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Localization of Generalized Frames in Hilbert Spaces: Asymptotic Behavior of Concentration and Toeplitz Operators, Sampling and Interpolation, and Density Results

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Abstract

Suppose $\mathcal{H}$ is a separable and complex Hilbert space with a generalized frame (also known as continuous frame) $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ indexed over a metric measure space (X, d, λ). We study the main properties of generalized frames and the operators defined by them, such as concentration operators and Toeplitz operators.

Imposing certain localization conditions to the generalized frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$, we describe the asymptotic behavior of concentration and Toeplitz operators, and derive important results about the distribution of their eigenvalues. Furthermore, working with multiple generalized frames in $\mathcal{H}$ intertwined by a localization conditions, we obtain very general density results.

Many examples and applications are shown, among others we obtain necessary density conditions for sampling and interpolation, and these conditions can be applied on classical spaces, such as the Paley-Wiener space, the Bargmann-Fock space, and Gabor systems.

Dedication

In the memory of my father.

To my loving wife Marisol, my warm mother Elena, my courageous father Ernesto, and my brother Erick and his joyful troop: Fernando, Kevin, Carlos, Christopher, and Paula. They motivated me to always move forward and are the reasons of my will.

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Introduction

In Linear Algebra we know the importance of bases in understanding the structure and the properties of a given vector space. By Zorn's Lemma, every vector space possesses a basis, however the mere existence of a basis is in many cases not enough for the understanding of the given vector space.

If the vector space is finite dimensional, the understanding of its bases is deep and extensive. One of the great successes of Linear Algebra is arguably the strength of its results regarding bases. As an example of a very well-understood fundamental property, given a finite dimensional vector space V, there exists an orthonormal basis $\{e_1, e_2, \ldots, e_N\} \subseteq V$, where $N = \dim(V)$.

On the contrary, if V is an infinite dimensional vector space, the scenario becomes blurry. For instance, the concept of basis (only finite linear combinations are allowed) becomes impractical, and there is a need to introduce the more suitable concept of Schauder bases (it allows infinite linear combinations, i.e., *series expansions*). But this new type of bases comes with its own expenses, there are various options in the literature how to generalize the concepts of spanning sets and linearly independent sets in the infinite dimensional setup. One problem arises from the technical difficulties that a series expansion introduces, such as completeness, conditional/unconditional convergence, stability, and uniqueness, and we may need to choose between one option over other depending on which of these issues are more important to deal with or to avoid.

In this context, it is clear that in order to give satisfactory answers to some fundamental questions in an infinite dimensional vector space, such as a characterization for spanning sets or linearly independent sets, among others questions, we need to specify certain assumptions on the infinite dimensional vector space. In this dissertation, our main vector space is an *infinite dimensional complex Hilbert space* $\mathcal{H}$, and we impose the conditions on $\mathcal{H}$ to be *separable*, which is related with the completeness difficulty. So, $\mathcal{H}$ has a countable orthonormal Schauder basis, and the so called *frames* and *Riesz sequences* will play the role of spanning sets and linearly independent sets in $\mathcal{H}$, respectively (see [15]).

One convenient feature of frames is that the series expansions generated by a frame converge unconditionally (regardless of the order). However, it is important to say that frames are highly overcomplete and there is no hope frames will deal with the uniqueness issue, i.e., there might be multiple series expansions generated by the frame associated to the same element in $\mathcal{H}$. However, such overcompleteness of frames is in many instances an advantage, first because frames are more abundant and easier to obtain than bases, and second because the redundancy in the series expansions generated by a frame implies such series expansions are *robust*, in the sense that we can drop many terms from a series expansion and still get a good representation of the element in $\mathcal{H}$ (see [33]). As a comparison, bases in $\mathcal{H}$ are complete, the series expansions converge unconditionally, the representation of an element in $\mathcal{H}$ is unique, all great qualities of bases, but on the downside, bases are very sensitive to the loss or corruption of any of the coefficients in a given series expansion.

Many more things can be said about frames, their properties, and their applications, but the fundamental idea we want to point out is the advantage in the use of such overcomplete systems, besides the loss in uniqueness of series expansions, such systems behave like bases and even better in certain aspects. Going one step further, generalizing the concept of frames we consider a broader family of systems called *generalized frames*, also known in the literature as *continuous frames*, or *coherent states*. These systems are the main object of study in this dissertation.

The system $\{k_x\}_{x\in(X,\lambda)} \subseteq \mathcal{H}$ is called a generalized frame for $\mathcal{H}$ (see Section 5.8 of [16]) if there exist constants $\alpha, \beta \in \mathbb{R}^+$ such that for any $f \in \mathcal{H}$ it holds

$$\alpha \|f\|^2 \leq \int_X |\langle f, k_x\rangle|^2 \, d\lambda(x) \leq \beta \|f\|^2 .$$

Furthermore, a *framed Hilbert space* is a triple $(\mathcal{H}, X, k)$ such that $\mathcal{H}$ is a complex and separable Hilbert space, (X, d, λ) is a metric measure space, and $k : X \to \mathcal{H}$ is a continuous function generating a generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$, $k(x) = k_x$ for $x \in X$.

In Chapter 1 we study the main properties of generalized frames, and how these systems include all classical frames. We illustrate how to construct new generalized frames, and we give many examples where generalized frames have been found and used in the past. Many of these examples are extremely important in research both theoretically and in applications, e.g., generalized frames in reproducing kernel Hilbert spaces, generalized frames associated to unitary representations, and of course all classical frames. Although there are some results about generalized frames in different contexts (see [5, 6, 16, 26]), they are considerably less studied than the classical discrete frames. One of our objectives is to contribute to the establishment of a general theoretical framework for generalized frames, specially for those ones satisfying almost orthogonality conditions.

In Chapter 2 we study the *concentration operator* with respect to a compact set Ω and associated to a generalized frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$, its properties and asymptotic behavior. It is denoted by C_Ω, and defined in a weak sense by

$$C_\Omega f = \int_\Omega \langle f, k_x\rangle\, k_x d\lambda(x), \quad f \in \mathcal{H}.$$

In a broad sense, the concentration operator is a generalization of an orthogonal projection onto a closed subspace. Any orthogonal projection onto a closed subspace is compact, self-adjoint, positive, and has spectrum $\{0,1\}$. The concentration operator C_Ω satisfies similar properties, specifically, its spectrum $\sigma(C_\Omega)$ is highly concentrated around 0 and 1 under the assumption that the given generalized frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ satisfies the so called *localization property* or *quasi-orthogonality property*. Intuitively the localization property means $\langle k_x, k_y\rangle$ is close to 1 if x, y are close, and $\langle k_x, k_y\rangle$ is close to 0 if x, y are far apart. An important remark is that the localization property is equivalent to certain inequality between the Schatten $1,2-$norms of C_Ω (see Section 2.2), and the inequality version was first used by Landau in [38].

The asymptotic behavior of the concentration operator is best described with an example, consider the simplest version of a concentration operator: an orthogonal projection onto a (closed) subspace of a finite dimensional vector space V. Let $\{e_i\}_{i=1}^N \subseteq V$ be an orthonormal basis, for a ball $B = B(a;r) \subseteq \mathbb{R}$ define the closed subspace $W(B) = \operatorname{span}\{e_i : i \in B\} \subseteq V$, then the concentration operator with respect to the orthonormal basis $\{e_i\}_{i=1}^N \subseteq V$ associated to the ball B is the orthogonal projection $P_{W(B)} : V \to W(B)$ onto $W(B)$, which is given by

$$P_{W(B)} f = \sum_{i\in B} \langle f, e_i\rangle\, e_i, \quad f \in V.$$

Recall that

$$f = \sum_{i=1}^N \langle f, e_i\rangle\, e_i, \quad f \in V,$$

so, $P_{W(B)}$ only takes into account the terms from the series expansions of f with respect to $\{e_i\}_{i=1}^{N} \subseteq V$ that are concentrated around B.

It is an interesting question to ask which is the largest subspace $W' \subseteq V$ such that $\left\|P_{W(B)}f\right\| \approx \|f\|$ for $f \in W'$, because in this case $P_{W(B)}f$ gives a good representation of $f \in W'$. Clearly, if $f \in W(B)$, then $P_{W(B)}f = f$. In fact, it is not difficult to prove that for any given $0 < \varepsilon < 1$, the maximum dimension of a subspace $W' \subseteq V$ such that $\left\|P_{W(B)}f\right\| \geq (1-\varepsilon)\|f\|$ for all $f \in W'$ is exactly $\dim(W(B))$, because assuming $\dim(W') > \dim(W(B))$, the Rank-Nullity Theorem implies there exists a nonzero element $f_0 \in W' \cap W(B)^{\perp}$, and hence $\left\|P_{W(B)}f_0\right\| = 0$. So, denoting by $\eta_1(\varepsilon, B)$ the maximum dimension of a subspace $W' \subseteq V$ such that $\left\|P_{W(B)}f\right\|^2 \geq 1-\varepsilon$ for all $f \in W'$ with $\|f\| = 1$, the previous reasoning shows that $\eta_1(\varepsilon, B) = \dim(W(B))$. Thus, when $B = B(a;r)$ varies the asymptotic behavior studied in Chapter 2 is in this case $\eta_1(\varepsilon, B(a;r)) \sim \lambda(B(a;r))$, for any $a \in \mathbb{R}$, when $r \to \infty$, where λ denotes the counting measure on $\{1, 2, \ldots, N\}$.

These ideas can be generalized to the infinite dimensional case, where the use of $\eta_1(\varepsilon, B)$ becomes crucial since the relationship $\eta_1(\varepsilon, B) = \dim(W(B))$ is not true in general. For example, when working with overcomplete systems such as a generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$, it may be the case that for a given ball B we obtain $W(B) = \overline{\text{span}}\{k_x : x \in B\} = \mathcal{H}$, so $W(B)$ is too large.

One important question concerning generalized frames is whether a given generalized frame can be discretized to a classical frame. This existence problem, known as the *discretization problem* for generalized frames, was first considered in [26], and recently solved by Freeman and Speegle for bounded generalized frames [27]. Thus, any bounded generalized frame can be *sampled* to a classical frame. In Chapter 3 we give necessary conditions that the sampled sequence must satisfy to be a classical frame.

These results in Chapter 3 give a very general necessary conditions in terms of *Beurling densities* for *sampling* and *interpolation* in a given framed Hilbert space $(\mathcal{H}, X, k)$. Such results generalize the well-known necessary conditions for spanning sets and linearly independent sets in a finite dimensional vector space V: if S is a spanning set in V, and I is a linear independent set in V, then $\#(I) \leq \dim(V) \leq \#(S)$. In the more general setup, given a generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$, we want to know how sparse (resp. tight) a subset $\Gamma \subseteq X$ can be and still retain the spanning property (resp. the linearly independent property); more explicitly, the sequence $\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{H}$ must be a frame (resp. a Riesz sequence) in which case Γ is called sampling (resp. interpolating).

Results of this kind were first proved by Beurling [10] who provided necessary and sufficient conditions for sampling and interpolation in the $1-$dimensional Paley-Wiener space $\mathcal{PW}_\alpha(\mathbb{R})$ in terms of certain natural density conditions. These conditions are still used today as a main tool for these types of problems. Beurling left the multidimensional case of the Paley-Wiener space open, noticing that in this more general setting we can only hope for a necessary condition (the sufficiency condition being clearly false). Soon afterwards his problem was solved by Landau [38]. These results of Beurling and Landau were later extended to many different contexts. Most notably, it was proved by Seip [56, 58, 61] that similar necessary conditions can be obtained for normalized reproducing kernels in the classical $1-$dimensional Bargmann-Fock space $\mathcal{F}^2_\alpha(\mathbb{C})$ as well as in the $1-$dimensional Bergman space $\mathcal{B}^2_\alpha(\mathbb{D})$. A more recent work closely related with this kind of results, was done by Lindholm [41]. Our theorems in Chapter 3 provide a unified treatment of many results of this type, and prove some new results too.

In Chapter 4 we further generalize the concentration operator by considering instead the so called *Toeplitz operator*. Recall $(\mathcal{H}, X, k)$ is a framed Hilbert space. A Toeplitz operator with *symbol* $a(x) \in L^1(X, \lambda)$, denoted by T_a, is defined in a weak sense by

$$T_a f = \int_X a(x) \langle f, k_x \rangle k_x d\lambda(x), \quad f \in \mathcal{H}.$$

In particular, if the symbol is the characteristic function $a(x) = \chi_\Omega(x)$, then the Toeplitz operator T_a becomes the concentration operator C_Ω. In this chapter, we study some properties and two different asymptotic behavior of Toeplitz operators.

It is important to remark the connection between a Toeplitz operator and spectral theory. Given an operator $T : V \to V$, one of the main ideas of spectral theory is to find a basis for V so that the operator behaves nicely with respect to such basis, specifically T becomes a multiplication operator. This is not always possible even if V is a finite dimensional vector space. However, it is possible for some classes of operators, e.g., if T is self-adjoint (a symmetric matrix), there exists an orthonormal basis for V of eigenvectors of T. In the infinite dimensional case, even the self-adjoint condition is not enough, but if $T : \mathcal{H} \to \mathcal{H}$ is a compact operator, then there exists an orthonormal basis for $\mathcal{H}$ of eigenvectors of T. In some sense, a Toeplitz operator $T_a : \mathcal{H} \to \mathcal{H}$ still has this nice behavior, it is a self-adjoint operator (in general non-compact), and its definition resembles a spectral resolution in terms of the generalized frame $\{k_x\}_{x \in X} \subseteq \mathcal{H}$.

It is also important to remark that a Toeplitz operator T_a can have multiple representations with respect to different generalized frames, even if T_a is compact, however these generalized frames appear naturally (in this respect, the results form Chapter 5 can be useful in order to compare two different generalized frames). On the other hand, it is not clear whether or under which conditions a self-adjoint operator on $\mathcal{H}$ can be expressed as a Toeplitz operator.

Finally, in Chapter 5 we extend our results from Chapter 3, so now instead of considering a generalized frame and its discretized classical frame, we consider two generalized frames. The main result [46, Theorem 3.2] states that two bounded generalized frames $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ and $\{g_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{G}$ satisfy some density results whenever they are intertwined by a certain *localization condition* (condition (ii) in Theorem 5.4, or condition (L) in [46, Theorem 3.2]). In general, such assumption (L) is difficult to fulfill, but as long as the specific assumptions on a particular setup imply (L), the conclusions from the theorem are far reaching. For example, we can apply this theorem to obtain the results in [28, 53], to recover the sampling and interpolation results from Chapter 3, and all their applications.

Chapter 1

Preliminaries: generalized frames

Throughout this chapter, we assume $\mathcal{H}$ is a separable and complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

1.1 Frames and Riesz sequences

1.1.1 Complete sequences and bases

Given $\{a_i\}_{i=1}^{\infty} \subseteq \mathbb{C}$ and $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, we say the *infinite series* $\sum_{i=1}^{\infty} a_i f_i$ is *(conditionally) convergent* if there exists $f \in \mathcal{H}$ such that

$$\left\| f - \sum_{i=1}^{n} a_i f_i \right\| \to 0 \text{ as } n \to \infty.$$

Triangle inequality guarantees the uniqueness of such element f. In this case we say f has a *series expansion* with respect to $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ with coefficients $\{a_i\}_{i=1}^{\infty} \subseteq \mathbb{C}$, denoted by

$$f = \sum_{i=1}^{\infty} a_i f_i.$$

In this definition, the order of the terms in the infinite sum is crucial, it is possible that after rearranging terms the new infinite series is not convergent anymore. Imposing the stronger condition that the infinite series $\sum_{i=1}^{\infty} a_i f_i$ is convergent for all rearrangements of the sum, i.e., if there exists

$f \in \mathcal{H}$ such that

$$f = \sum_{i=1}^{\infty} a_{\sigma(i)} f_{\sigma(i)}$$

for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, we say the infinite series is *unconditionally convergent.* Furthermore, we say the infinite series $\sum_{i=1}^{\infty} a_i f_i$ is *absolutely convergent* if

$$\sum_{i=1}^{\infty} |a_i| \, \|f_i\| < \infty.$$

It is well-known that absolute convergence implies unconditional convergence.

Definition 1.1. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is said to be a (Schauder) *basis* for $\mathcal{H}$ if for each $f \in \mathcal{H}$, there exists a unique series expansion

$$f = \sum_{i=1}^{\infty} a_i (f) \, f_i, \quad a_i (f) \in \mathbb{C}.$$

If such convergence is unconditional for each $f \in \mathcal{H}$, the sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called an *unconditional basis.* Furthermore, if the elements of the basis form an orthonormal set, i.e., $\langle f_i, f_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta (which is always achievable via the Gram-Schmidt process), then $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ forms an *orthonormal basis.*

It is well-known that every separable Hilbert space has an orthonormal basis (which is complete), and an orthonormal basis is always an unconditional basis [16, Theorem 3.2.2 and Corollary 3.2.3].

Theorem 1.1. [16, Theorem 3.2.4] *If $\mathcal{H}$ is a separable complex Hilbert space, then there exists an orthonormal basis for $\mathcal{H}$, and such basis is complete and an unconditional basis.*

Remark. Since all the examples that we will encounter only deal with unconditional convergence, from now on we take the convention that an infinite series is *convergent* if it is unconditionally convergent, and a basis for $\mathcal{H}$ always refers to an unconditional basis.

One of the basic themes of this dissertation is to study different families of elements in $\mathcal{H}$ that generalize the properties of orthonormal bases.

1.1.2 Bessel sequences

Definition 1.2. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *Bessel sequence* if there exists a constant $\beta \in \mathbb{R}^+$ such that for any $f \in \mathcal{H}$ it holds

$$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \beta \|f\|^2 .$$

Proposition 1.2. [16, Corollary 3.1.5] *If $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence, then $\sum_{i=1}^{\infty} a_i f_i$ converges unconditionally for all $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$.*

In particular, an orthonormal basis for $\mathcal{H}$ is a Bessel sequence, and also Riesz sequences and frames (defined below) are examples of Bessel sequences. The same conclusion from Proposition 1.2 is true in all these cases.

Let $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ (which exists due to Theorem 1.1), and let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be a Bessel sequence. Consider the linear map

$$A : \quad \operatorname{span}\{e_i\}_{i=1}^{\infty} \quad \rightarrow \quad \operatorname{span}\{f_i\}_{i=1}^{\infty}$$

given by $Ae_i = f_i$.

Due to Proposition 1.2, this map can be extended to a linear map from $\overline{\operatorname{span}}\{e_i\}_{i=1}^{\infty} = \mathcal{H}$ to $\overline{\operatorname{span}}\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ as follows: given $f \in \mathcal{H}$, write $f = \sum_{i=1}^{\infty} a_i e_i$ where $\{a_i = \langle f, e_i \rangle\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$, so A is defined by

$$\begin{aligned} A : \mathcal{H} \quad &\rightarrow \quad \mathcal{H} \\ f \quad &\mapsto \quad \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i. \end{aligned}$$

Since $\{\langle f, f_i \rangle\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ for all $f \in \mathcal{H}$ (by assumption $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence), then the series shown below converges. The adjoint of A is given by

$$\begin{aligned} A^* : \mathcal{H} \quad &\rightarrow \quad \mathcal{H} \\ f \quad &\mapsto \quad \sum_{i=1}^{\infty} \langle f, f_i \rangle e_i. \end{aligned}$$

The following theorem gives a full-characterization of Bessel sequences in terms of A.

Theorem 1.3. [16, Theorem 3.1.3] *Let* $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *be an orthonormal basis for* $\mathcal{H}$. *The sequence* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *is a Bessel sequence if and only if the map*

$$\begin{aligned} A : \mathcal{H} &\rightarrow \mathcal{H} \\ f &\mapsto \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i \end{aligned}$$

is a bounded linear operator on $\mathcal{H}$.

1.1.3 Riesz sequences and Riesz basis

Definition 1.3. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *Riesz sequence*, if there constants $\alpha_1, \beta_1 \in \mathbb{R}^+$ such that for any $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ it holds

$$\alpha_1 \sum_{i=1}^{\infty} |a_i|^2 \leq \left\| \sum_{i=1}^{\infty} a_i f_i \right\|^2 \leq \beta_1 \sum_{i=1}^{\infty} |a_i|^2.$$

If a Riesz sequence is complete, i.e., $\overline{\text{span}}\{f_i\}_{i=1}^{\infty} = \mathcal{H}$, we say $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *Riesz basis*.

In general Riesz sequences are not complete, however, restricting our attention to the closed subspace $\mathcal{F} := \overline{\text{span}}\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, a Riesz sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ becomes a Riesz basis for the Hilbert space $\mathcal{F}$. On the other hand, Riesz sequences are $\ell_2(\mathbb{N})$-*independent*, which means that given $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ such that $\sum_{i=1}^{\infty} a_i f_i = 0$, it implies $a_i = 0$ for all $i \in \mathbb{N}$.

Proposition 1.4. *If* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *is a Riesz sequence, then it is a* $\ell_2(\mathbb{N})$-*independent Bessel sequence.*

Proof. Clearly the map

$$\begin{aligned} B : \ell_2(\mathbb{N}) &\rightarrow \mathcal{H} \\ \{a_i\}_{i=1}^{\infty} &\mapsto \sum_{i=1}^{\infty} a_i f_i \end{aligned}$$

is linear. Furthermore, since $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence, $\|B\| \leq \sqrt{\beta_1}$ and so B is a bounded linear map. Applying Lemma 3.1.1 in [16] we conclude $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence.

Now suppose $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ such that $\sum_{i=1}^{\infty} a_i f_i = 0$, then

$$\alpha_1 \sum_{i=1}^{\infty} |a_i|^2 \leq \left\| \sum_{i=1}^{\infty} a_i f_i \right\|^2 = 0,$$

and hence $a_i = 0$ for all $i \in \mathbb{N}$. Therefore $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a $\ell_2(\mathbb{N})$-independent Bessel sequence. $\square$

We can give a full-characterization of Riesz sequences in terms of the operator A described in Theorem 1.3.

Theorem 1.5. *Let $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence if and only if the map*

$$\begin{aligned} A : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i \end{aligned}$$

is an injective bounded linear operator on $\mathcal{H}$ with closed range.

Remark. A bounded linear operator A is injective and has closed range if and only if its adjoint A^* is surjective.

Proof. ($\Rightarrow$) Suppose $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence. Due to Proposition 1.4, $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence, hence A is a bounded linear operator on $\mathcal{H}$ due to Theorem 1.3. Using the lower inequality of the Riesz sequence definition, for any $f \in \mathcal{H}$ it holds

$$\alpha_1 \|f\|^2 = \alpha_1 \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2 \leq \left\| \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i \right\|^2 = \|Af\|^2,$$

thus $\|A\| \geq \sqrt{\alpha_1} > 0$, i.e, A is bounded from below. It is well-known that a bounded linear operator is bounded from below if and only if it is injective and it has closed range.

($\Leftarrow$) Suppose A is an injective bounded linear operator on $\mathcal{H}$ with closed range. Then, A is a bounded linear operator which is also bounded from below, i.e., $0 < \sqrt{\alpha_1} \leq \|A\| \leq \sqrt{\beta_1}$ for some positive constants α_1 and β_1. Let $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$, and consider $f = \sum_{i=1}^{\infty} a_i e_i$. Since $f \in \mathcal{H}$, we obtain $\alpha_1 \|f\|^2 \leq \|Af\|^2 \leq \beta_1 \|f\|^2$, hence

$$\alpha_1 \sum_{i=1}^{\infty} |a_i|^2 \leq \left\| \sum_{i=1}^{\infty} a_i f_i \right\|^2 \leq \beta_1 \sum_{i=1}^{\infty} |a_i|^2 .$$

Therefore $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence. □

Theorem 1.6. [16, Theorem 3.3.2 and Corollary 3.3.4] *If $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a Riesz basis for $\mathcal{F} = \overline{\operatorname{span}} \{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, then there exists a unique sequence $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that for all $f \in \mathcal{F}$ it holds*

$$f = \sum_{i=1}^{\infty} \left\langle f, \widetilde{f}_i \right\rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle \widetilde{f}_i.$$

Furthermore, $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a Riesz basis for $\overline{\operatorname{span}} \left\{\widetilde{f}_i\right\}_{i=1}^{\infty} = \overline{\operatorname{span}} \{f_i\}_{i=1}^{\infty} = \mathcal{F}$, and $\{f_i\}_{i=1}^{\infty}$ and $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty}$ are biorthogonal, which means $\left\langle f_i, \widetilde{f}_j \right\rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta.

In this theorem, $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called the *dual Riesz sequence* associated to the Riesz sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, and vice-versa. As a trivial example, an orthonormal basis for $\mathcal{H}$ is a Riesz basis, and it is its own dual Riesz basis.

1.1.4 Frames

Definition 1.4. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame*, if there exists constants $\alpha, \beta \in \mathbb{R}^+$ such that for any $f \in \mathcal{H}$ it holds

$$\alpha\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \beta\|f\|^2.$$

Particularly, if $\alpha = \beta$ the sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *tight frame*, and in the case $\alpha = \beta = 1$, the sequence is called a *Parseval frame*.

We will often use the term *discrete frame* or *classical frame* when referring to a frame. Later we will introduce a more general concept of frames called *generalized frames* or *continuous frames*, which will include the classical frames as particular examples.

In contrast with Riesz sequences, frames are always complete as it is proved in the next proposition, however frames generally are not $\ell_2(\mathbb{N})-$independent. Intuitively, as elements of a vector space, frames are related to spanning sets, meanwhile Riesz sequences are related to linearly independent sets.

Proposition 1.7. *If* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *is a frame, then it is a complete Bessel sequence.*

Proof. Obviously $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence. Suppose $f \in \mathcal{H}$ is such that $\langle f, f_i \rangle = 0$ for all $i \in \mathbb{N}$. By the frame inequalities, $\|f\| = 0$, so $f = 0$. This guarantees $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is complete [16, Lemma 2.3.1]. □

Proposition 1.8. [16, Proposition 3.3.5] *If* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *is a Riesz basis, then it is a frame. In particular, an orthonormal basis for* $\mathcal{H}$ *is a Riesz basis and a frame.*

Similar as in Theorems 1.3 and 1.5, we can give a full-characterization of frames in terms of the same operator A. One interesting observation from Theorems 1.9 and 1.5 is that Riesz sequences and frames satisfy dual properties with respect to A.

Theorem 1.9. *Let* $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *be an orthonormal basis for* $\mathcal{H}$*. The sequence* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *is a frame if and only if the map*

$$\begin{aligned} A : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i \end{aligned}$$

is a surjective bounded linear operator on $\mathcal{H}$*.*

Remark. A bounded linear operator A is surjective if and only if its adjoint A^* is injective and has closed range. Therefore, this theorem combined with Theorem 1.5 says that frames and Riesz sequences are in some sense dual to each other.

Proof. (⇒) Suppose $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame. Due to Proposition 1.7, it is a Bessel sequence, so A is a bounded linear operator on $\mathcal{H}$ due to Theorem 1.3. Then A^*, the adjoint of A, is also a bounded linear operator on $\mathcal{H}$. Let $f \in \mathcal{H}$, recall $A^* f = \sum_{i=1}^{\infty} \langle f, f_i \rangle e_i$, then the lower inequality in the frame definition gives

$$\alpha \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 = \|A^* f\|^2 .$$

Hence $\|A^*\| \geq \sqrt{\alpha} > 0$, i.e., A^* is bounded from below. This implies A^* is injective and has closed range, therefore A is surjective.

(⇐) Suppose A is a surjective bounded linear operator on $\mathcal{H}$. Since A is a bounded linear operator, Theorem 1.3 implies $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Bessel sequence, so the upper inequality in the frame definition is satisfied. Furthermore, since A is surjective, its adjoint A^* is an injective bounded linear operator on $\mathcal{H}$ and it has closed range, hence A^* is bounded from below, so the lower inequality in the frame definition is satisfied. Therefore $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame. □

In particular, we can give a full-characterization for orthonormal bases in $\mathcal{H}$.

Theorem 1.10. [16, Theorem 3.2.7] *Let $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$ if and only if the map*

$$\begin{aligned} A : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i \end{aligned}$$

is a unitary bounded linear operator on $\mathcal{H}$.

Definition 1.5. Given a frame $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, the *analysis operator* associated to $\{f_i\}_{i=1}^{\infty}$ is the map

$$\begin{aligned} T : \mathcal{H} &\to \ell_2(\mathbb{N}) \\ f &\mapsto \{\langle f, f_i \rangle\}_{i=1}^{\infty}, \end{aligned}$$

the *synthesis operator* associated to $\{f_i\}_{i=1}^{\infty}$ is the map

$$\begin{aligned} T^* : \ell_2(\mathbb{N}) &\to \mathcal{H} \\ \{a_i\}_{i=1}^{\infty} &\mapsto \sum_{i=1}^{\infty} a_i f_i, \end{aligned}$$

and the *frame operator* associated to $\{f_i\}_{i=1}^{\infty}$ is the map $S = T^*T$

$$\begin{aligned} S : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i. \end{aligned}$$

The analysis operator T is a well-defined bounded linear operator because $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame, the synthesis operator T^* is a well-defined bounded linear operator, too [16, Theorem 3.1.3

and Corollary 3.1.5]. As the notation suggests, the synthesis operator is the adjoint operator of the analysis operator [16, Lemma 3.1.1].

Clearly the frame operator S is a well-defined bounded linear operator, and Lemma 5.1.6 in [16] shows that S is invertible, self-adjoint, and positive. Furthermore, Corollary 5.1.8 in [16] shows that $S = I$, the identity map on $\mathcal{H}$, whenever $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Parseval frame.

Definition 1.6. Let S be the frame operator associated to the frame $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, and let $\widetilde{f}_i = S^{-1} f_i$ for all $i \in \mathbb{N}$. The sequence $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame [16, Lemma 5.1.6], and it is called the *dual frame* associated to $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$.

Theorem 1.11. [16, Theorem 5.1.7] *Given a frame $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and its dual frame $\left\{\widetilde{f}_i\right\}_{i=1}^{\infty} \subseteq \mathcal{H}$, for any $f \in \mathcal{H}$ the following reconstruction formulas hold*

$$f = \sum_{i=1}^{\infty} \left\langle f, \widetilde{f}_i \right\rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle \widetilde{f}_i.$$

1.2 Generalized frames

In this section we introduce the concept of a *generalized frame*. The name *continuous frame* is also used in the literature. Unlike classical frames that always need to be sequences (discrete), the elements of a generalized frame are indexed by a measure space (X, λ).

Before defining this concept rigorously, we need some preliminary results. Recall $\mathcal{H}$ is a complex and separable Hilbert space, and assume (X, λ) is a measure space.

1.2.1 Weakly integrable functions

Definition 1.7. A function $z : X \to \mathcal{H}$ is called *weakly integrable* if for any $f \in \mathcal{H}$ the function

$$\begin{aligned} z_f : X &\to \mathbb{C} \\ x &\mapsto \langle z(x), f \rangle \end{aligned}$$

is integrable, i.e., $z_f \in L^1(X, \lambda)$ for all $f \in \mathcal{H}$.

Proposition 1.12. *For any fixed weakly integrable function $z : X \to \mathcal{H}$, the functional*

$$\begin{aligned} l_z : \mathcal{H} &\to \mathbb{C} \\ f &\mapsto \int_X z_f(x) d\lambda(x) = \int_X \langle z(x), f \rangle \, d\lambda(x) \end{aligned}$$

is a conjugate-linear functional. Moreover, if l_z is bounded, i.e., there exists a constant $c > 0$ such that $|l_z(f)| \leq c\|f\|$ for all $f \in \mathcal{H}$, then there exists a unique $g_z \in \mathcal{H}$ such that for all $f \in \mathcal{H}$

$$\begin{aligned} \int_X \langle z(x), f \rangle \, d\lambda(x) &= l_z(f) = \langle g_z, f \rangle \, , \\ \int_X \langle f, z(x) \rangle \, d\lambda(x) &= \overline{l_z(f)} = \langle f, g_z \rangle \, . \end{aligned}$$

Proof. First we will show l_z is conjugate-linear. By conjugate-linearity on the second component of the inner product on $\mathcal{H}$, and linearity of the integral, we obtain that for any $f_1, f_2 \in \mathcal{H}$ and any $\theta \in \mathbb{C}$

$$\begin{aligned} l_z(f_1 + \theta f_2) &= \int_X z_{f_1 + \theta f_2}(x) d\lambda(x) \\ &= \int_X \langle z(x), f_1 + \theta f_2 \rangle \, d\lambda(x) \\ &= \int_X \langle z(x), f_1 \rangle \, d\lambda(x) + \overline{\theta} \int_X \langle z(x), f_2 \rangle \, d\lambda(x) \\ &= l_z(f_1) + \overline{\theta} l_z(f_2). \end{aligned}$$

Second, under the assumption that l_z is bounded, by the Riesz's representation theorem applied to the bounded conjugate-linear functional l_z on the Hilbert space $\mathcal{H}$, there exists a unique $g_z \in \mathcal{H}$ such that $l_z(f) = \langle g_z, f \rangle$ for all $f \in \mathcal{H}$, and by taking the conjugate of $l_z(f)$

$$\begin{aligned} \int_X \langle f, z(x) \rangle \, d\lambda(x) &= \int_X \overline{\langle z(x), f \rangle} d\lambda(x) \\ &= \overline{\int_X \langle z(x), f \rangle \, d\lambda(x)} \\ &= \overline{l_z(f)} \\ &= \langle f, g_z \rangle \, . \end{aligned}$$

This completes the proof. □

Using this proposition, given a weakly integrable function z such that l_z is bounded, we can define its (vector valued) integral in the weak sense as follows.

Definition 1.8. Given a weakly integrable function $z : X \to \mathcal{H}$ such that $l_z : \mathcal{H} \to \mathbb{C}$ is bounded as described in the proposition 1.12, the *integral in weak sense* associated to z is the element $g_z \in \mathcal{H}$ related to l_z, given by the Riesz's representation theorem. Formally denoting

$$g_z = \int_X z(x) d\lambda(x)$$

the following relationships hold for any $f \in \mathcal{H}$

$$\begin{aligned}\left\langle \int_X z(x) d\lambda(x), f \right\rangle &= \int_X \langle z(x), f \rangle \, d\lambda(x), \\ \left\langle f, \int_X z(x) d\lambda(x) \right\rangle &= \int_X \langle f, z(x) \rangle \, d\lambda(x).\end{aligned}$$

1.2.2 Definition of generalized frames

Definition 1.9. A *generalized frame* (also known as *continuous frame*) is a function

$$\begin{aligned} k : X &\to \mathcal{H} \\ x &\mapsto k(x) \end{aligned}$$

such that there exist constants $\alpha, \beta \in \mathbb{R}^+$ satisfying

$$\alpha \|f\|^2 \leq \int_X |\langle f, k(x) \rangle|^2 \, d\lambda(x) \leq \beta \|f\|^2, \quad f \in \mathcal{H}.$$

Remark. If X is also a topological space (e.g., a metric space (X, d)) we will impose the additional condition on $k : X \to \mathcal{H}$ to be continuous.

To resemble the usual definition of a (discrete) frame, we will adopt the notation $k_x = k(x)$, $x \in X$, which gives a collection of elements in $\mathcal{H}$ generated by k; furthermore, this collection is complete for $\mathcal{H}$, i.e., $\overline{\operatorname{span}} \{k_x : x \in X\} = \mathcal{H}$.

Below we give an alternative definition of a generalized frame for $\mathcal{H}$, or more general, for a closed subspace $\mathcal{F} \subseteq \mathcal{H}$.

Definition 1.10. The collection $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is called a *generalized frame* for $\mathcal{H}$ if there exist constants $\alpha, \beta \in \mathbb{R}^+$ such that

$$\alpha \|f\|^2 \leq \int_X |\langle f, k_x\rangle|^2 \, d\lambda(x) \leq \beta \|f\|^2$$

for all $f \in \mathcal{H}$, in which case $\overline{\operatorname{span}}\{k_x : x \in X\} = \mathcal{H}$. Furthermore, given a closed subspace $\mathcal{F} \subseteq \mathcal{H}$, the collection $\{k_x\}_{x\in X} \subseteq \mathcal{F}$ is called a *generalized frame* for $\mathcal{F}$ if there exist constants $\alpha, \beta \in \mathbb{R}^+$ such that

$$\alpha \|f\|^2 \leq \int_X |\langle f, k_x\rangle|^2 \, d\lambda(x) \leq \beta \|f\|^2$$

for all $f \in \mathcal{F}$, in which case $\overline{\operatorname{span}}\{k_x : x \in X\} = \mathcal{F}$.

There are many generalized frames for any given separable Hilbert space $\mathcal{H}$, e.g., any orthonormal basis in $\mathcal{H}$ is a generalized frame for $\mathcal{H}$, where the index set $X = \mathbb{Z}$ is a measure space with respect to the counting measure. However, in many instances a Hilbert space $\mathcal{H}$ has one distinguished generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ attached to $\mathcal{H}$ as an integral part of its definition (e.g., reproducing kernel Hilbert spaces), we call them *framed Hilbert spaces.*

Definition 1.11. A *framed Hilbert space* is a triple $(\mathcal{H}, X, k)$ such that

1. $\mathcal{H}$ is a separable (complex) Hilbert space.
2. (X, λ) is a measure space.
3. $k : X \to \mathcal{H}$ is a function (assumed continuous if X is a topological space) generating a generalized frame $\{k_x\}_{x\in(X,\lambda)} \subseteq \mathcal{H}$.

From now on, we assume $\mathcal{H}$ is a framed Hilbert space, and $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ denotes the special generalized frame attached to $\mathcal{H}$.

Proposition 1.13. *Given $a \in L^2(X, \lambda)$ fixed, the functional*

$$\begin{aligned} l_a : \mathcal{H} &\to \mathbb{C} \\ f &\mapsto \int_X a(x)\langle f, k_x\rangle d\lambda(x) \end{aligned}$$

is linear and bounded. Moreover, there exists a unique $g_a \in \mathcal{H}$ such that

$$l_a(f) = \langle f, g_a \rangle$$

for all $f \in \mathcal{H}$.

Proof. First we will check l_a is linear. For any $f_1, f_2 \in \mathcal{H}$ and $\theta \in \mathbb{C}$, using linear properties of the inner product on $\mathcal{H}$ and integration, we get

$$\begin{aligned} l_a(f_1 + \theta f_2) &= \int_X a(x) \langle f_1 + \theta f_2, k_x \rangle \, d\lambda(x) \\ &= \int_X a(x) \left(\langle f_1, k_x \rangle + \theta \langle f_2, k_x \rangle \right) d\lambda(x) \\ &= \int_X a(x) \langle f_1, k_x \rangle d\lambda(x) + \theta \int_X a(x) \langle f_2, k_x \rangle d\lambda(x) \\ &= l_a(f_1) + \theta l_a(f_2). \end{aligned}$$

Second, we will check l_a is bounded. For any $f \in \mathcal{H}$, notice $\langle f, k_x \rangle \in L^2(X, \lambda)$ because $\{k_x\}_{x \in X}$ is a generalized frame, hence $a(x)\langle f, k_x \rangle \in L^2(X, \lambda)$ since it is assumed $a \in L^2(X, \lambda)$. Recall λ is a (positive) measure. By a classical inequality of integration and Cauchy-Schwarz inequality in $L^2(X, \lambda)$ we get

$$\begin{aligned} |l_a(f)| &= \left| \int_X a(x) \langle f, k_x \rangle d\lambda(x) \right| \\ &\leq \int_X |a(x)| |\langle f, k_x \rangle| d\lambda(x) \\ &= \langle |a(x)|, |\langle f, k_x \rangle| \rangle_{L^2} \\ &\leq \left(\int_X |a(x)|^2 d\lambda(x) \right)^{\frac{1}{2}} \left(\int_X |\langle f, k_x \rangle|^2 d\lambda(x) \right)^{\frac{1}{2}} \\ &\leq \|a\|_2 \left(\beta \|f\|^2 \right)^{\frac{1}{2}} \\ &= \left(\|a\|_2 \beta^{\frac{1}{2}} \right) \|f\|. \end{aligned}$$

This proves l_a is bounded and gives an estimate for its operator norm:

$$\|l_a\| \leq \|a\|_2 \beta^{\frac{1}{2}}.$$

Finally, by Riesz's representation theorem on $\mathcal{H}$, there exists some $g_a \in \mathcal{H}$ such that

$$l_a(f) = \langle f, g_a \rangle$$

for all $f \in \mathcal{H}$ $\square$

Notice that $l_a(f) = \int_X \left\langle f, \overline{a(x)} k_x \right\rangle d\lambda(x)$, where $\left\langle f, \overline{a(x)} k_x \right\rangle \in L^1(X, \lambda)$ for all $f \in \mathcal{H}$ due to the proof of the boundedness of l_a, so, $z_a(x) = \overline{a(x)} k_x$ is weakly integrable according to definition 1.7. This allows us to define a generalization of an infinite linear combination in the following way.

Definition 1.12. For any $a \in L^2(X, \lambda)$, the *integral in the weak sense* determined by a is

$$g_a = \int_X \overline{a(x)} k_x d\lambda(x)$$

where $g_a \in \mathcal{H}$ is given by the Riesz's representation theorem as described in proposition 1.13, and for any $f \in \mathcal{H}$ it holds

$$\begin{aligned} \left\langle f, \int_X \overline{a(x)} k_x d\lambda(x) \right\rangle &= \int_X a(x) \langle f, k_x \rangle d\lambda(x), \\ \left\langle \int_X \overline{a(x)} k_x d\lambda(x), f \right\rangle &= \int_X \overline{a(x)} \langle k_x, f \rangle d\lambda(x). \end{aligned}$$

1.2.3 The frame operator

Definition 1.13. The map

$$\begin{aligned} T : \mathcal{H} &\to L^2(X, \lambda) \\ f &\mapsto \langle f, k_x \rangle \end{aligned}$$

is called the *analysis operator* associated to the generalized frame $\{k_x\}_{x \in X} \subseteq \mathcal{H}$.

Notice that $Tf \in L^2(X, \lambda)$ for all $f \in \mathcal{H}$ because $\{k_x\}_{x \in X}$ is a generalized frame. The proof of this claim is as follows: for any $f \in \mathcal{H}$

$$\begin{aligned}\|Tf\|_2^2 &= \int_X |Tf(x)|^2 d\lambda(x) \\ &= \int_X |\langle f, k_x\rangle|^2 d\lambda(x) \\ &\leq \beta\|f\|^2 \\ &< \infty\end{aligned}$$

Proposition 1.14. *The analysis operator T is linear and bounded with $\|T\| \leq \beta^{\frac{1}{2}}$.*

Proof. First we will prove linearity: for any $f_1, f_2 \in \mathcal{H}$ and any $\theta \in \mathbb{C}$

$$\begin{aligned}T(f_1 + \theta f_2) &= \langle f_1 + \theta f_2, k_x\rangle \\ &= \langle f_1, k_x\rangle + \theta\langle f_2, k_x\rangle \\ &= T(f_1) + \theta T(f_2).\end{aligned}$$

The boundedness of T is given by the result proved above $\|Tf\|_2^2 \leq \beta\|f\|^2$ which implies

$$\|T\| \leq \beta^{\frac{1}{2}}.$$

□

Definition 1.14. The map

$$\begin{aligned}T^* : L^2(X, \lambda) &\to \mathcal{H} \\ a &\mapsto \int_X a(x) k_x d\lambda(x)\end{aligned}$$

is called the *synthesis operator* associated with the generalized frame $\{k_x\}_{x \in X} \subseteq \mathcal{H}$.

Remark. We have used the symbol T^* to denote the synthesis operator because, as we will prove later, it is the adjoint of the analysis operator T.

Remark. Using the notation of definition 1.12, the integral defining T^*a is an element of $\mathcal{H}$ given by the Riesz's representation theorem:

$$g_{\overline{a}} = \int_X a(x) k_x d\lambda(x).$$

Proposition 1.15. *The synthesis operator T^* is the adjoint of the analysis operator T.*

Proof. Let $a \in L^2(X, \lambda)$ and $f \in \mathcal{H}$. Then

$$\begin{aligned}
\langle a, Tf \rangle_{L^2} &= \overline{\langle Tf, a \rangle}_{L^2} \\
&= \overline{\int_X Tf(x)\overline{a(x)} d\lambda(x)} \\
&= \overline{\int_X \overline{a(x)} \langle f, k_x \rangle d\lambda(x)} \\
&= \overline{\int_X \langle f, a(x) k_x \rangle d\lambda(x)} \\
&= \overline{\left\langle f, \int_X a(x) k_x d\lambda(x) \right\rangle} \\
&= \langle T^* a, f \rangle.
\end{aligned}$$

This completes the proof. □

Proposition 1.16. *The synthesis operator T^* is linear and bounded with operator norm $\|T^*\| \leq \beta^{\frac{1}{2}}$.*

Proof. By the previous result, T^* is the adjoint of T which is linear and bounded, then T^* is also linear and bounded, moreover $\|T^*\| = \|T\| \leq \beta^{\frac{1}{2}}$ by proposition 1.14. □

Definition 1.15. The map $S := T^*T$ is called the *frame operator* associated to $\{k_x\}_{x \in X} \subseteq \mathcal{H}$, where

$$\begin{aligned}
S : \mathcal{H} &\to \mathcal{H} \\
f &\mapsto \int_X \langle f, k_x \rangle k_x d\lambda(x).
\end{aligned}$$

Theorem 1.17. *The frame operator S is an invertible self-adjoint bounded linear map. The inverse of the frame operator, S^{-1}, is also an invertible self-adjoint bounded linear map. Moreover, the inequalities $0 < \alpha \leq \|S\| \leq \beta$ and $0 < \frac{1}{\beta} \leq \|S^{-1}\| \leq \frac{1}{\alpha}$ hold.*

Proof. S is self-adjoint by definition because $S^* = (T^*T)^* = T^*T^{**} = T^*T = S$. Linearity and boundedness come from propositions 1.14 and 1.16: S is the composition of linear operators T and

T^*, so S is linear, and using a classical operator norm inequality we can estimate the operator norm of S as follows:

$$\begin{aligned} \|S\| &= \|T^*T\| \\ &\leq \|T^*\|\|T\| \\ &\leq \beta^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ &= \beta. \end{aligned}$$

This implies $\|Sf\| \leq \beta\|f\|$ for all $f \in \mathcal{H}$. We will prove now that S is bounded from below. For this, we will use the Cauchy-Schwarz inequality, the definition 1.12 on $\overline{\langle f, k_x \rangle} \in L^2(X, \lambda)$ and the lower bound of the generalized frame $\{k_x\}_{x\in X}$: for any $f \in \mathcal{H}$

$$\begin{aligned} \|Sf\|\|f\| &\geq |\langle f, Sf\rangle| \\ &= \left|\left\langle f, \int_X \langle f, k_x\rangle k_x d\lambda(x)\right\rangle\right| \\ &= \left|\int_X \langle f, \langle f, k_x\rangle k_x\rangle \, d\lambda(x)\right| \\ &= \left|\int_X \overline{\langle f, k_x\rangle} \, \langle f, k_x\rangle \, d\lambda(x)\right| \\ &= \left|\int_X |\langle f, k_x\rangle|^2 \, d\lambda(x)\right| \\ &= \int_X |\langle f, k_x\rangle|^2 \, d\lambda(x) \\ &\geq \alpha\|f\|^2. \end{aligned}$$

From here we can conclude $\|Sf\| \geq \alpha\|f\|$ whenever $\|f\| \neq 0$, but the same inequality trivially holds true when $\|f\| = 0$. Thus, $\|Sf\| \geq \alpha\|f\|$ for all $f \in \mathcal{H}$, which also implies $\|S\| \geq \alpha > 0$. Now we will apply some results from Functional Analysis: since S is bounded from below, then S is injective and has closed range, which implies S^* is surjective; but we have already proved that $S = S^*$, thus, S is surjective. Therefore, S is a bijection and hence invertible.

