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Properties of Some Markov Chains on Linear Extensions of Posets

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Properties of some Markov chains on linear extensions of posets

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Abstract

The Tsetlin library is a well-studied Markov model for how an arrangement of books on a library shelf evolves over time. It assumes that, given n books, one book is read and returned at the end of the shelf before another one is picked up. One of the most interesting properties of this Markov chain is that its spectrum can be computed exactly and its eigenvalues are linear in the transition probabilities. This result has been generalized in different ways by various people. In this work, we investigate three generalizations given by the extended promotion Markov chain on linear extensions of a poset introduced by Ayyer, Klee, and Schilling (2014), the generalization given by Brown and Diaconis (1998) and Bidigare, Hanlon, and Rockmore (1999) to random-to-back pop shuffles, and the generalization by Björner (2008, 2009) to hierarchies of libraries. We consider combining these results to hierarchies of libraries where the states are linear extensions of associated posets. We also expand Ayyer, Klee, and Schilling's result to a larger class of posets and derive convergence to stationarity.

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Chapter 1

Introduction

A *stochastic process* is a collection of random variables indexed by time. A finite Markov process, named after Andrey Markov, is a memoryless stochastic process with finite state space Ω and transition matrix M, indexed by the states. The memoryless property, often called the *Markov property*, means that the conditional probability of moving from state i to state j is the same, no matter what preceding steps were taken. A *probability distribution* on Ω is a map $P\colon \Omega \to \mathbb{R}$ such that $P(x) \geq 0$ for all $x \in \Omega$ and

$$\sum_{x \in \Omega} P(x) = 1.$$

We take the convention that the (i, j) entry gives the probability of going from i to j. This ensures that the rows sums of M are one, thus M is row stochastic. The *stationary distribution* for M is a row vector π such that $\pi M = \pi$. When considering Markov chains, a natural question that arises is to estimate the rate of convergence to the stationary distribution, also called an *invariant distribution.* A target distance to the stationary distribution is given and the number of steps required to reach this is called the *mixing time.*

A Markov chain is said to be *irreducible* if for every pair of states $x, y \in \Omega$ it is possible to move from x to y. In other words, a Markov chain is irreducible if its underlying directed graph is strongly connected. A Markov chain is said to be *aperiodic* if the greatest common divisor of lengths of all possible loops from any state to itself is one. The Perron-Frobenius theorem tells us that every irreducible aperiodic chains has a unique stationary distribution. More details about Markov chains, mixing times, and convergence to stationary results can be found at [18, 23], among others.

In this thesis, we study a few particular Markov chains. The first one relevant to our work is the Tsetlin library [29]. This is a model of the way books on a library shelf evolve over time. The state space consists of all $n!$ permutations on the labels of the books with transition probability x_i, a book i is picked up and put at the back of the shelf before another book is picked up. This Markov chain is a self-organizing system, meaning that over time the books used the most will end up at the end of the shelf. An interesting fact about this Markov chain is that the eigenvalues of the transition matrix are linear in the x_is. Hendricks [15, 16] found the stationary distribution, while the fact that the eigenvalues have an elegant formula, was discovered (independently) by Donnelly [12], Kapoor and Reingold [17], and Phatarfod [21].

The Tsetlin library has been generalized by many people in various ways. In particular, it has been generalized to walks on hyperplane arrangements [6, 11], left-regular bands [9], self-organizing libraries and complex hyperplane arrangements [7, 8], extended promotion operator [1, 2], $\mathcal{R}$-trivial monoids [4], among others.

There are three ways in which the Tsetlin library has been generalized that are relevant to our work. Ayyer, Klee and Schilling [1, 2] introduced the extended promotion Markov chain, which has the same number of moves as the Tsetlin library but a restricted state space. Instead of allowing for all permutations on n elements, the state space is the set of all linear extensions of a certain poset. If the poset is an antichain, the set of linear extensions are all $n!$ permutations and we get exactly the Tsetlin library result. Ayyer, Klee, and Schilling showed that if the poset is a rooted forest, then the eigenvalues of the transition matrix are linear in the transition probabilities. The results were proved using representation theory of $\mathcal{R}$-trivial monoids; however, Ayyer, Klee, and Schilling [1] also conjectured that there is a larger class of posets for which the same properties hold, but the associated monoid is not $\mathcal{R}$-trivial.

In Chapter 3, we expand this result to a larger class of posets. These results have already been published in [22]. A poset P is a *ladder* of rank k if $P = Q_1 \oplus \cdots \oplus Q_k$ where Q_i is an antichain of size 1 or 2 for all $i = 1, \ldots, k$ and $\oplus$ represents the ordinal sum of two posets. We show that the transition matrix of the Markov chain has eigenvalues that are linear in the transition probabilities in the case when P is a union of ordinal sums of a ladder and a forest.

We also answer some questions that arise when considering Markov chains, such as: *what is the stationary distribution and what is the rate of convergence?* We obtain the stationary distribution for the case when P is a union of an ordinal sum of a forest and a ladder. In the case when $P = F \oplus L$, we find an upper bound on the rate of convergence to the stationary distribution.

Bidigare, Hanlon, and Rockmore [6] and Brown and Diaconis [11] generalized the Tsetlin library model to pop shuffles. This generalization of the Tsetlin libary keeps the same state space of all permutations of n elements but changes the allowable moves. Instead of the moves being given by picking up only one book and placing it at the end of the shelf, multiple books can be picked up and placed at the end of the shelf by multiple readers while preserving their original order. This is a special case of a rich theory on random walks on the regions of hyperplane arrangements. In fact, Brown [9, 10] showed that random walks on the hyperplane arrangements are left-regular bands.

The third generalization of the Tsetlin library relevant to our work is the generalization by Björner [8] to self-organizing libraries. Consider a fixed rooted tree T whose leaves L are all of the same depth. The leaves correspond to books, the parents of the leaves to shelves, and so on. For each inner node (a node that is not a leaf) a total ordering of its children is given. A *local ordering* on the tree is given by a total ordering on the children of every inner node. A subset $E \subseteq L$ is chosen with some probability. A node $v \in T$ is *E-related* if some descendant of v is contained in E. A subset E of L acts on a local ordering of T by rearranging the order locally at each inner node so that the children having some descendant in E come last, otherwise the original order is preserved. Precisely, for $\pi = (\pi_v)_v$, a local ordering of T, and $E \subseteq L$, we have $E(\pi) = (E_v(\pi_v))_v$ where $E_v(\pi_v)$ is the linear ordering on the children of v where the E-related elements come last, in their original order. If the tree has depth 1, then this action is the pop shuffle. The main result of Björner [8] is that for a probability distribution on 2^L (the power set of L), the transition matrix of the induced random walk on local orderings of T has eigenvalues that are linear in the transition probabilities. Björner [7] further expands the results for hyperplane arrangements to complex hyperplane arrangements.

Chapter 4 combines these three generalizations of the Tsetlin library. Consider the same setup as Björner's hierarchies of libraries, where at each inner node, a poset on the children is given. For each inner node of depth $d-1$, the poset can be arbitrary. For all other inner nodes, the poset is the antichain. For each inner node, the total ordering is a linear extension on the poset associated to the node. The actions are given by subsets of the leaves, L. Instead of allowing for all subsets $E \subseteq L$, we restrict to subsets such that no two elements of E are siblings, i.e., have the same parent node.

In Section 4.1, we show that if all the associated posets are rooted forests, then the associated monoid is $\mathcal{R}$-trivial. Using existing theory of $\mathcal{R}$-trivial monoids, we show that the associated Markov chain has a transition matrix with eigenvalues that are linear in the transition probabilities. In Sections 4.4 and 4.5 we expand the result to the associated posets being a union of an ordinal sum of a forest and a ladder. In this case we give two different proofs. First, in Section 4.4 we show the associated monoid is in the class $\mathbf{DO(Ab)}$. The theory of walks on $\mathbf{DO(Ab)}$ monoids gives a way to find potential eigenvalues, but not specify the multiplicity (not even when it is zero). For that reason, in Section 4.5 we find a relationship between the transition matrices of a tree with known spectrum and the desired tree to compute the spectrum exactly.

Chapter 2

Background

2.1 Tsetlin library

The Tsetlin library [29] is a model for the way an arrangement of books on a library shelf evolves over time. In this Markov chain on permutations of n books, book i is picked up with probability x_i and put at the back of the shelf before another book is picked up. That is, if π and π' are two arrangements of the books, then the probability of transitioning from π to π' is x_i if π' is obtained from π by moving i to the end. Hendricks [15, 16] found the stationary distribution, while the fact that the eigenvalues have an elegant formula was discovered (independently) by Donnelly [12], Kapoor and Reingold [17], and Phatarfod [21].

A derangement of the set $[n] := \{1, \ldots, n\}$ is a permutation with no fixed points.

Example 2.1.1. *For $n = 3$, the permutations on three elements that have no fixed points are 231 and 312. Thus, there are two derangements of [3].*

The eigenvalues of the Tsetlin library model are given by the following theorem.

Theorem 2.1.2. *The distinct eigenvalues for the Tsetlin library model are indexed by $S \subseteq [n] = \{1, \ldots, n\}$:*

$$\lambda_S = \sum_{i \in S} x_i.$$

The multiplicity of λ_S is the number of derangements of $\{1, \ldots, n - |S|\}$.

Example 2.1.3. *Figure 2.1 gives the Tsetlin library Markov chain on 3 books. The edges with weight x_1 are dashed, the edges with weight x_2 are dotted, and the edges with weight x_3 are solid.*

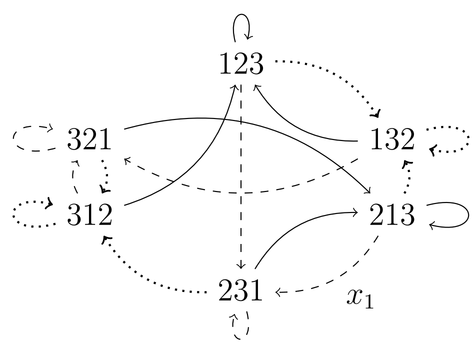


Figure 2.1: Tsetlin library Markov chain on 3 books.

The following table gives the subsets $S \subseteq [3]$, with the multiplicity and corresponding eigenvalue.

S	#Derangements of $\{1, \ldots, 3 - \|S\|\}$	Eigenvalue
$\emptyset$	2	0
$\{1\}$	1	x_1
$\{2\}$	1	x_2
$\{3\}$	1	x_3
$\{1,2\}$	0	$x_1 + x_2$
$\{1,3\}$	0	$x_1 + x_3$
$\{2,3\}$	0	$x_2 + x_3$
$\{1,2,3\}$	1	$x_1 + x_2 + x_3$

Table 2.1: Eigenvalues of the Tsetlin library Markov chain on 3 books.

Hence the eigenvalues of the Markov chain on 3 books are

$$x_1 + x_2 + x_3, x_1, x_2, x_3, 0, 0.$$

2.2 Hyperplane arrangements

This section is devoted to the extension by Bidigare, Hanlon, and Rockmore [6] and by Brown and Diaconis [11] who showed the Tsetlin library result is in fact a random walk on hyperplane arrangements. Consider the central hyperplane arrangement $\mathcal{A}_n$ (every hyperplane contains $\vec{0}$), called the *braid arrangement* consisting of the hyperplanes $\{H_{ij} \colon 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$ such that

$$H_{ij} = \{(x_1, \ldots, x_n) \colon x_i = x_j\}.$$

Fix a vector, $\bar{v} = (v_1, \ldots, v_n)$ such that $v_1 > \cdots > v_n$, notice that $\bar{v}$ is in the complement of $\mathcal{A}_n$. Every $H_{ij} \in \mathcal{A}_n$ partitions $\mathbb{R}^n$ into three parts: the hyperplane $H_{ij}^0 = H_{ij}$, the open half-space H_{ij}^+ of H_{ij} containing $\bar{v}$, i.e.,

$$H_{ij}^+ = \{(x_1, \ldots, x_n) \colon x_i > x_j\},$$

and the open half-space H_{ij}^- not containing $\bar{v}$, i.e.,

$$H_{ij}^- = \{(x_1, \ldots, x_n) \colon x_i < x_j\}.$$

Example 2.2.1. *Let $n = 3$. Then $\mathcal{A}_3$ consists of the three hyperplanes*

$$\left\{ \begin{array}{l} H_{12} = \{(x_1, x_2, x_3) \colon x_1 = x_2\}, \\ H_{13} = \{(x_1, x_2, x_3) \colon x_1 = x_3\}, \\ H_{23} = \{(x_1, x_2, x_3) \colon x_2 = x_3\}. \end{array} \right\}$$

Table 2.2 gives the half-spaces formed by each hyperplane.

H_{ij}^0	H_{ij}^+	H_{ij}^-
H_{12}	$\{(x_1, x_2, x_3) \colon x_1 > x_2\}$	$\{(x_1, x_2, x_3) \colon x_1 < x_2\}$
H_{13}	$\{(x_1, x_2, x_3) \colon x_1 > x_3\}$	$\{(x_1, x_2, x_3) \colon x_1 < x_3\}$
H_{23}	$\{(x_1, x_2, x_3) \colon x_2 > x_3\}$	$\{(x_1, x_2, x_3) \colon x_2 < x_3\}$

Table 2.2: Hyperplane arrangement half-spaces.

The *variety* of $\mathcal{A}_n$ is the union of its hyperplanes. A chamber of the hyperplane arrangement is a connected component of the complement. The set of chambers of the hyperplane arrangement $\mathcal{A}_n$ is denoted by $\mathcal{C}(\mathcal{A}_n)$. Let S_n be the set of all permutations on n elements. There is a cononical bijection:

$$\phi : \mathcal{C}(\mathcal{A}_n) \to S_n,$$

such that for $\sigma \in S_n$

$$\phi^{-1}(\sigma) = \{(x_1, \ldots, x_n) \colon x_{\sigma(1)} > \cdots > x_{\sigma(n)}\}.$$

Example 2.2.2. *Table 2.3 gives the nonempty chambers of $\mathcal{A}_3$ and the corresponding permutation label in S_3.*

Chamber	Permutation in S_3
$H_{12}^+ \cap H_{13}^+ \cap H_{23}^+$	123
$H_{12}^+ \cap H_{13}^+ \cap H_{23}^-$	132
$H_{12}^+ \cap H_{13}^- \cap H_{23}^-$	312
$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	213
$H_{12}^- \cap H_{13}^- \cap H_{23}^+$	231
$H_{12}^- \cap H_{13}^- \cap H_{23}^-$	321

Table 2.3: Chambers of $\mathcal{A}_3$.

For $H_{ij}^{\epsilon_{ij}}$, with $\epsilon_{ij} \in \{0, -, +\}$, the *faces* of $\mathcal{A}_n$ are the nonempty intersections $\bigcap H_{ij}^{\epsilon_{ij}}$, where this intersection is over all appropriate $H_{ij}^{\epsilon_{ij}}$ and the set of faces is denoted by $\mathcal{L}(\mathcal{A}_n)$. Let $L(\mathcal{A}_n)$ denote the intersection lattice (also called the *edge poset*), the set of intersections of hyperplanes from $\mathcal{A}_n$, ordered by reverse inclusion.

Example 2.2.3. *Figure 2.2 gives the intersection lattice of $\mathcal{A}_3$.*

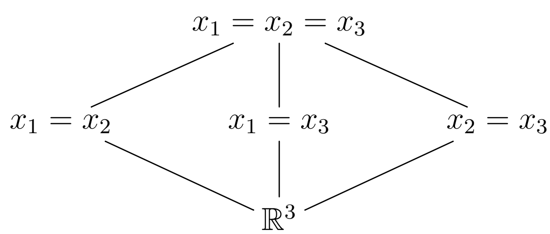


Figure 2.2: Intersection lattice of $\mathcal{A}_3$.

We have a left action of the faces on the chambers. For a face $F = \bigcap H_{ij}^{f_{ij}}$ and a chamber $C = \bigcap H_{ij}^{c_{ij}}$, define

$$F * C = \bigcap H_{ij}^{\epsilon_{ij}} \qquad \text{where } \epsilon_{ij} = \begin{cases} f_{ij} & \text{if } f_{ij} \neq 0, \\ c_{ij} & \text{if } f_{ij} = 0. \end{cases}$$

We say F acts on C with probability w_F. Geometrically, there is a metric d on the chambers defined by

$$d(C, C') = \# \text{ of hyperplanes separating } C \text{ and } C',$$

where $F * C =$ chamber adjacent to F that is closest to C under d.

Example 2.2.4. *Consider the hyperplane arrangement $\mathcal{A}_3$. Table 2.4 gives the nonempty faces of $\mathcal{A}_3$ and their weights if the action is the same as the Tsetlin library.*

Face	w_F
$H_{12}^0 \cap H_{13}^0 \cap H_{23}^0$	0
$H_{12}^0 \cap H_{13}^- \cap H_{23}^-$	0
$H_{12}^0 \cap H_{13}^+ \cap H_{23}^+$	w_3
$H_{12}^- \cap H_{13}^0 \cap H_{23}^+$	0
$H_{12}^+ \cap H_{13}^0 \cap H_{23}^-$	w_2
$H_{12}^+ \cap H_{13}^+ \cap H_{23}^0$	0
$H_{12}^- \cap H_{13}^- \cap H_{23}^0$	w_1
$H_{12}^+ \cap H_{13}^- \cap H_{23}^-$	0
$H_{12}^+ \cap H_{13}^+ \cap H_{23}^-$	0
$H_{12}^+ \cap H_{13}^- \cap H_{23}^-$	0
$H_{12}^- \cap H_{13}^- \cap H_{23}^-$	0
$H_{12}^- \cap H_{13}^- \cap H_{23}^+$	0
$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	0

Table 2.4: Nonempty faces of $\mathcal{A}_3$ and their weights corresponding to the Tsetlin library moves.

As an example, to verify why we associate the weight w_3 to the face

$$F = H_{12}^0 \cap H_{13}^+ \cap H_{23}^+,$$

the computations are in Table 2.5.

Permutation in S_n	Corresponding C	$F * C$	$F * C$ in S_n.
123	$H_{12}^+ \cap H_{13}^+ \cap H_{23}^+$	$H_{12}^+ \cap H_{13}^+ \cap H_{23}^+$	123
132	$H_{12}^+ \cap H_{13}^+ \cap H_{23}^-$	$H_{12}^+ \cap H_{13}^+ \cap H_{23}^+$	123
312	$H_{12}^+ \cap H_{13}^- \cap H_{23}^-$	$H_{12}^+ \cap H_{13}^+ \cap H_{23}^+$	123
213	$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	213
231	$H_{12}^- \cap H_{13}^- \cap H_{23}^+$	$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	213
321	$H_{12}^- \cap H_{13}^- \cap H_{23}^-$	$H_{12}^- \cap H_{13}^+ \cap H_{23}^+$	213

Table 2.5: Action of the face $F = H_{12}^0 \cap H_{13}^+ \cap H_{23}^+$ on the set of chambers of $\mathcal{A}_3$.

Figure 2.3 gives the hyperplane arrangement $\mathcal{A}_3$ where the chambers are labeled by their corresponding permutation in S_3 and the faces with nonzero weight w_1, w_2 and w_3 are given by dashed lines with the corresponding weights labeled.

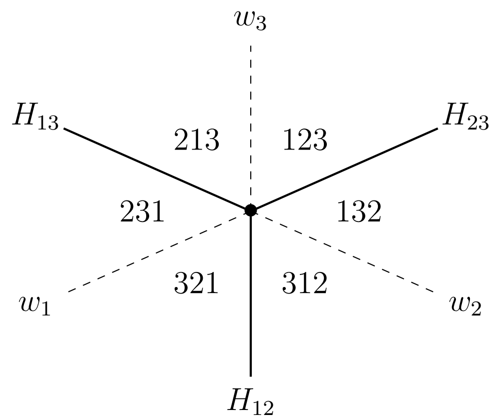


Figure 2.3: Hyperplane arrangement $\mathcal{A}_3$.

The Möbius function [26] on a poset P is defined by $\mu_P(X) = \mu_P(X, \widehat{1})$ where

$$\begin{cases} \mu_P(X,X) = 1 & \text{for all } X \in P, \\ \mu_P(X,Y) = -\sum_{X \leq Z < Y} \mu_P(X,Z) & \text{for } X \preceq Y, \\ \mu_P(X,Y) = 0 & \text{for } X \not\preceq Y. \end{cases}$$

Bigidare, Hanlon, and Rockmore [6] and Brown and Diaconis [11] proved the following theorem.

Theorem 2.2.5 ([6, 11])**.** *For any central hyperplane arrangement $\mathcal{A}$, let $\mathcal{L}(\mathcal{A})$ denote the set of faces of $\mathcal{A}$ (under reverse inclusion of closures) and $L(\mathcal{A})$ denote the edge poset with associated Möbius function $\mu_{L(\mathcal{A})}$. For a face F, let $\widehat{F}$ denote the transition matrix for the action of F on the chambers of $\mathcal{A}$. The face shuffle*

$$M = \sum_{F \in \mathcal{L}(\mathcal{A})} w_F \widehat{F}$$

is diagonalizable and the distinct eigenvalues can be indexed by $X \in L(\mathcal{A})$ *such that*

$$\lambda_X = \sum_{F:\, F \subseteq X} w_F.$$

The multiplicity of λ_X *is* $|\mu_{L(\mathcal{A})}(X)|$.

Example 2.2.6. *For* $n = 3$, *we verify Example 2.2.4 with Theorem 2.2.5. By Example 2.2.3, Figure 2.2 gives the edge poset. Table 2.6 gives the nonempty faces with nonzero weight, the multiplicities and the eigenvalues for the face shuffle.*

$X \in L(\mathcal{A})$	$F \subseteq X$ with $w_F \neq 0$	$\lvert\mu_{L(\mathcal{A}_3)}(X)\rvert$	λ_X
$\mathbb{R}^3$	$\emptyset$	2	0
$\{x_1 = x_2\}$	$F_3 = H_{12}^0 \cap H_{13}^+ \cap H_{23}^+$	1	x_3
$\{x_1 = x_3\}$	$F_2 = H_{12}^- \cap H_{13}^0 \cap H_{23}^+$	1	x_2
$\{x_2 = x_3\}$	$F_1 = H_{12}^- \cap H_{13}^- \cap H_{23}^0$	1	x_1
$\{x_1 = x_2 = x_3\}$	F_1, F_2, F_3	1	$x_1 + x_2 + x_3$

Table 2.6: Eigenvalues of the face shuffle.

Thus, the eigenvalues are $0, 0, x_1, x_2, x_3, x_1 + x_2 + x_3$, *which by Example 2.1.3 are precisely those of the Tsetlin library model.*

2.3 Pop shuffles

Bidigare, Hanlon, and Rockmore [6] and Brown and Diaconis [11] also introduced a pop shuffle on the elements of S_n. This generalizes the Tsetlin library result by allowing for multiple readers to check out multiple books each. After all readers have checked out their books, the first reader places their books back at the end of the shelf in the order they were originally found. The second reader then places their books at the end of the shelf in the order they were found. This continues until all readers have returned their books to the end of the shelf.

Example 2.3.1. *Say we have four books labeled* $1, 2, 3, 4$ *on one shelf and three readers* $R_1, R_2,$ *and* R_3*. Assume the current order of the books is 3142. Say* R_1 *checks out books* 2 *and* 3*,* R_2 *checks out book 4, and* R_3 *checks out books 1. Then* R_1 *puts back books* 2 *and* 3 *in the order they were originally found, i.e., 32. Then* R_2 *puts back book 4 at the end of the shelf, i.e., the shelf is 324. Finally* R_3 *puts back book 1 at the end of the shelf, so we have the arrangement 3241.*

To formally define pop shuffles, we need the notion of set partitions. For any set S, a *set partition* of S is a set of disjoint nonempty subsets of S whose union is S, we refer to each disjoint subset as a *block*. The set of set partitions of S is denoted by $\mathrm{Part}(S)$. An *ordered set partition* is a set partition with a linear ordering on the blocks. The set of ordered set partitions of S is denoted by $\mathrm{Part}^{\mathrm{ord}}(S)$. We will denote an ordered partition by $\underline{\beta}$ and its underlying set partition by β.

Example 2.3.2. *Let* $S = \{1, 2, 3\}$ *and* $\beta = \{12, 3\} \in \mathrm{Part}(S)$*. Then for*

$$\underline{\beta} \in \{(12, 3), (3, 12)\},$$

$\underline{\beta}$ *has underlying set partition* β*.*

The sets $\mathrm{Part}(S)$ and $\mathrm{Part}^{\mathrm{ord}}(S)$ are partially ordered sets ordered by reverse refinement. That is, $\alpha \leq \beta$ (resp. $\underline{\alpha} \leq \underline{\beta}$) if and only if each block of the partition α (resp. ordered partition $\underline{\alpha}$) is a union of blocks from β (resp. $\underline{\beta}$). In this case, β is said to be a *refinement* of α.

Example 2.3.3. *Let* $S = \{1, 2, 3\}$*.Then*

$$\mathrm{Part}(S) = \{\{1, 2, 3\}, \{12, 3\}, \{2, 13\}, \{1, 23\}, \{123\}\}$$

and $\mathrm{Part}^{\mathrm{ord}}(S)$ *is of size 13. Note that* $(12,3)$ *and* $(3,12)$ *are different ordered partitions which have the same underlying set partition,* $\{12,3\}$*. Similarly,* $(13,2)$ *and* $(2,13)$ *are different ordered partitions which both have* $\{13,2\}$ *as the underlying set partition. The lattice* $\mathrm{Part}(\{1,2,3\})$ *is given in Figure 2.4.*

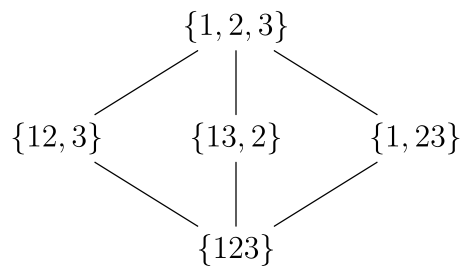


Figure 2.4: The lattice $\mathrm{Part}(\{1,2,3\})$.

