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Normal Domains Arising from Graph Theory

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Normal Domains arising from Graph Theory

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Abstract

Determining whether an arbitrary subring R of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a normal domain is, in general, a nontrivial problem, even in the special case of a monomial generated domain. First, we determine normality in the case where R is a monomial generated domain where the generators have he form $(x_i x_j)^{\pm 1}$. Using results for this special case we generalize to the case when R is a monomial generated domain where the generators have the form $x_i^{\pm 1} x_j^{\pm 1}$. In both cases, for the ring R, we consider the combinatorial structure that assigns an edge in a mixed directed signed graph to each monomial of the ring. We then use this relationship to provide a combinatorial characterization of the normality of R, and, when R is not normal, we use the combinatorial characterization to compute the normalization of R. Using this construction, we also determine when the ring R satisfies Serre's R_1 condition. We also discuss generalizations of this to directed graphs with a homogenizing variable and a special class of hypergraphs.

Dedication

Dedicated to my parents: Marc and Claudia Lipman, with love.

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Chapter 1

Introduction

In recent years there has been interest in modeling algebraic problems using combinatorial methods. This approach of using a combinatorial object to model the behavior of algebraic problems has led to powerful results, for example, Reisner's characterization of all Cohen-Macaulay Stanley-Reisner rings [3, Corollary 5.3.9]. One approach is to model the generators of an algebraic ring, or ideal, as the edges of a combinatorial graph. This model has led to characterizations of algebraic properties for these so-called *edge rings* and *edge ideals* in terms of the combinatorial behavior of the associated graphs.

Independently, Hibi and Ohsugi [38] and Simis, Vasconcelos, and Villarreal [49] (see also [4, 48, 49, 58–60]) gave a construction of the edge ring of a graph as a coordinate ring $k[G] \subseteq k[x_1, \ldots, x_n]$ of a toric variety over a field k from a graph G. Hibi and Ohsugi studied the case where G is connected, while Simis, Vasconcelos, and Villarreal considered a more general case. From the graph, they gave a combinatorial characterization of the normality of $k[G]$ in terms of G, additionally, when $k[G]$ is not a normal domain, they constructed the normalization of $k[G]$ from the combinatorial data.

Since the introduction of edge rings, there have been many papers studying other properties of the edge rings of graphs. Examples of algebraic properties that have been studied for edge rings include: studying invariants of the rings [12], depth [18, 19, 35, 57] determining when they are complete intersections [2, 28, 55], the Noether normalization

[1,31], determining multiplicities [13], determining when they are combinatorially pure [37], determining when the edge rings are strongly Koszul [22], and when the edge rings satisfy Serre's R_1 condition [20].

In addition to the papers studying the algebraic properties of the edge rings, there has been a lot of interest in studying the associated edge polytope of the the graph that was introduced as part of the construction of the edge ring in [38]. Some of the properties for edge polytopes studied have included: counting the number of edges of the polytope [23], determining when they are smooth Fano polytopes [25], extremal properties of the edge polytopes [56], when they are ample [17, 24], determining separating hyperplanes [21], and unimodular triangulations [36].

Edge rings of graphs have also been used in applications to other problems including: normality of $0-1$ polytopes [14,36], which can be thought of as the edge rings of hypergraphs, determing Gröbner bases of toric ideals from posets [43], the normality of Minkowski sums [26], geometric descriptions of holes in affine monoids [27], determining the circuits of toric ideals [42], the behavior of Ehrhart series [33, 34], constructing Gorenstein rings [40, 41], determining if a toric set is an affine toric variety [46], birationality of monomial subrings [50]. Applications appear even in coding theory [47] and statistical ranking [52].

In this dissertation, we generalize the work of the original papers [38, 49], constructing edge rings of finite signed graphs as quadratic-monomial generated domains in the Laurent polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we start with a signed graph, produce a coordinate ring for a toric variety, characterize the normality of the ring combinatorially, and give a combinatorial characterization of the normalization of the ring. This case is considerably more complicated than the situation in the previous work, because the negative powers allow for exponents to cancel. In order to address these difficulties, we introduce new proofs; in particular, our proofs are are more combinatorial and geometric in nature than in the previous work.

We determine the normality of a domain generated by monomials of the form $x_i^{\pm 1} x_j^{\pm 1}$ by reducing to the normality in the special case when the generators have the form $(x_i x_j)^{\pm 1}$.

For such a ring $k[G]$ in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, called the *edge ring*, we construct a signed graph G with vertices $\{1, \ldots, n\}$ by associating a signed edge to each monomial $(x_i x_j)^{\pm 1}$ in the ring. Following [38], the construction of a normal domain is done in several steps: first, from the graph G, we construct a polytope $\mathcal{P}_G$ in $\mathbb{R}^n$; then, from the polytope, we construct a normal semigroup $S_1 \subseteq \mathbb{Z}^n$ from the polytope; and, finally, from the semigroup S_1, we construct a normal ring $\mathcal{A}(\mathcal{P}_G) \subseteq k[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ containing $k[G]$. In fact, $\mathcal{A}(\mathcal{P}_G)$ is the normalization of $k[G]$, and, when $k[G]$ is not equal to $\mathcal{A}(\mathcal{P}_G)$ we use the combinatorial data from G to construct the generators of $\mathcal{A}(\mathcal{P}_G)$ over $k[G]$.

For the case where the edge ring $k[G]$ is generated by monomials of the form $x_i^{\pm 1} x_j^{\pm 1}$ in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we associate the ring to a mixed signed, directed graph G. We then reduce G to a larger signed graph $\widetilde{G}$ so that $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$ is isomorphic to the normalization of $k[G]$.

Observe that the this construction is distinct from the similar construction of an edge ideal of a graph. The edge ring and edge ideal are generated by the same elements, but one as a subring and the other as an ideal of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. For more details on the edge ideal, see, for example, [5, 8, 11, 15, 16, 18, 19, 35, 39, 45, 48, 53, 57].

Using the construction of the edge polytope of a signed graph G, $\mathcal{P}_G$, and the condition for Serre's R_1 condition presented in [61], we determine when the edge ring $k[G]$ satisfies Serre's R_1 condition. This is done by studying the lattice of exponents of $\mathcal{A}(\mathcal{P}_G)$ and sublattices associated with the the facets of the cone of $\mathcal{P}_G$. Then, in a similar manner to when we determined normality, we generalize this result to the edge rings for mixed signed directed graphs. As a consequence of this result, we can construct a set of toric coordinate rings that fail normality but satisfy Serre's R_1 condition. In particular, Serre's Criterion for Normality (see [10, Thm. 11.5], and [3, Thm. 2.2.22] for discussions and proof) shows that a ring is normal if and only if two conditions are met, R_1 and S_2. Thus, a non-normal ring which satisfies R_1 must fail S_2. Since, a ring is Cohen-Macaulay if and only if it satisifes S_ℓ for all ℓ [3, pp. 63], this allows us to construct rings which are not Cohen-Macaulay. Since normal toric varieties are Cohen-Macaulay [7, Thm. 9.2.9], these two results allow the construction of rings which are known to be Cohen-Macaulay, or are known to be not Cohen-Macaulay, and, in particular, fail S_2.

In the remaining chapters, we consider other combinatorial structures, directed graphs and hypergraphs, where we associate the directed edges and hyperedges to monomials in a monomial generated subring of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, in a similar manner to how we constructed the coordinate ring for mixed directed signed graphs. Note that Corollary 5.1.2 tells us that, using the construction given above for mixed directed signed graphs, the edge ring of a directed graph will always be normal. Thus, we consider a slightly different construction. Suppose the ring R is generated by monomials $x_i^{-1}x_js$ in the Laurent Polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, s]$ where the variable s is a *grading variable*. To each of these monomials we associate a directed edge (i, j) in the directed graph G with vertex set $\{1, 2, \ldots, n\}$. Using this new, graded, edge ring we give a combinatorial characterization of normality and the normalization when $k[G]$ is not normal.

Similar to the construction in [38], Kimura Terai and Yoshida [29] provide a construction for a toric coordinate ring $k[G] \subseteq k[x_1, \ldots, x_n]$ from a hypergraph G with vertex set $\{1, \ldots, n\}$. This is done by associating each hyperedge $e = \{v_1, \ldots, v_r\} \subseteq \{1, 2, \ldots, n\}$ to a monomial generator $x_{v_1} \cdots x_{v_r}$. This construction, and questions of normality are also asked by Martínez-Bernal, O'Shea, and Villarreal in [32] using both the edge ring and the edge ideal of the hypergraph. However, the edge ring studied is a homogenized edge ring in $k[x_1, \ldots, x_n, t]$, which will not be studied in this dissertation. Note that when using this construction, the exponents of the generating monomials will be 0-1 vectors in $\mathbb{Z}^d$. This construction is similar, but different, from the construction given by Lin and McCullough [30] and Hà and Lin [14]; in their construction; rather than associating the monomials with the hyperedges, they associate the monomials with the vertices. The hypergraph produced by this construction is described as the dual of the hypergraph given in [32]. Using the combinatorial behavior of the hypergraph G in [14] they provide a sufficient condition for normality, but demonstrate that it is not necessary. Takayama in [54] studied a special class of hypergraphs where the hyperedges correspond to the facets of a 2-dimensional simplicial complex. Using these hypergraphs, Takayama determined some sufficient normality conditions, properties of the Gröbner basis, and Koszulness by studying the 1-skeleton of the complex as a graph. Using a similar construction to the one given in [32], we study non-homogenized monomial-generated polynomial rings with generators of the form $(x_i x_j x_k)^{\pm 1}$ for $i \neq j \neq k$, or $(x_i x_j)^{\pm 1}$ and, in the case when the hyperedges are completely separable, hyperedges of arbitrary size.

Finally, in Chapter 8 we summarize the dissertation the results and the open questions. The main results are grouped by the nature of the result: dimension formulas, a normality conditions, normalization results, or Serre's R_1 conditions. The open questions from the end of Chapters 3 - 7 are then listed.

Chapter 2

Background and Notation

In this chapter, we recall notation, definitions, and results from graph theory, semi-group theory, and edge rings for use in this paper. Our notation for edge rings follows the notation of Hibi and Ohsugi [38].

2.1 Undirected Graphs

In this section, we cover the relevant definitions, notation and results for the graph theory used for the following chapters. In particular, we use the notation from [6]. We begin with basic definitions, see [6, Chapter 1]

Definition 2.1.1. A *graph* G is an ordered pair of finite sets (V, E), *vertices* V and *edges* E, here an edge is a two element subset of V. If $e = \{u, v\}$ is an edge of G we say u and v are the *endpoints* of e or are *incident* to e. For simplicity of notation, we denote an edge $\{u, v\}$ of G as uv. If u and v are vertices of G we say u is *adjacent* to v if there is an edge uv of G.

Definition 2.1.2. A *graph with loops* is a graph $G = (V, E)$ where E is a set of 2-element multisets of V. A *loop* of G is an edge $\{u, u\}$ for some vertex u in G. For simplicity of notation, we denote a loop $\{u, u\}$ of G as uu.

Definition 2.1.3. Let $G = (V, E)$ be a graph. A *path* P of G is a subgraph of G with vertex set $\{u_1, u_2, \ldots, u_n\}$ and edge set $\{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n\}$. A *cycle* C of G is a subgraph of G with vertex set $\{u_1, u_2, \ldots, u_n\}$ and edge set $\{u_1u_2, \ldots, u_{n-1}u_n, u_nu_1\}$. A *walk* W of G is a finite sequence of vertices $v_1, v_2, \ldots, v_n$ called the *vertex sequence* so that v_i is adjacent to v_{i+1} for $i = 1, \ldots, n$. We say a walk W is *closed* if $v_1 = v_n$.

Note that a path is a walk without repeating vertices and a cycle is a closed walk with no repeated vertices besides $v_1 = v_n$. These definitions extend to graphs with loops.

Definition 2.1.4. We say a graph $G = (V, E)$ is *connected* if, for every pair of vertices $u, v \in V$, there is a path P with vertex sequence $u = u_1, \ldots, u_n = v$ for some n in G. A *component* of G is a maximal connected subgraph of G.

Definition 2.1.5. We say a walk, path or cycle is *even* if the number of vertices in the vertex sequence, with multiplicity for walks, is even. Otherwise we say the walk, path or cycle is *odd*.

Definition 2.1.6. A graph $G = (V, E)$ is said to be *bipartite* if $V = L \cup R$ is a partition of V and for every edge $\{u, v\}$ of V, ($u \in L$ and $v \in R$) or ($u \in R$ and $v \in L$). Equivalently, a graph G is bipartite if every cycle of G is even.

Definition 2.1.7. A graph $G = (V, E)$ is said to be *acyclic* or a *forest* if G does not contain any cycles. If G is connected and acyclic then we say G is a *tree*.

Definition 2.1.8. The *neighborhood* or *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) \ : \ uv \in E(G)\}$, and the *neighborhood* of a set $S \in V(G)$ is the set $N(S) = \bigcup_{v \in S} N(v)$.

Definition 2.1.9. Given a graph G, a *matching* is a set of edges M so that each vertex lies in at most one element of M. A matching is said to be *perfect* if each vertex lies on exactly one edge in M.

The following theorem is derived from the Tutte-Berge Formula for b-matchings, see [44].

Theorem 2.1.10 (Hall's b-Matching condition)**.** Let $G = (V, E)$ be a bipartite multi-graph with vertex partition $L \cup R$. For each vertex $v \in V$, let b_v be a non-negative integer. There is a subgraph of G so that each vertex $v \in L$ has degree b_v and each vertex $u \in R$ has degree at most b_u if and only if for every subset S of L,

$$\sum_{v \in S} b_v \leq \sum_{u \in N(S)} b_u.$$

Proof. Clearly, if G has such a subgraph then the condition is met. Now assume that the condition is met for every set $S \subseteq L$, we construct a b-matching H. Let u be a vertex that does not have b_u chosen incident edges. Construct the alternating tree rooted at u using the neighbors with an edge that is either in or not in the H at each vertex. If we find a vertex v in R that does not have b_v incident edges in H, then we have an augmenting path. If the tree can not get any bigger then we have a subset S of L that does not match the criteria. Thus, by assumption, there is always an augmenting path and hence there is always a b-matching. □

Note that an easy corollary of this theorem is that we have a subgraph where each vertex $v \in V$ has degree b_v if and only if the condition can be applied to every $S \subseteq L$ and to every $S \subseteq R$.

2.1.1 Signed Graphs

Definition 2.1.11. A *signed graph* (G, sgn) is a finite graph $G = (V, E)$ and a *sign function* $\text{sgn} : E \to \{-1, +1\}$ where $\text{sgn}(e)$ denoted the *sign* of the edge $e \in E$. For notational convenience, an edge ij with $\text{sgn}(ij) = +1$ is denoted $+ij$, and an edge ij with $\text{sgn}(ij) = -1$ is denoted $-ij$. We omit the sign when it is understood from context.

Definition 2.1.12 (cf [38])**.** Let G be a signed graph with d vertices, possibly with loops, and without multiple edges. Define a map $\rho : E(G) \to \mathbb{R}^d$ as $\rho(e) = \text{sgn}(e)(e_i + e_j)$ where $e = +ij$ or $e = -ij$ is an edge of the graph. When e is a loop, $i = j$ and $\rho(ii) = 2\,\text{sgn}(ii)e_i$. Let $\rho(E(G))$ be the image of $E(G)$ and define the *edge polytope of* G as $\mathcal{P}_G := \text{conv}\{\rho(E(G))\}$.

Observe that, in [38], the graph G was not a signed graph, and, hence, all edges e in G have positive sign, i.e., $\rho(e) = (e_i + e_j)$.

2.2 Hypergraphs

Definition 2.2.1. A *hypergraph* G is a pair of finite sets (V, E), *vertices* V and *hyperedges* E, here a hyperedge is a subset of V. If $e = \{u_1, \ldots, u_r\}$ is a hyperedge of G, we say the vertices are *incident* to e. For simplicity of notation we denote a hyperedge $\{u_1, \ldots, u_r\}$ of G as $u_1 \cdots u_r$. If u and v are vertices of G, we say u is *adjacent* to v if there is a hyperedge containing u and v in G.

In this dissertation, we mainly study two types of hypergraphs, *totally separable* hypergraphs, and *separable* hypergraphs with 2-vertex and 3-vertex edges. In order to help clarify the notation, a hyperedge with two vertices will be referred to as an *edge*, an edge with the same vertex appearing two times is a *loop*, and a hyperedge with more than two, or an unspecified number of vertices will be referred to as a *hyperedge*. In the case when a hypergraph has only 2-vertex and 3-vertex hyperedges, the hypergraph will be denoted $G = (V, E_2 \cup E_3)$ where $V(G)$ is the set of vertices, $E_2(G)$ is the set of edges and $E_3(G)$ is the set of 3-vertex hyperedges.

Definition 2.2.2. A *component* of a hypergraph $G = (V, E)$ is a maximal connected sub-hypergraph. That is, replacing each hyperedge with a clique, the vertices in a component of the resulting graph induce a component in the hypergraph.

Definition 2.2.3. A hypergraph is said to be *totally separable* if, for every hyperedge e with more than two vertices, the vertices of e are pairwise in separate components of $G \setminus e$. A hypergraph is said to be *separable* if, for every hyperedge e with more than two vertices, $\mathrm{Comp}(G) < \mathrm{Comp}(G \setminus e)$.

Definition 2.2.4. A *signed hypergraph* (G, sgn) is a hypergraph $G = (V, E)$ and a *sign function* $\mathrm{sgn} : E \to \{-1, +1\}$ where $\mathrm{sgn}(e)$ denoted the *sign* of the edge, or hyperedge

$e \in E$. For notational convenience, a hyperedge e with $\mathrm{sgn}(e) = +1$ is denoted $+e$, and a hyperedge e with $\mathrm{sgn}(e) = -1$ is denoted $-e$, in particular for hyperedge $\{i, j, k\}$ is denoted $\pm ijk$.

Definition 2.2.5 (cf [38])**.** Let G be a signed hypergraph with d vertices, possibly with loops, and without multiple hyperedges. Define a map $\rho : E(G) \to \mathbb{R}^d$ as the sum $\rho(e) = \mathrm{sgn}(e) \sum_{i \in e} e_i$ where $\pm e$ is a hyperedge of the graph. When e is a loop, $i = j$ and $\rho(ii) = 2\,\mathrm{sgn}(ii)e_i$. Let $\rho(E(G))$ be the image of $E(G)$ and define the *edge polytope of* G as $\mathcal{P}_G := \mathrm{conv}\{\rho(E(G))\}$.

2.3 Algebra

This sections covers the relevant definitions and theorems from Algebra. In particular we will be using notation and definitions from [3] and [10].

Definition 2.3.1. ([10]) Given a commutative ring R, S is a *commutative algebra* over R, or *R-algebra*, if, S is a commutative algebra with a homomorphism $R \to S$.

2.3.1 Integral Domains

Definition 2.3.2. ([10]) If S is an R-algebra and $p(x)$ is a polynomial with coefficients in R, then we say that an element $s \in S$ *satisfies* p if $p(s) = 0$. The element s is called *integral over* R if it satisfies a monic polynomial with coefficients in R. If every element of S is integral over R we say that S itself is integral over R.

Definition 2.3.3. ([10]) Given an R-algebra S, the ring of all elements of S integral over R is called the *integral closure* of R in S. In the case when R is an integral domain and S is the its quotient field, then the subalgebra of elements of S integral over R is called the *normalization* of R. A domain equal to its own normalization is called a *normal domain*.

Let R be an integral domain, let $p, q \in R$, so that $q \neq 0$. Suppose the ratio $\frac{p}{q}$ is the root of a monic polynomial $f(x)$. Then $\frac{p}{q}$ is in the normalization of R. Moreover, if we can assume $q = 1$ for all such ratios, then R is a normal domain. Normal domains have many nice properties including that the set of normal domains is closed under intersection.

Proposition 2.3.4. If R and S are normal domains, then $R \cap S$ is also a normal domain.

Proof. Let a be an element of the fraction field of $R \cap S$, and suppose a satisfies a monic polynomial $f(x)$ with coefficients from $R \cap S$. Then a is an element of the fraction field of R. Observe that $f(x)$ is still a monic polynomial with coefficients in $R \cap S$ and hence in R. Thus, by the normality of R, $a \in R$. Similarly, $a \in S$ and hence $a \in R \cap S$. Thus, $R \cap S$ is a normal domain. □

2.3.2 Semigroups

Definition 2.3.5. A *semigroup* is a set, C, with an associative binary operation, we assume that the operation is commutative and has an identity element, zero. An *affine semigroup* C is a finitely generated semigroup which, for some n, is isomorphic to a subsemigroup of $\mathbb{Z}^n$. For a field k, the ring over k generated by elements $\{x_1^{c_1} \dots x_n^{c_n} = X^c \ : \ c = (c_1, \dots, c_n) \in C\}$ is called an *affine semigroup ring* and is denoted $k[C]$.

For the remainder of this section, we assume that C is a subsemigroup of $\mathbb{Z}^n$. Let $\mathbb{Z}C$ denote the smallest subgroup of $\mathbb{Z}^n$ that contains C, and denote $\mathbb{R}_+C := \mathbb{R}_+ \otimes_{\mathbb{Z}_+} C$ that is, the elements of $\mathbb{R}_+C$ are all positive linear combinations of elements of C.

Definition 2.3.6. An affine semigroup C is *normal* if it satisfies the following condition: if $mz \in C$ for some $z \in \mathbb{Z}C$ and m a positive integer, then $z \in C$.

Normality also has a geometric interpretation: consider the line segment between the origin and mz. The semigroup is normal if every point of $\mathbb{Z}^n$ that lies on this line segment is either in both C and $\mathbb{Z}C$ or in neither.

Proposition 2.3.7 (Gordan's Lemma [3, Proposition 6.1.2 and Theorem 6.1.4])**.**

a) If C is a normal semigroup, then $C = \mathbb{Z}C \cap \mathbb{R}_+C$.

b) Let G be a finitely generated subgroup of $\mathbb{Q}^n$ and D a finitely generated rational polyhedral cone in $\mathbb{R}^n$, equivalently, D is as the intersection of a finite number of halfspaces, where the linear equations defining the halfspaces have rational coefficients, and all the linear equations have constant zero. Then, $G \cap D$ is a normal semigroup.

c) Let C be an affine semigroup and k a field. C is a normal semigroup if and only if $k[C]$ is a normal domain.

2.3.3 Integral Closures of Edge Rings

Hibi and Ohsugi [38] gave a characterization normalization of quadratic monomial generated domains in $k[x_1, \ldots, x_n]$. In this section, we recall their construction and generalize it to disconnected graphs. We will connect the notation used by Hibi and Ohsugi in [38] with the notation used in [3].

Construction 2.3.8 (Generalization to $s = \pm 1$, cf [38])**.** Let $k[G]$ be a quadratic-monomial generated domain with generators of the form $(x_i x_j)^s$ in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $s = \pm 1$ then:

- Construct the graph G with vertex set $\{1, \ldots, n\}$ and edge ij with sign s, denoted $s \cdot ij$, for each monomial of the form $(x_i x_j)^s$ in $k[G]$.
- From the signed graph G, define the polytope
$$\mathcal{P}_G := \text{conv}\{\rho(E(G))\} = \left\{ \sum_{e \in E} a_e \rho(e) \ : \ \sum_{e \in E} a_e = 1 \right\}.$$
- Define the lattice $\mathcal{L}_G := \mathbb{Z}\rho\{E(G)\} = \left\{ \sum_{e \in G} z_e \rho(e) \ : \ z \in \mathbb{Z} \right\}$.
- Construct the semigroup $S_1 := \mathcal{L}_G \cap \text{cone}(\mathcal{P}_G) = \mathcal{L}_G \cap \{a \cdot p \ : \ a > 0, p \in \mathcal{P}_G\}$. By Proposition 2.3.7(b), S_1 is a normal semigroup.
- Construct the domain $\mathcal{A}(\mathcal{P}_G) := k[S_1]$. By Proposition 2.3.7(c), $\mathcal{A}(\mathcal{P}_G)$ is a normal domain.