Since S is linear, then S^{-1} is linear. Also, S invertible and self-adjoint gives S^{-1} is self-adjoint too, because $\left(S^{-1}\right)^* = (S^*)^{-1} = S^{-1}$. Finally, we can estimate the operator norm of S^{-1}

as follows: for any $f \in \mathcal{H}$, let $g = S^{-1}f$, then, by the previous reasoning,

$$\frac{1}{\beta}\|Sg\| \leq \|g\| \leq \frac{1}{\alpha}\|Sg\|,$$

which is equivalent to

$$\frac{1}{\beta}\|f\| \leq \|S^{-1}f\| \leq \frac{1}{\alpha}\|f\|.$$

This implies $0 < \frac{1}{\beta} \leq \|S^{-1}\| \leq \frac{1}{\alpha}$. □

1.2.4 Reconstruction formulas

Definition 1.16. The *generalized dual frame* associated to the generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is the collection $\left\{\widetilde{k_x}\right\}_{x\in X} \subseteq \mathcal{H}$, defined by $\widetilde{k_x} = S^{-1}k_x$ for all $x \in X$, where S is the frame operator associated to $\{k_x\}_{x\in X}$.

Remark. As the name suggests, the generalized dual frame $\left\{\widetilde{k_x}\right\}_{x\in X}$ is a generalized frame indeed, according with definition 1.9, but we need to check it.

Proposition 1.18. *The generalized dual frame* $\left\{\widetilde{k_x}\right\}_{x\in X} \subseteq \mathcal{H}$ *is a generalized frame.*

Proof. Due to theorem 1.17, S^{-1} is self-adjoint, then

$$\begin{aligned}\int_X \left|\left\langle f, \widetilde{k_x}\right\rangle\right|^2 d\lambda(x) &= \int_X \left|\left\langle f, S^{-1}k_x\right\rangle\right|^2 d\lambda(x)\\ &= \int_X \left|\left\langle \left(S^{-1}\right)^* f, k_x\right\rangle\right|^2 d\lambda(x)\\ &= \int_X \left|\left\langle S^{-1}f, k_x\right\rangle\right|^2 d\lambda(x).\end{aligned}$$

But $\{k_x\}_{x\in X}$ is a generalized frame, so

$$\alpha\|S^{-1}f\|^2 \leq \int_X \left|\left\langle S^{-1}f, k_x\right\rangle\right|^2 d\lambda(x) \leq \beta\|S^{-1}f\|^2.$$

Using the estimates on the operator norm of S^{-1} in theorem 1.17, we can conclude

$$\frac{\alpha}{\beta^2}\|f\|^2 \leq \int_X \left|\left\langle f, \widetilde{k_x}\right\rangle\right|^2 d\lambda(x) \leq \frac{\beta}{\alpha^2}\|f\|^2.$$

This completes the proof. □

Theorem 1.19. *Let $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ be a generalized frame for $\mathcal{H}$, and let $\left\{\widetilde{k_x}\right\}_{x\in X} \subseteq \mathcal{H}$ be its dual generalized frame (which is a generalized frame for $\mathcal{H}$, too). Then, for any $f \in \mathcal{H}$, the following reconstruction formulas hold:*

$$f = \int_X \left\langle f, \widetilde{k_x} \right\rangle k_x d\lambda(x) = \int_X \langle f, k_x \rangle \widetilde{k_x} d\lambda(x).$$

Proof. By theorem 1.17, the operator S^{-1} is self-adjoint, so, using the definition 1.15 of the frame operator S we get

$$\begin{aligned} f &= S\left(S^{-1}f\right) \\ &= \int_X \left\langle S^{-1}f, k_x \right\rangle k_x d\lambda(x) \\ &= \int_X \left\langle f, S^{-1}k_x \right\rangle k_x d\lambda(x) \\ &= \int_X \left\langle f, \widetilde{k_x} \right\rangle k_x d\lambda(x). \end{aligned}$$

For the second reconstruction formula we need to express S^{-1} explicitly. Consider the map

$$\begin{aligned} \hat{S} : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \int_X \left\langle f, \widetilde{k_x} \right\rangle \widetilde{k_x} d\lambda(x). \end{aligned}$$

Remark. Recall $\left\{\widetilde{k_x}\right\}_{x\in X}$ is a generalized frame, so $\left\langle f, \widetilde{k_x} \right\rangle \in L^2(X, \lambda)$ because of the range of the analysis operator associated to the generalized dual frame. Then, by definition 1.12 we have $\hat{S}f \in \mathcal{H}$ for all $f \in \mathcal{H}$ indeed.

Claim. $S^{-1} = \hat{S}$.

Since S is a bijection, it is enough to check if $\hat{S}$ is right inverse of S. For any $f \in \mathcal{H}$

$$\begin{aligned} S\left(\hat{S}f\right) &= \int_X \left\langle \hat{S}f, k_x \right\rangle k_x d\lambda(x) \\ &= \int_X \left\langle \int_X \left\langle f, \widetilde{k_y} \right\rangle \widetilde{k_y} d\lambda(y), k_x \right\rangle k_x d\lambda(x) \\ &= \int_X \left(\int_X \left\langle \left\langle f, \widetilde{k_y} \right\rangle \widetilde{k_y}, k_x \right\rangle d\lambda(y) \right) k_x d\lambda(x) \\ &= \int_X \left(\int_X \left\langle f, \widetilde{k_y} \right\rangle \left\langle \widetilde{k_y}, k_x \right\rangle d\lambda(y) \right) k_x d\lambda(x). \end{aligned}$$

Notice $\left\langle \widetilde{k_y}, k_x \right\rangle = \left\langle S^{-1}k_y, k_x \right\rangle = \left\langle k_y, S^{-1}k_x \right\rangle = \left\langle k_y, \widetilde{k_x} \right\rangle$ since S^{-1} is self-adjoint. Using this observation and the first reconstruction formula, the parenthesis inside of the last calculation of $S\left(\hat{S}f\right)$ can be simplified as follows

$$\begin{aligned}
\int_X \left\langle f, \widetilde{k_y} \right\rangle \left\langle \widetilde{k_y}, k_x \right\rangle d\lambda(y) &= \int_X \left\langle f, \widetilde{k_y} \right\rangle \left\langle k_y, \widetilde{k_x} \right\rangle d\lambda(y) \\
&= \int_X \left\langle \left\langle f, \widetilde{k_y} \right\rangle k_y, \widetilde{k_x} \right\rangle d\lambda(y) \\
&= \left\langle \int_X \left\langle f, \widetilde{k_y} \right\rangle k_y d\lambda(y), \widetilde{k_x} \right\rangle \\
&= \left\langle f, \widetilde{k_x} \right\rangle .
\end{aligned}$$

Thus, using the first reconstruction formula one more time

$$S\left(\hat{S}f\right) = \int_X \left\langle f, \widetilde{k_x} \right\rangle k_x d\lambda(x) = f.$$

Therefore $\hat{S} = S^{-1}$ as claimed. Now we can prove the second reconstruction formula as we did for the first one. By theorem 1.17, the operator S is self-adjoint, and using the explicit expression of S^{-1} given above we obtain

$$\begin{aligned}
f &= S^{-1}(Sf) \\
&= \int_X \left\langle Sf, \widetilde{k_x} \right\rangle \widetilde{k_x} d\lambda(x) \\
&= \int_X \left\langle f, S\widetilde{k_x} \right\rangle \widetilde{k_x} d\lambda(x) \\
&= \int_X \langle f, k_x \rangle \widetilde{k_x} d\lambda(x).
\end{aligned}$$

This completes the proof. □

Corollary 1.20. *Given a closed subspace $\mathcal{F} \subseteq \mathcal{H}$, let $\{k_x\}_{x\in X} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$, and let $\left\{\widetilde{k_x}\right\}_{x\in X} \subseteq \mathcal{F}$ be its generalized dual frame (which is a generalized frame for $\mathcal{F}$, too). Then the following formulas hold*

$$P_{\mathcal{F}}f = \int_X \left\langle f, \widetilde{k_x} \right\rangle k_x d\lambda(x) = \int_X \langle f, k_x \rangle \widetilde{k_x} d\lambda(x), \quad f \in \mathcal{H},$$

where $P_{\mathcal{F}} : \mathcal{H} \to \mathcal{F}$ denotes the orthogonal projection onto $\mathcal{F}$.

Proof. Recall $\mathcal{F} = \overline{\text{span}}\{k_x : x \in X\}$. Decompose $f \in \mathcal{H}$ as $f = f_1 + f_2$, where $f_1 \in \mathcal{F}$ and $f_2 \in \mathcal{F}^\perp$. On the one hand, Theorem 1.19 applied to $f_1 \in \mathcal{F}$ gives

$$f_1 = \int_X \langle f_1, k_x \rangle \widetilde{k_x} d\lambda(x).$$

On the other hand, notice that the linear functional

$$l(g) := \left\langle \int_X \langle f_2, k_x \rangle \widetilde{k_x} d\lambda(x), g \right\rangle = \int_X \langle f_2, k_x \rangle \left\langle \widetilde{k_x}, g \right\rangle d\lambda(x) \equiv 0, \quad g \in \mathcal{H},$$

since $\langle f_2, k_x \rangle = 0$ for all $x \in X$. Then the Riesz representation theorem gives

$$\int_X \langle f_2, k_x \rangle \widetilde{k_x} d\lambda(x) = 0 \in \mathcal{H}.$$

Therefore

$$\begin{aligned}
\int_X \langle f, k_x \rangle \widetilde{k_x} d\lambda(x) &= \int_X \langle f_1, k_x \rangle \widetilde{k_x} d\lambda(x) + \int_X \langle f_2, k_x \rangle \widetilde{k_x} d\lambda(x) \\
&= f_1 + 0 \\
&= P_{\mathcal{F}} f.
\end{aligned}$$

□

1.2.5 Generalized Parseval frames

Definition 1.17. A generalized frame $\{k_x\}_{x \in X} \subseteq \mathcal{H}$ is called a *generalized Parseval frame* if the frame constants α and β are equal to 1, and hence for any $f \in \mathcal{H}$

$$\|f\|^2 = \int_X |\langle f, k_x \rangle|^2 d\lambda(x).$$

For the rest of this section, we will assume $\{k_x\}_{x \in X} \subseteq \mathcal{H}$ is a generalized Parseval frame. Notice that the definition of a generalized Parseval frame is simply saying that $\|f\| = \|Tf\|_{L^2}$, so, in this case the analysis operator $T : \mathcal{H} \to L^2(X, \lambda)$ becomes an isometry.

Theorem 1.21. *The frame operator* $S : \mathcal{H} \to \mathcal{H}$ *associated to a generalized Parseval frame* $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ *is the identity map* $I : \mathcal{H} \to \mathcal{H}$. *Moreover, the generalized dual frame* $\left\{\widetilde{k_x}\right\}_{x\in X} \subseteq \mathcal{H}$ *coincides with the original generalized Parseval frame* $\{k_x\}_{x\in X}$ *and the reconstruction formulas become a single reconstruction formula*

$$f = \int_X \langle f, k_x \rangle\, k_x d\lambda(x).$$

Proof. By the definition of the frame operator and using the Parseval condition

$$\begin{aligned}
\langle Sf, f \rangle &= \left\langle \int_X \langle f, k_x \rangle\, k_x d\lambda(x), f \right\rangle \\
&= \int_X \langle f, k_x \rangle \langle k_x, f \rangle\, d\lambda(x) \\
&= \int_X |\langle f, k_x \rangle|^2\, d\lambda(x) \\
&= \|f\|^2 \\
&= \langle f, f \rangle .
\end{aligned}$$

Thus, $\langle (S-I)f, f \rangle = 0$ for all $f \in \mathcal{H}$. Since $S - I$ is a linear bounded map on $\mathcal{H}$, due to the generalized polarization identity, for any $f_1, f_2 \in \mathcal{H}$ it holds

$$\begin{aligned}
\langle (S-I)f_1, f_2 \rangle &= \frac{1}{4}\left[\langle (S-I)(f_1+f_2), f_1+f_2 \rangle - \langle (S-I)(f_1-f_2), f_1-f_2 \rangle \right.\\
&\quad \left. +i\, \langle (S-I)(f_1+if_2), f_1+if_2 \rangle - i\, \langle (S-I)(f_1-if_2), f_1-if_2 \rangle \right].
\end{aligned}$$

Notice that the right hand side of the last expression becomes zero, so $\langle (S-I)f_1, f_2 \rangle = 0$ for all $f_1, f_2 \in \mathcal{H}$. Taking $f_2 = (S-I)f_1$ we get $\|(S-I)f_1\|^2 = 0$ for all $f_1 \in \mathcal{H}$. This implies $S - I$ is the zero map on $\mathcal{H}$, therefore $S = I$. □

Since in the case of a generalized Parseval frame the analysis operator T is an isometry, T can be viewed as an embedding of $\mathcal{H}$ into $L^2(X, \lambda)$, and hence it is of interest to calculate the orthogonal projection from $L^2(X, \lambda)$ onto $T(\mathcal{H})$ in terms of the generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$. This follows as an application of Theorem 1.21.

Proposition 1.22. *The map*

$$\begin{aligned} P : L^2(X,\lambda) &\to T(\mathcal{H}) \subseteq L^2(X,\lambda) \\ a(x) &\mapsto \int_X a(y)\langle k_y, k_x\rangle d\lambda(y) \end{aligned}$$

is the orthogonal projection map from $L^2(X,\lambda)$ *onto* $T(\mathcal{H})$.

Remark. This is simply saying $P = TT^*$.

Proof. First we will show that for any $a(x) \in L^2(X,\lambda)$, the image $(Pa)(x) \in T(\mathcal{H})$ indeed. Applying the synthesis operator to a we get $g = T^*a = \displaystyle\int_X a(y)k_y d\lambda(y) \in \mathcal{H}$, and now applying the analysis operator to g we obtain $Tg \in T(\mathcal{H}) \subseteq L^2(X,\lambda)$. But

$$\begin{aligned} Tg &= \langle g, k_x\rangle \\ &= \left\langle \int_X a(y)k_y d\lambda(y), k_x \right\rangle \\ &= \int_X a(y)\langle k_y, k_x\rangle d\lambda(y) \\ &= (Pa)(x). \end{aligned}$$

So, $P = TT^* : L^2(X,\lambda) \to T(\mathcal{H}) \subseteq L^2(X,\lambda)$ as claimed. From here, P is self-adjoint because $P^* = T^{**}T^* = TT^* = P$. Moreover, $P^2 = (TT^*)(TT^*) = T(T^*T)T^* = TST^* = TIT^* = TT^* = P$ because the frame operator S associated to the generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is the identity map on $\mathcal{H}$ due to theorem 1.21.

To prove P is surjective, it is enough to prove P restricted to $T(\mathcal{H})$ is the identity map on $L^2(X,\lambda)$. Let $b \in T(\mathcal{H})$, say $b(x) = \langle h, k_x\rangle$ for some $h \in \mathcal{H}$, due to the reconstruction formula for generalized Parseval frames we obtain that

$$\begin{aligned} (Pb)(x) &= \int_X \langle h, k_y\rangle\langle k_y, k_x\rangle d\lambda(y) \\ &= \int_X \langle\langle h, k_y\rangle k_y, k_x\rangle\, d\lambda(y) \\ &= \left\langle \int_X \langle h, k_y\rangle k_y d\lambda(y), k_x \right\rangle \\ &= \langle h, k_x\rangle \\ &= b(x). \end{aligned}$$

Thus, P is surjective. Finally, for the orthogonality, because P is self-adjoint and $P|_{T(\mathcal{H})}$ is the identity map, for any $a \in L^2(X, \lambda)$ and for any $\langle f, k_x \rangle \in T(\mathcal{H})$,

$$\begin{aligned}
\langle a(x) - (Pa)(x), \langle f, k_x \rangle \rangle_{L^2} &= \langle a(x), \langle f, k_x \rangle \rangle_{L^2} - \langle (Pa)(x), \langle f, k_x \rangle \rangle_{L^2} \\
&= \langle a(x), \langle f, k_x \rangle \rangle_{L^2} - \langle a(x), P\left(\langle f, k_x \rangle\right) \rangle_{L^2} \\
&= \langle a(x), \langle f, k_x \rangle \rangle_{L^2} - \langle a(x), \langle f, k_x \rangle \rangle_{L^2} \\
&= 0.
\end{aligned}$$

Hence $(a(x) - (Pa)(x)) \perp \langle f, k_x \rangle$. □

1.3 Generalized frames in classical function spaces

1.3.1 Reproducing kernel Hilbert spaces (RKHS)

A *reproducing kernel Hilbert space*, abbreviated RKHS, is a Hilbert space of functions such that the point-evaluation $f \mapsto f(x)$ is a bounded functional on $\mathcal{H}$, for all $x \in X$ [63, Chapter 1, Section 1.4].

Let $\mathcal{H}$ be a separable and complex Hilbert space of complex valued functions $f : X \to \mathbb{C}$, where (X, μ) is a measure space such that the inner product on $\mathcal{H}$ is defined by

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} d\mu(x), \quad f, g \in \mathcal{H}.$$

In this case we say $\mathcal{H}$ is *embedded* in $L^2(X, \mu)$, denoted by $\mathcal{H} \subseteq L^2(X, \mu)$, since

$$\|f\|^2 = \langle f, f \rangle = \int_X |f(x)|^2 \, d\mu(x) = \|f\|^2_{L^2(X,\mu)}.$$

Also, there is a function $K(x, y) : X^2 \to \mathbb{C}$, called the *reproducing kernel* on $\mathcal{H}$, such that $K(x, \cdot), K(\cdot, y) \in \mathcal{H}$ for all $x, y \in X$, and the following *reproducing formula* holds for all $f \in \mathcal{H}$

$$f(x) = \int_X K(x, y) f(y) d\mu(y), \quad x \in X.$$

Proposition 1.23. *Let $\mathcal{H}$ be a RKHS embedded in $L^2(X,\mu)$, with a reproducing kernel $K(x,y)$. If $K_x := \overline{K(x,\cdot)} \in \mathcal{H}$, then $\{K_x\}_{x\in(X,\mu)} \subseteq \mathcal{H}$ forms a generalized Parseval frame for $\mathcal{H}$. Furthermore, $\{k_x\}_{x\in(X,\nu)} \subseteq \mathcal{H}$ forms a normalized generalized Parseval frame for $\mathcal{H}$, where $k_x = \dfrac{K_x}{\|K_x\|} \in \mathcal{H}$, and $d\nu(x) = \|K_x\|^2\, d\mu(x)$.*

Proof. By the reproducing formula, for all $x \in X$ and all $f \in \mathcal{H}$ it holds

$$f(x) = \int_X K(x,y)f(y)d\mu(y) = \int_X f(y)\overline{K_x(y)}d\mu(y) = \langle f, K_x\rangle .$$

Hence for all $f \in \mathcal{H}$

$$\|f\|^2 = \int_X |f(x)|^2\, d\mu(x) = \int_X |\langle f, K_x\rangle|^2\, d\mu(x).$$

Therefore $\{K_x\}_{x\in(X,\mu)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$. Furthermore, for all $f \in \mathcal{H}$

$$\begin{aligned}
\|f\|^2 &= \int_X |\langle f, K_x\rangle|^2\, d\mu(x) \\
&= \int_X |\langle f, \|K_x\|\, k_x\rangle|^2\, d\mu(x) \\
&= \int_X |\langle f, k_x\rangle|^2\, \|K_x\|^2\, d\mu(x) \\
&= \int_X |\langle f, k_x\rangle|^2\, d\nu(x).
\end{aligned}$$

Therefore $\{k_x\}_{x\in(X,\nu)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame for $\mathcal{H}$. □

Remark. If there is no confusion, we use the name *reproducing kernel* for the family $\{K_x\}_{x\in X} \subseteq \mathcal{H}$, and *normalized reproducing kernel* for the family $\{k_x\}_{x\in X} \subseteq \mathcal{H}$.

1.3.2 The Paley-Wiener space

For $\alpha \in \mathbb{R}^+$, we say that an entire function $f : \mathbb{C} \to \mathbb{C}$ has *exponential type* less than or equal to α, denoted $f \in \{\text{ e.t.} \leq \alpha\}$, if

$$\limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|} \leq \alpha \Leftrightarrow |f(z)| \leq Ce^{\alpha|z|} \text{ for some constant } C > 0.$$

In the following definition, the space $L^2(\mathbb{R})$ refers to the Lebesgue measure on $\mathbb{R}$, and a complex function $f : \mathbb{C} \to \mathbb{C}$ is said to be in $L^2(\mathbb{R})$ if its restriction to $\mathbb{R}$ is in $L^2(\mathbb{R})$.

Definition 1.18. Let $H(\mathbb{C})$ be the space of entire functions on $\mathbb{C}$, and $\alpha > 0$. The *Paley-Wiener space* $\mathcal{PW}_\alpha(\mathbb{R})$ is the subspace of $H(\mathbb{C})$ defined by

$$\mathcal{PW}_\alpha(\mathbb{R}) = \left\{ f : \mathbb{C} \to \mathbb{C} \text{ entire of e.t.} \leq \alpha : \int_{\mathbb{R}} |f(x)|^2 \, d(x) < \infty \right\}.$$

Recall that any entire function is completely determined by its values over $\mathbb{R}$, so, we can think on $\mathcal{PW}_\alpha(\mathbb{R})$ as a closed subset of $L^2(\mathbb{R})$. The Paley-Wiener Theorem ([51, Page 14], [55, Theorem 19.3], [63, Chapter 2, Theorem 18]) establishes an isometric bijection between $\mathcal{PW}_\alpha(\mathbb{R})$ and $L^2[-\alpha, \alpha]$ via the Fourier transform $\mathscr{F}$

$$\begin{aligned} \mathscr{F} : \mathcal{PW}_\alpha(\mathbb{R}) &\to L^2[-\alpha, \alpha] \\ f(x) &\mapsto F(t), \end{aligned}$$

where

$$\begin{aligned} F(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixt} d(x), \quad t \in [-\alpha, \alpha], \\ f(x) &= \int_{-\alpha}^{\alpha} F(t) e^{ixt} d(t), \quad x \in \mathbb{R}. \end{aligned}$$

Moreover, $\mathcal{PW}_\alpha(\mathbb{R})$ is a Hilbert space [63, Chapter 2, part two, section 5] with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} d(x), \quad f, g \in \mathcal{PW}_\alpha(\mathbb{R}),$$

where $d(x)$ is the Lebesgue measure on $\mathbb{R}$.

Proposition 1.24. [63, Chapter 2, Theorem 19] *The Paley-Wiener space $\mathcal{PW}_\alpha(\mathbb{R})$, $\alpha > 0$, is a RKHS with reproducing kernel given by*

$$K^\alpha(z, w) = \begin{cases} \dfrac{\sin \alpha (z - \overline{w})}{\pi (z - \overline{w})} & , \text{ if } z \neq \overline{w}, \quad z, w \in \mathbb{C}, \\ \dfrac{\alpha}{\pi} & , \text{ if } z = \overline{w}, \quad z, w \in \mathbb{C}. \end{cases}$$

For any $f \in \mathcal{PW}_\alpha(\mathbb{R})$ it holds

$$f(z) = \int_{\mathbb{R}} f(w) K^\alpha(z, w) d(w) = \langle f, K_z^\alpha \rangle,$$

where $K_z^\alpha(w) = \overline{K^\alpha(z,w)} \in \mathcal{PW}_\alpha(\mathbb{R})$ *for all* $z \in \mathbb{C}$.

As mentioned before, given $f \in \mathcal{PW}_\alpha(\mathbb{R})$, we only consider the restriction of f to $\mathbb{R}$, and also the integral on the reproducing kernel formula is computed over $\mathbb{R}$, thus we restrict our attention to $w, z \in \mathbb{R}$. In this case, the reproducing kernel $K_z^\alpha(w) \in \mathcal{PW}_\alpha(\mathbb{R})$ (its restriction to $\mathbb{R}$) satisfies

$$K_z^\alpha(w) = \begin{cases} \dfrac{\sin\alpha(z-w)}{\pi(z-w)} & \text{, if } z \neq w, \\ \dfrac{\alpha}{\pi} & \text{, if } z = w, \end{cases} \qquad \|K_z^\alpha\|^2 = \frac{\alpha}{\pi}, \qquad w, z \in \mathbb{R}.$$

Corollary 1.25. *Let* $d(z)$ *be the Lebesgue measure on* $\mathbb{R}$. *The family* $\{k_z^\alpha\}_{z\in(\mathbb{R},\lambda)} \subseteq \mathcal{PW}_\alpha(\mathbb{R})$ *forms a generalized Parseval frame for the Paley-Wiener space* $\mathcal{PW}_\alpha(\mathbb{R})$, *where* k_z^α *is the normalized reproducing kernel*

$$k_z^\alpha(w) = \frac{K_z^\alpha(w)}{\|K_z^\alpha\|} = \begin{cases} \dfrac{\sin\alpha(z-w)}{\sqrt{\pi\alpha}(z-w)} & \text{, if } z \neq w, \\ \sqrt{\dfrac{\alpha}{\pi}} & \text{, if } z = w, \end{cases} \qquad w, z \in \mathbb{R},$$

and λ *is the multiple constant of the Lebesgue measure on* $\mathbb{R}$ *given by*

$$d\lambda(z) = \|K_z^\alpha\|^2 d(z) = \frac{\alpha}{\pi} d(z).$$

Proof. This is a direct consequence of Propositions 1.23 and 1.24. □

For future reference, we equip $\mathbb{R}$ with the Euclidean metric d, and from Corollary 1.25, the natural measure to consider for such index set is (up to a constant) the Lebesgue measure on $\mathbb{R}$.

In summary, the generalized Parseval frame $\{k_z^\alpha\}_{z\in(\mathbb{R},d,\lambda)} \subseteq \mathcal{PW}_\alpha(\mathbb{R})$ is indexed by a metric measure space $(\mathbb{R}, d, \lambda)$, where d is the Euclidean metric on $\mathbb{R}$, and λ is essentially the Lebesgue measure on $\mathbb{R}$.

Finally, a larger class of Paley-Wiener spaces is defined in the following way. Let $S \subseteq \mathbb{R}$ with finite Lebesgue measure, and let $L^2(S)$ be the space of square integrable functions $F : S \to \mathbb{C}$ with respect to the Lebesgue measure on S. The map

$$U : F(t) \mapsto f(x) = \int_S F(t) e^{ixt} d(t), \quad x \in \mathbb{R},$$

is an isometric bijection between $L^2(S)$ and its image $U\left(L^2(S)\right) \subseteq L^2(\mathbb{R})$.

We define the *Paley-Wiener space* $\mathcal{PW}_S(\mathbb{R})$ as such image, i.e., $\mathcal{PW}_S(\mathbb{R}) := U\left(L^2(S)\right)$. In other words, $L^2(S) = \mathscr{F}(\mathcal{PW}_S(\mathbb{R}))$, where $\mathscr{F}$ is the Fourier transform.

The Paley-Wiener space $\mathcal{PW}_S(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}d(x), \quad f, g \in \mathcal{PW}_S(\mathbb{R}),$$

where $d(x)$ is the Lebesgue measure on $\mathbb{R}$.

Moreover, the point-evaluation $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{PW}_S(\mathbb{R})$ for all $x \in X$. Thus $\mathcal{PW}_S(\mathbb{R})$ is a RKHS. The precise formula for the reproducing kernel $K^S(x, y)$ is not available in this case, however the existence of the reproducing kernel is enough to guarantee the existence of a generalized Parseval frame in $\mathcal{PW}_S(\mathbb{R})$.

1.3.3 The Bargmann-Fock space

Given $z \in \mathbb{C}^n$, let $|z|$ denote the Euclidean norm of z in $\mathbb{C}^n$, i.e., $|z|^2 = z \cdot \overline{z} = \sum_{i=1}^{n} z_i\overline{z_i}$, where the operation used is the dot product on $\mathbb{C}^n$. It is customary to denote $z^2 = z \cdot z$. Also, given $z, w \in \mathbb{C}^n$, the inner product on $\mathbb{C}^n$ is $\langle z, w\rangle_{\mathbb{C}^n} = z \cdot \overline{w}$. Let $d(z)$ denote the Lebesgue measure on $\mathbb{C}^n$ (i.e., the Lebesgue measure on $\mathbb{R}^{2n}$). For $\alpha \in \mathbb{R}^+$ we define a new Borel measure on $\mathbb{C}^n$ by

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} d(z)$$

which is a probability measure on $\mathbb{C}^n$, $\lambda_\alpha(\mathbb{C}^n) = 1$.

Definition 1.19. Let $H(\mathbb{C}^n)$ be the space of entire functions on $\mathbb{C}^n$, and $\alpha > 0$. The $n-$dimensional *Bargmann-Fock space* $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is the subspace of $H(\mathbb{C}^n)$ defined by

$$\mathcal{F}^2_\alpha(\mathbb{C}^n) = H(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, d\lambda_\alpha) = \left\{ f : \mathbb{C}^n \to \mathbb{C} \text{ entire} : \int_{\mathbb{C}^n} |f(z)|^2 d\lambda_\alpha(z) < \infty \right\}.$$

The Bargmann-Fock space is a Hilbert space [66, Section 2.1] with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\lambda_\alpha(z), \quad f, g \in \mathcal{F}^2_\alpha(\mathbb{C}^n).$$

Furthermore, the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}^2_\alpha(\mathbb{C}^n)$ defined by

$$(Bf)(w) = \left(\frac{2\alpha}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{R}^n} f(v) e^{2\alpha v\cdot w - \alpha v^2 - \frac{\alpha}{2} w^2} d(v), \quad f(v) \in L^2(\mathbb{R}^n), \quad w \in \mathbb{C}^n,$$

is an isometric bijection between $L^2(\mathbb{R}^n)$ and $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ ([66, Theorem 6.8] and [29, Proposition 3.4.1, Theorem 3.4.3]).

Proposition 1.26. [66, Proposition 2.2] *The Bargmann-Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$, $\alpha > 0$, is a RKHS with reproducing kernel given by*

$$K^\alpha(z, w) = e^{\alpha z\cdot \overline{w}}, \quad z, w \in \mathbb{C}^n.$$

For any $f \in \mathcal{F}^2_\alpha(\mathbb{C}^n)$ it holds

$$f(z) = \int_{\mathbb{C}^n} f(w) K^\alpha(z, w) d\lambda_\alpha(w) = \langle f, K^\alpha_z \rangle,$$

where $K^\alpha_z(w) = \overline{K^\alpha(z, w)} \in \mathcal{F}^2_\alpha(\mathbb{C}^n)$ for all $z \in \mathbb{C}^n$.

Notice that the reproducing kernel $K^\alpha_z(w)$ satisfies

$$K^\alpha_z(w) = e^{\alpha \overline{z}\cdot w}, \qquad \|K^\alpha_z\|^2 = e^{\alpha|z|^2}, \qquad w, z \in \mathbb{C}^n.$$

In particular, taking $n = 1$, the reproducing kernel for the 1−dimensional Bargmann-Fock space $\mathcal{F}^2_\alpha := \mathcal{F}^2_\alpha(\mathbb{C})$ satisfies

$$K^\alpha_z(w) = e^{\alpha \overline{z} w}, \qquad \|K^\alpha_z\|^2 = e^{\alpha|z|^2}, \qquad w, z \in \mathbb{C}.$$

Corollary 1.27. *Let $d(z)$ be the Lebesgue measure on $\mathbb{C}^n$. The family $\{k^\alpha_z\}_{z\in(\mathbb{C}^n,\lambda)} \subseteq \mathcal{F}^2_\alpha(\mathbb{C}^n)$ forms a generalized Parseval frame for the Bargmann-Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$, where k^α_z is the normalized reproducing kernel*

$$k^\alpha_z(w) = \frac{K^\alpha_z(w)}{\|K^\alpha_z\|} = \frac{e^{\alpha\overline{z}\cdot w}}{e^{\frac{1}{2}\alpha|z|^2}}, \quad w, z \in \mathbb{C}^n,$$

and λ is the multiple constant of the Lebesgue measure on $\mathbb{C}^n$ given by

$$d\lambda(z) = \|K^\alpha_z\|^2 \, d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n d(z).$$

Proof. This is a direct consequence of Propositions 1.23 and 1.26. □

For future reference, we equip $\mathbb{C}^n$ with its Euclidean metric d, and from Corollary 1.27, the natural measure to consider for such index set is (up to a constant) the Lebesgue measure on $\mathbb{C}^n$.

In summary, the generalized Parseval frame $\{k_z^\alpha\}_{z\in(\mathbb{C}^n,d,\lambda)} \subseteq \mathcal{F}_\alpha^2(\mathbb{C}^n)$ is indexed by a metric measure space $(\mathbb{C}^n, d, \lambda)$, where d is the Euclidean metric on $\mathbb{C}^n$, and λ is essentially the Lebesgue measure on $\mathbb{C}^n$.

1.3.4 The Bergman space

Let $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, and let $d(z)$ be the Lebesgue measure on $\mathbb{C}^n$ (i.e., the Lebesgue measure on $\mathbb{R}^{2n}$). It is well-known that the volume of the unit ball $\mathbb{D}^n$ (its Lebesgue measure) is $\dfrac{\pi^n}{n!}$. For any $\alpha \in \mathbb{R}$ we can define a new positive Borel measure on $\mathbb{C}^n$ weighted by α as

$$dA_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha \frac{n!}{\pi^n} d(z),$$

where c_α is a positive constant (depending on n and α) described in terms of the Gamma function $\Gamma(\cdot)$ by

$$c_\alpha = \begin{cases} \dfrac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} & \text{, if } \alpha > -1, \\ 1 & \text{, if } \alpha \leq -1. \end{cases}$$

This measure is invariant under unitary transformations of the unit ball $\mathbb{D}^n$, i.e.,

$$\int_{\mathbb{D}^n} f(Uz)\, dA_\alpha(z) = \int_{\mathbb{D}^n} f(z)\, dA_\alpha(z)$$

for all $f \in L^1(\mathbb{D}^n, dA_\alpha)$ and all unitary transformations $U : \mathbb{D}^n \to \mathbb{D}^n$ [64, Equation 1.20]. In particular, taking $\alpha = -(n+1)$, the measure $dA_{-(n+1)}$ is called the *hyperbolic measure* on $\mathbb{D}^n$

$$d\tau_n(z) := \frac{n!}{\pi^n} \left(1 - |z|^2\right)^{-(n+1)} d(z)$$

which is invariant under automorphism of $\mathbb{D}^n$, i.e.,

$$\int_{\mathbb{D}^n} f(\psi(z))\, d\tau_n(z) = \int_{\mathbb{D}^n} f(z)\, d\tau_n(z)$$

for all $f \in L^1(\mathbb{D}^n, d\tau_n)$ and all $\psi \in \operatorname{Aut}(\mathbb{D}^n)$ [64, Equations 1.25 and 1.26]. Furthermore, when $n = 1$ and $\alpha = -2$, the hyperbolic measure is called the *Möbius invariant measure* on $\mathbb{D}$

$$d\tau_1(z) := \frac{1}{\pi}\left(1 - |z|^2\right)^{-2} d(z)$$

which is invariant under Möbius transformations of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, i.e.,

$$\int_{\mathbb{D}} f(\psi(z))\, d\tau_1(z) = \int_{\mathbb{D}} f(z)\, d\tau_1(z)$$

for all $f \in L^1(\mathbb{D}, d\tau_1)$ and all $\psi \in \operatorname{Aut}(\mathbb{D})$ [65, Equation 4.6], where $\psi(z) = e^{i\theta}\dfrac{a - z}{1 - \overline{a}z}$ for some $\theta \in \mathbb{R}$ and some $a \in \mathbb{D}$.

Going back to the general case, dA_α is a finite measure on $\mathbb{D}^n$ if and only if $\alpha > -1$ [65, Lemma 3.9], specifically dA_α is a probability measure on the unit ball, $A_\alpha(\mathbb{D}^n) = 1$ if and only if $\alpha > -1$. In particular, the hyperbolic measure $d\tau_n$ is not a finite measure.

Definition 1.20. Let $H(\mathbb{D}^n)$ be the space of holomorphic functions on $\mathbb{D}^n$, and $\alpha > -1$. The $n-$dimensional weighted *Bergman space* $\mathcal{B}^2_\alpha(\mathbb{D}^n)$ is the subspace of $H(\mathbb{D}^n)$ defined by

$$\mathcal{B}^2_\alpha(\mathbb{D}^n) = H(\mathbb{D}^n) \cap L^2(\mathbb{D}^n, dA_\alpha) = \left\{ f : \mathbb{D}^n \to \mathbb{C} \text{ holomorphic} : \int_{\mathbb{D}^n} |f(z)|^2\, dA_\alpha(z) < \infty \right\}.$$

The Bergman space $\mathcal{B}^2_\alpha(\mathbb{D}^n)$ is a Hilbert space [64, Corollary 2.5] with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}^n} f(z)\overline{g(z)} dA_\alpha(z), \quad f, g \in \mathcal{B}^2_\alpha(\mathbb{D}^n).$$

Proposition 1.28. [64, Theorems 2.2 and 2.7] *The weighted Bergman space $\mathcal{B}^2_\alpha(\mathbb{D}^n)$, $\alpha > -1$, is a RKHS with reproducing kernel given by*

$$K^\alpha(z, w) = \frac{1}{(1 - z \cdot \overline{w})^{n+1+\alpha}}, \quad z, w \in \mathbb{D}^n.$$

For any $f \in \mathcal{B}^2_\alpha(\mathbb{D}^n)$ it holds

$$f(z) = \int_{\mathbb{D}^n} f(w) K^\alpha(z, w) dA_\alpha(w) = \langle f, K^\alpha_z\rangle,$$

where $K^\alpha_z(w) = \overline{K^\alpha(z, w)} \in \mathcal{B}^2_\alpha(\mathbb{D}^n)$ for all $z \in \mathbb{D}^n$.

The reproducing kernel K_z^α satisfies

$$K_z^\alpha(w) = \frac{1}{(1 - \overline{z} \cdot w)^{n+1+\alpha}}, \qquad \|K_z^\alpha\|^2 = \frac{1}{\left(1 - |z|^2\right)^{n+1+\alpha}}, \qquad w, z \in \mathbb{D}^n.$$

In particular, taking $n = 1$, the reproducing kernel for the $1-$dimensional weighted Bergman space $\mathcal{B}_\alpha^2 := \mathcal{B}_\alpha^2(\mathbb{D})$ satisfies

$$K_z^\alpha(w) = \frac{1}{(1 - \overline{z}w)^{2+\alpha}}, \qquad \|K_z^\alpha\|^2 = \frac{1}{\left(1 - |z|^2\right)^{2+\alpha}}, \qquad w, z \in \mathbb{D}.$$

Corollary 1.29. *Let $d(z)$ be the Lebesgue measure on $\mathbb{C}^n$. The family $\{k_z^\alpha\}_{z \in (\mathbb{D}^n, \lambda)} \subseteq \mathcal{B}_\alpha^2(\mathbb{D}^n)$ forms a generalized Parseval frame for the weighted Bergman space $\mathcal{B}_\alpha^2(\mathbb{D}^n)$, where k_z^α is the normalized reproducing kernel*

$$k_z^\alpha(w) = \frac{K_z^\alpha(w)}{\|K_z^\alpha\|} = \frac{\left(1 - |z|^2\right)^{(n+1+\alpha)/2}}{(1 - \overline{z} \cdot w)^{n+1+\alpha}},$$

and λ is the multiple constant of the hyperbolic measure on $\mathbb{D}^n$ given by

$$d\lambda(z) = \|K_z^\alpha\|^2 \, dA_\alpha(z) = c_\alpha d\tau_n(z).$$

Proof. Direct consequence of Propositions 1.23 and 1.28. □

As Corollary 1.29 shows, the natural measure for the index set $\mathbb{D}^n$ is (up to a constant) the hyperbolic measure τ_n. For future reference, we also equip the index set $\mathbb{D}^n$ with an appropriate metric, such metric is the *Bergman metric* β on $\mathbb{D}^n$.

In summary, the generalized Parseval frame $\{k_z^\alpha\}_{z \in (\mathbb{D}^n, d, \lambda)} \subseteq \mathcal{B}_\alpha^2(\mathbb{D}^n)$ is indexed by a metric measure space $(\mathbb{D}^n, d, \lambda)$, where $d = \beta$ is the Bergman metric on $\mathbb{D}^n$, and λ is essentially the hyperbolic measure on $\mathbb{D}^n$.

For the rest of the subsection, we briefly discuss how the Bergman metric β is defined. For further details see Section 1.5 in [64]. Consider the *Bergman kernel*

$$K(z, w) = \frac{1}{(1 - z \cdot \overline{w})^{n+1}}, \quad z, w \in \mathbb{D}^n,$$

and construct the *Bergman matrix* on $\mathbb{D}^n$

$$B(z) = \frac{1}{n+1}\left[\ \frac{\partial^2}{\partial \overline{z_i}\partial z_j}\log K(z,z)\ \right]_{n\times n},$$

where as usual, given $z_j = x + iy \in \mathbb{C}$, the complex partial derivatives are defined in terms of real partial derivatives by $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, and $\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. This matrix $B(z)$ is symmetric, positive, invertible, and as a linear transformation on $\mathbb{C}^n$ satisfies

$$B(z) = \left(1-|z|^2\right)^{-2} P_z + \left(1-|z|^2\right)^{-1} Q_z,$$

where $P_z : \mathbb{C}^n \to \operatorname{span}\{z\}$ is the orthogonal projection onto $\operatorname{span}\{z\}$, and $Q_z : \mathbb{C}^n \to \operatorname{span}\{z\}^{\perp}$ is the orthogonal projection onto $\operatorname{span}\{z\}^{\perp}$ [64, Proposition 1.18].

Now, for a (piecewise) $\mathcal{C}^1$ curve $\gamma : [0,1] \to \mathbb{D}^n$, let

$$l(\gamma) = \int_0^1 \langle B(\gamma(t))\,\gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}\, dt,$$

where $\langle z, w\rangle = z \cdot \overline{w}$ denotes the inner product in $\mathbb{C}^n$. Using this function, we define the Bergman metric β as follows

$$\beta(z,w) = \inf\left\{ l(\gamma) : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve}, \ \gamma : [0,1] \to \mathbb{D}^n,\ \gamma(0) = z,\ \gamma(1) = w \right\}, \quad w, z \in \mathbb{D}^n.$$

To give an intuitive explanation to the definition of the Bergman metric, from the proof of Proposition 1.21 in [64] we obtain

$$\begin{aligned}
\int_0^1 \langle B(\gamma(t))\,\gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}\, dt &= \int_0^1 \left\langle \frac{P_{\gamma(t)}\gamma'(t)}{\left(1-|\gamma(t)|^2\right)^2} + \frac{Q_{\gamma(t)}\gamma'(t)}{1-|\gamma(t)|^2}, \gamma'(t)\right\rangle^{\frac{1}{2}} dt \\
&\geq \int_0^1 \left\langle \frac{P_{\gamma(t)}\gamma'(t)}{\left(1-|\gamma(t)|^2\right)^2}, \gamma'(t)\right\rangle^{\frac{1}{2}} dt \\
&= \int_0^1 \frac{\left|P_{\gamma(t)}\gamma'(t)\right|}{1-|\gamma(t)|^2} dt \\
&\geq \int_0^1 \frac{|\alpha'(t)|}{1-\alpha^2(t)} dt,
\end{aligned}$$

where $\alpha(t) = |\gamma(t)|$. Taking the infimum we reach an equality, and the very last integral resembles the length of a curve in the disc $\mathbb{D}$ with respect to its hyperbolic metric [65, page 67].

The Bergman metric β is invariant under automorphisms of $\mathbb{D}^n$, i.e.,

$$\beta\left(\psi(z), \psi(w)\right) = \beta(z, w)$$

for all $z, w \in \mathbb{D}^n$ and all $\psi \in \mathrm{Aut}\left(\mathbb{D}^n\right)$ [64, Proposition 1.20]. Furthermore, Proposition 1.21 in [64] gives

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\psi_z(w)|}{1 - |\psi_z(w)|},$$

where $\psi_z \in \mathrm{Aut}\left(\mathbb{D}^n\right)$ is the involutive automorphism of $\mathbb{D}^n$ that switches 0 and z [64, Equation 1.2].

There is another important metric on $\mathbb{D}^n$, it is called the *pseudo-hyperbolic* metric on $\mathbb{D}^n$, it is denoted by ρ and it is defined by

$$\rho(z, w) = \left|\psi_z(w)\right|,$$

where again $\psi_z \in \mathrm{Aut}\left(\mathbb{D}^n\right)$ is the involutive automorphism of $\mathbb{D}^n$ that switches 0 and z. The pseudo-hyperbolic metric is also invariant under automorphism of $\mathbb{D}^n$ [64, Corollary 1.22], and it satisfies

$$\rho(z, w) = \tanh \beta(z, w).$$

1.4 Generalized frames and unitary representations

In this section we construct many interesting generalized frames on $\mathcal{H}$ using the theory of unitary representations of groups on Hilbert spaces. Such approach gives rise to a very large family of generalized frames, and some of the examples studied in previous sections can be understood as particular cases under this setup.

1.4.1 LCH groups and left Haar measures

In this subsection X represents a topological space, if (X, d) is a metric space, the topology associated to X is the one generated by the open balls with respect to the metric d. We also impose algebraic structure on X, i.e., $(X, \cdot)$ is a group, which in general will be assumed non-Abelian.

Finally, (X, μ) is a measure space, where the measure μ will be assumed $\sigma-$finite (i.e., X can be decomposed as a countable union of sets with finite $\mu-$measure).

Definition 1.21. Let X be a topological space. If every point $x \in X$ has a compact neighborhood, X is called a *locally compact* space. Additionally, if X is a Hausdorff space, X is called a *locally compact Hausdorff* (LCH) space.

Definition 1.22. Let X be a LCH space, μ be a Borel measure on X, and $B \subseteq X$ be a Borel set. We say μ is *outer regular* on B if

$$\mu(B) = \inf \{\mu(U) : X \supseteq U \supseteq B, U \text{ open}\}.$$

We say μ is *inner regular* on B if

$$\mu(B) = \sup \{\mu(K) : K \subseteq B, K \text{ compact}\}.$$

We say the Borel measure μ is a *Radon measure* on X if μ is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Definition 1.23. Let X be a topological space. If $(X, \cdot)$ is a group such that the group operation $(x, y) \mapsto x \cdot y$ and the inverse operator $x \mapsto x^{-1}$ are both continuous ($x, y \in X$, and as usual $X \times X$ is equipped with the product topology), X is called a *topological group*. Additionally, if X is a Hausdorff space, or a LCH space, X is called a *Hausdorff group*, or a *LCH group*, respectively.

Remark. In the case of topological groups, the T_0, T_1, and T_2 (Hausdorff) conditions are equivalent [62, Theorem 3.4], so the assumption that X is a Hausdorff space is not very restrictive. On the other hand, we do not assume commutativity on the group $(X, \cdot)$ since there are many non-Abelian interesting examples in this setup.

Definition 1.24. Let $(X, \cdot)$ be a LCH group, and let μ be a Radon measure on X. Given $x \in X$ and $B \subseteq X$, denote by $xB := \{x \cdot b : b \in B\} \subseteq X$. We say the Radon measure μ is a *left Haar measure* on X if

$$\mu(xB) = \mu(B)$$

for all $x \in X$ and all Borel sets $B \subseteq X$, in which case we say μ is *left invariant*. Similarly we can define a *right Haar measure* on X.

Notice that if μ is a left Haar measure on X, we can create multiple left Haar measures on X as follows: fix $x \in X$ and define $\mu_x(B) := \mu(Bx)$, where $B \subseteq X$ is a Borel set, and $Bx := \{b \cdot x : b \in B\}$. Recall μ is left invariant, so the right invariance is not guaranteed. It is easy to check μ_x is also a left Haar measure on X. Furthermore, we can create a right Haar measure too: $\widetilde{\mu}(B) := \mu\left(B^{-1}\right)$, where $B \subseteq X$ is a Borel set, and $B^{-1} := \left\{b^{-1} : b \in B\right\}$. Proposition 4.3 in [62] shows $\widetilde{\mu}$ is a right Haar measure on X.