Let $\underline{\beta} = (B_1, \ldots, B_m)$ be an ordered partition of $[n]$ where each B_i is a block. Then $\underline{\beta}$ acts on an element of S_n, the symmetric group, by taking the elements in B_1 to the end of the permutation, while preserving the order in which they occur originally. The elements from B_2 are then placed after the elements from B_1, while preserving the original order of the elements from B_2. This is continued until the elements from B_m are placed at the end of the permutation. Such a move $\underline{\beta}$ is called an *elementary pop shuffle* and the action on a permutation π is denoted by $\underline{\beta}\pi$. The corresponding operator in the space of all linear transformations from $V(S_n)$ to $V(S_n)$ is $P_{\underline{\beta}}$ where $V(S_n)$ is the complex vector space on S_n.

Example 2.3.4. *Let* $\pi = 824793615$*, then for* $\underline{\beta} = (6, 25, 879, 134)$*,*

$$\underline{\beta}\pi = \pi' = 625879431.$$

Any $\mathbb{C}$-linear combination of elementary pop shuffles

$$\sum_{\underline{\beta} \in \mathrm{Part}^{\mathrm{ord}}([n])} x_{\underline{\beta}} P_{\underline{\beta}}$$

is called a *pop shuffle.*

Example 2.3.5. *Let* $n = 2$. *Then* $\mathrm{Part}^{\mathrm{ord}}([2]) = \{(1,2),(2,1),(12)\}$. *Assuming lexicographic ordering of the elements in* S_2*, we have*

$$P_{(1,2)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, P_{(2,1)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, P_{(12)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, a pop shuffle is given by

$$\begin{pmatrix} x_{(1,2)} + x_{(12)} & x_{(2,1)} \\ x_{(1,2)} & x_{(2,1)} + x_{(12)} \end{pmatrix}.$$

The *cycle partition* of a permutation $\pi \in S_n$ is the set partition formed from the orbits of π acting on the set $[n]$.

Example 2.3.6. *Let* $n = 9$. *For* $\pi = 824793615 = (18)(2)(3476)(59)$*, the cycle partition of* π *is* $\{18, 2, 3467, 59\}$. *In fact,* π *has cycle partition* $\{18, 2, 3467, 59\}$ *exactly for* π *in*

$$\left\{ \begin{matrix} (18)(2)(3476)(59), (18)(2)(3467)(59), (18)(2)(3647)(59), \\ (18)(2)(3674)(59), (18)(2)(3746)(59), (18)(2)(3764)(59) \end{matrix} \right\}.$$

In general, the number of permutations that have the same cycle partition as β *is*

$$\prod_{B \in \beta} (|B| - 1)!.$$

Theorem 2.3.7 ([6]). *The distinct eigenvalues of the pop shuffle*

$$\sum_{\underline{\beta}\in \mathrm{Part}^{\mathrm{ord}}([n])} x_{\underline{\beta}} P_{\underline{\beta}}$$

are indexed by set partitions $\alpha \in \mathrm{Part}([n])$. If λ_α denotes the eigenvalue associated to α, then

$$\lambda_\alpha = \sum_{\underline{\beta}:\ \beta\leq\alpha} x_{\underline{\beta}}.$$

The multiplicity of λ_α is the number of permutations with cycle partition equal to α.

Example 2.3.8. *For $n = 2$, $\mathrm{Part}([2]) = \{\{1,2\},\{12\}\}$. We have $\beta \leq \{1,2\}$ if and only if $\beta \in \{\{1,2\},\{12\}\}$ and $\beta \leq \{12\}$ if and only if $\beta = \{12\}$. Thus, we have the eigenvalues*

$$\lambda_{\{12\}} = x_{(12)}, \qquad \lambda_{\{1,2\}} = x_{(1,2)} + x_{(2,1)} + x_{(12)}.$$

The cycle permutations in S_2 are (12) and $(1)(2)$ which have cycle partitions $\{12\}$ and $\{1,2\}$, respectively. Thus, each eigenvalue has multiplicity 1.

Example 2.3.9. *Let $n = 3$ and $\alpha = \{12,3\}$. Then α is a refinement of β if and only if $\beta \in \{\{12,3\},\{123\}\}$. Thus, we have*

$$\lambda_\alpha = x_{(12,3)} + x_{(3,12)} + x_{(123)}.$$

The only permutation with cycle partition $\{12,3\}$ is $(12)(3)$, so the multiplicity of α is 1. In fact, we have

$$\prod_{B\in\alpha} = (|B|-1)! = (2-1)!(1-1)! = 1.$$

Example 2.3.10. *Let $n = 9$ and $\alpha = \{18,2,3467,59\}$ as in Example 2.3.6. Then λ_α*

has multiplicity 1!0!3!1!=6.

The pop shuffle can be seen as a specific case of the walk on the chambers of hyperplane arrangements from Section 2.2. Recall that $L(\mathcal{A}_n)$ denotes the intersection lattice, the set of intersections of hyperplanes from $\mathcal{A}_n$, ordered by reverse inclusion.

The following theorem gives an isomorphism between the intersection lattice and set partitions.

Theorem 2.3.11 ([20]). *There is an order-isomorphism* $\phi\colon L(\mathcal{A}_n) \to \mathrm{Part}([n])$.

Example 2.3.12. *Let* $n = 3$*, then Table 2.7 gives the edge in* $L(\mathcal{A}_3)$ *and its corresponding partition in* $\mathrm{Part}([3])$.

Edge	Corresponding set partition
$x_1 = x_2 = x_3$	$\{123\}$
$x_1 = x_2$	$\{12, 3\}$
$x_1 = x_3$	$\{13, 2\}$
$x_2 = x_3$	$\{1, 23\}$
$\mathbb{R}^3$	$\{1, 2, 3\}$

Table 2.7: Edge in the intersection lattice and corresponding partition in $\mathcal{A}_3$.

Recall that the set of faces $\mathcal{L}(\mathcal{A}_n)$ consists of nonempty intersections of the form $\bigcap H_{ij}^{\epsilon_{ij}}$ where $\epsilon_{ij} \in \{0, +, -\}$.

Theorem 2.3.13 ([6]). *There is a bijection* $\phi : \mathcal{L}(\mathcal{A}_n) \to \mathrm{Part}^{\mathrm{ord}}([n])$.

Example 2.3.14 (Pop Shuffle on 3 books)**.** *Consider the hyperplane arrangement* $\mathcal{A}_3$ *consisting of the hyperplanes* $\{H_{ij}\colon 1 \leq i < j \leq 3\}$ *in* $\mathbb{R}^3$ *as described in Section 2.2. Table 2.8 gives the faces of* $\mathcal{A}_3$ *and the corresponding ordered set partition.*

Face	Ordered Partition	Weight
$x_1 > x_2 > x_3$	$(1,2,3)$	$w_{(1,2,3)}$
$x_1 > x_3 > x_2$	$(1,3,2)$	$w_{(1,3,2)}$
$x_2 > x_1 > x_3$	$(2,1,3)$	$w_{(2,1,3)}$
$x_2 > x_3 > x_1$	$(2,3,1)$	$w_{(2,3,1)}$
$x_3 > x_1 > x_2$	$(3,1,2)$	$w_{(3,1,2)}$
$x_3 > x_2 > x_1$	$(3,2,1)$	$w_{(3,2,1)}$
$x_1 > x_2 = x_3$	$(1,23)$	$w_{(1,23)}$
$x_2 > x_1 = x_3$	$(2,13)$	$w_{(2,13)}$
$x_3 > x_1 = x_2$	$(3,12)$	$w_{(3,12)}$
$x_1 = x_2 > x_3$	$(12,3)$	$w_{(1,23)}$
$x_1 = x_3 > x_2$	$(13,2)$	$w_{(13,2)}$
$x_2 = x_3 > x_1$	$(23,1)$	$w_{(23,1)}$
$x_1 = x_2 = x_3$	(123)	$w_{(123)}$

Table 2.8: Faces, ordered set partitions of $\mathcal{A}_3$, and weights in the pop shuffle.

We can see that each chamber of $\mathcal{A}_3$ is characterized by its points satisfying $x_i > x_j > x_k$, where we label the chamber ijk with its corresponding permutation in S_3. Figure 2.5 gives the hyperplane arrangement $\mathcal{A}_3$ where the chambers are labeled by their corresponding permutation in S_3. The faces are also labeled by their weight that allows $F * C$ to correspond to a pop shuffle on three elements.

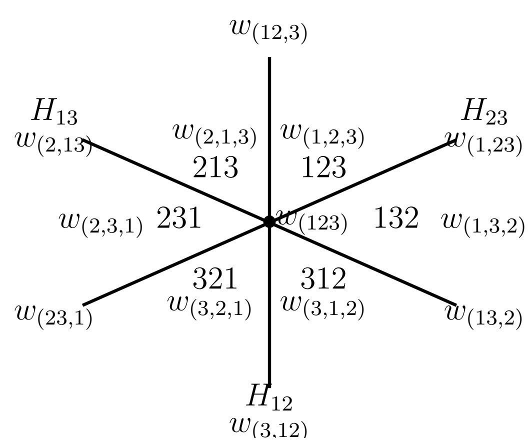


Figure 2.5: Pop shuffle hyperplane arrangement $\mathcal{A}_3$.

2.4 Bands

The Tsetlin library model was generalized by Brown to a class of monoids called left-regular bands [9] and subsequently to all bands [10]. A *left-regular band* (LRB) is a semigroup S (the associative property holds for the set S) such that for all $x, y \in S$,

$$x^2 = x \qquad \text{and} \qquad xyx = xy.$$

The hyperplane arrangements described in Sections 2.2 and 2.3 are LRBs. By Brown [9], S is a LRB if there are a lattice L with maximal element $\widehat{1}$ and a surjection $\operatorname{supp} : S \to L$ satisfying

$$\operatorname{supp} xy = \operatorname{supp} x \vee \operatorname{supp} y \qquad \text{and} \qquad xy = x \text{ if } \operatorname{supp} y \leq \operatorname{supp} x.$$

Let S be a LRB with support lattice L. An element $c \in S$ is called a *chamber* if $\operatorname{supp} c = \widehat{1}$. The set of chambers is denoted by C. An equivalent condition for $\operatorname{supp} c = \widehat{1}$ is $cx = c$ for all $x \in S$. For $X \in L$, c_X is the number of chambers in $S_{\geq X}$, i.e., the number of chambers $c \in C$ such that $c \geq x$, where $\operatorname{supp}(x) = X$.

Theorem 2.4.1 ([9])**.** *Let S be a finite LRB with identity, let $\{w_x\}$ be a probability distribution on S, and let P be the transition matrix of the random walk on chambers:*

$$P(c, d) = \sum_{xc=d} w_x$$

for $c, d \in C$. Then P is diagonalizable. It has an eigenvalue

$$\lambda_X = \sum_{\operatorname{supp} y \leq X} w_y$$

for each $X \in L$, with multiplicity m_X, given by

$$m_X = \sum_{Y \geq X} \mu(X, Y) c_Y$$

where μ is the Möbius function of the lattice L.

This generalizes the Tsetlin library model when we consider the free LRB with identity [9]. Namely, F_n be the free LRB with identity on n generators. Then $x \in F_n$ is of the form $(x_1, \ldots, x_\ell)$ such that $x_i \neq x_j$ for all $0 \leq i < j \leq \ell \leq n$ and $x_i \in [n]$. The multiplication action on two elements of F_n is defined by

$$(x_1, \ldots, x_\ell)(y_1, \ldots, y_m) = (x_1, \ldots, x_\ell, y_1, \ldots, y_m)^\wedge,$$

where the hat means that we omit any element that we have already seen, i.e., appears to the left of it.

If we let $w_x > 0$ for the n generators, i.e., $x = (x_1)$, and $w_x = 0$ for all other x, and walk on the ideal generated by $(y_1, \ldots, y_n)$, this is exactly the "move-to-front" Tsetlin library model. Notice, if we wanted to consider the "move-to-back" Tsetlin library model as described, then we could similarly consider the right-regular band $(x^2 = x, xyx = yx)$ and omit any element we have seen to the right.

Example 2.4.2. *Let $n = 3$. Thinking of the right-regular band (RRB), we have*

$$(x_1, x_2, x_3) \in \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

and a generator x of F_3 is in $\{(1), (2), (3)\}$. Table 2.9 gives the action of x.

(x_1, x_2, x_3)	x	$(x_1, x_2, x_3, x)^\wedge$
(1,2,3)	(1)	(2,3,1)
(1,3,2)	(1)	(3,2,1)
(2,1,3)	(1)	(2,3,1)
(2,3,1)	(1)	(2,3,1)
(3,1,2)	(1)	(3,2,1)
(3,2,1)	(1)	(3,2,1)
(1,2,3)	(2)	(1,3,2)
(1,3,2)	(2)	(1,3,2)
(2,1,3)	(2)	(1,3,2)
(2,3,1)	(2)	(3,1,2)
(3,1,2)	(2)	(3,1,2)
(3,2,1)	(2)	(3,1,2)
(1,2,3)	(3)	(1,2,3)
(1,3,2)	(3)	(1,2,3)
(2,1,3)	(3)	(2,1,3)
(2,3,1)	(3)	(2,1,3)
(3,1,2)	(3)	(1,2,3)
(3,2,1)	(3)	(2,1,3)

Table 2.9: Free RRB F_3 actions with nonzero weight.

The Markov chain formed by the action of the generators x on (x_i, x_j, x_k) where $x = (i)$ has weight w_i is given in Figure 2.6. The edges with weight w_1 are dashed, the edges with weight w_2 are dotted, and the edges with weight w_3 are solid.

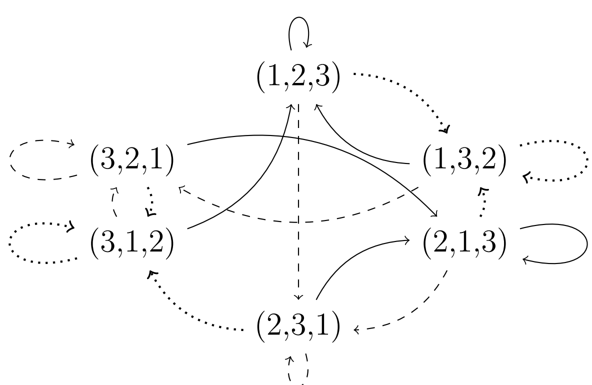


Figure 2.6: Markov chain of F_3.

This is precisely the Markov chain in Figure 2.1, the Tsetlin library Markov chain.

Brown [9] also showed that the hyperplane arrangements discussed in Sections 2.2 and 2.3 are LRB semigroups. Consider the braid arrangement $\mathcal{A}_n$ discussed in Section 2.3, The face semigroup $\mathcal{B}$ consists of ordered partitions $\beta = (B_1, \ldots, B_\ell)$ of the set $[n]$. Two ordered partitions $\underline{\beta}$ and $\underline{\alpha}$ are multiplied together by taking intersections of each block and ordering lexicographically. In other words, for $\underline{\beta} = (B_1, \ldots, B_\ell)$ and $\underline{\alpha} = (A_1, \ldots, A_k)$,

$$\underline{\beta} \circ \underline{\alpha} = (B_1 \cap A_1, \ldots, B_1 \cap A_k, \ldots, B_\ell \cap A_1, \ldots, B_\ell \cap A_k)^\wedge$$

where the hat means to omit empty intersections. Notice that

$$\underline{\beta} \circ \underline{\beta} = \underline{\beta} \text{ and } \underline{\beta} \circ \underline{\alpha} \circ \underline{\beta} = \underline{\beta} \circ \underline{\alpha},$$

so this defines a LRB with identity $\underline{\alpha} = (A_1)$.

Recall from Section 2.3 that the associated lattice is the lattice of set partitions $\mathrm{Part}([n])$ ordered by reverse refinement and the support map defined by

$$\mathrm{supp} : \mathcal{B} \to \mathrm{Part}([n])$$

"forgets" the order of the blocks.

Example 2.4.3. *Let $n = 2$. We verify Example 2.3.8 with the theory of left-regular bands. Let $\{w_{\underline{\beta}}\}$ be a probability distribution on $\mathcal{B}$. By Theorem 2.4.1, we have*

$$\lambda_{\{12\}} = w_{(12)} \qquad \textit{and} \qquad \lambda_{\{1,2\}} = w_{(1,2)} + w_{(2,1)} + w_{(12)}$$

with multiplicities

$$m_{\{12\}} = 1 \cdot 2 - 1 \cdot 1 = 1 \text{ and } m_{\{1,2\}} = 1 \cdot 1 = 1.$$

These are precisely the same eigenvalues as in Example 2.3.8.

Brown [10] also generalized this result to all bands. A *band* is an idempotent semigroup ($x^2 = x$ for all $x \in S$). In fact, for any band S, there is a semilattice L together with a surjection $\operatorname{supp} : S \to L$ such that

$$\operatorname{supp}(xy) = \operatorname{supp} x \vee \operatorname{supp} y \quad \text{and} \quad \operatorname{supp}(x) \geq \operatorname{supp}(y) \quad \Leftrightarrow \quad x = xyx.$$

By Brown [10], for any $X \in L$, let $x \in S$ such that $\operatorname{supp} x = X$ and c_X be the number of chambers in the band xS (number of c such that $cxs = c$). Define a family of integers m_X by

$$c_X = \sum_{Y \geq X} m_Y$$

for each $X \in L$.

Theorem 2.4.4 ([10])**.** *Let S be a finite band with at least one left identity. Let L be its support lattice, let C be the ideal of chambers, and let k be a field. For any element*

$$w = \sum_{x \in S} w_x x \in kS,$$

let T_w be the operator on kC given by left-multiplication by w. Then T_w has an eigenvalue

$$\lambda_X = \sum_{\operatorname{supp} y \leq X} w_y$$

for each $X \in L$, with multiplicity m_X.

2.5 Extended promotion operator

Consider a *naturally labeled* poset P on the set $[n]$, with partial order $\preceq$, where P is naturally labeled if $i \prec j$ in P implies $i < j$ as integers. A *linear extension* of P is a total ordering $\pi = \pi_1 \cdots \pi_n$ of its elements such that $\pi_i \prec \pi_j$ implies $i < j$. The set of linear extensions of P is denoted by $\mathcal{L}(P)$.

Example 2.5.1. *Let P be as in Figure 2.7.*

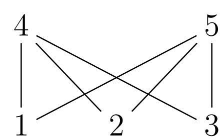


Figure 2.7: Poset on 5 vertices.

The set of linear extensions of P is

$$\mathcal{L}(P) = \left\{ \begin{array}{c} 12345, 12354, 13245, 13254, 21345, 21354, \\ 23145, 23154, 31245, 31254, 32145, 32154 \end{array} \right\}.$$

The moves used in the promotion Markov chain are a generalization of the Schützenberger's [25] promotion operator on $\mathcal{L}(P)$, hence the name, we recall the notion next.

There is a bijection $\partial : \mathcal{L}(P) \to \mathcal{L}(P)$ called the *promotion map* [1, 25, 27]. Let $x \in P$ be such that $\pi^{-1}(x) = 1$. Remove it and replace it by the minimum of all labels covering it, say y. Repeat this process with y until we get to a label that has no nodes covering it. Place the label $n + 1$ at that node. Now, decrease all labels by 1. Denote the new linear extension by $\partial\pi$.

Example 2.5.2. *Let P be as in Figure 2.8 with associated permutation $\pi = 123456789$.*

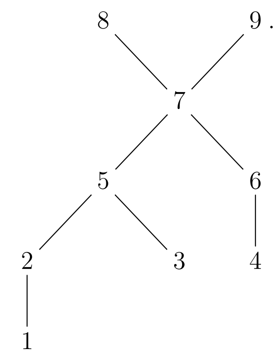


Figure 2.8: Partially ordered set, P.

Table 2.10 shows the promotion process step by step to get $\partial\pi = 134265798$.

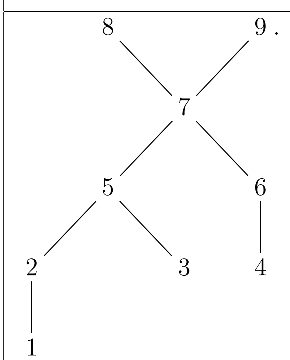
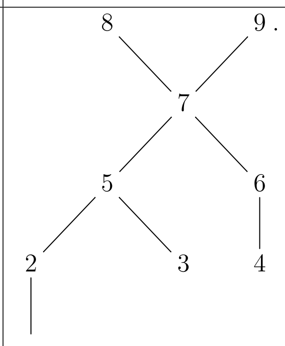
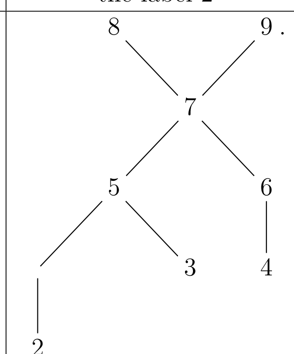
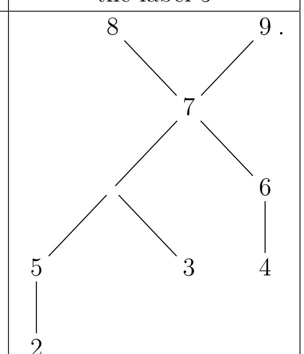
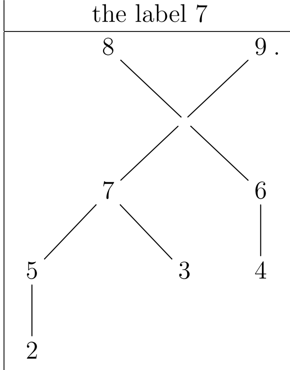
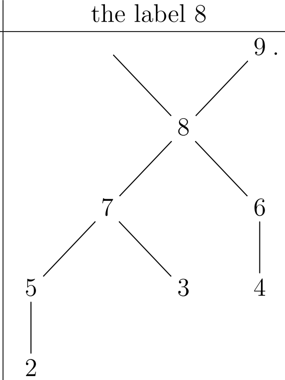
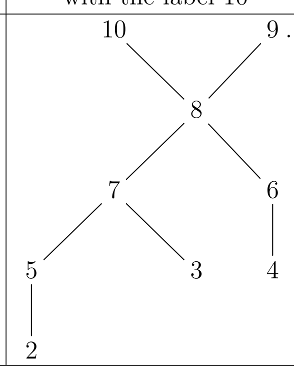
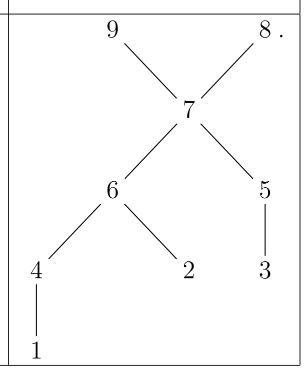
Original P	Remove the label 1	Replace the label 1 with the label 2	Replace the label 2 with the label 5
			
Replace the label 5 with the label 7	Replace the label 7 with the label 8	Replace the label 8 with with the label 10	Decrease all labels by 1
			

Table 2.10: Promotion process from $\pi = 123456789$ to $\partial\pi = 134265798$.

Ayyer, Klee, and Schilling [1, 2] introduced the idea of an extended promotion operator ∂_i on $\mathcal{L}(P)$. This generalizes Schützenberger's [25] promotion operator ∂, which can be expressed in terms of more elementary operators τ_i as shown in [14, 19]. Namely, for $i = 1, \dots, n$ and $\pi = \pi_1 \cdots \pi_n \in \mathcal{L}(P)$, let

$$\tau_i \pi = \begin{cases} \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are incomparable in } P, \\ \pi & \text{otherwise.} \end{cases}$$

In other words, τ_i acts nontrivially if the interchange of π_i and π_{i+1} yields a linear extension of P. The *extended promotion operator* ∂_i, $1 \leq i \leq n$, on $\mathcal{L}(P)$ is defined by

$$\partial_i = \tau_{n-1} \cdots \tau_{i+1} \tau_i$$

and, in particular, $\partial_1 = \partial$. Note that the operators act from the left; so τ_i is applied first, then τ_{i+1}, etc.

Example 2.5.3. *Let P be as in Figure 2.8, this corresponds to the linear extension* 123456789*. Then $\partial\pi =$ 134265798, as seen in Table 2.10. The posets corresponding to π and $\partial\pi$ are given in Figure 2.9. The linear extension corresponding to $\partial\pi$ is exactly $\partial_1\pi$.*

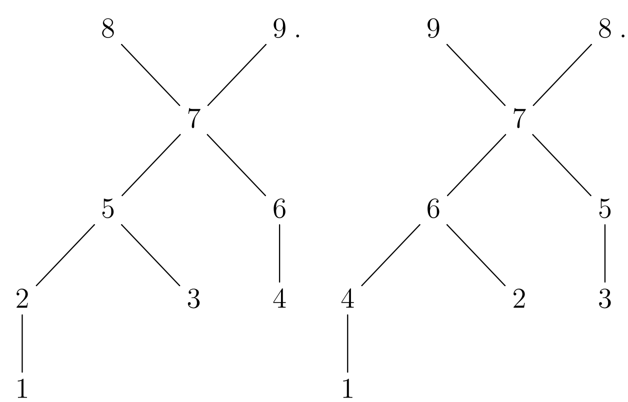


Figure 2.9: Linear extension π and linear extension $\partial\pi$.

The *promotion graph* of P is an edge-weighted directed graph G_P whose vertices are labeled by the elements of $\mathcal{L}(P)$. G_P contains a directed edge from π to π', with edge weight x_{π_i}, if and only if $\pi' = \partial_i \pi$. If $x_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} x_i = 1$, this gives rise to the promotion Markov chain on $\mathcal{L}(P)$, whose row stochastic transition matrix we will denote by M^P.

Example 2.5.4. *Consider the poset P from Figure 2.10.*

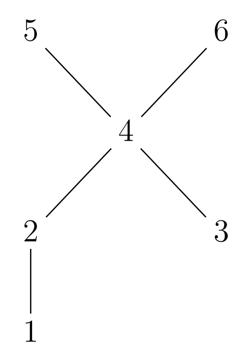


Figure 2.10: An example of an ordinal sum of a forest and a ladder.