This construction can be represented by the following sequence of objects:

$$k[G] \Rightarrow G \Rightarrow \mathcal{P}_G \Rightarrow S_1 = \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_G \Rightarrow \mathcal{A}(\mathcal{P}_G).$$

Moreover, the ring $k[G]$ can be recovered from the graph G as the semigroup ring $k[\mathbb{N}\rho(E(G))]$ where $\mathbb{N}\rho(E(G))$ is the semigroup consisting of all non-negative integer combinations of the elements of $\rho(E(G))$. Even though several signed graphs G can generate the same $k[G]$, this construction from G motivates the notation $k[G]$ for the edge ring.

The construction of Hibi and Ohsugi [38] gives an explicit combinatorial structure to the normalization of the edge ring. In particular, the construction allows the normalization of the edge ring to be computed explicitly from the structure of the graph. That is, they show that $k[G] \subseteq \mathcal{A}(\mathcal{P}_G)$ and provide explicit descriptions of the generators of $\mathcal{A}(\mathcal{P}_G)$ over $k[G]$ as elements of the fraction field of $k[G]$, hence $\mathcal{A}(\mathcal{P}_G)$ is the normalization of $k[G]$.

Briefly, we use the notation $\overline{R}$ to indicate the normalization of a domain R.

Lemma 2.3.9 ([49, Lemma 2.7])**.** Let G_1 and G_2 be graphs with vertices associated to disjoint sets of variables. Then $k[G_1 \cup G_2] \cong k[G_1] \otimes_k k[G_2]$ where $G_1 \cup G_2$ is the disjoint union of graphs and

$$\overline{k[G_1 \cup G_2]} \cong \overline{k[G_1]} \otimes_k \overline{k[G_2]} \cong \overline{k[G_1]k[G_2]},$$

where $\overline{k[G_1]k[G_2]}$ is the ring generated by the generators of $\overline{k[G_1]}$ and $\overline{k[G_2]}$.

Observe that Lemma 2.3.9 implies that the normality of $k[G]$ depends on the components of G individually. Hence, we can often restrict our attention in the remainder of this paper to connected components of G.

Chapter 3

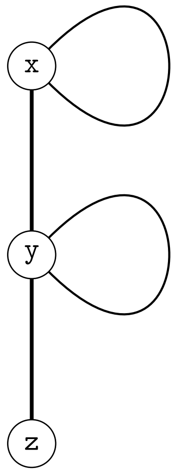
Edge Polytopes and Edge Rings

Recall we generate the edge rings in several steps: First, we construct a polytope $\mathcal{P}_G$ from the graph G. We then explore the geometry of $\mathcal{P}_G$ from the perspective of the combinatorics on G. This includes a characterization of the facets of $\mathcal{P}_G$ in terms of subgraphs of G. Next, we construct an affine semigroup from the polytope $\text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$ where $\mathcal{L}_d$ is a finitely generated subgroup of $\mathbb{Z}^d$, and hence of $\mathbb{Q}^d$. Using the standard construction from [3] we construct three integral domains $k[G]$, $k[\mathcal{P}_G]$ and $\mathcal{A}(\mathcal{P}_G)$, where $k[G]$, and $k[\mathcal{P}_G]$ are finitely generated domains, and $\mathcal{A}(\mathcal{P}_G)$ is a normal domain. Finally, we characterize when $k[G]$ is normal in particular when $k[G] = \mathcal{A}(\mathcal{P}_G)$, and when it is not we find the generators of the normalization of $k[G]$. To avoid confusion between the graph theory and the geometry, the words *vertices* and *edges* refer to the graphs, and *extremal points* and *faces* refer to the polytopes.

3.1 Geometric Results

Observation 3.1.1. This construction gives a covariant functor between finite loop graphs and integral polytopes, where the arrows are subgraph inclusions and subpolytope inclusions respectively.

Example 3.1.2. Let G be the graph on vertex set $V(G) = \{x, y, z\}$ and edge set $E(G) =$



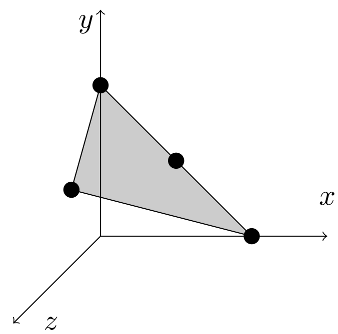


Figure 3.1: The graph G and Polytope $\mathcal{P}_G$. Observe that the point $(1, 1, 0)$ is not an extremal point of the polytope, this is the point associated with the edge xy in G. Moreover, every other edge is mapped to an extremal point of $\mathcal{P}_G$.

$\{xx, xy, yy, yz\}$. Then $\mathcal{P}_G$ is the triangle with vertices at $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 1, 1)$. This polytope has dimension 2 and is contained on the hyperplane $\{(x, y, z) \in \mathbb{R}^3 : x+y+z = 2\}$, see Figure 3.1. Observe that $\rho(xy)$ is not an extremal point of $\mathcal{P}_G$. In fact, $\rho(xy) = \frac{1}{2}\rho(xx) + \frac{1}{2}\rho(yy)$.

This naturally brings up the question: when is an edge of G an extremal point of $\mathcal{P}_G$?

Definition 3.1.3. ([38]) We say a G is *reduced* if G does not have vertices i and j such that ij, ii, and jj are all edges in G.

Note that, given a graph G, we can produce a reduced graph $\tilde{G}$ from G. This is done by deleting edges ij from G when we have loops ii and jj in G, for all vertices i and j of G. Observe that from the earlier example we know that $\mathcal{P}_{\tilde{G}}$ is the same as $\mathcal{P}_G$. It remains to be shown that this is the only situation when an edge does not produce an extremal point.

Proposition 3.1.4. (Generalized to disconnected graphs, [38, Proposition 1.2]) Every edge of G gives an extremal point if and only if G is a reduced graph.

Proof. Observe that all the points in $\rho((E(G))$ have non-negative coordinates. Thus, any convex combination has no cancellation. If $\rho(ij)$ is not an extremal point, it is a convex combination of extremal points that are non-zero only for i or j. Hence, if ij is a loop, that is $i = j$, there are no possible combinations. Otherwise, the only possible combination is $\rho(ij) = \frac{1}{2}\rho(ii) + \frac{1}{2}\rho(jj)$. □

Assumption 3.1.5. For the rest of this chapter, it is assumed that all graphs are reduced.

Now that the extremal points of $\mathcal{P}_G$ have been characterized, we study the dimension of $\mathcal{P}_G$. First, we consider the dimension produced by a connected graph. Then, we discuss how having more than one component influences the dimension of the polytope.

Observe that, in Example 3.1.2, the dimension of $\mathcal{P}_G$ is $|V| - 1$. This is shown by the containment in the hyperplane $\{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 2\}$. In fact, this hyperplane containing $\mathcal{P}_G$ generalizes to all graphs. Let $\mathbf{1}$ be the vector of all ones, $\mathbf{1}^*$ the dual vector, and consider the hyperplane $\{x \in \mathbb{R}^d : \langle \mathbf{1}^*, x\rangle = 2\}$ where $d = |V|$. As each edge maps to a point in this hyperplane, the polytope $\mathcal{P}_G$ is contained in the hyperplane as well. Hence, $\dim \mathcal{P}_G \leq d - 1$. This leaves the question, what is the actual dimension of $\mathcal{P}_G$?

Example 3.1.6. Let G and G' be graphs on $V(G) = V(G') = \{1, 2, 3, 4\}$ with edge sets $E(G) = \{12, 23, 34, 14\}$ and $E(G') = \{12, 23, 34, 14, 13\}$, see Figure 3.2. Observe that $E(G') = E(G) \cup \{13\}$. Then, $\mathcal{P}_G$ forms a square in $\mathbb{R}^4$ with dimension 2, and $\mathcal{P}_{G'}$ forms a pyramid with a square base in $\mathbb{R}^4$, and hence has dimension 3.

In fact, $\mathcal{P}_G$ is contained in hyperplanes $\{x \in \mathbb{R}^4 : x_1 + x_3 = 1\}$ and $\{x \in \mathbb{R}^4 : x_2 + x_4 = 1\}$, while $\mathcal{P}_{G'}$ is not. However, $\mathcal{P}_{G'} \subset \{x \in \mathbb{R}^4 : x_1 + x_3 \geq 1\}$. That is, $\mathcal{P}_{G'}$ is contained in one of the half spaces defined by $\{x \in \mathbb{R}^4 : x_1 + x_3 = 1\}$. In particular, $\mathcal{P}_G$ is a facet of $\mathcal{P}_{G'}$.

Observe that in the example, G is bipartite with partition $\{1, 3\}$ and $\{2, 4\}$. The two hyperplanes containing $\mathcal{P}_G$ correspond to this partition. That is, let a^* be the dual characteristic vector of $\{1, 3\}$ (or $\{2, 4\}$) then $\mathcal{P}_G \subset \{x \in \mathbb{R}^4 : \langle a^*, x\rangle = 1\}$. As before, this generalizes to all bipartite graphs.

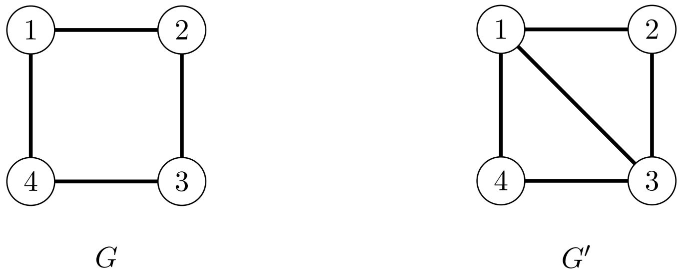


Figure 3.2: The graphs G and G'. The polytope associated with G is a rectangle in $\mathbb{R}^4$, and hence is 2-dimensional, while the polytope associated with G' is a pyramid with a square base in $\mathbb{R}^4$ and hence is 3-dimensional.

Definition 3.1.7. Let G be a finite graph. Let $\mathrm{Comp}(G)$ be the number of components of G, and let $\mathrm{BiComp}(G)$ be the number of components of G which are bipartite.

Lemma 3.1.8. Let G be a graph with $\mathrm{BiComp}(G) = k$. Then, $\mathcal{P}_G$ is contained in k linearly independent hyperplanes passing through the origin.

Proof. Let C be a bipartite component of G with partition $L \cup R$. Denote the dual characteristic vector of L as e_L^* and the dual characteristic vector of R as e_R^*. For every edge ij, either i is not in $L \cup R$ or i is in L or R and j is in the other. Observe that, $\mathcal{P}_G \subseteq \{x \in \mathbb{R}^d : \langle e_L^* - e_R^*, x\rangle = 0\}$. For each pair of bipartite components of G, these hyperplanes have non-zero coefficients on disjoint sets of variables; hence, these form a set of linearly independent hyperplanes containing $\mathcal{P}_G$. □

Proposition 3.1.9. (Generalized to disconnected graphs, [38]) Let G be a graph on d vertices with $\mathrm{BiComp}(G) = k$. Then, $\dim \mathcal{P}_G = d - k - 1$.

Proof. Construct $\mathcal{P}_G$ in $\mathbb{R}^{d+1}$ by $(a_1, \ldots, a_d) \mapsto (a_1, \ldots, a_d, 1)$ for all $(\alpha) = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Let $\mathcal{A}_G$ represent the affine span of $\mathcal{P}_G$ in $\mathbb{R}^{d+1}$. Now, construct the dual space in $\mathbb{R}^{d+1}$ of $\mathcal{P}_G$ as

$$\mathcal{A}_G^{\perp} = \{x \in \mathbb{R}^{d+1} : \langle a^*, x\rangle = 0 \text{ for all } a \in \mathcal{P}_G\}.$$

Observe that $\dim \mathcal{P}_G = \dim \mathcal{A}_G = d + 1 - \dim \mathcal{A}_G^{\perp}$.

Now, take H a maximal spanning forest of G. By Lemma 3.1.8 this gives a set of $\mathrm{BiComp}(G)$ orthogonal hyperplanes determined by $\langle e_L^* - e_R^*, x\rangle = 0$, where e_L^* and e_R^* are the dual characteristic vectors of the bipartite partition of a component of H. Since H is a forest, every component is bipartite, hence we have a hyperplane from each component. These hyperplanes are represented by vectors $(e_L - e_R, 0)^*$ in $(\mathbb{R}^{d+1})^*$. Also, there is the hyperplane determined by $\langle \mathbf{1}^*, x\rangle = 2$. This is represented by the dual vector $(\mathbf{1}, -2)^*$ in $(\mathbb{R}^{d+1})^*$. These hyperplanes form an independent set for $\mathcal{A}_H^{\perp}$.

Observe that for any vector $(v, 0)^* \in (\mathbb{R}^{d+1})^*$ so that $\langle v^*, \rho(ij)\rangle = 0$ for all $ij \in E(H)$ has $v_i = -v_j$. Thus, on a bipartite component of G with bipartition $L \cup R$ we have v restricted to $L \cup R$ is $a(e_L - e_R)^*$ for some $a \in \mathbb{R}$. Hence, every $(v, 0)^* \in (\mathbb{R}^{d+1})^*$ so that $\langle v^*, \rho(ij)\rangle = 0$ for all $ij \in E(H)$ can be written as a linear combination of the vectors $(e_L - e_R, 0)^*$. For a vector $(v, c)^* \in (\mathbb{R}^{d+1})^*$ that satisfies $\langle v^*, \rho(ij)\rangle = c \neq 0$ for all $ij \in E(H)$ we assume, without loss of generality that $c = 2$. Let $L \cup R$ be a bipartite component of H, and let $i \in L$, and let $(v')^* = v^* - (v_i + 1)(e_L - e_R)^*$, that is v' has 1 for the i coordinate. Since $\langle (v')^*, \rho(e)\rangle = 2$ this implies all the vertices adjacent to i in H also have coordinate 1 in v'. This implies that v^* is a sum of $(\mathbf{1}, \mathbf{-2})^*$ and $\{(e_L - e_R, 0)^* \ : \ L \cup R \text{ is a component of } H\}$, and thus this set of hyperplanes is also a basis of $\mathcal{A}_H^{\perp}$.

Since each component of H is a tree, if ij is an edge in H, then $\rho(ij)$ can not be written as a linear combination of $\rho(E(H))$ that does not use $\rho(ij)$. Since $\dim \mathcal{A}_H^{\perp} \geq \operatorname{Comp}(G) + 1$, $\dim \mathcal{A}_H \geq d - \operatorname{Comp}(G)$ and $\dim \mathcal{A}_H + \dim \mathcal{A}_H^{\perp} = d + 1$, $\dim \mathcal{P}_H = \dim \mathcal{A}_H = (d+1) - \dim \mathcal{A}_H^{\perp} = d - \operatorname{Comp}(G)$ in $\mathbb{R}^{d+1}$. Since H is a subgraph of G we have: $\mathcal{P}_H \subseteq \mathcal{P}_G$, and hence $\mathcal{A}_H \subseteq \mathcal{A}_G$ and $\mathcal{A}_G^{\perp} \subseteq \mathcal{A}_H^{\perp}$.

We iteratively add edges that are in G to H. The dimension of $\mathcal{A}_H^{\perp}$ decreases if and only if one of the orthogonal basis vectors is no longer perpendicular to the polytope. This happens if and only we add an edge e to a component H_i of H so that H_i is bipartite, but $H_i \cup \{e\}$ is not bipartite. Hence, when we have added all the edge in G to H, we have $\dim \mathcal{A}_G^{\perp} = \operatorname{BiComp}(G) + 1$, and hence, $\dim \mathcal{A}_G = d - 1 - \operatorname{BiComp}(G)$. □

Corollary 3.1.10. (Generalized to disconnected graphs, [38, Lemmas 1.4,1.5,1.6]) Let G be a reduced, graph on d vertices, and H a spanning subgraph of G, possibly with trivial components. Then, $\mathcal{P}_H$ is a $(d-k)$-simplex if and only if H has $d - k + 1$ edges and

BiComp$(H) = k - 1$.

Proof. If $\mathcal{P}_H$ is a $(d-k)$-simplex, then $\mathcal{P}_H$ must have exactly $d-k+1$ extremal points. Since H is reduced, each edge gives an extremal point in $\mathcal{P}_H$ and so H has exactly $d-k+1$ edges. As a consequence of Proposition 3.1.9, in order for $\dim \mathcal{P}_H = d-k$, BiComp$(H) = k-1$.

Conversely, if H has BiComp$(H) = k-1$ and has $d-k+1$ edges then $\mathcal{P}_H$ is a $d-k$ dimensional polytope with $d-k+1$ extremal points and hence is a $(d-k)$-simplex. $\square$

Thus, we have determined the minimal requirements for a subgraph to have a specified dimension. Since each face of the polytope is the convex hull of a set of extremal points a natural question is which sets of edges, that is which subgraphs, define facets of the polytope?

Example 3.1.11. Let G' be a graph on $V(G') = \{1,2,3,4\}$ and edge set $E(G') = \{12,23,34,14,13\}$, see G' from Figure 3.2. As mentioned above, $\mathcal{P}_{G'}$ is a four sided pyramid, and hence has dimension $4-1=3$. Recall the subgraph of G with edge set $\{12,23,34,14\}$ gives the base of the pyramid. We can now determine the other facets of the pyramid. $\{12,23,13\}$, $\{13,34,14\}$, $\{13,12,14\}$ and $\{13,23,34\}$ satisfy the conditions to be $d-2$-simplicies. That is, they are triangles. In fact these are the facets. For example, $\mathcal{P}_G \subseteq \{x \in \mathbb{R}^4 : x_1 + x_2 + x_3 \leq 2\}$, while $\rho(12), \rho(23), \rho(13) \in \{x \in \mathbb{R}^d : x_1 + x_2 + x_3 = 2\}$. This tells us that $\{12,23,13\}$ defines a face. The other faces are described similarly.

Observe, the edge 13 has the property that $\mathcal{P}_{G\setminus\{13\}}$ has dimension 2 while $\mathcal{P}_G$ has dimension 3. Similarly, $\mathcal{P}_{G\setminus\{34,14\}}$ has dimension 2. $G\setminus\{34\}$ and $G\setminus\{14\}$ both contain odd cycles and the associated polytopes have dimension 3.

Now, we ask how does $G\setminus\{34,14\}$ differ from $G\setminus\{12,34\}$? The resulting subgraph is connected and bipartite and so has dimension 2. However, this does not produce a facet. Our understanding of the geometry tells us that this subgraph is the polytope intersected with the hyperplane containing two opposite extreme points of the base as well as the point on the top of the pyramid. So, it is a diagonal rather than a face of the polytope.

The next set of results show how to determine a subgraph which corresponds to a facet. We begin by defining some combinatorial conditions on vertices and subgraphs. Observe that, from Proposition 3.1.9, we have two ways to find a subgraph that has an edge polytope with smaller dimension than the original graph's edge polytope. We either decrease the number of vertices, or we increase the number of bipartite components. The difficulty lies in choosing the subgraph so that it is a face.

Definition 3.1.12. Let G be a graph without trivial components.

- We say that vertex i is *regular* if $\mathrm{BiComp}(G \setminus i) = \mathrm{BiComp}(G)$.
- We say that vertex i is *ordinary* if $\mathrm{Comp}(G \setminus i) = \mathrm{Comp}(G)$.

If i is regular, then $\dim \mathcal{P}_{G\setminus i} = \dim \mathcal{P}_G - 1$. Let $\mathrm{BiComp}(G) = k$ and so $\mathcal{P}_G$ has dimension $d-k-1$, $G \setminus i$ has $\mathrm{BiComp}(G \setminus i) = k$ and so $\mathcal{P}_{G\setminus i}$ has dimension $(d-1)-k-1 = d-k-2$.

For i ordinary, $\dim \mathcal{P}_{G\setminus i} < \dim \mathcal{P}_G$, but the difference can be one or two. For example, if G is connected and not bipartite, but $G \setminus i$ is bipartite then $\dim \mathcal{P}_{G\setminus i} = \dim \mathcal{P}_G - 2$. However, if G is connected and bipartite then $\dim \mathcal{P}_{G\setminus i} = \dim \mathcal{P}_G - 1 = d - 3$.

Example 3.1.13. Let G' be a graph on $V(G') = \{1, 2, 3, 4\}$ and edge set $E(G') = \{12, 23, 34, 14, 13\}$, see Figure 3.2. Then vertices 2 and 4 are regular as $G \setminus 2$ has the odd cycle $\{13, 34, 14\}$, and, similarly, $G \setminus 4$ has odd cycle $\{13, 12, 23\}$.

Vertices 1 and 3 however, are ordinary. The dimension of edge polytope decreases by 2 if either 1 or 3 are deleted. This follows as $G \setminus 1$ has $E(G \setminus 1) = \{23, 34\}$ which is connected and bipartite on four vertices. Similarly, $G \setminus 3$ has $E(G \setminus 3) = \{12, 14\}$ which is also connected and bipartite on four vertices.

For the geometry, we make use of the following hyperplane defined for each vertex:

Definition 3.1.14. Let $i \in V(G)$,

$$\mathcal{H}_i := \{x \in \mathbb{R}^d : x_i = 0\}.$$

We also define the half space,

$$\mathcal{H}_i^{(+)} := \{x \in \mathbb{R}^d : x_i \geq 0\}.$$

Observe that $\mathcal{H}_i \cap \mathcal{P}_G \cong \mathcal{P}_{G\setminus i}$. That is, the polytope $\mathcal{P}_{G\setminus i}$ associated with $G \setminus i$ is contained in the hyperplane $\mathcal{H}_i$. Hence, $\mathcal{H}_i$ is a hyperplane that intersects $\mathcal{P}_G$. As long as $\deg_G(i) \neq 0$ we have $\mathcal{P}_G \cap \mathcal{H}_i \neq \mathcal{P}_G$. Moreover, $\mathcal{P}_G \subseteq \mathcal{H}_i^{(+)}$ for all i, since $\rho(e)$ has either $0, 1$ or 2 for each coordinate. Hence, $\mathcal{H}_i$ is a supporting hyperplane for each i with $deg(i) \neq 0$ and so, $\mathcal{P}_G \subseteq \bigcap_{i=1}^{d} \mathcal{H}_i^{(+)}$. However, not all the facets are given by the polytopes produced in this construction.

Example 3.1.15. Let G be the graph on $V(G) = \{1, 2, 3, 4\}$ with edge set $E(G) = \{12, 23, 34, 14, 13\}$, see graph G' in Figure 3.2. Recall the subgraph G of G' with edge set $\{12, 23, 34, 14\}$ is the base, see graph G in Figure 3.2. This facet is not given by any $\mathcal{H}_i \cap \mathcal{P}_G$.

Thus, we consider both non-trivial subgraphs as well as vertices for defining facets. We want a subgraph H of G so that $\dim \mathcal{P}_H < \dim \mathcal{P}_G$. That is, $\mathcal{P}_H$ is contained in a hyperplane that does not contain $\mathcal{P}_G$.

Suppose G is connected with H a spanning subgraph. For $\dim \mathcal{P}_H$ to be less $\dim \mathcal{P}_G$ the number of bipartite components in H must be greater than the number of bipartite components in G. Recall that we can write the hyperplane that contains a connected bipartite graph with partition $V = L \cup R$ as $\{x \in \mathbb{R}^d : \langle e_L^* - e_R^*, x\rangle = 0\}$, where e_R^* and e_L^* are the dual characteristic vectors of R and L, respectively. Suppose ij is an edge of G that is not in H and $\rho(ij) \notin \{x \in \mathbb{R}^d : \langle e_L^* - e_R^*, x\rangle = 0\}$. Then, without loss of generality, we may assume $\langle e_L^* - e_R^*, \rho(ij)\rangle > 0$, that is $i, j \in L$. In order for $\mathcal{P}_H$ to be a face of $\mathcal{P}_G$, $\langle e_L^* - e_R^*, \rho(ij)\rangle > 0$ must hold true for all edges ij in G such that $\rho(ij)$ is not on the hyperplane. Thus, the set R must be independent in G, as if ij is an edge with $i, j \in R$ then $\langle e_L^* - e_R^*, \rho(ij)\rangle < 0$.