The following theorems show that every LCH group X has a nonzero left Haar measure μ, $\operatorname{supp}(\mu) \neq \emptyset$. Moreover, all nonzero left Haar measures μ on a LCH group X are the same up to a multiplicative constant, as long as we assume (X, μ) is $\sigma-$finite.

Theorem 1.30. [25, Theorem 2.10] *Every locally compact group X has a left Haar measure μ (consequently, X also has a right Haar measure $\widetilde{\mu}$).*

Theorem 1.31. [62, Proposition 4.5] *If μ is a left Haar measure on the LCH group X, then $\mu(U) > 0$ for all nonempty open sets $U \subseteq X$.*

Theorem 1.32. [62, Theorem 4.7] *If μ and ν are $\sigma-$finite, nonzero, and left Haar measures on a LCH group X, then there exists a constant $a > 0$ such that $\mu = a\nu$.*

Let μ be a nonzero left Haar measure on the LCH group X. As mentioned before, for a fixed $x \in X$, μ_x is also a nonzero left Haar measure on X (recall $\mu_x(B) := \mu(Bx)$, $B \subseteq X$ Borel set). Applying Theorem 1.32, there exists a positive constant depending on x, say $a_x > 0$, such that $\mu_x = a_x\mu$, and similarly, if ν is another nonzero left Haar measure on X, then $\mu = a\nu$ for some $a > 0$. Then

$$\nu_x = a\mu_x = a\left(a_x\mu\right) = a_x\left(a\mu\right) = a_x\nu.$$

This says that the constants associated to μ_x and ν_x (when compared against μ and ν, respectively, via Theorem 1.32) are the same, such constant depends on x only.

Definition 1.25. Let X be a LCH group and μ a nonzero left Haar measure on X. For any $x \in X$, there exists a constant $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$. The map $\Delta : X \to \mathbb{R}^+$ is called the *modular function* on X. If $\Delta(x) = 1$ for all $x \in X$, the LCH group X is called *unimodular*.

Proposition 1.33. [62, Proposition 4.10] *The modular function $\Delta : X \to \mathbb{R}^{\times}$ is a continuous map and a homomorphism, where $\mathbb{R}^{\times}$ is the set of positive real numbers, which is a group under multiplication, and a metric space with respect the Euclidean distance.*

Proposition 1.34. [62, Propositions 4.13 and 4.14] *Let X be a LCH group. If X is Abelian, then X is unimodular. If X is compact, then X is unimodular.*

1.4.2 Unitary representations

Let X be a LCH group with nonzero left Haar measure μ, and as usual, let $\mathcal{H}$ be a (separable and complex) Hilbert space. Let $\mathcal{U}(\mathcal{H}) = \{U : \mathcal{H} \to \mathcal{H} \text{ s.t. } UU^* = U^*U = I\}$ the group of all unitary operators on $\mathcal{H}$ with respect the composition of mappings.

Definition 1.26. The map $\pi : X \to \mathcal{U}(\mathcal{H})$ is called a *unitary representation* of X on $\mathcal{H}$ if it a group homomorphism and it is *weakly continuous*, i.e., the map

$$\begin{aligned} \pi_{f,g} : X &\to \mathbb{C} \\ x &\mapsto \langle \pi(x)f, g\rangle \end{aligned}$$

is continuous for all $f, g \in \mathcal{H}$. The Hilbert space $\mathcal{H}$ is called the *representation space* of π.

Definition 1.27. A subspace $\mathcal{F} \subseteq \mathcal{H}$ is called *invariant* with respect to the unitary representation $\pi : X \to \mathcal{U}(\mathcal{H})$ of X on $\mathcal{H}$ if $\mathcal{F}$ is closed and satisfies

$$\pi(x)\mathcal{F} \subseteq \mathcal{F}$$

for all $x \in X$.

Clearly $\{0\}$ and $\mathcal{H}$ are invariant subspaces, they are called the *trivial invariant* subspaces of $\mathcal{H}$. A unitary representation $\pi : X \to \mathcal{U}(\mathcal{H})$ of X on $\mathcal{H}$ is called *irreducible* if it only has the trivial invariant subspaces.

Remark. It is well-known that all unitary representations of a LCH group X on a Hilbert space $\mathcal{H}$ can be decomposed into irreducible representations

Proposition 1.35. [62, Example 5.7] *Let X be a LCH group equipped with the nonzero left Haar measure μ. Consider $\mathcal{H} = L^2(X, \mu)$. For $x \in X$ denote by L_x the left translation by x, i.e., given $f(y) \in L^2(X, \mu)$, $(L_x f)(y) = f\left(x^{-1}y\right)$. Then the map*

$$\begin{aligned} L : X &\to \mathcal{U}\left(L^2(X, \mu)\right) \\ x &\mapsto L_x \end{aligned}$$

is a unitary representation of X on $L^2(X, \mu)$, called the left regular representation of X.

Definition 1.28. A unitary representation $\pi : X \to \mathcal{U}(\mathcal{H})$ is called *square-integrable* if there exists a nonzero element $\varphi \in \mathcal{H}$ satisfying the following *admissibility condition*

$$c_\varphi := \int_X |\langle \varphi, \pi(x)\varphi \rangle|^2 \, d\mu(x) < \infty.$$

Any element $\varphi \in \mathcal{H}$ such that $\|\varphi\| = 1$ and satisfying the admissibility condition is called an *admissible wavelet* for the square-integrable representation $\pi : X \to \mathcal{U}(\mathcal{H})$, and the constant c_φ is called the *wavelet constant* associated to the admissible wavelet φ. The set of all admissible wavelets associated to a unitary representation $\pi : X \to \mathcal{U}(\mathcal{H})$ (square-integrable or not) is denoted by $AW(\pi)$.

1.4.3 Resolution of the identity

The following theorems form the core of the entire section. Given a suitable unitary representation $\pi : X \to \mathcal{U}(\mathcal{H})$ of X on $\mathcal{H}$, we can construct generalized frames applying the following strategy.

First, we want to know if π is irreducible, usually this is done via Theorem 1.36 (Schur's Lemma). Otherwise, take an irreducible component.

Second, we want to know if π is square integrable, i.e., if there exists an admissible wavelet. This can be done by direct computations, or if X is unimodular (e.g., X is Abelian), Theorem 1.37 states that there are no admissible wavelets, or there are many.

Finally, Theorem 1.38 states that any admissible wavelet $\varphi \in \mathcal{H}$ generates a generalized frame $\{\pi(x)\varphi\}_{x \in (X,\mu)} \subseteq \mathcal{H}$, and adjusting the left Haar measure μ such system forms a generalized Parseval frame.

Theorem 1.36. [62, Theorem 5.2] *(Schur's lemma) Let X be a LCH group, and let $\pi : X \to \mathcal{U}(\mathcal{H})$ be a unitary representation of X on $\mathcal{H}$. The unitary representation π is irreducible if and only if the only bounded linear operators on $\mathcal{H}$ that commute with $\pi(x)$ for all $x \in X$ are scalar multiples of the identity operator in $\mathcal{H}$.*

Theorem 1.37. [62, Theorem 6.6] *Let X be a LCH group, and let $\pi : X \to \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of X on $\mathcal{H}$. If X is unimodular, then $AW(\pi) = \emptyset$ (there are not admissible wavelets whatsoever), or $AW(\pi) = \{f \in \mathcal{H} : \|f\| = 1\}$.*

Theorem 1.38. [62, Theorem 6.1] *Let X be a LCH group with nonzero left Haar measure μ, and let $\pi : X \to \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of X on $\mathcal{H}$. If $\pi : X \to \mathcal{U}(\mathcal{H})$ is square-integrable, then for any admissible wavelet $\varphi \in AW(\pi)$ it holds*

$$\langle f, g \rangle = \frac{1}{c_\varphi} \int_X \langle f, \pi(x)\varphi \rangle \langle \pi(x)\varphi, g \rangle \, d\mu(x).$$

This theorem says that $f = \frac{1}{c_\varphi} \int_X \langle f, \pi(x)\varphi \rangle \pi(x)\varphi d\mu(x)$ for any $f \in \mathcal{H}$, where the integral on the right hand side is understood in a weak sense. Therefore the the identity map $I : \mathcal{H} \to \mathcal{H}$ has the following *resolution formula*

$$I = \frac{1}{c_\varphi} \int_X \langle \cdot, \pi(x)\varphi \rangle \pi(x)\varphi d\mu(x).$$

1.4.4 Admissible wavelets and generalized Parseval frames

The resemblance between the resolution formula of the identity given by Theorem 1.38, and the frame operator associated to a generalized Parseval frame, which is the identity by Theorem 1.21, is not a coincidence. The following proposition clearly states the connection.

Proposition 1.39. *Let X be a LCH group with nonzero left Haar measure μ, and let $\pi : X \to \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of X on $\mathcal{H}$. If $\pi : X \to \mathcal{U}(\mathcal{H})$ is square-integrable, then any admissible wavelet $\varphi \in AW(\pi)$ generates a normalized generalized Parseval frame $\{\varphi_x\}_{x \in (X, \lambda_\varphi)} \subseteq \mathcal{H}$ for $\mathcal{H}$, where $\varphi_x = \pi(x)\varphi$, and $\lambda_\varphi = \frac{1}{c_\varphi}\mu$.*

Proof. Since $\varphi \in AW(\pi)$, then $\|\varphi\| = 1$, and since $\pi(x) : \mathcal{H} \to \mathcal{H}$ is unitary, then

$$\begin{aligned} \|\varphi_x\|^2 &= \langle \pi(x)\varphi, \pi(x)\varphi \rangle \\ &= \langle \varphi, \pi(x)^* \pi(x)\varphi \rangle \\ &= \langle \varphi, \varphi \rangle \\ &= 1. \end{aligned}$$

By Theorem 1.38, for any $f \in \mathcal{H}$

$$\begin{aligned} \|f\|^2 &= \frac{1}{c_\varphi} \int_X \langle f, \pi(x)\varphi \rangle \langle \pi(x)\varphi, f \rangle \, d\mu(x) \\ &= \int_X |\langle f, \varphi_x \rangle|^2 \, \frac{1}{c_\varphi} d\mu(x) \\ &= \int_X |\langle f, \varphi_x \rangle|^2 \, d\lambda_\varphi(x), \end{aligned}$$

therefore $\{\varphi_x\}_{x \in (X, \lambda_\varphi)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame for $\mathcal{H}$, and the frame operator coincides with the identity resolution formula

$$I = \int_X \langle \cdot, \varphi_x \rangle \, \varphi_x d\lambda_\varphi(x) = \frac{1}{c_\varphi} \int_X \langle \cdot, \pi(x)\varphi \rangle \, \pi(x)\varphi d\mu(x).$$

□

1.4.5 Wavelet transform and RKHS

The objective of this subsection is to give some insights of Theorem 1.38 and the left regular representation of X, Proposition 1.35. It turns out that $\mathcal{H}$ can be isometrically embedded in a reproducing kernel Hilbert space (RKHS) which is a subspace of $L^2(X, \mu)$.

We assume X is a LCH group with a nonzero left Haar measure μ, and $\pi : X \to \mathcal{U}(\mathcal{H})$ is an irreducible and square-integrable unitary representation of X on $\mathcal{H}$.

Definition 1.29. Given $\varphi \in AW(\pi)$, the *wavelet transform* associated to the admissible wavelet φ is the bounded linear operator

$$\begin{aligned} A_\varphi : \mathcal{H} &\rightarrow C(X) \cap L^2(X,\mu) \\ f &\mapsto \frac{1}{\sqrt{c_\varphi}} \langle f, \pi(\cdot)\varphi \rangle \end{aligned}$$

where $C(X)$ denotes the set of all continuous and complex-valued functions on X.

Remark. Clearly $A_\varphi f \in C(X)$ because π is weakly continuous given that it is a unitary representation. In contrast, $A_\varphi f \in L^2(X,\mu)$ is not trivial, it is a consequence of the irreducibility of π and it is one of the difficulties in the proof of Theorem 1.38 [62, see proofs of Theorem 6.1 and Lemma 6.3].

Proposition 1.40. [62, Theorem 7.6] *Let X be a LCH group with a nonzero left Haar measure μ, and let $\pi : X \rightarrow \mathcal{U}(\mathcal{H})$ be an irreducible and square-integrable unitary representation of X on $\mathcal{H}$. If $\varphi \in AW(\pi)$ and A_φ is the wavelet transform associated to the admissible wavelet φ, then A_φ is an isometry, its range $R(A_\varphi)$ is a closed subspace of $L^2(X,\mu)$, and a RKHS with reproducing kernel $\Phi(x,y) = \Phi\left(y^{-1}x\right)$, where*

$$\Phi := \frac{1}{\sqrt{c_\varphi}} \overline{A_\varphi \varphi}.$$

The reproducing kernel formula states that for any $F(\cdot) \in R(A_\varphi)$ and any fixed $x \in X$

$$F(x) = \langle F, \Phi_x \rangle_{L^2(X,\mu)},$$

where $\Phi_x(\cdot) = \overline{\Phi(x,\cdot)}$.

Proof. The range $R(A_\varphi) = A_\varphi(\mathcal{H})$ is closed as a subspace of $L^2(X,\mu)$ [62, Lemma 6.3], so it is a Hilbert space. We can prove $A_\varphi : \mathcal{H} \rightarrow L^2(X,\mu)$ is an isometry applying Theorem 1.38. For any $f, g \in \mathcal{H}$ we have

$$\begin{aligned} \langle f, g \rangle &= \frac{1}{c_\varphi} \int_X \langle f, \pi(x)\varphi \rangle \langle \pi(x)\varphi, g \rangle \, d\mu(x) \\ &= \int_X (A_\varphi f)(x) \overline{(A_\varphi g)(x)} d\mu(x) \\ &= \langle A_\varphi f, A_\varphi g \rangle_{L^2(X,\mu)}. \end{aligned}$$

It remains to prove the reproducing kernel formula, which is again a consequence of Theorem 1.38. For any $f \in \mathcal{H}$, i.e., for any $F = A_\varphi f \in R(A_\varphi)$, we have

$$\begin{aligned}
(A_\varphi f)(x) &= \frac{1}{\sqrt{c_\varphi}} \langle f, \pi(x)\varphi \rangle \\
&= \frac{1}{c_\varphi^{3/2}} \int_X \langle f, \pi(y)\varphi \rangle \langle \pi(y)\varphi, \pi(x)\varphi \rangle \, d\mu(y) \\
&= \frac{1}{c_\varphi^{3/2}} \int_X \langle f, \pi(y)\varphi \rangle \langle \varphi, \pi(y^{-1}x)\varphi \rangle \, d\mu(y) \\
&= \frac{1}{\sqrt{c_\varphi}} \int_X (A_\varphi f)(y) \overline{(A_\varphi \varphi)(y^{-1}x)} d\mu(y) \\
&= \int_X \Phi(x,y) (A_\varphi f)(y) d\mu(y) \\
&= \langle A_\varphi f, \Phi_x \rangle_{L^2(X,\mu)} .
\end{aligned}$$

□

1.4.6 Homogeneous spaces

In this subsection, X is a LCH group with left Haar measure μ and modular function Δ_X, and as usual let $\pi : X \to \mathcal{U}(\mathcal{H})$ be a unitary representation of X on $\mathcal{H}$. Let $Y \leq X$ be a closed subgroup of X which is also a LCH group with left Haar measure ξ and modular function Δ_Y. Let $q : X \to X/Y$ be the canonical quotient map, i.e., $q(x) = [x] = xY = \{xy : y \in Y\}$ is the left coset corresponding to $x \in X$.

We impose the quotient topology on X/Y, this is $U \subseteq X/Y$ is open in X/Y if and only if $q^{-1}(U) \subseteq X$ is open in X. Then the quotient map is a continuous and open map, and due to Proposition 2.2 in [25] we also have that X/Y is a LCH topological space. In the case when $Y \trianglelefteq X$ is a normal subgroup of X, then X/Y becomes a group and so a LCH group, but for the rest of the subsection we do not impose such normality assumption on Y.

In this setup X *acts* on X/Y (denoted $X \curvearrowright X/Y$) by *left multiplication*, i.e., the map

$$\begin{aligned}
X \times (X/Y) &\to X/Y \\
(x, [x']) &\mapsto x[x'] = [xx']
\end{aligned}$$

is continuous, and it satisfies the following properties

1. The map $X/Y \to X/Y$ given by $[x'] \mapsto x[x'] = [xx']$ is a homeomorphism of X/Y for each $x \in X$ fixed.

2. $x_1 (x_2[x']) = (x_1x_2)[x']$ for all $x_1, x_2 \in X$ and all $[x'] \in X/Y$.

Moreover, this action is $X-$*transitive*, which means that for every $[x], [y] \in X/Y$, there exists $z \in X$ such that $z[x] = [y]$. The $X-$transitive LCH topological space X/Y receives the name of *homogeneous space.*

We want to construct a (left) $X-$invariant Radon measure ν on X/Y, this means $\nu(xE) = \nu(E)$ for all $x \in X$ and all $E \subseteq X/Y$ Borel set, where $xE = \{x[e] : [e] \in E\}$. Recall $C_c(X)$ represents the space of complex-valued continuous functions on X with compact support, and similar definition for $C_c(X/Y)$. Consider the map $P : C_c(X) \to C_c(X/Y)$ defined by

$$(Pt)([x]) = \int_Y t(xy) d\xi(y), \quad t \in C_c(X).$$

This is a well-defined map due to the left invariance of the Haar measure ξ.

Theorem 1.41. [25, Theorem 2.51] *Suppose X is a LCH group with left Haar measure μ and modular function Δ_X, and let $Y \leq X$ be a closed subgroup of X which is a LCH group with left Haar measure ξ and modular function Δ_Y. There is a $X-$invariant Radon measure ν on the LCH topological space X/Y if and only if $\Delta_X|Y = \Delta_Y$. In this case, ν is unique up to a constant factor, and if this factor is suitable chosen we have*

$$\int_X t(x) d\mu(x) = \int_{X/Y} (Pt)([x]) d\nu([x]) = \int_{X/Y} \int_Y t(xy) d\xi(y) d\nu([x]), \quad t \in C_c(X).$$

Such equalities extend to any Borel $\mu-$measurable continuous function $t : X \to [0, \infty]$ [25, Lemma 2.66].

In particular, we are interested to apply Theorem 1.41 on the function $t : X \to [0, \infty)$ given by $t(x) = |\langle f, \pi(x)\varphi\rangle|^2$, where $\pi : X \to \mathcal{U}(\mathcal{H})$ is a unitary representation of X on $\mathcal{H}$, and $f, \varphi \in \mathcal{H}$ are fixed. Notice that t is continuous on X because π is a unitary representation.

Given $\varphi \in \mathcal{H}\backslash\{0\}$, consider the set $Y = \{y \in X : \pi(y)\varphi = c_y\varphi$ for some $c_y \in \mathbb{C}\}$. Exploiting the fact that π is a unitary representation we can prove Y is a closed subgroup of X. Note that $|c_y| = 1$ for any $y \in Y$ because $\|\varphi\| = \|\pi(y)\varphi\| = |c_y| \|\varphi\|$. Also Y is nonempty, $e \in Y$ since

$\pi(e)\varphi = I\varphi = \varphi$, where e is the identity element in X, and I the identity map on $\mathcal{H}$. Furthermore, for $y_1, y_2 \in Y$ we have $\pi\left(y_1 y_2^{-1}\right)\varphi = \pi\left(y_1\right)\pi\left(y_2^{-1}\right)\varphi = \pi\left(y_1\right)\pi\left(y_2\right)^*\varphi = c_{y_1}\overline{c_{y_2}}\varphi$, then $y_1 y_2^{-1} \in Y$. Then Y is a subgroup of X. Finally, as shown in Remark 5.1 of [62], the condition that π is weakly continuous implies π is strongly continuous, which means that $x \mapsto \pi(x)g$ is a continuous map for any fixed $g \in \mathcal{H}$, thus, taking a net $\{y_i\}_{i\in I} \subseteq Y$ such that $y_i \to y \in X$, by continuity we have $\pi(y_i)\varphi \to \pi(y)\varphi$ pointwise in $\mathcal{H}$. By the definition of Y, $\{\pi(y_i)\varphi : i \in I\} \subseteq \operatorname{span}\{\varphi\}$, where $\operatorname{span}\{\varphi\}$ is a $1-$dimensional subspace of $\mathcal{H}$ and thus closed. Hence $\pi(y)\varphi \in \operatorname{span}\{\varphi\}$, so $\pi(y)\varphi = c_y\varphi$ for some $c_y \in \mathbb{C}$, therefore Y is closed.

Corollary 1.42. *Let $\pi : X \to \mathcal{U}(\mathcal{H})$ be a unitary representation of X on $\mathcal{H}$, and pick $f, \varphi \in \mathcal{H}$. Suppose X is a LCH group with left Haar measure μ and modular function Δ_X, and let $Y \leq X$ be the closed subgroup of X defined by $Y = \{y \in X : \pi(y)\varphi = c_y\varphi$ for some $c_y \in \mathbb{C}\}$ which is a LCH group with left Haar measure ξ and modular function Δ_Y. If $\xi(Y) < \infty$ and $\Delta_X|Y = \Delta_Y$, then there is a $X-$invariant Radon measure ν on the LCH topological space X/Y such that*

$$\int_X |\langle f, \pi(x)\varphi\rangle|^2\, d\mu(x) = \int_{X/Y} |\langle f, \pi(x)\varphi\rangle|^2\, d\nu\left([x]\right).$$

In particular, if $\pi : X \to \mathcal{U}(\mathcal{H})$ is assumed to be an irreducible and square-integrable unitary representation of X on $\mathcal{H}$, then for any admissible wavelet $\varphi \in AW(\pi)$, both $\{\varphi_x\}_{x\in(X,\lambda_\varphi)} \subseteq \mathcal{H}$ and $\{\varphi_x\}_{[x]\in(X/Y,\sigma_\varphi)} \subseteq \mathcal{H}$ are generalized Parseval frames for $\mathcal{H}$, where $\varphi_x = \pi(x)\varphi$, $\lambda_\varphi = \frac{1}{c_\varphi}\mu$, and $\sigma_\varphi = \frac{1}{c_\varphi}\nu$.

Remark. Any choice of the representative in the equivalence class $[x] \in X/Y$ will work when generating $\{\varphi_x\}_{[x]\in(X/Y,\sigma_\varphi)}$. More precisely, we say $s : X/Y \to X$ is a *section* if $q \circ s$ is the identity map on X/Y, where $q : X \to X/Y$ is the canonical quotient map. So, this corollary shows that for any given section s, $\left\{\varphi_{s([x])}\right\}_{[x]\in(X/Y,\sigma_\varphi)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$.

Proof. Let $t : X \to [0, \infty)$ be the continuous function $t(x) = |\langle f, \pi(x)\varphi\rangle|^2$. By the definition of Y

$$\begin{aligned}
(Pt)([x]) &= \int_Y t(xy) d\xi(y) \\
&= \int_Y |\langle f, \pi(xy)\varphi\rangle|^2 d\xi(y) \\
&= \int_Y |\langle f, \pi(x)\pi(y)\varphi\rangle|^2 d\xi(y) \\
&= \int_Y |\langle f, c_y \pi(x)\varphi\rangle|^2 d\xi(y) \\
&= |\langle f, \pi(x)\varphi\rangle|^2 \int_Y |c_y|^2 d\xi(y) \\
&= |\langle f, \pi(x)\varphi\rangle|^2 \xi(Y).
\end{aligned}$$

It is important to note that the last expression is independent on the choice of the representative in the equivalence class $[x] \in X/Y$, this is a consequence of the left invariance of ξ. By Theorem 1.41, there exists a $X-$invariant Radon measure ν_1 on X/Y such that

$$\begin{aligned}
\int_X |\langle f, \pi(x)\varphi\rangle|^2 d\mu(x) &= \int_X t(x) d\mu(x) \\
&= \int_{X/Y} (Pt)([x]) d\nu_1([x]) \\
&= \int_{X/Y} |\langle f, \pi(x)\varphi\rangle|^2 \xi(Y) d\nu_1([x]) \\
&= \int_{X/Y} |\langle f, \pi(x)\varphi\rangle|^2 d\nu([x]),
\end{aligned}$$

where $\nu = \dfrac{1}{\xi(Y)}\nu_1$ is also a $X-$invariant Radon measure on X/Y. In the case when $\pi : X \to \mathcal{U}(\mathcal{H})$ is an irreducible and square-integrable unitary representation of X on $\mathcal{H}$, and $\varphi \in AW(\pi)$, the result follows from Proposition 1.39. □

This result is interesting because in some applications X/Y is not a group, or X/Y is a group but the restriction of the unitary representation $\pi|X/Y$ (by which we mean π restricted to $s(X/Y)$, where $s : X/Y \to X$ is a section) is not a unitary representation of X/Y on $\mathcal{H}$.

1.4.7 Unitary representations of the Heisenberg group

The objective of this subsection is to obtain similar results to Corollary 1.27, about the normalized reproducing kernel associated to the $n-$dimensional Bargmann-Fock space $\mathcal{F}_{\pi}^{2}\left(\mathbb{C}^{n}\right)$ (parameter $\alpha=\pi$), and the normalized generalized Parseval frame associated to such reproducing kernel, using the theory of homogeneous spaces and unitary representations on the Heisenberg group.

There are different definitions in the literature for the Heisenberg group, e.g., see [62, Chapter 17], and [29, Chapter 9]. We use the following definition for the the *full/reduced Heisenberg* group:

Definition 1.30. The *full Heisenberg group* $(\mathbb{H}_F^n,*)$ is the set $\mathbb{C}^n\times\mathbb{R}$ with non-commutativity group operation $*$ defined by

$$(z,t)*(z',t') \quad = \quad \left(z+z',t+t'+\frac{1}{2}\operatorname{Im}\left(z\cdot\overline{z'}\right)\right),$$

for $z,z'\in\mathbb{C}^n$, and $t,t'\in\mathbb{R}$, where $z\cdot\overline{z'}=(z_1,\ldots,z_n)\cdot\left(\overline{z_1'},\ldots,\overline{z_n'}\right)=\sum_{i=1}^{n}z_i\overline{z_i}$ is the dot product in $\mathbb{C}^n$. The *reduced Heisenberg group* $(\mathbb{H}_R^n,*)$ is the set $\mathbb{C}^n\times\mathbb{R}/\mathbb{Z}$ (we understand $\mathbb{R}/\mathbb{Z}$ as the interval $[0,1]$ with $0\sim 1$) with the same group operation.

Both $(\mathbb{H}_F^n,*)$ and $(\mathbb{H}_R^n,*)$ are unimodular LCH groups, with the Lebesgue measure μ on $\mathbb{C}^n\times\mathbb{R}$ or $\mathbb{C}^n\times\mathbb{R}/\mathbb{Z}$, being the respective left and right Haar measure [62, Proposition 17.3].

For the next two theorems, given $z\in\mathbb{C}^n$, let $z=x+iy$ where $x,y\in\mathbb{R}^n$. Also, the space $L^2\left(\mathbb{R}^n\right)$ refers to the Lebesgue measure on $\mathbb{R}^n$, and $\mathcal{F}_{\pi}^{2}\left(\mathbb{C}^{n}\right)$ is the $n-$dimensional Bargmann-Fock space with parameter $\alpha=\pi$.

Theorem 1.43. [29, Theorem 9.2.1], [62, Corollary 17.8] *The map* $\rho:\mathbb{H}_j^n\to\mathcal{U}\left(L^2\left(\mathbb{R}^n\right)\right)$, $j=F,R$, *given by*

$$\left(\rho\left(z,t\right)f\right)(w)=\left(\rho\left(x,y,t\right)f\right)(w)=e^{2\pi i\,t}e^{\pi i\,x\cdot y}e^{2\pi i\,y\cdot(w-x)}f\left(w-x\right),\quad f(w)\in L^2\left(\mathbb{R}^n\right),$$

is an irreducible unitary representation of $\mathbb{H}_j^n$ *on* $L^2\left(\mathbb{R}^n\right)$. *The map* ρ *is called the Schrödinger representation of the Heisenberg group. Such representation* ρ *is square-integrable for* $\mathbb{H}_R^n$ [62, Corollary 17.9], *but it is not square-integrable for* $\mathbb{H}_F^n$ [62, Theorem 17.12].

Theorem 1.44. [29, Theorem 9.2.1] *The map* $\beta : \mathbb{H}_j^n \to \mathcal{U}\left(\mathcal{F}_\pi^2(\mathbb{C}^n)\right)$, $j = F, R$, *given by*

$$(\beta(z,t) f)(w) = e^{2\pi i\, t} e^{\pi\, z\cdot w} e^{-\frac{\pi}{2}\, z\cdot \overline{z}} f(w - \overline{z}), \quad f(w) \in \mathcal{F}_\pi^2(\mathbb{C}^n),$$

is an irreducible unitary representation of $\mathbb{H}_j^n$ *on* $\mathcal{F}_\pi^2(\mathbb{C}^n)$. *The map* β *is called the Bargmann-Fock representation of the Heisenberg group. Such representation* β *is equivalent to the Schrödinger representation* ρ, *thus* β *is square-integrable for* $\mathbb{H}_R^n$ [62, Corollary 17.9], *but it is not square-integrable for* $\mathbb{H}_F^n$ [62, Theorem 17.12].

Proof. It only remains to justify β and ρ are equivalent, then the square-integrability condition will follow from Theorem 1.43. Given that the Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}_\pi^2(\mathbb{C}^n)$ defined by

$$(Bf)(w) = 2^{\frac{n}{4}} \int_{\mathbb{R}^n} f(v) e^{2\pi v\cdot w - \pi v\cdot v - \frac{\pi}{2} w\cdot w} d(v), \quad f(v) \in L^2(\mathbb{R}^n)$$

is an isometric bijection between $L^2(\mathbb{R}^n)$ and $\mathcal{F}_\pi^2(\mathbb{C}^n)$ (see [66, Theorem 6.8], [29, Proposition 3.4.1, Theorem 3.4.3]), then the Bargmann-Fock representation is equivalent to the Schrödinger representation ρ [29, Theorem 9.2.1]. □

Proposition 1.45. *Let* $\beta : \mathbb{H}_R^n \to \mathcal{U}\left(\mathcal{F}_\pi^2(\mathbb{C}^n)\right)$ *be the irreducible and square-integrable Bargmann-Fock unitary representation of the unimodular LCH reduced Heisenberg group* $\mathbb{H}_R^n$ *on the* $n-$*dimensional Bargmann-Fock space* $\mathcal{F}_\pi^2(\mathbb{C}^n)$. *If* $\varphi : \mathbb{C}^n \to \mathbb{C}$ *is the constant function* $\varphi \equiv 1$, *then* $\varphi \in AW(\beta)$, *and*

$$\{\beta(z,0)\varphi\}_{z\in(\mathbb{C}^n,\lambda)} \subseteq \mathcal{F}_\pi^2(\mathbb{C}^n)$$

is a normalized generalized Parseval frame for $\mathcal{F}_\pi^2(\mathbb{C}^n)$, *where* λ *is (up to a constant) the Lebesgue measure on* $\mathbb{C}^n$. *Furthermore*

$$(\beta(z,0)\varphi)(w) = e^{\pi\, z\cdot w} e^{-\frac{\pi}{2}\, z\cdot\overline{z}} = k_{\overline{z}}^{\pi}(w), \quad w, z \in \mathbb{C}^n,$$

where $k_{\overline{z}}^{\pi}(w)$ *is the normalized reproducing kernel for* $\mathcal{F}_\pi^2(\mathbb{C}^n)$ *from Corollary 1.27.*

Proof. The constant function $\varphi \equiv 1$ is entire, and

$$\|\varphi\|_{\mathcal{F}_\pi^2(\mathbb{C}^n)}^2 = \int_{\mathbb{C}^n} |\varphi(w)|^2\, d\lambda_\pi(w) = \int_{\mathbb{C}^n} (1)^2 \left(\frac{\pi}{\pi}\right)^n e^{-\pi w\cdot\overline{w}} d(w) = 1 < \infty,$$

where $d(w)$ is the Lebesgue measure on $\mathbb{C}^n$. Then $\varphi \in \mathcal{F}_\pi^2(\mathbb{C}^n)$, and $\|\varphi\|_{\mathcal{F}_\pi^2(\mathbb{C}^n)} = 1$. Theorem 1.37 implies that $\varphi \equiv 1$ is an admissible wavelet, i.e., $\varphi \in AW(\beta)$. By Proposition 1.39

$$\left\{\varphi_{(z,t)}\right\}_{(z,t)\in\left(\mathbb{H}_R^n,\lambda_\varphi\right)} \subseteq \mathcal{F}_\pi^2(\mathbb{C}^n)$$

is a normalized generalized Parseval frame for $\mathcal{F}_\pi^2(\mathbb{C}^n)$, where

$$\varphi_{(z,t)}(w) = (\beta(z,t)\varphi)(w) = e^{2\pi i\, t} e^{\pi\, z\cdot w} e^{-\frac{\pi}{2}\, z\cdot\overline{z}},$$

$\lambda_\varphi = \frac{1}{c_\varphi}\mu$, and μ is the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}/\mathbb{Z}$. Then, for any $f \in \mathcal{F}_\pi^2(\mathbb{C}^n)$ it holds

$$\|f\|^2_{\mathcal{F}_\pi^2(\mathbb{C}^n)} = \int_{\mathbb{H}_R^n} \left|\left\langle f, \varphi_{(z,t)}\right\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\lambda_\varphi(z,t).$$

Construct the set $Y = \{(z,t) \in \mathbb{H}_R^n : z = 0\} = \{0\} \times \mathbb{R}/\mathbb{Z}$. Clearly Y is non-empty, and for $(0,t_1),(0,t_2) \in Y$ we have $(0,t_1) * (0,t_2)^{-1} = (0,t_1) * (0,-t_1) = (0,t_1 - t_2) \in Y$, where the rest $t_1 - t_2$ is taken mod[1]. Then $(Y,*)$ is a subgroup of the Heisenberg group $(\mathbb{H}_R^n,*)$. Moreover, $(Y,*) \cong (\mathbb{R}/\mathbb{Z},+)$, and the last one is a unimodular LCH group such that its left and right Haar measure ξ is the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. Hence, Y is a unimodular LCH group with left and right Haar measure ξ satisfying $\xi(Y) = 1 < \infty$. Consider the quotient consisting on left cosets

$$\mathbb{H}_R^n/Y = \{(z,t) * Y : (z,t) \in \mathbb{H}_R^n\} = \{[(z,0)] : z \in \mathbb{C}^n\}.$$

Applying Corollary 1.42, there exists a (left) $\mathbb{H}_R^n-$invariant Radon measure ν on the LCH topological space $\mathbb{H}_R^n/Y$ such that for any $f \in \mathcal{F}_\pi^2(\mathbb{C}^n)$ it holds

$$\int_{\mathbb{H}_R^n} \left|\langle f, \beta(z,t)\varphi\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\mu(z,t) = \int_{\mathbb{H}_R^n/Y} \left|\langle f, \beta(z,0)\varphi\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\nu([(z,0)]),$$

and this implies

$$\|f\|^2_{\mathcal{F}_\pi^2(\mathbb{C}^n)} = \int_{\mathbb{H}_R^n/Y} \left|\left\langle f, \varphi_{(z,0)}\right\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\sigma_\varphi([z,0]),$$

where $\sigma_\varphi = \frac{1}{c_\varphi}\nu$, and ν is the (left) $\mathbb{H}_R^n-$invariant Radon measure on $\mathbb{H}_R^n/Y$. Hence

$$\left\{\varphi_{(z,0)}\right\}_{[(z,0)]\in\left(\mathbb{H}_R^n/Y,\sigma_\varphi\right)} \subseteq \mathcal{F}_\pi^2(\mathbb{C}^n)$$

is a normalized generalized Parseval frame for $\mathcal{F}_\pi^2(\mathbb{C}^n)$, where

$$\varphi_{(z,0)}(w) = (\beta(z,0)\varphi)(w) = e^{\pi\ z\cdot w} e^{-\frac{\pi}{2}\ z\cdot\overline{z}}.$$

We can say more about the (left) $\mathbb{H}_R^n-$invariant Radon measure σ_φ on the LCH topological space $\mathbb{H}_R^n/Y$. Notice that the left action $\mathbb{H}_R^n \curvearrowright \mathbb{H}_R^n/Y$ by left multiplication is defined by

$$(z,t) * [(z',0)] = [(z,t) * (z',0)] = [(z+z',0)],$$

so the left $\mathbb{H}_R^n-$invariance of σ_φ on $\mathbb{H}_R^n/Y$ can be understood as the left invariance of a Radon measure on the LCH topological space $\mathbb{C}^n$, such measure is unique [24, Theorem 7.2] and it is (up to a constant) the Lebesgue measure λ on $\mathbb{C}^n$.

Observe that the normalized reproducing kernel for $\mathcal{F}_\pi^2(\mathbb{C}^n)$ given in Corollary 1.27 is

$$k_{\overline{z}}^{\pi}(w) = e^{\pi\ z\cdot w} e^{-\frac{\pi}{2}\ z\cdot\overline{z}} = \varphi_{(z,0)}(w),$$

thus, for any $f \in \mathcal{F}_\pi^2(\mathbb{C}^n)$ it holds

$$\|f\|^2_{\mathcal{F}_\pi^2(\mathbb{C}^n)} = \int_{\mathbb{H}_R^n/Y} \left|\left\langle f, \varphi_{(z,0)}\right\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\sigma_\varphi([z,0]) = \int_{\mathbb{C}^n} \left|\langle f, k_{\overline{z}}^{\pi}\rangle_{\mathcal{F}_\pi^2(\mathbb{C}^n)}\right|^2 d\lambda(z),$$

where $d\lambda(z)$ is (up to a constant) the Lebesgue measure on $\mathbb{C}^n$.

Therefore $\left\{\varphi_{(z,0)}\right\}_{[(z,0)]\in\left(\mathbb{H}_R^n/Y,\sigma_\varphi\right)} \subseteq \mathcal{F}_\pi^2(\mathbb{C}^n)$ and $\{k_{\overline{z}}^{\pi}\}_{z\in(\mathbb{C}^n,\lambda)} \subseteq \mathcal{F}_\pi^2(\mathbb{C}^n)$ are the same generalized Parseval frame for $\mathcal{F}_\pi^2(\mathbb{C}^n)$. □

1.4.8 Unitary representation of the Blaschke group

The objective of this subsection is to obtain similar results to Corollary 1.29, about the normalized reproducing kernel associated to the unweighted $1-$dimensional Bergman space $\mathcal{B}_0^2(\mathbb{D})$ (parameters $\alpha = 0$, $n = 1$), and the normalized generalized Parseval frame associated to such reproducing kernel, using the theory of homogeneous spaces and unitary representations on the Blaschke group.

In the next definition, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the disc, and $\mathbb{T} = \left\{e^{i\theta} : \theta \in \mathbb{R}\right\}$ denotes the $1-$dimensional *torus*.

Definition 1.31. [44, Section 2.1] The *Blaschke group* $(\mathbb{B}, *)$ is the set $\mathbb{D}\times\mathbb{T}$ with non-commutativity group operation $*$ defined by

$$\left(z_1, e^{i\theta_1}\right) * \left(z_2, e^{i\theta_2}\right) = \left(\frac{z_1 e^{-i\theta_2} + z_2}{1 + z_1\overline{z_2}e^{-i\theta_2}}, e^{i\theta_1}\frac{e^{i\theta_2} + z_1\overline{z_2}}{1 + e^{i\theta_2}\overline{z_1}z_2}\right),$$

for $z_1, z_2 \in \mathbb{D}$ and $e^{i\theta_1}, e^{i\theta_2} \in \mathbb{T}$. The identity element is $(0, 1) \in \mathbb{B}$.

Let ξ be the Lebesgue measure on the torus $\mathbb{T}$, this is $\xi = m\circ t^{-1}$, where m is the normalized Lebesgue measure on $[-\pi, \pi)$, and $t : [-\pi, \pi) \to \mathbb{T}$ is given by $t : \theta \mapsto e^{i\theta}$. Also, let τ_1 be the Möbius invariant (hyperbolic) measure on $\mathbb{D}$. According with Section 2.1 in [44], $(\mathbb{B}, *)$ is an unimodular LCH group, with left and right Haar measure $\mu = (\pi\tau_1) \times \xi$, i.e., for $f : \mathbb{B} \to \mathbb{C}$

$$\int_{\mathbb{B}} f\left(z, e^{i\theta}\right) d\mu\left(z, e^{i\theta}\right) = \int_{\mathbb{T}}\int_{\mathbb{D}} f\left(z, e^{i\theta}\right) d\tau_1(z) d\xi\left(e^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{D}} \frac{f\left(z, e^{i\theta}\right)}{\left(1 - |z|^2\right)^2} d(z)d(\theta),$$

where as usual $d(z)$ is the Lebesgue measure on $\mathbb{C}$, and $d(\theta)$ is the Lebesgue measure on $\mathbb{R}$.

Theorem 1.46. [44, Theorems 3, 5, and 6] *The map* $\beta : \mathbb{B} \to \mathcal{U}\left(\mathcal{B}_0^2(\mathbb{D})\right)$ *given by*

$$\left(\beta\left(\left(z, e^{i\theta}\right)^{-1}\right) f\right)(w) = e^{i\theta}\frac{1 - |z|^2}{(1 - \overline{z}w)^2} f\left(e^{i\theta}\frac{w - z}{1 - \overline{z}w}\right), \quad f(w) \in \mathcal{B}_0^2(\mathbb{D}),$$

is an irreducible and square-integrable unitary representation of the Blaschke group $\mathbb{B}$ *on the Bergman space* $\mathcal{B}_0^2(\mathbb{D})$.

Proposition 1.47. *Let* $\beta : \mathbb{B} \to \mathcal{U}\left(\mathcal{B}_0^2(\mathbb{D})\right)$ *be the irreducible and square-integrable unitary representation of the unimodular LCH Blaschke group* $\mathbb{B}$ *on the unweighted* $1-$*dimensional Bergman space* $\mathcal{B}_0^2(\mathbb{D})$ *given by Theorem 1.46. If* $\varphi : \mathbb{D} \to \mathbb{C}$ *is the constant function* $\varphi \equiv 1$*, then* $\varphi \in AW(\beta)$*, and*

$$\left\{\beta\left((z, 1)^{-1}\right)\varphi\right\}_{z\in(\mathbb{D},\lambda)} \subseteq \mathcal{B}_0^2(\mathbb{D})$$

is a normalized generalized Parseval frame for $\mathcal{B}_0^2(\mathbb{D})$*, where* λ *is (up to a constant) the Möbius invariant (hyperbolic) measure on* $\mathbb{D}$*. Furthermore*

$$\left(\beta\left((z, 1)^{-1}\right)\varphi\right)(w) = \frac{1 - |z|^2}{(1 - \overline{z}w)^2} = k_z^0(w), \quad w, z \in \mathbb{D},$$

where $k_z^0(w)$ *is the normalized reproducing kernel for* $\mathcal{B}_0^2(\mathbb{D})$ *from Corollary 1.29.*

Proof. The constant function $\varphi \equiv 1$ is holomorphic on $\mathbb{D}$, and

$$\|\varphi\|^2_{\mathcal{B}^2_0(\mathbb{D})} = \int_{\mathbb{D}} |\varphi(w)|^2 \, dA_0(w) = \int_{\mathbb{D}} (1)^2 \frac{1}{\pi} d(w) = 1 < \infty,$$

where $d(w)$ is the Lebesgue measure on $\mathbb{C}$. Then $\varphi \in \mathcal{B}^2_0(\mathbb{D})$, and $\|\varphi\|_{\mathcal{B}^2_0(\mathbb{D})} = 1$. Theorem 1.37 implies that $\varphi \equiv 1$ is an admissible wavelet, i.e., $\varphi \in AW(\beta)$. By Proposition 1.39

$$\left\{\varphi_{(z,e^{i\theta})^{-1}}\right\}_{(z,e^{i\theta})^{-1} \in (\mathbb{B}, \lambda_\varphi)} \subseteq \mathcal{B}^2_0(\mathbb{D})$$

is a normalized generalized Parseval frame for $\mathcal{B}^2_0(\mathbb{D})$, where

$$\varphi_{(z,e^{i\theta})^{-1}}(w) = \left(\beta\left(\left(z,e^{i\theta}\right)^{-1}\right)\varphi\right)(w) = e^{i\theta}\frac{1-|z|^2}{(1-\overline{z}w)^2},$$

$\lambda_\varphi = \frac{1}{c_\varphi}\mu$, and μ is the Haar measure on $\mathbb{B}$. Then, for any $f \in \mathcal{B}^2_0(\mathbb{D})$ it holds

$$\|f\|^2_{\mathcal{B}^2_0(\mathbb{D})} = \int_{\mathbb{B}} \left|\left\langle f, \varphi_{(z,e^{i\theta})^{-1}}\right\rangle_{\mathcal{B}^2_0(\mathbb{D})}\right|^2 d\lambda_\varphi\left(\left(z,e^{i\theta}\right)^{-1}\right).$$

Construct the set $Y = \left\{\left(z,e^{i\theta}\right) \in \mathbb{B} : z = 0\right\} = \{0\} \times \mathbb{T}$. $(Y,*)$ is a subgroup of the Blaschke group $(\mathbb{B},*)$. Moreover, $(Y,*) \cong (\mathbb{T},\cdot)$, and the last one is a unimodular LCH group such that its left and right Haar measure ξ is the normalized Lebesgue measure on $\mathbb{T}$. Hence, Y is a unimodular LCH group with left and right Haar measure ξ satisfying $\xi(Y) = 1 < \infty$. Consider the quotient consisting on left cosets

$$\mathbb{B}/Y = \left\{\left(z,e^{i\theta}\right) * Y : \left(z,e^{i\theta}\right) \in \mathbb{B}\right\} = \{[(z,1)] : z \in \mathbb{D}\}.$$

Applying Corollary 1.42, there exists a (left) $\mathbb{B}-$invariant Radon measure ν on the LCH topological space $\mathbb{B}/Y$ such that for any $f \in \mathcal{B}^2_0(\mathbb{D})$ it holds

$$\int_{\mathbb{B}} \left|\left\langle f, \beta\left(\left(z,e^{i\theta}\right)^{-1}\right)\varphi\right\rangle_{\mathcal{B}^2_0(\mathbb{D})}\right|^2 d\mu\left(\left(z,e^{i\theta}\right)^{-1}\right) = \int_{\mathbb{B}/Y} \left|\left\langle f, \beta\left((z,1)^{-1}\right)\varphi\right\rangle_{\mathcal{B}^2_0(\mathbb{D})}\right|^2 d\nu\left(\left[(z,1)^{-1}\right]\right),$$

and this implies

$$\|f\|^2_{\mathcal{B}^2_0(\mathbb{D})} = \int_{\mathbb{B}/Y} \left|\left\langle f, \varphi_{(z,1)^{-1}}\right\rangle_{\mathcal{B}^2_0(\mathbb{D})}\right|^2 d\sigma_\varphi\left(\left[(z,1)^{-1}\right]\right),$$

where $\sigma_\varphi = \frac{1}{c_\varphi}\nu$, and ν is the (left) $\mathbb{B}-$invariant Radon measure on $\mathbb{B}/Y$. Hence

$$\left\{\varphi_{(z,1)^{-1}}\right\}_{[(z,1)^{-1}]\in(\mathbb{B}/Y,\sigma_\varphi)} \subseteq \mathcal{B}_0^2(\mathbb{D})$$

is a normalized generalized Parseval frame for $\mathcal{B}_0^2(\mathbb{D})$, where

$$\varphi_{(z,1)^{-1}}(w) = \left(\beta\left((z,1)^{-1}\right)\varphi\right)(w) = \frac{1-|z|^2}{(1-\overline{w}z)^2}.$$

We can say more about the (left) $\mathbb{B}-$invariant Radon measure σ_φ on the LCH topological space $\mathbb{B}/Y$. Notice that the left action $\mathbb{B} \curvearrowright \mathbb{B}/Y$ by left multiplication is defined by

$$\begin{aligned} \left(z, e^{i\theta}\right) * [(z',1)] &= \left[\left(z, e^{i\theta}\right) * (z',1)\right] \\ &= \left[\left(\frac{z+z'}{1+z\overline{z'}}, e^{i\theta}\frac{1+z\overline{z'}}{1+\overline{z}z'}\right)\right] \\ &= \left[\left(e^{i\theta}\frac{z+z'}{1+\overline{z}z'}, 1\right)\right] \\ &= \left[\left(\psi\left(z'\right),1\right)\right], \end{aligned}$$

where $\psi \in \mathrm{Aut}(\mathbb{D})$ is given as a composition of $\psi_1, \psi_2 \in \mathrm{Aut}(\mathbb{D})$ as shown below

$$\psi_1(w) = e^{i\theta}\frac{z-w}{1-\overline{z}w}, \quad \psi_2(w) = -w, \quad \psi(w) = \psi_1 \circ \psi_2(w) = e^{i\theta}\frac{z+w}{1+\overline{z}w}.$$

So, the left $\mathbb{B}-$invariance of σ_φ on $\mathbb{B}/Y$ can be understood as the left invariance of a Radon measure on the LCH topological space $\mathbb{D}$ under automorphisms of the disc, i.e., under Möbius transformations. Such measure is unique [24, Theorem 7.2] and it is (up to a constant) the Möbius invariant (hyperbolic) measure τ_1 on $\mathbb{D}$.