The linear extensions of P are

$$\mathcal{L}(P) = \{123456, 123465, 132456, 132465, 312456, 312465\}.$$

For $\pi = 312465 \in \mathcal{L}(P)$,

$$\partial_3 \pi = \tau_5 \tau_4 \tau_3 312465 = \tau_5 \tau_4 312465 = \tau_5 312465 = 312456.$$

Thus, since $\pi_3 = 2$, in G_P there is a directed edge from 312465 to 312456 with edge weight x_2. The promotion graph G_P is given in Figure 2.11.

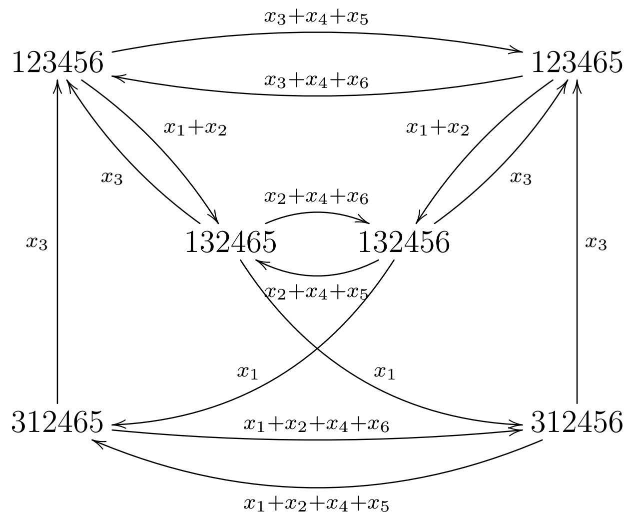


Figure 2.11: Promotion graph of the poset from Figure 2.10. Self-loops are omitted. Instead of multiple edges between vertices we have drawn only one edge with edge weights added.

With the lexicographic ordering of the elements of $\mathcal{L}(P)$, the transition matrix of the promotion Markov chain is given in Figure 2.12.

$$M^P = \begin{pmatrix} x_6 & x_3+x_4+x_5 & 0 & x_1+x_2 & 0 & 0 \\ x_3+x_4+x_6 & x_5 & x_1+x_2 & 0 & 0 & 0 \\ 0 & x_3 & x_6 & x_2+x_4+x_5 & 0 & x_1 \\ x_3 & 0 & x_2+x_4+x_6 & x_5 & x_1 & 0 \\ 0 & x_3 & 0 & 0 & x_6 & x_1+x_2+x_4+x_5 \\ x_3 & 0 & 0 & 0 & x_1+x_2+x_4+x_6 & x_5 \end{pmatrix}.$$

Figure 2.12: Transition matrix of the promotion Markov chain.

A *rooted tree* is a connected poset in which each vertex has at most one successor. A union of rooted trees is called a *rooted forest*. An *upset* (or *upper set*) S in a poset is a subset such that if $x \in S$ and $y \succeq x$, then $y \in S$. Consider a poset P with minimal element $\widehat{0}$ and maximal element $\widehat{1}$, then for each element $x \in P$, the *derangement number* of x is

$$d_x = \sum_{y \succeq x} \mu(x, y) f([y, \widehat{1}]),$$

where $f([y, \widehat{1}])$ is the number of maximal chains in the interval $[y, \widehat{1}]$ and μ is the Möbius function [26].

One of the main results in [1] is that for a rooted forest P, the characteristic polynomial of M^P factors into linear terms.

Theorem 2.5.5 ([1])**.** *Let P be a rooted forest of size n such that $|\mathcal{L}(P)| = N$ and let M^P be the transition matrix of the promotion Markov chain. Then*

$$\det(M - \lambda I_N) = \prod_{\substack{S \subseteq [n] \\ S \text{ upset in } P}} (\lambda - x_S)^{d_S}, \tag{2.1}$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upsets in P.

Example 2.5.6. *Let P be as in Example 2.5.4. The lattice of upsets of P is given in Figure 2.13, written as $S^{(f([S,\widehat{1}]), d_S)}$.*

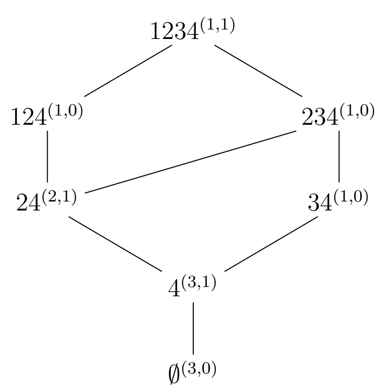


Figure 2.13: Lattice of upsets of P.

As an example of the computations,

$$\begin{aligned} d_{24} &= \sum_{T \succeq S} \mu(S,T) f([T, \widehat{1}]) \\ &= \mu(24,24) f([24,\widehat{1}]) + \mu(24,234) f([234,\widehat{1}]) + \mu(24,124) f([124,\widehat{1}]) \\ &\qquad + \mu(24,1234) f([1234,\widehat{1}]) \\ &= 1 \cdot 2 + -1 \cdot 1 + -1 \cdot 1 + 1 \cdot 1 = 1, \end{aligned}$$

and

$$\begin{aligned} d_{34} &= \sum_{T \succeq S} \mu(S,T) f([T, \widehat{1}]) \\ &= \mu(34,34) f([34,\widehat{1}]) + \mu(34,234) f([234,\widehat{1}]) + \mu(34,1234) f([1234,\widehat{1}]) \\ &= 1 \cdot 1 + -1 \cdot 1 + 0 \cdot 1 = 0 \end{aligned}$$

The upsets that have nonzero derangement number are precisely those in $\{4, 24, 1234\}$*. In fact, the derangement number is 1 in each case. Thus, the eigenvalues of* M^P *are*

$$\{x_4, x_2 + x_4, x_1 + x_2 + x_3 + x_4\}.$$

A linear extension π of a naturally labeled poset is called a *poset derangement* if it has no fixed points when considered as a permutation. Let $\mathfrak{d}_P$ be the number of poset derangements of the naturally labeled poset P.

Example 2.5.7. *Let P be as in Example 2.5.1. Then P has 2 poset derangements, namely, 23154 and 31254.*

If P is a union of chains, the eigenvalues of M^P have an alternate description.

Theorem 2.5.8 ([1])**.** *Let* $P = [n_1] + [n_2] + \cdots + [n_k]$ *be a union of chains of size n* *such that* $|\mathcal{L}(P)| = N$ *whose elements are labeled consecutively within chains. Then*

$$\det(M - \lambda I_N) = \prod_{\substack{S \subseteq [n] \\ S \text{ upset in } P}} (\lambda - x_S)^{\mathfrak{d}_{P \setminus S}}$$

where $\mathfrak{d}_\emptyset = 1$.

In particular, if $[n_i] = [1]$ for all $i = 1, \ldots, k$, then this gives us the Tsetlin library result.

Example 2.5.9. *Let* $P = [1] + [1] + [1]$. *Then* $P =$ $\overset{1}{\bullet} \quad \overset{2}{\bullet} \quad \overset{3}{\bullet}$ *and Table 2.11 gives the upsets in* P *and the corresponding multiplicities.*

Upset	$\mathfrak{d}_{P \setminus S}$
$\emptyset$	2
$\{1\}$	1
$\{2\}$	1
$\{3\}$	1
$\{1,2\}$	0
$\{1,3\}$	0
$\{2,3\}$	0
$\{1,2,3\}$	1

Table 2.11: Upsets of a union of chains and their poset derangement numbers.

Thus, the eigenvalues of the transition matrix are $0, 0, x_1, x_2, x_3, x_1 + x_2 + x_3$, *which by Example 2.1.3 are precicely those of the Tsetlin library model on 3 books.*

Not all posets have this nice property like rooted forests, for example, consider the poset in Figure 2.14

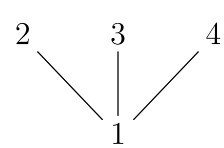


Figure 2.14: Poset P for which M^P has nonlinear eigenvalues.

the characteristic polynomial of M^P is

$$\begin{aligned}&(\lambda-(x_1+x_2+x_3+x_4))[-x_1^5+(-\lambda-(x_2+x_3+x_4))x_1^4\\&\quad+(-\lambda^2-(x_3+x_4)x_2-x_3x_4)x_1^3+(\lambda-x_4)(\lambda-x_3)(\lambda-x_2)x_1^2\\&\quad+(\lambda^3-(x_3x_4+(x_3+x_4)x_2)\lambda+2x_2x_3x_4)\lambda x_1\\&\quad+\lambda^2(\lambda-x_4)(\lambda-x_3)(\lambda-x_2)]\end{aligned}$$

which does not factor into linear terms.

However, the work of Ayyer *et al.* [1, 2] does not fully classify the posets with nice properties. For example, the poset from Figure 3.1 has eigenvalues $x_1+x_2+x_3+x_4, 0, x_3+x_4, -(x_1+x_2)$. Notice that, unlike in the case of forests, some of the eigenvalues contain negative coefficients. In view of this, they made the following conjecture.

If $x \prec y$, we say that y is a *successor* of x.

Conjecture 2.5.10 ([2]). *Let P be a poset of size n which is not a down forest and M^P be its promotion transition matrix. If M^P has eigenvalues which are linear in the parameters $x_1, \ldots, x_n$, then the following hold:*

(1) the coefficients of the parameters in the eigenvalues are only one of ± 1,

(2) each element of P has at most two successors,

(3) the only parameters whose coefficients in the eigenvalues are -1 are those which either have two successors or one of whose successors have two successors.

Even though we have not managed to fully classify the posets with nice properties, our results in Chapter 3 give further support to (1) and (2) from Conjecture 2.5.10, but show that (3) is not true (Example 3.0.2).

2.6 Self-organizing libraries

This section is devoted to the generalization by Björner [7, 8] to random-to-front shuffles on trees. For two sets A_1 and A_2, recall the *direct product*

$$A_1 \times A_2 = \{(a, b) \colon a \in A_1, b \in A_2\},$$

where we denote $A_1 \times \cdots \times A_k = \bigotimes_{i=1}^{k} A_i$. Now, consider a rooted tree T, whose leaves L all have the same depth d. Such a tree is called a *pure tree*. Let I be the set of inner nodes, i.e., nodes that are not leaves. Let C_v be the set of children of a node v and denote the set of linear orderings on C_v by $S(C_v)$. The local orderings of the tree T are given by a choice of linear order of C_v for each $v \in I$. The set of local orderings of T is denoted by

$$\mathcal{O}(T) \cong \bigotimes_{v \in I} S(C_v).$$

Example 2.6.1. *Consider the tree shown in Figure 2.15.*

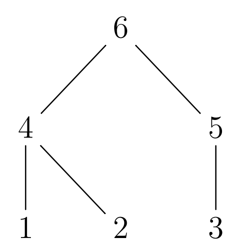


Figure 2.15: An example of a rooted tree of depth 2.

The sets of children of the nodes and their linear orderings are

$$C_4 = \{1, 2\},\ C_5 = \{3\},\ \textit{and}\ C_6 = \{4, 5\}\ \textit{and}$$
$$S(C_4) = \{12, 21\},\ S(C_5) = \{4\}\ \textit{and}\ S(C_6) = \{45, 54\}.$$

Thus, the set of local orderings is

$$\mathcal{O}(T) \cong \{(12,3,45),(12,3,54),(21,3,45),(21,3,54)\}.$$

For convenience we identify these with 12345, 12354, 21345, 21354.

For $E \subseteq L$, we say that a node $v \in T$ is *E-related* if v or one of its descendants is contained in E. The subset E of L acts on $\mathcal{O}(T)$ by rearranging the order locally at each inner node so that the children having some descendant in E come last, otherwise the original order is preserved.

Precisely, for a local ordering of T, $\pi = (\pi_v)_{v \in I}$, and $E \subseteq L$, let C_v^E be the set of E-related elements in C_v and $\underline{\beta}^E = (\underline{\beta}_v^E)_{v \in I}$ where $\underline{\beta}_v^E = (C_v \setminus C_v^E, C_v^E)$, i.e., the ordered set partition consisting of two blocks where one block is the E-related children of v. Then $\underline{\beta}^E \pi = (\underline{\beta}_v^E \pi_v)_{v \in I}$ where $\underline{\beta}_v^E \pi_v$ is the linear ordering of C_v obtained from performing the pop shuffle of the E-related elements to the end of the ordering.

Example 2.6.2. *Let T be as in Figure 2.15. Then, $L = \{1,2,3\}$,*

$$E \in \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

and the associated probabilities are

$$\{x_\emptyset, x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}\}.$$

For $E = \{1,3\}$, we have $C_4^E = \{1\}$, $C_5^E = \{3\}$, $C_6^E = \{4,5\}$, and $\underline{\beta}^E = ((2,1),(3),(45))$. Let $\pi = (12,3,54)$,

$$\underline{\beta}^E \pi = (21,3,54).$$

At each inner node v, we partition its children. Recall that the set partitions of C_v is denoted by $\mathrm{Part}(C_v)$. The set of local partitions of T is denoted by

$$\mathrm{Part}(T) = \bigotimes_{v \in I} \mathrm{Part}(C_v).$$

That is, a set partition $\alpha \in \mathrm{Part}(T)$ consists of a set partition α_v on C_v for each inner node. The partition associated to $E \subseteq L$, denoted by α_v^E, is formed by partitioning C_v into two blocks where one block contains all E-related elements and the other block contains the remaining elements. A subset $E \subseteq L$ is called *α-compatible* if α_v is a refinement of α_v^E for every $v \in I$.

Example 2.6.3. *Let T be as in Figure 2.15. Then*

$$\mathrm{Part}(T) = \{(\{12\}, \{3\}, \{45\}), (\{12\}, \{3\}, \{4,5\}), (\{1,2\}, \{3\}, \{45\}), (\{1,2\}, \{3\}, \{4,5\})\}.$$

Let $E = \{1,3\}$, then $\alpha^E = (\{1,2\}, \{3\}, \{45\})$. Thus, we have that E is α compatible for $\alpha \in \{(\{1,2\}, \{3\}, \{45\}), (\{1,2\}, \{3\}, \{4,5\})\}$.

The main result of Björner [8] is that the transition matrix of the induced random walk of the local orderings of T is diagonalizable and its eigenvalues are linear in the x_E's.

Theorem 2.6.4 ([8])**.** *Let T be a pure tree with leaves L. Furthermore, let $\{x_E\}_{E \subseteq L}$ be a probability distribution on 2^L and M the transition matrix of the induced random walk on local orderings of T:*

$$M(\pi, \pi') = \sum_{E:\, \underline{\beta}^E \pi = \pi'} x_E$$

for $\pi, \pi' \in \mathcal{O}(T)$. Then *(i) The matrix M is diagonalizable.*

(ii) For each $\alpha \in \mathrm{Part}(T)$ there is an eigenvalue

$$\lambda_\alpha = \sum_{E\colon E \text{ is } \alpha\text{-compatible}} x_E.$$

(iii) The multiplicity of the eigenvalue λ_α is

$$m_\alpha = \prod_{v \in I} \prod_{B \in \alpha_v} (|B| - 1)!. \tag{2.2}$$

(iv) These are all the eigenvalues of M.

Example 2.6.5. *Let T be as in Figure 2.15, $\pi = 12354$ and $E = \{1,3\}$, then $\underline{\beta}^E = ((2,1),(3),(45))$ and*

$$\pi' = \underline{\beta}^E \pi = 21354.$$

In fact, $\pi' = \underline{\beta}^E \pi$ for exactly $E \in \{\{1,3\},\{1\}\}$. Thus,

$$M(12354, 21354) = x_{13} + x_1.$$

Also, if $\alpha = \{\{12\},\{3\},\{4,5\}\}$, then E is α-compatible if and only if

$$E \in \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\}.$$

Thus, there is an eigenvalue of M

$$\lambda_\alpha = x_\emptyset + x_{12} + x_3 + x_{123}$$

with multiplicity $m_\alpha = 1$.

In Chapter 4, we consider expanding Björner's [8] result by allowing the case where not all permutations of the children of a node is allowed. In particular, we allow for the case where we associate a poset on the set of leaves instead of the antichain. We restrict E so that only one element from each set of siblings is allowed. The action we consider consists of applying the extended promotion operator on the leaves, and a pop shuffle on the other nodes.

Chapter 3

Properties of the Promotion Markov Chain on Linear Extensions

In this chapter, we study the promotion Markov chain on the set $\mathcal{L}(P)$ of linear extensions of a poset P, as defined in Section 2.5. This Markov chain was introduced by Ayyer *et al.* [1], where they showed that if the Hasse diagram of P is a rooted forest, then the transition matrix has eigenvalues which are linear in the transition probabilities. They noticed, however, that their result does not classify all posets with this nice property. The main goal of this chapter is to provide a larger class of posets for which the same result holds.

Ayyer *et al.* [4] extended the results of the Tsetlin library to the wider class of $\mathcal{R}$-trivial monoids and obtained the description of the eigenvalues of the promotion Markov chain for rooted forests as a consequence of the associated monoid being $\mathcal{R}$-trivial. Our results are about a class of posets whose components are an ordinal sum of a rooted forest and what we call a ladder. The associated monoid is not $\mathcal{R}$-trivial, so we can not use the same arguments as in the case of rooted forests to find its spectrum. However, we show that for these posets, the eigenvalues of the transition matrix are also linear in the probabilities x_i of the moves (Theorem 3.0.1). We also give a way to compute the eigenvalues explicitly (Theorem 3.2.2).

Let P and Q be two posets. The *direct sum* of P and Q is the poset $P+Q$ on their disjoint union such that $x \preceq y$ in $P+Q$ if either (a) $x, y \in P$ and $x \preceq y$ in P or (b) $x, y \in Q$ and $x \preceq y$ in Q. The *ordinal sum* $P \oplus Q$ is a poset on their union such that:

1. For $x, y \in P$, $x \preceq y \in P \oplus Q$ if and only if $x \preceq y \in P$.

2. For $x, y \in Q$, $x \preceq y \in P \oplus Q$ if and only if $x \preceq y \in Q$.

3. For all $x \in P$ and $y \in Q$, $x \preceq y$ in $P \oplus Q$.

We will say that the poset P is a *ladder* of rank k if $P = Q_1 \oplus \cdots \oplus Q_k$ where Q_i is an antichain of size 1 or 2 for all $i = 1, \ldots, k$. For example, the poset from Figure 3.1 is a ladder of rank 2, while the poset from Figure 2.10 is an ordinal sum of a forest on $\{1, 2, 3\}$ and a ladder on $\{4, 5, 6\}$.

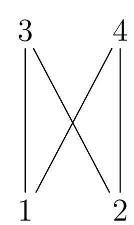


Figure 3.1: A ladder of rank 2.

Our main result is the following theorem.

Theorem 3.0.1. *Let F_i be a rooted forest and let L_i be a ladder for $i = 1, \ldots, k$. The eigenvalues of the promotion transition matrix M^P for $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$*

are linear in $x_1, \ldots, x_n$. Moreover, they can be explicitly computed using the formula for the eigenvalues of forests (Theorem 2.5.5) and Theorem 3.2.2.

The idea behind our proof is that the poset $P' = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ with $|L_i| = n_i$ can be obtained by starting with a poset $P = F_1 \oplus C_{n_1} + \cdots + F_k \oplus C_{n_k}$, where C_i is a chain of size i, and breaking covering relations in the chains C_i one by one. In Theorem 3.2.2, we show how the eigenvalues of the intermediary posets are related. Notice that P is a rooted forest. Therefore, using Theorem 3.2.2, the eigenvalues of $M^{P'}$ and their multiplicities can be obtained from the eigenvalues of M^P given by Theorem 2.5.5. If P' is just a union of ladders, as a starting point one could use the simpler description of the eigenvalues and their multiplicities for a union of chains given in Theorem 2.5.8.

Example 3.0.2. *Let P and P' be as in Figure 3.2.*

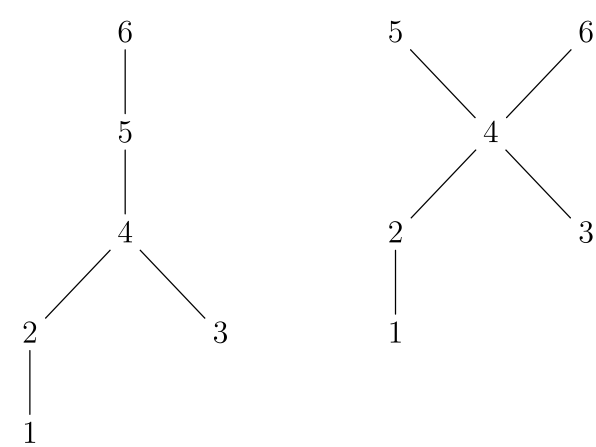


Figure 3.2: A poset P and a corresponding relaxed poset P'.

Then the promotion matrices M^P and $M^{P'}$ are given by:

$$M^P = \begin{pmatrix} x_3+x_4+x_5+x_6 & x_1+x_2 & \\ x_3 & x_2+x_4+x_5+x_6 & x_1 \\ x_3 & 0 & x_1+x_2+x_3+x_4+x_5+x_6 \end{pmatrix},$$

$$M^{P'} = \begin{pmatrix} x_6 & x_3+x_4+x_5 & 0 & x_1+x_2 & 0 & 0 \\ x_3+x_4+x_6 & x_5 & x_1+x_2 & 0 & 0 & 0 \\ 0 & x_3 & x_6 & x_2+x_4+x_5 & 0 & x_1 \\ x_3 & 0 & x_2+x_4+x_6 & x_5 & x_1 & 0 \\ 0 & x_3 & 0 & 0 & x_6 & x_1+x_2+x_4+x_5 \\ x_3 & 0 & 0 & 0 & x_1+x_2+x_4+x_6 & x_5 \end{pmatrix}.$$

The eigenvalues of M^P and $M^{P'}$ are given in Table 3.1.

Eigenvalue of M^P	Eigenvalue of $M^{P'}$
$x_4+x_5+x_6$	$x_4+x_5+x_6$ $-x_4$
$x_2+x_4+x_5+x_6$	$x_2+x_4+x_5+x_6$ $-x_2-x_4$
$x_1+x_2+x_3+x_4+x_5+x_6$	$x_1+x_2+x_3+x_4+x_5+x_6$ $-x_1-x_2-x_3-x_4$

Table 3.1: Eigenvalues of M^P and $M^{P'}$.

Notice that in the last eigenvalue of $M^{P'}$, x_1 appears with a negative coefficient, which contradicts property 3 from Conjecture 2.5.10.

Ayyer *et al.* [1] showed that the promotion Markov chain is irreducible and aperiodic and obtained the following result about its stationary distribution:

Theorem 3.0.3 ([1]). *The stationary state weight of the linear extension $\pi \in \mathcal{L}(P)$ for the discrete-time Markov chain for the promotion graph is proportional to*

$$w(\pi) = \prod_{i=1}^{n} \frac{1}{x_{\pi_1} + \cdots + x_{\pi_i}}.$$

Example 3.0.4. *Let P be as in Figure 3.1. Then Table 3.2 gives the linear extensions of P and the corresponding proportional stationary state weight.*

$\pi \in \mathcal{L}(P)$	$w(\pi)$
1234	$\dfrac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_3+x_4)}$
1243	$\dfrac{1}{x_1(x_1+x_2)(x_1+x_2+x_4)(x_1+x_2+x_3+x_4)}$
2134	$\dfrac{1}{x_2(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_3+x_4)}$
2143	$\dfrac{1}{x_2(x_1+x_2)(x_1+x_2+x_4)(x_1+x_2+x_3+x_4)}$

Table 3.2: Stationary state weights $w(\pi)$ for a ladder P.

These weights do not necessarily sum up to 1, which is remedied by multiplication by a suitable factor Z_P, known as the *partition function*. In [1], the authors found Z_P and in [2] they derived results about convergence to stationarity for rooted forests. In Section 3.3, we describe the partition function when $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ is a union of ordinal sums of forests and ladders, and derive convergence results for the case when $P = F \oplus L$.

The outline of the chapter is as follows: In Section 3.1 we first show that when P is a single ladder, the transition matrix is diagonalizable and we find its eigenfunctions. While the transition matrix of the Tsetlin library is diagonalizable, this is not true for general forests. Then we prove Theorem 3.0.1 in Section 3.2. In Section 3.3 we derive the partition function for our class of posets and convergence results for the case when P has a single component.

3.1 The case of one ladder

In this section we show that when P is a ladder, the promotion transition matrix M^P is diagonalizable and we explicitly describe its eigenvalues and eigenfunctions. We note that in general, M^P is not diagonalizable if P is a forest or a union of two or more ladders. Let I_n denote the identity matrix of size n and J_n be the anti-diagonal matrix of size n

$$J_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For two matrices, A and B, their *Kronecker product* is:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

Example 3.1.1. *For* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} x_5 & x_4 \\ x_5 & x_4 \end{pmatrix}$,

$$A \otimes B = \begin{pmatrix} x_5 & x_4 & 0 & 0 \\ x_5 & x_4 & 0 & 0 \\ 0 & 0 & x_5 & x_4 \\ 0 & 0 & x_5 & x_4 \end{pmatrix}.$$

Lemma 3.1.2. *Let P be a poset of size n and let Q be an antichain of size $j \in \{1, 2\}$.*

Then

$$M^{P \oplus Q} = M^P \otimes J_j + I_N \otimes M^Q,$$

where $N = |\mathcal{L}(P)|$.