Now, if we wish to increase the number of bipartite components in G by one so that all the edges deleted do not satisfy the bipartite hyperplane, what would this look like? Either we would delete edges in a non-bipartite component so that we leave exactly one bipartite component and possibly some non-bipartite components. Or, we would delete edges from a bipartite component so that we leave exactly two components behind.

Putting these together gives us the following definitions:

Definition 3.1.16. Let T be an independent set of G, G_T the bipartite graph on vertices $V(G_T) = T \cup N_G(T)$ that is T and all vertices adjacent to a vertex in T, and edge set $E(T, N_G(T))$ that is, all edges between vertices of T and vertices of $N_G(T)$ the set of neighbors of T, but not including the edges between vertices of $N_G(T)$.

Observe that G_T is a bipartite graph. Moreover, G_T satisfies the first condition we desired from our subgraphs, i.e. that one of the sets of the partition is an independent set.

Definition 3.1.17. Let $T \neq \emptyset$ be an independent set of vertices in a component G' of G.

- If G' is not bipartite, then T is *fundamental* if G_T is connected and either $T \cup N_G(T) = V(G')$ or every component of $G' \setminus V(G_T)$ has at least one odd cycle.
- If G' is bipartite then T is *acceptable* if G_T is connected and $G' \setminus V(G_T) \neq \emptyset$ is connected.

Observe that if T is acceptable, and $G' \setminus V(G_T)$ does not have any edges, then $G \setminus V(G_T)$ is an isolated vertex i and the supporting hyperplane of $\mathcal{P}_{G_T}$ is equal to the supporting hyperplane of $\mathcal{P}_{G \setminus i}$. That is, deleting all the edges incident to a vertex to increase the number of bipartite components geometrically is the same as deleting the vertex.

Example 3.1.18. Let G be a graph on vertex set $V(G) = \{1, 2, 3, 4\}$ and edge set $E(G) = \{14, 13, 34, 23\}$, see graph G in Figure 3.3. The set $T = \{1, 2\}$ is independent, the subgraph G_T has vertex set $\{1, 2, 3, 4\}$ and edge set $\{14, 13, 23, \}$, see graph G_T in Figure 3.3. Since G_T and G have the same vertex set, and G_T is connected, T is a fundamental set.

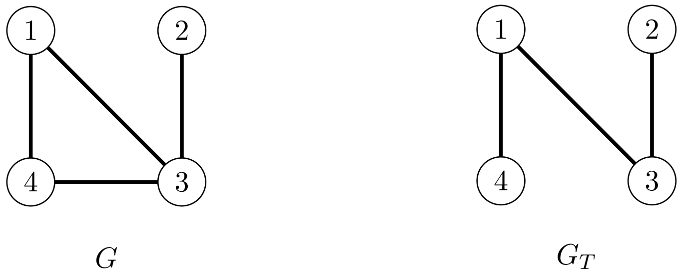


Figure 3.3: Graphs G and G_T. G_T is a subgraph of G that is a maximal bipartite subgraph so that the edges of G that are not in G_T go between vertices in the same part of G_T. Namely, the vertex partition of G_T is $\{1, 2\}$ and $\{3, 4\}$, with the edge 34 the only edge of G that is not in G_T.

Definition 3.1.19. Let $T \neq \emptyset$ be an independent set of vertices. Define the hyperplane,

$$\mathcal{H}_T := \left\{ x \in \mathbb{R}^d : \sum_{i \in T} x_i = \sum_{j \in N(T)} x_j \right\},$$

and the half space,

$$\mathcal{H}_T^{(-)} := \left\{ x \in \mathbb{R}^d : \sum_{i \in T} x_i \leq \sum_{j \in N(T)} x_j \right\}.$$

For all $e \in E(G)$, $\sum_{i \in T} x_i = 1$ if and only if $\sum_{j \in N(T)} x_j = 1$, otherwise $\sum_{i \in T} x_i = 0$. That is, each edge has either exactly one vertex in T or no vertices in T, this implies $\mathcal{P}_G \subseteq \mathcal{H}_T^{(-)}$. Thus,

$$\mathcal{P}_G \subseteq \bigcap_{T \text{ independent}} \mathcal{H}_T^{(-)}.$$

We now have a complete characterization of the facets of $\mathcal{P}_G$ in terms of the subgraphs and vertices of G.

Theorem 3.1.20. (Generalized to disconnected graphs, [38, Theorem 1.7]) Let $G = (V, E)$ be a finite graph on d vertices with no trivial components, $\mathcal{P}_G$ be the associated edge polytope, and $\mathcal{H}$ be a hyperplane in $\mathbb{R}^d$. If $\mathcal{H} \cap \mathcal{P}_G$ is a facet of $\mathcal{P}_G$, then one of the following must be true:

- for some fundamental or acceptable set T, $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_T \cap \mathcal{P}_G$, or

- for some regular or ordinary vertex i, $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_i \cap \mathcal{P}_G$.

Proof. Let $\mathcal{P}'_G = \mathcal{H} \cap \mathcal{P}_G$ be a facet of $\mathcal{P}_G$. Observe that $\dim \mathcal{P}'_G = \dim \mathcal{P}_G - 1$. Let $G' = (V', E')$ be the subgraph of $G = (V, E)$ induced by $\mathcal{P}'_G$. That is, an edge ij is in G' if and only if $\rho(ij) \in \mathcal{P}'_G$, and V' is the subset of vertices with degree at least one in G'. Hence, G' has either one fewer vertices than G or $\text{BiComp}(G') = \text{BiComp}(G) + 1$, but not both.

Suppose G' has one fewer vertices than G, say G' does not have vertex i in G. If i was in a bipartite component of G then i can not be a cut vertex, otherwise $\text{BiComp}(G) < \text{BiComp}(G \setminus i)$, and thus, is ordinary. If, i is a cut vertex then its deletion can not create more bipartite components in $G \setminus i$, and thus, is regular. Every edge e in $G \setminus i$, $\rho(e)$ is in $\mathcal{P}'_G$ since otherwise there is a hyperplane containing $\mathcal{P}'_G$ that does not contain $\rho(e)$. However, this implies that $\mathcal{H} \neq \mathcal{H}_i$ and so $\mathcal{P}'_G \subseteq \mathcal{H}_i \cap \mathcal{H}$, which implies $\dim \mathcal{P}'_G - 2 \leq \mathcal{P}_G$, which is a contradiction to $\mathcal{P}'_G$ being a facet. Thus, $\mathcal{P}'_G = \mathcal{H}_i \cap \mathcal{P}_G$.

Now, suppose G' has one more bipartite component than G. Then, one of the parts of this bipartite component must be independent in G, otherwise $\mathcal{P}'_G$ would not be a face of $\mathcal{P}_G$. Denote this independent set by T. Either this bipartite component was part of a larger bipartite component in G or it was part of a larger non-bipartite component in G. If the set T was part of a larger bipartite component, then T forms an acceptable set. If the set T was part of a larger non-bipartite component, then T forms a fundamental set. In either case, $\mathcal{P}'_G = \mathcal{H}_T \cap \mathcal{P}_G$. □

3.2 Algebraic Results

Let G be a finite graph, possibly with loops. We construct three rings from the edge polytope $\mathcal{P}_G$. To do this, we begin by defining a finitely generated lattice $\mathcal{L}_d$,

$$\mathcal{L}_d := \{x \in \mathbb{Z}^d : \langle \mathbf{1}^*, x \rangle \in 2\mathbb{Z}\}.$$

The normal domain $\mathcal{A}(\mathcal{P}_G)$ is constructed from cross sections of $\text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$. The second ring, $k[\mathcal{P}_G]$, is constructed from the lattice points of $\mathcal{P}_G$. The third ring, $k[G]$, is constructed from the extremal points of $\mathcal{P}_G$. We then characterize when these domains are equal and normal. In the event that $k[G]$ is not normal we provide the normalization.

Definition 3.2.1. Let $\mathcal{P}$ be a polytope and n a non-negative integer. Define

$$n\mathcal{P} := \{n\alpha : \alpha \in \mathcal{P}\}.$$

Note that for our edge polytope $\mathcal{P}_G$, the polytope $n\mathcal{P}_G$ corresponds to the scaled projection of $\mathcal{P}_G$ on to the hyperplane $\{x \in \mathbb{R}^d : \langle \mathbf{1}^*, x \rangle = 2n\}$ from the origin. Letting n range over $\mathbb{N}$ gives the cross sections of the cone at natural numbers.

A way of viewing $n\mathcal{P}_G \cap \mathbb{Z}^d$ is the set of rational points $(a_1, \ldots, a_d)$ in $\mathcal{P}_G$ so that the least common denominator of $a_1, \ldots, a_d$ divides n scaled from $\mathcal{P}_G$ to $n\mathcal{P}_G$. Taking the union of all $n\mathcal{P}_G \cap \mathbb{Z}^d$ over n gives the affine semigroup $S_1 = \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$. That is,

$$\bigcup_{n=0}^{\infty} \left(\mathbb{Z}^d \cap n\mathcal{P}_G\right) = \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d.$$

As a side note, if $\mathcal{P}$ is a d-dimensional integral polytope in $\mathbb{R}^n$ and $t > 0$ then

$$i(\mathcal{P}, t) = \#\{x \in t\mathcal{P} \cap \mathbb{Z}^n\}$$

is the *Ehrhart polynomial*, over t, of $\mathcal{P}$. It is known that the degree of $i(\mathcal{P}, t)$ is d and the leading coefficient of $i(\mathcal{P}, t)$ is the d-dimensional relative volume of $\mathcal{P}$ [51, Proposition 4.6.13]. Also, the constant term of $i(\mathcal{P}, t) = a_0 = 1$ [51, Corollary 4.6.11].

We now consider the affine semigroup $S_1 := \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$. Observe that this is an affine semigroup as any two points $\alpha, \beta \in \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$ satisfy $\alpha + \beta \in \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$ since $\mathcal{L}_d$ is a group, and adding two positive linear combinations together gives a positive linear combinations. It is affine since it contains the origin.

There are several properties of S_1 that we are interested in, for example S_1 is a normal semigroup. To see this observe that S_1, as defined, is the intersection between $\mathcal{L}_d$ a finitely generated subgroup of $\mathbb{Q}^d$, and cone($\mathcal{P}_G$) a rational cone defined by a finite number of half space intersections. Hence, by Proposition 2.3.7, S_1 is normal.

Another is that S_1 is generated by the points $(a_1, \ldots, a_d) \in S_1$ that satisfy

$$\gcd\left(a_1, \ldots, a_d, \frac{1}{2}\sum_{i=1}^{d} a_i\right) = 1.$$

This follows from the observation that each element of $n\mathcal{P}_G \cap \mathbb{Z}^d$ lies on a line segment, through the origin, and a rational point in $\mathcal{P}_G$ with least common denominator divisible by n. So, given a rational point $(b_1, \ldots, b_d) \in \mathcal{P}_G$ let n be the smallest integer so that $(nb_1, \ldots, nb_d) \in \mathcal{L}_d$. Then $(nb_1, \ldots, nb_d)$ satisfies

$$\gcd\left(nb_1, \ldots, nb_d, \frac{n}{2}(b_1 + \ldots + b_d)\right) = 1,$$

by the minimality of n. If the greatest common denominator is $c > 1$ then $\frac{n}{c}$ would be an integer so that $(\frac{n}{c}b_1, \ldots, \frac{n}{c}b_d) \in \mathcal{L}_d$, which would violated the minimality of n. The $\frac{n}{2}(b_1 + \ldots + b_d)$ term in the GCD expression is what gives the minimality of the generator. That is, $(2nb_1, \ldots, 2nb_d) = (a_1, \ldots, a_d)$ is also in $\mathcal{L}_d$, however $\gcd\left(a_1, \ldots, a_d, \frac{1}{2}\sum_{i=1}^{d} a_i\right) > 1$. This guarantees that the n used to produce $(a_1, \ldots, a_d)$ is as small as possible.

Combinatorially, the points in S_1 correspond to non-negative rational weights on the edges of G so that, for each vertex, the sum of the weights on incident edges is an integer and the vertex weights sum to an even integer. Of interest is the existence of a finite generating set for S_1. The generating set above is not finite in general. To construct a finite generating set for S_1 we take a set of elements that we know are in S_1, generate a new semigroup using them and determine the elements of S_1 that are not in the new semigroup. One set of points that are known to be in any generating set of S_1 is the extremal points of $\mathcal{P}_G$, another is the integral points of $\mathcal{P}_G$. So, when is one of these sets a generating set of

S_1? We can construct the semigroup using the extremal points of $\mathcal{P}_G$ as generators. Let,

$$S_3 := \left\{ x \in \mathbb{Z}^d : \sum_{ij \in E(G)} a_{ij}\rho(ij), a_{ij} \in \mathbb{N}\ \forall ij \in E(G) \right\}.$$

S_3 can be thought of as assigning non-negative integer weights to the edges of G.

Clearly, S_3 is a subsemigroup of S_1 as the extremal points of $\mathcal{P}_G$ are integral, and it is finitely generated. So, when is S_1 equal to S_3? That is, when are the extremal points of $\mathcal{P}_G$ a generating set of S_1? We address this question next.

Example 3.2.2. Let G be the four cycle on vertices $V(G) = \{1, 2, 3, 4\}$ with edge set $E(G) = \{12, 23, 34, 14\}$. Construct S_1 and S_3 as above. We show that in this case S_1 is equal to S_3. Let $(a_{12}, a_{23}, a_{34}, a_{14})$ be non-negative rational weights on the edges so that each vertex i has $a_{ij} + a_{ki}$ is an integer for the appropriate $j \neq k$. That is we have a point in S_1. We construct $(b_{12}, b_{23}, b_{34}, b_{14})$ as non-negative integer weights on the edges. First we observe that we can reduce a_{ij} to be between zero and one as follows: let $b'_{ij} = \lfloor a_{ij} \rfloor$ and $a'_{ij} = a_{ij} - b'_{ij}$. Note that if $a'_{ij} = 0$ for all ij then this set of edge weights is in S_3 as well as S_1, if not we will adjust the edge weights so the vertex weights remain constant to get a combination in S_3. Now, observe that $a'_{12} + a'_{14} = 1$ or $a'_{12} + a'_{14} = 0$ by choice of the size of the weights. If $a'_{12} + a'_{14} = 1$, then a'_{12} and a'_{14} are both non-zero, and hence a'_{23} and a'_{34} are also non-zero. Thus, setting $b_{12} = b_{34} = 1$, and $b_{23} = b_{14} = 0$ gives these vertex weights as well. Similarly, if $a'_{12} + a'_{14} = 0$ then all the a'_{ij} are zero and we have a point in S_3.

These weights gives a non-negative integer combination $b_{ij} + b'_{ij}$ so that the sums of the weights on the incident edges is the same as the original vertex weights. So in this example S_1 is equal to S_3.

Example 3.2.3. Let G be a graph on vertex set $V(G) = \{1, 2, 3\}$ with edge set $E(G) = \{11, 12, 23, 33\}$. Let $(a_{11}, a_{12}, a_{23}, a_{33}) = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$. Then, the associated weights on the vertices are $(x_1, x_2, x_3) = (1, 0, 1)$. However, there no non-negative integer weights on the edges that can give these vertex weights, the the reasoning for this is as follows: a weight of 1 or more on the loop $\{11\}$ would give a vertex weight of at least 2 on 1. However, any weight of 1 or more on $\{12\}$ would give a vertex weight of at least 1 on 2. Therefore, we can not have a set of non-negative integer edge weights that gives this set of vertex weights. Moreover, since the sum of the vertex weights is 2 this combination is in S_1. Thus, in this example S_3 is not equal to S_1.

Recall that given two distinct loops $\{ii\}$ and $\{jj\}$, the edge $\{ij\}$ is not an extremal point of the polytope $\mathcal{P}_G$. In the example above, the edge $\{13\}$ is such an edge; so the point $(x_1, x_2, x_3) = (1, 0, 1)$ is in S_1 but not S_3. However, if we added the edge $\{13\}$ to G then this is in S_3 as well as S_1. Adding this edge does not change the geometry of $\mathcal{P}_G$ only the combinatorics. This allows us to define a third semigroup, S_2, to be the semigroup generated by $\mathcal{P}_G \cap \mathbb{Z}^d$.

Observation 3.2.4. S_2 is equal to S_3 if and only if for every pair of loops ii and jj in G, ij is an edge of G. S_2 is equal to S_3 if and only if G is maximally unreduced, equivalently the set of vertices with loops in G form a clique.

This gives the inclusion relation $S_3 \subseteq S_2 \subseteq S_1$. We collect the definitions of the subgroups for convenience,

Definition 3.2.5. Let G be a finite graph possibly with loops. Then,

- $S_1 = \text{cone}(\mathcal{P}_G) \cap \mathcal{L}_d$,
- S_2 the semigroup generated by $\mathcal{P}_G \cap \mathbb{Z}^d$,
- S_3 the semigroup generated by $\rho(E(G))$.

Now, we define the associated integral domains for these three semigroups. We use the standard notation, where $k[S_i]$ is the polynomial subring of $k[t_1, \ldots, t_d]$ generated by monomials $t_1^{a_1} \cdots t_d^{a_d}$ for each $(a_1, \ldots, a_d)$ in S_i. Let t^a represent the monomial $t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $a = (a_1, \ldots, a_d)$.

Definition 3.2.6. Let $\mathcal{A}(\mathcal{P})_n$ be the vector space over k which is spanned by the monomials of the form t^a such that $a \in n\mathcal{P} \cap \mathbb{Z}^d$.

Definition 3.2.7. Let G be a graph and S_1, S_2 and S_3 as defined above. Define:

- the *Ehrhart* Polynomial ring:

$$k[S_1] = \mathcal{A}(\mathcal{P}_G) = \bigoplus_{n=0}^{\infty} \mathcal{A}(\mathcal{P}_G)_n,$$

- the *Polytope ring of* $\mathcal{P}_G$:

$$k[S_2] = k[\mathcal{P}_G] = \langle \mathcal{A}(\mathcal{P}_G)_1 \rangle,$$

- the *Edge ring of* G:

$$k[S_3] = k[G] = \langle t^a : a \in \rho(E(G)) \rangle.$$

Here $k[G]$ and $k[\mathcal{P}_G]$ are generated as subrings of $k[t_1, \dots, t_d]$.

Example 3.2.8. Consider the graph G on vertex set $V(G) = \{1, 2, 3, 4\}$, and edge set $\{12, 23, 13, 44\}$. As there is only one loop, S_2 is equal to S_3 and hence $k[G]$ is equal to $k[\mathcal{P}_G]$. Consider the point $(a_{12}, a_{23}, a_{13}, a_{44}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. This gives vertex weights $(1, 1, 1, 1)$, and thus is in S_1, and $t_1t_2t_3t_4$ is in $\mathcal{A}(\mathcal{P}_G)$. However, $(1, 1, 1, 1)$ is not in S_3 since every monomial in the field of fractions of $k[G]$ has t_4 with an even exponent. Thus, $t_1t_2t_3t_4$ is not only not in $k[G]$, it is not in the field of fractions of $k[G]$. This highlights the difficulty of the task to find monomials in the fraction field that are not in the ring which satisfy a monic polynomial. Denote the integral closure of $k[G]$ and $k[\mathcal{P}_G]$ as $N(k[G])$ and $N(k[\mathcal{P}_G])$, respectively. This example shows us that $k[G] \subseteq k[\mathcal{P}_G]$, and $N(k[G]) \subseteq N(k[\mathcal{P}_G]) \subseteq \mathcal{A}(\mathcal{P}_G)$.

This brings up the questions “when are these equal?” and “what are the generators?” From the above examples, we see that odd cycles are related to the behavior of the normalizations.

Example 3.2.9. Let G be the graph on vertex set $V(G) = \{1, 2\}$ with edge set $E(G) = \{11, 22\}$. Then $k[G]$ is isomorphic to $k[x^2, y^2]$, while $k[\mathcal{P}_G]$ is isomorphic to $k[x^2, y^2, xy]$. Observe that $k[x^2, y^2] \neq k[x^2, y^2, xy]$; moreover, the field of fractions $k(x^2, y^2) \neq k(x^2, y^2, xy)$ as xy is not in the fraction field of $k[x^2, y^2]$. This can be seen by observing that every monomial in $k(x^2, y^2)$ has an even x degree, and xy has an odd x degree and hence is not in $k(x^2, y^2)$. Thus, $N(k[G])$ is not equal to $N(k[\mathcal{P}_G])$.

This example shows us that the interactions of $k[G]$ and $k[\mathcal{P}_G]$ depend on the behavior of the loops, as predicted. Moreover, the behavior of $N(k[G])$ and $N(k[\mathcal{P}_G])$ also depend on the loops. This gives the lattice seen in Figure 3.4.

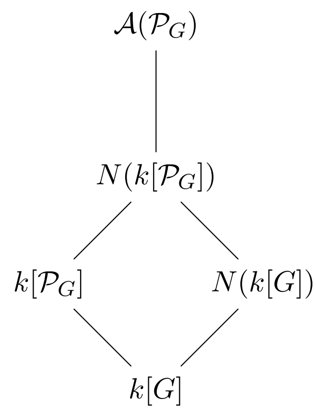


Figure 3.4: The lattice of subrings of $\mathcal{A}(\mathcal{P}_G)$. Showing the rings produces by the semirings S_1, S_2, S_3 and the associated normalizations.

Proposition 3.2.10. $k[G] = k[\mathcal{P}_G]$ if and only if for every pair of loops ii and jj in G, the edge ij is also in G.

Proof. Observe that if, for every pair of loops ii and jj in G, the edge ij is also in G then $\rho(E(G)) = \mathcal{P}_G \cap \mathbb{Z}^d$. That is, $k[G]$ and $k[\mathcal{P}_G]$ have the same generators and are, therefore, equal.

Suppose ii and jj lie in the same component and ij is not an edge in the graph. If the two rings were equal, then there must be an edge incident to i with positive integral weight that is not ii. However, this implies that there is a vertex with positive weight that is not i or j, and hence this is not the monomial $t_i t_j$. Now, suppose ii and jj lie in different components, then $\rho(ij)$ is in S_2, but $\rho(ij)$ is not in S_3. For each component of G, the sum of the vertex weights given by $\rho(e)$ is even. Hence, for each element of S_3, the sum of the vertex weights is even. Since, $\rho(ij)$ has an odd sum of vertex weights on the component containing i, $t_i t_j$ is not in $k[G]$, and so $k[G]$ is not equal to $k[\mathcal{P}_G]$. $\square$

Now, we define conditions on the odd cycles that give equality.

Definition 3.2.11. Let G be a graph,

- We say that G satisfies the *odd cycle condition* if for every pair of odd cycles C and C' of G, at least one of the following occurs: C and C' are in distinct components, have a vertex in common, or there is an edge ij in G so that i is a vertex of C and j is a vertex of C'.

- We say that G satisfies the *loop and cycle condition* if, in addition to the odd cycle condition all the loops are contained in a single component of G.

- We say that G satisfies the *strong odd cycle condition* if, in addition to the odd cycle condition, all the odd cycles occur in a single component of G.

The strong odd cycle condition implies the loop and cycle condition, which implies the odd cycle condition. Note, that in [38] all graphs are connected and all the conditions are equivalent.

Proposition 3.2.12. (Generalized to disconnected graphs, [38, Proposition 2.1]) If the edge ring $k[G]$ of a graph G is normal, then G satisfies the odd cycle condition.

Proof. Suppose not, that is $k[G]$ is normal and G does not satisfy the odd cycle condition. Then there are minimal odd cycles C_1 and C_2 which are vertex disjoint and do not have adjacent vertices contained in the same component. Let $|V(C_1) \cup V(C_2)| = 2n$. For each $ij \in E(C_1) \cup E(C_2)$ set $a_{ij} = \frac{1}{2}$ in the non-negative combination $\sum_{ij \in E} a_{ij}\rho(ij)$ and $a_{ij} = 0$ when $ij \notin E(C_1) \cup E(C_2)$. These weights are an integral point in $n\mathcal{P}_G$. The characteristic vector for $V(C_1) \cup V(C_2)$, since each vertex in $V(C_1)$ or $V(C_2)$ has two incident edges with coefficient $\frac{1}{2}$, we denote this vector, briefly, as α.