Observe that the normalized reproducing kernel for $\mathcal{B}_0^2(\mathbb{D})$ given in Corollary 1.29 is

$$k_z^0(w) = \frac{1-|z|^2}{(1-\overline{z}w)^2},$$

thus, for any $f \in \mathcal{B}_0^2(\mathbb{D})$ it holds

$$\|f\|^2_{\mathcal{B}_0^2(\mathbb{D})} = \int_{\mathbb{B}/Y}\left|\left\langle f, \varphi_{(z,1)^{-1}}\right\rangle_{\mathcal{B}_0^2(\mathbb{D})}\right|^2 d\sigma_\varphi\left([z,1]^{-1}\right) = \int_{\mathbb{D}}\left|\left\langle f, k_z^0\right\rangle_{\mathcal{B}_0^2(\mathbb{D})}\right|^2 d\lambda(z),$$

where $d\lambda(z)$ is (up to a constant) the Möbius invariant (hyperbolic) measure on $\mathbb{D}$.

Therefore $\left\{\varphi_{(z,1)^{-1}}\right\}_{[(z,1)^{-1}]\in(\mathbb{B}/Y,\sigma_{\varphi})} \subseteq \mathcal{B}_0^2(\mathbb{D})$ and $\left\{k_z^0\right\}_{z\in(\mathbb{D},\lambda)} \subseteq \mathcal{B}_0^2(\mathbb{D})$ are the same generalized Parseval frame for $\mathcal{B}_0^2(\mathbb{D})$. $\square$

Chapter 2

Concentration operators

Assume $(\mathcal{H}, X, k)$ is a framed Hilbert space, where the generalized frame attached to $\mathcal{H}$, $\{k_x\}_{x\in X} \subseteq \mathcal{H}$, is normalized, i.e., $\|k_x\| = 1$ for all $x \in X$. Furthermore, assume the index set (X, d, λ) is a metric measure space satisfying:

(S1) (X, d) is a locally compact and complete metric space.

(S2) (X, d) is a *length* metric space, which means that for any $x, y \in X$ it holds

$$d(x, y) = \inf \{\text{length}(\sigma) : \sigma \text{ is a rectifiable curve from } x \text{ to } y\}.$$

Recall that a *rectifiable curve* from x to y is a continuous function $\sigma : [0,1] \to X$ such that $\sigma(0) = x$, $\sigma(1) = y$ and

$$\text{length}(\sigma) = \inf \left\{ \sum_{i=1}^{n} d\left(\sigma(t_i), \sigma(t_{i-1})\right) : 0 = t_0 < t_1 < \cdots < t_n = 1 \text{ is a partition of } [0,1] \right\} < \infty.$$

Remark. As a consequence of S1 and S2, (X, d) is a *proper* metric space, which means that any closed ball is compact. Moreover, again as a consequence of S1 and S2, the closure of any open ball in X is the corresponding closed ball.

(S3) (X, λ) is a measure space with Radon measure λ such that any sphere (boundary of a ball) of the metric space (X, d) has measure zero, and two balls with equal radius have the same measure regardless of the center.

Remark. Notice that for any $z, z' \in X$, for any $r \geq 0$ it holds

$$\lambda\left(\overline{B(z;r)}\right) = \lambda\left(B(z;r)\right) = \lambda\left(B(z';r)\right).$$

2.1 The concentration operator

Definition 2.1. For any compact subset $\Omega \subseteq X$, the *concentration operator* on Ω with respect to the generalized frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$, denoted by C_Ω, is the map defined by the integral in the weak sense

$$\begin{aligned} C_\Omega : \mathcal{H} &\rightarrow \mathcal{H} \\ f &\mapsto \int_\Omega \langle f, k_x \rangle k_x d\lambda(x). \end{aligned}$$

Remark. A concentration operator can be understood as a Toeplitz operator (cf. Chapter 3) since $C_\Omega(f) = \int_\Omega \langle f, k_x \rangle k_x d\lambda(x) = \int_X \chi_\Omega(x) \langle f, k_x \rangle k_x d\lambda(x)$, where χ_Ω denotes the characteristic function on Ω. Also, a disclaimer about the notation, when $\Omega = \overline{B}$, where $B = B(a;R)$ is a ball in (X,d,λ), we often use C_B instead of the more appropriate $C_{\overline{B}}$, and there is no harm in doing so since we assume $\lambda(\partial B) = 0$.

Proposition 2.1. *The concentration operator $C_\Omega : \mathcal{H} \rightarrow \mathcal{H}$ with respect to the generalized frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$, is a positive self-adjoint bounded linear operator with $\|C_\Omega\| \leq \beta$.*

Proof. First, we will show C_Ω is linear. For any $f_1, f_2 \in \mathcal{H}$ and any $\theta \in \mathbb{C}$

$$\begin{aligned} C_\Omega(f_1 + \theta f_2) &= \int_\Omega \langle f_1 + \theta f_2, k_x \rangle k_x d\lambda(x) \\ &= \int_\Omega \langle f_1, k_x \rangle k_x d\lambda(x) + \theta \int_\Omega \langle f_2, k_x \rangle k_x d\lambda(x) \\ &= C_\Omega(f_1) + \theta C_\Omega(f_2). \end{aligned}$$

Second, we will use the identity valid on any Hilbert space $\|x\| = \sup_{\|y\|\leq 1} |\langle x, y \rangle|$ to prove C_Ω is bounded. Notice that for any $f, g \in \mathcal{H}$, by classical inequalities of integrals, Cauchy-Schwarz inequality on $L^2(X, \lambda)$, and the generalized frame definition imply

$$\begin{aligned}
|\langle C_\Omega f, g\rangle| &= \left|\left\langle \int_\Omega \langle f, k_x\rangle k_x d\lambda(x), g\right\rangle\right| \\
&= \left|\int_\Omega \langle \langle f, k_x\rangle k_x, g\rangle \, d\lambda(x)\right| \\
&= \left|\int_\Omega \langle f, k_x\rangle \langle k_x, g\rangle \, d\lambda(x)\right| \\
&\leq \int_\Omega |\langle f, k_x\rangle| \, |\langle k_x, g\rangle| \, d\lambda(x) \\
&= \int_X |\chi_\Omega(x) \langle f, k_x\rangle| \, |\chi_\Omega(x) \langle k_x, g\rangle| \, d\lambda(x) \\
&= \langle |\chi_\Omega(x) \langle f, k_x\rangle| \, , |\chi_\Omega(x) \langle k_x, g\rangle|\rangle_{L^2} \\
&\leq \left(\int_X |\chi_\Omega(x) \langle f, k_x\rangle|^2 \, d\lambda(x)\right)^{\frac{1}{2}} \left(\int_X |\chi_\Omega(x) \langle g, k_x\rangle|^2 \, d\lambda(x)\right)^{\frac{1}{2}} \\
&\leq \left(\int_X |\langle f, k_x\rangle|^2 \, d\lambda(x)\right)^{\frac{1}{2}} \left(\int_X |\langle g, k_x\rangle|^2 \, d\lambda(x)\right)^{\frac{1}{2}} \\
&\leq \left(\beta\|f\|^2\right)^{\frac{1}{2}} \left(\beta\|g\|^2\right)^{\frac{1}{2}} \\
&= \beta\|f\|\|g\|.
\end{aligned}$$

Then, C_Ω is a bounded operator with $\|C_\Omega\| \leq \beta$ because for any $f \in \mathcal{H}$ it holds

$$\|C_\Omega f\| = \sup_{\|g\|\leq 1} |\langle C_\Omega f, g\rangle| \leq \beta\|f\|.$$

Next, we will show C_Ω is self-adjoint. For any $f, g \in \mathcal{H}$

$$\begin{aligned}
\langle C_\Omega f, g\rangle &= \left\langle \int_\Omega \langle f, k_x\rangle k_x d\lambda(x), g\right\rangle \\
&= \int_\Omega \langle f, k_x\rangle \langle k_x, g\rangle \, d\lambda(x) \\
&= \int_\Omega \left\langle f, \overline{\langle k_x, g\rangle} k_x\right\rangle d\lambda(x) \\
&= \int_\Omega \langle f, \langle g, k_x\rangle k_x\rangle \, d\lambda(x) \\
&= \left\langle f, \int_\Omega \langle g, k_x\rangle k_x d\lambda(x)\right\rangle \\
&= \langle f, C_\Omega g\rangle .
\end{aligned}$$

Finally, C_Ω is positive because for any $f \in \mathcal{H}$

$$\begin{aligned} \langle C_\Omega f, f \rangle &= \int_\Omega \langle f, k_x \rangle \langle k_x, f \rangle \, d\lambda(x) \\ &= \int_\Omega |\langle f, k_x \rangle|^2 \, d\lambda(x) \\ &\geq 0. \end{aligned}$$

This completes the proof. □

Definition 2.2. Given a bounded linear map $A : \mathcal{H} \to \mathcal{H}$, the *singular values* associated to A are the eigenvalues of $(A^*A)^{\frac{1}{2}}$.

Remark. Notice that $(A^*A)^{\frac{1}{2}}$ is a positive self-adjoint bounded linear operator, so, any singular value of A has to be a nonnegative real number. In particular, if the singular values of A form a bounded sequence $\{\lambda_i\}_{i=1}^{\infty}$, without loss of generality we can write them as $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$.

Definition 2.3. Given a linear bounded map $A : \mathcal{H} \to \mathcal{H}$, we say A is *compact* if there exists a sequence of linear bounded maps with finite rank $\{A_n : \mathcal{H} \to \mathcal{H}\}_{n=1}^{\infty}$ such that $A_n \xrightarrow{\|\cdot\|} A$ as $n \to \infty$.

Remark. It is well-known that the eigenvalues of any compact operator form a countable set in $\mathbb{C}$ which is bounded and zero is the only possible accumulation point.

Definition 2.4. Given a compact operator $A : \mathcal{H} \to \mathcal{H}$ such that its singular values $\{\lambda_i\}_{i=1}^{\infty} \in \ell_p$ for some $1 \leq p \leq \infty$, define the *Schatten p−norm* of A, denoted by $\|A\|_{S_p}$, by the ℓ_p−norm of its singular values

$$\|A\|_{S_p} := \|\{\lambda_i\}_{i=1}^{\infty}\|_{\ell_p}.$$

The set of all compact operators on $\mathcal{H}$ with finite Schatten p−norm form a Banach space with respect to this norm, called the *Schatten p−space* and denoted by S_p.

Remark. It is well-known that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\infty$, where S_1 are called the *trace class* operators, S_2 are called the *Hilbert-Schmidt* class operators, and S_∞ denotes the *compact* operators.

Proposition 2.2. *The concentration operator $C_\Omega : \mathcal{H} \to \mathcal{H}$ with respect to the generalized frame $\{k_x\}_{x \in (X,d,\lambda)} \subseteq \mathcal{H}$, is a compact operator.*

Proof. By compactness of Ω, the continuous function $k : x \mapsto k_x$ generating the generalized frame

$\{k_x\}_{x\in X} \subseteq \mathcal{H}$ restricted to Ω is uniformly continuous

$$\begin{aligned} k|_{\Omega} : \Omega &\to \mathcal{H} \\ x &\mapsto k_x. \end{aligned}$$

So, for any $\varepsilon > 0$. there exists $\delta > 0$ such that for all $x, x' \in \Omega$ satisfying $d(x, x') < \delta$ it holds $\|k_x - k_{x'}\| < \frac{\varepsilon}{2}$. Again, since Ω is compact, Ω is totally bounded, so there exists a finite set $\{x_1, x_2, \ldots, x_m\} \subseteq \Omega$ such that $\Omega \subseteq \bigcup_{j=1}^{m} B\left(x_j; \frac{\delta}{4}\right)$. Consider the following construction

$$\begin{aligned} E_1 &= \Omega \cap B\left(x_1; \frac{\delta}{4}\right) \\ E_2 &= \Omega \cap \left[B\left(x_2; \frac{\delta}{4}\right) \backslash E_1\right] \\ &\vdots \\ E_m &= \Omega \cap \left[B\left(x_m; \frac{\delta}{4}\right) \backslash \bigcup_{j=1}^{n-1} E_j\right]. \end{aligned}$$

Then, $E_1, E_2, \ldots, E_m$ form a partition of Ω such that $\operatorname{diam}(E_j) \leq \frac{\delta}{2} < \delta$ for all $j \in \{1, 2, \ldots, m\}$. Let $x \in E_j$ for some $j \in \{1, 2, \ldots, m\}$, by the previous construction $\|k_x - k_{x_j}\| < \frac{\varepsilon}{2}$. Hence, using triangle inequality, Cauchy-Schwarz inequality, and $\|k_x\| = \|k_{x_j}\| = 1$, for any $f \in \mathcal{H}$ we get

$$\begin{aligned} \left\|\langle f, k_x\rangle k_x - \langle f, k_{x_j}\rangle k_{x_j}\right\| &= \left\|\langle f, k_x\rangle k_x - \langle f, k_{x_j}\rangle k_x + \langle f, k_{x_j}\rangle k_x - \langle f, k_{x_j}\rangle k_{x_j}\right\| \\ &\leq \left\|\langle f, k_x - k_{x_j}\rangle k_x\right\| + \left\|\langle f, k_{x_j}\rangle \left(k_x - k_{x_j}\right)\right\| \\ &= \left|\langle f, k_x - k_{x_j}\rangle\right| \|k_x\| + \left|\langle f, k_{x_j}\rangle\right| \left\|k_x - k_{x_j}\right\| \\ &\leq \|f\| \left\|k_x - k_{x_j}\right\| \|k_x\| + \|f\| \left\|k_{x_j}\right\| \left\|k_x - k_{x_j}\right\| \\ &< \varepsilon \|f\|. \end{aligned}$$

Define the following bounded linear map with finite rank

$$\begin{aligned} C_{\Omega,\varepsilon} : \mathcal{H} &\to \mathcal{H} \\ f &\mapsto \sum_{j=1}^{m} \lambda(E_j) \langle f, k_{x_j}\rangle k_{x_j}. \end{aligned}$$

This map has finite rank because by construction $\text{range}\,(C_{\Omega,\varepsilon}) \subseteq \text{span}\,\{k_{x_1}, k_{x_2}, \ldots, k_{x_m}\}$, where $\text{span}\,\{k_{x_1}, k_{x_2}, \ldots, k_{x_m}\}$ is a finite dimensional subspace of $\mathcal{H}$. Also, by the construction of the partition of Ω

$$\begin{aligned} C_\Omega(f) &= \int_\Omega \langle f, k_x\rangle\, k_x d\lambda(x) = \sum_{j=1}^m \int_{E_j} \langle f, k_x\rangle\, k_x d\lambda(x), \\ C_{\Omega,\varepsilon}(f) &= \sum_{j=1}^m \lambda\,(E_j)\,\langle f, k_{x_j}\rangle\, k_{x_j} = \sum_{j=1}^m \int_{E_j} \langle f, k_{x_j}\rangle\, k_{x_j} d\lambda(x). \end{aligned}$$

Then

$$\begin{aligned} \|(C_\Omega - C_{\Omega,\varepsilon})\,(f)\| &= \left\| \sum_{j=1}^m \int_{E_j} \left[\langle f, k_x\rangle\, k_x - \langle f, k_{x_j}\rangle\, k_{x_j}\right] d\lambda(x) \right\| \\ &\le \sum_{j=1}^m \left\| \int_{E_j} \left[\langle f, k_x\rangle\, k_x - \langle f, k_{x_j}\rangle\, k_{x_j}\right] d\lambda(x) \right\| \\ &\le \sum_{j=1}^m \int_{E_j} \left\|\langle f, k_x\rangle\, k_x - \langle f, k_{x_j}\rangle\, k_{x_j}\right\| d\lambda(x) \\ &< \sum_{j=1}^m \int_{E_j} \varepsilon\,\|f\|\, d\lambda(x) \\ &= \varepsilon\,\|f\| \sum_{j=1}^m \lambda\,(E_j) \\ &= \varepsilon\lambda\,(\Omega)\,\|f\|\,. \end{aligned}$$

From here, the operator norm of $C_\Omega - C_{\Omega,\varepsilon}$ satisfies

$$\|C_\Omega - C_{\Omega,\varepsilon}\| \le \varepsilon\lambda\,(\Omega)\,.$$

Since ε is arbitrary, taking for example $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, we have constructed a sequence of bounded linear operators on $\mathcal{H}$ with finite rank such that $C_{\Omega,\frac{1}{n}} \xrightarrow{\|\cdot\|} C_\Omega$ as $n \to \infty$. Therefore, C_Ω is a compact operator. □

Proposition 2.3. *The concentration operator* $C_\Omega : \mathcal{H} \to \mathcal{H}$ *with respect to the generalized frame* $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$*, is a trace class and thus a Hilbert-Schmidt class operator whenever* $\{k_x\}_{x\in(X,d,\lambda)}$ *is bounded. It holds*

$$\begin{aligned}\|C_\Omega\|_{S_1} &= \int_\Omega \|k_x\|^2 \, d\lambda(x), \\ \|C_\Omega\|_{S_2}^2 &= \int_\Omega \int_\Omega |\langle k_x, k_y\rangle|^2 \, d\lambda(x) d\lambda(y).\end{aligned}$$

Furthermore, if $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ *is a normalized generalized Parseval frame, it also holds*

$$\|C_\Omega\|_{S_1} = \int_X \int_\Omega |\langle k_x, k_y\rangle|^2 \, d\lambda(x) d\lambda(y) = \lambda(\Omega).$$

Proof. We will prove C_Ω is a trace class operator, which will imply that it is also a Hilbert-Schmidt class operator. Since C_Ω is self-adjoint, then $(C_\Omega^* C_\Omega)^{\frac{1}{2}} = C_\Omega$ and the singular values of C_Ω coincide with the eigenvalues of C_Ω.

Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues (and the singular values) of the positive self-adjoint operator C_Ω, by the spectral theorem applied on C_Ω, there exists an orthonormal basis $\{f_n\}_{n=1}^\infty \subseteq \mathcal{H}$ of eigenfunctions of C_Ω such that $C_\Omega f_n = \lambda_n f_n$ for all n. Before going further, it is important to emphasize that both the eigenvalues and the eigenfunctions of C_Ω depend on Ω, so, if we need to keep track of this dependence we will write instead $\lambda_n(\Omega)$ and $f_n(\Omega)$, $n = 1, 2, \ldots$

Thus, for $1 \le p < \infty$

$$\begin{aligned}\left(\sum_{n=1}^\infty |\lambda_n|^p\right)^{\frac{1}{p}} &= \left(\sum_{n=1}^\infty |\langle \lambda_n f_n, f_n\rangle|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^\infty |\langle C_\Omega f_n, f_n\rangle|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^\infty \langle C_\Omega f_n, f_n\rangle^p\right)^{\frac{1}{p}}.\end{aligned}$$

If the later expression is finite, then $C_\Omega \in S_p$ and $\|C_\Omega\|_{S_p} = (\sum_{n=1}^\infty \langle C_\Omega f_n, f_n\rangle^p)^{\frac{1}{p}}$.

We analyze the case $p = 1$. Recall that $\Omega \subseteq X$ is compact, so, $\lambda(\Omega) < \infty$. Using Tonelli's

Theorem and the Parseval identity with respect to the orthonormal basis $\{f_n\}$ we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} \langle C_\Omega f_n, f_n \rangle &= \sum_{i=1}^{\infty} \left(\int_\Omega |\langle f_n, k_x \rangle|^2 \, d\lambda(x) \right) \\
&= \int_\Omega \left(\sum_{i=1}^{\infty} |\langle f_n, k_x \rangle|^2 \right) d\lambda(x) \\
&= \int_\Omega \|k_x\|^2 d\lambda(x) \\
&\leq c\lambda(\Omega) \\
&< \infty.
\end{aligned}$$

If $\{k_y\}_{y \in (X,d,\lambda)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame, then

$$\|C_\Omega\|_{S_1} = \lambda(\Omega) = \int_X \int_\Omega |\langle k_x, k_y \rangle|^2 \, d\lambda(x) d\lambda(y).$$

It only remains to calculate the Schatten 2−norm of C_Ω. First, for a fixed n

$$\begin{aligned}
|\lambda_n|^2 &= \langle C_\Omega f_n, C_\Omega f_n \rangle \\
&= \left\langle \int_\Omega \langle f_n, k_x \rangle \, k_x d\lambda(x), \int_\Omega \langle f_n, k_y \rangle \, k_y d\lambda(y) \right\rangle \\
&= \int_\Omega \left\langle \langle f_n, k_x \rangle \, k_x, \int_\Omega \langle f_n, k_y \rangle \, k_y d\lambda(y) \right\rangle d\lambda(x) \\
&= \int_\Omega \int_\Omega \langle \langle f_n, k_x \rangle \, k_x, \langle f_n, k_y \rangle \, k_y \rangle \, d\lambda(y) d\lambda(x) \\
&= \int_\Omega \int_\Omega \overline{\langle k_x, f_n \rangle \, \langle f_n, k_y \rangle} \, \langle k_x, k_y \rangle \, d\lambda(y) d\lambda(x).
\end{aligned}$$

Then, taking the sum over n and using again Tonelli's theorem

$$\begin{aligned}
\|C_\Omega\|_{S_2}^2 &= \sum_{n=1}^{\infty}|\lambda_n|^2 \\
&= \sum_{n=1}^{\infty}\int_\Omega\int_\Omega \overline{\langle k_x, f_n\rangle\langle f_n, k_y\rangle}\,\langle k_x, k_y\rangle\, d\lambda(y)d\lambda(x) \\
&= \int_\Omega\int_\Omega \sum_{n=1}^{\infty}\overline{\langle k_x, f_n\rangle\langle f_n, k_y\rangle}\,\langle k_x, k_y\rangle\, d\lambda(y)d\lambda(x) \\
&= \int_\Omega\int_\Omega \overline{\left\langle \sum_{n=1}^{\infty}\langle k_x, f_n\rangle f_n, \sum_{m=1}^{\infty}\langle k_y, f_m\rangle f_m\right\rangle}\,\langle k_x, k_y\rangle\, d\lambda(y)d\lambda(x) \\
&= \int_\Omega\int_\Omega \overline{\langle k_x, k_y\rangle}\,\langle k_x, k_y\rangle\, d\lambda(y)d\lambda(x) \\
&= \int_\Omega\int_\Omega |\langle k_x, k_y\rangle|^2\, d\lambda(y)d\lambda(x).
\end{aligned}$$

This completes the proof. □

Remark. It is interesting that if μ is the counting measure on $\mathbb{N} \subseteq X$, i.e., $\mu(A) := \#(A \cap \mathbb{N})$ for any measure set $A \subseteq X$, then the orthonormal basis used to calculate the trace of C_Ω becomes a generalized Parseval frame $\{f_x\}_{x\in(X,\mu)} \subseteq \mathcal{H}$, and the following relationship holds assuming $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ is also a normalized generalized Parseval frame

$$\int_X \langle C_\Omega f_x, f_x\rangle\, d\mu(x) = \lambda(\Omega) = \int_X \langle C_\Omega k_x, k_x\rangle\, d\lambda(x).$$

2.2 Localization property and Schatten 1,2-norms

The concept of generalized frames encompasses so many different objects, some of them having rather different properties, that it is unlikely to obtain interesting results that remain valid for all o them. For this reason, we will impose certain *localization conditions* on the generalized frames. Nevertheless, the class of generalized frames satisfying such localization conditions is very large, and in most of the applications it is a natural assumption.

Furthermore, in this subsection we investigate a connection between a normalized generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ satisfying the condition (F) described below, and certain inequalities between the Schatten $1,2-$norms of a concentration operator C_Ω defined using such generalized frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$.

Intuitively, if the generalized Parseval frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ satisfies (F), then $\langle k_x, k_y\rangle$ decays very rapidly as x moves away from y, in a broad sense mimicking the behavior of an orthonormal set.

(F) For any $\varepsilon > 0$ there exists a $R > 0$ such that for all $r \geq R$ and all $z \in X$

$$\frac{1}{\lambda\left(B(z;r)\right)} \int_{B(z;r)^c} \int_{B(z;r)} |\langle k_x, k_y\rangle|^2 d\lambda(x) d\lambda(y) < \varepsilon.$$

Proposition 2.4. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Then, for any $z \in X$ and any $0 < \delta < 1$ there exists $R > 0$ independent on z such that for all $r \geq R$ it holds*

$$(1-\delta)\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2 \leq \|C_\Omega\|_{S_1},$$

where $\Omega = B(z;r)$ and $z \in X$ is arbitrary.

Remark. The proposition is stated as it is needed for future applications, however some clarifications are needed. On the one hand, the inequality $\|C_\Omega\|_{S_2}^2 \leq \|C_\Omega\|_{S_1}$ is in fact true for all $\Omega = B(z;r)$ and it does not require the assumption that the generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ satisfies the localization property (F). On the other hand, the localization property (F) is equivalent to the other inequality $(1-\delta)\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2$ for all $\Omega = B(z;r)$ with $r \geq R$, where $R > 0$ is a large enough radius depending on δ.

Proof. The inequality $\|C_\Omega\|_{S_2}^2 \leq \|C_\Omega\|_{S_1}$ is valid for all $\Omega = B(z;r)$ due to Proposition 2.3

$$\begin{aligned}
\|C_\Omega\|_{S_2}^2 &= \int_\Omega \int_\Omega |\langle k_x, k_y\rangle|^2 \, d\lambda(x) d\lambda(y) \\
&\leq \int_\Omega \int_X |\langle k_x, k_y\rangle|^2 d\lambda(y) d\lambda(x) \\
&= \|C_\Omega\|_{S_1}.
\end{aligned}$$

The other inequality, $(1-\delta)\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2$ for all $\Omega = B(z;r)$ with $r \geq R$, where $R > 0$ is a large enough radius depending on δ, will be proved to be equivalent to the localization property (F).

By Tonelli's Theorem and Proposition 2.3

$$\begin{aligned}
\frac{\|C_\Omega\|_{S_1}-\|C_\Omega\|_{S_2}^2}{\|C_\Omega\|_{S_1}} &= \frac{1}{\lambda(\Omega)}\left[\int_\Omega\int_X |\langle k_x,k_y\rangle|^2 d\lambda(y)d\lambda(x) - \int_\Omega\int_\Omega |\langle k_x,k_y\rangle|^2 d\lambda(x)d\lambda(y)\right] \\
&= \frac{1}{\lambda(\Omega)}\left[\int_X\int_\Omega |\langle k_x,k_y\rangle|^2 d\lambda(x)d\lambda(y) - \int_\Omega\int_\Omega |\langle k_x,k_y\rangle|^2 d\lambda(x)d\lambda(y)\right] \\
&= \frac{1}{\lambda(\Omega)}\int_{\Omega^c}\int_\Omega |\langle k_x,k_y\rangle|^2 d\lambda(x)d\lambda(y).
\end{aligned}$$

So, if the localization property (F) is valid, we can choose $R>0$ large enough such that for all $r \geq R$ and all $z \in X$ it holds

$$\frac{1}{\lambda(B(z;r))}\int_{B(z;r)^c}\int_{B(z;r)} |\langle k_x,k_y\rangle|^2 d\lambda(x)d\lambda(y) \quad < \quad \delta,$$

hence

$$\frac{\|C_\Omega\|_{S_1}-\|C_\Omega\|_{S_2}^2}{\|C_\Omega\|_{S_1}} \quad < \quad \delta$$

for all such balls. Therefore $(1-\delta)\,\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2$ for all $\Omega = B(z;r)$ with $r \geq R$. Clearly, this last reasoning works on the opposite direction, if there exists $R>0$ such that $(1-\delta)\,\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2$ for all $\Omega = B(z;r)$ with $r \geq R$, then the localization property (F) is valid. □

2.3 Useful inequalities

In this section we derive some useful inequalities valid in a Hilbert space $\mathcal{H}$ with multiple generalized frames defined in it. The index sets for the generalized frames are the measure spaces (X,μ) and (Y,ν) satisfying the usual assumptions.

Theorem 2.5. *Let $\mathcal{F} \subseteq \mathcal{H}$ be a closed subspace. Let $\{f_x\}_{x\in(X,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$ such that $\left\langle f_x, \widetilde{f_x}\right\rangle \leq 1$ for all $x \in \operatorname{supp}(\mu)$, and let $\{g_y\}_{y\in(Y,\nu)} \subseteq \mathcal{H}$ be a generalized Parseval frame for $\mathcal{H}$. Given $\Omega \subseteq X$, consider the concentration operator $C_\Omega : \mathcal{H} \to \mathcal{H}$ defined by $C_\Omega f = \int_\Omega \langle f, f_x\rangle\, \widetilde{f_x} d\mu(x)$, and the measurable function $Q(y) = \langle C_\Omega g_y, g_y\rangle$, $y \in Y$. If $\varepsilon > 0$ then*

$$\nu\left\{Q(y) \geq \varepsilon\right\} \leq \frac{\mu(\Omega)}{\varepsilon}.$$

Remark. $Q(y)$ is the so called *Berezin transform* associated to the concentration operator C_Ω with respect to $\{g_y\}_{y\in(Y,\nu)}$.

Proof. Since $\{f_x\}_{x\in(X,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F} \subseteq X$, the concentration operator C_Ω is a bounded, self-adjoint and positive operator. On the other hand, since $\{g_y\}_{y\in(Y,\nu)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$ and Tonelli's Theorem

$$\begin{aligned}
\int_Y Q(y)d\nu(y) &= \int_Y \left\langle \int_\Omega \langle g_y, f_x\rangle \widetilde{f}_x d\mu(x), g_y \right\rangle d\nu(y) \\
&= \int_Y \int_\Omega \langle g_y, f_x\rangle \left\langle \widetilde{f}_x, g_y \right\rangle d\mu(x)d\nu(y) \\
&= \int_\Omega \int_Y \left\langle \widetilde{f}_x, g_y \right\rangle \langle g_y, f_x\rangle d\nu(y)d\mu(x) \\
&= \int_\Omega \left\langle \widetilde{f}_x, f_x \right\rangle d\mu(x) \\
&\leq \mu(\Omega),
\end{aligned}$$

where the last inequality is due to the assumption $\left\langle f_x, \widetilde{f}_x \right\rangle \leq 1$ for all $x \in \operatorname{supp}(\mu)$. Applying Chebyshev's inequality we obtain the result

$$\nu\{Q(y) \geq \varepsilon\} \leq \frac{1}{\varepsilon}\int_Y Q(y)d\nu(y) \leq \frac{1}{\varepsilon}\mu(\Omega).$$

□

Corollary 2.6. *Let $\mathcal{F} \subseteq \mathcal{H}$ be a finite dimensional subspace. Let $\{g_i\}_{i=1}^\infty \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$. Consider $P_\mathcal{F}$, the orthogonal projection onto $\mathcal{F}$. If $\varepsilon > 0$ then*

$$\#\left\{g_i : \|P_\mathcal{F} g_i\|^2 \geq \varepsilon\right\} \leq \frac{\dim \mathcal{F}}{\varepsilon}.$$

Proof. Let $\{f_i\}_{i\in N} \subseteq \mathcal{F}$, $N = \{1, 2, \ldots, \dim \mathcal{F}\}$, be an orthonormal basis for $\mathcal{F}$, then $\{f_x\}_{x\in(X,\mu)} \subseteq \mathcal{F}$ is a generalized frame for $\mathcal{F}$ taking μ to be the counting measure on $N \subseteq X$ (X is a measure space containing N), i.e., $\mu(A) = \#(A \cap N)$ for any measurable set $A \subseteq X$. Similarly, $\{g_y\}_{y\in(Y,\nu)} \subseteq \mathcal{H}$, is a generalized Parseval frame for $\mathcal{H}$ taking ν to be the counting measure on $\mathbb{N} = \{1, 2, \ldots\} \subseteq Y$ (Y is a measure space containing $\mathbb{N}$). Notice that $\left\langle f_i, \widetilde{f}_i \right\rangle = 1$ for all $i \in \operatorname{supp}(\mu) = N$. Observe that the concentration operator C_Ω becomes $P_\mathcal{F}$ whenever $N \subseteq \Omega$ since for any $f \in \mathcal{H}$

$$C_\Omega f = \int_\Omega \langle f, f_x \rangle \widetilde{f}_x d\mu(x) = \sum_{i=1}^{\dim \mathcal{F}} \langle f, f_i \rangle f_i = P_\mathcal{F} f.$$

Also notice that $\|P_\mathcal{F} g\|^2 = \langle P_\mathcal{F} g, g \rangle$ for any $g \in \mathcal{H}$ by properties of orthogonal projections.

The conclusion follows from Theorem 2.5

$$\# \left\{ g_i : \|P_\mathcal{F} g_i\|^2 \geq \varepsilon \right\} = \nu \left\{ \langle C_\Omega g_y, g_y \rangle \geq \varepsilon \right\} \leq \frac{\mu(\Omega)}{\varepsilon} = \frac{\dim \mathcal{F}}{\varepsilon}.$$

□

2.4 Asymptotic behavior

In this section we assume $\{k_x\}_{x \in X} \subseteq \mathcal{H}$ is a generalized Parseval frame satisfying the localization property (F).

(F) For any $\varepsilon > 0$ there exists a $R > 0$ such that for all $r \geq R$ and all $z \in X$

$$\frac{1}{\lambda(B(z;r))} \int_{B(z;r)^c} \int_{B(z;r)} |\langle k_x, k_y \rangle|^2 d\lambda(x) d\lambda(y) < \varepsilon.$$

In order make sense of the next two theorems, recall that since $\{k_x\}_{x \in (X,d,\lambda)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$, by Theorem 1.21

$$f \quad = \quad \int_X \langle f, k_x \rangle k_x d\lambda(x), \quad f \in \mathcal{H}.$$

The definition of the concentration operator on $B = B(a;r)$ gives

$$C_B f \quad = \quad \int_B \langle f, k_x \rangle k_x d\lambda(x), \quad f \in \mathcal{H},$$

so, a natural questions to ask is *how close* is $C_B f$ to f in some sense. From the previous relationships, it is clear that for all $f \in \mathcal{H}$ it holds

$$\|f\|^2 = \int_X |\langle f, k_x \rangle|^2 d\lambda(x) \quad = \quad \langle C_B f, f \rangle + \int_{B^c} |\langle f, k_x \rangle|^2 d\lambda(x),$$

so, one way to interpret that $C_B f$ is close to f is say that the above integral on B^c is very small. This is stated precisely in the following definition.

Definition 2.5. Given $0 < \varepsilon < 1$ (we are usually interested on $\varepsilon \ll 1$), we say $f \in \mathcal{H}$ is a $\varepsilon-$*concentrated function* with respect to B if $\langle C_B f, f\rangle \geq (1-\varepsilon)\|f\|^2$. Given a subspace $\mathcal{F} \subseteq \mathcal{H}$, we say $\mathcal{F}$ is an $\varepsilon-$*concentrated subspace* with respect to B if f is $\varepsilon-$concentrated with respect to B, for all nonzero $f \in \mathcal{F}$. On the other hand, we say $\mathcal{F}$ is a *not at all* $\varepsilon-$*concentrated subspace* with respect to B if f is not $\varepsilon-$concentrated with respect to B, for all nonzero $f \in \mathcal{F}$.

Under this convention, Theorem 2.7 (resp. Theorem 2.8) stated below gives the asymptotic behavior, when B varies, of the dimensions of finite dimensional $G \subseteq \mathcal{H}$ such that G is $\varepsilon-$concentrated subspace (resp. $G^{\perp}$ is not at all $\varepsilon-$concentrated subspace) with respect to B.

Theorem 2.7. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Given $0<\varepsilon<1$ and $B := B(a;r)$, let $\eta_1(\varepsilon, B)$ be the maximum dimension of an $\varepsilon-$concentrated subspace $G \subseteq \mathcal{H}$ with respect to B, i.e., $\langle C_B g, g\rangle \geq (1-\varepsilon)\|g\|^2$ for all nonzero $g \in G$. Then*

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_1(\varepsilon, B(a;r))}{\lambda(B(a;r))} \quad \leq \quad 1.$$

Proof. Given $B := B(a;r)$, let $\{\lambda_i(B)\}_{i=1}^{\infty}$ be the eigenvalues of C_B, and $\{f_i(B)\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ such that $f_i(B)$ is an eigenfunction of C_B associated to $\lambda_i(B)$. As we did before, we abbreviate these expressions by λ_i and f_i, and in this case it holds $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Notice the enumeration starts at 1 this time.

Consider a $n-$dimensional subspace G such that $\langle C_B g, g\rangle \geq 1-\varepsilon$ for all $g \in G$ with $\|g\| = 1$. Then $n \leq \eta_1(\varepsilon, B)$. By the Weyl-Courant lemma

$$\begin{aligned}
\lambda_n &= \max_{\{G'\subseteq\mathcal{H}:\dim G'=n\}} \min_{\{g\in G':\|g\|=1\}} \langle C_B g, g\rangle \\
&\geq \min_{\{g\in G:\|g\|=1\}} \langle C_B g, g\rangle \\
&\geq 1-\varepsilon.
\end{aligned}$$

In particular, if G is optimal, i.e., $n = \eta_1(\varepsilon, B)$, then $\lambda_{\eta_1(\varepsilon,B)} \geq 1-\varepsilon$. Choose γ' such that

$1-\varepsilon<\gamma'<1$, then

$$\begin{aligned}\eta_1(\varepsilon,B) &\le \#\{i:\lambda_i\ge 1-\varepsilon\}\\ &= \#\{i:\lambda_i>\gamma'\}+\#\{i:\gamma'\ge\lambda_i\ge 1-\varepsilon\}\\ &\le \sum_{\{i:\lambda_i>\gamma'\}}\frac{\lambda_i}{\gamma'}+\sum_{\{i:\gamma'\ge\lambda_i\ge 1-\varepsilon\}}\frac{\lambda_i}{1-\varepsilon}\\ &\le \frac{1}{\gamma'}\sum_{i=1}^{\infty}\lambda_i+\frac{1}{1-\varepsilon}\sum_{\{i:\lambda_i\le\gamma'\}}\lambda_i.\end{aligned}$$

By Proposition 2.4, for any $0<\delta<1$ there exists $R>0$ large enough such that for all balls $B=B(a;r)$ with $r\ge R$ it holds $(1-\delta)\left\|C_B\right\|_{S_1}\le\left\|C_B\right\|_{S_2}^2$. Then for such balls

$$\begin{aligned}(1-\delta)\sum_{i=1}^{\infty}\lambda_i &\le \sum_{i=1}^{\infty}\lambda_i^2\\ &= \sum_{\{i:\lambda_i>\gamma'\}}\lambda_i^2+\sum_{\{i:\lambda_i\le\gamma'\}}\lambda_i^2\\ &\le \left[\sum_{i=1}^{\infty}\lambda_i-\sum_{\{i:\lambda_i\le\gamma'\}}\lambda_i\right]+\gamma'\sum_{\{i:\lambda_i\le\gamma'\}}\lambda_i,\end{aligned}$$

which implies

$$\sum_{\{i:\lambda_i\le\gamma'\}}\lambda_i\le\frac{\delta}{1-\gamma'}\sum_{i=1}^{\infty}\lambda_i.$$

Hence, for all balls $B(a;r)$ with $r\ge R$ it holds

$$\eta_1(\varepsilon,B)\le\left(\frac{1}{\gamma'}+\frac{\delta}{(1-\varepsilon)(1-\gamma')}\right)\sum_{i=1}^{\infty}\lambda_i.$$

Proposition 2.3 gives $\left\|C_B\right\|_{S_1}=\lambda(B)$. On the other hand, for all $0<\varepsilon_1\ll\max\left\{\frac{1}{2},\varepsilon\right\}$, we can choose γ' close to 1 such that $1-\varepsilon<1-\varepsilon_1<\gamma'<1$, and then we can choose $\delta>0$ small enough such that $\delta<\varepsilon_1(1-\varepsilon)(1-\gamma')$. Then

$$\begin{aligned}\frac{1}{\gamma'}+\frac{\delta}{(1-\varepsilon)(1-\gamma')} &< \frac{1}{1-\varepsilon_1}+\varepsilon_1\\ &= 1+\varepsilon_1\left(\frac{1}{1-\varepsilon_1}+1\right)\\ &< 1+3\varepsilon_1.\end{aligned}$$

So, for any $0 < \varepsilon_1 \ll \max\left\{\frac{1}{2}, \varepsilon\right\}$ there exists an $R > 0$ large enough such that for all balls $B = B(a; r)$ with $r \geq R$ it holds

$$\frac{\eta_1(\varepsilon, B)}{\lambda(B)} \leq \frac{1}{\gamma'} + \frac{\delta}{(1-\varepsilon)(1-\gamma')} < 1 + 3\varepsilon_1.$$

Therefore letting $R \to \infty$, thus $\varepsilon_1 \to 0$, $\gamma' \to 1$, and $\delta \to 0$

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_1(\varepsilon, B(a; r))}{\lambda(B(a; r))} \quad \leq \quad 1.$$

□

Theorem 2.8. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Given $0 < \varepsilon < 1$ and $B := B(a; r)$, let $\eta_2(\varepsilon, B)$ be the minimum dimension of a finite dimensional subspace $G \subseteq \mathcal{H}$ such that its orthogonal complement is not at all $\varepsilon-$concentrated subspace with respect to B, i.e., $\langle C_B g, g\rangle < (1-\varepsilon)\|g\|^2$ for all nonzero $g \in G^{\perp}$. Then*

$$\liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_2(\varepsilon, B(a; r))}{\lambda(B(a; r))} \quad \geq \quad 1.$$

Alternatively, let $\eta_2'(\varepsilon, B)$ be the minimum dimension of G such that $\langle C_B g, g\rangle \leq (1-\varepsilon)\|g\|^2$ for all nonzero $g \in G^{\perp}$. Then

$$\liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_2'(\varepsilon, B(a; r))}{\lambda(B(a; r))} \quad \geq \quad 1.$$

Proof. Given $B := B(a; r)$, let $\{\lambda_i(B)\}_{i=0}^{\infty}$ be the eigenvalues of C_B, and $\{f_i(B)\}_{i=0}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ such that $f_i(B)$ is an eigenfunction of C_B associated to $\lambda_i(B)$. For short we write λ_i and f_i if there is no room for confusion. Recall $1 \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq 0$. Notice the enumeration starts at 0 this time.

Clearly $\eta_2'(\varepsilon, B) \leq \eta_2(\varepsilon, B)$ because the family of finite dimensional subspaces G satisfying $\langle C_B g, g\rangle \leq 1-\varepsilon$ for all $g \in G^{\perp}$ with $\|g\| = 1$ contains the family of finite dimensional subspaces G satisfying $\langle C_B g, g\rangle < 1-\varepsilon$ for all $g \in G^{\perp}$ with $\|g\| = 1$. Then, it is enough to consider the case of $\eta_2'(\varepsilon, B)$.