Proof. First, let $Q = \bullet a$. Then $M^Q = \begin{pmatrix} x_a \end{pmatrix}$ and

$$\mathcal{L}(P \oplus Q) = \{\pi a \colon \pi \in \mathcal{L}(P)\}.$$

One can readily see that $\pi a \xrightarrow{x_a} \pi a$ and $\pi a \xrightarrow{x_j} \pi' a$ in the promotion graph $G_{P \oplus Q}$ if and only if $\pi \xrightarrow{x_j} \pi'$ in G_P, $j = 1, \ldots, n$. Therefore,

$$M^{P \oplus Q} = M^P + x_a I_N = M^P \otimes J_1 + I_N \otimes M^Q.$$

Now, let $Q = \bullet a \quad \bullet b$. Then $M^Q = \begin{pmatrix} x_b & x_a \\ x_b & x_a \end{pmatrix}$ and

$$\mathcal{L}(P \oplus Q) = \{\pi ab, \pi ba \colon \pi \in \mathcal{L}(P)\}.$$

The matrix $M^{P \oplus Q}$ is of size $2N$ with blocks

$$\begin{array}{c|cc} & \pi ab & \pi ba \\ \hline \pi ab & x_b & x_a \\ \pi ba & x_b & x_a \end{array}$$

on the diagonal. Furthermore, for $j \neq a, b$, if $\pi \xrightarrow{x_j} \pi'$ in M^P, then in $M^{P \oplus Q}$ we have

$$\begin{array}{c} \\ \pi ab \\ \pi ba \end{array} \begin{array}{c} \begin{array}{cc} \pi' ab & \pi' ba \end{array} \\ \begin{pmatrix} 0 & x_j \\ x_j & 0 \end{pmatrix} \end{array}$$

Thus, $M^{P \oplus Q} = M^P \otimes J_2 + I_N \otimes M^Q$. □

Example 3.1.3. *Consider $P \oplus Q$ given in Figure 3.3,*

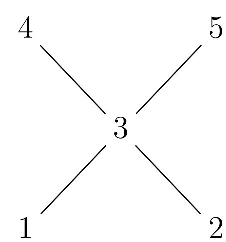
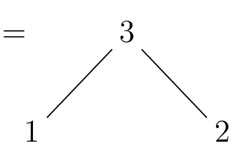


Figure 3.3: Ordinal sum $P \oplus Q$.

where $P =$ 

and $Q =$ 

. Notice that

$$M^{P \oplus Q} = \begin{pmatrix} x_5 & x_4 + x_2 + x_3 & 0 & x_1 \\ x_5 + x_2 + x_3 & x_4 & x_1 & 0 \\ 0 & x_2 & x_5 & x_4 + x_1 + x_3 \\ x_2 & 0 & x_5 + x_1 + x_3 & x_4 \end{pmatrix}.$$

Also, $M^P = \begin{pmatrix} x_2 + x_3 & x_1 \\ x_2 & x_1 + x_3 \end{pmatrix}$ *and* $M^Q = \begin{pmatrix} x_5 & x_4 \\ x_5 & x_4 \end{pmatrix}$*. Thus,*

$$\begin{aligned} M^P \otimes J_2 + I_2 \otimes M^Q &= \begin{pmatrix} 0 & x_2 + x_3 & 0 & x_1 \\ x_2 + x_3 & 0 & x_1 & 0 \\ 0 & x_2 & 0 & x_1 + x_3 \\ x_2 & 0 & x_1 + x_3 & 0 \end{pmatrix} + \begin{pmatrix} x_5 & x_4 & 0 & 0 \\ x_5 & x_4 & 0 & 0 \\ 0 & 0 & x_5 & x_4 \\ 0 & 0 & x_5 & x_4 \end{pmatrix} \\ &= \begin{pmatrix} x_5 & x_4 + x_2 + x_3 & 0 & x_1 \\ x_5 + x_2 + x_3 & x_4 & x_1 & 0 \\ 0 & x_2 & x_5 & x_4 + x_1 + x_3 \\ x_2 & 0 & x_5 + x_1 + x_3 & x_4 \end{pmatrix} \\ &= M^{P \oplus Q}. \end{aligned}$$

Corollary 3.1.4. *Let* $P = Q_1 \oplus \cdots \oplus Q_k$ *be a rank* k *ladder and let*

$$B_i = \begin{cases} \begin{pmatrix} x_{b_i} & x_{a_i} \\ x_{b_i} & x_{a_i} \end{pmatrix} & \text{if } Q_i = \bullet a_i \quad \bullet b_i \\ (x_{a_i}) & \text{if } Q_i = \bullet a_i \,. \end{cases}$$

Then $M^P = \sum_{t=1}^{k} I_{|Q_1|} \otimes \cdots \otimes I_{|Q_{t-1}|} \otimes B_t \otimes J_{|Q_{t+1}|} \otimes \cdots \otimes J_{|Q_k|}$.

Proof. Since $M^{Q_i} = B_i$, the claim follows by iteratively applying Lemma 3.1.2. □

Example 3.1.5. *Let P be as in Figure 3.3. Then by Example 3.1.3,*

$$M^P = \begin{pmatrix} x_5 & x_4+x_2+x_3 & 0 & x_1 \\ x_5+x_2+x_3 & x_4 & x_1 & 0 \\ 0 & x_2 & x_5 & x_4+x_1+x_3 \\ x_2 & 0 & x_5+x_1+x_3 & x_4 \end{pmatrix}.$$

Notice that $Q_1 = \overset{1}{\bullet} \qquad \overset{2}{\bullet}$, $Q_2 = \overset{3}{\bullet}$, *and* $Q_3 = \overset{4}{\bullet} \qquad \overset{5}{\bullet}$. *Then we have*

$$B_1 = \begin{pmatrix} x_2 & x_1 \\ x_2 & x_1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} x_3 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} x_5 & x_4 \\ x_5 & x_4 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} M^P &= B_1 \otimes J_1 \otimes J_2 + I_2 \otimes B_2 \otimes J_2 + I_2 \otimes I_1 \otimes B_3 \\ &= \begin{pmatrix} x_2 & x_1 \\ x_2 & x_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (x_3) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_5 & x_4 \\ x_5 & x_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_2 & 0 & x_1 \\ x_2 & 0 & x_1 & 0 \\ 0 & x_2 & 0 & x_1 \\ x_2 & 0 & x_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x_3 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & x_3 & 0 \end{pmatrix} + \begin{pmatrix} x_5 & x_4 & 0 & 0 \\ x_5 & x_4 & 0 & 0 \\ 0 & 0 & x_5 & x_4 \\ 0 & 0 & x_5 & x_4 \end{pmatrix} \\ &= \begin{pmatrix} x_5 & x_4+x_2+x_3 & 0 & x_1 \\ x_5+x_2+x_3 & x_4 & x_1 & 0 \\ 0 & x_2 & x_5 & x_4+x_1+x_3 \\ x_2 & 0 & x_5+x_1+x_3 & x_4 \end{pmatrix}. \end{aligned}$$

To describe the eigenvalues and eigenfunctions of M^P for a ladder

$$P = Q_1 \oplus \cdots \oplus Q_k,$$

we consider the set of vectors v and corresponding scalars c^v that can be obtained as follows in Algorithm 1:

$c_0 = 0$ **for** $i = 1$ **to** k **do**

  if $|Q_i| = 1$ **then**

    $v_i = (1)$ $c_i = c_{i-1} + x_{a_i}$

  end

  if $|Q_i| = 2$ **then**

    $v_i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

    $c_i = c_{i-1} + x_{a_i} + x_{b_i}$

    or

    $v_i = \begin{pmatrix} -x_{a_i} \\ x_{b_i} \end{pmatrix} - c_{i-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

    $c_i = -c_{i-1}$

  end

end

$v = v_1 \otimes \cdots \otimes v_k$

$c^v = c_k$

Algorithm 1: Algorithm for finding the eigenvalues and eigenfunctions of a ladder.

Example 3.1.6. *Let P be as in Figure 3.3. The eigenfunctions v that can be generated this way for P and their corresponding scalar c^v are given in Table 3.3.*

v	c^v
$\begin{pmatrix}1\\1\end{pmatrix} \otimes (1) \otimes \begin{pmatrix}1\\1\end{pmatrix}$	$x_1 + x_2 + x_3 + x_4 + x_5$
$\begin{pmatrix}1\\1\end{pmatrix} \otimes (1) \otimes \left[\begin{pmatrix}-x_4\\x_5\end{pmatrix} - (x_1 + x_2 + x_3)\begin{pmatrix}1\\-1\end{pmatrix}\right]$	$-(x_1 + x_2 + x_3)$
$\begin{pmatrix}-x_1\\x_2\end{pmatrix} \otimes (1) \otimes \begin{pmatrix}1\\1\end{pmatrix}$	$x_3 + x_4 + x_5$
$\begin{pmatrix}-x_1\\x_2\end{pmatrix} \otimes (1) \otimes \left[\begin{pmatrix}-x_4\\x_5\end{pmatrix} - x_3\begin{pmatrix}1\\-1\end{pmatrix}\right]$	$-x_3$

Table 3.3: Eigenfunctions and corresponding eigenvalue of M^P.

As an example of the computations, since $|Q_1| = 2$, choose $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$. Then $c_1 = x_1 + x_2$. Then since $|Q_2| = 1$, we have $v_2 = (1)$ and $c_2 = c_1 + x_3 = x_1 + x_2 + x_3$. Finally, since $|Q_3| = 2$, we can choose

$$v_3 = \begin{pmatrix}-x_4\\x_5\end{pmatrix} - c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-x_4\\x_5\end{pmatrix} - (x_1 + x_2 + x_3)\begin{pmatrix}1\\-1\end{pmatrix},$$

and $c_3 = -c_2 = -(x_1 + x_2 + x_3)$.

Theorem 3.1.7. *If $P = Q_1 \oplus \cdots \oplus Q_k$ is a ladder, then M^P is diagonalizable. In particular, the eigenvalues of M^P are exactly the scalars c^v that can be obtained using Algorithm 1 with corresponding eigenfunctions v.*

Proof. Let $\widetilde{v_i} = J_{|Q_i|}v_i$. In view of Corollary 3.1.4, it's sufficient to prove that for $0 \leq m \leq k-1$,

$$\sum_{t=k-m}^{k} I_{|Q_1|} \otimes \cdots \otimes I_{|Q_{k-t}|} \otimes B_{k-t+1} \otimes J_{|Q_{k-t+2}|} \otimes \cdots \otimes J_{|Q_k|}(v_1 \otimes \cdots \otimes v_k)$$
$$= c_{m+1}v_1 \otimes \cdots \otimes v_m \otimes \widetilde{v}_{m+1} \otimes \cdots \otimes \widetilde{v}_k.$$

For $m = 0$,

$$\sum_{t=k}^{k} I_{|Q_1|} \otimes \cdots \otimes I_{|Q_{k-t}|} \otimes B_{k-t+1} \otimes J_{|Q_{k-t+2}|} \otimes \cdots \otimes J_{|Q_k|}(v_1 \otimes \cdots \otimes v_k)$$

$$= B_1 v_1 \otimes \widetilde{v}_2 \otimes \cdots \otimes \widetilde{v}_k$$

$$= \begin{cases} (x_{a_1}) v_1 \otimes \widetilde{v}_2 \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_1 = (1) \\ (x_{a_1} + x_{b_1}) v_1 \otimes \widetilde{v}_2 \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -c_0 v_1 \otimes \widetilde{v}_2 \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_1 = \begin{pmatrix} -x_{a_1} \\ x_{b_1} \end{pmatrix} \end{cases}$$

$$= c_1 v_1 \otimes \widetilde{v}_2 \otimes \cdots \otimes \widetilde{v}_k.$$

Using the induction hypothesis, we have

$$\sum_{t=k-m}^{k} I_{|Q_1|} \otimes \cdots \otimes I_{|Q_{k-t}|} \otimes B_{k-t+1} \otimes J_{|Q_{k-t+2}|} \otimes \cdots \otimes J_{|Q_k|}(v_1 \otimes \cdots \otimes v_k)$$

$$= v_1 \otimes \cdots \otimes v_m \otimes B_{m+1} v_{m+1} \otimes \widetilde{v}_{m+2} \otimes \cdots \otimes \widetilde{v}_k$$

$$+ c_m v_1 \otimes \cdots \otimes v_m \otimes \widetilde{v}_{m+1} \otimes \widetilde{v}_{m+2} \otimes \cdots \otimes \widetilde{v}_k$$

$$= v_1 \otimes \cdots \otimes v_m \otimes (B_{m+1} v_{m+1} + c_m \widetilde{v}_{m+1}) \otimes \widetilde{v}_{m+2} \otimes \cdots \otimes \widetilde{v}_k$$

$$= \begin{cases} v_1 \otimes \cdots \otimes (c_m + x_{a_{m+1}}) v_{m+1} \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_{m+1} = (1) \\ v_1 \otimes \cdots \otimes (c_m + x_{a_{m+1}} + x_{b_{m+1}}) v_{m+1} \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_{m+1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_1 \otimes \cdots \otimes v_m \otimes (-c_m v_{m+1}) \otimes \widetilde{v}_{m+2} \otimes \cdots \otimes \widetilde{v}_k & \text{if } v_{m+1} = \begin{pmatrix} -x_{a_{m+1}} \\ x_{b_{m+1}} \end{pmatrix} -c_m \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$$

$$= c_{m+1} v_1 \otimes \cdots \otimes v_m \otimes \widetilde{v}_{m+1} \otimes \cdots \otimes \widetilde{v}_k.$$

□

3.2 Proof of Theorem 3.0.1

For a poset P, let R_P be the set of all pairs (a, b) for which P can be written in the form

$$P = Q' \oplus a \oplus b \oplus Q'' + P_2.$$

Example 3.2.1. *Let P be as in Figure 3.4, then* $R_P = \{(3, 4), (4, 5), (9, 10)\}$.

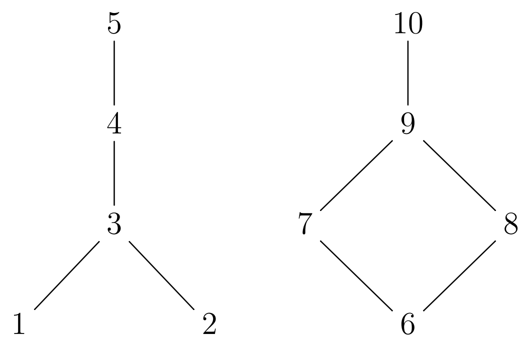


Figure 3.4: Poset with P with $R_P = \{(3, 4), (4, 5), (9, 10)\}$.

Throughout this section we will assume that $R_P \neq \emptyset$ and for a pair $(a, b) \in R_P$, we will denote by P' the poset $P \backslash \{(a, b)\}$, i.e., the poset whose Hasse diagram is obtained from the Hasse diagram of P by deleting the edge that represents the covering relation $a \prec b$. We will say that M^P has the *upset property* if its characteristic polynomial factors into linear terms and for each eigenvalue $x^{\mathfrak{s}} = \sum c_k^{\mathfrak{s}} x_k$ of M^P and a pair $(a, b) \in R_P$, the following two conditions are true:

(a) $x_a \in x^{\mathfrak{s}} \implies x_b \in x^{\mathfrak{s}}$ and $c_a^{\mathfrak{s}} = c_b^{\mathfrak{s}}$

(b) $x_b \in x^{\mathfrak{s}}, x_a \notin x^{\mathfrak{s}} \implies x_k \notin x^{\mathfrak{s}}$ for $k \prec a$.

Here and throughout this thesis, we will use $x_k \in x^{\mathfrak{s}}$ to denote that x_k appears in $x^{\mathfrak{s}}$ with a nonzero coefficient. Note that the matrix M^P can be written as

$M^P = \sum x_i G_i$, where G_i are the matrices corresponding to the extended promotion operators ∂_i.

Theorem 3.2.2. *Let $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and $P' = P \setminus \{(a,b)\}$. Suppose the G_i are simultaneously upper-triangularizable matrices. If M^P has the upset property then so does $M^{P'}$. In particular, for each eigenvalue $x^{\mathfrak{s}} = \sum c_k^{\mathfrak{s}} x_k$ of M^P, $M^{P'}$ has two eigenvalues given by*

$$\begin{cases} x^{\mathfrak{s}}, \sum_{k \not\preceq_P b} c_k^{\mathfrak{s}} x_k - \sum_{k \prec_P a} c_k^{\mathfrak{s}} x_k & \text{if } x_a, x_b \in x^{\mathfrak{s}} \text{ or } x_a, x_b \notin x^{\mathfrak{s}}, \\ x^{\mathfrak{s}}, x^{\mathfrak{s}} - c_b^{\mathfrak{s}} x_b + c_b^{\mathfrak{s}} x_a & \text{if } x_a \notin x^{\mathfrak{s}}, x_b \in x^{\mathfrak{s}}. \end{cases}$$

Remark 3.2.3. *The assumption that the G_i's are simultaneously upper-triangularizable is stronger than asking that the characteristic polynomial M^P factors into linear terms. We do not know whether this stronger assumption is necessary, but we need it in our proof.*

Notice that each poset $F_1 \oplus L_1 + \cdots F_k \oplus L_k$ for forests F_i and ladders L_i, can be obtained starting from a forest in which the upper parts of the tree components are chains and breaking covering relations in the chains. Moreover, the transition matrix of a forest satisfies the assumptions of Theorem 3.2.2 because, as proved in [1], the monoid generated by the matrices G_i is $\mathcal{R}$-trivial and the eigenvalues of the transition matrix are supported on the upsets of the forest (Theorem 2.5.5). Therefore, Theorem 3.0.1 follows from Theorem 3.2.2.

Example 3.2.4. *Let P and P' be the posets given in Figure 3.5. With our notation, $P' = P \setminus \{(5,6)\}$.*

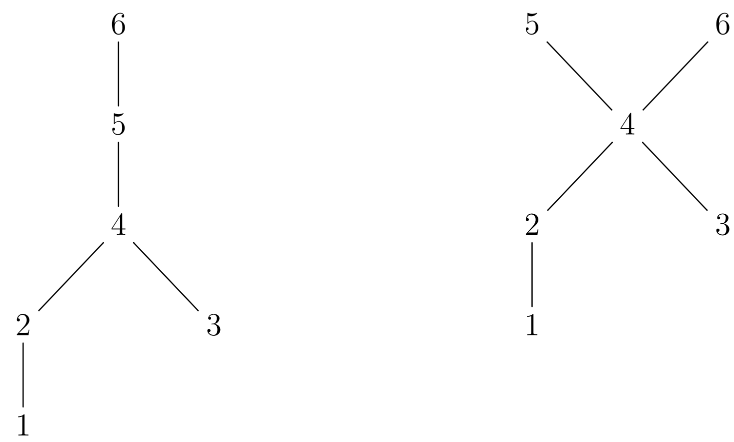


Figure 3.5: A forest P, and the associated poset P', obtained by breaking a covering relation.

Note that $R_P = \{(4,5),(5,6)\}$, $a = 5$, $b = 6$. *By Theorem 2.5.5, the eigenvalues of* M^P *are*

$$x_4 + x_5 + x_6, x_2 + x_4 + x_5 + x_6, x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

and by Theorem 3.2.2, the eigenvalues of $M^{P'}$ *are*

$$x_4+x_5+x_6, -x_4, x_2+x_4+x_5+x_6, -(x_2+x_4), x_1+x_2+x_3+x_4+x_5+x_6, -(x_1+x_2+x_3+x_4).$$

Notice that in the last eigenvalue, x_1 *shows up with a negative coefficient, which contradicts the conjectured Property* (3) *from Conjecture 2.5.10.*

The rest of this section is devoted to the proof of Theorem 3.2.2 which is based on several lemmas that we prove first. For the posets P and P' described at the beginning of this section, and $\pi \in \mathcal{L}(P)$, let $\widehat{\pi} \in \mathcal{L}(P')$ be the linear extension of P' obtained by interchanging a and b. Then

$$\mathcal{L}(P') = \{\pi, \widehat{\pi} \colon \pi \in \mathcal{L}(P)\}.$$

Recall that G_P is the promotion graph of the poset P. The graphs G_P and G'_P are closely related as described in the following lemma.

Lemma 3.2.5. *Let $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and let $P' = P \setminus \{(a,b)\}$:*

(1) If $k \prec a$ and $\pi \xrightarrow{x_k} \widetilde{\pi}$ in G_P, then $\pi \xrightarrow{x_k} \widehat{\widetilde{\pi}}$ and $\widehat{\pi} \xrightarrow{x_k} \widetilde{\pi}$ in $G_{P'}$.

(2) If $k \not\preceq a, b$ and $\pi \xrightarrow{x_k} \widetilde{\pi}$ in G_P, then $\pi \xrightarrow{x_k} \widetilde{\pi}$ and $\widehat{\pi} \xrightarrow{x_k} \widehat{\widetilde{\pi}}$ in $G_{P'}$.

(3) If $\pi \xrightarrow{x_a} \widetilde{\pi}$ in G_P, then $\pi \xrightarrow{x_a} \widehat{\widetilde{\pi}}$ and $\widehat{\pi} \xrightarrow{x_b} \widetilde{\pi}$ in $G_{P'}$.

(4) If $\pi \xrightarrow{x_b} \widetilde{\pi}$ in G_P, then $\pi \xrightarrow{x_b} \widetilde{\pi}$ and $\widehat{\pi} \xrightarrow{x_a} \widehat{\widetilde{\pi}}$ in $G_{P'}$.

Proof. Notice that the structure of P and P' implies that for $x \neq a, b$, $x \prec a$ (respectively, $a \prec x$) if and only if $x \prec b$ (respectively, $b \prec x$). Let $m = \partial_{\pi^{-1}(k)}$. We split the analysis into four cases.

(1) If $k \prec a$, then $\pi \in \mathcal{L}(P)$ is of the form $\pi = A_1 k A_2 a B b C$. Because of the structure of P', we have that for every x in A_2, $k \prec x$ implies $k \prec a$. Therefore, $\widetilde{\pi} = \partial_m \pi = A_1(\partial_1 k A_2) B a (\partial_1 b C)$. In $\mathcal{L}(P')$, however, since a and b are incomparable, $\partial_m \pi = A_1(\partial_1 k A_2) B b (\partial_1 a C) = \widehat{\widetilde{\pi}}$. The last equality is true because $\partial_1 a C$ can be obtained from $\partial_1 b C$ by replacing b with a. Also,

$$\partial_m \widehat{\pi} = \partial_m A_1 k A_2 b B a C = A_1(\partial_1 k A_2) B a (\partial_1 b C) = \widetilde{\pi}.$$

(2) If $k \not\preceq a, b$, there are three possible subcases:

(2a). If $\pi = AaBbC_1kC_2$, then $\widetilde{\pi} = \partial_m \pi = AaBbC_1(\partial_1 k C_2)$. But then, clearly, in $\mathcal{L}(P')$, $\partial_m \pi = \widetilde{\pi}$ as well. Also, $\partial_m \widehat{\pi} = \partial_m AbBaC_1kC_2 = AbBaC_1(\partial_1 k C_2) = \widehat{\widetilde{\pi}}$.

(2b). If $\pi = AaB_1kB_2bC$, the analysis is similar to the previous case. Namely, $\widetilde{\pi} = \partial_m \pi = AaB_1(\partial_1 k B_2 b C)$. So, in $\mathcal{L}(P')$, $\partial_m \pi = \widetilde{\pi}$ as well. Also, $\partial_m \widehat{\pi} = \partial_m AbB_1kB_2aC = AbB_1(\partial_1 k B_2 a C) = \widehat{\widetilde{\pi}}$.

(2c). If $\pi = A_1kA_2aBbC$, then notice that $\partial_1 kA_2$ ends with an element c which is also incomparable with a. Therefore, c will swap with a and a will precede b in $\widetilde{\pi}$. Hence, in $\mathcal{L}(P')$, $\partial_m\pi = \widetilde{\pi}$ as well. Now it's not hard to see that $\partial_m\widehat{\pi} = \partial_m A_1kA_2bBaC = \widehat{\widetilde{\pi}}$.

(3) Let $\pi = AaBbC$. The elements in B are incomparable to both a and b and therefore, for $m = \partial_{\pi^{-1}(a)}$, $\widetilde{\pi} = \partial_m\pi = ABa(\partial_1 bC)$. However, in $\mathcal{L}(P')$, a and b can swap, so $\partial_m\pi = ABb(\partial_1 aC) = \widehat{\widetilde{\pi}}$. Also,

$$\partial_m\widehat{\pi} = \partial_m AbBaC = ABa(\partial_1 bC) = \widetilde{\pi}.$$

(4) In this case for $\pi = AaBbC \in \mathcal{L}(P)$, $\widetilde{\pi} = \partial_m\pi = AaB(\partial_1 bC)$. So, in $\mathcal{L}(P')$, $\partial_m\pi = \widetilde{\pi}$ as well and $\partial_m\widehat{\pi} = \partial_m AbBaC = AbB(\partial_1 aC) = \widehat{\widetilde{\pi}}$. $\square$

Let P be a poset of size n of the form $P = Q' \oplus a \oplus b \oplus Q'' + P_2$. For the transition matrix M^P of size m, we will denote by $\partial_{a,b}M^P$ the $2m \times 2m$ matrix obtained by replacing each entry of M^P by a 2×2 block using the linear extension of the map:

$$\begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_k\right) \end{array} \longmapsto \begin{array}{c|cc} & \widetilde{\pi} & \widehat{\widetilde{\pi}} \\ \hline \pi & 0 & x_k \\ \widehat{\pi} & x_k & 0 \end{array} \text{ for } k \prec a \qquad \begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_a\right) \end{array} \longmapsto \begin{array}{c|cc} & \widetilde{\pi} & \widehat{\widetilde{\pi}} \\ \hline \pi & 0 & x_a \\ \widehat{\pi} & x_b & 0 \end{array}$$

$$\begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_k\right) \end{array} \longmapsto \begin{array}{c|cc} & \widetilde{\pi} & \widehat{\widetilde{\pi}} \\ \hline \pi & x_k & 0 \\ \widehat{\pi} & 0 & x_k \end{array} \text{ for } k \not\preceq b \qquad \begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_b\right) \end{array} \longmapsto \begin{array}{c|cc} & \widetilde{\pi} & \widehat{\widetilde{\pi}} \\ \hline \pi & x_b & 0 \\ \widehat{\pi} & 0 & x_a \end{array}$$

In particular, a zero entry goes to a 2×2 block of zeros.