Thus, $t^\alpha \in k[t_1, \ldots, t_d]$, however, $t^\alpha \notin k[G]$ as $k[G]$ is an algebra over k generated by $t_i t_j$. In fact, there is no positive integral assignment to the edges that gives an odd vertex sum on each cycle and zero everywhere else. That is, the weights on each edge in the cycle would give an even number to the sum of the vertex weights on each cycle, so to have an odd total on a cycle would require a positive weight on a non-cycle edge. This would give a non-zero vertex weight for a vertex not in the cycle. Since this can not happen t^α is not in $k[G]$.

However, we can write $(t^\alpha)^2 = \prod_{ij \in E(C_1) \cup E(C_2)} t_i t_j \in k[G]$, as each variable shows up exactly twice and since the product is a product of monomials produced by the edges, it is in $k[G]$. That is, each vertex has weight two in the cycles and assigning the edges weight one on the odd cycles gives the desired vertex weights, since each vertex is incident to exactly two edges. This implies that t^α is integral over $k[G]$.

Now, we show that t^α is in the quotient field of $k[G]$. Let P be a walk in G with vertex sequence $\{i_1, i_2, \ldots, i_{2m}\}$, where $i_1 \in V(C_1)$ and $i_{2m} \in V(C_2)$, with edges $i_j i_{j+1}$, for $j = 1, \ldots, 2m-1$. Note that, for some m, such a walk P exists since the cycles are in the same component, we can add edges from one of the cycles as needed to get the correct parity. We can write,

$$t_1 t_{2m} = (t_{i_1} t_{i_2}) \cdot (t_{i_2} t_{i_3})^{-1} \cdot \ldots \cdot (t_{i_{2m-2}} t_{i_{2m-1}})^{-1} \cdot (t_{i_{2m-1}} t_{i_{2m}}).$$

That is, every other monomial has a negative exponent. This gives $t_1 t_{2m}$ as a monomial in the field of fractions of $k[G]$. Thus, we can write t^α as a product of quadratic polynomials in the quotient field of $k[G]$. In particular, we take a perfect matching on $(C_1 \setminus i_1) \cup (C_2 \setminus i_{2m})$. Then we observe that the product of the monomials associated with the edges in this matching together with $t_1 t_{2m}$ give t^α. Since t^α is a product of terms in the quotient field of $k[G]$ it is in the quotient field itself. Hence there is a monic polynomial with coefficients from $k[G]$, namely $z^2 - (t^\alpha)^2$ which has a root over the quotient field of $k[G]$ which is not in $k[G]$. Therefore, $k[G]$ is not a normal domain. □

Observe that this proof depends on being able to build a path from C_1 to C_2. In the original paper [38], the graph G is assumed to be connected and such a path always exists for any two disjoint cycles C_1 and C_2.

Now, we construct a normal domain and we show that this is, in fact, the integral closure of $k[G]$. Observe that if we let L be the subgroup of $\mathbb{Z}^d$ generated by $\rho(E(G))$, then cone$(\mathcal{P}_G) \cap L$ satisfies the conditions of Gordan's Lemma, [3, Lemma 6.1.2] and thus is a normal semigroup. Therefore, to show $k[G] = N(k[G])$ it suffices to show that the semigroup S_3 is equal to cone$(\mathcal{P}_G) \cap L$.

First, we give several reduction conditions:

Theorem 3.2.13. Let G be a graph and, for each edge ij in G, let a_{ij} be an integral edge weight. The following are equivalent:

1. $\prod_{ij \in E(g)} (t_i t_j)^{a_{ij}} = 1$,
2. $\sum_{j \sim i} a_{ij} = 0$ for all vertices i, summed over all j adjacent to i,
3. There is a multiset of alternating walks, allowing loops, so that a_{ij} is the signed sum of the occurrences of ij in the walks.
4. There is a multiset of alternating walks, allowing loops, that are consistent with respect to the sign on the edges, and each edge shows up at most twice per walk so that a_{ij} is the signed sum of occurrences of ij in the walks.

Proof. First, we show that the first condition holds if and only if the second condition holds. Let a_{ij} be an integral edge weight for all ij in G. $\prod_{ij \in E(g)} (t_i t_j)^{a_{ij}} = 1$, if and only if for each t_i the sum of the exponents is zero. The sum of the exponents of t_i is zero if and only if $\sum_{j \sim i} a_{ij} = 0$ for all vertices i, as these are the exponents of all the terms with t_i.

Second, we show that the second condition implies the fourth condition. Let $\sum_{j \sim i} a_{ij} = 0$ for all vertices i. Construct a minimal closed walk of G on the edges with $a_{ij} \neq 0$ with alternating signs of the edge weights. As $\sum_{j \sim i} a_{ij} = 0$, if $a_{ij} > 0$ then there is some jk so that $a_{jk} < 0$. Observe that if ij occurs three times in the walk then there is a smaller closed alternating walk, which contradicts the minimality of the walk. That is, if the sequence of vertices is $\{i_0, i_1, \ldots, u, i, j, \ldots, i, j, k, \ldots, i_r = i_0\}$ then $\{i, j, k, \ldots, i\}$ is a smaller closed walk. In particular, if i, j appears twice then the subwalk containing the first appearance of ij and the second appearance of i in the pair of i, j is closed and, as the next edge is ij, alternating. Since i and j have two orderings, the edge ij shows up at most twice in the walk. Add this walk to the multiset of walks, and, for each ij, if ij occurs m_{ij} times in the walk, set $a'_{ij} = a_{ij} - sgn(ij) \cdot m_{ij}$, if the loop ii occurs m_{ii} times in the walk, set $a'_{ii} = a_{ii} - 2sgn(ii) \cdot m_{ii}$. By repeating this construction on the adjusted edge weights. As each walk reduces the sum of the absolute values of the edge weights, this process terminates when all edges have weight zero. Also, as the walks are alternating, $\sum_{j \sim i} a'_{ij} = 0$ is maintained for each vertex i.

Third, we show that the fourth condition implies the third condition. Trivially, if there is a multiset of alternating walks, allowing loops, that are consistent with sign on the edges, and each edge shows up at most twice per walk so that a_{ij} is the signed sum of occurrences of ij in the walks, then there is a multiset of alternating walks, allowing loops, so that a_{ij} is the signed sum of the occurrences of ij in the walks. Namely, the original set of walks.

Finally, we show that the third condition implies the second condition. Now, assume there is a multiset of alternating walks, allowing loops, so that a_{ij} is the signed sum of the occurrences of ij in the walks, and consider $\sum_{j \sim i} a_{ij}$. For each walk containing ij, an occurrence of ij in the walk gives an edge ki immediately preceding ij in the walk. Hence, ij and ki have different sign and so in the signed sum of i cancel. If ii is a loop in the walk then there are edges ki and ij in the walk immediately preceding and following ii. Hence ii has a different sign from ki and ij and hence in the signed sum of i cancel. Thus, $\sum_{j \sim i} a_{ij} = 0$. □

The following two special cases of this result are used in the rest of this section. We also strengthen the result slightly to guarantee that any particular edge does not have a negative weight.

Corollary 3.2.14. Let G be a graph with $\alpha = \sum_{ij \in E} b_{ij} \rho(ij) \in \mathcal{P}_G$ and G' a subgraph of G.

1. If G' has an even number of edges and an Eulerian Tour, then we may assume that

$b_{ij} = 0$ for at least one $ij \in E(G') \setminus \{e\}$, for any edge e of G'.

2. [38, Lemma 2.5] If $G' = C_1 \cup C_2 \cup \{e\}$ where e is a bridge connecting vertex disjoint odd cycles C_1 and C_2 of G, then we may assume that $b_{ij} = 0$ for at least one $ij \in E(C_1) \cup E(C_2)$.

Proof. Each of the G' subgraphs can be written as an alternating walk where each edge shows up at most twice. In the first case, the walk is the Eulerian tour. In the second case, each edge in C_1 and C_2 appears once and the bridge e twice. Moreover, we can choose the signs so that any particular edge is positive. From Theorem 3.2.13, this gives weights $\{a_{ij}\}$ so that $\sum_{j\sim i} a_{ij} = 0$. A constant multiple of each of these edges still satisfies this condition. Set $\delta = min\left\{\frac{b_{ij}}{m_{ij}} : a_{ij} < 0\right\}$ where m_{ij} is the number of times ij appears in the tour, and $b'_{ij} = b_{ij} + \delta \cdot a_{ij}$, observe that for some ij not equal to e, $b'_{ij} = 0$. Moreover,

$$\alpha = \sum_{ij\in E} b_{ij} = \sum_{ij\in E} b_{ij} + \delta \sum_{ij\in E} a_{ij} = \sum_{ij\in E} b'_{ij}.$$

Thus, we may assume that for at lesat one edge in $E(G) \setminus \{e\}$ there is an edge with weight 0. □

Example 3.2.15. Let G be the four cycle on vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 34, 14\}$. Suppose we have edge weights $(b_{12}, b_{23}, b_{34}, b_{14}) = \left(\frac{1}{2}, \frac{3}{4}, 1, \frac{1}{2}\right)$. This is an example of an Eulerian tour with an even number of edges, suppose we wish to keep edge 12 with a positive weight. Then, $(1, -1, 1, -1)$ are the weights for the alternating closed walk. $\delta = \min\left\{\frac{3}{4}, \frac{1}{2}\right\} = \frac{1}{2}$, thus the adjusted weights are $(b'_{12}, b'_{23}, b'_{34}, b'_{14}) = \left(1, \frac{1}{4}, \frac{3}{2}, 0\right)$, see Figure 3.5 for the graph, original, and adjusted edge weights.

Example 3.2.16. Let G be a graph with two odd cycles connected by a bridge, on vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 13, 34, 44\}$. Suppose we have edge weights $(b_{12}, b_{23}, b_{13}, b_{34}, b_{44}) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, 1\right)$. This gives vertex weights $\left(\frac{4}{3}, \frac{3}{2}, \frac{7}{6}, \frac{4}{3}\right)$. Also, suppose we want to keep the edge 34 with a positive weight. Then, $(1, -1, -1, 2, -2)$ are the weights for the alternating closed walk. Observe that 44 occurs, once but since it is a loop, it

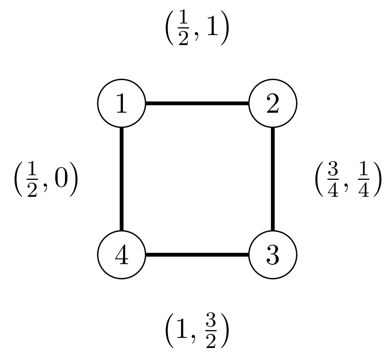


Figure 3.5: The graph G edge label for ij is (a_{ij}, a'_{ij}). That is, two different edge weights that give the same vertex weights. The cycle itself gives an alternating Eulerian tour with edges 12 and 34 positive and 23 and 14 negative. The smallest edge weight on a negative edge is $\frac{1}{2}$ and so $a'_{ij} = a_{ij} \pm \frac{1}{2}$, as needed.

has weight -2 for the alternating closed walk. $\delta = \min\left\{\frac{1}{2}, \frac{1}{3}, 1\right\} = \frac{1}{3}$. The adjusted weights are $(b'_{12}, b'_{23}, b'_{13}, b'_{34}, b'_{44}) = \left(\frac{4}{3}, \frac{1}{6}, 0, 1, \frac{1}{3}\right)$. Notice that that the vertex weights remain $\left(\frac{4}{3}, \frac{3}{2}, \frac{7}{6}, \frac{4}{3}\right)$, see Figure 3.6 for the graph, original, and adjusted edge weights.

Theorem 3.2.17. Let G be a graph that satisfies the odd cycle condition, and let $\alpha = \sum_{e \in E(G)} a_e \rho(e)$ where $a_e \geq 0$ for all edges e in G. Then, there is a subgraph H of G with trees and unicyclic graphs as components so that $\alpha = \sum_{e \in E(H)} b_e \rho(e)$ for some $b_e \geq 0$ for all edges e in H.

Proof. Let G' be the subgraph of G defined by edge set $\{e : a_e > 0\}$. Suppose we have a component H of G' that is not a tree or a unicyclic graph, and let C be a cycle of H. If C is even, then we can apply Corollary 3.2.14 and delete an edge from the component. If C is odd, then there is a path connecting two vertices of C or there is a cycle C' edge disjoint from C in the same component.

First, assume that there is a path P connecting vertices i and j of C. There are two paths connecting i and j in C, since C is odd, one path is even and one is odd. Thus, P and one of these paths gives an even cycle. By Corollary 3.2.14 we may delete an edge from this component.

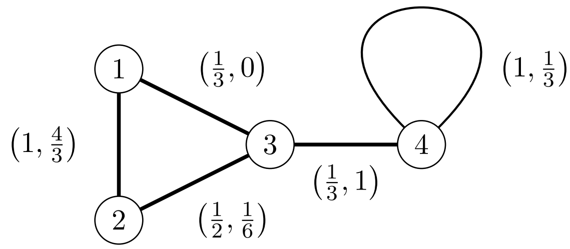


Figure 3.6: The graph G edge label for ij is (a_{ij}, a'_{ij}). That is, two different edge weights that give the same vertex weights. The alternating Eulerian tour with edges 13, 23 and the loop 44 positive and edge 12 and 34 negative, with 34 negative with multiplicity two. The smallest edge weight on a negative edge is $\frac{1}{3}$ ans so $a'_{ij} = a_{ij} \pm \frac{1}{3}$ as needed, except on 34 where the multiplicity gives $a'_{14} = a_{14} - \frac{2}{3} = \frac{1}{3}$.

Observe that if there are two odd cycles C and C' that are edge disjoint but not vertex disjoint, then either there is a path connecting two vertices in C, or C and C' have exactly one vertex in common. In the former case, apply the previous argument, for the latter case observe that two odd cycles with a single vertex in common has an even Eulerian tour. Thus, by Corollary 3.2.14 we can delete an edge.

Now, assume that there is a cycle C' vertex disjoint from C in the same component as C, we may assume C' is an odd cycle, otherwise it is an even Eulerian tour. Since C and C' are in the same component, there is a path P, edge disjoint from C and C', connecting a vertex i in C to a vertex j in C'. If P is a single edge we apply Corollary 3.2.14 and delete an edge from C or C'. Thus, we may assume P contains at least two edges. As G satisfies the odd cycle condition, C and C' are vertex disjoint in the same component of G there is an edge uv in G so that u is in C and v is in C'. Note that we may assume uv is not in G', as otherwise we can apply an earlier case. Then, there is an even cycle containing P, uv, a subpath of C connecting i and u and a subpath of C' connecting j and v. Corollary 3.2.14 allows us to delete an edge of this cycle, that is not uv, note that uv has a non-zero weight at this point so we have not changed the size of the edge set of G'. If the edge is in P, then we apply Corollary 3.2.14 to delete an edge. If the edge is not in P, we may assume it is in C'. Then, e, P and the remaining subpath of C' connecting j and u form a path connecting two of the vertices of C, by applying the previous case and delete an edge as before.

In all of these cases, we can find a subgraph of G with fewer edges than G'. Thus, applying this process iteratively produces smaller graphs so long as there is a component that not a tree or unicyclic. Hence, there is a subgraph H of G with weights $b_e > 0$ that satisfies $\alpha = \sum_{e \in E(H)} b_e \rho(e)$, and every component of H is either a tree or unicyclic. $\square$

This is not a surprising result, because the polytope $\mathcal{P}_G$ with k bipartite components has $\dim \mathcal{P}_G = d - 1 - k$. If the subgraph H has only trees and unicyclic components, with odd cycles, then $\dim \mathcal{P}_H \leq d - 1 - k$. In [38] the authors construct a triangulation of $\mathcal{P}_G$ using this observation and get a similar result for the connected case.

If G satisfies the odd cycle condition, what does this tell us? This tells us that any point inside cone($\mathcal{P}_G$) can be written as a non-negative linear combination of edges. The positive edges form a graph with trees and unicyclic components. In particular, if the point p is an integral point this implies each edge that is not in a cycle has integral weight. If we add in the condition that p is in the group generated by $\rho(E(G))$, then we get that each edge has an non-negative integer in the linear combination giving p. That is, the associated monomial is in the edge ring.

Theorem 3.2.18. (Generalized to disconnected graphs, [38, Corollary 2.3]) Given a graph G, $k[G]$ is normal if and only if G satisfies the odd cycle condition.

Proof. Proposition 3.2.12 proves that if $k[G]$ is normal, then G satisfies the odd cycle condition. We now prove the converse. Let p be a point in the group generated by $\rho(E(G))$ and in cone($\mathcal{P}_G$) by non-negative edge weights a_e for each edge $e \in E(G)$. By Theorem 3.2.17, we can write p as a positive linear combination of edges of a subgraph H of G with trees and unicyclic graphs as components. Let i be a vertex of H with degree 1, and let ij be the associated edge in H. Let p_i be the value of the i coordinate of p, $a_{ij} = p_i$, note that $p' = p - a_{ij}\rho(ij)$ is still in the group and the cone since this is the only possible coefficient of a non-zero linear combination that gives p. Therefore, we may assume that H contains only cycles as components. Note that we can assume all the cycles are odd since each even cycle is an Eulerian tour and we can apply Corollary 3.2.14. Now, suppose we have a set of a_{ij} that gives $p = \sum_{ij \in E} a_{ij}\rho(ij)$ as a non-negative linear combination. Then let $p' = p - \sum_{ij \in E} \lfloor a_{ij} \rfloor \rho(ij)$ has to be in the group generated by $\rho(E(G))$ as well as the cone. Now, each vertex in H is one or zero for p' however, as the sum of vertex weights of each cycle must be even, odd cycles have a vertex with weight zero. If vertex u has weight zero, then $a_{uv} - \lfloor a_{uv} \rfloor$ and $a_{uw} - \lfloor a_{uw} \rfloor$ are both zero as well, when v and w adjacent to u in H. Thus, v and w have weight zero as well as, otherwise, they have a single positive edge weight, which is not integral. Repeating this process gives every vertex in the odd cycles have weight zero in p'. This gives a positive integral combination of p. Hence, x^p is a monomial in $k[G]$. Thus, $k[G]$ is equal to the affine semigroup ring generated by the group and the lattice, and hence, normal. □

This characterizes when $k[G]$ is normal. However, we still have to determine when $k[G]$ is equal to $N(k[\mathcal{P}_G])$ and $\mathcal{A}(\mathcal{P}_G)$. Recall that $N(k[\mathcal{P}_G])$ is the affine semigroup ring generated by the integer points of $\mathcal{P}_G$, and $\mathcal{A}(\mathcal{P}_G)$ the normal semigroup ring generated by the even lattice and cone$(\mathcal{P}_G)$. Also, recall that $k[G] \subseteq k[\mathcal{P}] \subseteq \mathcal{A}(\mathcal{P}_G)$.

Corollary 3.2.19. Let G be a finite graph, possibly with loops then:

- $k[G] = N(k[\mathcal{P}_G])$ if and only if G satisfies the loop and cycle condition,
- $k[G] = \mathcal{A}(\mathcal{P}_G)$ if and only if G satisfies the strong odd cycle condition.

Proof. In both of these cases, we assume that G satisfies the odd cycle condition. $N(k[\mathcal{P}_G])$ and $\mathcal{A}(\mathcal{P}_G)$ are both normal, and Theorem 3.2.18 tell us that the odd cycle condition is necessary for $k[G]$ to be normal.

- $k[G] = N(k[\mathcal{P}_G])$ if and only if $k[G] = k[\mathcal{P}_G]$ since otherwise $N(k[\mathcal{P}_G])$ contains a quadratic monomial, which is not in $k[G]$, this is true if and only if for every pair of loops ii and jj in G the edge ij is also in G by Proposition 3.2.10. Note that this is equivalent to loop and cycle condition. Hence, $k[G] = N(k[\mathcal{P}_G])$ if and only if G satisfies the loop and cycle condition.

- Now, suppose $k[G]$ does not satisfy the strong cycle condition. Then, there are two odd cycles C_1 and C_2 in G that are not in the same component. Let $a_e = \frac{1}{2}$ for every edge in C_1 and C_2, and zero otherwise. Then, $\sum_{ij \in E} a_{ij} \in 2\mathbb{Z}$ hence $\alpha = \sum_{ij \in E} a_{ij}\rho(ij)$ is in $\mathcal{A}(\mathcal{P}_G)$, however α is not in $k[G]$ as each component has odd total vertex weight. Now, suppose G satisfies the strong odd cycle condition. By Theorem 3.2.18, $k[G]$ is normal, since G satisfies the odd cycle condition, and hence $k[G] = N(k[G]) = \mathcal{A}(\mathcal{P}_G)$.

□

This gives the characterization of when a graph is normal, and when it has a predetermined geometric structure. However, we would also like to characterize the integral closure when the graph does not satisfy the odd cycle condition. In Proposition 3.2.12, we constructed a polynomial that was in the field of fractions and the root of a monic polynomial but was not in the original polynomial ring. In particular, this monomial is associated with two odd cycles that were vertex disjoint and did not have an edge connecting them.

Definition 3.2.20. We say a pair $\Pi = \{C, C'\}$ of minimal odd cycles in a component of G that are vertex disjoint are *exceptional* if there exists no edge connecting C and C' in G. Given an exceptional pair $\Pi = \{C, C'\}$ we write

$$\frac{1}{2}\rho(\Pi) = \frac{1}{2}\sum_{ij \in C} \rho(ij) + \frac{1}{2}\sum_{ij \in C'} \rho(ij),$$

and $M_\Pi = (\prod_{i \in V(C)} t_i)(\prod_{j \in V(C')} t_j) \in k[t_1, \ldots, t_d]$.

By minimal we mean that C and C' are without chords. That is, the vertex set of C does not contain, as a subset, the vertex set of a smaller cycle. Note that $\frac{1}{2}\rho(\Pi)$ is precisely the characteristic vector of the vertices in $C \cup C'$. Observe that C and C' violate the odd cycle condition. In the paper [38] there was no need for the containment of the odd cycles in the same component. Also, note that the monomial associated with $\frac{1}{2}\rho(\Pi)$ is precisely M_Π.

Theorem 3.2.21. (Generalized to disconnected graphs, [38, Theorem 2.2]) Let G be a finite graph possibly with loops and $k[G]$ be the edge ring. Let $\Pi_1 = \{C_1, C_1'\}, \ldots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs of minimal odd cycles in G. Then the normalization of $k[G]$ is generated by the monomials $M_{\Pi_1} \ldots, M_{\Pi_q}$ as an algebra over $k[G]$

Proof. It suffices to show that, for any point p in the group L generated by $\rho(E(G))$ and in the cone $\text{cone}(\mathcal{P}_G)$, p can be written as a positive integer combination of elements of $\rho(E(G))$ and $\{\frac{1}{2}\rho(\Pi_k)\}$ for a subset of the exceptional pairs of G. Suppose p is such a point.

We apply the proof of Theorem 3.2.17 with the additional case for when the two odd cycles C and C' are vertex disjoint and do not have an edge connecting them. Note that if one of the cycles, say C, is not minimal then there is an edge e connecting two of the vertices of C. This gives an even cycle on the vertices of C using e and a subpath of C. Apply Corollary 3.2.14 to this even cycle, increasing the coefficient of $\rho(e)$ in the linear combination that gives p and setting the coefficient of another edge in C to be zero. This gives a strictly smaller odd cycle. Thus, we may assume C and C' are minimal. Let Π_k be the pair representing $\{C, C'\}$.

As before, set $\delta = \min\{a_e : e \in E(C) \cup E(C')\}$, and write $p' = p - \delta\frac{1}{2}\rho(\Pi_k)$. Observe that p' has strictly fewer edges with non-zero coefficients. Apply Corollary 3.2.14 on p', as before. This process terminates when all the components are trees or unicyclic graphs. Then we apply the proof Theorem 3.2.18. This process does not depend on if the graph satisfied the odd cycle condition, only that we could assume the edges with positive coefficients defined components that were trees and unicyclic graphs. The proof terminates with a non-negative integer combination of $\rho(E(G))$ giving p.