Consider a $n-$dimensional subspace G such that $\langle C_B g, g\rangle \leq 1-\varepsilon$ for all $g \in G^{\perp}$ with $\|g\| = 1$, then $n \geq \eta_2'(\varepsilon, B)$. If $\{g_1, \ldots, g_n\}$ is a basis for G, by the rank-nullity Theorem there exists a nontrivial solution to the linear system

$$\sum_{i=0}^{n} \langle f_i, g_j \rangle \, t_i = 0, \quad j = 1, \ldots, n.$$

Such nontrivial solution, say $\widehat{t_0}, \ldots, \widehat{t_n} \in \mathbb{C}$, determines a nonzero function $\widehat{g} = \sum_{i=0}^{n} \widehat{t_i} f_i$ such that $\langle \widehat{g}, g_j \rangle = 0$ for all $j = 1, \ldots, n$, so $\widehat{g} \in G^{\perp}$. It is clear that $\widehat{g}$ can be chosen such that $\|\widehat{g}\|^2 = \sum_{i=0}^{n} \left|\widehat{t_i}\right|^2 = 1$. By the assumption on G

$$\begin{aligned} 1 - \varepsilon &\geq \langle C_B \widehat{g}, \widehat{g} \rangle \\ &= \left\langle \sum_{i=0}^{n} \lambda_i \widehat{t_i} f_i, \sum_{k=0}^{n} \widehat{t_k} f_k \right\rangle \\ &= \sum_{i=0}^{n} \left|\widehat{t_i}\right|^2 \lambda_i \\ &\geq \lambda_n \sum_{i=0}^{n} \left|\widehat{t_i}\right|^2 \\ &= \lambda_n. \end{aligned}$$

In particular if G is optimal, i.e. $n = \eta_2' (\varepsilon, B)$, we conclude $\lambda_{\eta_2'(\varepsilon,B)} \leq 1 - \varepsilon$, hence

$$\eta_2' (\varepsilon, B) \quad \geq \quad \# \{ i : \lambda_i > 1 - \varepsilon \} \geq \sum_{\{i : \lambda_i > 1 - \varepsilon\}} \lambda_i.$$

Next we apply Proposition 2.4, for any $0 < \delta < 1$ there exists $R > 0$ large enough such that for all balls $B = B(a; r)$ with $r \geq R$ it holds $(1 - \delta) \|C_B\|_{S_1} \leq \|C_B\|_{S_2}^2$. Then for such balls

$$\begin{aligned} (1 - \delta) \sum_{i=0}^{\infty} \lambda_i &\leq \sum_{i=0}^{\infty} \lambda_i^2 \\ &= \sum_{\{i:\lambda_i > 1-\varepsilon\}} \lambda_i^2 + \sum_{\{i:\lambda_i \leq 1-\varepsilon\}} \lambda_i^2 \\ &\leq \sum_{\{i:\lambda_i > 1-\varepsilon\}} \lambda_i^2 + (1 - \varepsilon) \sum_{\{i:\lambda_i \leq 1-\varepsilon\}} \lambda_i \\ &= \sum_{\{i:\lambda_i > 1-\varepsilon\}} \lambda_i + (1 - \varepsilon) \left[\sum_{i=0}^{\infty} \lambda_i - \sum_{\{i:\lambda_i > 1-\varepsilon\}} \lambda_i \right], \end{aligned}$$

which implies

$$\sum_{\{i:\lambda_i > 1-\varepsilon\}} \lambda_i \geq \left(1 - \frac{\delta}{\varepsilon} \right) \sum_{i=0}^{\infty} \lambda_i.$$

From Proposition 2.3 we also have $\|C_B\|_{S_1} = \lambda(B)$. Combining these results, we conclude that for all balls $B = B(a;r)$ with $r \geq R$ it holds

$$\frac{\eta_2'(\varepsilon, B)}{\lambda(B)} \quad \geq \quad 1 - \frac{\delta}{\varepsilon}.$$

Therefore letting $R \to \infty$ and thus $\delta \to 0$

$$\liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_2(\varepsilon, B(a;r))}{\lambda(B(a;r))} \geq \liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_2'(\varepsilon, B(a;r))}{\lambda(B(a;r))} \geq 1.$$

□

Corollary 2.9. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Given $0 < \varepsilon < 1$, and $B := B(a;r)$, let*

$$\eta_3(\varepsilon, B) := \#\{\lambda_i \in \sigma(C_B) : \lambda_i \geq 1 - \varepsilon\},$$

where $\sigma(C_B)$ denotes the spectrum of the concentration operator C_B. Then

$$\lim_{r\to\infty} \sup_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} = \lim_{r\to\infty} \inf_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} = 1.$$

Proof. Given $B := B(a;r)$, let $\{\lambda_i(B)\}_{i=1}^{\infty}$ be the eigenvalues of C_B, and $\{f_i(B)\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ such that $f_i(B)$ is an eigenfunction of C_B associated to $\lambda_i(B)$. As we did before, we abbreviate these expressions by λ_i and f_i, and in this case it holds $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Since C_B is a compact operator, 0 is the only accumulation point, then $\eta_3(\varepsilon, B) < \infty$. Let $G(\varepsilon, B) := \operatorname{span}\{f_i : \lambda_i \geq 1 - \varepsilon\}$, so $\dim G(\varepsilon, B) = \eta_3(\varepsilon, B)$. Suppose $g \in G(\varepsilon, B)^{\perp}$ with $\|g\| = 1$. Then

$$\begin{aligned} g &= \sum_{i=\eta_3(\varepsilon,B)+1}^{\infty} a_i f_i, \\ \|g\| &= \sum_{i=\eta_3(\varepsilon,B)+1}^{\infty} |a_i|^2 = 1. \end{aligned}$$

Thus for all $g \in G(\varepsilon, B)^{\perp}$ with $\|g\| = 1$ it holds

$$\begin{aligned}
\langle C_B g, g \rangle &= \left\langle \sum_{i=\eta_3(\varepsilon,B)+1}^{\infty} \lambda_i a_i f_i, \sum_{j=\eta_3(\varepsilon,B)+1}^{\infty} a_j f_j \right\rangle \\
&= \sum_{i=\eta_3(\varepsilon,B)+1}^{\infty} \lambda_i |a_i|^2 \\
&\leq \lambda_{\eta_3(\varepsilon,B)+1} \sum_{i=\eta_3(\varepsilon,B)+1}^{\infty} |a_i|^2 \\
&< 1 - \varepsilon.
\end{aligned}$$

Hence $\eta_3(\varepsilon, B) \geq \eta_2(\varepsilon, B)$. Therefore, by Theorem 2.8

$$\liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} \geq \liminf_{r\to\infty} \inf_{a\in X} \frac{\eta_2(\varepsilon, B(a;r))}{\lambda(B(a;r))} \geq 1.$$

On the other hand, suppose $g \in G(\varepsilon, B)$ with $\|g\| = 1$. Then

$$\begin{aligned}
\langle C_B g, g \rangle &= \left\langle \sum_{i=1}^{\eta_3(\varepsilon,B)} \lambda_i a_i f_i, \sum_{j=1}^{\eta_3(\varepsilon,B)} a_j f_j \right\rangle \\
&= \sum_{i=1}^{\eta_3(\varepsilon,B)} \lambda_i |a_i|^2 \\
&\geq \lambda_{\eta_3(\varepsilon,B)} \sum_{i=1}^{\eta_3(\varepsilon,B)} |a_i|^2 \\
&\geq 1 - \varepsilon.
\end{aligned}$$

Hence $\eta_3(\varepsilon, B) \leq \eta_1(\varepsilon, B)$. Therefore, by Theorem 2.7

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} \leq \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_1(\varepsilon, B(a;r))}{\lambda(B(a;r))} \leq 1.$$

We immediately conclude

$$\lim_{r\to\infty} \sup_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} = \lim_{r\to\infty} \inf_{a\in X} \frac{\eta_3(\varepsilon, B(a;r))}{\lambda(B(a;r))} = 1.$$

□

Corollary 2.10. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Given $0<\varepsilon_1<\varepsilon_2<1$, and $B := B(a;r)$, let*

$$\eta_4(\varepsilon_1,\varepsilon_2,B) := \#\{\lambda_i \in \sigma(C_B) : 1-\varepsilon_2 \le \lambda_i < 1-\varepsilon_1\},$$

where $\sigma(C_B)$ denotes the spectrum of the concentration operator C_B. Then

$$\lim_{r\to\infty}\sup_{a\in X}\frac{\eta_4(\varepsilon_1,\varepsilon_2,B(a;r))}{\lambda(B(a;r))} = 0.$$

Proof. First notice that $\eta_4(\varepsilon_1,\varepsilon_2,B) = \eta_3(\varepsilon_2,B) - \eta_3(\varepsilon_1,B)$, then by Corollary 2.9

$$\begin{aligned}
\limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_4(\varepsilon_1,\varepsilon_2,B(a;r))}{\lambda(B(a;r))} &\le \limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_3(\varepsilon_2,B(a;r))}{\lambda(B(a;r))} + \limsup_{r\to\infty}\sup_{a\in X}\frac{-\eta_3(\varepsilon_1,B(a;r))}{\lambda(B(a;r))}\\
&= \limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_3(\varepsilon_2,B(a;r))}{\lambda(B(a;r))} - \liminf_{r\to\infty}\inf_{a\in X}\frac{\eta_3(\varepsilon_1,B(a;r))}{\lambda(B(a;r))}\\
&= 0.
\end{aligned}$$

□

2.5 Applications: asymptotic behavior on pseudospectra

The objective of this section is to give a proof of a theorem similar to the main result in [4], in the context of the concentration operators under a very general setup.

In order to apply the results from the previous section, again we assume $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property (F).

Definition 2.6. Given an orthonormal set $G' := \{g_i\}_{i\in I} \subseteq \mathcal{H}$, we say the set $\{g_j\}_{j\in J} \subseteq \mathcal{H}$ is a *complement for an orthonormal basis* associated to G' if $\{g_j\}_{j\in J}$ is an orthonormal basis for $(\operatorname{span} G')^{\perp}$.

Remark. In order to avoid any confusion, notice that G' in the previous definition is just a set, not a subspace; also, given an orthonormal set G' there are many ways how to complete it to become an orthonormal basis for $\mathcal{H}$.

Theorem 2.11. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Let $G \subseteq \mathcal{H}$ denote an arbitrary finite dimensional subspace. Given $0 < \varepsilon < 1$ and $B := B(a;r)$, let $\eta_5(\varepsilon, B)$ be the maximum dimension of G such that $\langle C_B g, g\rangle - \|C_B g\|^2 \geq \varepsilon$ for all $g \in G$ with $\|g\| = 1$, and $\eta_6(\varepsilon, B)$ be the minimum dimension of G such that $\langle C_B g, g\rangle - \|C_B g\|^2 < \varepsilon$ for all $g \in G^\perp$ with $\|g\| = 1$. Then*

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_5(\varepsilon, B(a;r))}{\lambda(B(a;r))} = \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_6(\varepsilon, B(a;r))}{\lambda(B(a;r))} \quad = \quad 0.$$

Alternatively, let $G' \subseteq \mathcal{H}$ denote a finite orthonormal set, and G'' denote a complement for an orthonormal basis associated to G'. Given $0 < \varepsilon < 1$ and $B := B(a;r)$, let $\eta_5'(\varepsilon, B)$ be the maximum number of elements in G' such that $\langle C_B g, g\rangle - \|C_B g\|^2 \geq \varepsilon$ for all $g \in G'$, and $\eta_6'(\varepsilon, B)$ be the minimum number of elements in G' such that there exists G'' satisfying $\langle C_B g, g\rangle - \|C_B g\|^2 < \varepsilon$ for all $g \in G''$. Then

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_5'(\varepsilon, B(a;r))}{\lambda(B(a;r))} = \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_6'(\varepsilon, B(a;r))}{\lambda(B(a;r))} \quad = \quad 0.$$

Proof. Given $B := B(a;r)$, let $\{\lambda_i(B)\}_{i=1}^\infty$ be the eigenvalues of C_B, and $\{f_i(B)\}_{i=1}^\infty \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ such that $f_i(B)$ is an eigenfunction of C_B associated to $\lambda_i(B)$. As we did before, we abbreviate these expressions by λ_i and f_i, and in this case it holds $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Recall $0 < \varepsilon < 1$ and define

$$\begin{aligned} \mathcal{E}(\varepsilon, B) &:= \operatorname{span}\left\{f_i : \frac{\varepsilon}{3} \leq \lambda_i < 1 - \frac{\varepsilon}{3}\right\}, \\ \mathcal{F}(\varepsilon, B) &:= \left\{g \in \mathcal{H} : \left\|P_{\mathcal{E}(\varepsilon,B)} g\right\|^2 < \frac{\varepsilon}{3}\right\}, \end{aligned}$$

where $P_{\mathcal{E}(\varepsilon,B)}$ denotes the orthogonal projection onto $\mathcal{E}(\varepsilon, B)$. Notice that $\mathcal{E}(\varepsilon, B)$ is a finite dimensional subspace since $\dim \mathcal{E}(\varepsilon, B) = \eta_4\left(\frac{\varepsilon}{3}, 1 - \frac{\varepsilon}{3}, B\right) < \infty$. If $g \in \mathcal{F}(\varepsilon, B)$ with $\|g\| = 1$, then $\langle C_B g, g\rangle - \|C_B g\|^2 < \varepsilon$ because writing

$$g = \sum_{i=1}^\infty a_i f_i$$

we have

$$\begin{aligned}
\|g\|^2 &= \sum_{i=1}^{\infty} |a_i|^2 = 1, \\
\left\|P_{\mathcal{E}(\varepsilon,B)} g\right\|^2 &= \sum_{\left\{i: \frac{\varepsilon}{3} \leq \lambda_i < 1-\frac{\varepsilon}{3}\right\}} |a_i|^2 < \frac{\varepsilon}{3},
\end{aligned}$$

which imply

$$\begin{aligned}
\langle C_B g, g\rangle - \|C_B g\|^2 &= \left\langle \sum_{i=1}^{\infty} \lambda_i a_i f_i, \sum_{j=1}^{\infty} a_j f_j \right\rangle - \left\langle \sum_{i=1}^{\infty} \lambda_i a_i f_i, \sum_{j=1}^{\infty} \lambda_j a_j f_j \right\rangle \\
&= \sum_{i=1}^{\infty} \lambda_i |a_i|^2 - \sum_{i=1}^{\infty} \lambda_i^2 |a_i|^2 \\
&= \sum_{i=1}^{\infty} \lambda_i (1-\lambda_i) |a_i|^2 \\
&= \sum_{\left\{i:\lambda_i < \frac{\varepsilon}{3}\right\}} \lambda_i (1-\lambda_i) |a_i|^2 + \sum_{\left\{i: \frac{\varepsilon}{3} \leq \lambda_i < 1-\frac{\varepsilon}{3}\right\}} \lambda_i (1-\lambda_i) |a_i|^2 + \sum_{\left\{i:\lambda_i \geq 1-\frac{\varepsilon}{3}\right\}} \lambda_i (1-\lambda_i) |a_i|^2 \\
&\leq \frac{\varepsilon}{3} \sum_{\left\{i:\lambda_i < \frac{\varepsilon}{3}\right\}} |a_i|^2 + \sum_{\left\{i: \frac{\varepsilon}{3} \leq \lambda_i < 1-\frac{\varepsilon}{3}\right\}} |a_i|^2 + \frac{\varepsilon}{3} \sum_{\left\{i:\lambda_i \geq 1-\frac{\varepsilon}{3}\right\}} |a_i|^2 \\
&\leq \frac{\varepsilon}{3} \|g\|^2 + \sum_{\left\{i: \frac{\varepsilon}{3} \leq \lambda_i < 1-\frac{\varepsilon}{3}\right\}} |a_i|^2 + \frac{\varepsilon}{3} \|g\|^2 \\
&< \varepsilon.
\end{aligned}$$

On the one hand this implies that $\eta_6(\varepsilon, B) \leq \dim \mathcal{E}(\varepsilon, B)$ since $\mathcal{E}(\varepsilon, B)^{\perp} \subseteq \mathcal{F}(\varepsilon, B)$. So, for any $g \in \mathcal{E}(\varepsilon, B)^{\perp}$ with $\|g\| = 1$ it holds $\langle C_B g, g\rangle - \|C_B g\|^2 < \varepsilon$. Thus applying Corollary 2.10 and noting that $\eta_6'(\varepsilon, B) \leq \eta_6(\varepsilon, B)$ by definition, we conclude

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_6'(\varepsilon, B(a;r))}{\lambda(B(a;r))} \leq \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_6(\varepsilon, B(a;r))}{\lambda(B(a;r))} \leq \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_4\left(\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3}, B(a;r)\right)}{\lambda(B(a;r))} = 0.$$

On the other hand, if g is such that $\langle C_B g, g\rangle - \|C_B g\|^2 \geq \varepsilon$ with $\|g\| = 1$, then $g \in \mathcal{F}(\varepsilon, B)^c$. Suppose $G' = \{g_1, g_2, \ldots, g_n\}$ is an orthonormal set such that $\langle C_B g, g\rangle - \|C_B g\|^2 \geq \varepsilon$ for all $g \in G'$. Clearly $n \leq \eta_5'(\varepsilon, B)$. By the reasoning above, $G' \subseteq \mathcal{F}(\varepsilon, B)^c$. Thus, applying Corollary 2.6

$$n \leq \frac{\dim \mathcal{E}(\varepsilon, B)}{\frac{\varepsilon}{3}}.$$

In particular, if G' is optimal we conclude

$$\eta_5'(\varepsilon, B) \le \frac{\eta_4\left(\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3}, B\right)}{\frac{\varepsilon}{3}}.$$

Applying again Corollary 2.10 and noting $\eta_5(\varepsilon, B) \le \eta_5'(\varepsilon, B)$ by definition, we conclude

$$\limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_5(\varepsilon, B(a;r))}{\lambda(B(a;r))} \le \limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_5'(\varepsilon, B(a;r))}{\lambda(B(a;r))} \le \frac{3}{\varepsilon}\limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_4\left(\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3}, B(a;r)\right)}{\lambda(B(a;r))} = 0.$$

□

The following is a result similar to the upper inequality of Theorem 1 in [4].

Theorem 2.12. *Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame satisfying the localization property (F). Let $G' \subseteq \mathcal{H}$ denote a finite orthonormal set. Given $0 < \varepsilon < 1$ and $B := B(a;r)$, let $\eta_7(\varepsilon, B)$ be the maximum number of elements in G' such that $\|C_B g - g\|^2 \le \varepsilon$ for all $g \in G'$. Then*

$$\limsup_{r\to\infty}\sup_{a\in X}\frac{\eta_7(\varepsilon, B(a;r))}{\lambda(B(a;r))} \quad\le\quad \frac{1}{1-2\varepsilon}.$$

Proof. Let $G' = \{g_1, g_2, \ldots, g_n\}$ be an orthonormal set such that $\|C_B g - g\|^2 \le \varepsilon$ for all $g \in G'$. Then $n \le \eta_7(\varepsilon, B(a;r))$. We split G' in two sets

$$\begin{aligned} G_1' &:= \left\{g \in G' : \langle C_B g, g\rangle - \|C_B g\|^2 \ge \varepsilon\right\}, \\ G_2' &:= \left\{g \in G' : \langle C_B g, g\rangle - \|C_B g\|^2 < \varepsilon\right\}. \end{aligned}$$

Notice that $\#G_1' \le \eta_5'(\varepsilon, B(a;r))$. On the other hand, for any $g \in G_2'$ it holds $\langle C_B g, g\rangle \ge 1-2\varepsilon$. Then

$$(1-2\varepsilon)\#G_2' \le \sum_{g\in G_2'}\langle C_B g, g\rangle \le \operatorname{tr}(C_B) = \|C_B\|_{S_1} = \lambda(B),$$

where the last step is due to Proposition 2.3. Thus $n = \#G_1' + \#G_2' \le \eta_5'(\varepsilon, B(a;r)) + \frac{1}{1-2\varepsilon}\lambda(B)$. In particular if G' is optimal

$$\eta_7(\varepsilon, B(a;r)) \quad\le\quad \eta_5'(\varepsilon, B(a;r)) + \frac{1}{1-2\varepsilon}\lambda(B).$$

Applying Theorem 2.11 we conclude

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_7\left(\varepsilon, B(a;r)\right)}{\lambda\left(B(a;r)\right)} \leq \limsup_{r\to\infty} \sup_{a\in X} \frac{\eta_5'\left(\varepsilon, B(a;r)\right)}{\lambda\left(B(a;r)\right)} + \frac{1}{1-2\varepsilon} = \frac{1}{1-2\varepsilon}.$$

□

Chapter 3

Sampling and interpolation

Let $(\mathcal{H}, X, k)$ be a framed Hilbert space. Assume the index set (X, d, λ) is a metric measure space satisfying (S1)-(S3) as in the previous chapter, and impose the additional condition

(S4) (X, d, λ) is a *doubling* metric measure space, which means that there exists a constant $c_1 > 0$ such that for all $x \in X$ and all $r > 0$ it holds

$$\lambda\left(B(x; 2r)\right) \leq c_1 \lambda\left(B(x; r)\right).$$

Due to [14], since (X, d, λ) is metric measure space which is doubling and length space, then (X, d, λ) has the *annular decay* property, which states that there exist constants $c_2 > 0$ and $0 < a < 1$ such that for all $z \in X$, for all $R > 0$, for all $0 < \delta < R$ it holds

$$\lambda\left(B(z; R) \backslash B(z; R - \delta)\right) \leq c_2 \left(\frac{\delta}{R}\right)^a \lambda\left(B(z; R)\right).$$

Furthermore, assume the generalized frame $\{k_x\}_{x \in X} \subseteq \mathcal{H}$ is normalized and satisfies the localization property (F) from the previous chapter. Finally, assume the generalized frame satisfies the following *mean value property*:

(F2) For all $f \in \mathcal{H}$, for all $x \in X$, for all $R > 0$, there exists a constant α_R depending on R only, such that

$$\left|\langle f, k_x \rangle\right|^2 \leq \alpha_R \int_{B(x;R)} \left|\langle f, k_y \rangle\right|^2 d\lambda(y).$$

3.1 Technical lemmas

We start with two technical lemmas which depend on the geometric assumptions of the metric measure space X: the annular decay property and the doubling property (S4).

Definition 3.1. For any subset $\Gamma \subseteq X$ let $\delta_0 := \inf\{d(z,z') : z, z' \in \Gamma, z \neq z'\}$. If $\delta_0 > 0$, we say Γ is δ_0-*separated.*

Lemma 3.1. *Let $\Gamma \subseteq X$ be δ_0-separated, and let (X, d, λ) be a metric measure space satisfying the doubling property (S4). Then, for any $\varepsilon > 0$ there exists $R > 0$ such that for all $r \geq R$ and all $z \in X$ it holds*

$$\frac{\#\left(\Gamma \cap \left[B\left(z; r+\frac{\delta_0}{2}\right) \setminus B\left(z;r\right)\right]\right)}{\lambda\left(B\left(z;r\right)\right)} < \varepsilon.$$

Proof. Let $z \in X$. For a given $r > \frac{\delta_0}{2} > 0$ let $N = \#\left(\Gamma \cap \left[B\left(z; r+\frac{\delta_0}{2}\right) \setminus B\left(z;r\right)\right]\right)$, so

$$\Gamma \cap \left[B\left(z; r+\frac{\delta_0}{2}\right) \setminus B\left(z;r\right)\right] \quad = \quad \{x_1, x_2, \ldots, x_N\}.$$

Let $i \in \{1, 2, \ldots, N\}$. By construction, $r \leq d\left(x_i; z\right) < r + \frac{\delta_0}{2}$, so for any $w \in B\left(x_i; \frac{\delta_0}{2}\right)$, by triangle inequality we obtain that

$$\begin{aligned} d(z;w) &\leq d(z;x_i) + d(x_i, w) \\ &< r + \frac{\delta_0}{2} + \frac{\delta_0}{2} \\ &= r + \delta_0, \\ d(z;w) &\geq d(z;x_i) - d(x_i, w) \\ &> r - \frac{\delta_0}{2}. \end{aligned}$$

Then, $B\left(x_i; \frac{\delta_0}{2}\right) \subseteq B\left(z; r+\delta_0\right) \setminus B\left(z; r-\frac{\delta_0}{2}\right)$ for all $i \in \{1, 2, \ldots, N\}$. Also, since Γ is δ_0-separated, $B\left(x_1; \frac{\delta_0}{2}\right), B\left(x_2; \frac{\delta_0}{2}\right), \ldots, B\left(x_N; \frac{\delta_0}{2}\right)$ are mutually disjoint balls. From here, we get

$$\bigsqcup_{i=1}^{N} B\left(x_i; \frac{\delta_0}{2}\right) \quad \subseteq \quad B\left(z; r+\delta_0\right) \setminus B\left(z; r-\frac{\delta_0}{2}\right).$$

Thus, using the sub-additivity of λ and applying the annular decay property of the metric measure space (X,d,λ) we obtain

$$\begin{aligned} N\lambda\left(B\left(z;\frac{\delta_0}{2}\right)\right) &= \sum_{i=1}^{N}\lambda\left(B\left(x_i;\frac{\delta_0}{2}\right)\right) \\ &\leq \lambda\left(B\left(z;r+\delta_0\right)\backslash B\left(z;r-\frac{\delta_0}{2}\right)\right) \\ &= \lambda\left(B\left(z;r+\delta_0\right)\backslash B\left(z;r+\delta_0-\frac{3\delta_0}{2}\right)\right) \\ &\leq c_2\left(\frac{\frac{3}{2}\delta_0}{r+\delta_0}\right)^a\lambda\left(B\left(z;r+\delta_0\right)\right). \end{aligned}$$

Now, taking $r \geq \delta_0$ such that $2r \geq r+\delta_0$, applying the doubling property of the metric measure space (X,d,λ) and again sub-additivity of λ, from the previous inequality we obtain

$$\begin{aligned} \frac{\#\left(\Gamma\cap\left[B\left(z;r+\frac{\delta_0}{2}\right)\backslash B\left(z;r\right)\right]\right)}{\lambda\left(B\left(z;r\right)\right)} &= \frac{N}{\lambda\left(B\left(z;r\right)\right)} \\ &\leq \frac{c_2\left(\frac{\frac{3}{2}\delta_0}{r+\delta_0}\right)^a\lambda\left(B\left(z;r+\delta_0\right)\right)}{\lambda\left(B\left(z;\frac{\delta_0}{2}\right)\right)\lambda\left(B\left(z;r\right)\right)} \\ &\leq c_2\left(\frac{\frac{3}{2}\delta_0}{r+\delta_0}\right)^a\frac{1}{\lambda\left(B\left(z;\frac{\delta_0}{2}\right)\right)}\frac{\lambda\left(B\left(z;2r\right)\right)}{\lambda\left(B\left(z;r\right)\right)} \\ &\leq \frac{c_1c_2}{\lambda\left(B\left(z;\frac{\delta_0}{2}\right)\right)}\left(\frac{\frac{3}{2}\delta_0}{r+\delta_0}\right)^a \\ &< \varepsilon \end{aligned}$$

for all $r > R$, where R is a constant independent on z given by

$$R = \max\left\{\frac{3}{2}\delta_0\left(\frac{\varepsilon\lambda\left(B\left(z;\frac{\delta_0}{2}\right)\right)}{c_1c_2}\right)^{-\frac{1}{a}}-\delta_0,\delta_0\right\}.$$

This completes the proof. □

Lemma 3.2. *Let Γ be δ_0-separated, and let (X,d,λ) be a metric measure space satisfying the doubling property (S4). Then, for any $\varepsilon > 0$, there exists $R > \frac{\delta_0}{2} > 0$ such that for all $r \geq R$ and all $z \in X$ it holds*

$$\frac{\#\left(\Gamma\cap\left[B(z;r)\setminus B\left(z;r-\frac{\delta_0}{2}\right)\right]\right)}{\lambda\left(B(z;r)\right)} < \varepsilon.$$

Proof. By Lemma 3.1, there exists $R_1 > \frac{\delta_0}{2} > 0$ such that for all $r_1 \geq R_1$ and all $z \in X$ it holds

$$\frac{\#\left(\Gamma \cap \left[B\left(z; r_1 + \frac{\delta_0}{2}\right) \setminus B\left(z; r_1\right)\right]\right)}{\lambda\left(B(z; r_1)\right)} < \varepsilon.$$

Define $r = r_1 + \frac{\delta_0}{2}$, the last inequality says that for all $r \geq R_1 + \frac{\delta_0}{2}$ it holds

$$\frac{\#\left(\Gamma \cap \left[B\left(z; r\right) \setminus B\left(z; r - \frac{\delta_0}{2}\right)\right]\right)}{\lambda\left(B\left(z; r - \frac{\delta_0}{2}\right)\right)} < \varepsilon.$$

But $\left[\lambda\left(B\left(z; r\right)\right)\right]^{-1} \leq \left[\lambda\left(B\left(z; r - \frac{\delta_0}{2}\right)\right)\right]^{-1}$ since $B\left(z; r - \frac{\delta_0}{2}\right) \subseteq B\left(z; r\right)$. Therefore, for all $z \in X$, and for all $r \geq R$, where $R = R_1 + \frac{\delta_0}{2}$, it holds

$$\frac{\#\left(\Gamma \cap \left[B(z; r) \setminus B\left(z; r - \frac{\delta_0}{2}\right)\right]\right)}{\lambda\left(B(z; r)\right)} < \varepsilon.$$

This completes the proof. □

3.2 Sampling

Definition 3.2. The subset $\Gamma \subseteq X$ is called *sampling* if Γ is countable and $\{k_x\}_{x \in \Gamma}$ is a frame in the usual sense, i.e. there exists constants $\hat{\alpha}, \hat{\beta} \in \mathbb{R}^+$ such that for any $f \in \mathcal{H}$ it holds

$$\hat{\alpha}\|f\|^2 \leq \sum_{x \in \Gamma} \left|\langle f, k_x \rangle\right|^2 \leq \hat{\beta}\|f\|^2.$$

For the rest of this section, we assume $\Gamma \subseteq X$ is δ_0−separated and sampling. Furthermore, given a compact set $\Omega = \overline{B(z; r)} \subseteq X$, let $\lambda_0 \geq \lambda_1 \geq \cdots$ be the eigenvalues of the concentration operator C_Ω, and f_n be the eigenfunction associated to λ_n, for all n. By the spectral theorem, we can choose $\{f_n\}_{n=0}^{\infty} \subseteq \mathcal{H}$ such that they form an orthonormal basis for $\mathcal{H}$.

For any n, since $C_\Omega f_n = \lambda_n f_n$ then $\langle C_\Omega f_n, f_n \rangle = \langle \lambda_n f_n, f_n \rangle = \lambda_n \|f_n\|^2 = \lambda_n$. Thus

$$\begin{aligned} \lambda_n = \langle C_\Omega f_n, f_n \rangle &= \left\langle \int_{B(z;r)} \langle f_n, k_x \rangle \, k_x d\lambda(x), f_n \right\rangle \\ &= \int_{B(z;r)} \langle f_n, k_x \rangle \langle k_x, f_n \rangle \, d\lambda(x) \\ &= \int_{B(z;r)} |\langle f_n, k_x \rangle|^2 d\lambda(x). \end{aligned}$$

Theorem 3.3. *Let $\Gamma \subseteq X$ be δ_0-separated and sampling for $\mathcal{H}$, and let $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ be a generalized frame satisfying the Mean Value Property (F2). Define $N = N(z;r) := \#\left(\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)\right)$ and assume $N \neq 0$ (this is true for a large enough r), then $\lambda_N\left(B(z;r)\right) \leq \gamma$ for some positive constant γ independent on z and r. Moreover, if $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is assumed to be a generalized Parseval frame, then $\gamma < 1$.*

Proof. Let $\Omega = \overline{B(z;r)}$ and say that $\{\lambda_n\left(B(z;r)\right)\}_{n=0}^{\infty}$ and $\{f_n\left(B(z;r)\right)\}_{n=0}^{\infty}$ are the eigenvalues and eigenfunctions of the concentration operator C_Ω. For simplicity we write λ_n and f_n instead, but these depend on z and r. Let $\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right) = \{x_1, x_2, \ldots, x_N\}$, and consider the linear system

$$\sum_{i=0}^{N} t_i \left\langle f_i, k_{x_j} \right\rangle = 0, \quad j = 1, 2, \ldots, N.$$

By the rank-nullity Theorem, there exists a non trivial solution $\widehat{t_0}, \widehat{t_1}, \ldots, \widehat{t_N}$. Consider the linear combination $\widehat{f} = \sum_{n=0}^{N} \widehat{t_n} f_n$ of eigenfunctions of C_Ω, clearly $\left\|\widehat{f}\right\|^2 = \sum_{n=0}^{N} |\widehat{t_n}|^2 > 0$. Furthermore, by construction $\left\langle \widehat{f}, k_{x_j} \right\rangle = 0$ for all $j = 1, 2, \ldots, N$.

It will be useful to do some calculations for $\left\langle C_\Omega \widehat{f}, \widehat{f} \right\rangle$. By the definition of the concentration operator

$$\left\langle C_\Omega \widehat{f}, \widehat{f} \right\rangle = \int_{B(z;r)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y).$$

On the other hand, by the definition of $\widehat{f}$ as a linear combination of eigenfunctions of C_Ω

$$\begin{aligned}
\left\langle C_\Omega \widehat{f}, \widehat{f} \right\rangle &= \left\langle C_\Omega \left(\sum_{n=0}^{N} \widehat{t_n} f_n \right), \sum_{m=0}^{N} \widehat{t_m} f_m \right\rangle \\
&= \left\langle \sum_{n=0}^{N} \widehat{t_n} C_\Omega f_n, \sum_{m=0}^{N} \widehat{t_m} f_m \right\rangle \\
&= \left\langle \sum_{n=0}^{N} \widehat{t_n} \lambda_n f_n, \sum_{m=0}^{N} \widehat{t_m} f_m \right\rangle \\
&= \sum_{n=0}^{N} \lambda_n |\widehat{t_n}|^2.
\end{aligned}$$

Next, by construction we have $\left\langle \widehat{f}, k_x \right\rangle = 0$ for all $x \in \Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)$. So

$$\sum_{x\in\Gamma} \left|\left\langle \widehat{f}, k_x \right\rangle\right|^2 = \sum_{x \in \Gamma \setminus B\left(z; r + \frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_x \right\rangle\right|^2,$$

and since Γ is sampling we conclude

$$\hat{\alpha}\left\|\widehat{f}\right\|^2 \leq \sum_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_x \right\rangle\right|^2 \leq \hat{\beta}\left\|\widehat{f}\right\|^2 .$$

Also, since $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a generalized frame for $\mathcal{H}$ satisfying the Mean Value Property (F2), for any $x \in \Gamma\backslash B\left(z; r+\frac{\delta_0}{2}\right)$ we have

$$\left|\left\langle \widehat{f}, k_x \right\rangle\right|^2 \leq \alpha_{\delta_0} \int_{B\left(x;\frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y),$$

where α_{δ_0} is a positive constant depending on δ_0 only. Notice that the elements of $\{k_x\}_{x\in\Gamma}$ are δ_0-separated, so $B\left(x; \frac{\delta_0}{2}\right) \cap B\left(x'; \frac{\delta_0}{2}\right) = \emptyset$ for any $x, x' \in \Gamma$, $x \neq x'$, which implies

$$X\backslash B(z;r) \supseteq \bigcup_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} B\left(x;\frac{\delta_0}{2}\right) = \bigsqcup_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} B\left(x;\frac{\delta_0}{2}\right),$$

where the expression on the right denotes a disjoint union. Hence,

$$\begin{aligned}
\hat{\alpha}\sum_{n=0}^{N}\left|\widehat{t_n}\right|^2 &= \hat{\alpha}\left\|\widehat{f}\right\|^2 \\
&\leq \sum_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_x \right\rangle\right|^2 \\
&\leq \alpha_{\delta_0} \sum_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} \int_{B\left(x;\frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) \\
&= \alpha_{\delta_0} \int_{\bigsqcup_{x\in\Gamma\backslash B\left(z;r+\frac{\delta_0}{2}\right)} B\left(x;\frac{\delta_0}{2}\right)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) \\
&\leq \alpha_{\delta_0} \int_{X\backslash B(z;r)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) \\
&= \alpha_{\delta_0} \int_{X} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) - \alpha_{\delta_0} \int_{B(z;r)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) \\
&\leq \alpha_{\delta_0}\beta\left\|\widehat{f}\right\|^2 - \alpha_{\delta_0} \int_{B(z;r)} \left|\left\langle \widehat{f}, k_y \right\rangle\right|^2 d\lambda(y) \\
&= \alpha_{\delta_0}\beta\sum_{n=0}^{N}\left|\widehat{t_n}\right|^2 - \alpha_{\delta_0}\sum_{n=0}^{N}\lambda_n\left|\widehat{t_n}\right|^2 \\
&\leq \alpha_{\delta_0}\beta\sum_{n=0}^{N}\left|\widehat{t_n}\right|^2 - \alpha_{\delta_0}\lambda_N\sum_{n=0}^{N}\left|\widehat{t_n}\right|^2 .
\end{aligned}$$

Therefore,

$$\lambda_N\left(B(z;r)\right) \le \beta - \frac{\hat{\alpha}}{\alpha_{\delta_0}} =: \gamma.$$

In particular, if $\{k_x\}_{x\in X}$ is a generalized Parseval frame, then $\alpha = \beta = 1$, so

$$\lambda_N\left(B(z;r)\right) \le 1 - \frac{\hat{\alpha}}{\alpha_{\delta_0}} =: \gamma < 1.$$

In both cases γ is a constant which is independent on z and r. This completes the proof. □

For the rest of this section, we assume $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property (F).

Lemma 3.4. *In addition to the hypotheses of Theorem 3.3, suppose $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property (F). Then, for any $z \in X$ and any $0 < \delta < 1$ there exists a $R > 0$ independent on z such that for all $r \ge R$ it holds*

$$\frac{\#\left(\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)\right)}{\lambda\left(B(z;r)\right)} \ge 1 - \frac{\delta}{1-\gamma},$$

where $0 < \gamma < 1$ is the special constant from Theorem 3.3.

Proof. We can deduce this lemma as a consequence of Theorem 3.3 and the main inequality in the proof of Theorem 2.8. This inequality says that for any given $0 < \delta < 1$ there exists $R > 0$ such that for all $z \in X$ and all $r \ge R$ it holds

$$\frac{\eta_2'\left(\varepsilon, B(z;r)\right)}{\lambda\left(B(z;R)\right)} \ge 1 - \frac{\delta}{\varepsilon},$$

where $0 < \varepsilon < 1$ is fixed, and $\eta_2'(\varepsilon, B)$ is the minimum dimension of a finite dimensional subspace $G \subseteq \mathcal{H}$ such that $\langle C_B g, g\rangle \le 1 - \varepsilon$ for all $g \in G^{\perp}$ with $\|g\| = 1$.

Pick $\varepsilon = 1 - \gamma$, where γ is the special constant from Theorem 3.3 (recall $0 < \gamma < 1$). For any ball $B = B(z;r)$, let $N := \#\left(\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)\right)$ and define the finite dimensional subspace $G_B := \operatorname{span}\{f_0, f_1, \ldots, f_{N-1}\} \subseteq \mathcal{H}$, where as usual $\{f_i\}_{i=0}^{\infty} \subseteq \mathcal{H}$ is an orthonormal basis generated by eigenfunctions of C_B, f_i is an eigenfunction corresponding to the eigenvalue λ_i, and $1 \ge \lambda_0 \ge \lambda_1 \ge \cdots \ge 0$.

As a consequence of Theorem 3.3, for any $g \in G_B^{\perp}$ with $\|g\| = 1$, i.e., $g = \sum_{i=N}^{\infty} a_i f_i$ with

$\sum_{i=N}^{\infty} |a_i|^2 = 1$, we obtain

$$\begin{aligned}\langle C_B g, g\rangle &= \sum_{i=N}^{\infty} \lambda_i |a_i|^2 \\ &\leq \lambda_N \sum_{i=N}^{\infty} |a_i|^2 \\ &\leq \gamma,\end{aligned}$$

then, $\eta_2'(1-\gamma, B) \leq \dim(G_B) = N = \#\left(\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)\right)$.

Therefore, applying the main inequality in the proof of Theorem 2.8, for all $z \in X$ and all $r \geq R$ it holds

$$\begin{aligned}\frac{\#\left(\Gamma \cap B\left(z; r + \frac{\delta_0}{2}\right)\right)}{\lambda(B(z;r))} &\geq \frac{\eta_2'(1-\gamma, B(z;r))}{\lambda(B(z;r))} \\ &\geq 1 - \frac{\delta}{1-\gamma}.\end{aligned}$$

□

Alternative direct proof. Due to Proposition 2.1, for any compact Ω the concentration operator C_Ω satisfies $\|C_\Omega\| \leq \beta = 1$, since we are assuming the generalized frame is Parseval. Consider the eigenvalues of C_Ω, $\lambda_0(\Omega) \geq \lambda_1(\Omega) \geq \cdots \geq 0$, and say $f_n(\Omega)$ is the eigenfunction associated to $\lambda_n(\Omega)$, $n = 0, 1, \ldots$. Then

$$\lambda_n \|f_n\| = \|\lambda_n f_n\| = \|C_\Omega f_n\| \leq \|f_n\|.$$

Since $\|f_n\| \neq 0$ for all n, then $0 \leq \lambda_n \leq 1$ for all n.

Given $z \in X$ and $0 < \delta < 1$, let $R > 0$ obtained by Proposition 2.4 which is independent on z. For any $r \geq R$, let $\Omega = B(z;r)$ and say $\{\lambda_n(\Omega)\}_{n=1}^{\infty}$ are the eigenvalues of C_Ω. The first inequality of the Proposition 2.4 states that

$$(1-\delta)\sum_{n=0}^{\infty} \lambda_n(\Omega) \leq \sum_{n=0}^{\infty} \lambda_n^2(\Omega).$$

We can split the sum on the right hand side with respect to the constant $0 < \gamma < 1$ given by

Theorem 3.3 (which is independent on z and r), and take advantage of $0 \le \lambda_n(\Omega) \le 1$ for all n

$$\begin{aligned}
(1-\delta)\sum_{n=0}^{\infty}\lambda_n(\Omega) &\le \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n^2(\Omega) + \sum_{\{n:\lambda_n(\Omega)\le\gamma\}}\lambda_n^2(\Omega) \\
&\le \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega) + \sum_{\{n:\lambda_n(\Omega)\le\gamma\}}\gamma\lambda_n(\Omega) \\
&= \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega) + \gamma\left[\sum_{n=0}^{\infty}\lambda_n(\Omega) - \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega)\right].
\end{aligned}$$

This implies

$$\sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega) \ge \left(1-\frac{\delta}{1-\gamma}\right)\sum_{n=0}^{\infty}\lambda_n(\Omega).$$

Now, for this particular $\Omega = B(z;r)$ consider $N = \#\left(\Gamma \cap B\left(z; r+\frac{\delta_0}{2}\right)\right)$ as in Theorem 3.3, and recall $1 \ge \lambda_0(\Omega) \ge \lambda_1(\Omega) \ge \cdots \ge \lambda_N(\Omega) \ge \cdots \ge 0$. Theorem 3.3 states that $1 > \gamma \ge \lambda_N(\Omega) \ge \lambda_{N+1}(\Omega) \ge \cdots \ge 0$. Combining these inequalities, and noticing that $\{n : \lambda_n(\Omega) > \gamma\} \subseteq \{0, 1, \ldots, N-1\}$, we obtain

$$\begin{aligned}
\#\left(\Gamma \cap B\left(z; r+\frac{\delta_0}{2}\right)\right) &= N \\
&\ge \#\{n : \lambda_n(\Omega) > \gamma\} \\
&\ge \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega),
\end{aligned}$$

where the last step is because $0 \le \lambda_n(\Omega) \le 1$ for all n. Finally, since $\sum_{n=0}^{\infty}\lambda_n(\Omega) = \lambda(\Omega)$ due to Proposition 2.3, we conclude

$$\begin{aligned}
\#\left(\Gamma \cap B\left(z; r+\frac{\delta_0}{2}\right)\right) &\ge \sum_{\{n:\lambda_n(\Omega)>\gamma\}}\lambda_n(\Omega) \\
&\ge \left(1-\frac{\delta}{1-\gamma}\right)\sum_{n=0}^{\infty}\lambda_n(\Omega) \\
&= \left(1-\frac{\delta}{1-\gamma}\right)\lambda(\Omega),
\end{aligned}$$

which is true for all $r \ge R$ and for all $z \in X$. □

Definition 3.3. The *lower density* of Γ, denoted by $D^{-}(\Gamma)$, is defined by

$$D^{-}(\Gamma) = \liminf_{r\to\infty} \inf_{z\in X} \frac{\#\left(\Gamma \cap B(z;r)\right)}{\lambda\left(B(z;r)\right)}.$$

The theorem below is the main result of this section. Recall that all the assumptions (S1)-(S4) for the indexing metric measure space (X, d, λ) of the framed Hilbert space $\mathcal{H}$, and the assumptions (F) and (F2) for the normalized generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ remain valid.

Theorem 3.5. *Let $\mathcal{H}$ be a framed Hilbert space with indexing set (X, d, λ). If $\Gamma \subseteq X$ is sampling and δ_0-separated, then the lower density of Γ satisfies*

$$D^{-}(\Gamma) \geq 1.$$

Proof. Let $\varepsilon > 0$ and $z \in X$. Choose $0 < \delta < 1$ small enough such that $\frac{\delta}{1-\gamma} < \frac{\varepsilon}{2}$, where $0 < \gamma < 1$ is the constant described in Theorem 3.3, independent on z and r. On the one hand, due to Lemma 3.4, for this particular $\delta > 0$ there exists $R_1 > 0$ such that for all $r \geq R_1$ and for all $z \in X$ it holds

$$\frac{\#\left(\Gamma \cap B\left(z; r+\frac{\delta_0}{2}\right)\right)}{\lambda\left(B(z;r)\right)} \geq 1 - \frac{\delta}{1-\gamma} > 1 - \frac{\varepsilon}{2}.$$

On the other hand, according to Lemma 3.1, there exists $R_2 > 0$ such that for all $r \geq R_2$ and all $z \in X$ it holds

$$\frac{\#\left(\Gamma \cap \left[B\left(z; r+\frac{\delta_0}{2}\right) \backslash B\left(z;r\right)\right]\right)}{\lambda\left(B\left(z;r\right)\right)} < \frac{\varepsilon}{2}.$$

Take $R = \max\{R_1, R_2\}$, so for all $r \geq R$ and for all $z \in X$ we get

$$\begin{aligned} \frac{\#\left(\Gamma \cap B\left(z;r\right)\right)}{\lambda\left(B\left(z;r\right)\right)} &= \frac{\#\left(\Gamma \cap B\left(z; r+\frac{\delta_0}{2}\right)\right)}{\lambda\left(B\left(z;r\right)\right)} - \frac{\#\left(\Gamma \cap \left[B\left(z; r+\frac{\delta_0}{2}\right) \backslash B\left(z;r\right)\right]\right)}{\lambda\left(B\left(z;r\right)\right)} \\ &> \left(1 - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} \\ &= 1 - \varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} D^{-}(\Gamma) &= \sup_{r'\geq 0}\inf_{r\geq r'}\inf_{z\in X}\frac{\#\left(\Gamma\cap B(z;r)\right)}{\lambda\left(B(z;r)\right)} \\ &\geq \inf_{r\geq R}\inf_{z\in X}\frac{\#\left(\Gamma\cap B(z;r)\right)}{\lambda\left(B(z;r)\right)} \\ &\geq 1-\varepsilon. \end{aligned}$$

Therefore, since $\varepsilon > 0$ is arbitrary, we conclude that

$$D^{-}(\Gamma) \geq 1.$$

This completes the proof. □

3.3 Interpolation

Recall, we say $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence if there constants $\alpha_1, \beta_1 \in \mathbb{R}^+$ such that for any $\{a_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ it holds

$$\alpha_1 \sum_{i=1}^{\infty} |a_i|^2 \leq \left\| \sum_{i=1}^{\infty} a_i f_i \right\|^2 \leq \beta_1 \sum_{i=1}^{\infty} |a_i|^2.$$

Definition 3.4. The sequence $\Gamma \subseteq X$ is called *interpolating* if $\{k_x\}_{x\in\Gamma}$ is a Riesz sequence.

For the rest of this section, we consider $\Gamma \subseteq X$ to be δ_0-separated and interpolating. Let $\Omega = \overline{B(z;r)}$, as before let $\{\lambda_n(B(z;r))\}_{n=1}^{\infty}$ be the eigenvalues of the concentration operator C_Ω.