Corollary 3.2.6. *Let $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and let $P' = P \setminus \{(a,b)\}$. Then $M^{P'} = \partial_{a,b} M^P$ in an appropriate basis of $\mathcal{L}(P')$.*

Example 3.2.7. *Let P and P' be as in Example 3.2.4. Then*

$$L(P) = \{123456, 132456, 312456\},$$
$$\mathcal{L}(P') = \{123456, 123465, 132456, 132465, 312456, 312465\},$$

$$M^P = \begin{pmatrix} x_3+x_4+x_5+x_6 & x_1+x_2 & 0 \\ x_3 & x_2+x_4+x_5+x_6 & x_1 \\ x_3 & 0 & x_1+x_2+x_4+x_5+x_6 \end{pmatrix}$$

and

$$M^{P'} = \begin{pmatrix} x_6 & x_3+x_4+x_5 & 0 & x_1+x_2 & 0 & 0 \\ x_3+x_4+x_6 & x_5 & x_1+x_2 & 0 & 0 & 0 \\ 0 & x_3 & x_6 & x_2+x_4+x_5 & 0 & x_1 \\ x_3 & 0 & x_2+x_4+x_6 & x_5 & x_1 & 0 \\ 0 & x_3 & 0 & 0 & x_6 & x_1+x_2+x_4+x_5 \\ x_3 & 0 & 0 & 0 & x_1+x_2+x_4+x_6 & x_5 \end{pmatrix}.$$

For a complex matrix S, denote by $\partial S = S \otimes I_2$. If E is an elementary matrix of size k corresponding to a row operation R, then ∂E corresponds to performing a corresponding operation to 2 rows on a matrix of size $2k$.

Lemma 3.2.8. *Let S be a matrix with complex entries and M a matrix whose entries are homogeneous degree-1 polynomials in $x_1, \ldots, x_n$. Then*

$$(\partial S)(\partial_{a,b} M) = \partial_{a,b}(SM) \quad \text{and} \quad (\partial_{a,b} M)(\partial S) = \partial_{a,b}(MS).$$

Proof. Notice that the definition of $\partial_{a,b}M$ can be restated as

$$\partial_{a,b}M = M\Big|_{\substack{x_k=0\\k\not\preceq a}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + M\Big|_{\substack{x_k=0\\k\preceq b}} \otimes I_2 + \frac{1}{x_a}M\Big|_{\substack{x_k=0\\k\neq a}} \otimes \begin{pmatrix} 0 & x_a \\ x_b & 0 \end{pmatrix}$$
$$+ \frac{1}{x_b}M\Big|_{\substack{x_k=0\\k\neq b}} \otimes \begin{pmatrix} x_b & 0 \\ 0 & x_a \end{pmatrix}.$$

The claim follows since for a complex matrix S independent of the x_i's,

$$SM\Big|_{\substack{x_k=0\\k\not\preceq a}} = (SM)\Big|_{\substack{x_k=0\\k\not\preceq a}},$$

etc. □

Lemma 3.2.9. *Let M be a matrix whose entries are homogeneous degree-1 polynomials in $x_1, \ldots, x_n$ and let S be a complex matrix such that $T = SMS^{-1}$ is upper-triangular. Then the eigenvalues of $\partial_{a,b}M$ are the same as the eigenvalues of $\partial_{a,b}T$.*

Proof. Note that $(\partial S)^{-1} = (S \otimes I_2)^{-1} = S^{-1} \otimes I_2 = \partial(S^{-1})$. By Lemma 3.2.8 we get

$$\partial_{a,b}T = \partial_{a,b}(SMS^{-1}) = (\partial S)(\partial_{a,b}M)(\partial S^{-1}) = (\partial S)(\partial_{a,b}M)(\partial S)^{-1}.$$

Therefore, $\partial_{a,b}M$ and $\partial_{a,b}T$ are similar and thus have the same eigenvalues. □

Proof of Theorem 3.2.2. By Corollary 3.2.6, $M^{P'} = \partial_{a,b}M^P$. Let S be the matrix that simultaneously upper-triagularizes the matrices G_i. Then $T = SMS^{-1}$ is an upper triangular matrix whose diagonal entries are the eigenvalues $x^{\mathfrak{s}}$ of M^P. By Lemma 3.2.9 the eigenvalues of $M^{P'}$ are the same as the eigenvalues of $\partial_{a,b}T$ which is block upper-triangular with 2×2 blocks $\partial_{a,b}x^{\mathfrak{s}}$ on the main diagonal. Note that

$$\partial_{a,b}x^{\mathfrak{s}} = \begin{pmatrix} c_b^{\mathfrak{s}}x_b + \sum\limits_{k \not\prec a,b} c_k^{\mathfrak{s}}x_k & c_a^{\mathfrak{s}}x_a + \sum\limits_{k \prec a} c_k^{\mathfrak{s}}x_k \\ c_a^{\mathfrak{s}}x_b + \sum\limits_{k \prec a} c_k^{\mathfrak{s}}x_k & c_b^{\mathfrak{s}}x_a + \sum\limits_{k \not\prec a,b} c_k^{\mathfrak{s}}x_k \end{pmatrix}.$$

Since by assumption, M^P has the upset property, there are only two cases: $c_a^{\mathfrak{s}} = c_b^{\mathfrak{s}} = c$ and $c_a^{\mathfrak{s}} = 0, c_b^{\mathfrak{s}} \neq 0$. In the former case,

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \partial_{a,b}x^{\mathfrak{s}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x^{\mathfrak{s}} & 0 \\ 0 & \sum\limits_{k \not\prec a,b} c_k^{\mathfrak{s}}x_k - \sum\limits_{k \prec a} c_k^{\mathfrak{s}}x_k \end{pmatrix}.$$

In the latter case, by the upset property we also have that $\sum_{k \prec a} c_k^{\mathfrak{s}}x_k = 0$, and therefore

$$\partial_{a,b}x^{\mathfrak{s}} = \begin{pmatrix} c_b^{\mathfrak{s}}x_b + \sum\limits_{k \not\prec a,b} c_k^{\mathfrak{s}}x_k & 0 \\ 0 & c_b^{\mathfrak{s}}x_a + \sum\limits_{k \not\prec a,b} c_k^{\mathfrak{s}}x_k \end{pmatrix}.$$

This also shows that there is a real matrix S' such that $S'(\partial_{a,b}T)(S')^{-1}$ is upper triangular. Consequently, $S'(\partial S)M^{P'}(S'(\partial S))^{-1}$ is upper triangular, which means that the matrices G_i' such that $M^{P'} = \sum x_k G_i'$ are simultaneously upper-triangularizable.

Finally, notice that $R_{P'} \subset R_P$ and if $(a', b') \in R_{P'}$ then $\{a', b'\} \cup \{a, b\} = \emptyset$ and either $a', b' \prec a$ or $a', b' \not\prec b$. So, by inspection, the eigenvalues of $M^{P'}$ satisfy conditions (a) and (b) from the definition of the upset property.

□

Example 3.2.10. *Let P and P' be as in Example 3.2.4. Then $M^{P'} = \partial_{5,6} M^P$. For $S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, we have*

$$\partial S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

and thus

$$T = SM^P S^{-1} = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & x_1 + x_2 & 0 \\ 0 & x_4 + x_5 + x_6 & x_1 \\ 0 & 0 & x_2 + x_4 + x_5 + x_6 \end{pmatrix}$$

and

$$\partial S M^{P'} (\partial S)^{-1} = \begin{pmatrix} x_6 & x_1 + x_2 + x_3 + x_4 + x_5 & 0 & x_1 + x_2 & 0 & 0 \\ x_1 + x_2 + x_3 + x_4 + x_6 & x_5 & x_1 + x_2 & 0 & 0 & 0 \\ 0 & 0 & x_6 & x_4 + x_5 & 0 & x_1 \\ 0 & 0 & x_4 + x_6 & x_5 & x_1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & x_2 + x_4 + x_5 \\ 0 & 0 & 0 & 0 & x_2 + x_4 + x_6 & x_5 \end{pmatrix}$$

$$= \partial_{5,6} T.$$

3.3 Partition function and convergence rates

The stationary distribution for the promotion Markov chain is given by Theorem 3.0.3. Here we find the partition function in the case when P is a union of ordinal sums of forests and ladders.

Theorem 3.3.1. *Let $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ be a poset of size n where F_i is a forest and L_i is a ladder for $i = 1, \ldots, k$. Let $L_i = Q^i_1 \oplus \cdots \oplus Q^i_{t_i}$ where $Q^i_j = \bullet a^i_j \qquad \bullet b^i_j$ or $Q^i_j = \bullet a^i_j$. The partition function for the promotion graph is is given by*

$$Z_P = \prod_{i=1}^{n} x_{\preccurlyeq i} \prod_{Q^i_j \colon |Q^i_j|=2} \frac{x_{\preccurlyeq a^i_j \cup b^i_j}}{x_{\preccurlyeq a^i_j} + x_{\preccurlyeq b^i_j}}, \tag{3.1}$$

where $x_{\preccurlyeq a_{i_j} \cup b_{i_j}} = \sum\limits_{s \preccurlyeq a_{i_j} \text{ or } s \preccurlyeq b_{i_j}} x_s.$

Proof. By Theorem 3.0.3, we need to show that $w'(\pi) := w(\pi) Z_P$ with

$$w(\pi) = \prod_{i=1}^{n} \frac{1}{x_{\pi_1} + \cdots + x_{\pi_i}} \tag{3.2}$$

satisfies $\sum\limits_{\pi \in \mathcal{L}(P)} w'(\pi) = 1.$

We will use induction on the size of P. One can readily check that this is true if $n = 1$. Assume it is true for posets of this form of size $n - 1$ and let P be as described in the assumptions. If $\pi = \pi_1 \cdots \pi_n$, then π_n is an element in one of the top levels of P, i.e., $\pi_n \in Q^i_{t_i}$ for some $i \in [k]$. Therefore,

$$\sum_{\pi\in\mathcal{L}(P)} w'(\pi) = \sum_{i:|Q^i_{t_i}|=2}\left(\sum_{\sigma\in\mathcal{L}(P\setminus\{a^i_{t_i}\})} w'(\sigma a^i_{t_i}) + \sum_{\sigma\in\mathcal{L}(P\setminus\{b^i_{t_i}\})} w'(\sigma b^i_{t_i})\right)$$
$$+ \sum_{i:|Q^i_{t_i}|=1}\sum_{\sigma\in\mathcal{L}(P\setminus\{a^i_{t_i}\})} w'(\sigma a^i_{t_i}).$$

By (3.1) and (3.2), if $|Q^i_{t_i}| = 2$ then

$$w'(\sigma a^i_{t_i}) = w'(\sigma)\frac{x_{\preccurlyeq a^i_{t_i}}}{x_1+\cdots+x_n}\cdot\frac{x_{\preccurlyeq a^i_{t_i}\cup b^i_{t_i}}}{x_{\preccurlyeq a^i_{t_i}} + x_{\preccurlyeq b^i_{t_i}}},$$
$$w'(\sigma b^i_{t_i}) = w'(\sigma)\frac{x_{\preccurlyeq b^i_{t_i}}}{x_1+\cdots+x_n}\cdot\frac{x_{\preccurlyeq a^i_{t_i}\cup b^i_{t_i}}}{x_{\preccurlyeq a^i_{t_i}} + x_{\preccurlyeq b^i_{t_i}}},$$

and if $|Q^i_{t_i}| = 1$,

$$w'(\sigma a^i_{t_i}) = w'(\sigma)\frac{x_{\preccurlyeq a^i_{t_i}}}{x_1+\cdots+x_n}.$$

Hence, using the induction hypothesis, we get

$$\sum_{\pi\in\mathcal{L}(P)} w'(\pi) = \sum_{i:|Q^i_{t_i}|=2}\left(\frac{x_{\preccurlyeq a^i_{t_i}}}{x_1+\cdots+x_n}\cdot\frac{x_{\preccurlyeq a^i_{t_i}\cup b^i_{t_i}}}{x_{\preccurlyeq a^i_{t_i}} + x_{\preccurlyeq b^i_{t_i}}} + \frac{x_{\preccurlyeq b^i_{t_i}}}{x_1+\cdots+x_n}\cdot\frac{x_{\preccurlyeq a^i_{t_i}\cup b^i_{t_i}}}{x_{\preccurlyeq a^i_{t_i}} + x_{\preccurlyeq b^i_{t_i}}}\right)$$
$$+ \sum_{i:|Q^i_{t_i}|=1}\frac{x_{\preccurlyeq a^i_{t_i}}}{x_1+\cdots+x_n}$$
$$= \sum_{i:|Q^i_{t_i}|=2}\frac{x_{\preccurlyeq a^i_{t_i}\cup b^i_{t_i}}}{x_1+\cdots+x_n} + \sum_{i:|Q^i_{t_i}|=1}\frac{x_{\preccurlyeq a^i_{t_i}}}{x_1+\cdots+x_n}$$
$$= 1.$$

□

Example 3.3.2. *Let P be as in Figure 3.1. We calculated the stationary state weights in Example 3.0.4. By Theorem 3.3.1, the partition function is*

$$Z_p = x_1 \cdot x_2 \cdot (x_1+x_2+x_3) \cdot (x_1+x_2+x_4) \cdot \frac{x_1+x_2}{x_1+x_2} \cdot \frac{x_1+x_2+x_3+x_4}{(x_1+x_2+x_3)+(x_1+x_2+x_4)}.$$

Using Table 3.2 in Example 3.0.4, we get

$$\begin{aligned}\sum_{\pi\in\mathcal{L}(P)} w(\pi)Z_p &= \tfrac{x_2}{x_1+x_2}\tfrac{x_1+x_2+x_3}{(x_1+x_2+x_3)+(x_1+x_2+x_4)} + \tfrac{x_2}{x_1+x_2}\tfrac{x_1+x_2+x_4}{(x_1+x_2+x_3)+(x_1+x_2+x_4)} \\ &\quad + \tfrac{x_1}{x_1+x_2}\tfrac{x_1+x_2+x_4}{(x_1+x_2+x_3)+(x_1+x_2+x_4)} + \tfrac{x_1}{x_1+x_2}\tfrac{x_1+x_2+x_3}{(x_1+x_2+x_3)+(x_1+x_2+x_4)} \\ &= 1.\end{aligned}$$

For the case $P = F \oplus L$, we can make an explicit statement about the rate of convergence to stationary and the mixing time. Let P^k be the distribution after k steps and and $\mathbb{P}^k$ be the k-th convolution power of the distribution $\mathbb{P}$. The rate of convergence is the total variation distance from stationary after k steps, that is,

$$\|P^k - w\|_{TV} = \frac{1}{2}\sum_{\pi\in\mathcal{L}(P)} |\mathbb{P}^k(\pi) - w(\pi)|$$

where w is the stationary distribution. We will use the following theorem:

Theorem 3.3.3 ([3])**.** *Let M be a monoid acting on a set Ω and let $\mathbb{P}$ be a probability distribution on M. Let $\mathcal{M}$ be the Markov chain with state set Ω such that the transition probability from x to y is the probability that $mx = y$ if m is chosen according to $\mathbb{P}$. Assume that $\mathcal{M}$ is irreducible and aperiodic with stationary distribution w and that some element of M acts as a constant map on Ω. Letting P^k be the distribution* *of $\mathcal{M}$ after k steps and $\mathbb{P}^k$ be the k-th convolution power of $\mathbb{P}$, we have that*

$$\|P^k - w\|_{TV} \leq \mathbb{P}^k(M \setminus C),$$

where C is the set of elements of M acting as constants on Ω.

In our case, the monoid (a set with an associative multiplication and an identity element) acting on $\mathcal{L}(P)$ is $\mathcal{M}^{\widehat{\partial}}$ generated by the operators $\widehat{\partial}_i$ defined by the promotion graph G_P. That is, for $\pi, \pi' \in \mathcal{L}(P)$, $\widehat{\partial}_i \pi = \pi'$ if and only if $\pi' = \partial_{\pi^{-1}(i)}\pi$. In what follows, it will be helpful to have the following alternate description of $\widehat{\partial}_i$.

Lemma 3.3.4. *Let $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$, where F_i is a forest and L_i is a ladder. For $\pi \in \mathcal{L}(P)$, $\widehat{\partial}_k \pi$ is the linear extension of P obtained from π by moving the letter k to the last position and reordering the letters $j \succeq k$, swapping the original order of incomparable elements at the same level of a ladder L_i.*

Proof. By the definition of $\widehat{\partial}_i$, we have $\widehat{\partial}_i \pi = \tau_{n-1} \cdots \tau_{k+1}\tau_k \pi$, where $k = \pi^{-1}(i)$. The transpositions start swapping i with the elements that follow it until an element $j \succeq i$ is reached. Then j is swapped with the elements that follow it, etc. So, the elements j that begin the new series of swaps are the ones that are in the ladder above i. Moreover, the two elements in this ladder will be swapped themselves because they are incomparable. □

Example 3.3.5. *Let P be the poset on* $[9]$ *with covering relations* $1 \prec 2$, $2 \prec 4$, $3 \prec 4$, $4 \prec 5$, $4 \prec 6$, $7 \prec 8$, *and* $7 \prec 9$. *To compute* $\widehat{\partial}_3 371824695$, *we first move* 3 *to the end of the word to obtain* 718246953. *Then we reorder the elements* $\{3, 4, 5, 6\}$ *to form a linear extension, but in the process we swap the order of* 5 *and* 6. *Since* 6 *appears to the left of* 5, *we now place* 5 *to the left of* 6. *This way we get* $\widehat{\partial}_3 371824695 = 718234956$.

For $x \in \mathcal{M}^{\widehat{\partial}}$, let $\operatorname{im}(x) = \{x\pi \colon \pi \in \mathcal{L}(P)\}$. Let $\operatorname{rfactor}(x)$ be the maximal common right factor of the elements in $\operatorname{im}(x)$ and let $\operatorname{Rfactor}(x) = \{i \colon i \in \operatorname{rfactor}(x)\}$.

Lemma 3.3.6. *Let $P = F \oplus L$ be a poset of size n, where F is a rooted forest and L is a ladder. Then*

(a) $\operatorname{Rfactor}(x) \subseteq \operatorname{Rfactor}(\widehat{\partial}_i x)$ *for all $x \in \mathcal{M}^{\widehat{\partial}}$ and $i = 1, \ldots, n$,*

(b) $\operatorname{Rfactor}(x) \subsetneq \operatorname{Rfactor}(\widehat{\partial}_k x)$ *for k maximal in $P \setminus \operatorname{Rfactor}(x)$.*

Proof. Let $x \in \mathcal{M}^{\widehat{\partial}}$. Each $\pi \in \operatorname{im}(x)$ is of the form $\pi = \pi' \operatorname{rfactor}(x)$. We consider two cases. If $i \in \operatorname{Rfactor}(x)$, then $\widehat{\partial}_i \pi = \pi' \widehat{\partial}_i \operatorname{rfactor}(x)$ and therefore, clearly, $\operatorname{Rfactor}(x) \subseteq \operatorname{Rfactor}(\widehat{\partial}_i x)$. Suppose now $i \notin \operatorname{Rfactor}(x)$. Since $\operatorname{Rfactor}(x)$ is an upset of P, the poset $P \setminus \operatorname{Rfactor}(x)$ is also of the form $P \setminus \operatorname{Rfactor}(x) = F' \oplus L'$, for a forest F' and a ladder L'. Notice that if $P \setminus \operatorname{Rfactor}(x)$ has one maximal element then we get a contradiction on the maximality of $\operatorname{rfactor}(x)$. Therefore, either $L' = \emptyset$ or $L' \neq \emptyset$ and $P \setminus \operatorname{Rfactor}(x)$ has two maximal elements. If $L' = \emptyset$, i.e., $P \setminus \operatorname{Rfactor}(x)$ is a forest, for every $i \in P \setminus \operatorname{Rfactor}(x)$, then the set $\{j \in P \setminus \operatorname{Rfactor}(x) \colon i \preceq j\}$ is a chain and has a unique maximal element k_i. Then, by Lemma 3.3.4, $\operatorname{Rfactor}(x) \cup \{k_i\} \subset \operatorname{Rfactor}(\widehat{\partial}_i x)$. On the other hand, if $L' \neq \emptyset$ and $P \setminus \operatorname{Rfactor}(x)$ has two maximal elements, a and b, then each $\pi \in \operatorname{im}(x)$ is of the form $\pi = \pi''\, a\, b\, \operatorname{rfactor}(x)$ or $\pi = \pi''\, b\, a\, \operatorname{rfactor}(x)$ and both these forms appear in $\operatorname{im}(x)$. Hence $\operatorname{Rfactor}(\widehat{\partial}_a x) \supseteq \operatorname{Rfactor}(x) \cup \{a\}$, $\operatorname{Rfactor}(\widehat{\partial}_b x) \supseteq \operatorname{Rfactor}(x) \cup \{b\}$, and for $i \neq a, b$, $\operatorname{Rfactor}(\widehat{\partial}_i x) = \operatorname{Rfactor}(x)$. □

Theorem 3.3.7. *Let $P = F \oplus L$ be a poset of size n, where F is a forest and L is a ladder. Let $p_x = \min\{x_i \colon 1 \leq i \leq n\}$. Then for $k \geq (n-1)/p_x$, the distance to stationary distribution of the promotion Markov chain satisfies*

$$\|P^k - \omega\|_{TV} \leq \exp\left(-\frac{(kp_x - (n-1))^2}{2kp_x}\right).$$

Proof. Similar to the proof by Ayyer *et al.* [1], for $m \in \mathcal{M}^{\widehat{\partial}}$, let

$$u(m) = n - |\text{Rfactor}(m)|.$$

The statistic u has the following three properties:

(1) $u(m'm) \leq u(m)$ for all $m, m' \in \mathcal{M}^{\widehat{\partial}}$;

(2) if $u(m) > 0$, then there exists $\widehat{\partial}_i \in \mathcal{M}^{\widehat{\partial}}$ such that $u(\widehat{\partial}_i m) < u(m)$;

(3) $u(m) = 0$ if and only if m acts as a constant on $\mathcal{L}(P)$.

The first two properties follow from Lemma 3.3.6, while $u(m) = 0$ if and only if rfactor(m) is a linear extension of P which is equivalent to m being a constant map. Furthermore, for the identity map ϵ, $u(\epsilon) \leq n$.

A step $m_i \to m_{i+1}$ in the left random walk on $\mathcal{M}^{\widehat{\partial}}$ is successful if

$$u(m_{i+1}) < u(m_i).$$

Property (1) of u implies that the step is not successful if and only if

$$u(m_i) = u(m_{i+1}),$$

and by Property (2), each step has probability at least p_x to be successful. Therefore, the probability that $n \geq u(m) > 0$ after k steps of the left random walk on $\mathcal{M}^{\widehat{\partial}}$ is bounded above by the probability of having at most $n-1$ successes in k Bernoulli trials with success probability p_x. Using Theorem 3.3.3 and Chernoff's inequality,

$$\|P^k - \omega\|_{TV} \leq \exp\left(-\frac{(kp_x - (n-1))^2}{2kp_x}\right),$$

where the inequality holds for $p_x k > n - 1$. $\square$

The *mixing time* is the number of steps k until $\|P^k - \omega\|_{TV} \leq e^{-c}$. Using Theorem 3.3.7, it suffices to have

$$(kp_x - (n-1))^2 \geq 2kp_x c,$$

so the mixing time is at most $\dfrac{2(n+c-1)}{p_x}$. If the probability distribution

$$\{x_i \colon 1 \leq i \leq n\}$$

is uniform, then p_x is of order $\dfrac{1}{n}$ and the mixing time is of order at most n^2.

Chapter 4

Self-Organizing Libraries with a Poset Structure on the Leaves

In this chapter, we generalize Björner's [8] results of hierarchies of libraries. Consider a rooted tree T whose leaves are all at the same depth, d. Suppose that at each inner node v, a poset P_v on the children is given; we refer to these as *leaf posets*.

We will consider a Markov chain, where the objects are elements of $\mathcal{L}(T)$, and the set of total orderings of T are of the form

$$\mathcal{L}(T) \cong \bigotimes_{v \in I} \mathcal{L}(P_v)$$

where $\mathcal{L}(P_v)$ is the set of linear extensions of P_v. The actions are given by certain subsets of the leaves, L. Specifically, let

$$\mathcal{A}(L) = \{E \subseteq L\colon \text{ no two elements of } E \text{ are siblings}\}.$$

We will use the same notion of E-related vertices as before, that is a node $v \in T$ is

E-related if some descendant of v is contained in E. Also, recall that

$$C_v^E = \{v \in C_v \colon v \text{ is } E\text{-related}\},$$

where C_v is the set of children of the node v. An element E of $\mathcal{A}(L)$ acts on $\mathcal{L}(T)$ with probability x_E in the following way: Let $\pi = (\pi_v)_{v\in I}$ be a given ordering. Then $\widehat{\partial}_E\pi = (\widehat{\partial}_{E_v}\pi_v)_{v\in I}$ where

$$\widehat{\partial}_{E_v}\pi_v = \begin{cases} \widehat{\partial}_{C_v^E}\pi_v & \text{if depth}(v) = d-1 \\ \underline{\beta}_v^E\pi_v & \text{otherwise.} \end{cases}$$

In other words, the ordering is rearranged locally at each inner node so that the elements of E are promoted and the E-related elements not in E are pop shuffled.

We use the convention that the leaves have the same depth for ease of notation and to follow Björner [8], but this condition is not necessary. One could consider a tree with the leaves at varying depths. Now, if we want to move a node that is a leaf not of depth d, we can let that node be in E. In Björner's results, an inner node is inherently moved by having a descendant in E, so in order to get the same result with a non-pure tree, we can just extend the node to depth d with a chain.

Furthermore, even though the set $\mathcal{A}(L)$ is restricted to one sibling, we can still get Björner's result by extending each leaf by adding a single child to it. For each leaf l, label its child l_c. Then the eigenvalues of the extended tree will be in the form $\sum x_E$ where E contains elements of the form l_c. To get Björner's results, simply replace l_c with l in E.

Example 4.0.1. *Let T be given as in Figure 4.1.*

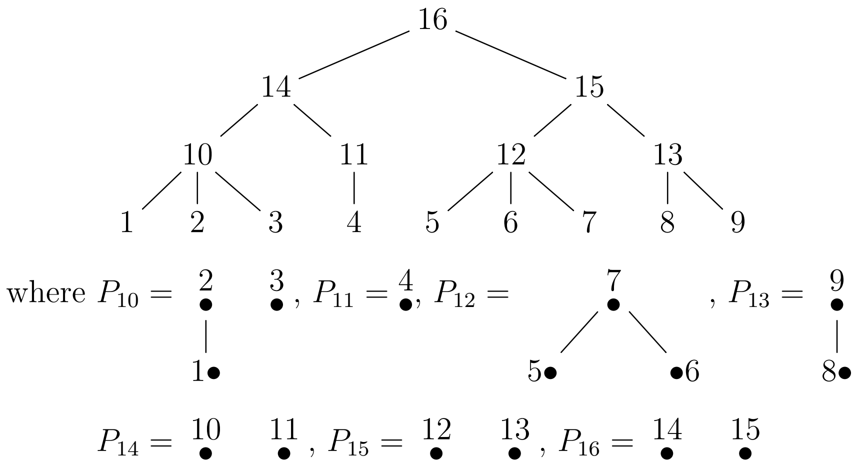


Figure 4.1: A rooted tree and its leaf posets.

