.0037	-0.0035	-0.0179	-0.0363
9	8.9358	-0.0107	-0.0055	-0.0032	0.0175
10	9.0551	0.0025	0.0001	-0.0038	-0.0115
11	9.0721	0.0044	-0.0046	0.0002	0.0232
12	8.9775	-0.0061	-0.0106	-0.0059	0.0344
13	8.9855	-0.0052	0.0175	0.0013	-0.0822
14	9.1189	0.0096	-0.0011	-0.0144	-0.0376
15	9.0106	-0.0024	-0.0061	-0.0031	0.0207
16	8.8466	-0.0206	-0.0049	-0.0064	0.0055
17	8.9312	-0.0112	0.0028	0.0241	0.0558
18	9.0933	0.0068	0.0067	-0.0022	-0.0393
19	9.2432	0.0234	-0.0073	0.0062	0.0538
20	8.9915	-0.0045	-0.0079	-0.0401	-0.0822
21	9.0292	-0.0003	-0.0139	-0.0148	0.0237
22	9.0810	0.0054	-0.0142	-0.0119	0.0345
23	8.9923	-0.0044	-0.0048	-0.0202	-0.0364
24	9.0692	0.0041	0.0111	0.0190	0.0013
25	9.1107	0.0087	0.0031	0.0022	-0.0083
26	9.2436	0.0234	0.0022	-0.0044	-0.0237
27	9.0740	0.0046	0.0004	-0.0091	-0.0292
28	9.1066	0.0082	0.0045	0.0144	0.0200
29	9.0341	0.0002	0.0025	0.0289	0.0723
30	8.3228	-0.0786	0.0174	0.0138	-0.0444

Table 3.6: Simulated Prices of Seller -1 using Certainty Equivalent Pricing Policy

Sample	Prices	Price Err.	Parameter Err. ($\beta_{1,0}$,$\beta_{1,1}$,$\beta_{1,-1}$)			Sufficient Conditions
1	6.1238	-0.0009	0.0032	-0.0107	0.0012	0.4547
2	6.1149	-0.0023	-0.0067	0.0040	-0.0094	2.2406
3	6.1280	-0.0002	0.0031	-0.0039	0.0036	0.7406
4	6.1124	-0.0027	-0.0035	0.0278	0.0046	1.0051
5	6.1037	-0.0041	0.0069	0.0526	0.0310	2.3962
6	6.1292	0.0000	-0.0010	-0.0309	-0.0131	0.2925
7	6.1114	-0.0029	0.0014	0.0146	0.0077	0.7340
8	6.1065	-0.0037	-0.0069	0.0347	0.0017	1.0192
9	6.0842	-0.0073	-0.0079	0.0072	-0.0102	0.6182
10	6.0596	-0.0113	-0.0027	0.0310	0.0070	0.9757
11	6.0835	-0.0074	-0.0022	0.0275	0.0066	2.7271
12	6.1564	0.0045	0.0038	-0.0041	0.0046	0.4451
13	6.1681	0.0064	-0.0016	0.0247	0.0065	0.4774
14	6.1621	0.0054	0.0057	-0.0372	-0.0045	0.5304
15	6.1505	0.0035	-0.0034	-0.0121	-0.0100	1.0673
16	6.1151	-0.0023	-0.0042	-0.0051	-0.0086	1.5500
17	6.0816	-0.0077	0.0004	-0.0463	-0.0161	0.5263
18	6.0483	-0.0132	0.0009	-0.0372	-0.0122	0.5787
19	6.1330	0.0007	-0.0014	-0.0102	-0.0059	0.7509
20	6.1742	0.0074	0.0109	0.0031	0.0192	0.5041
21	6.1080	-0.0034	-0.0072	-0.0454	-0.0281	0.3180
22	6.0960	-0.0054	-0.0027	0.0474	0.0132	0.4535
23	6.1981	0.0113	0.0029	-0.0234	-0.0039	0.7957
24	6.1592	0.0049	-0.0026	0.0358	0.0089	0.7839
25	6.1513	0.0036	-0.0046	0.0020	-0.0068	0.3712
26	6.1161	-0.0021	-0.0030	0.0226	0.0035	1.5336
27	6.1583	0.0048	0.0148	-0.0190	0.0172	0.5521
28	6.1380	0.0015	0.0022	-0.0114	-0.0005	1.6712
29	6.2163	0.0142	0.0015	0.0134	0.0074	3.9975
30	6.0709	-0.0095	-0.0049	-0.0150	-0.0133	11.6727