Theorem 3.6. *Let $\Gamma \subseteq X$ be δ_0-separated and interpolating, and let $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ be a generalized frame satisfying the Mean Value Property (F2). Define $N = N(z;r) := \#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)$ and assume $N \neq 0$ (which is true for a large enough r). Then $\lambda_N(B(z;r)) \geq \gamma$ for some positive constant γ independent on z and r. Moreover, if $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is assumed to be a generalized Parseval frame, then $\gamma < 1$.*

Proof. Let $\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right) = \{x_1, x_2, \ldots, x_N\} \subseteq X$. Since Γ is δ_0-separated, $B\left(x_i; \frac{\delta_0}{2}\right) \cap B\left(x_j; \frac{\delta_0}{2}\right) = \emptyset$ for any $x_i \neq x_j \in \Gamma$, and by construction $x_j \in B\left(z; r - \frac{\delta_0}{2}\right)$ for $j = 1, 2 \ldots, N$, so $B\left(x_j; \frac{\delta_0}{2}\right) \subseteq$

$B(z;r)$ for $j=1,2,\ldots,N$. Hence

$$\bigcup_{j=1}^{N} B\left(x_j;\frac{\delta_0}{2}\right)=\bigsqcup_{j=1}^{N} B\left(x_j;\frac{\delta_0}{2}\right)\subseteq B(z;r).$$

Since Γ is interpolating, $\{k_{x_i}\}_{x_i\in\Gamma}\subseteq\mathcal{H}$ is a Riesz sequence in $\mathcal{H}$. Denote by $\{g_j\}_{j=1}^{\infty}\subseteq\mathcal{H}$ the dual Riesz sequence in $\mathcal{H}$ associated with $\{k_{x_i}\}_{x_i\in\Gamma}$, such that $\langle k_{x_i},g_j\rangle=\delta_{ij}$ and $\|g_j\|\leq\sqrt{C}$, where $C>0$ is the upper Riesz constant of $\{g_j\}_{j=1}^{\infty}$. Define $F=\operatorname{span}\{g_1,g_2,\ldots,g_N\}\subseteq\mathcal{H}$. For any $f\in F$, $f=\sum_{j=1}^{N} c_j g_j$ for some $\{c_j\}_{j=1}^{N}\subseteq\mathbb{C}$, but by bi-orthogonality

$$\begin{aligned}\langle f,k_{x_i}\rangle &= \left\langle\sum_{j=1}^{N} c_j g_j,k_{x_i}\right\rangle\\ &= \sum_{j=1}^{N} c_j\langle g_j,k_{x_i}\rangle\\ &= c_i\langle g_i,k_{x_i}\rangle\\ &= c_i.\end{aligned}$$

Thus, for any $f\in F$ it holds

$$f=\sum_{j=1}^{N}\langle f,k_{x_j}\rangle g_j.$$

Also notice that $\{g_j\}_{j=1}^{N}\subseteq F$ is a Riesz sequence (and also a Riesz basis) for F because $\{g_i\}_{i=1}^{\infty}$ is a Riesz sequence for $\mathcal{H}$, so $\|\sum_{j=1}^{N} a_j g_j\|^2$ can be controlled using the same Riesz constants just by considering $\{a_j\}_{j=1}^{\infty}\in\ell_2(\mathbb{N})$ as a truncated sequence such that $a_j=0$ for all $j>N$. Using these observations and the Mean Value Property (F2), for any $f \in F$ it holds

$$\begin{aligned}
\|f\|^2 &= \left\| \sum_{j=1}^{N} \left\langle f, k_{x_j} \right\rangle g_j \right\|^2 \\
&\leq C \sum_{j=1}^{N} \left| \left\langle f, k_{x_j} \right\rangle \right|^2 \\
&\leq C \sum_{j=1}^{N} \alpha_{\delta_0} \int_{B\left(x_j; \frac{\delta_0}{2}\right)} \left| \left\langle f, k_y \right\rangle \right|^2 d\lambda(y) \\
&= C\alpha_{\delta_0} \int_{\bigsqcup_{j=1}^{N} B\left(x_j; \frac{\delta_0}{2}\right)} \left| \left\langle f, k_y \right\rangle \right|^2 d\lambda(y) \\
&\leq C\alpha_{\delta_0} \int_{B(z;r)} \left| \left\langle f, k_y \right\rangle \right|^2 d\lambda(y).
\end{aligned}$$

But for $\Omega = \overline{B(z;r)}$, the concentration operator C_Ω gives

$$\begin{aligned}
\langle C_\Omega f, f \rangle &= \left\langle \int_{B(z;r)} \langle f, k_y \rangle k_y d\lambda(y), f \right\rangle \\
&= \int_{B(z;r)} \langle f, k_y \rangle \langle k_y, f \rangle d\lambda(y) \\
&= \int_{B(z;r)} \left| \langle f, k_y \rangle \right|^2 d\lambda(y)
\end{aligned}$$

Combining this result with the previous inequality, we obtain that for any $f \in F$ such that $\|f\| = 1$ the following holds

$$\langle C_\Omega f, f \rangle \geq \frac{\|f\|^2}{C\alpha_{\delta_0}} = \frac{1}{C\alpha_{\delta_0}} =: \gamma.$$

Notice that $\gamma > 0$ is a constant independent of z and r. Now we apply the Weyl-Courant lemma to C_Ω to calculate its N^{th}-eigenvalue running over all subspaces $F' \subseteq \mathcal{H}$ of dimension N

$$\begin{aligned}
\lambda_N(\Omega) &= \max_{\{F' \subseteq \mathcal{H} : \dim F' = N\}} \min_{\{f \in F' : \|f\| = 1\}} \langle C_\Omega f, f \rangle \\
&\geq \min_{\{f \in F : \|f\| = 1\}} \langle C_\Omega f, f \rangle \\
&\geq \gamma.
\end{aligned}$$

In particular, if $\{k_x\}_{x \in X}$ is a generalized Parseval frame, then all the eigenvalues of the concentration operator C_Ω are less than or equal to 1, which implies $1 \geq \gamma$. To prove that the inequality is strict consider the following reasoning. By the mean value property and the Parseval identity, for any $x \in X$

$$
\begin{aligned}
1 &= \|k_x\|^4 \\
&= |\langle k_x, k_x\rangle|^2 \\
&\leq \alpha_{\delta_0} \int_{B\left(x;\frac{\delta_0}{2}\right)} |\langle k_x, k_y\rangle|^2 \, d\lambda(y) \\
&\leq \alpha_{\delta_0} \int_{B\left(x;\frac{\delta_0}{2}\right)} |\langle k_x, k_y\rangle|^2 \, d\lambda(y) + \alpha_{\delta_0} \int_{X\setminus B\left(x;\frac{\delta_0}{2}\right)} |\langle k_x, k_y\rangle|^2 \, d\lambda(y) \\
&= \alpha_{\delta_0} \int_X |\langle k_x, k_y\rangle|^2 \, d\lambda(y) \\
&= \alpha_{\delta_0} \|k_x\|^2 \\
&= \alpha_{\delta_0}
\end{aligned}
$$

But there exists some $x \in X$ such that $\int_{B\left(x;\frac{\delta_0}{2}\right)} |\langle k_x, k_y\rangle|^2 \, d\lambda(y) < \int_X |\langle k_x, k_y\rangle|^2 \, d\lambda(y)$, or equivalently, such that $\int_{X\setminus B\left(x;\frac{\delta_0}{2}\right)} |\langle k_x, k_y\rangle|^2 \, d\lambda(y) > 0$. This implies $\alpha_{\delta_0} > 1$ must hold. Also, $C \geq 1$ because taking $x_1 \in \Gamma$, $1 = \|k_{x_1}\|^2 \leq C|1|^2$ since Γ is interpolating. Therefore $\gamma < 1$. □

For the rest of this section, we assume $\{k_x\}_{x\in X}$ is a normalized generalized Parseval frame satisfying the localization property (F).

Lemma 3.7. *In addition to the hypotheses of Theorem 3.6, assume $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the Localization Property (F). Then, for any $z \in X$ and any $0 < \delta < 1$ there exists $R > \frac{\delta_0}{2} > 0$ independent of z such that for all $r \geq R$ it holds*

$$
\frac{\#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)}{\lambda\left(B(z;r)\right)} \leq \frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')},
$$

where $0 < \gamma < 1$ is the special constant from Theorem 3.6, and γ' satisfies $\gamma < \gamma' < 1$.

Proof. We can deduce this lemma as a consequence of Theorem 3.6 and the main inequality in the proof of Theorem 2.7. This inequality says that for any given $0 < \delta < 1$ there exists $R > 0$ such that for all $z \in X$ and all $r \geq R$ it holds

$$
\frac{\eta_1\left(\varepsilon, B(z;r)\right)}{\lambda\left(B(z;r)\right)} \leq \frac{1}{\gamma'} + \frac{\delta}{(1-\varepsilon)(1-\gamma')},
$$

where $0 < \varepsilon < 1$ is fixed, γ' has been chosen such that $1-\varepsilon < \gamma' < 1$, and $\eta_1(B, \varepsilon)$ is the maximum dimension of a finite dimensional subspace $G \subseteq \mathcal{H}$ such that $\langle C_B g, g\rangle \geq 1-\varepsilon$ for all $g \in G$ with $\|g\| = 1$.

Pick $\varepsilon = 1-\gamma$, where γ is the special constant from Theorem 3.6 (recall $0 < \gamma < 1$). For any ball $B = B(z;r)$ with $r > \frac{\delta_0}{2}$, let $N := \#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)$ and define the finite dimensional subspace $G_B := \text{span}\{f_1, f_2, \dots, f_N\}$, where as usual $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is an orthonormal basis generated by eigenfunctions of C_B, f_i is an eigenfunction corresponding to the eigenvalue λ_i, and $1 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq 0$.

As a consequence of Theorem 3.6, for any $g \in G_B$ with $\|g\| = 1$, i.e., $g = \sum_{i=1}^{N} a_i f_i$ with $\sum_{i=1}^{N} |a_i|^2 = 1$, we obtain

$$\begin{aligned} \langle C_B g, g\rangle &= \sum_{i=1}^{N} \lambda_i |a_i|^2 \\ &\geq \lambda_N \sum_{i=1}^{N} |a_i|^2 \\ &\geq \gamma, \end{aligned}$$

then $\eta_1(1-\gamma, B) \geq \dim(G_B) = N = \#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)$.

Therefore, applying the main inequality in the proof of Theorem 2.7, for all $z \in X$ and all $r \geq R$ it holds

$$\begin{aligned} \frac{\#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)}{\lambda(B(z;r))} &\leq \frac{\eta_1(1-\gamma, B(z;r))}{\lambda(B(z;r))} \\ &\leq \frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')}. \end{aligned}$$

□

Alternative direct proof. Using the same reasoning as in the proof of Lemma 3.4, due to Proposition 2.4, for any $0 < \delta < 1$ there exists $R > \frac{\delta_0}{2} > 0$ such that for all $\Omega = B(z;r)$ with $r \geq R$ the concentration operator C_Ω satisfies $(1-\delta)\|C_\Omega\|_{S_1} \leq \|C_\Omega\|_{S_2}^2$, so, calling its eigenvalues $\lambda_n(B(z;r)) =$

λ_n (recall $1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots \geq 0$ for all n) we obtain that for any $0 < \gamma' < 1$

$$\begin{aligned}
(1-\delta)\sum_{n=1}^{\infty}\lambda_n &\leq \sum_{n=1}^{\infty}\lambda_n^2 \\
&= \sum_{\{n:\lambda_n>\gamma'\}}\lambda_n^2 + \sum_{\{n:\gamma'\geq\lambda_n\}}\lambda_n^2 \\
&\leq \sum_{\{n:\lambda_n>\gamma'\}}\lambda_n + \sum_{\{n:\gamma'\geq\lambda_n\}}\gamma'\lambda_n \\
&= \left(\sum_{n=1}^{\infty}\lambda_n - \sum_{\{n:\gamma'\geq\lambda_n\}}\lambda_n\right) + \gamma' \sum_{\{n:\gamma'\geq\lambda_n\}}\lambda_n.
\end{aligned}$$

This implies that for all $B(z;r)$ with $r \geq R$, and all γ' such that $0 < \gamma' < 1$

$$\sum_{\{n:\gamma'\geq\lambda_n(B(z;r))\}}\lambda_n\left(B(z;r)\right) \leq \frac{\delta}{1-\gamma'}\sum_{n=1}^{\infty}\lambda_n\left(B(z;r)\right).$$

Let $N = \#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)$. If $N = 0$ the inequality of Lemma 3.7 becomes trivial. Otherwise, by Theorem 3.6, $1 \geq \lambda_1 \geq \cdots \geq \lambda_N \geq \gamma$, which combined with the previous reasoning, for all γ' such that $0 < \gamma < \gamma' < 1$ gives

$$\begin{aligned}
\#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right) &= N \\
&\leq \#\left\{n : \lambda_n \geq \gamma\right\} \\
&= \#\left\{n : \lambda_n > \gamma'\right\} + \#\left\{n : \gamma' \geq \lambda_n \geq \gamma\right\} \\
&\leq \sum_{\{n:\lambda_n>\gamma'\}}\frac{\lambda_n}{\gamma'} + \sum_{\{n:\gamma'\geq\lambda_n\geq\gamma\}}\frac{\lambda_n}{\gamma} \\
&\leq \frac{1}{\gamma'}\sum_{n=1}^{\infty}\lambda_n + \frac{1}{\gamma}\sum_{\{n:\gamma'\geq\lambda_n\}}\lambda_n \\
&\leq \frac{1}{\gamma'}\sum_{n=1}^{\infty}\lambda_n + \frac{\delta}{\gamma(1-\gamma')}\sum_{n=1}^{\infty}\lambda_n \\
&= \left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')}\right)\sum_{n=1}^{\infty}\lambda_n.
\end{aligned}$$

Finally, due to Proposition 2.3, $\lambda\left(B(z;r)\right) = \sum_{n=1}^{\infty}\lambda_n\left(B(z;r)\right)$, so, the inequality of the Lemma 3.7 holds in this case too. This completes the proof. $\square$

Definition 3.5. The *upper density* of Γ, denoted by $D^+(\Gamma)$, is defined by

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{z\in X} \frac{\#\left(\Gamma \cap B(z;r)\right)}{\lambda\left(B(z;r)\right)}.$$

The theorem below is the main result of this section. Recall that all the assumptions (S1)-(S4) for the indexing metric measure space (X, d, λ) of the framed Hilbert space $\mathcal{H}$, and the assumptions (F) and (F2) for the normalized generalized Parseval frame $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ remain valid.

Theorem 3.8. *Let $\mathcal{H}$ be a framed Hilbert space with indexing set (X, d, λ). If $\Gamma \subseteq X$ is interpolating and δ_0-separated, then the upper density of Γ satisfies*

$$D^+(\Gamma) \leq 1.$$

Proof. Let $0 < \varepsilon < \frac{1}{2}$, let $z \in X$. Consider $0 < \gamma < 1$, the constant given by Theorem 3.6. Choose γ' such that $0 < \gamma < \gamma' < 1$, and $\gamma' > 1 - \varepsilon$. Choose δ such that $0 < \delta < 1$, and $\frac{\delta}{\gamma(1-\gamma')} < \frac{\varepsilon}{2}$. By Lemma 3.7, there exists $R_1 > \frac{\delta_0}{2} > 0$ such that for all $r_1 \geq R_1$ it holds

$$\frac{\#\left(\Gamma \cap B\left(z; r_1 - \frac{\delta_0}{2}\right)\right)}{\lambda\left(B(z;r_1)\right)} \leq \frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')} < \frac{1}{1-\varepsilon} + \frac{\varepsilon}{2}.$$

By Lemma 3.2, there exists $R_2 > \frac{\delta_0}{2} > 0$ such that for all $r_2 \geq R_2$ it holds

$$\frac{\#\left(\Gamma \cap \left[B(z;r_2) \setminus B\left(z; r_2 - \frac{\delta_0}{2}\right)\right]\right)}{\lambda\left(B(z;r_2)\right)} < \frac{\varepsilon}{2}.$$

Then, defining $R = \max\{R_1, R_2\}$, for all $r \geq R$ it holds

$$\begin{aligned}
\frac{\#\left(\Gamma \cap B(z;r)\right)}{\lambda\left(B(z;r)\right)} &= \frac{\#\left(\Gamma \cap B\left(z; r - \frac{\delta_0}{2}\right)\right)}{\lambda\left(B(z;r)\right)} + \frac{\#\left(\Gamma \cap \left[B(z;r) \setminus B\left(z; r - \frac{\delta_0}{2}\right)\right]\right)}{\lambda\left(B(z;r)\right)} \\
&< \frac{1}{1-\varepsilon} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= 1 + \varepsilon\left(\frac{1}{1-\varepsilon} + 1\right) \\
&< 1 + 3\varepsilon
\end{aligned}$$

Hence

$$\begin{aligned} D^+(\Gamma) &= \inf_{r' \geq 0} \sup_{r \geq r'} \sup_{z \in X} \frac{\#\left(\Gamma \cap B(z;r)\right)}{\lambda\left(B(z;r)\right)} \\ &\leq \sup_{r \geq R} \sup_{z \in X} \frac{\#\left(\Gamma \cap B(z;r)\right)}{\lambda\left(B(z;r)\right)} \\ &\leq 1 + 3\varepsilon \end{aligned}$$

Since $0 < \varepsilon < \frac{1}{2}$ is arbitrary, we conclude that

$$D^+(\Gamma) \leq 1.$$

This completes the proof. □

3.4 Applications: classical function spaces

The Paley-Wiener space $\mathcal{PW}_\alpha(\mathbb{R})$ is a framed Hilbert space (see Corollary 1.25), and it satisfies conditions (S1)-(S4), (F) and (F2), so, Theorems 3.5 and 3.8 can be applied to obtain necessary density conditions for sampling and interpolation. These density results for the Paley-Wiener space were proved in essence by Beurling, and later generalized by Landau [38]. It is important to mention that the proof of our main results from Chapter 2 are related with Landau's method, which was also used by Lindholm in [41].

The $1-$dimensional Bargmann-Fock space $\mathcal{F}^2_\alpha(\mathbb{C})$ is a framed Hilbert space (see Corollary 1.27), and it satisfies conditions (S1)-(S4), (F) and (F2), so, Theorems 3.5 and 3.8 can be applied to obtain necessary density conditions for sampling and interpolation. However, in this case there are stronger density results (if and only if condition) due to Seip [56] and [61].

The $1-$dimensional Bergman space $\mathcal{B}^2_\alpha(\mathbb{D})$ is a framed Hilbert space (see Corollary 1.29). This space is problematic for our approach since it does not satisfy the assumptions on Theorems 3.5 and 3.8, the Möbius invariant measure on $\mathbb{D}$ is not doubling, and thus the annular decay property is not guaranteed. However, Seip was able to bypass these inconveniences given the rich structure of this space, and proved stronger density results (if and only if condition) in [58].

Chapter 4

Toeplitz operators

Assume $(\mathcal{H}, X, k)$ is a framed Hilbert space, where $\{k_x\}_{x\in X} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame for $\mathcal{H}$ satisfying the mean value property (F2) from the previous chapter.

4.1 Basic results

Definition 4.1. Given $a \in L^1(X, \lambda)$, the *Toeplitz operator* $T_a : \mathcal{H} \to \mathcal{H}$ with *symbol* a and with respect to the generalized Parseval frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$, is defined in weak sense by

$$T_a f = \int_X a(x) \langle f, k_x \rangle k_x d\lambda(x), \quad f \in \mathcal{H}.$$

Proposition 4.1. *The Toeplitz operator T_a with symbol $a \in L^1(X, \lambda)$ is a well-defined bounded linear operator.*

Proposition 4.2. *The Toeplitz operator T_a with symbol $a \in L^1(X, \lambda)$ is self-adjoint, and it is positive provided that the symbol a is nonnegative. Moreover, $\|T_a\| \leq 1$ whenever $|a(x)| \leq 1$ for $\lambda-$a.e. x, and $\{k_x\}_{x\in(X,\lambda)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$.*

Definition 4.2. Let $\{k_x\}_{x\in(X,\lambda)} \subseteq \mathcal{H}$ be a generalized Parseval frame for $\mathcal{H}$. Given a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, the *Berezin transform* associated to T with respect to $\{k_x\}_{x\in(X,\lambda)}$ is defined by the function

$$\widetilde{T}(x) = \langle Tk_x, k_x \rangle, \quad x \in (X, \lambda).$$

In the case of a Toeplitz operator T_a, we use the notation $\widetilde{T_a}(x) = \widetilde{a}(x)$.

The following results give relationships between a bounded linear operator T and its Berezin transform $\widetilde{T}$ (with respect to a generalized Parseval frame) under a very general setup. We are particularly interested in the case when the operator T is a Toeplitz operator T_a.

Proposition 4.3. *Given a trace class operator $T : \mathcal{H} \to \mathcal{H}$, the trace of T can be calculated using its Berezin transform associated to T with respect a generalized Parseval frame $\{k_x\}_{x\in(X,\lambda)} \subseteq \mathcal{H}$ as follows*

$$\operatorname{tr}(T) = \int_X \widetilde{T}(y) d\lambda(y) = \int_X \langle Tk_y, k_y\rangle \, d\lambda(y).$$

Proof. Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$, then applying properties of generalized Parseval frames and orthonormal basis, we obtain

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{n=1}^{\infty} \langle Tf_n, f_n\rangle \\
&= \sum_{n=1}^{\infty} \left\langle \int_X \langle Tf_n, k_x\rangle k_x d\lambda(x), \int_X \langle f_n, k_y\rangle k_y d\lambda(y) \right\rangle \\
&= \sum_{n=1}^{\infty} \int_X \int_X \langle Tf_n, k_x\rangle \langle k_x, k_y\rangle \langle k_y, f_n\rangle \, d\lambda(x) d\lambda(y) \\
&= \int_X \int_X \langle k_x, k_y\rangle \sum_{n=1}^{\infty} \langle k_y, f_n\rangle \langle Tf_n, k_x\rangle \, d\lambda(x) d\lambda(y) \\
&= \int_X \int_X \langle k_x, k_y\rangle \langle Tk_y, k_x\rangle \, d\lambda(x) d\lambda(y) \\
&= \int_X \langle Tk_y, k_y\rangle \, d\lambda(y).
\end{aligned}$$

□

The following theorem assumes the mean value property (F2) is valid.

Theorem 4.4. [40, Chapters 3, 4, and 5] *If the metric measure space (X, d, λ) satisfies the conditions*

S1) λ is a positive Borel measure with respect to d.

S2) (X, d, λ) has the following covering property: for any $r > 0$ there exist $N > 0$ and a collection of Borel sets $\{F_n\}_{n=1}^{\infty} \subseteq X$ such that

i) $X = \bigsqcup_{n=1}^{\infty} F_n$, disjoint union.

ii) Any $x \in X$ is contained in at most N sets of $\{G_n\}_{n=1}^{\infty} \subseteq X$, where G_n is a $r-$neighborhood of F_n, $G_n := \{x \in X : d(x, F_n) \leq r\}$.

iii) $\operatorname{diam}(F_n) \leq r$ *for all n.*

iv) There exists constants $A_r, B_r \geq 0$ such that $A_r \leq \lambda(F_n) \leq \lambda(G_n) \leq B_r$ for all n.

Then the following criteria hold between the Toeplitz operator $T_a : \mathcal{H} \to \mathcal{H}$ and its Berezin transform $\widetilde{a} : X \to \mathbb{R}$:

1. *T_a is bounded if and only if $\widetilde{a}$ is bounded.*

2. *T_a is compact if and only if $\widetilde{a}(x) \to 0$ as $x \to \infty$ (this means $d(x, y) \to \infty$ for any fixed $y \in X$).*

3. *$T_a \in S_p$ if and only if $\widetilde{a} \in L^p(X, \lambda)$.*

Note that if the symbol $a \in L^1(X, \lambda)$ of the Toeplitz operator T_a is such that $0 \leq a(x) \leq 1$ for $\lambda-$a.e. x, and $\widetilde{a}(x) = \langle T_a k_x, k_x \rangle \to 0$ as $x \to \infty$, then T_a is positive and compact with $\|T_a\| \leq 1$. Thus T_a has a sequence of eigenvalues $1 \geq \lambda_1(a) \geq \lambda_2(a) \geq \cdots \geq 0$, and there exists an orthonormal basis for $\mathcal{H}$ consisting of eigenfunction $\{f_i(a)\}_{i=1}^{\infty} \subseteq \mathcal{H}$, where $f_i(a)$ is an eigenfunction corresponding to $\lambda_i(a)$. If there is no room for confusion, we write λ_i and f_i instead.

Proposition 4.5. *If T_a is a compact Toeplitz operator with symbol $0 \leq a(x) \leq 1$ for $\lambda-$a.e. x, then T_a is a trace class and a Hilbert-Schmidt class operator, satisfying*

$$\begin{aligned}
\|T_a\|_{S_1} &= \int_X \int_X a(x) \left|\langle k_x, k_y \rangle\right|^2 d\lambda(x) d\lambda(y) = \|a\|_{L^1(X,\lambda)} = \|\widetilde{a}\|_{L^1(X,\lambda)}, \\
\|T_a\|_{S_2}^2 &= \int_X \int_X a(x) a(y) \left|\langle k_x, k_y \rangle\right|^2 d\lambda(x) d\lambda(y) = \|a\,\widetilde{a}\|_{L^1(X,\lambda)}.
\end{aligned}$$

Proof. Since T_a is self-adjoint, its eigenvalues coincide with its singular values, then

$$\begin{aligned}
\|T_a\|_{\mathrm{tr}} &= \|T_a\|_{S_1} \\
&= \sum_{i=1}^{\infty} \lambda_i \\
&= \sum_{i=1}^{\infty} \langle T_a f_i, f_i \rangle \\
&= \sum_{i=1}^{\infty} \left\langle \int_X a(x) \langle f_i, k_x \rangle k_x d\lambda(x), f_i \right\rangle \\
&= \sum_{i=1}^{\infty} \int_X a(x) \left|\langle f_i, k_x \rangle\right|^2 d\lambda(x) \\
&= \int_X a(x) \left(\sum_{i=1}^{\infty} \left|\langle f_i, k_x \rangle\right|^2 \right) d\lambda(x) \\
&= \int_X a(x) \|k_x\|^2 d\lambda(x) \\
&= \int_X a(x) d\lambda(x) \\
&< \infty.
\end{aligned}$$

So T_a is a trace class operator satisfying $\mathrm{tr}(T_a) = \|T_a\|_{S_1} = \|a\|_{L^1(X,\lambda)} = \|\widetilde{a}\|_{L^1(X,\lambda)}$, where the last equality is due to Proposition 4.3. Moreover, by Tonelli's Theorem we also have

$$\begin{aligned}
\int_X \int_X a(x) \left|\langle k_y, k_x \rangle\right|^2 d\lambda(x) d\lambda(y) &= \int_X a(x) \left(\int_X \left|\langle k_y, k_x \rangle\right|^2 d\lambda(y) \right) d\lambda(x) \\
&= \int_X a(x) \|k_x\|^2 d\lambda(x) \\
&= \mathrm{tr}(T_a).
\end{aligned}$$

On the other hand $\|T_a\|_{S_2}^2 = \sum_{i=1}^{\infty} \lambda_i^2 \le \sum_{i=1}^{\infty} \lambda_i < \infty$, and also

$$\begin{aligned}
\|T_a\|_{S_2}^2 &= \sum_{i=1}^{\infty} \langle T_a f_i, T_a f_i \rangle \\
&= \sum_{i=1}^{\infty} \left\langle \int_X a(x) \langle f_i, k_x \rangle k_x d\lambda(x), \int_X a(y) \langle f_i, k_y \rangle k_y d\lambda(y) \right\rangle \\
&= \sum_{i=1}^{\infty} \int_X \int_X a(x)a(y) \langle f_i, k_x \rangle \langle k_x, k_y \rangle \langle k_y, f_i \rangle d\lambda(x) d\lambda(y) \\
&= \int_X \int_X a(x)a(y) \langle k_x, k_y \rangle \left(\sum_{i=1}^{\infty} \langle k_y, f_i \rangle \langle f_i, k_x \rangle \right) d\lambda(x) d\lambda(y) \\
&= \int_X \int_X a(x)a(y) \left| \langle k_x, k_y \rangle \right|^2 d\lambda(x) d\lambda(y) \\
&= \int_X a(y) \widetilde{a}(y) d\lambda(y).
\end{aligned}$$

□

The assumptions on the symbol $a(x)$ guarantee that $a(x)^m$, $m \in \mathbb{N} = \{1, 2, \ldots\}$, satisfies the same assumptions. Then T_{a^m} is a trace class operator. On the other hand, T_a^m is also a trace class operator since such class of operators is a two-sided ideal.

Furthermore, since there is an orthonormal basis of eigenfunctions associated to the self-adjoint Toeplitz operator T_a (as usual, $1 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge 0$ are the eigenvalues of T_a, and f_i is an eigenfunction associated to λ_i such that $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is an orthonormal basis), there is a *continuous functional calculus* associated to T_a as follows:

$$T_a f = \sum_{i=1}^{\infty} \lambda_i \langle f, f_i \rangle f_i, \quad f \in \mathcal{H},$$

allows us to define a bounded linear operator $h(T_a) : \mathcal{H} \to \mathcal{H}$ by

$$h(T_a) f = \sum_{i=1}^{\infty} h(\lambda_i) \langle f, f_i \rangle f_i, \quad f \in \mathcal{H},$$

where $h : [0,1] \to \mathbb{C}$ is continuous (notice the spectrum $\sigma(T_a) \subseteq [0,1]$). Even further, if $h : [0,1] \to \mathbb{C}$ is bounded but not necessarily continuous, $h(T_a) : \mathcal{H} \to \mathcal{H}$ as defined above is still a bounded linear operator due to the *extended functional calculus* associated to T_a.

Proposition 4.6. *If T_a is a compact Toeplitz operator with symbol $0 \leq a(x) \leq 1$ for $\lambda-$a.e. x, then all the operators T_a^m and T_{a^m} for $m \in \mathbb{N}$ are positive, trace class, and compact. They satisfy*

$$\begin{aligned} \operatorname{tr}(T_a^m) &= \int_X a(x) \left\langle T_a^{m-1} k_x, k_x \right\rangle d\lambda(x) = \|T_a\|_{S_m}^m, \\ \operatorname{tr}(T_{a^m}) &= \int_X a(x)^m d\lambda(x). \end{aligned}$$

Consequently

$$\operatorname{tr}(T_a) \geq \operatorname{tr}\left(T_a^2\right) \geq \cdots \geq \operatorname{tr}(T_a^m) \geq \cdots \geq 0,$$

$$\operatorname{tr}(T_a) \geq \operatorname{tr}(T_{a^2}) \geq \cdots \geq \operatorname{tr}(T_{a^m}) \geq \cdots \geq 0.$$

Proof. First we show the results for T_a^m. Notice that $T_a^m = T_a T_a^{m-1}$, $m \geq 1$, so T_a^m is a trace class operator since T_a is a trace class operator, and such class is a two-sided ideal. Same reasoning for compactness. Furthermore, since $T_a^m = h(T_a)$ for $h(t) = t^m$ which is continuous and it is defined on $[0, 1]$, the continuous functional calculus on T_a (which is compact and self-adjoint) gives the following spectral resolution for T_a^m

$$T_a^m f = \sum_{i=1}^{\infty} \lambda_i^m \langle f, f_i \rangle f_i, \quad f \in \mathcal{H},$$

where $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ are the eigenvalues of T_a and $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ consisting of eigenfunctions of T_a. From here, since $h(t) = t^m$ is real and nonnegative, then T_a^m is self-adjoint and positive because for any $f, g \in \mathcal{H}$

$$\begin{aligned} \langle T_a^m f, g \rangle &= \left\langle \sum_{i=1}^{\infty} \lambda_i^m \langle f, f_i \rangle f_i, \sum_{j=1}^{\infty} \langle g, f_j \rangle f_j \right\rangle \\ &= \sum_{i=1}^{\infty} \lambda_i^m \langle f, f_i \rangle \langle f_i, g \rangle \\ &= \langle f, T_a^m g \rangle, \end{aligned}$$

given that the eigenvalues of T_a satisfy $\lambda_i \geq 0$ for all $i \in \mathbb{N}$. Thus T_a^m is self-adjoint. In particular, for $f \in \mathcal{H}$

$$\langle T_a^m f, f \rangle = \sum_{i=1}^{\infty} \lambda_i^m |\langle f, f_i \rangle|^2 \geq 0,$$

so T_a^m is positive. Furthermore, for $1 \leq n \leq m$

$$0 \leq \operatorname{tr}\left(T_a^m\right) = \left\|T_a^m\right\|_{S_1} = \sum_{i=1}^{\infty} \lambda_i^m = \left\|T_a\right\|_{S_m}^m \leq \operatorname{tr}\left(T_a^n\right).$$

Additionally to this, applying Proposition 4.3 on T_a^m and Fubini's Theorem

$$\begin{aligned}
\operatorname{tr}\left(T_a^m\right) &= \int_X \widetilde{T_a^m}(y) d\lambda(y) \\
&= \int_X \left\langle T_a^m k_y, k_y \right\rangle d\lambda(y) \\
&= \int_X \left\langle T_a\left(T_a^{m-1} k_y\right), k_y \right\rangle d\lambda(y) \\
&= \int_X \left\langle \int_X a(x) \left\langle T_a^{m-1} k_y, k_x \right\rangle k_x d\lambda(x), k_y \right\rangle d\lambda(y) \\
&= \int_X \int_X a(x) \left\langle T_a^{m-1} k_y, k_x \right\rangle \left\langle k_x, k_y \right\rangle d\lambda(x) d\lambda(y) \\
&= \int_X a(x) \int_X \left\langle k_x, k_y \right\rangle \left\langle k_y, \left(T_a^{m-1}\right)^* k_x \right\rangle d\lambda(y) d\lambda(x) \\
&= \int_X a(x) \left\langle \int_X \left\langle k_x, k_y \right\rangle k_y d\lambda(y), \left(T_a^{m-1}\right)^* k_x \right\rangle d\lambda(x) \\
&= \int_X a(x) \left\langle k_x, \left(T_a^{m-1}\right)^* k_x \right\rangle d\lambda(x) \\
&= \int_X a(x) \left\langle T_a^{m-1} k_x, k_x \right\rangle d\lambda(x).
\end{aligned}$$

Next we show the results for T_{a^m}, $m \in \mathbb{N}$. Notice that for $f, g \in \mathcal{H}$ we have

$$\begin{aligned}
\left\langle T_{a^m} f, g \right\rangle &= \left\langle \int_X a(x)^m \left\langle f, k_x \right\rangle k_x d\lambda(x), g \right\rangle \\
&= \int_X a(x)^m \left\langle f, k_x \right\rangle \left\langle k_x, g \right\rangle d\lambda(x) \\
&= \left\langle f, \int_X a(x)^m \left\langle g, k_x \right\rangle k_x d\lambda(x) \right\rangle \\
&= \left\langle f, T_{a^m} g \right\rangle,
\end{aligned}$$

since $a(x) \geq 0$ a.e., so T_{a^m} is self-adjoint. In particular, for $f \in \mathcal{H}$ we have

$$\left\langle T_{a^m} f, f \right\rangle = \int_X a(x)^m \left|\left\langle f, k_x \right\rangle\right|^2 d\lambda(x) \geq 0,$$

then T_{a^m} is positive. Observe that for any $y \in X$

$$\begin{aligned} 0 \le \widetilde{T_{a^m}}(y) &= \langle T_{a^m} k_y, k_y \rangle \\ &= \int_X a(x)^m \left| \langle k_y, k_x \rangle \right|^2 d\lambda(x) \\ &\le \int_X a(x) \left| \langle k_y, k_x \rangle \right|^2 d\lambda(x) \\ &= \langle T_a k_y, k_y \rangle \\ &= \widetilde{T_a}(y), \end{aligned}$$

since $0 \le a(x) \le 1$ a.e. Applying Theorem 4.4, T_a compact implies $\widetilde{T_a}(y) \to 0$ as $y \to \infty$, which combined with the inequality above implies $\widetilde{T_{a^m}}(y) \to 0$ as $y \to \infty$. Hence T_{a^m} is compact. Applying Proposition 4.3 on T_{a^m} and Fubini's Theorem

$$\begin{aligned} \operatorname{tr}\left(T_{a^m}\right) &= \int_X \widetilde{T_{a^m}}(y) d\lambda(y) \\ &= \int_X \int_X a(x)^m \left| \langle k_y, k_x \rangle \right|^2 d\lambda(x) d\lambda(y) \\ &= \int_X a(x)^m d\lambda(x), \end{aligned}$$

so using again $0 \le a(x) \le 1$ a.e., for any $1 \le n \le m$

$$0 \le \operatorname{tr}\left(T_{a^m}\right) = \int_X a(x)^m d\lambda(x) \le \int_X a(x)^n d\lambda(x) = \operatorname{tr}\left(T_{a^n}\right).$$

□

Proposition 4.7. *If T_a is a compact Toeplitz operator with symbol $0 \le a(x) \le 1$ for $\lambda-$a.e. x, then for any $m \in \mathbb{N}$*

$$\begin{aligned} 1 \ge \langle T_a k_x, k_x \rangle &\ge \langle T_a^m k_x, k_x \rangle \ge & \langle T_a k_x, k_x \rangle^m \ge 0, \\ 1 \ge \langle T_a k_x, k_x \rangle &\ge \langle T_{a^m} k_x, k_x \rangle \ge & \langle T_a k_x, k_x \rangle^m \ge 0. \end{aligned}$$

Remark. As positive operators the following order relationships hold

$$I \ge T_a \ge \cdots \ge T_a^m \ge \cdots \ge 0, \qquad I \ge T_a \ge \cdots \ge T_{a^m} \ge \cdots \ge 0.$$

Proof. Due to Proposition 4.6, the operators T_a^m and T_{a^m}, $m \in \mathbb{N}$, are positive. Using some of the calculations in the proof of Proposition 4.6, we get $0 \leq \langle T_a^m k_x, k_x \rangle \leq \langle T_a k_x, k_x \rangle \leq 1$ which implies $I \geq T_a \geq \cdots \geq T_a^m \geq \cdots \geq 0$. This is because

$$\begin{aligned} 0 \leq \langle T_a^m k_x, k_x \rangle &= \sum_{i=1}^{\infty} \lambda_i^m \left| \langle k_x, f_i \rangle \right|^2 \\ &\leq \sum_{i=1}^{\infty} \lambda_i \left| \langle k_x, f_i \rangle \right|^2 = \langle T_a k_x, k_x \rangle \\ &\leq \sum_{i=1}^{\infty} \left| \langle k_x, f_i \rangle \right|^2 = 1, \end{aligned}$$

Similarly $0 \leq \langle T_{a^m} k_x, k_x \rangle \leq \langle T_a k_x, k_x \rangle \leq 1$ holds and hence $I \geq T_a \geq \cdots \geq T_{a^m} \geq \cdots \geq 0$ because

$$\begin{aligned} 0 \leq \langle T_{a^m} k_x, k_x \rangle &= \int_X a(y)^m \left| \langle k_x, k_y \rangle \right|^2 d\lambda(y) \\ &\leq \int_X a(y) \left| \langle k_x, k_y \rangle \right|^2 d\lambda(y) = \langle T_a k_x, k_x \rangle \\ &\leq \int_X \left| \langle k_x, k_y \rangle \right|^2 d\lambda(y) = 1, \end{aligned}$$

Next we prove the inequalities $\langle T_a^m k_x, k_x \rangle \geq \langle T_a k_x, k_x \rangle^m$ and $\langle T_{a^m} k_x, k_x \rangle \geq \langle T_a k_x, k_x \rangle^m$. The case $m = 1$ is trivial, and if $m > 1$ we can apply Hölder's inequality [65, Proposition 1.31]. Let m' be the conjugate index of $m > 1$, i.e., $\frac{1}{m} + \frac{1}{m'} = 1$. Then

$$\begin{aligned} \langle T_a k_x, k_x \rangle &= \int_X a(y) \left| \langle k_x, k_y \rangle \right|^{\frac{2}{m}} \left| \langle k_x, k_y \rangle \right|^{\frac{2}{m'}} d\lambda(y) \\ &\leq \left(\int_X a(y)^m \left| \langle k_x, k_y \rangle \right|^2 d\lambda(y) \right)^{\frac{1}{m}} \left(\int_X \left| \langle k_x, k_y \rangle \right|^2 d\lambda(y) \right)^{\frac{1}{m'}} \\ &= \langle T_{a^m} k_x, k_x \rangle^{\frac{1}{m}}, \end{aligned}$$

and also

$$\begin{aligned} \langle T_a k_x, k_x \rangle &= \sum_{i=1}^{\infty} \lambda_i \left| \langle k_x, f_i \rangle \right|^{\frac{2}{m}} \left| \langle k_x, f_i \rangle \right|^{\frac{2}{m'}} \\ &\leq \left(\sum_{i=1}^{\infty} \lambda_i^m \left| \langle k_x, f_i \rangle \right|^2 \right)^{\frac{1}{m}} \left(\sum_{i=1}^{\infty} \left| \langle k_x, f_i \rangle \right|^2 \right)^{\frac{1}{m'}} \\ &= \langle T_a^m k_x, k_x \rangle^{\frac{1}{m}}. \end{aligned}$$

□

4.2 Asymptotic behavior 1

In this section we assume all the Toeplitz operators in consideration are positive and compact, with spectrum contained in $[0,1]$.

Proposition 4.8. *Suppose $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of compact Toeplitz operators such that their symbols satisfy $0 \leq a_n(x) \leq 1$ for $\lambda-$a.e. x, for all n. The following localization holds*

$$\frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}} \to 1 \text{ as } n \to \infty.$$

if and only if for any $0 < \delta < 1$, there exists $N = N(\delta)$ such that for all $n \geq N$ it holds

$$(1-\delta) \|T_{a_n}\|_{S_1} \leq \|T_{a_n}\|_{S_2}^2 \leq \|T_{a_n}\|_{S_1}.$$

Proof. The inequality $\|T_{a_n}\|_{S_2}^2 \leq \|T_{a_n}\|_{S_1}$ is true for all n due to the assumption $0 \leq a_n \leq 1$ and Proposition 4.5. Moreover, $\|T_{a_n}\|_{S_1} = \|a_n\|_{L^1}$ and $\|T_{a_n}\|_{S_2}^2 = \|a_n \widetilde{a_n}\|_{L^1}$ for all n due to the same Proposition 4.5. Thus

$$1 \geq \frac{\|T_{a_n}\|_{S_2}^2}{\|T_{a_n}\|_{S_1}} = \frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}}.$$

From here, it is clear that the localization condition is equivalent to say that for any $0 < \delta < 1$, there exists $N = N(\delta)$ such that for all $n \geq N$ it holds $(1-\delta) \|T_{a_n}\|_{S_1} \leq \|T_{a_n}\|_{S_2}^2$. □

The next results can be proved using the same scheme as in the proofs of Theorems 2.7 and 2.8, and Corollaries 2.9 and 2.10, respectively, where we make use of Proposition 4.8 instead of Proposition 2.4.

We assume $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of Toeplitz operators satisfying the assumptions of Proposition 4.8.

Theorem 4.9. *Suppose $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of Toeplitz operators such that*

1. *For all $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ is compact, trace class, and $0 \leq a_n(x) \leq 1$.*

2. *The following localization condition holds:*

$$\frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}} \to 1 \text{ as } n \to \infty.$$

Let $G \subseteq \mathcal{H}$ denote an arbitrary finite dimensional subspace. Given $0 < \varepsilon < 1$ and n, let $\vartheta_1(\varepsilon, n)$ be the maximum dimension of G such that $\langle T_{a_n} g, g\rangle \geq 1 - \varepsilon$ for all $g \in G$ with $\|g\| = 1$. Then

$$\limsup_{n\to\infty} \frac{\vartheta_1(\varepsilon, n)}{\mathrm{tr}(T_{a_n})} \leq 1.$$

Proof. For any $n \in \mathbb{N}$, let $\{\lambda_i(n)\}_{i=1}^{\infty}$ be the eigenvalues of T_{a_n}, $1 \geq \lambda_1(n) \geq \lambda_2(n) \geq \cdots \geq 0$, and let $\{f_i(n)\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ of eigenfunctions of T_{a_n}, where $f_i(n)$ corresponds to $\lambda_i(n)$. The conclusion follows from the proof of Theorem 2.7 but applying Proposition 4.8 instead of Proposition 2.4. □

Theorem 4.10. *Suppose $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of Toeplitz operators such that*

1. *For all $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ is compact, trace class, and $0 \leq a_n(x) \leq 1$.*

2. *The following localization condition holds:*

$$\frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}} \to 1 \text{ as } n \to \infty.$$

Let $G \subseteq \mathcal{H}$ denote an arbitrary finite dimensional subspace. Given $0 < \varepsilon < 1$ and n, let $\vartheta_2(\varepsilon, n)$ be the minimum dimension of G such that $\langle T_{a_n} g, g\rangle < 1 - \varepsilon$ for all $g \in G^{\perp}$ with $\|g\| = 1$. Then

$$\liminf_{n\to\infty} \frac{\vartheta_2(\varepsilon, n)}{\mathrm{tr}(T_{a_n})} \geq 1.$$

Proof. Follows from Proposition 4.8 and the proof of Theorem 2.8. □

Corollary 4.11. *Suppose $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of Toeplitz operators such that*

1. *For all $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ is compact, trace class, and $0 \le a_n(x) \le 1$.*
2. *The following localization condition holds:*

$$\frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}} \to 1 \text{ as } n \to \infty.$$

Given $0 < \varepsilon < 1$, and n, let

$$\vartheta_3(\varepsilon, n) := \#\{\lambda_i \in \sigma(T_{a_n}) : \lambda_i \ge 1 - \varepsilon\},$$

where $\sigma(T_{a_n})$ denotes the spectrum of the Toeplitz operator T_{a_n}. Then

$$\lim_{n \to \infty} \frac{\vartheta_3(\varepsilon, n)}{\operatorname{tr}(T_{a_n})} = 1.$$

Proof. By Theorems 4.9 and 4.10, and using the same scheme as in the proof of Corollary 2.9, we obtain

$$\limsup_{n \to \infty} \frac{\vartheta_3(\varepsilon, n)}{\operatorname{tr}(T_{a_n})} = \liminf_{n \to \infty} \frac{\vartheta_3(\varepsilon, n)}{\operatorname{tr}(T_{a_n})} = 1,$$

and the conclusion follows. □

Corollary 4.12. *Suppose $\{T_{a_n}\}_{n=1}^{\infty}$ is a sequence of Toeplitz operators such that*

1. *For all $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ is compact, trace class, and $0 \le a_n(x) \le 1$.*
2. *The following localization condition holds:*

$$\frac{\|a_n \widetilde{a_n}\|_{L^1}}{\|a_n\|_{L^1}} \to 1 \text{ as } n \to \infty.$$

Given $0 < \varepsilon_1 < \varepsilon_2 < 1$, and n, let

$$\vartheta_4(\varepsilon_1, \varepsilon_2, n) := \#\{\lambda_i \in \sigma(T_{a_n}) : 1 - \varepsilon_2 \le \lambda_i < 1 - \varepsilon_1\},$$

where $\sigma(T_{a_n})$ denotes the spectrum of the Toeplitz operator T_{a_n}. Then

$$\lim_{n\to\infty} \frac{\vartheta_4(\varepsilon_1, \varepsilon_2, n)}{\operatorname{tr}(T_{a_n})} = 0.$$

Proof. By Theorems 4.9 and 4.10, and using the same scheme as in the proof of Corollary 2.10, we obtain

$$\limsup_{n\to\infty} \frac{\vartheta_4(\varepsilon_1, \varepsilon_2, n)}{\operatorname{tr}(T_{a_n})} = 0,$$

and the conclusion follows. □

4.3 Asymptotic behavior 2

Proposition 4.13. *If T_a is a compact Toeplitz operator with symbol $0 \le a(x) \le 1$ for $\lambda-$a.e. x, then for any $m \in \mathbb{N}$*

$$|\operatorname{tr}(T_a^m) - \operatorname{tr}(T_{a^m})| \quad \le \quad \frac{m(m-1)}{2} \left(\|[a - \widetilde{a}]\, a\|_{L^1} + \|[a - \widetilde{a}]\, \widetilde{a}\|_{L^1} \right).$$

Proof. If $m = 1$ the statement is trivial. If $m = 2$, then according with Proposition 4.6

$$\begin{aligned} \left|\operatorname{tr}\left(T_a^2\right) - \operatorname{tr}(T_{a^2})\right| &= \left| \int_X a(x)\widetilde{a}(x) d\lambda(x) - \int_X a(x)^2 d\lambda(x) \right| \\ &= \left| \int_X a(x) \left[\widetilde{a}(x) - a(x)\right] d\lambda(x) \right| \\ &\le \|[a - \widetilde{a}]\, a\|_{L^1} \\ &\le \|[a - \widetilde{a}]\, a\|_{L^1} + \|[a - \widetilde{a}]\, \widetilde{a}\|_{L^1}, \end{aligned}$$

so the result holds in this case too. Next we proceed by induction. Assume $m \ge 3$ and that the result holds for $m-1$. Applying Propositions 4.6 and 4.7, and using $0 \le a(x) \le 1$ a.e., we obtain

$$
\begin{aligned}
& \left|\operatorname{tr}\left(T_a^m\right) - \operatorname{tr}\left(T_{a^m}\right)\right| \\
= \; & \left| \int_X \left[a(x) \left\langle T_a^{m-1} k_x, k_x \right\rangle - a(x)\widetilde{a}(x)^{m-1} + a(x)\widetilde{a}(x)^{m-1} - a(x)^m \right] d\lambda(x) \right| \\
\le \; & \left| \int_X a(x) \left[\left\langle T_a^{m-1} k_x, k_x \right\rangle - \widetilde{a}(x)^{m-1} \right] d\lambda(x) \right| + \left| \int_X a(x) \left[\widetilde{a}(x)^{m-1} - a(x)^{m-1} \right] d\lambda(x) \right| \\
\le \; & \int_X \left[\left\langle T_a^{m-1} k_x, k_x \right\rangle - \widetilde{a}(x)^{m-1} \right] d\lambda(x) + \int_X \left| \widetilde{a}(x)^{m-1} - a(x)^{m-1} \right| d\lambda(x) \\
= \; & \int_X \left[\left\langle T_a^{m-1} k_x, k_x \right\rangle - a(x)^{m-1} \right] d\lambda(x) + \int_X \left[a(x)^{m-1} - \widetilde{a}(x)^{m-1} \right] d\lambda(x) \\
& + \int_X \left| \widetilde{a}(x)^{m-1} - a(x)^{m-1} \right| d\lambda(x) \\
\le \; & \left| \operatorname{tr}\left(T_a^{m-1}\right) - \operatorname{tr}\left(T_{a^{m-1}}\right) \right| + 2 \int_X \left| \widetilde{a}(x)^{m-1} - a(x)^{m-1} \right| d\lambda(x) \\
\le \; & \left| \operatorname{tr}\left(T_a^{m-1}\right) - \operatorname{tr}\left(T_{a^{m-1}}\right) \right| \\
& + 2 \int_X \left| \widetilde{a}(x) - a(x) \right| \left[\left|\widetilde{a}(x)\right|^{m-2} + \left|a(x)\right|^{m-3} \left|\widetilde{a}(x)\right| + \cdots + \left|\widetilde{a}(x)\right|^{m-2} \right] d\lambda(x) \\
\le \; & \left| \operatorname{tr}\left(T_a^{m-1}\right) - \operatorname{tr}\left(T_{a^{m-1}}\right) \right| + (m-1) \int_X \left| \widetilde{a}(x) - a(x) \right| \left(|a(x)| + |\widetilde{a}(x)| \right) d\lambda(x) \\
\le \; & \frac{(m-1)(m-2)}{2} \left(\left\| [a - \widetilde{a}]\, a \right\|_{L^1} + \left\| [a - \widetilde{a}]\, \widetilde{a} \right\|_{L^1} \right) + (m-1) \left(\left\| [a - \widetilde{a}]\, a \right\|_{L^1} + \left\| [a - \widetilde{a}]\, \widetilde{a} \right\|_{L^1} \right) \\
= \; & \frac{m(m-1)}{2} \left(\left\| [a - \widetilde{a}]\, a \right\|_{L^1} + \left\| [a - \widetilde{a}]\, \widetilde{a} \right\|_{L^1} \right).
\end{aligned}
$$

□

The following is a generalization to the main result, Theorem 2.1, in [22].