Therefore, $\rho(E(G))$ and $\{\frac{1}{2}\rho(\Pi_k)\}_k$ give a generating set for $L \cap \text{cone}(\mathcal{P}_G)$. Since $L \cap \text{cone}(\mathcal{P}_G)$ is a normal semigroup the associated monomials generate $N(k[G])$. Hence, $N(k[G])$ is generated by $\{t_i t_j : ij \in E(G)\} \cup \{M_{\Pi_k}\}_{k=1}^{r}$. □

Note that this can be thought of from the perspective of determining the normalizer of any polynomial subring of $k[t_1, \ldots, t_d]$ generated by quadratic monomials.

Definition 3.2.22. Let R be a subring of $k[t_1, \ldots, t_d]$ generated by $\{t_i t_j\}$ for some set of pairs i and j. Then let G_R be the *graph induced by* R. That is, ij is an edge of G_R if and only if $t_i t_j$ is a generator of R.

Corollary 3.2.23. Let R be a subring of $k[t_1, \ldots, t_d]$ generated by a set of quadratic monomials $\{t_i t_j\}$. Then the integral closure of R is generated by $\{M_{\Pi_1}, \ldots, M_{\Pi_r}\}$ as an algebra over R, where Π_i is an exceptional pair of odd cycles in G_R.

Proof. It suffices to observe that $R = k[G_R]$, and thus, by Theorem 3.2.21, the integral closure of R is given by $\rho(E(G_R))$ and by $\{M_{\Pi_k}\}$ the exceptional odd cycle pairs of G_R. □

Example 3.2.24. Let R be the subalgebra of $k[t_1, t_2, t_3, t_4]$ generated by monomials $\{t_1^2, t_1t_2, t_2t_3, t_3t_4, t_4^2\}$. This corresponds to the graph G_R on vertex set $\{1, 2, 3, 4\}$ with edge set $\{11, 12, 23, 34, 44\}$. This graph does not satisfy the odd cycle condition, as the loops 11 and 44 do not have a vertex in common, and 14 is not an edge in the graph. Theorem 3.2.21 gives that $N(R)$ the integral closure of R is generated by $\{t_1t_4\}$ as an algebra over R. Thus, $N(R)$ is generated by $\{t_1^2, t_1t_2, t_2t_3, t_3t_4, t_4^2, t_1t_4\}$, as a subalgebra of $k[t_1, t_2, t_3, t_4]$.

3.3 Serre's R_1 Condition for Graphs

There are many equivalent definitions of normality. One such definition follows from two of Serre's conditions, and is referred to as *Serre's Normality Criterion* in [3, pp. 71].

Definition 3.3.1. A finite module over a Noetherian ring R satisfies *Serre's condition* S_n if $\operatorname{depth}(M_p) \geq \min(n, \dim M_p)$ for all $p \in \operatorname{Spec} R$.

By an easy extension of the definition of the Cohen-Macaulay we can describe Cohen-Macaulaydom in terms of the S_n conditions.

Proposition 3.3.2. A ring R is Cohen-Macaulay if and only if it satisfies S_n for all n.

Definition 3.3.3. A Noetherian ring R satisfies *Serre's condition* R_n if $R_\mathfrak{p}$ is a regular local ring for all prime ideals $\mathfrak{p}$ in R with $\dim R_\mathfrak{p} \leq n$

Theorem 3.3.4. [3, Theorem 2.2.22] A Noetherian ring R is normal if and only if it satisfies R_1 and S_2.

A result of Hochester gives a connection between a normal toric coordinate ring and Cohen-Macaulay toric coordinate ring.

Theorem 3.3.5. [3, Theorem 6.3.5(a)] Let C be a normal semigroup, and k a field, then $k[C]$ is a Cohen-Macaulay ring.

Moreover, we have a characterization of Serre's R conditions for an affine semigroup ring in terms of the behavior of the semigroup.

Theorem 3.3.6. [61, Theorem 2.7] An affine semigroup ring $R = k[S]$ satisfies Serre's R_ℓ condition if and only if for each positive integer $k \leq \ell$ and any face F of cone(S) so that height(P_F) $= k$ there exists facets $F_1, \ldots F_k$ of cone(S) so that $F = F_1 \cap \ldots \cap F_k$ and the following conditions hold:

- there exist $\gamma_1, \ldots, \gamma_k \in S$ so that $\sigma_i(\gamma_j) = \delta_i^j$ for all $1 \leq i, j \leq k$ and,
- $\mathbb{Z}(S \cap F_1 \cap \ldots \cap F_k) = \mathbb{Z}S \cap H_1 \cap \ldots \cap H_k$.

Where, $P_F = \langle x^\alpha \mid \alpha \in S \setminus F \rangle$, H_i a supporting hyperplane for σ_i a linear form on $\mathbb{R}^n$ and $F_i = \text{cone}(S) \cap H_i$

The $\ell = 1$ case can be seen as a special case of this much more general theorem.

Proposition 3.3.7. [20, Proposition 3.2] Let S be an affine semigroup, k a field and $k[S]$ the associated semigroup ring. $k[S]$ satisfies Serre's R_1 condition if and only if every facet F of cone(S) satisfies:

- there exists $x \in S$ so that $\sigma_F(x) = 1$ where σ_F is a support form for F taking integer values on $\mathbb{Z}S$,
- $\mathbb{Z}(S \cap F) = \mathbb{Z}S \cap H$ where H is the supporting hyperplane of F.

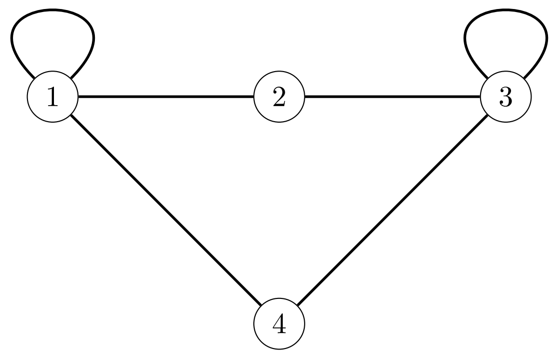


Figure 3.7: An example of a graph which satisfies Proposition 3.3.8 but fails normality. Note that in order to delete a single bipartite component and leaves behind two non-bipartite components, we must include vertices 2 and 4 in the bipartite component. However, since there is no edge between 2 and 4 we can not find such a subgraph.

Determining when a graph satisfies this gives a condition for an edge ring to satisfy Serre's R_1 condition, rewritten using the notation and concepts we use in Chapter 4 this condition is:

Proposition 3.3.8. [20, Theorem 2.1] Let G be a finite, non-bipartite graph. $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with exactly one bipartite component $L \cup R$ that is not bipartite in G, obtained from G be deleting edges incident to L and not R and satisfies $\text{BiComp}(G') = \text{BiComp}(G)$, has $\text{Comp}(G') \leq \text{Comp}(G) + 1$.

Example 3.3.9. Let G be a graph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{11, 12, 23, 34, 14, 33\}$, see Figure 3.7. This graph fails the odd cycle condition as it has two loops and do not have an edge between them. However, it satisfies Serre's R_1 condition as any bipartite component obtained from the graph by deleting edges from one component, leaves a connected component behind.

Example 3.3.10. Let G be a graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set $\{11, 12, 23, 33, 24, 25, 45\}$, see Figure 3.8. This graph fails the odd cycle condition as it has two loops and does not have an edge between them. However, to form a bipartite component formed by deleting edges incident to one of the parts must contain the vertex 2. However, if we delete the vertex 2 from the graph we have a bipartite component containing 4 and 5. Including

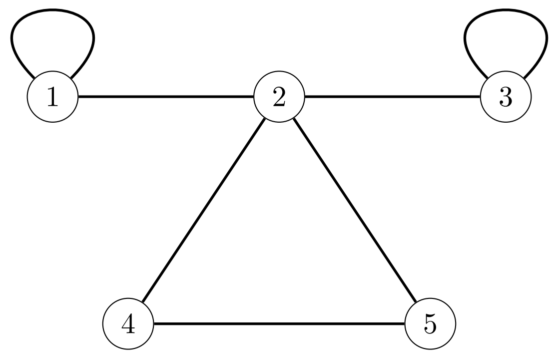


Figure 3.8: An example of a graph which satisfies Proposition 3.3.8 but fails normality. Note that any bipartite subgraph that separates two odd cycles must include vertex 2. However, vertices 4 and 5 form a bipartite subgraph if we delete all the edges incident to 2. Thus, we must include 4 and 5 in the bipartite subgraph, however, 2, 4, and 5 form an odd cycle and the subgraph is no longer bipartite. Thus, the graph satisfies R_1 but is not normal.

them in the bipartite component produces a graph which is not bipartite. Thus, the graph satisfies Serre's R_1 condition.

Chapter 4

Signed Graphs

4.1 Geometric Properties

In this section, we relate the combinatorial properties of a signed graph G to the geometric properties of $\mathcal{P}_G$ the edge polytope of the signed graph. In particular, we characterize the extremal points and dimension of $\mathcal{P}_G$ in terms of of G. These are required to determine the dimension of the facets of the cone containing $\mathcal{P}_G$, which can be used for determining which of Serre's R conditions [3, pp. 71] are satisfied [20].

4.1.1 Extremal Points of the Polytope

In this section, we characterize the edges of G corresponding to extremal points of the polytope $\mathcal{P}_G$. When all the edges of G correspond to extremal points of $\mathcal{P}_G$, the generators of $\mathcal{L}_G$ are the ray generators of cone($\mathcal{P}_G$). That is, together with Theorem 4.2.22, the extremal points of $\mathcal{P}_G$ give a Hilbert basis for S_1. The following result and definition are generalizations and adaptions from Hibi and Ohsugi [38].

Definition 4.1.1. A graph G is *reduced* if G does not have vertices i and j so that $\pm ii$, $\pm ij$, and $\pm jj$ are edges of G (all with the same sign).

Theorem 4.1.2. Every edge of G gives an extremal point of the polytope $\mathcal{P}_G$ if and only if G is reduced.

Proof. Assume first that G is not reduced. Then, there exist vertices i and j so that ii, ij and jj are edges in G with the same sign. Observe that $\rho(ij) = \frac{1}{2}(\rho(ii) + \rho(jj))$. Thus, $\rho(ij)$ is not an extremal point of $\mathcal{P}_G$.

Now, assume G is reduced, but the edge $e = ij$ of G $\rho(ij)$ is not an extremal point of $\mathcal{P}_G$. Without loss of generality, we assume that $\text{sgn}(e) = +1$. If $i = j$ then $(\rho(e))_i = 2$, but for all other edges, $\rho(E(G))_i \leq 1$. Therefore, $\rho(ii)$ can not be a nontrivial convex combination of $\rho(E(G))$.

Now, assume $i \neq j$. Since G is reduced, assume, without loss of generality, that $+ii$ is not an edge in E. Since $\rho(+ij)$ is not extremal, there exist $\lambda_i \geq 0$ so that $\sum_k \lambda_i = 1$ and the convex combination $\sum_k \lambda_k \rho(e_k) = \rho(e)$, where $e_k \neq +ij$ for all k. As $\rho(e)_i = 1$, $\sum_k \lambda_k \rho(e_k)_i = 1$ as well. Since all for all k, $\rho(e_k)_i \leq 1$ we know that every edge e_k in the combination with $\lambda_k \neq 0$ must have $\rho(e_k)_i = 1$. That is, every edge must be a positive edge with i as an end point. Since every edge in the combination contains i as an end point, every edge is a positive edge, and $+ij$ is not one of these edges, then none of them have j as the other end point. Hence, $\sum_k \lambda_k \rho(e_k)_j = 0 \neq \rho(e)_j = 1$, a contradiction. □

4.1.2 Dimension of the Polytope

In this section, we compute the dimension of $\mathcal{P}_G$ by constructing a collection of independent hyperplanes which contain the polytope. In particular, we determine a collection of independent hyperplanes which determine the affine span of $\mathcal{P}_G$. More precisely, we associate the hyperplane $\{x \in \mathbb{R}^n \ : \ \langle v^*, x \rangle = c\}$ with the vector (v^*, c) in $\mathbb{P}^n(\mathbb{R}^*)$, where R^* is the dual of $\mathbb{R}$, and compute the codimension of the span of the vectors $\{(v^*, c)\}$ in $(\mathbb{R}^{n+1})^*$.

Without loss of generality, by scaling, we may assume that $c = 0$ or $c = 1$ for our hyperplanes. Moreover, for notational convenience, we assume the dual vectors use the standard dual basis, that is $v^* = \sum_i \lambda_i e_i^*$ where $\{e_i^*\}$ is the basis dual to $\{e_i\}$. We may also view $(v^*)_i$ as a vertex weight on vertex i of G. More precisely, if the hyperplane $\{w \ : \ \langle v^*, w \rangle = c\}$ contains $\rho(E)$, then, for all $ij \in E$, $(v^*)_i + (v^*)_j = \text{sgn}(ij) \cdot c$.

4.1.2.1 Bipartite Hyperplanes

Suppose G is a signed graph and H a bipartite component of H. Let $V(H) = L \cup R$ be the unique bipartition of H, i.e., every edge of H has one endpoint in L and the other in R. We write $e_L^* = \sum_{i \in L} e_i^*$ and $e_R^* = \sum_{j \in R} e_j^*$ for the dual characteristic vectors for L and R. With this notation, every edge ij in H has $\langle e_L^*, \rho(ij) \rangle = \langle e_R^*, \rho(ij) \rangle = \operatorname{sgn}(ij)$. Hence, $\langle e_L^* - e_R^*, \rho(ij) \rangle = 0$ for every edge in G. If G has multiple bipartite components, then observe that the supports of the hyperplanes for distinct bipartite components are disjoint.

Definition 4.1.3. Let G be a signed graph; we write $\operatorname{BiComp}(G)$ for the number of bipartite components of G.

Since the supports for the hyperplanes for disjoint bipartite components are disjoint, observe that there are at least $\operatorname{BiComp}(G)$ independent hyperplanes that contain $\mathcal{P}_G$.

Example 4.1.4. Let G be the graph in Figure 4.1, i.e., G has vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set
$\{+12, -23, +34, +45, -56, +16\}$. This is a bipartite graph with bipartition $\{1, 3, 5\}$ and $\{2, 4, 6\}$. Hence $\mathcal{P}_G$ is contained in the hyperplane defined by $\langle (e_1^* + e_3^* + e_5^*) - (e_2^* + e_4^* + e_6^*), x \rangle = 0$.

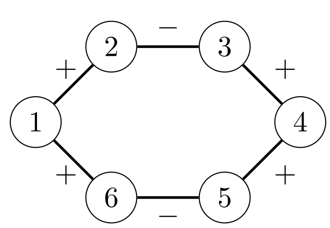


Figure 4.1: The signed graph G where $\mathcal{P}_G$ is contained in two independent hyperplanes, one due to the biparition and the hyperplane associated to the stratified partition $\mathcal{A} = \{A_0 = \{1, 2, 6\}, A_1 = \{3, 5\}, A_2 = \{4\}\}$. The hyperplane corresponding to the bipartition is $\langle (e_1^* + e_3^* + e_5^*) - (e_2^* + e_4^* + e_6^*), x \rangle = 0$ and the hyperplane corresponding to the stratified partition is $\left\langle \frac{1}{2}(e_1^* + e_2^* + e_6^*) - \frac{3}{2}(e_3^* + e_5^*) + \frac{5}{2}(e_6^*), x \right\rangle = 1$. All hyperplanes which contain $\mathcal{P}_G$ are linear combinations of these two planes; thus, $\dim \mathcal{P}_G = 6 - 1 - 1 = 4$.

Lemma 4.1.5. Let G be a signed graph with bipartite components $G_1, G_2, \ldots, G_k$, and associated dual vectors $e^*_{L_1} - e^*_{R_1}, \ldots, e^*_{L_k} - e^*_{R_k}$. Suppose $\mathcal{P}_G$ is contained by the hyperplane defined by $\langle v^*, x\rangle = 0$, then v^* is a linear combination of $e^*_{L_1} - e^*_{R_1}, \ldots, e^*_{L_k} - e^*_{R_k}$.

Proof. The claim is trivial when $v^* = 0$; so, we assume that $v^* \neq 0$. Suppose G is a connected graph. Observe that for all edges ij, $\langle v^*, \rho(ij)\rangle = \mathrm{sgn}(ij)\,((v^*)_i + (v^*)_j) = 0$; it follows that, $(v^*)_i = -(v^*)_j$. Since $v^* \neq 0$, there is some i such that $(v^*)_i = c \neq 0$. From connectivity and the observation that neighboring vertices have opposite signs, it follows that for all vertices j, $(v^*)_j = \pm c$. Define L to be the set of vertices whose weight is c and R to be the set of vertices whose weight is $-c$. L and R form a partition of G and they cannot contain any edges because the signs of the endpoints of an edge have opposite signs. Therefore, L and R form the bipartition of G and $v^* = \pm c(e^*_L - e^*_R)$. For a disconnected graph we see that v^* will be a linear combination of these dual vectors. □

4.1.2.2 Stratified Hyperplanes

Next, we construct hyperplanes whose inner product with edges of G is a non-zero constant. Recall, we have assumed that when $c \neq 0$, we scale the hyperplane so that $c = 1$, i.e., $\langle v^*, x\rangle = 1$.

Example 4.1.6. Let G be the signed graph in Figure 4.2, i.e., G contains an odd cycle C containing vertex i as well as vertices k and r which are not in C. Suppose that all of the edges in C are positive, and the edges $-ik$ and $+kr$ are also in the graph. If these are the only edges of the graph, we may construct a hyperplane that contains $\mathcal{P}_G$ of the form $\langle v^*, x\rangle = 1$ as follows:

Let $+ij$ and $+is$ be edges in the cycle C, since $\langle v^*, e_i + e_j\rangle = \langle v^*, e_j + e_s\rangle = 1$ it follows that $(v^*)_i + (v^*)_j - (v^*)_i - (v^*)_s = 0$ and hence $(v^*)_j = (v^*)_s$. Since C is an odd cycle we can repeat this argument to obtain $(v^*)_i = (v^*)_j$, and, hence, $2(v^*)_i = 1$. So, every vertex of C has weight $\frac{1}{2}$.

Now, consider the edge $-ik$. Since $(v^*)_i = \frac{1}{2}$, we know that k must have weight

$\frac{-3}{2}$, since $\langle v^*, -e_i - e_k \rangle = \frac{-1}{2} - (v^*)_k = 1$. Similarly, r has weight $\frac{5}{2}$, since $\langle v^*, e_k + e_r \rangle = \frac{-3}{2} + (v^*)_r = 1$.

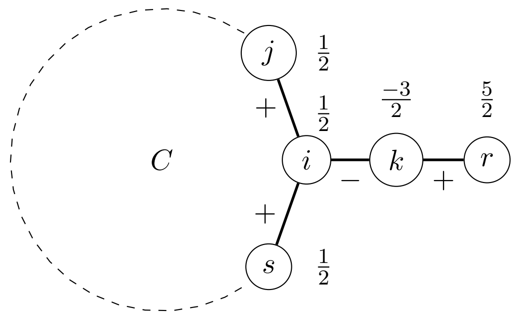


Figure 4.2: A signed graph G containing an odd cycle C with all positive edges as well as two other edges and vertices. Let v^* be determined by the weights in the diagram; then the hyperplane $\langle v^*, x \rangle = 1$ contains $\mathcal{P}_G$ since the weights of the endpoints of an edge e must equal $\mathrm{sgn}(e)$.

We can continue the discussion in Example 4.1.6 to more general graphs containing an odd cycle of positive edges as follows: Suppose that the hyperplane $\langle v^*, x \rangle = 1$ contains $\mathcal{P}_G$. Then, all the vertices on the odd cycle have weight $\frac{1}{2}$, any vertex connected to the odd cycle by a positive edge must have weight $\frac{1}{2}$, and any vertex connected to the odd cycle by a negative edge must have weight $\frac{-3}{2}$.

Alternatively, observe also that if the odd cycle had negative edges, the weights on the vertices of the cycle would be $\frac{-1}{2}$. The weights of the vertices adjacent to the odd cycle would also change following the pattern in the discussion above. Instead of detailing various cases, we extend this discussion and construction to all graphs.

Definition 4.1.7. Let G be a signed graph, $\mathcal{A} = \{A_0, A_1, \ldots\}$ a partition of the vertices of a component H of G where, for each $r \in \mathbb{N}$, A_r is called the r^{th}-*level* of H. We say H is *positive stratified* with *positive stratification* $\mathcal{A}$ if:

1. there are only edges between vertices in A_0 and between consecutive levels of the partition,

2. the signs of the edges between levels are the same and alternate between levels,

3. the edges between vertices in A_0 are positive,

4. the edges between A_0 and A_1 are negative.

For a visual representation of a positive stratified partition with four parts see Figure 4.3.

A partition $\mathcal{A}$ is a *negative stratification* if the first two conditions are the same but the edges between vertices in A_0 are negative, and edges between A_0 and A_1 are positive. The partition $\mathcal{A} = \{A_0, A_1 \ldots\}$ of the vertex set of G is a *stratification* if, for each component H, the induced partition $\mathcal{A}_H = \{A_0 \cap H, A_1 \cap H, \ldots\}$ is a stratified partition of H. Note that, in a stratified partition, one component can have a stratified partition, while another has a negative stratified partition. That is, different components can have different signs in their stratified partitions.

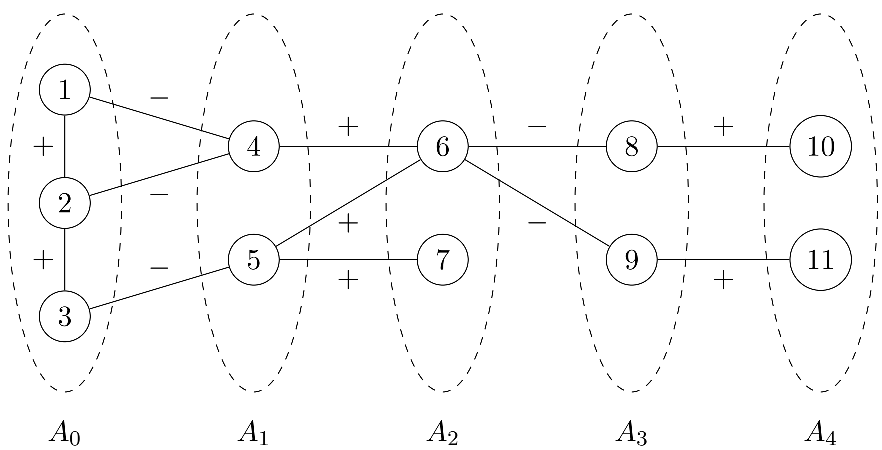


Figure 4.3: A positive stratified graph with five levels. There are positive edges between the nodes of A_0, but no other level can have internal edges. All other edges are between neighboring levels in the stratification. Between any two neighboring levels, the edges have one sign, which alternates between pairs of neighbors.