3.4 Pricing policy proposed in [12]

We first set $T_1 = rand([T/2])$ and $T_2 = T_1 + 5 \times [\sqrt{T}]$, where $T = 100,000$ that represents total number of time periods. If $k \in (T_1, T_2)$, we randomly select price $p_i^k \in [p_{i,l}, p_{i,h}]$ according to uniform distribution. The results for seller A and B are listed in as follows, from which we see that all of the 30 sample paths converge to NE price, and the relative parameter errors are less than 0.01.

Table 3.7: Simulated Prices of Seller 1

Sample	Prices	Price Err.	Parameter Err. ($\beta_{1,0}$,$\beta_{1,1}$,$\beta_{1,-1}$)		
1	9.0287	-0.0004	0.0002	0.0007	0.0014
2	9.0342	0.0002	0.0000	-0.0001	-0.0005
3	9.0395	0.0008	-0.0006	-0.0001	0.0028
4	9.0305	-0.0002	-0.0004	-0.0009	-0.0006
5	9.0321	0.0000	0.0008	0.0007	-0.0019
6	9.0309	-0.0001	0.0002	0.0001	-0.0008
7	9.0345	0.0003	0.0000	-0.0001	-0.0003
8	9.0285	-0.0004	0.0002	-0.0002	-0.0017
9	9.0348	0.0003	0.0003	0.0004	-0.0001
10	9.0307	-0.0002	0.0007	0.0003	-0.0026
11	9.0285	-0.0004	0.0001	0.0006	0.0015
12	9.0349	0.0003	0.0000	0.0000	0.0001
13	9.0334	0.0001	0.0003	0.0002	-0.0011
14	9.0331	0.0001	-0.0005	0.0000	0.0027
15	9.0304	-0.0002	-0.0005	0.0003	0.0030
16	9.0378	0.0006	-0.0015	-0.0006	0.0055
17	9.0335	0.0001	0.0002	0.0004	-0.0002
18	9.0246	-0.0009	-0.0009	-0.0004	0.0033
19	9.0328	0.0001	-0.0005	-0.0003	0.0018
20	9.0280	-0.0005	-0.0003	0.0002	0.0020
21	9.0302	-0.0002	-0.0007	-0.0002	0.0027
22	9.0324	0.0000	0.0000	-0.0001	-0.0004
23	9.0312	-0.0001	-0.0001	0.0005	0.0020
24	9.0241	-0.0009	0.0003	0.0001	-0.0011
25	9.0261	-0.0007	-0.0007	-0.0007	0.0014
26	9.0319	0.0000	0.0008	0.0006	-0.0023
27	9.0322	0.0000	-0.0001	-0.0001	-0.0001
28	9.0325	0.0000	-0.0004	-0.0001	0.0015
29	9.0342	0.0002	0.0001	-0.0006	-0.0021
30	9.0364	0.0005	0.0004	0.0009	0.0007