The transition matrix M of the corresponding Markov chain is a 48×48 matrix. That is $\mathcal{L}(T)$ is of size $3 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 48$. Also, $\mathcal{A}(L)$ is of size $4 \cdot 2 \cdot 4 \cdot 3 = 96$. For $\pi = (123, 4, 657, 8\ 9, 11\ 10, 12\ 13, 15\ 14)$ and $E = \{2, 6\}$,

$$\widehat{\partial}_E = (\widehat{\partial}_2, id, \widehat{\partial}_6, id), \qquad \underline{\beta}^E = ((11, 10), (13, 12), (14\ 15)) \qquad \textit{and}$$

$$\pi' = \widehat{\partial}_E \pi = (132, 4, 567, 89, 11\ 10, 13\ 12, 15\ 14).$$

In fact, $\widehat{\partial}_E \pi = \pi'$ if and only if $E \in \{\{2, 4, 6\}, \{2, 6\}, \{1, 6\}, \{1, 4, 6\}\}$. So the entry of M in row π and column π' is $x_{246} + x_{26} + x_{146} + x_{16}$.

In the case where all the leaf posets are rooted forests, we use theory of $\mathcal{R}$-trivial monoids to prove our results. In Section 4.1 we give background of $\mathcal{R}$-trivial monoids. In Section 4.2 we prove the associated monoid is $\mathcal{R}$-trivial and apply the theory of $\mathcal{R}$-trivial monoids to find the eigenvalues and their multiplicities. In the case when the leaf posets are a union of ordinal sums of a forest and a ladder, we consider two different treatments. In Section 4.3 we give background of the class **DO(Ab)** [28]. In Section 4.4 we show the associated monoid $\mathcal{M}$ is in the class **DO(Ab)** and we use theory of the class of **DO(Ab)** to find the eigenvalues. However, this theory does not give explicit multiplicities. As an alternate treatment in Section 4.5 we take a similar approach to Section 3.2 and find a relationship between two trees T_P and $T_{P'}$ where the underlying tree structure is the same, but the leaf posets are such that $P' = P \setminus \{(a, b)\}$.

4.1 Background on $\mathcal{R}$-trivial monoids

In this section, we look at the case when the leaf posets are rooted forests. We give the definition of left and right order on $\mathcal{M}$ as introduced by Green [13], and we adopt the same convention as Ayyer *et al.* [1, 2].

For $x, y \in \mathcal{M}$, the left and right order is defined by

$$\begin{aligned} x \leq_{\mathcal{R}} y \text{ if } y = xu \text{ for some } u \in \mathcal{M}, \\ x \leq_{\mathcal{L}} y \text{ if } y = ux \text{ for some } u \in \mathcal{M}. \end{aligned} \tag{4.1}$$

A monoid $\mathcal{M}$ is said to be $\mathcal{R}$-trivial if $y\mathcal{M} = x\mathcal{M}$ implies $x = y$.

Definition 4.1.1 ([24]). *A finite monoid $\mathcal{M}$ is said to be weakly ordered if there is a finite upper semi-lattice $(L^{\mathcal{M}}, \preceq)$ together with two maps* $\operatorname{supp}, \operatorname{des}\colon \mathcal{M} \to L^{\mathcal{M}}$ *satisfying the following axioms:*

1. supp *is a surjective monoid morphism, that is,* $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$ *for all $x, y \in \mathcal{M}$ and* $\operatorname{supp}(\mathcal{M}) = L^{\mathcal{M}}$.
2. *If $x, y \in \mathcal{M}$ are such that $xy \leq_{\mathcal{R}} x$, then* $\operatorname{supp}(y) \preceq \operatorname{des}(x)$.
3. *If $x, y \in \mathcal{M}$ are such that* $\operatorname{supp}(y) \preceq \operatorname{des}(x)$, *then $xy = x$.*

The following theorem connects weakly ordered with $\mathcal{R}$-trivial.

Theorem 4.1.2. *[5, Theorem 2.18] Let $\mathcal{M}$ be a finite monoid. Then $\mathcal{M}$ is weakly ordered if and only if $\mathcal{M}$ is $\mathcal{R}$-trivial.*

Remark 6.4 from Ayyer, *et al.* [1] gives the following result about the upper semi-lattice $L^{\mathcal{M}}$ of an $\mathcal{R}$-trivial monoid $\mathcal{M}$.

Remark 4.1.3 ([1])**.** *The upper semi-lattice $L^{\mathcal{M}}$ and the maps* supp, des *for an $\mathcal{R}$-trivial monoid $\mathcal{M}$ can be constructed as follows:*

1. *$L^{\mathcal{M}}$ is the set of left ideals $\mathcal{M}e$ generated by the idempotents $e \in \mathcal{M}$, ordered by reverse inclusion.*
2. $\operatorname{supp}\colon \mathcal{M} \to L^{\mathcal{M}}$ *is defined as* $\operatorname{supp}(x) = \mathcal{M}x^{\omega}$*, where ω is such that $(x^{\omega})^2 = x^{\omega}$.*
3. $\operatorname{des}\colon \mathcal{M} \to L^{\mathcal{M}}$ *is defined as* $\operatorname{des}(x) = \operatorname{supp}(e)$*, where e is some maximal element is the set $\{y \in \mathcal{M} \colon xy = x\}$ with respect to the preorder $\leq_{\mathcal{R}}$.*

In fact, for an $\mathcal{R}$-trivial monoid there always exists an exponent of x such that $x^{\omega}x = x^{\omega}$.

We apply the following theorem of $\mathcal{R}$-trivial monoids to prove Theorem 4.2.1. Let $\mathcal{C}$ be the set of chambers, that is, the set of maximal elements in $\mathcal{M}$ under $\geq_{\mathcal{R}}$. For $X \in L^{\mathcal{M}}$, define c_X to be the number of chambers in $\mathcal{M}_{\geq X}$. This is precisely the number of $c \in \mathcal{C}$ such that $c \geq_{\mathcal{R}} x$, where $x \in \mathcal{M}$ is any fixed element such that $\operatorname{supp}(x) = X$.

Theorem 4.1.4 ([1, 4, 28])**.** *Let $\{w_x\}$ be a probability distribution on $\mathcal{M}$, a finite $\mathcal{R}$-trivial monoid, that acts on the state space Ω. Let M be the transition matrix for the random walk of $\mathcal{M}$ on Ω driven by the w_x's. For each $X \in L^{\mathcal{M}}$ and x such that*

$\operatorname{supp}(x) = X$, *$M$ has an eigenvalue*

$$\lambda_X = \sum_{\substack{y \\ \operatorname{supp}(y) \preceq X}} w_y$$

with multiplicity m_X recursively defined by

$$\sum_{Y \succeq X} m_Y = c_X.$$

Equivalently,

$$m_X = \sum_{Y \succeq X} \mu(X,Y) c_Y,$$

where μ is the Möbius function of $L^{\mathcal{M}}$.

The class of $\mathcal{R}$-trivial monoids forms a pseudovariety (closed under finite direct products, submonoids, and quotients); this property has been cited in the literature [4] without an explicit proof. For completeness, we prove closure under direct product and submonoids, both properties that we use later.

Lemma 4.1.5. *For $\mathcal{M}$ and $\mathcal{N}$ $\mathcal{R}$-trivial monoids, $\mathcal{M} \times \mathcal{N}$ is an $\mathcal{R}$-trivial monoid.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{R}$-trivial monoids. Then by definition for $x, y \in \mathcal{M}$, $y\mathcal{M} = x\mathcal{M}$ implies $x = y$ and for $w, z \in \mathcal{N}$, $w\mathcal{N} = z\mathcal{N}$ implies $w = z$. Thus, for $(x,w),(y,z) \in \mathcal{M} \times \mathcal{N}$ if $(x,w)(\mathcal{M} \times \mathcal{N}) = (y,z)(\mathcal{M} \times \mathcal{N})$, then $x\mathcal{M} = y\mathcal{M}$ and $w\mathcal{N} = z\mathcal{N}$. Thus $(x,w) = (y,z)$. □

The semilattice on $\mathcal{M} \times \mathcal{N}$ is $L = L^{\mathcal{M}} \times L^{\mathcal{N}}$ where $L^{\mathcal{M}}$ is the semilattice on $\mathcal{M}$ and $L^{\mathcal{N}}$ is the semilattice on $\mathcal{N}$.

Lemma 4.1.6. *A submonoid of an $\mathcal{R}$-trivial monoid is $\mathcal{R}$-trivial.*

Proof. Let $\mathcal{N} \subseteq \mathcal{M}$ be a submonoid of an $\mathcal{R}$-trivial monoid. Then for all $x, y \in \mathcal{N}$, $xy \in \mathcal{N}$. Let $x, y \in \mathcal{N}$ such that $x\mathcal{N} = y\mathcal{N}$. Then by definition, we have $u, v \in \mathcal{N}$ such that $xu = y$ and $yv = x$. Thus, we have $y\mathcal{M} = xu\mathcal{M} \subseteq x\mathcal{M}$ and $x\mathcal{M} = yv\mathcal{M} \subseteq y\mathcal{M}$. So $y\mathcal{M} = x\mathcal{M}$ and since $\mathcal{M}$ is $\mathcal{R}$-trivial, we have $x = y$. So $\mathcal{N}$ is $\mathcal{R}$-triival. $\square$

We define the semilattice $L^{\mathcal{N}} = \{\mathrm{supp}_{\mathcal{M}}(x) \colon x \in \mathcal{N}\} \subseteq L^{\mathcal{M}}$.

4.2 When the leaf posets are rooted forests

In the case that the leaf posets are all rooted forests, we prove the associated monoid is $\mathcal{R}$-trivial and we use the theory of $\mathcal{R}$-trivial monoids to give us our main theorem.

Theorem 4.2.1. *Let T be described as above where all leaf posets are rooted forests and let M be the transition matrix of the random walk on $\mathcal{L}(T)$:*

$$M(\pi, \pi') = \sum_{E \colon \widehat{\partial}_E \pi = \pi'} x_E$$

for $\pi, \pi' \in \mathcal{L}(T)$. Then for an upset S of P, the union of the leaf posets, and $\alpha \in$ $\mathrm{Part}(T \setminus L)$*, there is an eigenvalue*

$$\lambda_{(S,\alpha)} = \sum_{\substack{E \subseteq S,\ E \in \mathcal{A}(L) \\ E \text{ is } \alpha\text{-compatible}}} x_E$$

with multiplicity $m_{(S,\alpha)} = d_S m_\alpha$, where $d_S = d_{S_1} \cdots d_{S_k}$ for $S_i = S \cap P_{v_i}$ and d_{S_i} is the derangement number of S_i in the lattice of upsets of P_{v_i} as in (2.1) and m_α is the multiplicity as given in (2.2). Furthermore, this is the complete list of eigenvalues.

Example 4.2.2. *Let T be given as in Figure 4.2.*

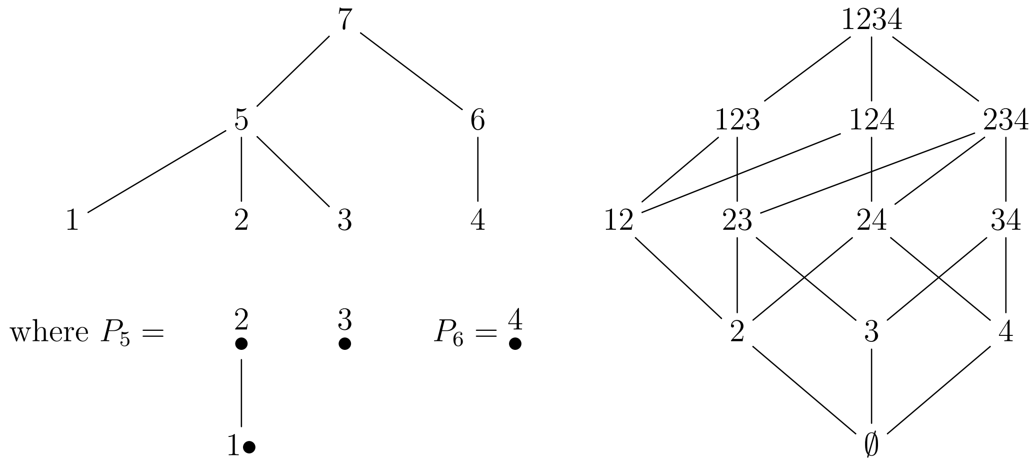
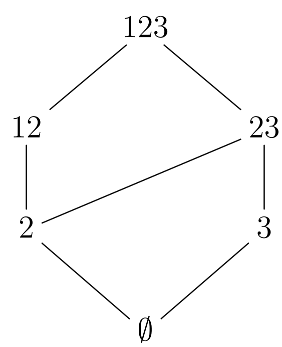


Figure 4.2: A rooted tree with its leaf posets and associated lattice of upsets.

Notice that the lattice of upsets on P_5 is



and the lattice of upsets on P_6 is

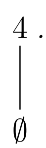


Table 4.1 shows that $d_{S_5} = 1$ if and only if $S_5 \in \{123, 2, \emptyset\}$ and $d_{S_6} = 1$ if and only if $S_6 = 4$.

Upset S in P	Upset S_5	Upset S_6	d_{S_5}	d_{S_6}	d_S
$\emptyset$	$\emptyset$	$\emptyset$	1	0	0
2	2	$\emptyset$	1	0	0
3	3	$\emptyset$	0	0	0
4	$\emptyset$	4	1	1	1
12	12	$\emptyset$	0	0	0
23	23	$\emptyset$	0	0	0
24	2	4	1	1	1
34	3	4	0	1	0
123	123	$\emptyset$	1	0	0
124	12	4	0	1	0
234	23	4	0	1	0
1234	123	4	1	1	1

Table 4.1: Multiplicities in the lattice of upsets.

Furthermore, the partitions of $T \backslash L$ that have multiplicity 1 are exactly $\alpha \in \{\{5,6\},\{56\}\}$. Thus, the eigenvalues of T are

$$\lambda_{(1234,\{56\})} = x_{14} + x_{24} + x_{34},$$

$$\lambda_{(1234,\{5,6\})} = x_{14} + x_{24} + x_{34} + x_1 + x_2 + x_3 + x_4,$$

$$\lambda_{(24,\{56\})} = x_{24},$$

$$\lambda_{(24,\{5,6\})} = x_{24} + x_2 + x_4,$$

$$\lambda_{(4,\{56\})} = 0,$$

$$\lambda_{(4,\{5,6\})} = x_4,$$

with multiplicity $m_{(S,\alpha)} = 1$ for all pairs (S, α) given above.

In order to prove Theorem 4.2.1, we will first prove that if all the leaf posets are rooted forests, then the associated monoid $\mathcal{M}^T$ is $\mathcal{R}$-trivial. We then apply Theorem 4.1.4 which gives the explicit eigenvalues for $\mathcal{R}$-trivial monoids. In order to use Theorem 4.1.4 we also describe the associated upper semilattice.

Theorem 4.2.3. *$\mathcal{M}^T$ is $\mathcal{R}$-trivial if each leaf poset is a forest.*

Proof. For v_i of depth $d-1$, let $\mathcal{M}_{v_i} = \mathcal{M}_{v_i}^{\widehat{\partial}}$ where $\mathcal{M}_{v_i}^{\widehat{\partial}}$ is the monoid generated by $\{\widehat{\partial}_j : j \in C_{v_i}\}$ as defined by Ayyer *et al.* [1, 2]. Ayyer *et al.* proved that $\mathcal{M}_{v_i}^{\widehat{\partial}}$ is $\mathcal{R}$-trivial.

Let $\mathcal{M}^{\text{top}} = \text{Part}^{\text{ord}}(T \setminus L)$. By Björner [8], $\mathcal{M}^{\text{top}}$ is a left-regular band with component-wise composition defined as: if $\underline{\alpha} = (a_1, \ldots, a_\ell)$ and $\underline{\beta} = (b_1, \ldots, b_m)$, then

$$\underline{\alpha} \circ \underline{\beta} = (a_i \cap b_j),$$

where the blocks are ordered by the indices (i, j) in lexicographic order. Since $\mathcal{M}^{\text{top}}$ is a left-regular band, it is also an $\mathcal{R}$-trivial monoid.

Thus, for

$$\mathcal{M}^T \subseteq \mathcal{M}_{v_1} \times \cdots \times \mathcal{M}_{v_k} \times \mathcal{M}^{\text{top}} = \mathcal{M},$$

$\mathcal{M}^T$ is submonoid of a product of $\mathcal{R}$-trivial monoids and is therefore $\mathcal{R}$-trivial. □

We can think of acting with the larger monoid $\mathcal{M}$, by allowing for the following probabilities:

$$\text{Prob}(\widehat{\partial}_E, \underline{\alpha}) = \begin{cases} x_E & \text{if } \underline{\alpha} = \underline{\alpha}^E \\ 0 & \text{otherwise.} \end{cases}$$

That is, to find the eigenvalues, we will use the embedding of $\mathcal{M}^T$ into $\mathcal{M}$.

In general, $\mathcal{M}^T$ is a submonoid of $\mathcal{M}$, but we have equality in the special case where all the leaf posets are rooted trees, which we explain next. One can notice that $\{\underline{\alpha}^E : E \in \mathcal{A}(L)\}$ generates $\text{Part}^{\text{ord}}(T \setminus L)$. Furthermore, since P_{v_i} is a rooted tree, $\widehat{\partial}_{r_i}$ is the identity element for the root r_i of P_{v_i}. Let $(\widehat{\partial}_E, \underline{\alpha}^F)$ be a generator of $\mathcal{M}$. We have the following algorithm to find $(\widehat{\partial}_G, \underline{\alpha}^G), (\widehat{\partial}_H, \underline{\alpha}^H) \in \mathcal{M}^T$ such that $(\widehat{\partial}_G, \underline{\alpha}^G) \cdot (\widehat{\partial}_H, \underline{\alpha}^H) = (\widehat{\partial}_E, \underline{\alpha}^F) \in \mathcal{M}$.

```
Let F ∈ A(L) be the set that forms α^F.
Let G = ∅
Let H = ∅
for i = 1 to k do
    Let {s} = F ∩ E ∩ C_{v_i} if |{s}| = 1 then
        G = G ∪ {s}
        H = H ∪ {r_i}
    else
        Let {ℓ} = E ∩ C_{v_i}
        Let {t} = F ∩ C_{v_i}
        if |{ℓ}| = 1 then
            H = H ∪ {ℓ}
        end
        if |{ℓ}| = 0 then
            H = H ∪ {r_i}
        end
        if |{t}| = 1 then
            G = G ∪ {r_i}
        end
    end
end
```

Algorithm 2: Algorithm for finding the decomposition of $(\widehat{\partial}_E, \underline{\alpha}^F)$.

Notice that $\underline{\alpha}^H$ is the identity block ordered partition (the partition with just one block) since H contains an element from every set of children of the leaves. Furthermore, notice that $\underline{\alpha}^G = \underline{\alpha}^F$ since for every $t \in F$, if $t \notin E$, we replaced t with the root of the associated poset containing t and for all $\ell \in E$ not in F, we added ℓ

to H. That is, we have

$$\widehat{\partial}_G \widehat{\partial}_H = \widehat{\partial}_E \text{ and } \underline{\alpha}^G \circ \underline{\alpha}^H = \underline{\alpha}^G = \underline{\alpha}^F.$$

Example 4.2.4. *Let T be as given in Figure 4.3.*

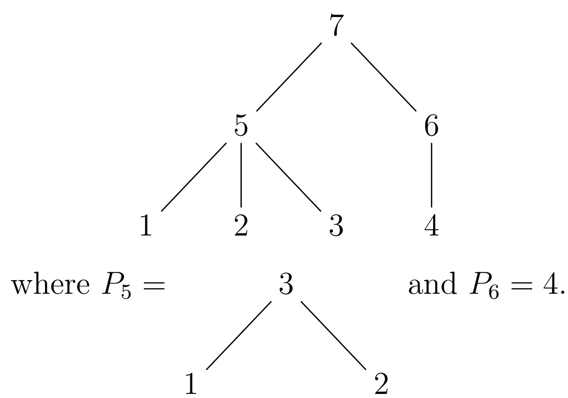


Figure 4.3: A rooted tree with rooted tree as the leaf poset.

Let $(\widehat{\partial}_1, (56)) \in \mathcal{M}$, but notice that $(\widehat{\partial}_1, (56)) \notin \mathcal{M}^T$. However, for $F = \{2, 4\}$, $\underline{\alpha}^F = (56)$. We have, $E \cap F = \emptyset$, thus by the previous algorithm we have

$$H = \{1, 4\} \textit{ and } G = \{3, 4\},$$

where

$$\widehat{\partial}_G \widehat{\partial}_H = \widehat{\partial}_1 \textit{ and } (56) \circ (56) = (56) = \underline{\alpha}^F.$$

Since $\mathcal{M}^T$ is $\mathcal{R}$-trivial, the remainder of this section is devoted to the proof of Theorem 4.2.1.

In order to introduce the associated semilattice, we need the notion of Rfactor(x). Let rfactor(x) be the largest common right factor of all $\pi \in \text{im}(x)$. In other words, for $\pi \in \text{im}(x)$, $\pi = \pi'\text{rfactor}(x)$ and there is no bigger rfactor(x) such that this is true.

Ayyer *et al.* [1, 2] proved that for the $\mathcal{R}$-trivial monoid $\mathcal{M}_{v_i}^{\widehat{\partial}}$ its associated semilattice is

$$\{\mathrm{Rfactor}(x) \colon x \in \mathcal{M}_{v_i}^{\widehat{\partial}}, x = x^2\}$$

where $\mathrm{Rfactor}(x) = \{j \colon j \in \mathrm{rfactor}(x)\}$. Furthermore the support map is defined by

$$\mathrm{supp} \colon \mathcal{M}^{\widehat{\partial}_v} \to L^{\mathcal{M}_v}$$

$$x \mapsto I_{x^\omega}$$

where $I_{x^\omega} = \{j \colon x^\omega \widehat{\partial}_j = x^\omega\}$ is maximal and ω is such that $x^\omega x = x^\omega$.

The associated $\mathcal{R}$-trivial monoid is

$$\mathcal{M}^{\widehat{\partial}} = \mathcal{M}_{v_1}^{\widehat{\partial}} \times \cdots \times \mathcal{M}_{v_k}^{\widehat{\partial}},$$

where the associated semilattice is defined by $L^{\mathcal{M}^{\hat{\partial}}} = L^{\mathcal{M}_{v_1}} \times \cdots \times L^{\mathcal{M}_{v_k}}$ and the support map is defined component-wise.

For $\mathcal{M}^{\mathrm{top}}$, the associated $\mathcal{R}$-trivial monoid is $\mathrm{Part}^{\mathrm{ord}}(T \setminus L)$ and the associated semilattice is $\mathrm{Part}(T \setminus L)$.

The support map is defined by

$$\mathrm{supp} \colon \mathrm{Part}^{\mathrm{ord}}(T \setminus L) \to \mathrm{Part}(T \setminus L)$$

$$\underline{\alpha} \mapsto \alpha.$$

Thus, for

$$\mathcal{M}^T \subseteq \mathcal{M}_{v_1} \times \cdots \times \mathcal{M}_{v_k} \times \mathcal{M}^{\mathrm{top}} = \mathcal{M}^{\widehat{\partial}} \times \mathrm{Part}^{\mathrm{ord}}(T \setminus L) = \mathcal{M},$$

$\mathcal{M}^T$ is an $\mathcal{R}$-trivial monoid. Recall that we can think of acting with the larger monoid $\mathcal{M}$, by allowing for the following probabilities:

$$\text{Prob}(\widehat{\partial}_E, \underline{\alpha}) = \begin{cases} x_E & \text{if } \underline{\alpha} = \underline{\alpha}^E \\ 0 & \text{otherwise.} \end{cases}$$

That is, we consider the associated semilattice

$$L^{\mathcal{M}} = L^{\mathcal{M}^{\hat{\partial}}} \times \text{Part}(T \setminus L).$$

The supp map is defined by

$$\text{supp} : \mathcal{M} \to L^{\mathcal{M}}$$

where $\text{supp}(x, \underline{\alpha}) = (\text{supp}(x^1), \ldots, \text{supp}(x^k), \text{supp}(\underline{\alpha}))$, i.e., the support map is defined by the support map on each component.

In order for the multiplicities of Theorem 4.2.1 and Theorem 4.1.4 to match, we need to show that for the pairs (S, α) if $(S, \alpha) = (\text{Rfactor}(x), \alpha)$ for $S = (S_1, \ldots, S_k)$, then $m_{(S,\alpha)} = d_{S_1} \cdots d_{S_k} m_\alpha$. Otherwise, we need to show that $m_{(S,\alpha)} = 0$.

First, we show that $m_{(S,\alpha)} = d_{S_1} \cdots d_{S_k} m_\alpha$.

Let $c_{(S,\alpha)}$ be the number of maximal elements $(x, \underline{\alpha}) \in \mathcal{M}$ with $(x, \underline{\alpha}) \geq_{\mathcal{R}} (s, \underline{\beta})$ for some $(s, \underline{\beta})$ with $\text{supp}((s, \underline{\beta})) = (S, \alpha)$. Recall that in order for $\text{supp}(s, \underline{\beta}) = (S, \alpha)$, we must have $\text{supp}(s_i) = S_i$ and $\text{supp}(\underline{\beta}) = \alpha$. In fact, we have $(x, \underline{\alpha}) \geq_{\mathcal{R}} (s, \underline{\beta})$ exactly when $x^i \geq_{\mathcal{R}} s_i$ for all $i = 1, \ldots, k$ and $\underline{\alpha} \geq_{\mathcal{R}} \underline{\beta}$. From Ayyer *et al.* [1, 2], we have that c_{S_i} is the number of maximal elements $x^i \in \mathcal{M}_{v_i}$ with $x^i \geq_R s_i$ for some s_i with $\text{supp}(s_i) = S_i$. Define c_α to be α such that $\underline{\alpha} \geq_{\mathcal{R}} \underline{\beta}$ where $\text{supp}(\underline{\beta}) = \alpha$. Thus, we have that $c_{(S,\alpha)} = c_{S_1} \cdots c_{S_k} c_\alpha$.