Theorem 4.14. *Suppose that* $\{T_{a_n}\}_{n=1}^{\infty}$ *is a sequence of Toeplitz operators such that*

1. *For all* $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ *is compact, trace class, and* $0 \leq a_n(x) \leq 1$.

2. *The following localization condition holds for any* $m \in \mathbb{N}$*:*

$$\frac{\|[a_n - \widetilde{a_n}]\, a\|_{L^1} + \|[a_n - \widetilde{a_n}]\, \widetilde{a}\|_{L^1}}{\|a_n^m\|_{L^1}} \to 0 \ \text{as}\ n \to \infty.$$

Then, for any $h_1 : [0,1] \to \mathbb{C}$ *continuous it holds*

$$\lim_{n\to\infty} \frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)} = 1,$$

where $h(t) = t\, h_1(t)$.

Proof. Suppose $m > 1$ is fixed. Due to assumption (2), for $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ the following inequality holds

$$\frac{\|[a_n - \widetilde{a_n}]\, a\|_{L^1} + \|[a_n - \widetilde{a_n}]\, \widetilde{a}\|_{L^1}}{\|a_n^m\|_{L^1}} < \frac{2\varepsilon}{m(m-1)},$$

then, by Propositions 4.6 and 4.13, we obtain that for all $n \geq N$

$$\begin{aligned} \left| \frac{\operatorname{tr}\left(T_{a_n}^m\right)}{\operatorname{tr}\left(T_{a_n^m}\right)} - 1 \right| &= \left| \frac{\operatorname{tr}\left(T_{a_n}^m\right) - \operatorname{tr}\left(T_{a_n^m}\right)}{\operatorname{tr}\left(T_{a_n^m}\right)} \right| \\ &\leq \frac{m(m-1)}{2} \left(\frac{\|[a_n - \widetilde{a_n}]\, a\|_{L^1} + \|[a_n - \widetilde{a_n}]\, \widetilde{a}\|_{L^1}}{\|a_n^m\|_{L^1}} \right) \\ &< \varepsilon. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)} = 1,$$

for the monomials $h(t) = t^m$ with $m > 1$, and the same result trivially holds when $m = 1$.

For the general case, consider $h(t) = t\, h_1(t)$, where $h_1(t) : [0,1] \to \mathbb{C}$ is continuous on $[0,1]$. Fix $n \in \mathbb{N}$. Recall T_{a_n} is compact and trace class due to assumption (1) and Proposition 4.5, also $h_1(T_{a_n})$ is a bounded linear operator defined by the continuous functional calculus associated to T_{a_n}. Thus $h(T_{a_n}) = T_{a_n} h_1(T_{a_n})$ is trace class (and compact) for all $n \in \mathbb{N}$, because such class is a two-sided ideal.

Next, since $h(a_n(x)) = a_n(x)\, h_1(a_n(x))$, then $|h(a_n(x))| \leq a_n(x)\, \|h_1\|_\infty$ for all $x \in X$, where $\|h_1\|_\infty = \sup_{t\in[0,1]} |h_1(t)| < \infty$ because h_1 is continuous on $[0,1]$. Then

$$\begin{aligned}\left|\widetilde{h(a_n)}(x)\right| &= \left|\int_X h(a_n(y))\, |\langle k_x, k_y\rangle|^2\, d\lambda(y)\right| \\ &\leq \|h_1\|_\infty \int_X a_n(y)\, |\langle k_x, k_y\rangle|^2\, d\lambda(y) \\ &= \|h_1\|_\infty\, \widetilde{a_n}(x).\end{aligned}$$

Applying Theorem 4.4, since T_{a_n} is compact and trace class, then $\widetilde{a_n}(x) \in L^1(X,\lambda)$ satisfying $\widetilde{a_n}(x) \to 0$ as $x \to \infty$, so the above inequality implies $\widetilde{h(a_n)}(x) \in L^1(X,\lambda)$ and $\widetilde{h(a_n)}(x) \to 0$ as $x \to \infty$, thus $T_{h(a_n)}$ is compact and trace class for all $n \in \mathbb{N}$.

Therefore, the Weierstrass Approximation Theorem and the linearity of the trace give

$$\lim_{n\to\infty} \frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)} = 1,$$

because the polynomials being dense in $C[0,1]$ implies that we can approximate the continuous function $h(t) = t\, h_1(t)$ using the monomials $\{t^m\}_{m=1}^{\infty}$, carrying the conclusion from the monomials to h.

□

The following is a generalization to Corollaries 2.2 and 2.3 in [22]. We use $\vartheta_3(\varepsilon, n)$ and $\vartheta_4(\varepsilon_1, \varepsilon_2, n)$ as defined in Corollaries 4.11 and 4.12: given $0 < \varepsilon < 1$, $0 < \varepsilon_1 < \varepsilon_2 < 1$, and n

$$\begin{aligned}\vartheta_3(\varepsilon, n) &:= \#\{\lambda_i \in \sigma(T_{a_n}) : \lambda_i \geq 1 - \varepsilon\}, \\ \vartheta_4(\varepsilon_1, \varepsilon_2, n) &:= \#\{\lambda_i \in \sigma(T_{a_n}) : 1 - \varepsilon_2 \leq \lambda_i < 1 - \varepsilon_1\},\end{aligned}$$

where $\sigma(T_{a_n})$ denotes the spectrum of the Toeplitz operator T_{a_n}.

Corollary 4.15. *Suppose that* $\{T_{a_n}\}_{n=1}^{\infty}$ *is a sequence of Toeplitz operators such that*

1. *For all* $n \in \mathbb{N}$, $T_{a_n} : \mathcal{H} \to \mathcal{H}$ *is compact, trace class, and* $0 \leq a_n(x) \leq 1$.

2. *The following localization condition holds for any* $m \in \mathbb{N}$:

$$\frac{\left\| \left[a_n - \widetilde{a_n}\right] a \right\|_{L^1} + \left\| \left[a_n - \widetilde{a_n}\right] \widetilde{a} \right\|_{L^1}}{\|a_n^m\|_{L^1}} \to 0 \text{ as } n \to \infty.$$

If $0 < \varepsilon < 1$ *is such that* $\lambda(\{a_n = 1 - \varepsilon\}) = 0$ *for all* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \frac{\vartheta_3(\varepsilon, n)}{\lambda(\{a_n \geq 1 - \varepsilon_1\})} = 1.$$

If $0 < \varepsilon_1 < \varepsilon_2 < 1$ *are such that* $\lambda(\{a_n = 1 - \varepsilon_j\}) = 0$, $j = 1, 2$, *for all* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \frac{\vartheta_4(\varepsilon_1, \varepsilon_2, n)}{\lambda(\{1 - \varepsilon_2 \leq a_n < 1 - \varepsilon_1\})} = 1.$$

Proof. Consider $h_1(t) = \frac{1}{t}\chi_{[1-\varepsilon,1]}(t)$ and $h(t) = th_1(t) = \chi_{[1-\varepsilon,1]}(t)$. Since h_1 is bounded on $[0,1]$, repeating the same argument as in the proof of Theorem 4.14, we can show that for all $n \in \mathbb{N}$, both $h(T_{a_n})$ and $T_{h(a_n)}$ are trace class (and compact) operators on $\mathcal{H}$.

As usual, let $1 \geq \lambda_1(n) \geq \lambda_2(n) \geq \cdots \geq 0$ be the eigenvalues of T_{a_n}, and $\{f_i(n)\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be an orthonormal basis generated by eigenfunctions of T_{a_n}, where $\lambda_i(n)$ corresponds to $f_i(n)$. The extended functional calculus associated to T_{a_n} defines $h(T_{a_n}) : \mathcal{H} \to \mathcal{H}$ by

$$h(T_{a_n}) f = \sum_{i=1}^{\infty} h(\lambda_i(n)) \langle f, f_i(n) \rangle f_i(n) = \sum_{\lambda_i(n) \geq 1-\varepsilon} \langle f, f_i(n) \rangle f_i(n), \quad f \in \mathcal{H},$$

then

$$\operatorname{tr}(h(T_{a_n})) = \sum_{\lambda_i(n) \geq 1-\varepsilon} (1) = \vartheta_3(\varepsilon, n).$$

On the other hand, since $T_{h(a_n)} : \mathcal{H} \to \mathcal{H}$ is defined by

$$T_{h(a_n)} f = \int_X h(a_n(x)) \langle f, k_x \rangle k_x d\lambda(x) = \int_X \chi_{\{a_n \geq 1-\varepsilon\}}(x) \langle f, k_x \rangle k_x d\lambda(x), \quad f \in \mathcal{H},$$

then

$$\operatorname{tr}\left(T_{h(a_n)}\right) = \lambda(\{a_n \geq 1 - \varepsilon_1\}).$$

Thus

$$\lim_{n\to\infty}\frac{\vartheta_3(\varepsilon,n)}{\lambda(\{a_n\geq 1-\varepsilon_1\})}=\lim_{n\to\infty}\frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)}.$$

Since h is not continuous on $[0,1]$, we cannot apply Theorem 4.14 directly on h, instead, we need to use some continuous perturbations of h as in the proof of Corollary 2.2 in [22].

Let $h_\rho^-, h_\rho^+ \in C[0,1]$ such that $0 \leq h_\rho^- \leq h \leq h_\rho^+ \leq 1$, and $h_\rho^- = h_\rho^+$ except on $B(1-\varepsilon;\rho)$, for some $\rho > 0$ small. As before, $h_\rho^-(T_{a_n})$, $T_{h_\rho^-(a_n)}$, $h_\rho^+(T_{a_n})$, $T_{h_\rho^+(a_n)}$, are trace class (and compact) operators on $\mathcal{H}$, for all $n\in\mathbb{N}$, for all $\rho>0$ small. Clearly $\operatorname{tr}\left(h_\rho^-(T_{a_n})\right) \leq \operatorname{tr}(h(T_{a_n})) \leq \operatorname{tr}\left(h_\rho^+(T_{a_n})\right)$, $\operatorname{tr}\left(T_{h_\rho^-(a_n)}\right) \leq \operatorname{tr}\left(T_{h(a_n)}\right) \leq \operatorname{tr}\left(T_{h_\rho^+(a_n)}\right)$, and

$$\begin{aligned}
\operatorname{tr}(h(T_{a_n})) &= \lim_{\rho\to 0}\operatorname{tr}\left(h_\rho^-(T_{a_n})\right) = \lim_{\rho\to 0}\operatorname{tr}\left(h_\rho^+(T_{a_n})\right),\\
\operatorname{tr}\left(T_{h(a_n)}\right) &= \lim_{\rho\to 0}\operatorname{tr}\left(T_{h_\rho^-(a_n)}\right) = \lim_{\rho\to 0}\operatorname{tr}\left(T_{h_\rho^+(a_n)}\right),
\end{aligned}$$

for all $n\in\mathbb{N}$. Applying Theorem 4.14 on h_ρ^+ and h_ρ^- we obtain

$$\lim_{n\to\infty}\left[\operatorname{tr}\left(h_\rho^+(T_{a_n})\right)-\operatorname{tr}\left(h_\rho^-(T_{a_n})\right)\right]=\lim_{n\to\infty}\left[\operatorname{tr}\left(T_{h_\rho^+(a_n)}\right)-\operatorname{tr}\left(T_{h_\rho^-(a_n)}\right)\right],$$

for all $\rho>0$ small. Then

$$\begin{aligned}
0 \leq \lim_{\rho\to 0}\lim_{n\to\infty}\left[\operatorname{tr}\left(h_\rho^+(T_{a_n})\right)-\operatorname{tr}\left(h_\rho^-(T_{a_n})\right)\right] &= \lim_{\rho\to 0}\lim_{n\to\infty}\left[\operatorname{tr}\left(T_{h_\rho^+(a_n)}\right)-\operatorname{tr}\left(T_{h_\rho^-(a_n)}\right)\right]\\
&= \lim_{\rho\to 0}\lim_{n\to\infty}\int_X\left(h_\rho^+-h_\rho^-\right)(a_n(x))\,d\lambda(x)\\
&\leq \lim_{\rho\to 0}\lim_{n\to\infty}\lambda(\{1-\varepsilon-\rho<a_n<1-\varepsilon+\rho\})\\
&= \lim_{n\to\infty}\lim_{\rho\to 0}\lambda(\{1-\varepsilon-\rho<a_n<1-\varepsilon+\rho\})\\
&= \lim_{n\to\infty}\lambda(\{a_n=1-\varepsilon\})\\
&= 0,
\end{aligned}$$

where interchange on the limits is justified by the Monotone Convergence Theorem, and the last step is due to the assumption $\lambda(\{a_n=1-\varepsilon\})=0$ for all $n\in\mathbb{N}$. Hence

$$\lim_{n\to\infty}\operatorname{tr}(h(T_{a_n}))=\lim_{n\to\infty}\operatorname{tr}\left(T_{h(a_n)}\right),$$

therefore

$$\lim_{n\to\infty} \frac{\vartheta_3(\varepsilon,n)}{\lambda(\{a_n \geq 1-\varepsilon_1\})} = \lim_{n\to\infty} \frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)} = 1.$$

The other case follows immediately from here, for $0 < \varepsilon_1 < \varepsilon_2 < 1$ we have

$$\begin{aligned}
\lim_{n\to\infty} \vartheta_4(\varepsilon_1,\varepsilon_2,n) &= \lim_{n\to\infty} [\vartheta_3(\varepsilon_2,n) - \vartheta_3(\varepsilon_1,n)] \\
&= \lim_{n\to\infty} [\lambda(\{a_n \geq 1-\varepsilon_2\}) - (\{a_n \geq 1-\varepsilon_1\})] \\
&= \lim_{n\to\infty} \lambda(\{1-\varepsilon_2 \leq a_n < 1-\varepsilon_1\}).
\end{aligned}$$

□

4.4 Applications: concentration operators

The objective of this section is to apply the results for Toeplitz operators developed in the previous sections to concentration operators under the assumption that $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property (F).

Given $a \in X$ fixed, let $a_n = \chi_{B(a;n)}$ for $n \in \mathbb{N}$, so the Toeplitz operator T_{a_n} becomes the concentration operator $C_{B(a;n)}$. For any $f \in \mathcal{H}$ it holds

$$T_{a_n} f = \int_X a_n(x) \langle f, k_x \rangle k_x d\lambda(x) = \int_{B(a;n)} \langle f, k_x \rangle k_x d\lambda(x) = C_{B(a;n)} f,$$

and the Berezin transform $\widetilde{a_n}$ becomes

$$\widetilde{a_n}(x) = \int_X a_n(y) \left|\langle k_x, k_y \rangle\right|^2 d\lambda(y) = \int_{B(a;n)} \left|\langle k_x, k_y \rangle\right|^2 d\lambda(y).$$

First, we are interested to apply Theorems 4.9, 4.10, and their corollaries, so we need to check that conditions (1) and (2) in these results are satisfied. Clearly $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$, and Propositions 2.2 and 2.3 guarantee condition (1). Also, since $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property $(F3)$, then Propositions 2.4, 4.5, and 4.8 guarantee condition (2).

Thus, the conclusions from Theorems 4.9, 4.10, and Corollaries 4.11, 4.12 are valid, which are essentially the same (slightly weaker) as the conclusions from Theorems 2.7, 2.8, and Corollaries

2.9, 2.10 given that $\operatorname{tr}(T_{a_n}) = \operatorname{tr}\left(C_{B(a;n)}\right) = \lambda(B(a;n))$.

Next, we are interested to apply Theorem 4.14 and Corollary 4.15, such results are stated below.

Theorem 4.16. *Suppose $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame satisfying the localization property $(F3)$. Given $a \in X$ fixed, let $a_n = \chi_{B(a;n)}$ for $n \in \mathbb{N}$. Then, for any $h_1 : [0,1] \to \mathbb{C}$ continuous it holds*

$$\lim_{n\to\infty} \frac{\operatorname{tr}(h(T_{a_n}))}{\operatorname{tr}\left(T_{h(a_n)}\right)} = 1,$$

where $h(t) = t\, h_1(t)$.

Proof. It follows from Theorem 4.14, it is enough to check that conditions (1) and (2) in this theorem are satisfied. Clearly $0 \le a_n \le 1$ for all $n \in \mathbb{N}$, and Propositions 2.2 and 2.3 guarantee condition (1) in Theorem 4.14 is satisfied. It remains to show condition (2) in Theorem 4.14 is satisfied. Notice that $0 \le a_n \le 1$ implies

$$\begin{aligned}
\|(a_n - \widetilde{a_n})\, a_n\|_{L^1(X)} + \|(a_n - \widetilde{a_n})\, \widetilde{a_n}\|_{L^1(X)} &\le 2\,\|a_n - \widetilde{a_n}\|_{L^1(X)} \\
&= 2\int_X |a_n(x) - \widetilde{a_n}(x)|\, d\lambda(x) \\
&= 2\int_X \left| a_n(x) - \int_X a_n(y)\, |\langle k_x, k_y\rangle|^2\, d\lambda(y) \right| d\lambda(x) \\
&= 2\int_X \left| \int_X [a_n(x) - a_n(y)]\, |\langle k_x, k_y\rangle|^2\, d\lambda(y) \right| d\lambda(x) \\
&\le 2\int_X \int_X |a_n(x) - a_n(y)|\, |\langle k_x, k_y\rangle|^2\, d\lambda(y) d\lambda(x) \\
&\le 4\int_{B(a;n)^c} \int_{B(a;n)} |\langle k_x, k_y\rangle|^2\, d\lambda(y) d\lambda(x).
\end{aligned}$$

The last step is true because $|a_n(x) - a_n(y)| = \left|\chi_{B(a;n)}(x) - \chi_{B(a;n)}(y)\right|$, so

$$|a_n(x) - a_n(y)| = \begin{cases} 1, & \text{if } x \in B(a;n) \text{ and } y \in B(a;n)^c, \text{ or vice versa,} \\ 0, & \text{if } x, y \in B(a;n), \text{ or } x, y \in B(a;n)^c. \end{cases}$$

Hence splitting X and applying Fubini's Theorem gives

$$\begin{aligned}
&\int_X \int_X |a_n(x) - a_n(y)| \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x) \\
=\ & \int_{B(a;n)^c} \int_{B(a;n)} \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x) + \int_{B(a;n)} \int_{B(a;n)^c} \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x) \\
=\ & 2 \int_{B(a;n)^c} \int_{B(a;n)} \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x).
\end{aligned}$$

On the other hand, for any $m \in \mathbb{N}$ fixed

$$\begin{aligned}
\|a_n^m\|_{L^1(X)} &= \int_X \left|\chi_{B(a;n)}(x)\right|^m d\lambda(x) \\
&= \int_{B(a;n)} d\lambda(x) \\
&= \lambda\left(B(a;n)\right).
\end{aligned}$$

Finally, due to the localization property (F), for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $z \in X$ (in particular $z = a$) it holds

$$\frac{1}{\lambda\left(B(z;n)\right)} \int_{B(z;n)^c} \int_{B(z;n)} \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x) < \frac{\varepsilon}{4}.$$

Thus, for all $n \geq N$ we obtain

$$\begin{aligned}
\frac{\|(a_n - \widetilde{a_n})\, a_n\|_{L^1(X)} + \|(a_n - \widetilde{a_n})\, \widetilde{a_n}\|_{L^1(X)}}{\|a_n^m\|_{L^1(X)}} &\leq \frac{4}{\lambda\left(B(a;n)\right)} \int_{B(a;n)^c} \int_{B(a;n)} \left|\langle k_x, k_y\rangle\right|^2 d\lambda(y) d\lambda(x) \\
&< \varepsilon.
\end{aligned}$$

Therefore condition (2) in Theorem 4.14 is satisfied. □

Similarly, we can apply Corollary 4.15 to this setup and obtain essentially the same conclusions as in Corollaries 2.9 and 2.10. Recall, given $0 < \varepsilon < 1$, $0 < \varepsilon_1 < \varepsilon_2 < 1$, and n

$$\begin{aligned}
\vartheta_3\left(\varepsilon, n\right) &:= \#\left\{\lambda_i \in \sigma\left(T_{a_n}\right) : \lambda_i \geq 1 - \varepsilon\right\}, \\
\vartheta_4\left(\varepsilon_1, \varepsilon_2, n\right) &:= \#\left\{\lambda_i \in \sigma\left(T_{a_n}\right) : 1 - \varepsilon_2 \leq \lambda_i < 1 - \varepsilon_1\right\},
\end{aligned}$$

where $\sigma\left(T_{a_n}\right)$ denotes the spectrum of the Toeplitz operator T_{a_n}.

Corollary 4.17. *Suppose* $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ *is a normalized generalized Parseval frame satisfying the localization property* $(F3)$*. Given* $a \in X$ *fixed, let* $a_n = \chi_{B(a;n)}$ *for* $n \in \mathbb{N}$*.*

If $0 < \varepsilon < 1$*, then*

$$\lim_{n\to\infty} \frac{\vartheta_3(\varepsilon, n)}{\lambda(B(a;n))} = 1.$$

If $0 < \varepsilon_1 < \varepsilon_2 < 1$ *, then*

$$\lim_{n\to\infty} \vartheta_4(\varepsilon_1, \varepsilon_2, n) = 0.$$

Proof. As in the proof of the previous theorem, conditions (1) and (2) in Corollary 4.15 are satisfied. If $0 < \varepsilon < 1$, then $\lambda(\{a_n = 1-\varepsilon\}) = 0$ for all $n \in \mathbb{N}$, because $a_n = \chi_{B(a;n)}$. By Corollary 4.15

$$\lim_{n\to\infty} \frac{\vartheta_3(\varepsilon, n)}{\lambda(B(a;n))} = \lim_{n\to\infty} \frac{\vartheta_3(\varepsilon, n)}{\lambda(\{a_n \geq 1-\varepsilon_1\})} = 1.$$

If $0 < \varepsilon_1 < \varepsilon_2 < 1$, then $\lambda(\{a_n = 1-\varepsilon_j\}) = 0$, $j = 1, 2$, and $\lambda(\{1-\varepsilon_2 \leq a_n < 1-\varepsilon_1\}) = 0$ for all $n \in \mathbb{N}$, because $a_n = \chi_{B(a;n)}$. Applying again Corollary 4.15

$$\lim_{n\to\infty} \vartheta_4(\varepsilon_1, \varepsilon_2, n) = \lim_{n\to\infty} \lambda(\{1-\varepsilon_2 \leq a_n < 1-\varepsilon_1\}) = 0.$$

□

4.5 Applications: Gabor-Toeplitz localization operators

The objective of this section is to derive as a consequence of Theorem 4.14 and Corollary 4.15 the main result Theorem 2.1 and Corollaries 2.2 and 2.3 in [22].

Given $\mathbb{R}^n$ equipped with the $n-$dimensional Lebesgue measure, let $\phi \in L^2(\mathbb{R}^n)$ be a fixed square integrable *window* such that $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, and consider $\phi_{(q,p)}$, the *phase-space shift* of ϕ by $(q, p) \in \mathbb{R}^{2n}$ defined by

$$\phi_{(q,p)}(x) = e^{2\pi ipx}\phi(x-q).$$

$\mathbb{R}^{2n}$ is called the *phase-space*. For any $f(x) \in L^2(\mathbb{R}^n)$, the *Gabor transform* $F(q, p)$ of $f(x)$ with respect to ϕ is defined by

$$F(q,p) = \left\langle f(x), \phi_{(q,p)}(x)\right\rangle = \int_{\mathbb{R}^n} f(x)e^{2\pi ipx}\phi(x-q)dx.$$

In particular, we will make use of the Gabor transform $\Phi(q,p)$ of the window $\phi(x)$ with respect to itself

$$\Phi(q,p) = \left\langle \phi(x), \phi_{(q,p)}(x) \right\rangle = \int_{\mathbb{R}^n} \phi(x) e^{2\pi i p x} \phi(x-q) dx.$$

Now we accommodate these definitions to our setup. Let the phase-space $X = \mathbb{R}^{2n}$ be a measure space equipped with the usual $2n-$dimensional Lebesgue measure, and let $\mathcal{H} = L^2(\mathbb{R}^n)$ (recall $\mathbb{R}^n$ is equipped with the $n-$dimensional Lebesgue measure) be a Hilbert space. It can be shown that $\left\{\phi_{(q,p)}(x)\right\}_{(q,p)\in\mathbb{R}^{2n}} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame for $\mathcal{H}$ due to the *Gabor reproducing formula*

$$f = \int_{\mathbb{R}^{2n}} F(q,p)\phi_{(q,p)} dq dp = \int_{\mathbb{R}^{2n}} \left\langle f, \phi_{(q,p)} \right\rangle \phi_{(q,p)} dq dp, \quad f \in \mathcal{H}.$$

This means that the Gabor reproducing formula is the frame operator associated to the generalized Parseval frame $\left\{\phi_{(q,p)}(x)\right\}_{(q,p)\in\mathbb{R}^{2n}} \subseteq \mathcal{H}$, which is the identity map. Also notice $F(q,p) \in L^2\left(\mathbb{R}^{2n}\right)$ whenever $f(x) \in L^2\left(\mathbb{R}^2\right)$. Moreover, the map

$$\begin{aligned} L^2(\mathbb{R}^n) &\to L^2\left(\mathbb{R}^{2n}\right) \\ f(x) &\mapsto F(q,p) \end{aligned}$$

is an isometry since

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^n)} &= \left\langle \int_{\mathbb{R}^{2n}} \left\langle f, \phi_{(q,p)} \right\rangle \phi_{(q,p)} dq dp, f \right\rangle \\ &= \int_{\mathbb{R}^{2n}} \left|\left\langle f, \phi_{(q,p)} \right\rangle\right|^2 dq dp \\ &= \int_{\mathbb{R}^{2n}} |F(q,p)|^2 \, dq dp \\ &= \|F\|_{L^2(\mathbb{R}^{2n})}. \end{aligned}$$

Given $b(q,p) \in L^1\left(\mathbb{R}^{2n}\right)$ nonnegative and bounded, define the *Gabor-Toeplitz localization operator* $T_b : \mathcal{H} \to \mathcal{H}$ by

$$T_b f = \int_{\mathbb{R}^{2n}} b(q,p) \left\langle f, \phi_{(q,p)} \right\rangle \phi_{(q,p)} dq dp, \quad f \in \mathcal{H}.$$

This operator is trace class and compact. Also, its Berezin transform is

$$\widetilde{b}(q,p) = \left\langle T_b\left(\phi_{(q,p)}\right), \phi_{(q,p)}\right\rangle = \int_{\mathbb{R}^{2n}} b\left(q',p'\right) \left|\left\langle \phi_{(q,p)}, \phi_{(q',p')}\right\rangle\right|^2 dq'dp', \quad (q,p) \in \mathbb{R}^{2n}.$$

Theorem 4.18. [22, Theorem 2.1] *Let $\phi \in L^2\left(\mathbb{R}^n\right)$, $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, and $b \in L^1\left(\mathbb{R}^{2n}\right)$ with $0 \leq b \leq 1$. Then for any continuous function h defined on the closed interval $[0,1]$, the following asymptotic formula holds:*

$$\lim_{R\to\infty} \frac{\operatorname{tr}\left(T_{b_R} h\left(T_{b_R}\right)\right)}{R^{2n}} = \int_{\mathbb{R}^{2n}} b(\eta) h\left(b(\eta)\right) d\eta,$$

where $b_R(\eta) = b\left(\frac{\eta}{R}\right)$, for $R > 0$.

Proof. First notice that $T_{b_R} h\left(T_{b_R}\right)$ and $T_{b_R h(b_R)}$ are trace class and compact operators for every $R > 0$. Doing a change of variables

$$\begin{aligned}
\operatorname{tr}\left(T_{b_R h(b_R)}\right) &= \int_{\mathbb{R}^{2n}} b_R(\zeta) h\left(b_R(\zeta)\right) d\zeta \\
&= \int_{\mathbb{R}^{2n}} b\left(\frac{\zeta}{R}\right) h\left(b\left(\frac{\zeta}{R}\right)\right) d\zeta \\
&= R^{2n} \int_{\mathbb{R}^{2n}} b(\eta) h\left(b(\eta)\right) d\eta.
\end{aligned}$$

We need to prove

$$\lim_{R\to\infty} \frac{\operatorname{tr}\left(T_{b_R} h\left(T_{b_R}\right)\right)}{\operatorname{tr}\left(T_{b_R h(b_R)}\right)} = 1.$$

The strategy is to apply Theorem 4.14 in order to show that the last expression is true. Condition (1) in Theorem 4.14 is satisfied, so it only remains to show that condition (2) in the same Theorem 4.14 is also satisfied. Fix $R > 0$ and $m \in \mathbb{N}$. First notice that

$$\begin{aligned}
\left\|\left[b_R - \widetilde{b_R}\right] b_R\right\|_{L^1(\mathbb{R}^{2n})} &= \int_{\mathbb{R}^{2n}} \left|b_R(\zeta) - \widetilde{b_R}(\zeta)\right| \left|b_R(\zeta)\right| d\zeta \\
&= \int_{\mathbb{R}^{2n}} \left|b_R(\zeta) \int_{\mathbb{R}^{2n}} \left|\langle \phi_\zeta, \phi_\eta\rangle\right|^2 d\eta - \int_{\mathbb{R}^{2n}} b_R(\eta) \left|\langle \phi_\zeta, \phi_\eta\rangle\right|^2 d\eta\right| \left|b_R(\zeta)\right| d\zeta \\
&= \int_{\mathbb{R}^{2n}} \left|\int_{\mathbb{R}^{2n}} \left[b_R(\zeta) - b_R(\eta)\right] \left|\langle \phi_\zeta, \phi_\eta\rangle\right|^2 d\eta\right| \left|b_R(\zeta)\right| d\zeta \\
&\leq \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left|\langle \phi_\zeta, \phi_\eta\rangle\right|^2 \left|b_R(\zeta) - b_R(\eta)\right| \left|b_R(\zeta)\right| d\eta d\zeta.
\end{aligned}$$

Also notice that $\langle \phi_\zeta, \phi_\eta\rangle = \langle \phi, \phi_{\eta-\zeta}\rangle = \Phi\left(\eta - \zeta\right)$, then, from the previous inequality and after changing variables twice, we obtain

$$\begin{aligned}
\left\|\left[b_R-\widetilde{b_R}\right]b_R\right\|_{L^1(\mathbb{R}^{2n})} &\leq \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta''-\zeta''\right)\right|^2 \left|b\left(\frac{\zeta''}{R}\right)-b\left(\frac{\eta''}{R}\right)\right|\left|b\left(\frac{\zeta''}{R}\right)\right| d\eta'' d\zeta'' \\
&= R^{2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} R^{2n}\left|\Phi\left(R\left(\eta'-\zeta'\right)\right)\right|^2 \left|b(\zeta')-b(\eta')\right|\left|b(\zeta')\right| d\eta' d\zeta' \\
&= R^{2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\frac{\eta}{R}+\zeta\right)\right|\left|b(\zeta)\right| d\eta d\zeta.
\end{aligned}$$

Similarly

$$\left\|\left[b_R-\widetilde{b_R}\right]\widetilde{b_R}\right\|_{L^1(\mathbb{R}^{2n})} \leq R^{2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\frac{\eta}{R}+\zeta\right)\right|\left|\widetilde{b}(\zeta)\right| d\eta d\zeta,$$

and clearly

$$\|b_R^m\|_{L^1(\mathbb{R}^{2n})} = R^{2n}\int_{\mathbb{R}^{2n}} b(\eta)^m d\eta = R^{2n}\operatorname{tr}\left(T_{b^m}\right),$$

where T_{b^m} is known to be trace class and compact. Recall $0 \leq b \leq 1$. Then $2\left|\Phi\left(\eta\right)\right|^2\left|b(\zeta)\right|$ and $2\left|\Phi\left(\eta\right)\right|^2\left|\widetilde{b}(\zeta)\right|$ are dominating functions in $L^1\left(\mathbb{R}^{4n}\right)$. Hence the Lebesgue Dominated Theorem gives

$$\begin{aligned}
\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\frac{\eta}{R}+\zeta\right)\right|\left|b(\zeta)\right| d\eta d\zeta &\to \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\zeta\right)\right|\left|b(\zeta)\right| d\eta d\zeta = 0, \\
\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\frac{\eta}{R}+\zeta\right)\right|\left|\widetilde{b}(\zeta)\right| d\eta d\zeta &\to \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \left|\Phi\left(\eta\right)\right|^2 \left|b(\zeta)-b\left(\zeta\right)\right|\left|\widetilde{b}(\zeta)\right| d\eta d\zeta = 0,
\end{aligned}$$

as $R \to \infty$. Therefore the condition (2) in Theorem 4.14 is satisfied

$$\frac{\left\|\left[b_R-\widetilde{b_R}\right]b_R\right\|_{L^1(\mathbb{R}^{2n})}+\left\|\left[b_R-\widetilde{b_R}\right]\widetilde{b_R}\right\|_{L^1(\mathbb{R}^{2n})}}{\|b_R^m\|_{L^1(\mathbb{R}^{2n})}} \to 0$$

as $R \to 0$. $\square$

Corollary 4.19. [22, Corollaries 2.2 and 2.3] *Let* $\phi \in L^2(\mathbb{R}^n)$, $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, *and* $b \in L^1(\mathbb{R}^{2n})$ *with* $0 \leq b \leq 1$. *Denote by* λ *the Lebesgue measure on* $\mathbb{R}^{2n}$. *Let* $b_R(\eta) = b\left(\frac{\eta}{R}\right)$ *for any* $R > 0$ *fixed, and let* $1 \geq \lambda_0(R) \geq \lambda_1(R) \geq \cdots \geq 0$ *be the eigenvalues of* T_{b_R}.

If $0 < \delta < 1$ *is such that* $\lambda(\{b = \delta\}) = 0$, *then*

$$\lim_{R\to\infty} \frac{\#\{\lambda_i(R) > \delta\}}{R^{2n}} = \lambda(\{b > \delta\}).$$

If $0 < \delta_1 < \delta_2 < 1$ *is such that* $\lambda(\{b = \delta_j\}) = 0$, $j = 1, 2$, *then*

$$\lim_{R\to\infty} \frac{\#\{\delta_1 < \lambda_i(R) < \delta_2\}}{R^{2n}} = \lambda(\{\delta_1 < b < \delta_2\}).$$

Proof. Follows from Corollary 4.15 and the proof of Theorem 4.18. □

Chapter 5

General density results

Let $\mathcal{H}$ be a complex and separable Hilbert space. We assume any generalized frame in $\mathcal{H}$ under consideration, say $\{h_x\}_{x\in X} \subseteq \mathcal{H}$, satisfies that its index set (X, d, μ) is a metric measure space with μ a Borel measure with respect to the metric d, and $\operatorname{supp}(\mu) \neq \emptyset$.

5.1 Main theorem

Proposition 5.1. *Let $\{h_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a generalized frame for $\mathcal{H}$. Then for all $x \in X$ it holds*

$$\left\langle \widetilde{h_x}, h_x \right\rangle = \left\langle h_x, \widetilde{h_x} \right\rangle = \int_X \left| \left\langle h_x, \widetilde{h_y} \right\rangle \right|^2 d\lambda(y) = \int_X \left| \left\langle \widetilde{h_x}, h_y \right\rangle \right|^2 d\lambda(y) \geq 0.$$

Proof. Fix $x \in X$. Denote by S^{-1} the inverse of the frame operator associated to the generalized frame $\{h_x\}_x$. Recall S^{-1} is a self-adjoint operator, so $\left\langle \widetilde{h_x}, h_x \right\rangle = \left\langle h_x, \widetilde{h_x} \right\rangle$. Moreover

$$\begin{aligned}
\left\langle h_x, \widetilde{h_x} \right\rangle &= \langle h_x, S^{-1}(h_x) \rangle \\
&= \left\langle h_x, \int_X \left\langle h_x, \widetilde{h_y} \right\rangle \widetilde{h_y} d\lambda(y) \right\rangle \\
&= \int_X \left\langle \widetilde{h_y}, h_x \right\rangle \left\langle h_x, \widetilde{h_y} \right\rangle d\lambda(y) \\
&= \int_X \left| \left\langle h_x, \widetilde{h_y} \right\rangle \right|^2 d\lambda(y) \\
&= \int_X \left| \left\langle \widetilde{h_x}, h_y \right\rangle \right|^2 d\lambda(y) \\
&\geq 0.
\end{aligned}$$

This completes the proof. □

From now on, we consider $\mathcal{F}, \mathcal{G} \subseteq \mathcal{H}$, closed subspaces of the Hilbert space $\mathcal{H}$, and denote by $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ to the orthogonal projection onto $\mathcal{K} = \mathcal{F}, \mathcal{G}$.

Proposition 5.2. *Let $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$, and let $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ be a generalized frame for $\mathcal{G}$, such that*

$$\begin{aligned} 0 \leq c_f &:= \inf_{y\in \operatorname{supp}\mu} \left|\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle\right| \leq \sup_{y\in \operatorname{supp}\mu} \left|\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle\right| =: C_f < \infty, \\ 0 \leq c_g &:= \inf_{y\in \operatorname{supp}\nu} \left|\left\langle P_{\mathcal{F}}\widetilde{g}_y, g_y\right\rangle\right| \leq \sup_{y\in \operatorname{supp}\nu} \left|\left\langle P_{\mathcal{F}}\widetilde{g}_y, g_y\right\rangle\right| =: C_g < \infty. \end{aligned}$$

Then the following statements hold:

1) For any $\mu-$measurable set $\Omega \subseteq X$

(a)

$$\int_X \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle d\mu(y)d\nu(x) = \int_{\Omega\cap \operatorname{supp}(\mu)} \left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle d\mu(y).$$

(b)

$$\int_X \int_\Omega \left|\langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle\right| d\mu(y)d\nu(x) \geq \int_{\Omega\cap \operatorname{supp}(\mu)} \left|\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle\right| d\mu(y).$$

(c) If $\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle = 1$ for all $y \in \operatorname{supp}\mu$, then

$$\mu(\Omega) = \int_X \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle d\mu(y)d\nu(x).$$

(d) If $\mathcal{G} = \mathcal{H}$, then

$$c_f\mu(\Omega) \leq \int_X \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle d\mu(y)d\nu(x) \leq C_f\mu(\Omega).$$

(e)

$$c_f\,\mu(\Omega) \leq \int_X \int_\Omega \left|\langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle\right| d\mu(y)d\nu(x).$$

(f)

$$\left|\int_X \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g}_x\right\rangle d\mu(y)d\nu(x)\right| \leq C_f\,\mu(\Omega).$$

2) For any $\nu-$*measurable set* $\Omega \subseteq X$

(a)

$$\int_X \int_\Omega \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle d\nu(y) d\mu(x) = \int_{\Omega \cap \operatorname{supp}(\nu)} \langle P_{\mathcal{F}} \widetilde{g_y}, g_y \rangle \, d\nu(y).$$

(b)

$$\int_X \int_\Omega \left| \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle \right| d\nu(y) d\mu(x) \geq \int_{\Omega \cap \operatorname{supp}(\nu)} \left| \langle P_{\mathcal{F}} \widetilde{g_y}, g_y \rangle \right| d\nu(y).$$

(c) If $\langle P_{\mathcal{F}} \widetilde{g_y}, g_y \rangle = 1$ *for all* $y \in \operatorname{supp} \nu$*, then*

$$\nu(\Omega) = \int_X \int_\Omega \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle d\nu(y) d\mu(x).$$

(d) If $\mathcal{F} = \mathcal{H}$*, then*

$$c_g \nu(\Omega) \leq \int_X \int_\Omega \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle d\nu(y) d\mu(x) \leq C_g \nu(\Omega).$$

(e)

$$c_g \, \nu(\Omega) \leq \int_X \int_\Omega \left| \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle \right| d\nu(y) d\mu(x).$$

(f)

$$\left| \int_X \int_\Omega \langle f_x, g_y \rangle \left\langle \widetilde{g_y}, \widetilde{f_x} \right\rangle d\nu(y) d\mu(x) \right| \leq C_g \, \nu(\Omega).$$

Remark: We are particularly interested in the cases where $\left\langle f_x, \widetilde{f_x} \right\rangle = 1$ for all $x \in \operatorname{supp}(\mu)$, as in generalized Parseval frames, or Riesz bases; or in cases where $\left| \langle P_{\mathcal{F}} \widetilde{g_y}, g_y \rangle \right| \leq 1$ for all $x \in \operatorname{supp}(\nu)$, as in frames.