Let G be a connected signed graph with positive stratified partition $\mathcal{A}$. We now construct a dual vector $e^*_{\mathcal{A}}$ from this partition such that the hyperplane given by $\langle e^*_{\mathcal{A}}, x\rangle = 1$

contains $\mathcal{P}_G$. In particular, we assign a weight $(e^*_{\mathcal{A}})_i$ to each vertex i so that $\langle e^*_{\mathcal{A}}, x\rangle = 1$. Using the motivation provided in Example 4.1.6, we attach weight $\frac{1}{2}$ to each vertex in A_0, weight $\frac{-3}{2}$ to each vertex in A_1, and, more generally, to each vertex in A_r, assign weight $(-1)^r\left(\frac{1}{2}+r\right)$. Let $e^*_{A_r} = \sum_{i\in A_r} e^*_i$ be the dual characteristic vector of A_r, and let $e^*_{\mathcal{A}} := \sum_{r=0}^{n}(-1)^r\left(\frac{1}{2}+r\right)e^*_{A_r}$. Observe that with this choice of weights, $\langle e^*_{\mathcal{A}}, -e_i - e_j\rangle = 1$ for all edges ij with $i \in A_0$ and $j \in A_1$. More generally, observe that if $+ij$ is an edge so that $i \in A_{2r}$ and $j \in A_{2r-1}$ then $\langle e^*_{\mathcal{A}}, \rho(ij)\rangle = \left(\frac{1}{2}+2r\right) + (-1)\left(\frac{1}{2}+2r-1\right) = 1$. A similar calculation holds for $-ij$ with $i \in A_{2r}$ and $j \in A_{2r+1}$. If G were, instead, negatively stratified, then the weight on A_r would be given by $(-1)^{r+1}\left(\frac{1}{2}+r\right)$.

Example 4.1.8. Let G be the graph in Figure 4.1, i.e., G has vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{+12, -23, +34, +45, -56, +16\}$. Then $\mathcal{A} = \{A_0 = \{1, 2, 6\}, A_1 = \{3, 5\}, A_2 = \{4\}\}$ is a positive stratified partition of G and $\mathcal{B} = \{B_0 = \{1\}, B_1 = \{2, 6\}, B_2 = \{3, 5\}, B_3 = \{4\}\}$ is a negative stratified partition of G. The associated dual vectors are $v^*_{\mathcal{A}} = \frac{1}{2}(e^*_1 + e^*_2 + e^*_6) - \frac{3}{2}(e^*_3 + e^*_5) + \frac{5}{2}(e^*_4)$ and $v^*_{\mathcal{B}} = \frac{-1}{2}(e^*_1) + \frac{3}{2}(e^*_2 + e^*_6) - \frac{5}{2}(e^*_3 + e^*_5) + \frac{7}{2}(e^*_4)$. Observe that $v^*_{\mathcal{A}} - v^*_{\mathcal{B}} = (e^*_1 + e^*_3 + e^*_5) - (e^*_2 + e^*_4 + e^*_6)$, which is the hyperplane corresponding to the bipartition of G.

Example 4.1.9. Let G be the graph in Figure 4.4, i.e., G has vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{+12, -23, +34, -45, +56, -16\}$. If 1 is in A_ℓ, for some $\ell \geq 0$, then, as the edges have different sign, one of the vertices adjacent to 1 must be in $A_{\ell+1}$. Suppose without loss of generality that $2 \in A_{\ell+1}$, note that if $\ell = 0$ then $6 \in A_\ell$, otherwise $6 \in A_{\ell-1}$. Since the edges alternate in sign this implies, $3 \in A_{\ell+2}$, $4 \in A_{\ell+3}$, $5 \in A_{\ell+4}$ and $6 \in A_{\ell+5}$. However, 6 can not be in both A_ℓ or $A_{\ell-1}$ and $A_{\ell+5}$ as the sets are disjoint. Therefore, G does not have a stratified partition.

Observe that a signed graph is stratified if and only if each of its components is stratified. Therefore, since each stratified partition of a component gives a dual vector $e^*_{\mathcal{A}_H}$, we can write $e^*_{\mathcal{A}} = \sum_H e^*_{\mathcal{A}_H}$, where the sum is over the components of G. Note that the vector $e^*_{\mathcal{A}}$ depends on our choice of $e^*_{\mathcal{A}_H}$, and, hence, the stratification for each component

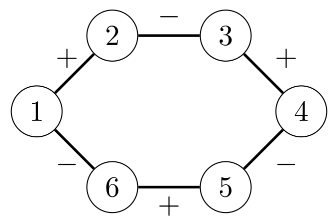


Figure 4.4: The signed graph G is not stratified. In particular, if $1 \in A_\ell$, then the levels of $2, \cdots, 6$ are uniquely determined. Following this argument, 1 and 6 can neither be in neighboring levels nor both in A_0. Therefore, a stratification does not exist because the edge -16 contradicts the condition in the definition of a stratification that there can be no edges between levels whose difference is more than 1. This is an example of a minimal non-stratified graph from Observation 4.1.15.

H. Therefore, our discussion focuses on connected components of G.

We now show that the every hyperplane given by an expression of the form $\langle v^*, x\rangle = 1$ can be obtained from a stratified partition. We begin with two direct corollaries of Lemma 4.1.5:

Corollary 4.1.10. Let G be a signed graph with bipartite components $G_1, \ldots, G_k$, whose associated dual vectors are $e^*_{L_1} - e^*_{R_1}, \ldots, e^*_{L_k} - e^*_{R_k}$. If $\langle v^*, x\rangle = 1$ and $\langle u^*, x\rangle = 1$ are hyperplanes that contain $\mathcal{P}_G$, then $v^* - u^*$ is a linear combination of $e^*_{L_1} - e^*_{R_1}, \ldots, e^*_{L_k} - e^*_{R_k}$.

Corollary 4.1.11. Let G be a signed graph which does not have any bipartite components. Then G has at most one stratified partition.

Proof. The construction above shows that each stratified partition $\mathcal{A}$ gives rise to a distinct dual vector $e^*_{\mathcal{A}}$. By Corollary 4.1.10, the dual vectors for two stratified partitions differ by a linear combination of dual vectors from bipartite components. However, since there are no bipartite components, the dual vectors from the stratified partitions must be equal. Hence, the stratified partitions are also equal. □

The next sequence of results proves that for a signed graph G, there exists a hyperplane $\langle u^*, x\rangle = 1$ containing $\mathcal{P}_G$ if and only if G admits a stratified partition. Therefore, every hyperplane containing $\mathcal{P}_G$ of the form $\langle u^*, x\rangle = 1$ is a linear combination of a hyperplane from a stratified partition and the hyperplanes from the bipartite components of G.

Lemma 4.1.12. Let G be a connected signed graph so that the hyperplane defined by $\langle v^*, x\rangle = 1$ contains $\mathcal{P}_G$. Then, v^* is uniquely determined by $(v^*)_i$ for any vertex i.

Proof. We construct the weights for the vertices in terms of the value of $(v^*)_i$. For each neighbor j of i, $\langle v^*, \rho(ij)\rangle = 1$, and so $\text{sgn}(ij)((v^*)_i + (v^*)_j) = 1$. Therefore, $(v^*)_j = \text{sgn}(ij) - (v^*)_i$. By connectivity, for any vertex k, there is a path in the graph between i and k; we can calculate $(v^*)_k$ by applying the calculation above to each pair of neighbors along the path. This calculation is independent of the choice of path because if two paths resulted in different values for $(v^*)_k$, then, since each choice is unique, these different values would contradict the assumed existence of v^*. $\square$

In particular, a stratified partition $\mathcal{A} = \{A_0, \ldots, A_n\}$, is determined uniquely by $i \in A_r$ and the sign of the partition. We address the cases of stratifications of bipartite and non-bipartite components of the graph G separately in the following lemmas.

Lemma 4.1.13. Let G be a connected signed graph with C an odd cycle so that every edge of C has the same sign s and a hyperplane given by $\langle v^*, x\rangle = 1$ that contains $\mathcal{P}_G$. Then G admits a stratified partition, and v^* is the dual vector of a unique stratified partition with sign s and $C \subseteq A_0$.

Proof. Without loss of generality, assume $s = +1$ and $\langle v^*, x\rangle = 1$ contains $\mathcal{P}_G$. If $+ij$ and $+jk$ are edges of C then $\langle v^*, \rho(+ij) - \rho(+jk)\rangle = 0$. Thus, $(v^*)_i + (v^*)_j - (v^*)_j - (v^*)_k = 0$ and hence $(v^*)_i = (v^*)_k$. Since C is an odd cycle, this implies that for every vertex r of C, $(v^*)_r = (v^*)_i$. Moreover, from $(v^*)_i + (v^*)_j = 2(v^*)_i = 1$, it follows that $(v^*)_i = \frac{1}{2}$. Since the value of $(v^*)_i$ is uniquely specified, it follows from Lemma 4.1.12 that there is a unique dual vector u^* so that the hyperplane $\langle u^*, x\rangle = 1$ contains $\mathcal{P}_G$. There is at most one stratification, since each stratification $\mathcal{A}$ induces a hyperplane $e^*_{\mathcal{A}}$ so that the hyperplane $\langle e^*_{\mathcal{A}}, x\rangle = 1$ contains $\mathcal{P}_G$, and, by uniqueness, this dual vector must be v^*.

We now construct a stratification of G: Let A_r be the set of vertices i of G such that $(v^*)_i = (-1)^r \left(\frac{1}{2} + r\right)$. We observe that A_0 is not empty as A_0 includes the vertices of C which have weight $\frac{1}{2}$. By a case-by-case analysis, we observe that any neighbor of a vertex in A_r must be in A_s where $s = r \pm 1$ for $r \neq 0$ and $s = 0, 1$ for $r = 0$. For instance, if vertex i is in A_r, $-ij$ is an edge of G, and r is odd, then since $\langle v^*, \rho(-ij)\rangle = 1$, it follows that $-(v^*)_i - (v^*)_j = -(-1)^r \left(\frac{1}{2} + r\right) - (v^*)_j = 1$. Then, $(v_j)^* = (-1)^{r-1} \left(\frac{1}{2} + (r-1)\right)$ and $j \in A_{r-1}$. Note that if r were even, this would imply that $j \in A_{r+1}$, for $r \neq 0$. The other cases are similar. Therefore, G has a unique stratified partition $\mathcal{A}$ with sign $s = +1$ and $v^* = e^*_{\mathcal{A}}$. □

The following lemma allows us to reduce stratifications of non-bipartite graphs which do not have an odd cycle with all edges of the same sign to stratifications of smaller graphs which satisfy the conditions of Lemma 4.1.13. By inducting on the size of the graph, this reduction extends Lemma 4.1.13 to all non-bipartite graphs. The intuition behind this can be obtained from the commutative diagram in Figure 4.5.

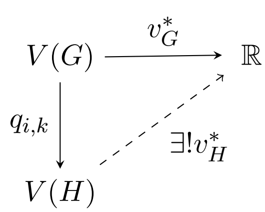


Figure 4.5: Let G be a signed graph and v^*_G a dual vector; then, v^*_G may be interpreted as a map from from the set of vertices V of G to $\mathbb{R}$. Suppose, in addition that for two vertices i and k, $(v^*_G)_i = (v^*_G)_k$. Let H be be the signed graph formed by identifying i and k and $q_{i,k}$, the corresponding quotient map. Then, the vertex weights on the endpoints of edges are preserved by $q_{i,k}$. Moreover, there is a unique dual vector v^*_H on H such that for every edge e in G, $\langle v^*_G, \rho_G(e)\rangle = \langle v^*_H, \rho_H \circ q_{i,k}(e)\rangle$.

Lemma 4.1.14. Let G be a connected signed graph with vertices i, j, and k, and edges $\pm ij$, $\pm jk$ of the same sign. Let H be the signed graph formed by identifying vertices i and k and identifying duplicated edges with the same sign. Then, there is a bijection between hyperplanes of the form $\langle v^*_G, x\rangle = 1$ which contain $\mathcal{P}_G$ and hyperplanes of the form

$\langle v_H^*, x\rangle = 1$ which contain $\mathcal{P}_H$.

Proof. Let v_G^* be any dual vector on G (at this point, we do not assume $\langle v_G^*, x\rangle = 1$ contains $\mathcal{P}_G$). Since, by assumption, $\text{sgn}(ij) = \text{sgn}(jk)$, it follows that $\langle v_G^*, \rho(ij) - \rho(jk)\rangle = 0$, and, hence, $(v_G^*)_i = (v_G^*)_k$. Let $q_{i,k} : G \to H$ be the quotient map and i' be the vertex of H formed by identifying i and k. Observe that if i is adjacent to k, then H has a loop at i'.

Suppose first that a hyperplane given by $\langle v_G^*, x\rangle = 1$ contains $\mathcal{P}_G$. Then, we construct the dual vector v_H^* by setting $(v_H^*)_r = (v_G^*)_r$ for all vertices $r \neq i, k$ and $(v_H^*)_{i'} = (v_G^*)_i = (v_G^*)_k$. Since $q_{i,k}$ is surjective, and, for all edges e in G, the endpoints of $q_{i,k}(e)$ have the same weights as the endpoints of e, $\langle v_H^*, \rho \circ q_{i,k}(e)\rangle = \langle v_G^*, \rho(e)\rangle = 1$.

On the other hand, suppose now that the hyperplane given by $\langle v_H^*, x\rangle = 1$ contains $\mathcal{P}_H$. We construct v_G^* by $(v_H^*)_r = (v_G^*)_r$ for all vertices $r \neq i, k$, and $(v_H^*)_{i'} = (v_G^*)_i = (v_G^*)_k$. Since $q_{i,k}$ is surjective, and, for all edges e in G, the endpoints of $q_{i,k}(e)$ have the same weights as the endpoints of e, $\langle v_G^*, \rho(e)\rangle = \langle v_H^*, \rho \circ q_{i,k}(e)\rangle = 1$.

Since the two constructions are inverses of each other, there is a bijection between dual vectors, and hence a bijection between the hyperplanes. □

Observation 4.1.15. Observe that, under the identification in Lemma 4.1.14, i and k must be in the same level for all stratifications of G, and, if G is bipartite, then i and k are in the same bipartition of G. Therefore, G has as many stratifications as H, and G is bipartite if and only if H is bipartite.

We next characterize the minimal connected signed graphs under the identifications in Lemma 4.1.14. Observe that the vertices must have at most two neighbors (counting itself in the case of a loop) since, otherwise, two of the neighbors could be contracted. Therefore minimal elements under the identifications in Lemma 4.1.14 consist of one of the following: (1) an alternating cycle, (2) an alternating path, possibly with loops at the ends. The loops, if they exist, have the opposite sign than the incident edge from the path. Note that the only minimal graphs which are bipartite are those without loops.

Finally, we characterize which minimal graphs have stratifications. Let H be a minimal graph and j is a vertex of H. Suppose that H has a stratification, j has two neighbors i and k, i is not the base of a loop (so $j \neq i, k$), and j is in level r in the stratification. Observe that i and k cannot be in the same level of the stratification because otherwise, the incident edges would have the same sign and they could be contracted. Therefore, if r is nonzero, then one neighbor of r is in level $r-1$ and the other is in $r+1$. If r is zero, one neighbor is in level 0 and the other is in level 1, see Figure 4.4. Therefore, the only minimal graphs which have a stratified partition are paths which have at most one loop. In the case of a path with a loop, the vertex with the loop must be in level 0. Therefore, this identification provides an effective way to identify when a graph admits a stratification (and bipartition).

Lemma 4.1.16. Let G be a connected bipartite graph with bipartition $V = L \cup R$. Suppose $\mathcal{P}_G$ is contained in the hyperplane given by $\langle v^*, x \rangle = 1$. Then G admits a stratified partition $\mathcal{A}$, and the dual vector v^* is a linear combination of the dual vector $e^*_{\mathcal{A}}$ of the stratified partition and the dual vector $e^*_L - e^*_R$ of the bipartition.

Proof. Fix a vertex i of G and let $c = \frac{1}{2} - (v^*)_i$. Let $a^* = v^* + c(e^*_L - e^*_R)$. Let $a^* = v^* + c(e^*_L - e^*_R)$. Observe that $(a^*)_i = \frac{1}{2}$ and the hyperplane given by $\langle a^*, x \rangle = 1$ contains $\mathcal{P}_G$.

We now construct a stratification of G: Let A_r be the set of vertices i of G such that $(a^*)_i = (-1)^r \left(\frac{1}{2} + r\right)$. We observe that A_0 is not empty as A_0 includes the vertex i which has weight $\frac{1}{2}$. By a case-by-case analysis, we observe that any neighbor of a vertex in A_r must be in A_s where $s = r \pm 1$ for $r \neq 0$ and $s = 0, 1$ for $r = 0$. This construction then proceeds identically to the construction in Lemma 4.1.13. Therefore, G has a stratified partition with sign $s = +1$.

By Lemma 4.1.12, there is a unique dual vector u^* with $(u^*)_i = \frac{1}{2}$. Since both a^* and the dual vector $e^*_{\mathcal{A}}$ associated to the stratified partition have value $\frac{1}{2}$ at i, they must be equal. Therefore, v^* is a linear combination of $e^*_{\mathcal{A}}$ and $e^*_L - e^*_R$. □

Note that not all dual vectors v^* such that the hyperplane $\langle v^*, x\rangle = 1$ are dual vectors associated to a stratified partition. For example, for a bipartite graph G with bipartition $L \cup R$, and a vector given by a stratified partition v_G^*, then for any c, $\langle v_G^* + c(e_L^* - e_R^*), x\rangle = 1$ contains $\mathcal{P}_G$; if, however, c is not dyadic, then then the vertex weights cannot match those from the dual vector to a stratified partition.

Definition 4.1.17. Let G be a signed graph; we write $\mathrm{Strat}(G) = 1$ if every component of G has a stratified partition and $\mathrm{Strat}(G) = 0$ otherwise.

Corollary 4.1.18. Let G be a signed graph, then there exists a hyperplane of the form $\langle v^*, x\rangle = 1$ containing $\mathcal{P}_G$ if and only if G admits a stratification.

Proof. By Observation 4.1.15, it is enough to consider alternating cycles and alternating paths, possibly with loops at the ends. By Lemma 4.1.13, the alternating paths with loops at the ends admit a hyperplane of the form $\langle v^*, x\rangle = 1$ if and only if they have a stratification. Similarly, by Lemma 4.1.16, alternating cycles and paths without loops admit a hyperplane of the form $\langle v^*, x\rangle = 1$ if and only if they have a stratification. □

4.1.2.3 Dimension Formula

We now connect the dimension of $\mathcal{P}_G$ with the number of bipartite components $\mathrm{BiComp}(G)$ and $\mathrm{Strat}(G)$.

Theorem 4.1.19. Let G be a signed graph on n vertices, then $\dim \mathcal{P}_G = n - \mathrm{BiComp}(G) - \mathrm{Strat}(G)$.

Proof. We compute the dimension by finding a maximal linearly independent set of hyperplanes that contain $\mathcal{P}_G$. Let $\mathcal{H}$ be the collection of hyperplanes of the form $\langle e_L^* - e_R^*, x\rangle = 0$ for each bipartite component G and $\langle e_{\mathcal{A}}^*, x\rangle = 1$ if G admits a stratified partition $\mathcal{A}$. Recall that the hyperplanes for bipartite components are independent from each other because their supports are disjoint. Moreover, each of these bipartite dual vectors is independent of the dual vector from the stratified partition because the hyperplane $\langle e_{\mathcal{A}}^*, x\rangle = 1$ is the only hyperplane with a nonzero constant.

Now, we show that any other hyperplane containing $\mathcal{P}_G$ can be written as a linear combination of the hyperplanes in $\mathcal{H}$. Suppose that $\langle v^*, x\rangle = c$ contains $\mathcal{P}_G$. If c is zero, we know, by Lemma 4.1.5, that v^* is a linear combination of the dual vectors from the bipartite components of G. If c is nonzero, we may replace v^* by $\frac{1}{c}\cdot v^*$ without changing the hyperplane; then $\langle v^*, x\rangle = 1$ contains $\mathcal{P}_G$. By Corollary 4.1.18, it follows that G admits a stratification $\mathcal{A}$. Then, by Lemma 4.1.13 or 4.1.16, v^* is a linear combination of the dual vectors $e^*_{\mathcal{A}}$ and the dual vectors from the bipartite components.

Hence the codimension of $\mathcal{P}_G$ is $\operatorname{BiComp}(G) + \operatorname{Strat}(G)$ which gives the formula:

$$\dim \mathcal{P}_G = n - \operatorname{BiComp}(G) - \operatorname{Strat}(G).$$

□

Example 4.1.20. Let G be the graph in Figure 4.1, i.e., G has vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{+12, -23, +34, +45, -56, +16\}$. Observe that G is a connected bipartite graph that admits a stratified partition, hence, $n = 6$, $\operatorname{BiComp}(G) = 1$, and $\operatorname{Strat}(G) = 1$. Thus, by Theorem 4.1.19, $\dim \mathcal{P}_G = n - \operatorname{BiComp}(G) - \operatorname{Strat}(G) = 4$.

4.2 Algebraic Properties

In this section, we classify the normality of $k[G]$ in terms of combinatorial properties of the graph G. Additionally, when $k[G]$ is not normal, we use the structure of G to determine the additional elements needed to normalize $k[G]$. The combinatorial structure is similar to the odd cycle condition of [38], but is more nuanced due to the more general class of signed graphs.

Definition 4.2.1. Let G be a signed graph, possibly with loops and $\mathcal{L}_G$ the subgroup of $\mathbb{Z}^n$ generated by $\rho(E(G))$. Then, let

- $S_1 := \operatorname{cone}(\mathcal{P}_G) \cap \mathcal{L}_G = \mathbb{R}_+\rho(E(G)) \cap \mathbb{Z}\rho(E(G))$,
- $S_2 := \mathbb{Z}_+\rho(E(G))$, i.e., the semigroup generated by $\rho(E(G))$.

Note that $k[S_2] = k[G]$ and that $S_2 \subseteq S_1$.

Definition 4.2.2. Let G be a signed graph, S_1 and S_2 as defined above, and let $\mathcal{A}(\mathcal{P}_G)_n$ be the vector space over k which is spanned by the monomials of the form x^a such that $a \in n\mathcal{P} \cap \mathbb{Z}^d$. then we define:

- the *Ehrhart* Polynomial ring:

$$k[S_1] := \mathcal{A}(\mathcal{P}_G) = \sum_{n=0}^{\infty} \mathcal{A}(\mathcal{P}_G)_n,$$

- the *Edge ring of* G:

$$k[S_2] = k[G] := \langle t^a : a \in \rho(E(G)) \rangle.$$

Where $k[G]$ and $\mathcal{A}(\mathcal{P}_G)$ are generated as subrings of $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$.

Notation 4.2.3. Throughout the remainder of the chapter ring notation or semigroup notation will be used depending on which is clearer in the particular context. Recall that the coordinates in $\mathbb{Z}^n$ correspond to the exponents of the x_i's in the ring.

Our goal throughout the remainder of this section is to use combinatorial properties of G to relate $k[G]$ with $\mathcal{A}(\mathcal{P}_G)$. In particular, we compute generators of $\mathcal{A}(\mathcal{P}_G)$ over $k[G]$ in terms of properties of the graph.

4.2.1 Non-normality

In this section, we define the main combinatorial property, called the odd cycle condition, for signed graphs. Moreover, we show that if a signed graph G does not satisfy this property, then $k[G]$ is not normal.

Example 4.2.4. Let G be the signed graph in Figure 4.6(a), i.e. G has vertex set $\{1, 2, 3\}$ and edge set $\{+11, +12, +23, -33\}$. Then, the monomial $x_1 x_3^{-1}$ is in $\mathcal{A}(\mathcal{P}_G)$, but it is not in $k[G]$. We demonstrate this by showing $(1, 0, -1) \in S_1 \setminus S_2$ since the elements of the S_i's represent the exponents of the variables.

To show that $(1,0,-1) \in S_1$ we give two ways of writing $(1,0,-1)$: one in $\text{cone}(\mathcal{P}_G)$ and the other in $\mathcal{L}_G$. To show that $(1,0,-1) \in \text{cone}(\mathcal{P}_G)$, we show that $(1,0,-1)$ can be written as a nonnegative real combination of the edges of G:

$$(1,0,-1) = \frac{1}{2}\rho(+11) + \frac{1}{2}\rho(-33) = \frac{1}{2}(2,0,0) + \frac{1}{2}(0,0,2).$$

Similarly, to show that $(1,0,-1) \in \mathcal{L}_G$, we show that $(1,0,-1)$ can be written as a integral combination of the edges of G:

$$(1,0,-1) = \rho(+12) - \rho(+23) = (1,1,0) - (0,1,1).$$

Thus, $(1,0,-1) \in S_1$ and hence $x_1x_3^{-1} \in \mathcal{A}(\mathcal{P}_G)$.