Table 3.8: Simulated Prices of Seller -1

Sample	Prices	Price Err.	Parameter Err.($\beta_{-1,0}$,$\beta_{-1,-1}$,$\beta_{-1,1}$)		
1	6.1279	-0.0002	-0.0001	0.0010	0.0002
2	6.1244	-0.0008	-0.0001	0.0013	0.0003
3	6.1302	0.0002	0.0003	-0.0007	0.0001
4	6.1296	0.0001	-0.0002	0.0018	0.0004
5	6.1300	0.0002	0.0002	-0.0012	0.0000
6	6.1283	-0.0001	0.0002	-0.0001	0.0003
7	6.1287	-0.0001	0.0000	0.0006	0.0002
8	6.1294	0.0001	0.0001	-0.0008	-0.0001
9	6.1311	0.0003	0.0000	-0.0003	-0.0001
10	6.1295	0.0001	0.0002	0.0007	0.0006
11	6.1244	-0.0008	0.0002	-0.0010	0.00002
12	6.1285	-0.0001	0.0005	-0.0027	-0.0001
13	6.1328	0.0006	-0.0009	0.0007	-0.0012
14	6.1271	-0.0003	0.0005	-0.0003	0.0007
15	6.1275	-0.0003	-0.0007	0.0016	-0.0006
16	6.1321	0.0005	0.0001	-0.0012	-0.0002
17	6.1284	-0.0001	-0.0011	0.0004	-0.0016
18	6.1281	-0.0001	-0.0002	0.0030	0.0008
19	6.1283	-0.0001	-0.0003	-0.0005	-0.0006
20	6.1295	0.0001	0.0005	-0.0005	0.0006
21	6.1276	-0.0002	0.0001	-0.0003	0.0000
22	6.1281	-0.0002	0.0002	-0.0012	-0.0002
23	6.1296	0.0001	0.0002	0.0015	0.0008
24	6.1274	-0.0003	-0.0001	0.0017	0.0005
25	6.1291	0.0000	0.0003	0.0001	0.0005
26	6.1274	-0.0003	-0.0001	0.0019	0.0005
27	6.1299	0.0001	-0.0005	0.0011	-0.0004
28	6.1289	0.0000	0.0006	0.0000	0.0009
29	6.1299	0.0001	0.0000	-0.0001	0.0000
30	6.1294	0.0001	-0.0001	-0.0008	-0.0005

In the previous experiment, prices are chosen randomly for times periods in $[T_1, T_2]$ and the prices converge to those in NE. In the current experiment, we randomly generate $\sqrt{T}$ time periods, and for each of these time periods, price is choose according to uniform distribution. Table 3.9 and Table 3.10 present the results. It's clearly shown that the parameter estimates are very good but none of the simulated price is acceptable. According to Proposition 2.2.1, if the parameters converge to the true value, the average prices are supposed to converge to NE price as well. The reason for such large price errors, we believe, is that the price convergence is very slow, and and $T = 100000$ is not big enough.

Table 3.9: Simulated Prices of Seller 1

Sample	Average prices	Price Err.	Parameter Err. ($\beta_{-1,0}$,$\beta_{-1,-1}$,$\beta_{-1,1}$)		
1	8.8769	-0.0172	0.0002	0.0002	-0.0002
2	8.8665	-0.0183	0.0001	0.0002	0.0001
3	8.8688	-0.0181	0.0001	0.0002	0.0004
4	8.8716	-0.0178	0.0004	0.0008	0.0003
5	8.8723	-0.0177	0.0001	0.0001	-0.0004
6	8.8704	-0.0179	0.0003	0.0005	-0.0001
7	8.8761	-0.0173	-0.0005	-0.0007	0.0003
8	8.8709	-0.0179	0.0002	0.0003	-0.0002
9	8.8711	-0.0178	0.0004	0.0006	0.0000
10	8.8646	-0.0186	0.0004	0.0007	0.0002
11	8.8784	-0.0170	-0.0006	-0.0009	0.0003
12	8.8760	-0.0173	0.0000	0.0001	0.0002
13	8.8753	-0.0174	0.0002	0.0004	0.0003
14	8.8742	-0.0175	-0.0001	-0.0003	-0.0005
15	8.8687	-0.0181	0.0002	0.0003	-0.0001
16	8.8746	-0.0175	-0.0003	-0.0005	0.0001
17	8.8725	-0.0177	0.0000	0.0000	0.0002
18	8.8754	-0.0174	0.0000	-0.0001	-0.0001
19	8.8711	-0.0178	0.0005	0.0009	0.0001
20	8.8765	-0.0172	0.0000	0.0002	0.0004
21	8.8753	-0.0174	-0.0001	-0.0001	0.0000
22	8.8713	-0.0178	-0.0001	-0.0002	-0.0001
23	8.8715	-0.0178	0.0002	0.0004	0.0003
24	8.8719	-0.0178	-0.0001	-0.0001	0.0001
25	8.8718	-0.0178	0.0004	0.0007	0.0003
26	8.8693	-0.0180	0.0001	0.0000	-0.0003
27	8.8744	-0.0175	0.0003	0.0004	-0.0003
28	8.8628	-0.0188	0.0004	0.0006	-0.0002
29	8.8707	-0.0179	-0.0001	-0.0001	0.0002
30	8.8722	-0.0177	-0.0002	-0.0004	-0.0001