Furthermore, since the Möbius function is multiplicative, for $S = (S_1, \dots, S_k)$ and $T = (T_1, \dots, T_k)$ we have

$$\mu((S, \alpha), (T, \beta)) = \mu(S_1, T_1) \cdots \mu(S_k, T_k)\mu(\alpha, \beta).$$

Hence, we have

$$m_{(S,\alpha)} = \sum_{(T,\beta) \geq (S,\alpha)} \mu((S, \alpha), (T, \beta))c_{(T,\beta)} = \sum_{\beta \geq \alpha} \mu(\alpha, \beta)c_\beta \prod_{i=1}^{k} \sum_{T_i \geq S_i} \mu(S_i, T_i)c_{T_i},$$

where Ayyer *et al.* showed that $\sum_{T_i \geq S_i} \mu(S_i, T_i)c_{T_i} = d_{S_i}$.

Björner [8] showed that by applying Theorem 1 of Brown [9], we get precisely that

$$m_\alpha = \sum_{\beta \geq \alpha} \mu(\alpha, \beta)c_\beta.$$

Thus, we have

$$m_{(S,\alpha)} = d_{S_1} \cdots d_{S_k} m_\alpha.$$

Let $(S, \alpha) \in L^{\mathcal{M}}$. Then $(S, \alpha) = (\text{Rfactor}(x), \alpha)$ for some $x \in \mathcal{M}_{v_1}^{\widehat{\partial}} \times \cdots \times \mathcal{M}_{v_k}^{\widehat{\partial}}$. That is, $S_i = \text{Rfactor}(x^i)$ for $i = 1, \dots, k$ for $x^i \in \mathcal{M}_{v_i}^{\widehat{\partial}}$. Ayyer *et al.* showed that $d_{S_i} > 0$ for precisely such S_i. Thus $m_{(S,\alpha)} > 0$ for S such that $S = \text{Rfactor}(x)$.

Let (S, α) be such that S is an upset of P that is not $\text{Rfactor}(x)$ for some x. Then there exists some component of S, say S_j such that S_j is not $\text{Rfactor}(x^j)$ where x^j is the component of x that acts on a linear extension of P_{v_j}. Thus, by Ayyer *et al.*, we have that $d_{S_j} = 0$. Thus, $m_{(S,\alpha)} = 0$ since $m_{(S,\alpha)} = d_{S_1} \cdots d_{S_k} m_\alpha$.

Example 4.2.5. *Let T be as in Example 4.2.2. Then $\mathcal{M}_5^{\widehat{\partial}}$ is generated by $\{\widehat{\partial}_1, \widehat{\partial}_2, \widehat{\partial}_3\}$, $\mathcal{M}_6^{\widehat{\partial}}$ is generated by $\{\widehat{\partial}_4\}$, and $\mathcal{M}^{top} = \{(56), (5, 6), (6, 5)\}$. We also have the associated semilattice in Figure 4.4. For this particular example, $\mathcal{M}^T = \mathcal{M}$.*

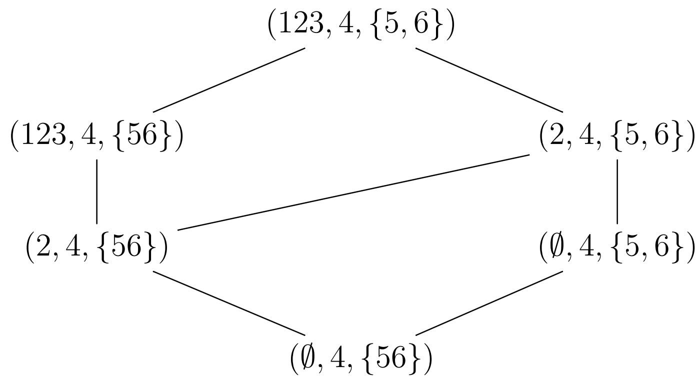


Figure 4.4: Upper semilattice $L^{\mathcal{M}}$ of the monoid $\mathcal{M}$.

For $\widehat{\partial} = \widehat{\partial}_{24}\widehat{\partial}_3\widehat{\partial}_4$ *and* $\alpha = (6,5)$, *we have*

$$\mathrm{supp}(x) = (\mathrm{supp}(\widehat{\partial}_2\widehat{\partial}_3), \mathrm{supp}(\widehat{\partial}_4^2), \mathrm{supp}(6,5)) = (\{1,2,3\}, \{4\}, \{5,6\}).$$

4.3 Background on the class DO(Ab)

First, we need to introduce the notion of a $\mathcal{J}$-class. Let $x, y \in S$ for a semigroup S. Then we have

$$x \leq_{\mathcal{J}} y \quad \text{if } x = uyv \text{ for some } u, v \in S.$$

We say that x and y are in the same $\mathcal{J}$-class if $x \leq_{\mathcal{J}} y$ and $y \leq_{\mathcal{J}} x$. In particular, x and y are $\mathcal{J}$-equivalent if and only if $SxS = SyS$, i.e., if they generate the same two-sided ideal. Furthermore, a $\mathcal{J}$-class is an *orthodox semigroup* if the idempotents ($\{x \colon x^2 = x\}$) form a subsemigroup. For a finite semigroup S and an idempotent element $x \in S$, the *maximal subgroup* is the group of units ($\{u \colon \exists v, uv = vu = \mathrm{id}\}$) of the submonoid xSx. Note that this depends only on the $\mathcal{J}$-class of x up to isomorphism. A semigroup is *regular* if for each element x in S there exists y such that $yxy = y$. The class of $\mathbf{DO(Ab)}$ consists of all finite semigroups whose regular $\mathcal{J}$-classes are orthodox semigroups and whose maximal subgroups are abelian.

The following theorem from Steinberg [28] gives equivalent conditions for S in the class $\mathbf{DO(Ab)}$.

Theorem 4.3.1 ([28])**.** *Let S be a finite semigroup. Then the following are equivalent*:

1. $S \in \mathbf{DO}(\mathbf{Ab})$*;*

2. *every irreducible complex representation of S is a homomorphism $\phi : S \to \mathbb{C}$;*

3. *every complex representation of S is equivalent to one by upper triangular matrices;*

4. *S admits a faithful complex representation by upper triangular matrices.*

The following theorem from Steinberg [28] gives an explicit representation of the eigenvalues for the left random walk on a minimal left ideal of a semigroup in the class $\mathbf{DO(Ab)}$. One can note that this does not give the multiplicities, and it is possible for the multiplicity zero to occur.

Theorem 4.3.2 ([28])**.** *Let $S \in \mathbf{DO}(\mathbf{Ab})$ with generating set X and let L be a minimal left ideal. Assume that S has left identity. Choose a maximal subgroup H_J, with identity e_J, for each regular $\mathcal{J}$-class J. Let $\{w_x\}_{x \in X}$ be a probability distribution on X. Then the transition matrix for the left random walk on L can be placed in upper triangular form over $\mathbb{C}$. Moreover, there is an eigenvalue $\lambda_{(J,\chi)}$ for each regular $\mathcal{J}$-class J and irreducible character χ of H_J given by the formula*

$$\lambda_{(J,\chi)} = \sum_{x \in X, x \geq_{\mathcal{J}} J} w_x \cdot \chi(e_J x e_J).$$

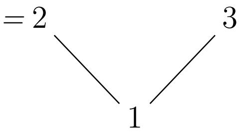
In order to prove Theorem 4.4.1, we need the following lemma which tells us that the direct product of **DO(Ab)** is still **DO(Ab)**.

Lemma 4.3.3. *For S and T two semigroups in the class* **DO**(**Ab**)*, $S \times T$ is in* **DO**(**Ab**)*.*

Proof. Since S and T are in the class **DO(Ab)**, the regular $\mathcal{J}$-classes are orthodox semigroups whose maximal subgroups are abelian. Let J_S be a $\mathcal{J}$-class of S and J_T be a $\mathcal{J}$-class of T with corresponding maximal subgroups H_{J_S} and H_{J_T}. Consider $s, s' \in J_S$ and $t, t' \in J_T$. Then we have $s \leq_{\mathcal{J}} s'$, $s' \leq_{\mathcal{J}} s$, $t \leq_{\mathcal{J}} t'$, and $t' \leq_{\mathcal{J}} t$. Thus, we have the following relations:

$$(s,t) \leq_{\mathcal{J}} (s',t') \text{ and } (s',t') \leq_{\mathcal{J}} (s,t)$$
$$(s,t') \leq_{\mathcal{J}} (s',t) \text{ and } (s',t) \leq_{\mathcal{J}} (s,t')$$
$$(s',t) \leq_{\mathcal{J}} (s,t') \text{ and } (s,t') \leq_{\mathcal{J}} (s',t).$$

That is, for $s, s' \in J_S$ and $t, t' \in J_T$, $(s,t), (s',t), (s,t')$, and (s',t') are in the same $\mathcal{J}$-class of $S \times T$. Now, for $(x,y), (w,z)$ in the same $\mathcal{J}$-class of $S \times T$, we have $(x,y) \leq_{\mathcal{J}} (w,z)$ and $(w,z) \leq_{\mathcal{J}} (x,y)$. So $x \leq_{\mathcal{J}} w$, $w \leq_{\mathcal{J}} x$, $y \leq_{\mathcal{J}} z$, and $z \leq_{\mathcal{J}} y$, i.e., x and w are in the same $\mathcal{J}$-class of S, J_S and y and z are in the same $\mathcal{J}$-class of T, J_T. Precisely, we have that the $\mathcal{J}$-classes of $S \times T$ all have the form $J_S \times J_T$. Since J_S and J_T are orthodox semigroups, we deduce that $J_S \times J_T$ is also an orthodox semigroup. Furthermore, since H_{J_S} is a maximal subgroup of J_S and H_{J_T} is a maximal subgroup of J_T, we have that $H_{J_S} \times H_{J_T}$ will be a maximal subgroup of $J_S \times J_T$. Finally, since H_{J_S} and H_{J_T} are abelian, we have that $H_{J_S} \times H_{J_T}$ is also abelian. Thus, $S \times T$ is in **DO(Ab)**. □

Example 4.3.4. *Let* $P_1 =$ 

and $P_2 = 4$. *Let* S *be the semigroup generated by* $\{\widehat{\partial}_1, \widehat{\partial}_2, \widehat{\partial}_3, id\}$ *and let* T *be the semigroup generated by* $\{\widehat{\partial}_4\}$. *Then the* $\mathcal{J}$*-classes of* S *are*

$$\{\widehat{\partial}_2, \widehat{\partial}_3\} \text{ and } \{\widehat{\partial}_1, id\}$$

and the $\mathcal{J}$*-class of* T *is* $\{\widehat{\partial}_4\}$. *The corresponding maximal subgroups of* S *are*

$$\{\widehat{\partial}_2\} \text{ and } \{\widehat{\partial}_1, id\}$$

and the maximal subgroup of T *is* $\{\widehat{\partial}_4\}$. *For the semigroup* $S \times T$, *the* $\mathcal{J}$*-classes are*

$$\{(\widehat{\partial}_2, \widehat{\partial}_4), (\widehat{\partial}_3, \widehat{\partial}_4)\} \text{ and } \{(\widehat{\partial}_1, \widehat{\partial}_4), (id, \widehat{\partial}_4)\}.$$

The corresponding maximal subgroups of $S \times T$ *are*

$$\{(\widehat{\partial}_2, \widehat{\partial}_4)\} \text{ and } \{(\widehat{\partial}_1, \widehat{\partial}_4), (id, \widehat{\partial}_4)\}.$$

In order to prove Theorem 4.4.3, which says that $\mathcal{M}^T$ is in $\mathbf{DO(Ab)}$, we need Theorem 4.3.1 from Steinberg [28] and Theorem 4.5.2.

4.4 When the leaf posets are unions of an ordinal sum of a forest and a ladder: an algebraic treatment

The theory of **DO(Ab)** leads us to the following theorem.

Theorem 4.4.1. *Let T be as described above where the leaf poset has the form*

$$F_1 \oplus L_1 + \cdots + F_t \oplus L_t$$

and let M be the transition matrix of the random walk on $\mathcal{L}(T)$:

$$M(\pi, \pi') = \sum_{E:\, \widehat{\partial}_E \pi = \pi'} x_E$$

for $\pi, \pi' \in \mathcal{L}(T)$. Choose a maximal subgroup H_J of each $\mathcal{J}$-class J of the underlying monoid $\mathcal{M}$ with identity e_J, and let $\alpha \in \mathrm{Part}(T \setminus L)$. Then there is an eigenvalue

$$\lambda_{((J,\chi),\alpha)} = \sum_{\substack{E \in \mathcal{A}(L) \\ \widehat{\partial}_E \geq_{\mathcal{J}} J \\ E \text{ is } \alpha\text{-compatible}}} x_E \chi(e_J \widehat{\partial}_E e_J),$$

where χ is an irreducible character of H_J. The eigenvalue $\lambda_{((J,\chi),\alpha)}$ can show up with multiplicity zero; however, there are no other eigenvalues.

Example 4.4.2. *Let T be given as in Figure 4.5.*

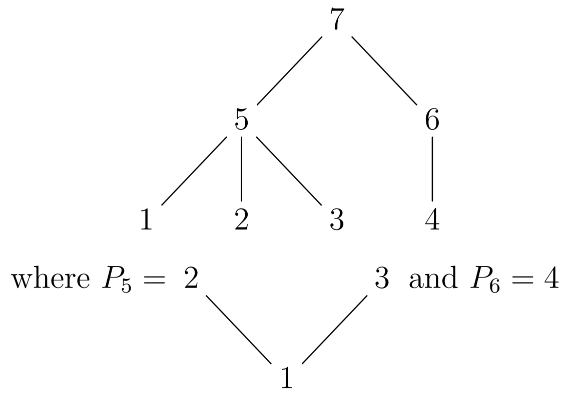


Figure 4.5: A rooted tree with a ladder as the leaf poset.

The underlying semigroup of P is generated by $\{\widehat{\partial}_E \colon E \in \mathcal{A}(L)\}$. The $\mathcal{J}$-classes of the underlying monoid $\mathcal{M}$ are

$$J = \{(\widehat{\partial}_1, \widehat{\partial}_4), (id, \widehat{\partial}_4)\} \text{ and } J' = \{(\widehat{\partial}_2, \widehat{\partial}_4), (\widehat{\partial}_3, \widehat{\partial}_4)\}$$

and the corresponding maximal subgroups are

$$H_J = J \text{ and } H_{J'} = \{(\widehat{\partial}_2, \widehat{\partial}_4)\}.$$

The identity element of each maximal subgroup is

$$e_J = (id, \widehat{\partial}_4) \text{ and } e_{J'} = (\widehat{\partial}_2, \widehat{\partial}_4).$$

The character table for H_J is

	$(id, \widehat{\partial}_4)$	$(\widehat{\partial}_1, \widehat{\partial}_4)$
$\chi^{(1)}$	1	1
$\chi^{(2)}$	1	-1

and the character table for $H_{J'}$ is

$$\begin{array}{c|c} & (\widehat{\partial}_2, \widehat{\partial}_4) \\ \hline \chi^{(1)} & 1 \end{array}.$$

We have $\widehat{\partial}_E \geq_{\mathcal{J}} J$ *if and only if* $\widehat{\partial}_E \geq_{\mathcal{J}} y$ *for all* $y \in J$. *For example, if* $J = \{(\widehat{\partial}_1, \widehat{\partial}_4), (id, \widehat{\partial}_4)\}$, *then* $\widehat{\partial}_E \geq_{\mathcal{J}} J$ *for* $E \in \{\{1\}, \{4\}, \{1,4\}\}$. *Furthermore, for* J' *as given above,* $\widehat{\partial}_E \geq_{\mathcal{J}} J'$ *for* $E \in \mathcal{A}(L)$. *Thus, the possible eigenvalues are*

$$\lambda_{(J,\chi^{(1)}),\{56\}} = x_{14}$$

$$\lambda_{(J,\chi^{(1)}),\{5,6\}} = x_{14} + x_1 + x_4$$

$$\lambda_{(J,\chi^{(2)}),\{56\}} = -x_{14}$$

$$\lambda_{(J,\chi^{(2)}),\{5,6\}} = -x_1 - x_{14} + x_4$$

$$\lambda_{(J',\chi^{(1)}),\{56\}} = x_{14} + x_{24} + x_{34}$$

$$\lambda_{(J',\chi^{(1)}),\{5,6\}} = x_1 + x_2 + x_3 + x_4 + x_{14} + x_{24} + x_{34}.$$

Explicit calculations show that the eigenvalues $\lambda_{(J,\chi^{(1)}),\{56\}}$ *and* $\lambda_{(J,\chi^{(1)}),\{5,6\}}$ *have multiplicity zero and the remaining eigenvalues have multiplicity one.*

In order to apply Theorem 4.4.1, we first prove that the associated monoid $\mathcal{M}$ is in **DO(Ab)** if all the leaf posets have the form $F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ where each F_i is a rooted forest and L_i is a ladder.

Theorem 4.4.3. *For* T *be as described above,* $\mathcal{M}$ *is in the class* **DO**(**Ab**).

Proof. Let G_j be the matrix corresponding to $\widehat{\partial}_j$ with a 1 in the position of x_j and a zero otherwise. Let $\mathcal{M}_v^{\widehat{\partial}}$ be the monoid generated by the matrices G_j such that $j \in C_v$. Every element in the monoid can be written as a product of the G_j's. By Theorem 3.2.2, there exists a matrix U that simultaneously upper triangularizes all

G_j. If we conjugate every element by U, then all elements of $\mathcal{M}_v^{\widehat{\partial}}$ will be upper triangular. Thus, this is a representation of $\mathcal{M}_v^{\widehat{\partial}}$ by upper triangular matrices. That is, we have

$$\widehat{\partial}_j \mapsto U G_j U^{-1}.$$

Thus, Condition (4) of Theorem 4.3.1 holds. Hence, $\mathcal{M}_v^{\widehat{\partial}}$ is in $\mathbf{DO(Ab)}$.

Since $\mathcal{M}^{\text{top}}$ is a left-regular band, Steinberg [28] guarantees that every band is in the class $\mathbf{DO(Ab)}$.

Since each $\mathcal{M}_{v_i}^{\widehat{\partial}}$ and $\mathcal{M}^{\text{top}}$ is in $\mathbf{DO(Ab)}$ and by the Lemma 4.3.3 we have that a direct product of elements of $\mathbf{DO(Ab)}$ is $\mathbf{DO(Ab)}$, we have that $\mathcal{M}$ is in $\mathbf{DO(Ab)}$. □

The rest of this section is devoted to the proof of Theorem 4.4.1.

Since Theorem 4.4.3 tells us that $\mathcal{M}$ is in the class $\mathbf{DO(Ab)}$, in order to prove Theorem 4.4.1, we just need to verify that it coincides with Theorem 4.3.2. The generating set X of $\mathcal{M} = \mathcal{M}_{v_1}^{\widehat{\partial}} \times \cdots \times \mathcal{M}_{v_k}^{\widehat{\partial}} \times \mathcal{M}^{\text{top}}$ is $(\widehat{\partial}_E, \underline{\alpha}^E)$ where $E \in \mathcal{A}(L)$ and $\underline{\alpha}^E \in \text{Part}^{\text{ord}}(T \setminus L)$. Furthermore, $\underline{\alpha}$ and $\underline{\beta}$ are in the same $\mathcal{J}$-class if their underlying set partitions α and β are equal. Thus, the $\mathcal{J}$-classes of $\mathcal{M}$ are of the form (J, J_α) where $J_\alpha = \{\underline{\beta} \colon \beta = \alpha\}$. Notice that every element of J_α is idempotent since $\underline{\alpha} \circ \underline{\alpha} = \underline{\alpha}$ and the maximal subgroups of J_α are one element, namely $\underline{\alpha}$. So the maximal subgroups are of the form $(H_J, \underline{\alpha})$. The identity element of $(H_J, \underline{\alpha})$ is $(e_J, \underline{\alpha})$, where e_J is the identity on H_J.

Let $\{x_E\}_{E \in \mathcal{A}(L)}$ be a probability distribution. There is an eigenvalue $\lambda_{((J, J_\alpha), \chi)}$

for each regular $\mathcal{J}$-class (J, J_α) and irreducible character χ of $(H_J, \underline{\alpha})$ given by

$$\begin{aligned}
\lambda_{((J,J_\alpha),\chi)} &= \sum_{\substack{(\widehat{\partial}_E, \underline{\beta}^E) \in X \\ (\widehat{\partial}_E, \underline{\beta}^E) \geq_{\mathcal{J}} (J, J_\alpha)}} x_E \chi((e_J, \underline{\alpha})(\widehat{\partial}_E, \underline{\beta}^E)(e_J, \underline{\alpha})) \\
&= \sum_{\substack{E \in \mathcal{A}(L) \\ \widehat{\partial}_E \geq_{\mathcal{J}} J \\ \underline{\beta}^E \geq_{\mathcal{J}} J_\alpha}} x_E \chi((e_J \widehat{\partial}_E e_J, \underline{\alpha} \circ \underline{\beta}^E \circ \underline{\alpha})).
\end{aligned}$$

In fact, $\underline{\beta}^E \geq_{\mathcal{J}} J_\alpha$ if and only if $\underline{\beta}^E \geq_{\mathcal{J}} \underline{\alpha}$ for all $\underline{\alpha}$ with underlying set partition α. Furthermore, $\underline{\beta}^E \geq_{\mathcal{J}} \underline{\alpha}$ if and only if $\underline{\alpha}$ is a refinement of $\underline{\beta}^E$. This is true for all α, so E is α-compatible. Since α is a refinement of β^E, we have $\underline{\alpha} \circ \underline{\beta}^E \circ \underline{\alpha} = \underline{\alpha}$. Notice since J_α is generated by α, we have

$$\begin{aligned}
\lambda_{((J,J_\alpha),\chi)} = \lambda_{((J,\chi),\alpha)} &= \sum_{\substack{E \in \mathcal{A}(L) \\ \widehat{\partial}_E \geq_{\mathcal{J}} J \\ E \text{ is } \alpha\text{-compatible}}} x_E \chi((e_J \widehat{\partial}_E e_J, \underline{\alpha})) \\
&= \sum_{\substack{E \in \mathcal{A}(L) \\ \widehat{\partial}_E \geq_{\mathcal{J}} J \\ E \text{ is } \alpha\text{-compatible}}} x_E \chi(e_J \widehat{\partial}_E e_J) \chi(\underline{\alpha}) \\
&= \sum_{\substack{E \in \mathcal{A}(L) \\ \widehat{\partial}_E \geq_{\mathcal{J}} J \\ E \text{ is } \alpha\text{-compatible}}} x_E \chi(e_J \widehat{\partial}_E e_J).
\end{aligned}$$

We will see in the next section that these characters will all be ± 1.

4.5 When the leaf posets are unions of an ordinal sum of a forest and a ladder: a combinatorial treatment

The following theorem gives us a more general result. The explicit multiplicities are given in Theorem 4.5.2.

Theorem 4.5.1. *Let T be a described above and let $P_{v_i} = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ where each F_j is a rooted forest and L_j is a ladder and v_i has depth $d-1$. The eigenvalues of the transition matrix M^T are linear in the x_E's. Moreover, they can be explicitly computed using the formula for the case when P_{v_i} is a rooted forest of depth $d-1$ (Theorem 4.2.1) and Theorem 4.5.2.*

Theorem 4.4.1 does not give explicit eigenvalues for the case when the leaf posets are unions of an ordinal sum of a forest and a ladder, but Theorem 4.5.2 gives an explicit way to calculate the eigenvalues. The proofs and definitions in this section are in the same spirit as the proofs and definitions in Section 3.2. We first need the notion of the upset property on a tree T. Instead of having only a single element as in Section 3.2, let $E \in \mathcal{A}(L)$. Recall $P = P_{v_1} + \cdots + P_{v_k}$ where P_{v_i} has the upset property for all i. Let R_P be the set of all pairs (a, b) for which P can be written in the form

$$P = Q' \oplus a \oplus b \oplus Q'' + P_2,$$

and denote by P' the poset $P \setminus \{(a, b)\}$. We will say that M^{T_P} has the *upset property* if its characteristic polynomial factors into linear terms, and for each eigenvalue $x^{\mathfrak{s}} = \sum c^{\mathfrak{s}}_E x_E$ of M^T, the pair $(a, b) \in R_P$, and for each subset $E_k = E \cup \{k\}$ of $\mathcal{A}(L)$ the following two conditions hold:

(a) $x_{E_a} \in x^{\mathfrak{s}} \implies x_{E_b} \in x^{\mathfrak{s}}$ and $c^{\mathfrak{s}}_{E_a} = c^{\mathfrak{s}}_{E_b}$

(b) $x_{E_b} \in x^{\mathfrak{s}}, x_{E_a} \notin x^{\mathfrak{s}} \implies x_{E_k} \notin x^{\mathfrak{s}}$ for $k \prec a$.

Similar to the case of Section 3.2, we will use $x_E \in x^{\mathfrak{s}}$ to denote that x_E appears in $x^{\mathfrak{s}}$ with a nonzero coefficient.

Let T_P and $T_{P'}$ be two trees that have the same underlying structure, but whose leaf posets satisfy $P' = P \setminus \{(a, b)\}$.