We now argue that $(1,0,-1)$ is not in S_2 because it cannot be written as a nonnegative integral combination of the edges of G. In particular, if $(1,0,-1)$ were in S_2, then we could write $(1,0,-1) = \sum_{e \in G} z_e \rho(e)$ for $z_e \in \mathbb{Z}_{\geq 0}$. Since the only two edges incident to vertex 1 are $+11$ and $+12$, the coefficient of one of these must be at least one. However, the vectors $(2,0,0)$ and $(1,1,0)$ contain coordinates that are strictly larger than the same coordinates in $(1,0,-1)$ and there is no way to cancel these coordinates. Therefore, $(1,0,-1)$ cannot be written as a nonnegative integral combination of the edges of G, $(1,0,-1) \notin S_2$, and $x_1x_3^{-1} \notin k[G]$.

Observe that $(x_1x_3^{-1})^2 = (x_1^2)(x_3^{-1})^2 \in k[G]$, and, hence, the monomial $z^2 - (x_1x_3^{-1})^2$ has the rational solution $x_1x_3 = \frac{x_1^2(x_2x_3)(x_3^{-1})}{(x_1x_2)}$. Since $x_1x_3^{-1}$ is not in $k[G]$, is in the fraction field of $k[G]$, and is a solution to a monic polynomial with coefficients in $k[G]$, $k[G]$ is not normal.

Example 4.2.5. Let H be the signed graph in Figure 4.6, i.e. H has vertex set $\{1,2,3\}$ and edge set $\{+11,+12,-23,-33\}$. Observe that H is the same graph as G from Example 4.2.4, except that edge $+23$ has been replaced by -23. In this case, we can see that $(1,0,-1) \in S_1$ by a similar set of coefficients as the example above. On the other hand, $(1,0,-1) \in S_2$

Figure 4.6: The edge ring for the graph G is not normal. In particular, $(x_1 x_3^{-1})^2$ is in $k[G]$, but its square root is not in $k[G]$. By changing the sign on the edge 23, the edge ring for the graph H is normal. In this case, $x_1 x_3^{-1}$ can be formed by observing that since the pair of edges $+12$ and -23 differ in signs, the corresponding powers on x_2 cancel.

since $(1, 0, -1)$ can be written as a nonnegative integral combination of the edges.

$$(1, 0, -1) = \rho(+12) + \rho(-23).$$

Therefore, $x_1 x_3^{-1}$ is not a witness to non-normality; in fact, $k[H]$ is normal.

Examples 4.2.4 and 4.2.5 illustrate a subtlety in detecting normality in the case of a signed graph since the normality depends not only on the structure of the graph, but also the signs of the edges. These examples lead to the odd cycle condition for signed graphs.

Definition 4.2.6. Let G be a signed graph, we say that G satisfies the *odd cycle condition* if for every pair of odd cycles C and C' of G, at least one of the following occurs:

1. C and C' are in distinct components,
2. C and C' have a vertex in common,
3. C and C' have a sign alternating i, j-path in G, where i is a vertex of C and j is a vertex of C'.

Note that the definition of the odd cycle condition extends the definition from [38]. In particular, the graphs that Hibi and Ohsugi consider have only positive edges, so the only possible alternating paths are of length one. By restricting to that case, the odd cycle condition of [38] follows. We now show that the odd cycle condition is a necessary condition on G for $k[G]$ to be normal.

Definition 4.2.7. Let G be a signed graph, C be a cycle in G, and $\{u_1, \ldots, u_n\}$ be the vertices of C. Define the *signature* of vertex u_i of C as the average of the signs of the edges incident to i in C, i.e., $\mathrm{sig}_C(u_i) = \frac{1}{2}(\mathrm{sgn}(u_{i-1}u_i) + \mathrm{sgn}(u_i u_{i+1}))$, where the subscripts are computed modulo n. Observe that the signature of a vertex is one of 1, 0, and -1.

Observation 4.2.8. Let G be a signed graph containing cycle C. Suppose that $W \subseteq C$ is an alternating path with vertices $\{i_0, \ldots, i_n\}$ and edges $\{i_0 i_1, \ldots, i_{n-1} i_n\}$. Then,

$$\prod_{j=1}^{n} (x_{i_{j-1}} x_{i_j})^{\mathrm{sgn}(i_{j-1} i_j)} = x_{i_0}^{\mathrm{sgn}(i_0 i_1)} x_{i_n}^{\mathrm{sgn}(i_{n-1} i_n)}.$$

The intermediate terms cancel because each x_j appears in two terms with opposite signs. Moreover, if i_0 and i_n have nonzero signature in C, then, $\mathrm{sgn}(i_0 i_1) = \mathrm{sig}_C(i_0)$ and $\mathrm{sgn}(i_{n-1} i_n) = \mathrm{sig}_C(i_n)$, so the product equals $x_0^{\mathrm{sig}_C(i_0)} x_n^{\mathrm{sig}_C(i_n)}$.

Theorem 4.2.9. Let G be a signed graph which does not satisfy the odd cycle condition, then $k[G]$ is not normal.

Proof. Let C and C' be a pair of odd cycles that violate the odd cycle condition. Our goal is to show that

$$\prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell \in C'} x_\ell^{\mathrm{sig}_{C'}(\ell)} \tag{4.1}$$

is in the fraction field of $k[G]$, but not $k[G]$. To do this, we write this product in several ways:

First, observe that this product can be constructed by putting the weight of $\frac{1}{2}$ on all the edges of $C \cup C'$. In particular,

$$\prod_{\pm k\ell \in C \cup C'} (x_k x_\ell)^{\frac{1}{2}\mathrm{sgn}(k\ell)} = \prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell \in C'} x_\ell^{\mathrm{sig}_{C'}(\ell)}.$$

Since all the exponents (weights) are positive rational numbers, Expression (4.1) is in $k[\mathrm{cone}(\mathcal{P}_G)]$.

On the other hand, since C and C' do not satisfy the odd cycle condition, C and

C' are in the same component, and every path between C and C' is not alternating. Since C is an odd cycle, by a parity argument, there exists a vertex i in C such that the incident edges have the same sign. Similarly, there is a vertex j in C' such that the incident edges have the same sign. Since C and C' are in the same component, let P be any path between i and j.

We choose a walk W in G between i and j with the following properties: if $\text{sig}_C(i) = \text{sig}_{C'}(j)$, then W has odd length, and if $\text{sig}_C(i) = -\text{sig}_{C'}(j)$, then W has even length. Such a walk can be constructed by considering $W = P$ or $W = P \cup C$, depending on the parity of P.

Suppose that the vertices of W are $\{i = i_0, \ldots, i_n = j\}$ such that the edges of W are $\{i_0i_1, i_1i_2, \ldots, i_{n-1}i_n\}$ (possibly with repeated vertices or edges). Let $a_1, \ldots, a_n$ be ± 1 so that $a_\ell \, \text{sgn}(i_{\ell-i}i_\ell) = (-1)^{\ell-1} \text{sig}_C(i)$ for $1 \leq l \leq n$. In other words, $a_1 \, \text{sgn}(i_0i_1) = \text{sig}_C(i)$, $a_2 \, \text{sgn}(i_1i_2) = -\text{sig}_C(i)$, and, due to choice of the length of the walk, $a_n \, \text{sgn}(i_{n-1}i_n) = (-1)^n \, \text{sig}(C)(i) = \text{sig}_{C'}(j)$. Therefore, while the signs on W may not alternate, the products $a_\ell \, \text{sgn}(i_{\ell-1}i_\ell)$ alternate. Therefore, in the following product, the intermediate terms cancel:

$$\prod_{\ell=1}^{n} (x_{i_{\ell-1}} x_{i_\ell})^{a_\ell \, \text{sgn}_C(i_{l-1}i_l)} = x_i^{\text{sig}_C(i)} x_j^{\text{sig}_{C'}(j)}. \tag{4.2}$$

Consider $C \setminus i$, which is a path with an even number of vertices. Let $\{u_1, \ldots, u_{2m}\}$ be the vertices of $C \setminus i$, in order. We now construct the product

$$\prod_{\ell \in C \setminus i} x_\ell^{\text{sig}_C(\ell)}. \tag{4.3}$$

Observe that, since the signature is a weighted sum of the signed edges, we can rearrange a sum of signatures into a sum over edges as follows:

$$\sum_{\ell \in C} \text{sig}_C(\ell) = \sum_{\ell \in C} \frac{1}{2} \left(\sum_{\pm j\ell} \text{sgn}(j\ell) \right) = \sum_{e \in C} \text{sgn}(e). \tag{4.4}$$

Since $\mathrm{sig}_C(i) = \frac{1}{2}\left(\sum_{\pm ji \in C} \mathrm{sgn}(ij)\right)$ we can write a similar equation for $C \setminus i$:

$$\sum_{\ell \in C \setminus i} \mathrm{sig}_C(\ell) = \mathrm{sig}_C(i) + \sum_{k=1}^{2m-1} \mathrm{sgn}(u_k u_{k+1}). \tag{4.5}$$

One way to interpret Equation 4.4 is the sign of each incident edge has weight one half in $\mathrm{sig}_C(\ell)$, and weight one in $\mathrm{sgn}(e)$. Hence, $C \setminus i$ is removing one half of the weight of each edge incident to i, the remaining weight of the edges incident to i will be $\mathrm{sig}_C(i)$, and the other edges are unaffected and still have weight $\mathrm{sgn}(e)$. Since the right-hand-side of Equation (4.5) contains an even number of terms which are all equal to ± 1, we know that both sides of the equation are even. Hence, using a parity argument, we see that there are an even number of vertices in $C \setminus i$ with nonzero signature. Let $\{u_{j_1}, \ldots, u_{j_{2p}}\}$ be the vertices of $C \setminus i$ with nonzero signature, in order around C. Then, there are disjoint alternating paths $\{W_1, \ldots, W_p\}$ such that W_k ends at $u_{j_{2k-1}}$ and $u_{j_{2k}}$; these paths are alternating because otherwise there would be a vertex with nonzero signature between $u_{j_{2k-1}}$ and $u_{j_{2k}}$. By Observation 4.2.8, for each W_k, we assign an edge weight of 1 on the edges of W_k so that the corresponding ring element is $x_{j_{2k-1}}^{\mathrm{sig}_C(u_{j_{2k-1}})} x_{j_{2k}}^{\mathrm{sig}_C(u_{j_{2k}})}$. By multiplying these products over all paths, the product is Expression (4.3). Similarly, we can construct $\prod_{\ell \in C' \setminus j} x_\ell^{\mathrm{sig}_{C'}(\ell)}$. By multiplying Equation (4.2) with Expression (4.3) as well as the corresponding expression for j, the result is

$$\left(x_i^{\mathrm{sig}_C(i)} x_j^{\mathrm{sig}_{C'}(j)}\right) \prod_{\ell \in C \setminus i} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell \in C' \setminus j} x_\ell^{\mathrm{sig}_{C'}(\ell)} = \prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell \in C'} x_\ell^{\mathrm{sig}_{C'}(\ell)}.$$

Since all the exponents are integers, Expression (4.1) is in $k[\mathcal{L}_G]$. Therefore, Expression (4.1) is in $\mathcal{A}(\mathcal{P}_G)$.

Since the exponent of the monomial in Expression (4.1) is in $\mathcal{L} + G$, the monomial is in $k[\mathcal{L}_G]$, and hence is in the fraction field of $k[G]$; we now show that this expression is not in $k[G]$. Suppose, for contradiction, that there are nonnegative integer weights on the edges of G whose corresponding ring element is Expression (4.1).

Let $\mathcal{E}$ be the multiset of the edges of G with multiplicity according to their weights. All vertices other than in C or C' with nonzero signature do not appear in Expression (4.1); therefore, for any such vertex v, the number of positive or negative edges in $\mathcal{E}$ incident to v must be equal. Therefore, we can partition $\mathcal{E}$ into a collection of alternating closed walks and alternating walks whose endpoints are points in C or C' with nonzero signature. With this construction, there are four types of walks: (1) closed walks, (2) walks with both endpoints in C, (3) walks with both endpoints in C', and (4) walks with one endpoint in C and the other in C'. Since C and C' violate the odd cycle condition, there are no walks of Type (4). Similarly, Type (1) closed walks correspond to the identity in the ring and can be discarded. Therefore, we reduce to the case where $\mathcal{E}$ can be partitioned into walks of Types (2) and (3). By Observation 4.2.8, for any walk of Type (2), with endpoints i_0, and i_n, the corresponding ring element is $x_{i_0}^{\mathrm{sig}_C(i_0)} x_{i_n}^{\mathrm{sig}_C(i_n)}$; a similar statement holds for walks of Type (3). We established above that C and C' each have an odd number of vertices with nonzero signature. However, since each walk of Type (2) introduces a monomial of even total degree, it is impossible for the product $\prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(\ell)}$, which is of odd total degree to be the product. This is a contradiction, so Expression (4.1) cannot be in $k[G]$, and, hence, $k[G]$ is not normal. □

The odd cycle condition is also a sufficient condition for $k[G]$ to be normal. We prove this throughout the remainder of this section; first, we must prove a few structural results in the next section.

Observation 4.2.10. The proof of Theorem 4.2.9 includes an important construction that we collect here: If C is an odd cycle in a graph G and i is a vertex of C such that $\mathrm{sig}_C(i) \neq 0$, then there are edge weights of 0 and 1 on C whose product is

$$\prod_{\ell \in C \setminus i} x_\ell^{\mathrm{sig}_C(\ell)}.$$

4.2.2 Structural Results

In this section, we explicitly translate between certain combinatorial graph properties and ring properties. In particular, we provide the combinatorial property such that the corresponding product in $\mathcal{A}(\mathcal{P}_G)$ is 1.

Definition 4.2.11. Let G be a signed graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. The *signed incidence matrix* is an $n \times m$ matrix with $M_{a,b} = 2\,\mathrm{sgn}(e_b)$ if a is the endpoint of the loop e_b, $M_{a,b} = \mathrm{sgn}(e_b)$ if a is an end point of e_b, and $M_{a,b} = 0$ otherwise.

Note that the non-zero entries in row a of M correspond to the edges of G incident to vertex a, and the non-zero entries of column b of M correspond to the end points of edge b. Moreover, identifying the edge weights and vertex weights of a graph with coordinates of vector spaces, the matrix M represents the linear map ρ so that if a is a vector of edge weights, then Ma represents $\sum_{e \in E} a_e \rho_e$.

Example 4.2.12. Let H be the graph in Figure 4.6(b), i.e. H has vertex set $\{1, 2, 3\}$ and edge set $\{+11, +12, -23, -33\}$. The signed incidence matrix for this graph (where the edges are listed in order) is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Theorem 4.2.13. Let G be a signed graph, and, for each edge ij in G, let a_{ij} be an integer. The following are equivalent:

1. $\prod_{\pm ij \in E} (x_i x_j)^{\mathrm{sgn}(ij) a_{ij}} = 1$.

2. For all vertices i, $\sum_{j \sim i} \mathrm{sgn}(ij) a_{ij} = 0$ where the sum is taken over all j adjacent to i.

3. If M is the signed incidence matrix of G and $a = (a_k)$ the vector of integral edge weights, then $Ma = 0$.

4. There is a multiset of closed walks allowing loops with weights $w_{ij} = \pm 1$ so that the products $w_{ij}\operatorname{sgn}(ij)$ alternate along the closed walks and a_{ij} is the sum of the weights of the occurrences of edge ij in the closed walks.

For clarity, we split the proof of Theorem 4.2.13 into the following three lemmas:

Lemma 4.2.14. Let G be a signed graph, and, for each edge ij in G, let a_{ij} be an integer. $\prod_{\pm ij\in G}(x_i x_j)^{\operatorname{sgn}(ij)a_{ij}} = 1$, if and only if $\sum_{j\sim i}\operatorname{sgn}(ij)a_{ij} = 0$ for all vertices i, summed over all j adjacent to i.

Proof. The product $\prod_{\pm ij\in G}(t_i t_j)^{\operatorname{sgn}(ij)a_{ij}} = 1$ if and only if, for each i, the exponent of x_i in the product is zero. For each i, the exponent of x_i in the product is $\sum_{j\sim i}\operatorname{sgn}(ij)a_{ij}$, where the sum is taken over all j adjacent to i. □

Lemma 4.2.15. Let G be a signed graph, and, for each edge ij in G, let a_{ij} be an integer. Let M be the signed incidence matrix of G. The sum $\sum_{j\sim i}\operatorname{sgn}(ij)a_{ij} = 0$ for all vertices i, where the sum is taken over all j adjacent to i, if and only if the product $Ma = 0$, where a the vector of integral edge weights.

Proof. The i^{th} entry of Ma is $(Ma)_i = \sum_{j\sim i}\operatorname{sgn}(ij)a_{ij}$. Hence, $\sum_{j\sim i}\operatorname{sgn}(ij)a_{ij} = 0$, for all vertices i, where the sum is taken over all j adjacent to i if and only if $Ma = 0$. □

Lemma 4.2.16. Let G be a signed graph, and, for each edge ij in G, let a_{ij} be an integer. The sum $\sum_{j\sim i}\operatorname{sgn}(ij)a_{ij} = 0$ for all vertices i, where the sum is taken over all j adjacent to i, if and only if there is a multiset of closed walks allowing loops with weights $w_{ij} = \pm 1$ so that the products $w_{ij}\operatorname{sgn}(ij)$ alternate along the closed walks and a_{ij} is the sum of the weights of the occurrences of edge ij in the closed walks

Proof. If such a collection of closed walks exist, then, let ji and ik be consecutive edges along one such closed walk. Since the weighted walks are alternating, $w_{ji}\operatorname{sgn}(ji)+w_{jk}\operatorname{sgn}(jk) = 0$. Therefore, for a fixed vertex i, the weighted sum over all incident edges is 0 because the edges come in alternating pairs. By reorganizing this sum to combine repeated edges and using the assumption about the weighted sums of the edges, the sum over all edges incident to i is precisely the sum $\sum_{j\sim i} \text{sgn}(ij)a_{ij}$, which is zero. Since i is arbitrary, the first statement follows.

On the other hand, suppose that the given sums are zero for all vertices. Let $\mathcal{E}$ be the multiset of pairs of signed edges of G and weights; in particular, for each jk in G, the edge-weight pair $(jk, \text{sgn}(a_{jk}))$ occurs $|a_{jk}|$ times. Therefore, the sum of the weights of the occurrences of jk is a_{jk}. For any vertex i, since the sum $\sum_{j\sim i} \text{sgn}(ij)a_{ij}$ is zero by assumption, there are the number of edges incident to i in $\mathcal{E}$ where $w_{ij}\,\text{sgn}(ij)$ is positive or negative must be equal. Therefore, we can partition $\mathcal{E}$ into a collection of cycles with $w_{jk}\,\text{sgn}(jk)$ alternating. □

Corollary 4.2.17. Let G be a signed graph. There are nonzero edge weights a_{ij} on the edges of G satisfying the equivalence of Theorem 4.2.13 if

1. G has an even number of edges and an Eulerian tour or,
2. $G = C_1 \cup C_2 \cup P$ where P is a path connecting odd cycles C_1 and C_2 of G, where C_1, C_2 and the interior of P are all vertex disjoint.

Proof. For both cases, we construct walks that satisfy Condition 4 of Theorem 4.2.13.

1. Choose an Eulerian tour of G; we assign $a_e = \pm 1$ so that $a_e\,\text{sgn}(e)$ is alternating in this tour. This gives a closed alternating walk that satisfies Condition 4 of Theorem 4.2.13.
2. Consider the walk formed by walking around C_1 and C_2 once each and the path P between them twice. We now assign $w_e = \pm 1$ so that $w_e\,\text{sgn}(e)$ is alternating. Since C_1 and C_2 are alternating except where P meets C_1 and C_2, the resulting walk and weights satisfy Condition 4 of Theorem 4.2.13. Moreover, observe that for an edge e in the path P it appears two times in the closed walk, with an odd number of edges between the occurances, and thus w_e has the same sign both times it occurs in the

walk. Hence, we can choose the nonzero edge weights so $a_e = w_e$ for edges in the cycles and $a_e = 2w_e$ for edges in the path.

□

We use the nonzero edge weights from Corollary 4.2.17 in order to write elements of S_1 in several ways, using different generators. By adding multiples of the edge weights from Corollary 4.2.17, we can reduce edge weights to cases with a fewer non-integral edge weights, or a minimal structure.

4.2.3 Reductions in S_1

In this section, we use Theorem 4.2.13 and Corollary 4.2.17 to rewrite elements of S_1 in alternate ways. These reductions allow us to rewrite elements of S_1 in canonical ways in order to determine the normality of $k[G]$. In Section 4.2.4, the reductions in this section are used to complete the proof that the odd cycle condition is necessary and equivalent to the normality of $k[G]$.

Lemma 4.2.18. Let G be a signed graph, and for each edge e of G, assume a_e is a positive weight. Suppose that G contains an even cycle C. Then there is a proper subgraph H of G with fewer cycles and positive edge weights b_e so that $\sum_{e \in G} a_e \rho(e) = \sum_{e \in H} b_e \rho(e)$.

Proof. Since C has an even number of edges and an Eulerian tour, by Corollary 4.2.17, there exists weights $w_e = \pm 1$ so that $w_e \operatorname{sgn}(e)$ is alternating around C. By Condition 3 in Theorem 4.2.13, $\sum_{e \in C} w_e \rho(e) = 0$. Suppose that ij is an edge of C so that a_{ij} is minimized. Then, for each e in C add $-a_{ij} w_{ij} w_e$ to its weight, i.e., we add a scaled copy of the weights on C from above to cancel the weight on a_{ij}. Therefore, the edges e in C have weight $-a_{ij} w_{ij} w_e + a_e$ and the edges e in $G \setminus C$ have weight a_e. By the minimality of a_{ij}, all of the weights are nonnegative, and edge ij has weight 0. Let H be the subgraph of G whose edges have nonzero weights. Observe that C is not contained in H, so H is a proper subgraph with fewer cycles. Moreover, the weights on G and H differ by $\sum_{e \in C} w_e \rho(e)$, which is zero, so the given equality holds. □

Lemma 4.2.19. Let G be a signed graph, and for each edge e of G, assume a_e is a positive weight. Suppose that G contains a pair of odd cycles $\{C, C'\}$ which are not disjoint. Then, there is proper subgraph H of G with fewer cycles and positive edge weights b_e so that $\sum_{e\in G} a_e\rho(e) = \sum_{e\in H} b_e\rho(e)$.

Proof. Since C and C' are not disjoint, they either have an edge in common or a vertex in common. Suppose first that C and C' have an edge e in common. Then, we claim that $C \cup C'$ contains an even cycle. More precisely, there exist vertices i and j of C and a path P' in C' whose interior is vertex disjoint from C. There are two paths P_1 and P_2 in C from i to j; since C is an odd cycle, one of P_1 and P_2 is of odd length while the other is of even length. Therefore, one of $P_1 \cup P'$ and $P_2 \cup P'$ is an even cycle. We can apply Lemma 4.2.18 to this even cycle to find a proper subgraph H of G with the desired properties.

Suppose now that C and C' do not share an edge; therefore C and C' must share a vertex. Observe that $C \cup C'$ is connected, has an even number of edges, and each vertex has even degree, thus $C \cup C'$ has an Eulerian tour. Since $C \cup C'$ has an even number of edges and an Eulerian tour, by Corollary 4.2.17, there exists weights $w_e = \pm 1$ so that $w_e \operatorname{sgn}(e)$ is alternating on the tour of $C \cup C'$. By Condition 3 in Theorem 4.2.13, $\sum_{e\in C\cup C'} w_e\rho(e) = 0$. Suppose that ij is an edge of $C \cup C'$ so that a_{ij} is minimized. Then, for each e in $C \cup C'$ add $-a_{ij}w_{ij}w_e$ to its weight, i.e., we add a scaled copy of the weights on the Eulerian tour from above to cancel the weight on a_{ij}. Therefore, the edges e in $C \cup C'$ have weight $-a_{ij}w_{ij}w_e + a_e$ and the edges e in $G \setminus (C \cup C')$ have weight a_e. By the minimality of a_{ij}, all of the weights are nonnegative, and edge ij has weight 0. Let H be the subgraph of G whose edges have nonzero weights. Observe that either C or C' is not contained in H, so H is a proper subgraph with fewer cycles. Moreover, the weights on G and H differ by $\sum_{e\in C\cup C'} w_e\rho(e)$, which is zero, so the given equality holds. □

Lemma 4.2.20. Let G be a signed graph, and for each edge e of G, assume a_e is a positive weight. Suppose that G contains a pair of odd cycles $\{C, C'\}$ which are disjoint and are connected by a path P. Then, there is a proper subgraph H of G, with fewer cycles or more components and positive edge weights b_e so that $\sum_{e\in G} a_e\rho(e) = \sum_{e\in H} b_e\rho(e)$.