Proof. We will prove the statements in (1). The statements in (2) are completely analogous. Let $\Omega \subseteq X$ be a $\mu-$measurable set. By Corollary 1.20

$$P_{\mathcal{G}} h = \int_X \langle h, \widetilde{g_x} \rangle \, g_x d\nu(x) = \int_X \langle h, g_x \rangle \, \widetilde{g_x} d\nu(x),$$

for any $h \in \mathcal{H}$. So, applying Fubini's theorem, (1a) holds since

$$\begin{aligned}
\int_X \int_\Omega \langle g_x, f_y \rangle \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle d\mu(y) d\nu(x) &= \int_\Omega \left[\int_X \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle \langle g_x, f_y \rangle \, d\nu(x) \right] d\mu(y) \\
&= \int_\Omega \left\langle \int_X \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle g_x d\nu(x), f_y \right\rangle d\mu(y) \\
&= \int_\Omega \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y) \\
&= \int_{\Omega \cap \text{supp}\,(\mu)} \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y).
\end{aligned}$$

From here we obtain that (1f) holds, since by assumption $\left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| \leq C_f$ for all $y \in \text{supp}\,(\mu)$, so

$$\left| \int_{\Omega \cap \text{supp}\,(\mu)} \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y) \right| \leq \int_{\Omega \cap \text{supp}\,(\mu)} \left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| d\mu(y) \leq C_f \, \mu\left(\Omega\right).$$

Similarly, by Tonelli's theorem and integral inequalities, (1b) holds since

$$\begin{aligned}
\int_X \int_\Omega \left| \langle g_x, f_y \rangle \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle \right| d\mu(y) d\nu(x) &= \int_\Omega \int_X \left| \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle \langle g_x, f_y \rangle \right| d\nu(x) d\mu(y) \\
&\geq \int_\Omega \left| \int_X \left\langle \widetilde{f}_y, \widetilde{g}_x \right\rangle \langle g_x, f_y \rangle \, d\nu(x) \right| d\mu(y) \\
&= \int_\Omega \left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| d\mu(y) \\
&= \int_{\Omega \cap \text{supp}\,(\mu)} \left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| d\mu(y).
\end{aligned}$$

And from here (1e) follows immediately since by assumption $\left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| \geq c_f$ for all $y \in \text{supp}\,(\mu)$, then

$$\int_{\Omega \cap \text{supp}\,(\mu)} \left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle \right| d\mu(y) \geq c_f \, \mu\left(\Omega\right).$$

If $\left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle = 1$ for all $y \in \text{supp}\,(\mu)$, by (1a) we obtain that (1c) holds since

$$\int_{\Omega \cap \text{supp}\,(\mu)} \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y) = \mu\left(\Omega \cap \text{supp}\,(\mu)\right) = \mu\left(\Omega\right).$$

If $\mathcal{G} = \mathcal{H}$, then $P_{\mathcal{G}}$ is simply the identity map on $\mathcal{H}$, hence $\left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle = \left\langle \widetilde{f}_y, f_y \right\rangle \geq 0$ by Proposition

5.1. Thus by (1a) we obtain that (1d) holds since

$$c_f\,\mu\,(\Omega) = \int_{\Omega\cap \operatorname{supp}(\mu)} c_f\,d\mu(y) \le \int_{\Omega\cap \operatorname{supp}(\mu)} \left\langle \widetilde{f}_y, f_y \right\rangle d\mu(y) \le \int_{\Omega\cap \operatorname{supp}(\mu)} C_f\,d\mu(y) = C_f\,\mu\,(\Omega)\,.$$

This completes the proof. □

Proposition 5.3. *Let $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$, and let $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ be a generalized frame for $\mathcal{G}$. Then, for any $\mu,\nu-$measurable set $\Omega \subseteq X$ the following identity holds*

$$\begin{aligned}\int_\Omega \left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y \right\rangle d\mu(y) - \int_\Omega \left\langle P_{\mathcal{F}} g_x, \widetilde{g_x} \right\rangle d\nu(x) \quad &= \quad \int_{\Omega^c}\int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) \\ &\quad - \int_{\Omega^c}\int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y).\end{aligned}$$

Proof. Recall that $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ denotes the orthogonal projection onto the closed subspace $\mathcal{K} \subseteq \mathcal{H}$, $\mathcal{K} = \mathcal{F}, \mathcal{G}$. By Corollary 1.20

$$\begin{aligned} P_{\mathcal{G}}\widetilde{f}_y \quad &= \quad \int_X \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle g_x d\nu(x), \\ P_{\mathcal{F}} g_x \quad &= \quad \int_X \langle g_x, f_y\rangle\, \widetilde{f}_y d\mu(y), \end{aligned}$$

then, by Fubini's Theorem we obtain that for any $\mu,\nu-$measurable set $\Omega \subseteq X$

$$\begin{aligned} \int_\Omega \left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y \right\rangle d\mu(y) \quad &= \quad \int_\Omega \int_X \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &= \quad \int_\Omega \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) + \int_\Omega \int_{\Omega^c} \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &= \quad \int_X \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) - \int_{\Omega^c} \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &\quad + \int_\Omega \int_{\Omega^c} \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &= \quad \int_\Omega \int_X \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) - \int_{\Omega^c} \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &\quad + \int_{\Omega^c} \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) \\ &= \quad \int_\Omega \left\langle P_{\mathcal{F}} g_x, \widetilde{g_x} \right\rangle d\nu(x) - \int_{\Omega^c} \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \\ &\quad + \int_{\Omega^c} \int_\Omega \langle g_x, f_y\rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x). \end{aligned}$$

□

Theorem 5.4. *Let $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$, and let $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ be a generalized frame for $\mathcal{G}$, such that they satisfy the following conditions:*

i) Boundedness condition: $n_f := \sup_{y\in \operatorname{supp}(\mu)} \left\| \widetilde{f_y} \right\| < \infty$, $n_g := \sup_{y\in \operatorname{supp}(\nu)} \|\widetilde{g_y}\| < \infty$.

ii) Localization condition: for any $\varepsilon > 0$, there exists $R > 0$ such that for all $a \in X$, for all $r \geq R$

$$\int_{B(a;r)^c} \int_{B(a;r)} |\langle f_x, g_y\rangle|^2 \, d\nu(y) d\mu(x) \leq \varepsilon \, (\mu+\nu)\,(B(a;r))\,,$$
$$\int_{B(a;r)^c} \int_{B(a;r)} |\langle g_x, f_y\rangle|^2 \, d\mu(y) d\nu(x) \leq \varepsilon \, (\mu+\nu)\,(B(a;r))\,.$$

Then the following statements hold:

1) If $|\langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle| \leq 1$ for all $x \in \operatorname{supp}(\nu)$, then

$$\begin{aligned}
\liminf_{r\to\infty} \inf_{a\in X} \frac{\nu\,(B(a;r))}{\mu\,(B(a;r))} \quad &\geq \quad \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\mu\,(B(a;r))} \left| \int_{B(a;r)} \left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle d\mu(y) \right|, \\
\limsup_{r\to\infty} \sup_{a\in X} \frac{\nu\,(B(a;r))}{\mu\,(B(a;r))} \quad &\geq \quad \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\mu\,(B(a;r))} \left| \int_{B(a;r)} \left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle d\mu(y) \right|.
\end{aligned}$$

2) If $\left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle \geq 1$ for all $y \in \operatorname{supp}(\mu)$, then

$$\begin{aligned}
\limsup_{r\to\infty} \sup_{a\in X} \frac{\mu\,(B(a;r))}{\nu\,(B(a;r))} \quad &\leq \quad \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\nu\,(B(a;r))} \left| \int_{B(a;r)} \langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle \, d\nu(x) \right|, \\
\liminf_{r\to\infty} \inf_{a\in X} \frac{\mu\,(B(a;r))}{\nu\,(B(a;r))} \quad &\leq \quad \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\nu\,(B(a;r))} \left| \int_{B(a;r)} \langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle \, d\nu(x) \right|.
\end{aligned}$$

Proof. Due to Proposition 5.3 and triangle inequality, for any $\mu,\nu-$measurable ball $B \subseteq X$ it holds

$$\begin{aligned}
\left| \int_B \left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle d\mu(y) \right| \quad &\leq \quad \left| \int_B \langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle \, d\nu(x) \right| + \left| \int_{B^c} \int_B \langle g_x, f_y\rangle \left\langle \widetilde{f_y}, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \right| \\
&\quad + \left| \int_{B^c} \int_B \langle g_x, f_y\rangle \left\langle \widetilde{f_y}, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) \right|.
\end{aligned}$$

Let $\varepsilon > 0$. By assumption (ii) there exists $R > 0$ such that for all balls $B := B(a;r)$, $a \in X$, $r \geq R$

$$\begin{aligned}
\int_{B^c} \int_B |\langle g_x, f_y\rangle|^2 \, d\nu(x) d\mu(y) \quad &\leq \quad \varepsilon^2 \,(\mu+\nu)\,(B)\,, \\
\int_{B^c} \int_B |\langle g_x, f_y\rangle|^2 \, d\mu(y) d\nu(x) \quad &\leq \quad \varepsilon^2 \,(\mu+\nu)\,(B)\,.
\end{aligned}$$

Recall that $\left\{\widetilde{f_x}\right\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ is a generalized frame for $\mathcal{F}$, let $\widetilde{\beta_f}$ be its upper constant. For any $g \in \mathcal{H}$, we can write $g = P_{\mathcal{F}}g + P_{\mathcal{F}^\perp}g$, so $\left\langle \widetilde{f_y}, g \right\rangle = \left\langle \widetilde{f_y}, P_{\mathcal{F}}g \right\rangle$ for all $y \in X$, since $\left\langle \widetilde{f_y}, P_{\mathcal{F}^\perp}g \right\rangle = 0$ given that $\widetilde{f_y} \in \mathcal{F}$. Then

$$\int_X \left|\left\langle \widetilde{f_y}, g \right\rangle\right|^2 d\mu(y) = \int_X \left|\left\langle \widetilde{f_y}, P_{\mathcal{F}}g \right\rangle\right|^2 d\mu(y) \leq \widetilde{\beta_f}\left\|P_{\mathcal{F}}g\right\|^2 \leq \widetilde{\beta_f}\left\|g\right\|^2.$$

Combining this inequality with assumptions (i) and (ii), Tonelli's theorem, and Cauchy-Schwarz inequality, we obtain that for any $B := B(a;r)$, $a \in X$, $r \geq R$

$$\begin{aligned}
&\left|\int_{B^c}\int_B \langle g_x, f_y\rangle \left\langle \widetilde{f_y}, \widetilde{g_x}\right\rangle d\nu(x)d\mu(y)\right| \\
\leq\;& \int_{B^c}\int_B |\langle g_x, f_y\rangle| \left|\left\langle \widetilde{f_y}, \widetilde{g_x}\right\rangle\right| d\nu(x)d\mu(y) \\
=\;& \int_{B}\int_{B^c} |\langle g_x, f_y\rangle| \left|\left\langle \widetilde{f_y}, \widetilde{g_x}\right\rangle\right| d\mu(y)d\nu(x) \\
\leq\;& \int_B \left(\int_{B^c} |\langle g_x, f_y\rangle|^2 d\mu(y)\right)^{1/2} \left(\int_{B^c} \left|\left\langle \widetilde{f_y}, \widetilde{g_x}\right\rangle\right|^2 d\mu(y)\right)^{1/2} d\nu(x) \\
\leq\;& \int_B \left(\int_{B^c} |\langle g_x, f_y\rangle|^2 d\mu(y)\right)^{1/2} \left(\int_{X} \left|\left\langle \widetilde{f_y}, \widetilde{g_x}\right\rangle\right|^2 d\mu(y)\right)^{1/2} d\nu(x) \\
=\;& \int_B \left(\int_{B^c} |\langle g_x, f_y\rangle|^2 d\mu(y)\right)^{1/2} \left(\int_{X} \left|\left\langle \widetilde{f_y}, P_{\mathcal{F}}\widetilde{g_x}\right\rangle\right|^2 d\mu(y)\right)^{1/2} d\nu(x) \\
\leq\;& \int_B \left(\int_{B^c} |\langle g_x, f_y\rangle|^2 d\mu(y)\right)^{1/2} \left(\widetilde{\beta_f}\|\widetilde{g_x}\|^2\right)^{1/2} d\nu(x) \\
\leq\;& \left(\int_B\int_{B^c} |\langle g_x, f_y\rangle|^2 d\mu(y)d\nu(x)\right)^{1/2} \left(\int_B \widetilde{\beta_f}\|\widetilde{g_x}\|^2 d\nu(x)\right)^{1/2} \\
=\;& \left(\int_{B^c}\int_{B} |\langle g_x, f_y\rangle|^2 d\nu(x)d\mu(y)\right)^{1/2} \left(\int_B \widetilde{\beta_f}\|\widetilde{g_x}\|^2 d\nu(x)\right)^{1/2} \\
\leq\;& \left(\varepsilon^2\,(\mu+\nu)(B)\right)^{1/2} \left(\widetilde{\beta_f} n_g^2\,\nu(B)\right)^{1/2} \\
\leq\;& \varepsilon\, n_g \sqrt{\widetilde{\beta_f}}\,(\mu+\nu)(B).
\end{aligned}$$

Similarly, if $\widetilde{\beta_g}$ denotes the upper constant of the generalized frame $\{\widetilde{g_x}\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$, for any

$B := B(a;r)$, $a \in X$, $r \geq R$, we obtain

$$\begin{aligned}
& \left| \int_{B^c} \int_B \langle g_x, f_y \rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) \right| \\
\leq \; & \int_{B^c} \int_B |\langle g_x, f_y \rangle| \left| \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle \right| d\mu(y) d\nu(x) \\
= \; & \int_B \int_{B^c} |\langle g_x, f_y \rangle| \left| \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle \right| d\nu(x) d\mu(y) \\
\leq \; & \int_B \left(\int_{B^c} |\langle g_x, f_y \rangle|^2 \, d\nu(x) \right)^{1/2} \left(\int_{B^c} \left| \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle \right|^2 d\nu(x) \right)^{1/2} d\mu(y) \\
\leq \; & \int_B \left(\int_{B^c} |\langle g_x, f_y \rangle|^2 \, d\nu(x) \right)^{1/2} \left(\int_X \left| \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle \right|^2 d\nu(x) \right)^{1/2} d\mu(y) \\
= \; & \int_B \left(\int_{B^c} |\langle g_x, f_y \rangle|^2 \, d\nu(x) \right)^{1/2} \left(\int_X \left| \left\langle P_{\mathcal{G}} \widetilde{f}_y, \widetilde{g_x} \right\rangle \right|^2 d\nu(x) \right)^{1/2} d\mu(y) \\
\leq \; & \int_B \left(\int_{B^c} |\langle g_x, f_y \rangle|^2 \, d\nu(x) \right)^{1/2} \left(\widetilde{\beta_g} \| \widetilde{f}_y \|^2 \right)^{1/2} d\mu(y) \\
\leq \; & \left(\int_B \int_{B^c} |\langle g_x, f_y \rangle|^2 \, d\nu(x) d\mu(y) \right)^{1/2} \left(\int_B \widetilde{\beta_g} \| \widetilde{f}_y \|^2 d\mu(y) \right)^{1/2} \\
= \; & \left(\int_{B^c} \int_B |\langle g_x, f_y \rangle|^2 \, d\mu(y) d\nu(x) \right)^{1/2} \left(\int_B \widetilde{\beta_g} \| \widetilde{f}_y \|^2 d\mu(y) \right)^{1/2} \\
\leq \; & \left(\varepsilon^2 \, (\mu + \nu)\,(B) \right)^{1/2} \left(\widetilde{\beta_g} n_f^2 \, \mu\,(B) \right)^{1/2} \\
\leq \; & \varepsilon \, n_f \sqrt{\widetilde{\beta_g}} \; (\mu + \nu)\,(B)\,.
\end{aligned}$$

Now we can prove the statement (1), in this case we assume $|\langle P_{\mathcal{F}} g_x, \widetilde{g_x} \rangle| \leq 1$ for all $x \in \operatorname{supp}(\nu)$. We conclude that for all balls $B = B(a;r)$, $a \in X$, $r \geq R$

$$\begin{aligned}
\left| \int_B \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y) \right| \; \leq \; & \left| \int_B \langle P_{\mathcal{F}} g_x, \widetilde{g_x} \rangle \, d\nu(x) \right| + \left| \int_{B^c} \int_B \langle g_x, f_y \rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\nu(x) d\mu(y) \right| \\
& + \left| \int_{B^c} \int_B \langle g_x, f_y \rangle \left\langle \widetilde{f}_y, \widetilde{g_x} \right\rangle d\mu(y) d\nu(x) \right| \\
\leq \; & \nu\,(B) + \varepsilon \, n_g \sqrt{\widetilde{\beta_f}} \; (\mu + \nu)\,(B) + \varepsilon \, n_f \sqrt{\widetilde{\beta_g}} \; (\mu + \nu)\,(B)\,.
\end{aligned}$$

Hence, for all balls $B = B(a;r)$, $a \in X$, $r \geq R$

$$\frac{1}{\mu\,(B)} \left| \int_B \left\langle P_{\mathcal{G}} \widetilde{f}_y, f_y \right\rangle d\mu(y) \right| \; \leq \; \left(1 + \varepsilon \, n_g \sqrt{\widetilde{\beta_f}} + \varepsilon \, n_f \sqrt{\widetilde{\beta_g}} \right) \frac{\nu\,(B)}{\mu\,(B)} + \left(\varepsilon \, n_g \sqrt{\widetilde{\beta_f}} + \varepsilon \, n_f \sqrt{\widetilde{\beta_g}} \right).$$

Therefore

$$
\begin{aligned}
\liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle d\mu(y)\right| &\leq \liminf_{r\to\infty}\inf_{a\in X}\frac{\nu\left(B(a;r)\right)}{\mu\left(B(a;r)\right)},\\
\limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle d\mu(y)\right| &\leq \limsup_{r\to\infty}\sup_{a\in X}\frac{\nu\left(B(a;r)\right)}{\mu\left(B(a;r)\right)}.
\end{aligned}
$$

Similarly, to prove statement (2) assume $\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle \geq 1$ for all $y\in\operatorname{supp}(\mu)$. Then for all balls $B=B(a;r)$, $a\in X$, $r\geq R$

$$
\begin{aligned}
\mu(B)\leq\int_B\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle d\mu(y) &= \left|\int_B\left\langle P_{\mathcal{G}}\widetilde{f}_y, f_y\right\rangle d\mu(y)\right|\\
&\leq \left|\int_B\left\langle P_{\mathcal{F}}g_x, \widetilde{g_x}\right\rangle d\nu(x)\right| + \left|\int_{B^c}\int_B\left\langle g_x, f_y\right\rangle\left\langle \widetilde{f}_y, \widetilde{g_x}\right\rangle d\nu(x)d\mu(y)\right|\\
&\quad + \left|\int_{B^c}\int_B\left\langle g_x, f_y\right\rangle\left\langle \widetilde{f}_y, \widetilde{g_x}\right\rangle d\mu(y)d\nu(x)\right|\\
&\leq \left|\int_B\left\langle P_{\mathcal{F}}g_x, \widetilde{g_x}\right\rangle d\nu(x)\right| + \varepsilon\, n_g\sqrt{\widetilde{\beta_f}}\;(\mu+\nu)(B)\\
&\quad +\varepsilon\, n_f\sqrt{\widetilde{\beta_g}}\;(\mu+\nu)(B).
\end{aligned}
$$

Hence, for all balls $B=B(a;r)$, $a\in X$, $r\geq R$

$$
\left(1-\varepsilon\, n_g\sqrt{\widetilde{\beta_f}}-\varepsilon\, n_f\sqrt{\widetilde{\beta_g}}\right)\frac{\mu(B)}{\nu(B)} \leq \frac{1}{\nu(B)}\left|\int_B\left\langle P_{\mathcal{F}}g_x, \widetilde{g_x}\right\rangle d\nu(x)\right| + \left(\varepsilon\, n_g\sqrt{\widetilde{\beta_f}}+\varepsilon\, n_f\sqrt{\widetilde{\beta_g}}\right).
$$

Therefore

$$
\begin{aligned}
\limsup_{r\to\infty}\sup_{a\in X}\frac{\mu\left(B(a;r)\right)}{\nu\left(B(a;r)\right)} &\leq \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\nu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{F}}g_x, \widetilde{g_x}\right\rangle d\nu(x)\right|,\\
\liminf_{r\to\infty}\inf_{a\in X}\frac{\mu\left(B(a;r)\right)}{\nu\left(B(a;r)\right)} &\leq \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\nu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{F}}g_x, \widetilde{g_x}\right\rangle d\nu(x)\right|.
\end{aligned}
$$

This completes the proof. □

Corollary 5.5. *Let* $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ *be a generalized frame for* $\mathcal{F}$*, and let* $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ *be a generalized frame for* $\mathcal{G}$*, such that they satisfy the boundedness condition (i) and the localization condition (ii) in Theorem 5.4. If* $|\langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle| \le 1$ *for all* $x \in \mathrm{supp}\,(\nu)$*, and* $\left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle \ge 1$ *for all* $y \in \mathrm{supp}\,(\mu)$*, then*

$$\begin{aligned}\liminf_{r\to\infty} \inf_{a\in X} \frac{\nu\,(B(a;r))}{\mu\,(B(a;r))} &\ge 1,\\ \limsup_{r\to\infty} \sup_{a\in X} \frac{\mu\,(B(a;r))}{\nu\,(B(a;r))} &\le 1.\end{aligned}$$

Proof. Since $|\langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle| \le 1$ for all $x \in \mathrm{supp}\,(\nu)$, by Theorem 5.4(1) we obtain

$$\begin{aligned}\liminf_{r\to\infty} \inf_{a\in X} \frac{\nu\,(B(a;r))}{\mu\,(B(a;r))} &\ge \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\mu\,(B(a;r))} \left| \int_{B(a;r)} \left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle d\mu(y) \right|\\ &= \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\mu\,(B(a;r))} \int_{B(a;r)} \left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle d\mu(y)\\ &\ge \liminf_{r\to\infty} \inf_{a\in X} \frac{1}{\mu\,(B(a;r))} \int_{B(a;r)} d\mu(y)\\ &= 1,\end{aligned}$$

where the last calculations are valid due to the assumption $\left\langle P_{\mathcal{G}} \widetilde{f_y}, f_y \right\rangle \ge 1$ for all $y \in \mathrm{supp}\,(\mu)$. Similarly, by Theorem 5.4(2)

$$\begin{aligned}\limsup_{r\to\infty} \sup_{a\in X} \frac{\mu\,(B(a;r))}{\nu\,(B(a;r))} &\le \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\nu\,(B(a;r))} \left| \int_{B(a;r)} \langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle\, d\nu(x) \right|\\ &\le \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\nu\,(B(a;r))} \int_{B(a;r)} |\langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle|\, d\nu(x)\\ &\le \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\nu\,(B(a;r))} \int_{B(a;r)} d\nu(x)\\ &= 1,\end{aligned}$$

where the last calculations are valid due to the assumption $|\langle P_{\mathcal{F}} g_x, \widetilde{g_x}\rangle| \le 1$ for all $x \in \mathrm{supp}\,(\nu)$. □

5.2 Applications: sampling and interpolating sequences

The hypotheses from Chapter 3 are assumed to be valid in this section. The objective of this section is to prove Theorems 3.5 and 3.8 as a consequence of Corollary 5.5 (alternatively we can apply Theorem 5.4 and get slightly better results).

Let (X, d, λ) be a metric measure space satisfying (S1)-(S4), and let $\mathcal{H}$ be a separable framed Hilbert space. Consider a normalized generalized Parseval frame $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ satisfying (F), (F2), and such that $\operatorname{supp}(\lambda) \neq \emptyset$. Denote by $\left\{\widetilde{k_x}^{\lambda}\right\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ its generalized dual frame.

Let $\Gamma \subseteq X$ be δ_0-separated sequence, consider the closed subspace $\mathcal{K} := \overline{\operatorname{span}}\,\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{H}$. Denote by σ the counting measure on Γ, i.e., $\sigma(A) := \#\,(A \cap \Gamma)$ for any measurable set $A \subseteq X$. If $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{K}$ is a generalized frame for $\mathcal{K}$, its dual generalized frame $\left\{\widetilde{k_x}^{\sigma}\right\}_{x\in(X,d,\sigma)} \subseteq \mathcal{K}$ is also a generalized frame for $\mathcal{K}$ and satisfies $\overline{\operatorname{span}}\left\{\widetilde{k_\gamma}^{\sigma}\right\}_{\gamma\in\Gamma} = \mathcal{K}$.

If Γ is sampling, $\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{K} = \mathcal{H}$ is a frame for $\mathcal{H}$, then $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{H}$ is a generalized frame for $\mathcal{H}$. If Γ is interpolating, $\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{K}$ is a Riesz-basis for $\mathcal{K}$ and hence a frame for $\mathcal{K}$, then $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{K}$ is a generalized frame for $\mathcal{K}$ with $\operatorname{supp}(\sigma) = \Gamma$.

Theorem 5.6. *(Same as Theorem 3.5) Let (X, d, λ) be a metric measure space satisfying (S1)-(S4). Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame for $\mathcal{H}$ satisfying (F) and (F2). If $\Gamma \subseteq X$ is sampling and δ_0−separated, then the lower density of Γ satisfies*

$$D^-(\Gamma) = \liminf_{r\to\infty} \inf_{z\in X} \frac{\#\,(\Gamma \cap B(z; r))}{\lambda\,(B(z; r))} \geq 1.$$

Proof. Since $\Gamma \subseteq X$ is sampling, then $\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{H}$ is a frame, so, denoting by σ the counting measure on Γ we have that $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{H}$ is a generalized frame for $\mathcal{H}$. In the notation of Corollary 5.5, the generalized frames $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ and $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{H}$ correspond to $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ and $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ respectively.

First we check the boundedness condition (i) in Corollary 5.5. On the one hand, since $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ is a normalized generalized Parseval frame for $\mathcal{H}$, then $k_y = \widetilde{k_y}^{\lambda}$ for all $y \in X$ because the frame operator is the identity map on $\mathcal{H}$, and hence $\sup_{y\in X}\left\|\widetilde{k_y}^{\lambda}\right\| = \sup_{y\in X}\|k_y\| = 1 < \infty$. On the other hand, $\sup_{y\in\operatorname{supp}(\sigma)}\left\|\widetilde{k_y}^{\sigma}\right\| \leq \frac{1}{\alpha_\sigma}\sup_{y\in\operatorname{supp}(\sigma)}\|k_y\| = \frac{1}{\alpha_\sigma} < \infty$, where α_σ is the lower constant of the frame $\{k_\gamma\}_{\gamma\in\Gamma}$.

Next we check the localization condition (ii) in Corollary 5.5. Fix $\varepsilon > 0$. Recall $\delta_0 > 0$ is the separation constant of Γ. Due to the annular decay property and the doubling property (S4), for all $a \in X$ and all $r > \frac{\delta_0}{2} > 0$ (this means $2r > r + \frac{\delta_0}{2}$) we have

$$\begin{aligned}
\lambda\left(B\left(a; r, r+\frac{\delta_0}{2}\right)\right) &\leq c_2\left(\frac{\frac{\delta_0}{2}}{r+\frac{\delta_0}{2}}\right)^a \lambda\left(B\left(a; r+\frac{\delta_0}{2}\right)\right) \\
&\leq c_2\left(\frac{\delta_0}{2r+\delta_0}\right)^a \lambda\left(B\left(a; r+\frac{\delta_0}{2}\right)\right) \\
&\leq c_2\left(\frac{\delta_0}{2r}\right)^a \lambda\left(B\left(a; 2r\right)\right) \\
&\leq c_1 c_2\left(\frac{\delta_0}{2r}\right)^a \lambda\left(B\left(a; r\right)\right),
\end{aligned}$$

and also

$$\begin{aligned}
\lambda\left(B\left(a; r-\frac{\delta_0}{2}, r\right)\right) &\leq c_2\left(\frac{\frac{\delta_0}{2}}{r}\right)^a \lambda\left(B\left(a; r\right)\right) \\
&= c_2\left(\frac{\delta_0}{2r}\right)^a \lambda\left(B\left(a; r\right)\right).
\end{aligned}$$

The mean value property (F2) establishes the existence of a constant $\alpha_{\delta_0} > 0$ such that for all $x, y \in X$ it holds

$$|\langle k_x, k_y\rangle|^2 \leq \alpha_{\delta_0} \int_{B\left(y;\frac{\delta_0}{2}\right)} |\langle k_x, k_z\rangle|^2 \, d\lambda(z).$$

Due to the localization property (F), there exists $R_1 > 0$ such that for all $a \in X$ and all $r \geq R_1$ it holds

$$\int_{B(a;r)^c} \int_{B(a;r)} |\langle k_x, k_z\rangle|^2 \, d\lambda(z) d\lambda(x) < \frac{\varepsilon}{2\alpha_{\delta_0}} \lambda\left(B\left(a; r\right)\right).$$

Pick $R > 0$ such that

$$R > \max\left\{\frac{\delta_0}{2}, \frac{\delta_0}{2}\left(\frac{\varepsilon}{2\alpha_{\delta_0} c_1 c_2}\right)^{-\frac{1}{a}}, \frac{\delta_0}{2}\left(\frac{\varepsilon}{2\alpha_{\delta_0} c_2}\right)^{-\frac{1}{a}}, R_1\right\},$$

then, for all $a \in X$ and all $r \geq R$ it holds

$$\begin{aligned}
\int_{B(a;r)^c}\int_{B(a;r)} \left|\langle k_x, k_y\rangle\right|^2 d\sigma(y)d\lambda(x) &= \int_{B(a;r)^c}\left(\sum_{\gamma\in B(a;r)\cap\Gamma}\left|\langle k_x, k_\gamma\rangle\right|^2\right)d\lambda(x)\\
&\leq \int_{B(a;r)^c}\left(\sum_{\gamma\in B(a;r)\cap\Gamma}\alpha_{\delta_0}\int_{B\left(\gamma;\frac{\delta_0}{2}\right)}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(z)\right)d\lambda(x)\\
&\leq \alpha_{\delta_0}\int_{B(a;r)^c}\int_{B\left(a;r+\frac{\delta_0}{2}\right)}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(z)d\lambda(x)\\
&= \alpha_{\delta_0}\int_{B(a;r)^c}\int_{B(a;r)}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(z)d\lambda(x)\\
&\quad +\alpha_{\delta_0}\int_{B(a;r)^c}\int_{B\left(a;r,r+\frac{\delta_0}{2}\right)}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(z)d\lambda(x)\\
&< \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)\\
&\quad +\alpha_{\delta_0}\int_{X}\int_{B\left(a;r,r+\frac{\delta_0}{2}\right)}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(z)d\lambda(x)\\
&= \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)\\
&\quad +\alpha_{\delta_0}\int_{B\left(a;r,r+\frac{\delta_0}{2}\right)}\int_{X}\left|\langle k_x, k_z\rangle\right|^2 d\lambda(x)d\lambda(z)\\
&= \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)+\alpha_{\delta_0}\int_{B\left(a;r,r+\frac{\delta_0}{2}\right)}\left\|k_z\right\|^2 d\lambda(z)\\
&= \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)+\alpha_{\delta_0}\,\lambda\left(B\left(a;r,r+\frac{\delta_0}{2}\right)\right)\\
&\leq \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)+\alpha_{\delta_0}c_1c_2\left(\frac{\delta_0}{2r}\right)^a\lambda\left(B\left(a;r\right)\right)\\
&\leq \frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)+\frac{\varepsilon}{2}\lambda\left(B\left(a;r\right)\right)\\
&\leq \varepsilon\left(\lambda+\sigma\right)\left(B\left(a;r\right)\right).
\end{aligned}$$

Similarly, for all $a \in X$ and all $r \geq R$ it also holds

$$\begin{aligned}
\int_{B(a;r)^c} \int_{B(a;r)} |\langle k_x, k_y \rangle|^2 \, d\lambda(y) d\sigma(x) &= \sum_{\gamma \in B(a;r)^c \cap \Gamma} \int_{B(a;r)} |\langle k_\gamma, k_y \rangle|^2 \, d\lambda(y) \\
&= \int_{B(a;r)} \sum_{\gamma \in B(a;r)^c \cap \Gamma} |\langle k_\gamma, k_y \rangle|^2 \, d\lambda(y) \\
&\leq \int_{B(a;r)} \left(\sum_{\gamma \in B(a;r)^c \cap \Gamma} \alpha_{\delta_0} \int_{B\left(\gamma; \frac{\delta_0}{2}\right)} |\langle k_z, k_y \rangle|^2 \, d\lambda(z) \right) d\lambda(y) \\
&\leq \alpha_{\delta_0} \int_{B(a;r)} \int_{B\left(a; r - \frac{\delta_0}{2}\right)^c} |\langle k_z, k_y \rangle|^2 \, d\lambda(z) d\lambda(y) \\
&= \alpha_{\delta_0} \int_{B(a;r)} \int_{B(a;r)^c} |\langle k_z, k_y \rangle|^2 \, d\lambda(z) d\lambda(y) \\
&\quad + \alpha_{\delta_0} \int_{B(a;r)} \int_{B\left(a; r - \frac{\delta_0}{2}, r\right)} |\langle k_z, k_y \rangle|^2 \, d\lambda(z) d\lambda(y) \\
&< \frac{\varepsilon}{2} \lambda \left(B \left(a; r \right) \right) \\
&\quad + \alpha_{\delta_0} \int_{B\left(a; r - \frac{\delta_0}{2}, r\right)} \int_X |\langle k_z, k_y \rangle|^2 \, d\lambda(y) d\lambda(z) \\
&= \frac{\varepsilon}{2} \lambda \left(B \left(a; r \right) \right) + \alpha_{\delta_0} \lambda \left(B \left(a; r - \frac{\delta_0}{2}, r \right) \right) \\
&\leq \frac{\varepsilon}{2} \lambda \left(B \left(a; r \right) \right) + \alpha_{\delta_0} c_2 \left(\frac{\delta_0}{2r} \right)^a \lambda \left(B \left(a; r \right) \right) \\
&\leq \frac{\varepsilon}{2} \lambda \left(B \left(a; r \right) \right) + \frac{\varepsilon}{2} \lambda \left(B \left(a; r \right) \right) \\
&\leq \varepsilon \left(\lambda + \sigma \right) \left(B \left(a; r \right) \right).
\end{aligned}$$

Finally, since $\{k_x\}_{x \in X}$ is a normalized generalized Parseval frame, then $\left\langle P_{\mathcal{H}} \widetilde{k_x}^{\lambda}, k_x \right\rangle = \langle k_x, k_x \rangle = 1$ for all $x \in X$, and since $\{k_\gamma\}_{\gamma \in \Gamma}$ is a frame, it is well-known that $\left| \left\langle P_{\mathcal{H}} k_\gamma, \widetilde{k_\gamma}^{\sigma} \right\rangle \right| = \left\langle k_\gamma, \widetilde{k_\gamma}^{\sigma} \right\rangle \leq 1$ for all $\gamma \in \Gamma = \operatorname{supp}(\sigma)$. Therefore, by Corollary 5.5

$$D^-(\Gamma) = \liminf_{r \to \infty} \inf_{a \in X} \frac{\# \left(B(a;r) \cap \Gamma \right)}{\lambda \left(B(a;r) \right)} = \liminf_{r \to \infty} \inf_{a \in X} \frac{\sigma \left(B(a;r) \right)}{\lambda \left(B(a;r) \right)} \geq 1.$$

This completes the proof. □

Theorem 5.7. *(Same as Theorem 3.8) Let (X, d, λ) be a metric measure space satisfying (S1)-(S4). Let $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ be a normalized generalized Parseval frame for $\mathcal{H}$ satisfying (F) and (F2). If $\Gamma \subseteq X$ is interpolating and δ_0-separated, then the upper density of Γ satisfies*

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{z\in X} \frac{\#(\Gamma \cap B(z;r))}{\lambda(B(z;r))} \leq 1.$$

Proof. Since $\Gamma \subseteq X$ is interpolating, then $\{k_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{K} \subseteq \mathcal{H}$ is a Riesz-sequence for $\mathcal{H}$ and a Riesz-basis (thus a frame) for $\mathcal{K}$ (recall $\mathcal{K} = \overline{\operatorname{span}}\{k_\gamma\}_{\gamma\in\Gamma}$), so $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{K}$ is a generalized frame for $\mathcal{K}$, where σ denotes the counting measure on Γ.

Following the same strategy as in the previous theorem, we apply Corollary 5.5, this time the generalized frames $\{k_x\}_{x\in(X,d,\sigma)} \subseteq \mathcal{K}$ and $\{k_x\}_{x\in(X,d,\lambda)} \subseteq \mathcal{H}$ correspond to $\{f_x\}_{x\in(X,d,\mu)} \subseteq \mathcal{F}$ and $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ respectively. The proof that the boundedness condition (i) and the localization condition (ii) in Corollary 5.5 are satisfied is identical as in the proof of Theorem 5.6.

Since $\{k_\gamma\}_{\gamma\in\Gamma}$ is a Riesz-sequence, then $\left\langle P_{\mathcal{H}}\widetilde{k_\gamma}^{\sigma}, k_\gamma\right\rangle = \left\langle \widetilde{k_\gamma}^{\sigma}, k_\gamma\right\rangle = 1$ for all $\gamma \in \Gamma = \operatorname{supp}(\sigma)$. Also, since $\{k_x\}_{x\in X}$ is a normalized generalized Parseval frame, then $k_x = \widetilde{k_x}^{\lambda}$ for all $x \in X$, so $\left|\left\langle P_{\mathcal{K}}k_x, \widetilde{k_x}^{\lambda}\right\rangle\right| \leq \|P_{\mathcal{K}}k_x\| \|k_x\| \leq 1$ for all $x \in X$. Therefore, by Corollary 5.5

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{a\in X} \frac{\#(B(a;r) \cap \Gamma)}{\lambda(B(a;r))} = \limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a;r))}{\lambda(B(a;r))} \leq 1.$$

This completes the proof. □

5.3 Applications: The density theorem

The objective of this section is to prove the main result, Theorem 2.2 in [28], as a consequence of the Theorem 5.4.

Consider a metric measure space (X, d, μ) and a reproducing kernel Hilbert space $\mathcal{H}$ with reproducing kernel $k(x, y)$ such that $\mathcal{H}$ is isometrically embedded in $L^2(X, \mu)$. For a fixed $x \in X$ denote $k_x := \overline{k(x, \cdot)} \in \mathcal{H}$. For any function $f \in \mathcal{H}$, it holds

$$f(x) = \int_X k(x,y) f(y) d\mu(y) \quad = \quad \int_X \overline{k_x(y)} f(y) d\mu(y) = \langle f, k_x\rangle .$$

In particular, $k(x,y) = \langle k_y, k_x \rangle$. Moreover, the relationship above implies that

$$\|f\|^2 = \int_X |f(y)|^2 \, d\mu(y) \quad = \quad \int_X \langle f, k_y \rangle \langle k_y, f \rangle \, d\mu(y) = \int_X |\langle f, k_y \rangle|^2 \, d\mu(y),$$

which shows that $\{k_x\}_{x \in (X,d,\mu)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$. Hence, the frame operator is the identity map, so, $\widetilde{k_x}^{\mu} = k_x$ for all $x \in X$. Also notice that for all $x \in X$

$$k(x,x) = \langle k_x, k_x \rangle = \left\langle k_x, \widetilde{k_x}^{\mu} \right\rangle = \|k_x\|^2 = \left\| \widetilde{k_x}^{\mu} \right\|^2 .$$

First we prove a proposition, which in essence states that the localization assumptions in [28] imply the localization condition (ii) in Theorem 5.4, under the regularity assumption that the annular decay property holds. In order to agree with the results in [28] but without going into much technicalities, we say that (X, d, μ) satisfies the *general annular decay property* if for any $\varepsilon > 0$ there exists $R > 0$ such that for all $a \in X$, for all $0 < \rho < R$, and all $r \geq R$ it holds

$$\mu\left(B(a; r, r + \rho)\right) < \varepsilon \mu\left(B(a; r)\right).$$

Remark. The doubling property (S4) together with the annular decay property from Chapter 3 imply the general annular decay property as stated above.

Proposition 5.8. *Let $\{f_x\}_{x \in (X,d,\mu)} \subseteq \mathcal{F}$ be a generalized frame for $\mathcal{F}$, $\{g_x\}_{x \in (X,d,\nu)} \subseteq \mathcal{G}$ be a generalized frame for $\mathcal{G}$, such that they satisfy the following conditions:*

i) Boundedness condition: $n_f := \sup_{y \in X} \left\| \widetilde{f_y} \right\| < \infty$, and $n_g := \sup_{y \in X} \|\widetilde{g_y}\| < \infty$.

ii) Special localization condition: for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\begin{aligned} \sup_{x \in X} \int_{B(x;R)^c} |\langle f_x, g_y \rangle|^2 \, d\nu(y) \quad &< \quad \varepsilon, \\ \sup_{x \in X} \int_{B(x;R)^c} |\langle g_x, f_y \rangle|^2 \, d\mu(y) \quad &< \quad \varepsilon. \end{aligned}$$

iii) Both (X, d, μ) and (X, d, ν) satisfy the general annular decay property.

Then, the localization condition (ii) in Theorem 5.4 holds.

Remark. We can substitute the assumption (iii) by the weaker assumption (iii)' shown below, the proof of the proposition is almost the same.

iii)' (X, d, μ) satisfies the general annular decay property, and (X, d, ν) satisfies that for any $\varepsilon > 0$ there exists $R > 0$ such that for all $a \in X$, for all $0 < \rho < R$, and all $r \geq R$ it holds

$$\nu\left(B(a; r, r+\rho)\right) < \varepsilon\mu\left(B(a; r)\right).$$

Proof. Let $\varepsilon > 0$. By the special localization condition (ii), there exists $\rho > 0$ such that for all $x \in X$

$$\int_{B(x;\rho)^c} |\langle g_x, f_y \rangle| \, d\mu(y) \quad < \quad \varepsilon.$$

If $x \in B(a; r)$ then $B(a; r+\rho)^c \subseteq B(x; \rho)^c$, then

$$\int_{B(a;r)} \int_{B(a;r+\rho)^c} |\langle g_x, f_y \rangle| \, d\mu(y) d\nu(x) \leq \int_{B(a;r)} \int_{B(x;\rho)^c} |\langle g_x, f_y \rangle| \, d\mu(y) d\nu(x) < \varepsilon\nu\left(B(a; r)\right).$$

Due to the general annular decay property (iii) on (X, d, μ), there exists $R' > \rho > 0$ such that for all $a \in X$ and all $r \geq R'$ it holds

$$\mu\left(B(a; r, r+\rho)\right) \quad < \quad \varepsilon\mu\left(B(a; r)\right),$$

then, using the boundedness condition (i), Tonelli's Theorem, and the fact that $\{g_x\}_{x\in(X,d,\nu)} \subseteq \mathcal{G}$ is a continuous frame for $\mathcal{G}$, we obtain

$$\int_{B(a;r)} \int_{B(a;r,r+\rho)} |\langle g_x, f_y \rangle| \, d\mu(y) d\nu(x) \lesssim \int_{B(a;r,r+\rho)} \|f_y\|^2 \, d\mu(y) \lesssim \varepsilon\mu\left(B(a; r)\right).$$

Combining these results we conclude that one of the inequalities in the localization condition (ii), Theorem 5.4, holds: for all $a \in X$ and all $r \geq R'$

$$\int_{B(a;r)^c} \int_{B(a;r)} |\langle f_x, g_y \rangle|^2 \, d\nu(y) d\mu(x) \quad \lesssim \quad \varepsilon\left(\mu + \nu\right)\left(B(a; r)\right).$$

Similar reasoning for the other inequality. □

Theorem 5.9. [28, Theorem 2.2] *Let (X, d, μ) be a metric measure space satisfying the general annular decay property. Let $\mathcal{H}$ be a reproducing kernel Hilbert space embedded in $L^2(X, \mu)$ and having a reproducing kernel such that $k(x, y) = \langle k_y, k_x \rangle$, thus $\{k_x\}_{x \in (X,d,\mu)} \subseteq \mathcal{H}$ is a generalized Parseval frame for $\mathcal{H}$. Let $\Gamma \subseteq X$ such that $\{k_\gamma\}_{\gamma \in \Gamma}$ is a Bessel sequence for $\mathcal{H}$.*

If the following conditions hold

i) There exist constants $C_1, C_2 > 0$ such that for all $x \in X$

$$C_1 \leq k(x, x) \leq C_2.$$

ii) Weak localization of the kernel: for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{x \in X} \int_{B(x;R)^c} |k(x, y)|^2 \, d\mu(y) < \varepsilon^2.$$

iii) Homogeneous approximation property: for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{x \in X} \sum_{\gamma \in \Gamma \cap B(x;R)^c} |k(x, \gamma)|^2 < \varepsilon^2.$$

Then, the following results hold

1) If $\Gamma \subseteq X$ is sampling, then

$$\begin{aligned} \liminf_{r\to\infty} \inf_{a \in X} \frac{\#\left(\Gamma \cap B(a; r)\right)}{\mu\left(B(a; r)\right)} &\geq \liminf_{r\to\infty} \inf_{a \in X} \frac{1}{\mu\left(B(a; r)\right)} \int_{B(a;r)} k(y, y) d\mu(y) \\ \limsup_{r\to\infty} \sup_{a \in X} \frac{\#\left(\Gamma \cap B(a; r)\right)}{\mu\left(B(a; r)\right)} &\geq \limsup_{r\to\infty} \sup_{a \in X} \frac{1}{\mu\left(B(a; r)\right)} \int_{B(a;r)} k(y, y) d\mu(y) \end{aligned}$$

2) If $\Gamma \subseteq X$ is interpolating, then

$$\begin{aligned} \liminf_{r\to\infty} \inf_{a \in X} \frac{\#\left(\Gamma \cap B(a; r)\right)}{\mu\left(B(a; r)\right)} &\leq \liminf_{r\to\infty} \inf_{a \in X} \frac{1}{\mu\left(B(a; r)\right)} \int_{B(a;r)} k(y, y) d\mu(y) \\ \limsup_{r\to\infty} \sup_{a \in X} \frac{\#\left(\Gamma \cap B(a; r)\right)}{\mu\left(B(a; r)\right)} &\leq \limsup_{r\to\infty} \sup_{a \in X} \frac{1}{\mu\left(B(a; r)\right)} \int_{B(a;r)} k(y, y) d\mu(y) \end{aligned}$$

Proof. By assumption (i), $\sup_{x\in X}\left\|\widetilde{k_x}^{\mu}\right\| \le \sqrt{C_2}$. Denote by σ the counting measure generated by Γ, $\operatorname{supp}(\sigma)=\Gamma$. Let $\mathcal{K}=\overline{\operatorname{span}}\,\{k_\gamma\}_{\gamma\in\Gamma}$, so $\{k_x\}_{x\in(X,d,\sigma)}\subseteq\mathcal{K}$ is a generalized frame for $\mathcal{K}$. Recall that $\left\|\widetilde{k_\gamma}^{\sigma}\right\| \le \frac{1}{\alpha_\sigma}\|k_\gamma\|$ for all $\gamma\in\Gamma=\operatorname{supp}(\sigma)$, where α_σ is the lower constant of the generalized frame $\{k_x\}_{x\in(X,d,\sigma)}$. Again by assumption (i) we conclude $\sup_{x\in\operatorname{supp}(\sigma)}\left\|\widetilde{k_x}^{\sigma}\right\| \le \frac{1}{\alpha_\sigma}\sqrt{C_2}$. Hence, the boundedness condition (i) in Theorem 5.4 (and in Proposition 5.8) holds taking $\{f_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{F}$ as the generalized Parseval frame $\{k_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{H}$, and $\{g_x\}_{x\in(X,d,\nu)}\subseteq\mathcal{G}$ as the generalized frame $\{k_x\}_{x\in(X,d,\sigma)}\subseteq\mathcal{K}$, or the other way around.

If $\{f_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{F}$ corresponds to the generalized Parseval frame $\{k_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{H}$, and $\{g_x\}_{x\in(X,d,\nu)}\subseteq\mathcal{G}$ corresponds to the generalized frame $\{k_x\}_{x\in(X,d,\sigma)}\subseteq\mathcal{K}$, then

$$\begin{aligned}\sup_{x\in X}\int_{B(x;R)^c}|\langle f_x,g_y\rangle|^2\,d\nu(y) &= \sup_{x\in X}\int_{B(x;R)^c}|\langle k_x,k_y\rangle|^2\,d\sigma(y)=\sup_{x\in X}\sum_{\gamma\in\Gamma\cap B(x;R)^c}|k(x,\gamma)|^2,\\ \sup_{x\in X}\int_{B(x;R)^c}|\langle g_x,f_y\rangle|^2\,d\mu(y) &= \sup_{x\in X}\int_{B(x;R)^c}|\langle k_x,k_y\rangle|^2\,d\mu(y)=\sup_{x\in X}\int_{B(x;R)^c}|k(x,y)|^2\,d\mu(y).\end{aligned}$$

So, the weak localization of the kernel and the homogeneous approximation property, i.e., assumptions (ii) and (iii), imply that the special localization condition (ii) in Proposition 5.8 is satisfied, and the same conclusion follows if the correspondence between generalized frames is interchanged.

Additionally, since (X,d,μ) satisfies the general annular decay property, and $\{k_\gamma\}_{\gamma\in\Gamma}\subseteq\mathcal{H}$ is a Bessel sequence for $\mathcal{H}$, Lemmas 3.4 and 3.7 in [28] imply that (X,d,μ) and (X,d,σ) satisfy the condition (iii)' in Proposition 5.8. Hence, Proposition 5.8 guarantees that the localization condition (ii) in Theorem 5.4 is satisfied.

In order to prove (1), assume Γ is sampling, in this case $\mathcal{K}=\mathcal{H}$. Make the correspondence $\{f_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{F}$ with $\{k_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{H}$, and $\{g_x\}_{x\in(X,d,\nu)}\subseteq\mathcal{G}$ with $\{k_x\}_{x\in(X,d,\sigma)}\subseteq\mathcal{H}$. Notice that

$$\left|\left\langle P_{\mathcal{H}}k_\gamma,\widetilde{k_\gamma}^{\sigma}\right\rangle\right|=\left\langle k_\gamma,\widetilde{k_\gamma}^{\sigma}\right\rangle\le 1$$

for all $\gamma\in\Gamma=\operatorname{supp}(\sigma)$, since $\{k_\gamma\}_{\gamma\in\Gamma}$ is a frame. Applying Theorem 5.4(1)

$$\begin{aligned}\liminf_{r\to\infty}\inf_{a\in X}\frac{\#(\Gamma\cap B(a;r))}{\mu(B(a;r))} &= \liminf_{r\to\infty}\inf_{a\in X}\frac{\sigma(B(a;r))}{\mu(B(a;r))}\\ &\ge \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu(B(a;r))}\left|\int_{B(a;r)}\left\langle P_{\mathcal{H}}\widetilde{k_y}^{\mu},k_y\right\rangle d\mu(y)\right|\\ &= \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu(B(a;r))}\int_{B(a;r)}k(y,y)d\mu(y),\end{aligned}$$

$$\begin{aligned}
\limsup_{r\to\infty}\sup_{a\in X}\frac{\#\left(\Gamma\cap B(a;r)\right)}{\mu\left(B(a;r)\right)} &= \limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma\left(B(a;r)\right)}{\mu\left(B(a;r)\right)} \\
&\geq \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{H}}\widetilde{k_y}^{\mu}, k_y\right\rangle d\mu(y)\right| \\
&= \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)} k(y,y)d\mu(y).
\end{aligned}$$

In order to prove (2), assume Γ is interpolating. Make the correspondence $\{f_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{F}$ with $\{k_x\}_{x\in(X,d,\sigma)}\subseteq\mathcal{K}$, and $\{g_x\}_{x\in(X,d,\nu)}\subseteq\mathcal{G}$ with $\{k_x\}_{x\in(X,d,\mu)}\subseteq\mathcal{H}$. Notice that

$$\left\langle P_{\mathcal{H}}\widetilde{k_\gamma}^{\sigma}, k_\gamma\right\rangle = \left\langle \widetilde{k_\gamma}^{\sigma}, k_\gamma\right\rangle = 1$$

for all $\gamma\in\Gamma=\operatorname{supp}(\sigma)$, since $\{k_\gamma\}_{\gamma\in\Gamma}$ is a Riesz-sequence. Applying Theorem 5.4(2)

$$\begin{aligned}
\limsup_{r\to\infty}\sup_{a\in X}\frac{\#\left(\Gamma\cap B(a;r)\right)}{\mu\left(B(a;r)\right)} &= \limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma\left(B(a;r)\right)}{\mu\left(B(a;r)\right)} \\
&\leq \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{K}}k_y, \widetilde{k_y}^{\mu}\right\rangle d\mu(y)\right| \\
&\leq \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)}\left|\left\langle P_{\mathcal{K}}k_y, \widetilde{k_y}^{\mu}\right\rangle\right| d\mu(y) \\
&= \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)}\left|\left\langle P_{\mathcal{K}}k_y, k_y\right\rangle\right| d\mu(y) \\
&\leq \limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)} k(y,y)d\mu(y),
\end{aligned}$$

$$\begin{aligned}
\liminf_{r\to\infty}\inf_{a\in X}\frac{\#\left(\Gamma\cap B(a;r)\right)}{\mu\left(B(a;r)\right)} &= \liminf_{r\to\infty}\inf_{a\in X}\frac{\sigma\left(B(a;r)\right)}{\mu\left(B(a;r)\right)} \\
&\leq \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\left|\int_{B(a;r)}\left\langle P_{\mathcal{K}}k_y, \widetilde{k_y}^{\mu}\right\rangle d\mu(y)\right| \\
&\leq \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)}\left|\left\langle P_{\mathcal{K}}k_y, \widetilde{k_y}^{\mu}\right\rangle\right| d\mu(y) \\
&= \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)}\left|\left\langle P_{\mathcal{K}}k_y, k_y\right\rangle\right| d\mu(y) \\
&\leq \liminf_{r\to\infty}\inf_{a\in X}\frac{1}{\mu\left(B(a;r)\right)}\int_{B(a;r)} k(y,y)d\mu(y).
\end{aligned}$$

$\square$

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