Theorem 4.5.2. *Let T_P and $T_{P'}$ be as described above where the leaf poset is of the form $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and $P' = P \setminus \{(a, b)\}$. Suppose $M^T = \sum x_E G_E$, where the G_E are complex simultaneously upper-triangularizable matrices. If the characteristic polynomial of M^{T_P} has the upset property then so does $M^{T_{P'}}$. In particular, for each eigenvalue $x^{\mathfrak{s}} = \sum c^{\mathfrak{s}}_E x_E$ of M^{T_P} and for each set E_a, $M^{T_{P'}}$ has two eigenvalues given by*

$$\begin{cases} x^{\mathfrak{s}}, \displaystyle\sum_{k \not\succeq_P b} c^{\mathfrak{s}}_{E_k} x_{E_k} - \sum_{k \prec_P a} c^{\mathfrak{s}}_{E_k} x_{E_k} & \text{if } x_{E_a}, x_{E_b} \in x^{\mathfrak{s}} \text{ or } x_{E_a}, x_{E_b} \notin x^{\mathfrak{s}} \\ x^{\mathfrak{s}}, x^{\mathfrak{s}} - c^{\mathfrak{s}}_{E_b} x_{E_b} + c^{\mathfrak{s}}_{E_b} x_{E_a} & \text{if } x_{E_a} \notin x^{\mathfrak{s}}, x_{E_b} \in x^{\mathfrak{s}}. \end{cases}$$

Exactly as in the case of Section 3.2, for each leaf poset of the form $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ where F_i is a forest and L_i is a ladder, P can be obtained by starting from a forest in which the upper parts of the tree components are chains and then breaking covering relations in the chains. Furthermore, the transition matrix of T_P where the leaf posets are forests satisfies the assumptions of Theorem 4.5.2, because the monoid generated by the matrices G_E is $\mathcal{R}$-trivial and the eigenvalues of the transition matrix are supported on the upsets of the tree (Theorem 4.2.1). Thus, Theorem 4.5.1 follows directly from Theorem 4.5.2.

Example 4.5.3. *We can now verify Example 4.4.2 with Theorem 4.5.2. Let* T_P *and* T'_P *be given in Figure 4.6.*

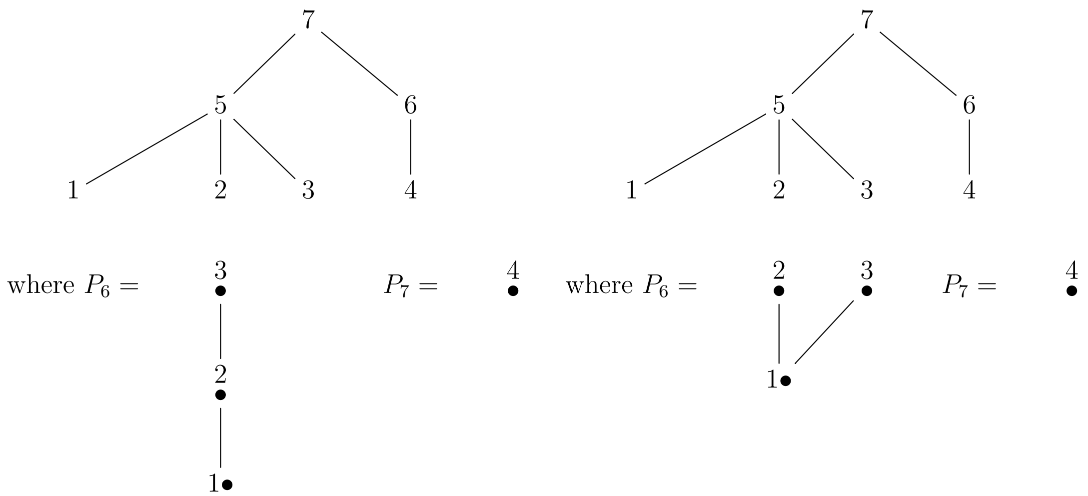
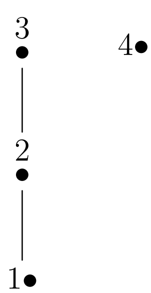
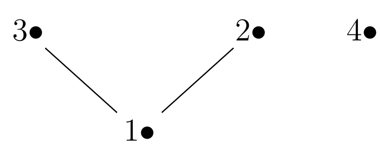


Figure 4.6: The trees T_P and T'_P where $P' = P \setminus \{(2,3)\}$.

Notice that the leaf posets are $P =$  *and* $P' =$  .

Since P *is a rooted forest, we can use Theorem 4.2.1 to compute the eigenvalues of* T_P*, and they are*

$$\{x_{14} + x_{24} + x_{34}, x_{14} + x_{24} + x_{34} + x_1 + x_2 + x_3 + x_4\}.$$

Now, by Theorem 4.5.2, we compute the eigenvalues of $T_{P'}$ *to be*

$$\{x_{14} + x_{24} + x_{34}, -x_{14}, x_1 + x_2 + x_3 + x_4 + x_{14} + x_{24} + x_{34}, x_4 - (x_1 + x_{14})\},$$

which are precisely the eigenvalues from Example 4.4.2 with nonzero multiplicity.

The rest of this section is devoted to the proof of Theorem 4.5.2 which is based on several lemmas that we prove first. For the rooted trees T_P and $T_{P'}$ described at the beginning of this section, the matrices M^{T_P} and $M^{T_{P'}}$ are closely related as described in the following lemma, which is analogous to Lemma 3.2.5.

Lemma 4.5.4. *Let $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and let $P' = P \setminus \{(a,b)\}$ where M^{T_P} has leaf poset P and $M^{T_{P'}}$ has leaf poset P'.*

(1) If $k \in E$ and $k \prec a$ and $\pi \overset{x_E}{\to} \widetilde{\pi}$ in G_{T_P}, then $\pi \overset{x_E}{\to} \widehat{\widetilde{\pi}}$ and $\widehat{\pi} \overset{x_E}{\to} \widetilde{\pi}$ in $G_{T_{P'}}$.

(2) If $k \in E$ and $k \not\preceq a, b$ and $\pi \overset{x_E}{\to} \widetilde{\pi}$ in G_{T_P}, then $\pi \overset{x_E}{\to} \widetilde{\pi}$ and $\widehat{\pi} \overset{x_E}{\to} \widehat{\widetilde{\pi}}$ in $G_{T_{P'}}$.

(3) If $a \in E$, $E_b = E \setminus \{a\} \cup \{b\}$ and $\pi \overset{x_E}{\to} \widetilde{\pi}$ in G_{T_P}, then $\pi \overset{x_E}{\to} \widehat{\widetilde{\pi}}$ and $\widehat{\pi} \overset{x_{E_b}}{\to} \widetilde{\pi}$ in $G_{T_{P'}}$.

(4) If $b \in E$ and $E_a = (E \setminus \{b\}) \cup \{a\}$ and $\pi \overset{x_E}{\to} \widetilde{\pi}$ in G_{T_P}, then $\pi \overset{x_E}{\to} \widetilde{\pi}$ and $\widehat{\pi} \overset{x_{E_a}}{\to} \widehat{\widetilde{\pi}}$ in $G_{T_{P'}}$.

Proof. Notice that if we relax the relation between a and b in C_v for some v, then we have two linear extensions of P'_v, namely π_v and $\widehat{\pi_v}$, for every linear extension π_v of P_v. Furthermore, for every $w \neq v$, we have $\mathcal{L}(P_w) = \mathcal{L}(P'_w)$ and $\pi \overset{x_E}{\to} \widetilde{\pi}$ in G_{T_P}, implies that $\pi_x \overset{x_E}{\to} \widetilde{\pi}_x$ in G_{T_P} for all $x \in I$. In $G_{T_{P'}}$ we have $\pi_w \overset{x_E}{\to} \widetilde{\pi}_w$ for $w \neq v$. Thus, we only need to consider the linear extension $\pi_v \in \mathcal{L}(P'_v)$. Furthermore, by the structure of $\mathcal{A}(L)$, if $k \in E$, then k is the only child of v in E. Thus, $\widehat{\partial}_E \pi_v = \widehat{\partial}_k \pi_v$.

The proof of the case of π_v follows exactly from the proof of Lemma 3.2.5. □

Let T_P be a rooted tree with a size-n leaf poset $P = Q' \oplus a \oplus b \oplus Q'' + P_2$. For the size-$m$ transition matrix M^{T_P}, we denote by $\partial_{a,b} M^{T_P}$ the $2m \times 2m$ matrix obtained by replacing each entry of M^{T_P} by a 2×2 block using the linear extension of the map:

$$\begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_{E_k} \right) \end{array} \longmapsto \begin{array}{c} \\ \pi \\ \widehat{\pi} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \widetilde{\pi} & \widehat{\widetilde{\pi}} \end{array} \\ \begin{pmatrix} & x_{E_k} \\ x_{E_k} & \end{pmatrix} \end{array} \text{for } k \prec a \qquad \begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_{E_a} \right) \end{array} \longmapsto \begin{array}{c} \\ \pi \\ \widehat{\pi} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \widetilde{\pi} & \widehat{\widetilde{\pi}} \end{array} \\ \begin{pmatrix} & x_{E_a} \\ x_{E_b} & \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_{E_k} \right) \end{array} \longmapsto \begin{array}{c} \\ \pi \\ \widehat{\pi} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \widetilde{\pi} & \widehat{\widetilde{\pi}} \end{array} \\ \begin{pmatrix} x_{E_k} & \\ & x_{E_k} \end{pmatrix} \end{array} \text{for } k \not\preceq b \qquad \begin{array}{cc} & \widetilde{\pi} \\ \pi & \left(x_{E_b} \right) \end{array} \longmapsto \begin{array}{c} \\ \pi \\ \widehat{\pi} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \widetilde{\pi} & \widehat{\widetilde{\pi}} \end{array} \\ \begin{pmatrix} x_{E_b} & \\ & x_{E_a} \end{pmatrix} \end{array}$$

For a complex matrix S, define $\partial S = S \otimes I_2$. So, if $\mathcal{E}$ is an elementary matrix of size k corresponding to a row operation R, then $\partial\mathcal{E}$ describes performing a corresponding operation to 2 rows on a matrix of size $2k$. Note that the remaining proofs in this section follow exactly from the analogous proofs in Section 3.2.

Corollary 4.5.5. *Let T_P be a rooted tree with leaf poset $P = Q' \oplus a \oplus b \oplus Q'' + P_2$ and $T_{P'}$ be the tree with leaf poset $P' = P \setminus \{(a, b)\}$. Then $M^{T_{P'}} = \partial_{a,b} M^{T_P}$.*

Lemma 4.5.6. *Let S be a matrix with complex entries and M a matrix whose entries are homogeneous degree-1 polynomials in the x_E's where $E \in \mathcal{A}(L)$. Then*

$$(\partial S)(\partial_{a,b} M) = \partial_{a,b}(SM) \quad \textit{and} \quad (\partial_{a,b} M)(\partial S) = \partial_{a,b}(MS).$$

Lemma 4.5.7. *Let M be a matrix whose entries are homogeneous degree-1 polynomials in the x_E's and let S be a complex matrix such that $U = SMS^{-1}$ is upper triangular. Then the eigenvalues of $\partial_{a,b} M$ are the same as the eigenvalues of $\partial_{a,b} U$.*

The rest of this section is devoted to the proof of Theorem 4.5.2.

By Corollary 3.2.6, $M^{T_{P'}} = \partial_{a,b} M^{T_P}$. Let S be the matrix that simultaneously upper-triagularizes the G_E's. Then $U = SMS^{-1}$ is an upper triangular matrix whose diagonal entries are the eigenvalues $x^{\mathfrak{s}}$ of M^{T_P}. By Lemma 3.2.9 the eigenvalues of $M^{T_{P'}}$ are the same as the eigenvalues of $\partial_{a,b} U$ which is block upper-triangular with 2×2 blocks $\partial_{a,b} x^{\mathfrak{s}}$ on the main diagonal, where

$$\partial_{a,b} x^{\mathfrak{s}} = \begin{pmatrix} c^{\mathfrak{s}}_{E_b} x_{E_b} + \sum\limits_{k \not\prec a,b} c^{\mathfrak{s}}_{E_k} x_{E_k} & c^{\mathfrak{s}}_{E_a} x_{E_a} + \sum\limits_{k \prec a} c^{\mathfrak{s}}_{E_k} x_{E_k} \\ c^{\mathfrak{s}}_{E_a} x_{E_b} + \sum\limits_{k \prec a} c^{\mathfrak{s}}_{E_k} x_{E_k} & c^{\mathfrak{s}}_{E_b} x_{E_a} + \sum\limits_{k \not\prec a,b} c^{\mathfrak{s}}_{E_k} x_{E_k} \end{pmatrix}.$$

Since by assumption, M^{T_P} has the upset property, there are only two cases:

$$c^{\mathfrak{s}}_{E_a} = c^{\mathfrak{s}}_{E_b} = c \text{ and } c^{\mathfrak{s}}_{E_a} = 0, c^{\mathfrak{s}}_{E_b} \neq 0.$$

In the former case,

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \partial_{a,b} x^{\mathfrak{s}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x^{\mathfrak{s}} & 0 \\ 0 & \sum\limits_{k \not\prec a,b} c^{\mathfrak{s}}_{E_k} x_{E_k} - \sum\limits_{k \prec a} c^{\mathfrak{s}}_{E_k} x_{E_k} \end{pmatrix}.$$

In the latter case, by the upset property we also have that $\sum_{k \prec a} c^{\mathfrak{s}}_{E_k} x_{E_k} = 0$, and therefore

$$\partial_{a,b} x^{\mathfrak{s}} = \begin{pmatrix} c^{\mathfrak{s}}_{E_b} x_{E_b} + \sum\limits_{k \not\prec a,b} c^{\mathfrak{s}}_{E_k} x_{E_k} & 0 \\ 0 & c^{\mathfrak{s}}_{E_b} x_{E_a} + \sum\limits_{k \not\prec a,b} c^{\mathfrak{s}}_{E_k} x_{E_k} \end{pmatrix}.$$

This also shows that there is a real matrix S' such that $S'(\partial_{a,b} U)(S')^{-1}$ is upper triangular. Consequently, $S'(\partial S) M^{T_{P'}} (S'(\partial S))^{-1}$ is upper triangular, which means that the matrices G'_E such that $M^{T_{P'}} = \sum x_{E_k} G'_E$ are simultaneously upper-triangularizable.

Finally, notice that $R_{P'} \subset R_P$ and if $(a', b') \in R_{P'}$ then $\{a', b'\} \cup \{a, b\} = \emptyset$

and either $a', b' \prec a$ or $a', b' \not\preceq b$. So, by inspection, the eigenvalues of $M^{T_{P'}}$ satisfy the conditions (a) and (b) from the definition of the upset property.

Chapter 5

Future Directions and Discussion

The posets of the form $P = F_1 \oplus L_1 + \cdots + F_k \oplus L_k$ discussed in Chapter 3 are not the only ones to have the nice property that the eigenvalues of their promotion matrices are linear in the x_i's. In fact, we conjecture the following; where A_i is an antichain of size i.

Conjecture 5.0.1. *The characteristic polynomial for the promotion matrix M^P of any poset P whose Hasse diagram is contained in $A_k \oplus A_2$ factors into linear terms.*

As a justification of this conjecture, we will prove the special case when P is $A_k \oplus A_2$ with one missing edge.

Theorem 5.0.2. *The characteristic polynomial of M^P for the poset*

$$P = (A_k \oplus A_2) \setminus \{(k, k+1)\}$$

which is assumed to be labeled naturally is

$$\det(M^P - \lambda I) = (x_{k+2} - \lambda)^{(k-1)!} \prod_{U \subseteq [k]} (x_u + x_{k+1} + x_{k+2} - \lambda)^{d_{k-|U|}} \prod_{U \subseteq [k-1]} (-x_u - \lambda)^{d_{k-|U|} + d_{k-|U|-1}}, \tag{5.1}$$

where d_i is the number of derangements in the symmetric group S_i.

Proof. Let σ_i represent the permutation of $[k]$ in which k is in the i-th position, and let σ be the permutation of $[k-1]$ obtained by deleting k. Consider $M^P - \lambda I$ for $P = (A_k \oplus A_2) \setminus \{(k, k+1)\}$.

M^P can be split into blocks $B_{\sigma_i}^{\pi_j}$ of size $2 \times 2, 2 \times 3, 3 \times 2$, or 3×3 as follows:

- If $i = k$, the three rows of $B_{\sigma_k}^{\pi_j}$ correspond to the linear extensions $\sigma_k(k+1)(k+2)$, $\sigma_k(k+2)(k+1)$, and $\sigma(k+1)k(k+2)$.
- If $i \neq k$, the two rows of $B_{\sigma_i}^{\pi_j}$ correspond to the linear extensions $\sigma_i(k+1)(k+2)$ and $\sigma_i(k+2)(k+1)$.

The columns of $B_{\sigma_i}^{\pi_j}$ are indexed analogously depending of whether $j = k$ or $j \neq k$. Let the transition matrix of the Tsetlin library for k books be $M^{A_k} = \left(a_{\sigma_i}^{\pi_j}\right)_{i,j=1}^{k}$.

Then

$$B_{\sigma_i}^{\pi_j} = \begin{cases} a_\sigma^\pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \delta_{\sigma,\pi} \begin{pmatrix} x_{k+2}-\lambda & x_{k+1} & x_k \\ x_k + x_{k+2} & x_{k+1}-\lambda & 0 \\ 0 & x_{k+1} & x_k + x_{k+2} - \lambda \end{pmatrix} & \text{if } i, j = k, \\ a_{\sigma_i}^{\pi_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \delta_{\sigma_i,\pi_j} \begin{pmatrix} x_{k+2}-\lambda & x_{k+1} \\ x_{k+2} & x_{k+1}-\lambda \end{pmatrix} & \text{if } i, j \neq k, \\ a_{\sigma_k}^{\pi_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } i = k, j \neq k, \\ a_{\sigma_i}^{\pi_k} \delta_{\sigma,\pi} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \text{if } i \neq k, j = k, \end{cases}$$

where $\delta_{x,y}$ is the Kronecker delta function.

If we subtract the first two rows of $B_{\sigma_k}^*$ from $B_{\sigma_i}^*$ for all $i \neq k$, the block change is given by

$$B_{\sigma_i}^{\pi_j} \mapsto B_{\sigma_i}^{\pi_j} = \begin{cases} -\delta_{\sigma,\pi} \begin{pmatrix} x_{k+2}-\lambda & x_{k+1} & 0 \\ x_{k+2} & x_{k+1}-\lambda & 0 \end{pmatrix} & \text{if } j = k, \\ a_{\sigma_i}^{\pi_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \delta_{\sigma_i,\pi_j} \begin{pmatrix} x_{k+2}-\lambda & x_{k+1} \\ x_{k+2} & x_{k+1}-\lambda \end{pmatrix} - a_{\sigma_k}^{\pi_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } j \neq k. \end{cases}$$

If we then add the columns of $B_*^{\pi_j}$ to the first two columns of $B_*^{\pi_k}$ for $j =$

$1, \dots, k-1$, the block change is given by

$$B_{\sigma_i}^{\pi_k} \mapsto B_{\sigma_i}^{\pi_k}$$
$$= \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } i \neq k, \\ a_\sigma^\pi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \delta_{\sigma_i, \pi_j} \begin{pmatrix} x_{k+2} - \lambda & x_{k+1} & x_k \\ x_k + x_{k+2} & x_{k+1} - \lambda & 0 \\ 0 & x_{k+1} & x_k + x_{k+2} - \lambda \end{pmatrix} & \text{if } i = k. \end{cases}$$

If the blocks are ordered so that $B_{\sigma_k}^{\pi_k}$ are in the upper left, this yields a block upper triangular matrix with one block B_u of size $3(k-1)! \times 3(k-1)!$ consisting of the 3×3 blocks $B_{\sigma_k}^{\pi_k}$ and another block B_ℓ consisting of the 2×2 blocks $B_{\sigma_i}^{\pi_j}$ for $i, j \neq k$. Next, we compute the determinant of each of these blocks separately. The upper block B_u is similar in structure to $M^{A_{k-1}} - \lambda I$. Namely, B_u can be obtained from $M^{A_{k-1}} - \lambda I$ by the substitutions

$$x_m \mapsto \begin{pmatrix} 0 & x_m & 0 \\ x_m & 0 & 0 \\ 0 & x_m & 0 \end{pmatrix} \text{ and } \quad -\lambda \mapsto \Lambda = \begin{pmatrix} x_{k+2} - \lambda & x_{k+1} & x_k \\ x_k + x_{k+2} & x_{k+1} - \lambda & 0 \\ 0 & x_{k+1} & x_k + x_{k+2} - \lambda \end{pmatrix}.$$

In other words,

$$B_u = M^{A_{k-1}} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + I_{k-1} \otimes \begin{pmatrix} x_{k+2} - \lambda & x_{k+1} & x_k \\ x_k + x_{k+2} & x_{k+1} - \lambda & 0 \\ 0 & x_{k+1} & x_k + x_{k+2} - \lambda \end{pmatrix}.$$

By [9], $SM^{A_{k-1}}S^{-1}$ is diagonal for some matrix S. Therefore,

$$(S \otimes I_3) B_u (S^{-1} \otimes I_3)$$

is block diagonal with blocks $\begin{pmatrix} x_{k+2}-\lambda & x_{k+1}+x_U & x_k \\ x_k+x_{k+2}+x_U & x_{k+1}-\lambda & 0 \\ 0 & x_{k+1}+x_U & x_k+x_{k+2}-\lambda \end{pmatrix}$ corresponding to $(x_U-\lambda)$ in $M^{A_{k-1}}-\lambda I$. Thus,

$$\det B_u = \prod_{U\subseteq[k-1]} (-x_U-\lambda)^{d_{k-1-|U|}}(x_k+x_{k+1}+x_{k+2}+x^U-\lambda)^{d_{k-1-|U|}}(x_{k+2}-\lambda)^{d_{k-1-|U|}}.$$

For the lower block B_ℓ, first notice that there are similarities between M^P and M^{A_k}. The entries in $M^{A_k}=\left(a_{\sigma_i}^{\pi_j}\right)_{i,j=1}^k$ are only zero or x_i for some $i\in\{1,\ldots,k\}$. We perform the following row and column operations on $M^{A_k}-\lambda I=\left(m_{\sigma_i}^{\pi_j}\right)_{i,j=1}^k$. If we subtract the rows σ_i from σ_k for all $i\neq k$, the entries change to

$$m_{\sigma_i}^{\pi_j}\mapsto m_{\sigma_i}^{\pi_j}=\begin{cases}-\lambda\delta_{\sigma,\pi} & \text{if } j=k,\\ a_{\sigma_i}^{\pi_j}-a_{\sigma_k}^{\pi_j}+\lambda\delta_{\sigma_i,\pi_j} & \text{if } j\neq k.\end{cases}$$

If we then add the columns π_j to the column π_k for $j=1,\ldots,k$, the entries become

$$m_{\sigma_i}^{\pi_k}\mapsto m_{\sigma_i}^{\pi_k}=\begin{cases}0 & \text{if } i\neq k,\\ a_{\sigma_i}^{\pi_j}+\lambda\delta_{\sigma_i,\pi_j} & \text{if } i=k.\end{cases}$$

Notice that the matrices for the row and column operations are inverses of each other, so that the resulting matrix is similar to $M^{A_k}-\lambda I$. Moreover, if we order the linear extensions of A_k so that $a_{\sigma_k}^{\pi_k}$ is in the upper-left corner, the resulting matrix is block upper triangular matrix, where for the lower block b_ℓ, we have

$$B_\ell = b_\ell\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}+I\otimes\begin{pmatrix}x_{k+2}&x_{k+1}\\x_{k+2}&x_{k+1}\end{pmatrix}.$$

The block b_ℓ does not contain x_k and is of size $(k-1)(k-1)! \times (k-1)(k-1)!$, which is the sum of the multiplicities of all eigenvalues of M^{A_k} whose support does not contain x_k.

Since $M^{A_k} - \lambda I$ is diagonalizable, so is b_ℓ. Let S be such that $S b_\ell S^{-1}$ is diagonal. Then $(S \otimes I_2) B_\ell (S^{-1} \otimes I_2)$ is a block diagonal matrix with blocks $\begin{pmatrix} x_{k+2} - \lambda & x_{k+1} + x_U \\ x_{k+2} + x_U & x_{k+1} - \lambda \end{pmatrix}$ for every eigenvalue x_U of M^{A_k} whose support does not contain x_k. This gives

$$\det B_\ell = \prod_{U \subseteq [k-1]} (-x_U - \lambda)^{d_{k-|U|}} (x_{k+1} + x_{k+2} + x_U - \lambda)^{d_{k-|U|}}.$$

Since $\det(M^P - \lambda I) = \det B_u \det B_\ell$, Equation (5.1) follows. $\square$

Example 5.0.3. *Let $P = (A_2 \oplus A_2) \setminus \{(1,4)\}$, as shown in Figure 5.1.*

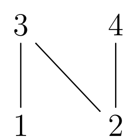


Figure 5.1: Poset $(A_2 \oplus A_2) \setminus \{(1,4)\}$.

The eigenvalues of M^P are

$$x_1 + x_2 + x_3 + x_4, \qquad 0, \qquad x_3 + x_4, -x_2, x_3.$$

Another question would be to investigate similar results to those in Chapter 4 when the leaf poset has this form. We do not have nice representation theory results for a poset of this form and it does not fall into the class that it is a relaxation of a rooted forest, so we can not apply any of the theorems from Chapter 4. However, we can see with the following example that Chapter 4 also does not classify all the leaf posets.

Example 5.0.4. *Let T_P be as in Figure 5.2.*

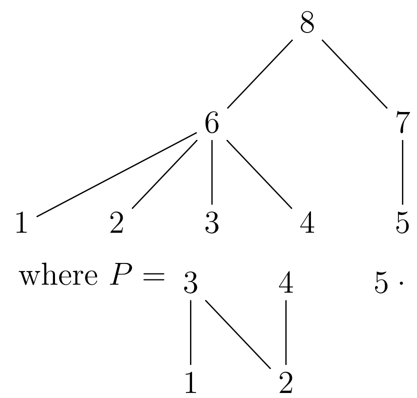


Figure 5.2: Tree T_P with $A_2 \oplus A_2 \setminus \{(1,4)\} + P_2$ as a leaf poset.

Then the eigenvalues of T_P are:

0,

x_{35},

$x_{35} + x_{45}$,

$-x_{25}$,

$x_{15} + x_{25} + x_{35} + x_{45}$,

x_5,

$x_3 + x_5 + x_{35}$,

$x_3 + x_4 + x_5 + x_{35} + x_{45}$,

$x_5 - x_2 - x_{25}$,

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{15} + x_{25} + x_{35} + x_{45}$.

which are all linear in the x_E's.

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