Table 3.10: Simulated Prices of Seller -1

Sample	Average prices	Price Err.	Parameter Err. ($\beta_{-1,0}$,$\beta_{-1,-1}$,$\beta_{-1,1}$)			Sufficient Condition
1	5.5038	-0.1020	0.0003	-0.0017	0.0000	0.0066
2	5.4879	-0.1046	0.0003	-0.0012	0.0000	-0.0031
3	5.4946	-0.1035	0.0000	0.0005	0.0001	0.0049
4	5.4977	-0.1030	-0.0001	0.0006	0.0000	-0.0066
5	5.5060	-0.1017	0.0002	-0.0008	0.0001	-0.0030
6	5.4932	-0.1037	-0.0002	0.0008	-0.0001	0.0006
7	5.4943	-0.1036	0.0000	0.0000	-0.0001	0.0014
8	5.4943	-0.1036	0.0002	-0.0008	0.0000	0.0012
9	5.5014	-0.1024	-0.0004	0.0018	0.0000	0.0003
10	5.5041	-0.1020	-0.0002	0.0010	-0.0001	0.0038
11	5.5018	-0.1023	-0.0002	0.0008	0.0000	-0.0014
12	5.5042	-0.1020	-0.0002	0.0005	-0.0001	-0.0035
13	5.5008	-0.1025	0.0001	-0.0006	-0.0001	-0.0013
14	5.4979	-0.1030	0.0002	-0.0007	0.0001	-0.0005
15	5.4930	-0.1038	-0.0004	0.0015	0.0000	-0.0005
16	5.5083	-0.1013	0.0002	-0.0006	0.0000	-0.0039
17	5.5023	-0.1023	-0.0001	0.0002	-0.0001	-0.0002
18	5.5042	-0.1019	-0.0003	0.0013	0.0000	-0.0013
19	5.5028	-0.1022	0.0001	-0.0005	0.0000	-0.0023
20	5.5079	-0.1013	-0.0001	0.0003	0.0001	0.0002
21	5.5057	-0.1017	0.0000	-0.0001	0.0000	-0.0007
22	5.4968	-0.1032	-0.0002	0.0011	0.0001	0.0001
23	5.4953	-0.1034	-0.0003	0.0015	0.0000	0.0024
24	5.4969	-0.1031	-0.0002	0.0007	-0.0001	0.0050
25	5.5016	-0.1024	0.0001	-0.0009	-0.0001	-0.0052
26	5.4935	-0.1037	-0.0001	0.0006	0.0000	-0.0015
27	5.5056	-0.1017	0.0000	0.0004	0.0001	0.0035
28	5.4791	-0.1060	0.0000	0.0000	0.0000	-0.0017
29	5.5028	-0.1022	0.0000	0.0001	0.0000	0.0022
30	5.4937	-0.1037	-0.0001	0.0006	0.0000	-0.0015

Chapter 4

Conclusions and Discussion

We use numerical experiments to study convergence of prices under different pricing policies, and our results, in particular, show that

(i) RCEP gives good approximations for parameters and prices.

(ii) CVP is not optimal in the duopoly case.

(iii) The convergence of the average prices under the policy proposed in [12] is slow.

Under the RCEP policy, the price subprocesses $\{p_i^t, t \in \mathcal{K}_i\}$ are discrete time Markov processes. So one of our future directions is to rigorously prove the optimality of RCEP using techniques from Markov processes.

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