Proof. By Corollary 4.2.17, there exists nonzero weights w_e on $C \cup C' \cup P$ so that $\sum_{e\in C\cup C'\cup P} w_e\rho(e) = 0$. Let ij be the edge of C such that $\left|\frac{a_{ij}}{w_{ij}}\right|$ is minimized. Then, for each e in $C \cup C' \cup P$ add $-\frac{a_{ij}}{w_{ij}}w_e$ to its weight, i.e., we add a scaled copy of the weights from above to cancel the weight on a_{ij}. Therefore, the edges e in $C \cup C' \cup P$ have weight $-\frac{a_{ij}}{w_{ij}}w_e + a_e$ and the edges e in $G \setminus (C \cup C' \cup P)$ have weight a_e. By the minimality of a_{ij}, all of the weights are nonnegative, and edge ij has weight 0. Let H be the subgraph of G whose edges have nonzero weights. Observe that the weights on G and H differ by $\sum_{e\in C\cup C'} w_e\rho(e)$, which is zero, so the given equality holds. If ij is in C or C', then C or C' is not contained in H, so H is a proper subgraph with fewer cycles. On the other hand, if ij is in P, then either C or C' are in distinct components, or the number of distinct minimal paths between C and C' is reduced. By inducting on the number of distinct minimal paths, the result holds. □

Corollary 4.2.21. Let G be a signed graph, and for each edge e of G, assume a_e is a positive weight. Then there is a subgraph H where each component of H is a tree or a unicyclic graph with an odd cycle. Moreover, there are positive edge weights b_e on H so that $\sum_{e\in G} a_e\rho(e) = \sum_{e\in H} b_e\rho(e)$.

Proof. We proceed by inducting on the number of cycles and components of G. If G has an even cycle, we apply Lemma 4.2.18. If G has two odd cycles, then we apply either Lemma 4.2.19 or Lemma 4.2.20, whichever is appropriate. In each of these Lemmas, the proper subgraph of G has either fewer cycles or more components. The base case for this induction occurs when each component of H is a tree or a unicyclic graph. □

4.2.4 Sufficiency of Odd Cycle Condition

In this section, we use Corollary 4.2.21, to show that the odd cycle condition in the graph G is equivalent to the normality of the ring $k[G]$.

Theorem 4.2.22. Let G be a signed graph, then $k[G]$ is normal if and only if G satisfies the odd cycle condition.

Proof. The contrapositive of Theorem 4.2.9 shows that a graph which is normal must satisfy the odd cycle condition. We, therefore, prove the other direction. Suppose that G is a signed graph which satisfies the odd cycle condition. Suppose that α is in S_1; then we may write $\alpha = \sum_{e \in G} a_e \rho(e)$ where $a_e \geq 0$. Our goal is to rewrite α as a sum $\alpha = \sum_{e \in G} b_e \rho(e)$ where the b_e's are non-negative integral edge weights.

Let G' be the subgraph of G consisting of those edges where a_e is positive. Then, we apply Corollary 4.2.21 to find a graph H whose components are trees or unicyclic graphs with odd cycles. Additionally, there are positive edge weights c_e on H so that $\alpha = \sum_{e \in H} c_e \rho(e)$.

Consider $\alpha' = \sum_{e \in H}(c_e - \lfloor c_e \rfloor)\rho(e)$. Observe that α' is in $\operatorname{cone}(\mathcal{P}_G)$ as all edge weights are nonnegative. On the other hand, α and α' differ by an element of $S_2 \subseteq \mathcal{L}_G$ as $\lfloor c_e \rfloor$ is a non-negative integer. Since $\alpha \in S_1$, α is also in $\mathcal{L}_G$. Since $\mathcal{L}_G$ is a group, by closure, α' is in $\mathcal{L}_G$. Hence, $\alpha' \in S_1$. We proceed by proving that α' is in S_2 because if α' is in S_2, then $\alpha = \alpha' + \sum_{e \in H} \lfloor c_e \rfloor \rho(e)$ is also in S_2.

Let H' be the subgraph of H whose edges have nonzero weight in α'. Let $c'_e = c_e - \lfloor c_e \rfloor$ be the edge weights on H', and observe that the edges of H' have weights strictly between 0 and 1, by construction. We claim that H' is a collection of disjoint cycles. Since α' is in S_1, $\sum_{e \in H'} c'_e \rho(e)$ can be written with integral weights, for any vertex i, the sum $\sum_{j \sim i} \operatorname{sgn}(ij) c'_{ij}$ over neighbors of i must be integral. Hence, no component of H' can have any leaves because if i were a leaf of a component, then $\sum_{j \sim i} \operatorname{sgn}(ij) c'_{ij}$ would have exactly one nonzero term $\operatorname{sgn}(ik) c'_{ik}$. In this case, c'_{ij} must be integral, but this is a contradiction. Therefore, every vertex of H' is in a cycle, and, since the components of H are unicyclic, it must be that H' is a union of disjoint cycles.

Let C be a cycle in H'. Suppose that j, i, and k are vertices in C, in order. By the argument above, $\operatorname{sgn}(ij) c'_{ij} + \operatorname{sgn}(ik) c'_{ik}$ must be an integer. If $\operatorname{sgn}(ij) = -\operatorname{sgn}(ik)$, i.e., $\operatorname{sig}_C(i) = 0$, then $c'_{ij} - c'_{ik}$ must be an integer. Because c'_e is restricted to be between 0 and 1 for each edge e of C, it must be that $c'_{ij} = c'_{ik}$, and the difference is zero. If $\operatorname{sig}(ij) = \operatorname{sig}(ik)$, i.e., $\mathrm{sig}_C(i) \neq 0$, then $c'_{ij} + c'_{ik}$ must be an integer. Because c'_e is restricted to be between 0 and 1 for each edge e of C, it follows that $c'_{ij} + c'_{ik} = 1$, so that $c'_{ik} = 1 - c'_{ij}$. Fix an edge e in C; by this argument the edges of C can be partitioned into two classes: those with weight c'_e and those with weight $1 - c'_e$. In the proof of Theorem 4.2.9, we concluded that there are an odd number of vertices in C with nonzero signature. Therefore, by a parity argument, there must either be a vertex i of C with signature 0 whose incident edges have weights c'_e and $1 - c'_e$ or a vertex i of C with nonzero signature whose incident edges have the same weight. In each of these cases, we conclude that $c'_e = \frac{1}{2}$; therefore, all edges in C have weight $\frac{1}{2}$. Let α'_C be the restriction of α' to C, i.e., $\alpha'_C = \sum_{e \in C} c'_e \rho(e)$. Then,

$$x^{\alpha'_C} = \prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(\ell)},$$

which are the same types of products considered in Theorem 4.2.9. Therefore, we observe that normality of $k[G]$ is completely based on these types of products on cycles.

Let G_1 be a component of G and $\{C_1, \cdots, C_m\}$ the cycles of H' contained in G_1. We claim that the number of these cycles is even. Let $\alpha'_{G_1} = \sum_{e \in G_1} c'_e \rho(e)$ and $\alpha'_{C_\ell} = \sum_{e \in C_\ell} c'_e \rho(e)$ be the restrictions of α' to G_1 and the cycles $\{C_1, \cdots, C_m\}$. Since α' is in S_1, α'_{G_1} is also in S_1; therefore, α'_{G_1} is in $\mathcal{L}_G$, so $\alpha'_{G_1} = \sum_{e \in G_1} d_e \rho(e)$ where d_e is an integer. Since $\rho(e) = \mathrm{sgn}(e)(e_i + e_j)$ where e_i and e_j are the basis elements corresponding to the endpoints i and j of e, the sum of the coefficients of the basis vectors in α'_{G_1} is $\sum_{e \in G_1} 2 d_e \, \mathrm{sgn}(e)$, which is even. On the other hand, in the proof of Theorem 4.2.9, we concluded that there are an odd number of vertices in C with nonzero signature, so that the sum of the coefficients in α'_{C_ℓ} is odd. For the parities to match, there must be an even number of cycles of H' in G_1.

Let C and C' be any two cycles of H in the same component of G. It is sufficient to show that $\alpha'_C + \alpha'_{C'}$ is in S_2. Once this is shown, since C and C' are arbitrary and there are an even number of cycles per component of G, it follows that α' is in S_2 as it is the sum over such cycles. We now prove the claim: since C and C' are disjoint odd cycles in the same component of G, by the odd cycle condition, there is an alternating path between C and C'. Let P be an alternating path between a vertex of C and a vertex of C' which is edge disjoint from C and C'. We may extend P to a path P' whose endpoints on C and C' have nonzero signature as follows: Let i and j be the endpoints of P, where i is in C and j is in C'. If $\mathrm{sig}_C(i)$ is zero, then i is incident to an edge of each sign in C and thus we may add a path of C starting at i and going to a vertex i' in C to P so that the new path is still alternating. Since C has an odd number of edges there is a vertex with nonzero signature, and thus such a path exists. Let i' and j' be the endpoints of P', where i' is in C and j' is in C'. Assign a weight of 1 to all edges in P; by Observation 4.2.8, the intermediate terms in $\sum_{e\in P'} \mathrm{sgn}(e)$ cancel, and the result of the sum is $\mathrm{sig}_C(i')e_{i'} + \mathrm{sig}_{C'}(j')e_{j'}$. In order to show that $\alpha'_C + \alpha'_{C'}$ is in S_2, it remains to show that

$$\sum_{\ell\in C\setminus i'} \mathrm{sig}_C(\ell) + \sum_{\ell\in C'\setminus j'} \mathrm{sig}_{C'}(\ell)$$

is in S_2 as well. In terms of ring elements, this is precisely the product

$$\prod_{\ell\in C\setminus i'} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell\in C'\setminus j'} x_\ell^{\mathrm{sig}_{C'}(\ell)},$$

which appeared in Theorem 4.2.9. By Observation 4.2.10, there are weights of 1 and 0 on the edges of C and C' to achieve this product. Therefore, this product is also in $k[G]$; from here, we conclude that α is in S_2. □

Observation 4.2.23. We highlight the following observation in the proof of Theorem 4.2.22: The normality of a graph G depends entirely on the existence in $k[G]$ of products of the form

$$\prod_{\ell\in C\setminus i} x_\ell^{\mathrm{sig}_C(\ell)} \prod_{\ell\in C'\setminus j} x_\ell^{\mathrm{sig}'_C(\ell)}$$

where C and C' are odd cycles, $\mathrm{sig}_C\, i \neq 0$, and $\mathrm{sig}_{C'}(j) \neq 0$.

The characterization of the normality of toric varieties in Theorem 4.2.22 can be strengthened by describing the generators $\mathcal{A}(\mathcal{P}_G)$ over $k[G]$.

Definition 4.2.24. Let G be a signed graph. We say a pair $\Pi = \{C, C'\}$ of odd cycles in a component of G that are vertex disjoint are *exceptional* if there does not exist an alternating path connecting C and C' in G. Given an exceptional pair $\Pi = \{C, C'\}$, let

$$\frac{1}{2}\rho(\Pi) = \frac{1}{2}\sum_{\pm ij \in C} \rho(ij) + \frac{1}{2}\sum_{\pm ij \in C'} \rho(ij),$$

and

$$M_\Pi = x^{\frac{1}{2}\rho(\Pi)} = \prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(x_\ell)} \prod_{\ell \in C'} x_\ell^{\mathrm{sig}_{C'}(x_\ell)}$$

in $k[x_1, \ldots, x_n]$.

Observe that for a pair of exceptional odd cycles $\Pi = \{C, C'\}$, the monomial M_Π is precisely the monomial which appears in Theorems 4.2.9 and 4.2.22.

Corollary 4.2.25. Let G be a signed graph and $k[G]$ the edge ring of G. Let $\Pi_1 = \{C_1, C_1'\}, \ldots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs of odd cycles in G. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by the monomials $M_{\Pi_1} \ldots, M_{\Pi_q}$ as an algebra over $k[G]$.

Proof. In the proof of Theorem 4.2.22, proving that some $\alpha \in S_1$ is in S_2 amounted to proving that $\alpha'_C + \alpha'_{C'}$ is in S_2 as well for a pair of odd cycles C and C' in the same component of G (where α' are the reduced weights determined in Theorem 4.2.22). If C and C' are not an exceptional pair, then the proof of Theorem 4.2.22 shows that $\alpha'_C + \alpha'_{C'}$ is in S_2. Therefore, extending S_2 by $\alpha'_C + \alpha'_{C'}$ when C and C' are an exceptional pair extends S_2 to S_1. Since $x^{\alpha'_C + \alpha'_{C'}} = M_{\{C,C'\}}$, this extension extends $k[G]$ by M_Π where Π is exceptional. $\square$

4.3 Serre's R_1 Condition for Signed Graphs

Recall Proposition 3.3.7 gives a geometric characterization of when a semigroup ring satisfies R_1 in terms of the behavior of the lattice, and facets of the cone of an affine semigroup. To determine the facets of cone($\mathcal{P}_G$) it suffices to find the faces of $\mathcal{P}_G$ which have an additional supporting hyperplane with constant zero, i.e. a subspace that contains a face of $\mathcal{P}_G$, but not all of $\mathcal{P}_G$. In particular, note that every face of cone($\mathcal{P}_G$) contains a unique face of $\mathcal{P}_G$.

Proposition 4.3.1. For a graph G, any facet F of cone($\mathcal{P}_G$) is given by a subgraph G' of G with one more bipartite component, $L \cup R$, than G obtained by deleting positive edges incident to a vertex in L and a vertex not in R, and negative edges incident to a vertex in R, but not a vertex in L.

Proof. Every facet of cone($\mathcal{P}_G$) is contained in a supporting hyperplane, is a subspace and thus is defined by an equality $\langle v^*, x\rangle = 0$ which does not contain cone($\mathcal{P}_G$). That is, there is a subgraph G' which every edge satisfies $\langle v^*, x\rangle = 0$ and every edge in G but not in G' satisfies $\langle v^*, x\rangle > 0$. By Lemma 4.1.5 we know that v^* is a linear combination of dual vectors associated to bipartite components of G'.

Let G' be a subgraph of G that defines a facet of cone($\mathcal{P}_G$). From Theorem 4.1.19 and Lemma 4.1.5, we know that G' has exactly one more bipartite component than G, say $L \cup R$, and thus a supporting hyperplane for the facet is given by $\langle e_L^* - e_R^*, x\rangle = 0$. Observe that any edge, where the end points are not in L or R, satisfies the equation $\langle e_L^* - e_R^*, x\rangle = 0$, and any edge with one end point in L and one in R also satisfies the equation. Hence, the only edges which do not satisfy the equation are edges with exactly one end point in $L \cup R$. In order for this to be a facet we require all the deleted edges to lie in one half space, and hence all the deleted positive edges must be incident to, without loss of generality, L, and the negative edges R. □

Note that, using the notation from Proposition 3.3.7, the support form found in Proposition 4.3.1 $\sigma_{L,R}(x) = \langle e_L^* - e_R^*, x\rangle$, in fact satisfies the requirement for the existence for an integral support form, σ_F of the facet F. The integrality of $\sigma_{L,R}$ follows from $\sigma_{L,R}$ being integral on the generators of the lattice. The hyperplane $H_{L,R} = \{x \in \mathbb{R}^d \ : \ \sigma_{L,R}(x) = 0\}$ and the associated facet, $F = \text{cone}(\mathcal{P}_{G'})$ also satisfy the requirements to apply Proposition

3.3.7.

Lemma 4.3.2. For any facet defined by the subgraph G' with new bipartite component $L \cup R$ the support form $\sigma_{L,R}$, or $\frac{1}{2}\sigma_{L,R}$ satisfies the first condition of Proposition 3.3.7: there exists $x \in S_2$ so that $\sigma_F(x) = 1$ where σ_F is a support form for F taking integer values on $\mathbb{Z}S$.

Proof. Let G and G' be as stated. Suppose the edge $e = \text{sgn}(ij)ij$ is an edge in G but not in G'. By Proposition 4.3.1, we may assume, without loss of generality, that $\text{sgn}(ij) = +1$ and $i \in L$. One of two cases happens, $\sigma_{L,R}(e) = 1$ if $j \notin L$ or $\sigma_{L,R}(e) = 2$ if $j \in L$. Thus, $\sigma_{L,R}(x)$ takes on three values, 0,1 or 2.

If there is an edge e so that $\sigma_{L,R}(e) = 1$, then the support form satisfies the first condition of Proposition 3.3.7. Otherwise, the linear form $\frac{1}{2}\sigma_{L,R}$ takes on values 0,1 for the edges of G and hence satisfies the first condition of Proposition 3.3.7. □

In order to determine which subgraphs of G satisfy the second condition, we need to determine the lattice for G. In Lemma 4.3.3 and Lemma 4.3.4 we determine the lattices of bipartite and nonbipartite components.

Lemma 4.3.3. Let $G = L \cup R$ be a connected bipartite graph, then $\mathbb{Z}S_2 = \mathbb{Z}\rho(E(G))$ is the sublattice of $\mathbb{Z}^d$ where $\alpha \in \mathbb{Z}S_2$ if and only if,

$$\sum_{i \in L} \alpha_i = \sum_{j \in R} \alpha_j.$$

Proof. For any integral combination of vectors associated with edges, each edge adds the same amount to each of the summations, and hence each lies within this sublattice. Now, let $\alpha \in \mathbb{Z}^d$ be a vector so that $\sum_{i \in L} \alpha_i = \sum_{j \in R} \alpha_j$. Since G is connected there is a path any pair of vertices $i \in L$ and $j \in R$. This path has odd length since every edge has a vertex in L and a vertex in R. Assign weights of $+1$ and -1 to the edges of the path so that the product of the weights and the signs are alternating along the path. Using these weights we obtain the vector $e_i + e_j$, or $-e_i - e_j$ depending on the choice of initial weight. By using such paths we can decompose the weights of α into pairs and obtain a sum of edge weights that gives us α. $\square$

Lemma 4.3.4. Let G be a connected nonbipartite graph, then $\mathbb{Z}S_2 = \mathbb{Z}\rho(E(G))$ is the sublattice of $\mathbb{Z}^d$ where $\alpha \in \mathbb{Z}S_2$ if and only if

$$\sum_{i \in V} \alpha_i \in 2\mathbb{Z}.$$

Proof. Let G be a connected nonbipartite graph. Note that, $\mathbb{Z}S_2$ is contained in the lattice given by the equality, thus, it suffices to show that all such vectors are in $\mathbb{Z}S_2$. Since G is not bipartite, there exist two walks, one of odd length, one of even length between any two vertices allowing repetition, that is including closed walks by picking the same vertex twice. Assign $+1$ and -1 edge weights so that the product of the edge weights and the signs of the edges are alternating along the walks. Using these edge weights we can obtain ± 1 on both vertices or $+1$ on one vertex and -1 on the other up to choice of which walk, and the initial edge weight. By decomposing the vertex weights into pairs and using the appropriate walks we can obtain the vector by using integral edge weights. Thus, $\mathbb{Z}S_2$ is the even lattice. $\square$

Observation 4.3.5. Since the components of a graph G are vertex disjoint, the lattice $\mathbb{Z}S_2 = \mathbb{Z}\rho(E(G))$ can be obtained by the lattices of the components:

$$\mathbb{Z}\rho(E(G)) = \bigoplus_{H \text{ component} of G} \mathbb{Z}\rho(E(H)).$$

Recall the second condition of Proposition 3.3.7, $\mathbb{Z}(S \cap F) = \mathbb{Z}S \cap H$ where H is the supporting hyperplane of facet F, To determine when this condition is satisfied, we need to determine when these two lattices are equal. By Proposition 4.3.1, if a subgraph G' determines a facet of cone$(\mathcal{P}_G)$ then BiComp(G') = BiComp$(G) + 1$, this implies there are two cases, either the number of non-bipartite components in G' is greater than the number of non-bipartite components of G or it is not.

Lemma 4.3.6. Let G' be a subgraph of G that defines a facet of cone($\mathcal{P}_G$) with new bipartite component $L \cup R$, so that G' has more non-bipartite components than G. Then G does not satisfy Serre's R_1 condition.

Proof. Let G and G' be as above. In order for the number of non-bipartite components of G' to be greater than the number of non-bipartite components of G, G must contain a pair of odd cycles that were in the same component of G but are now in distinct components of G'. Let i and j be vertices of two such odd cycles. By Lemma 4.3.4, we know that $e_i + e_j \in \mathbb{Z}\rho(E(G))$. Since i and j are not in $L \cup R$, $\langle e_L^* - e_R^*, e_i + e_j \rangle = 0$ and, hence, the vector lies in the hyperplane of the facet. However, $e_i + e_j$ is not in $\mathbb{Z}(S_1 \cap F)$ as i and j are in distinct components, and violate Lemma 4.3.4. Hence, G does not satisfy the conditions in Proposition 3.3.7, and thus does not satisfy Serre's R_1 condition. □

Lemma 4.3.7. Let G' be a subgraph of G that defines a facet of cone($\mathcal{P}_G$) with new bipartite component $L \cup R$, so that G' has at most as many non-bipartite components as G. Then, G does satisfy the second condition in Proposition 3.3.7.

Proof. Let G and G' be as stated above. There are two cases, the bipartite component was obtained by deleting edges from a component which includes a cut, or the component was obtained by deleting edges from a non-bipartite component, but $\text{Comp}(G') = \text{Comp}(G)$.

Any set of vertex weights that are in the lattice of G' is in the lattice of G intersected with the hyperplane. Thus, it suffices to show that every set of vertex wights in the lattice of G, that are also in the hyperplane, are also in the lattice of G'. Suppose the bipartite component is $L \cup R$ and $\text{Comp}(G') = \text{Comp}(G) + 1$. In this case the support form $\sigma_{L,R}$ guarantees that $\sum_{i \in L} \alpha_i = \sum_{j \in R} \alpha_j$ for any vector $\alpha \in \mathbb{Z}\rho(E(G)) \cap H$. Thus if the other component is not bipartite, the sum of the vertex weights is even, as the sum of the vertex weights of the bipartite component is even. Thus, it is in the lattice of G'. If the other component is bipartite then, in the original lattice, the sum of the vertex weights in one part is equal to the sum of the vertex weights of the other part. Subtracting an equal amount, $\sum_{i \in L} \alpha_i = \sum_{j \in R} \alpha_j$ from this equality implies the remaining sum is equal and hence in the lattice of G'. Hence, $\mathbb{Z}(S_1 \cap F) = \mathbb{Z}S_1 \cap H$. Similarly, if the component was not bipartite and was made into a bipartite component, then use the previous case where the nonbipartite component is thought of as the empty graph. □

Putting Proposition 4.3.1, Lemma 4.3.2, Lemma 4.3.6 and Lemma 4.3.7 together gives the characterization of when a signed graph satisfies Serre's R_1 condition.

Theorem 4.3.8. Let G be a graph, then $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with a unique new bipartite component $L \cup R$ obtained from G by deleting positive edges incident to L and not to R, and negative edges incident to R and not to L satisfies:

$$\mathrm{Comp}(G') - \mathrm{BiComp}(G') \leq \mathrm{Comp}(G) - \mathrm{BiComp}(G).$$

As a consequence of this theorem, we can construct signed graphs which satisfy Serre's R_1 condition, and are not normal. By Theorem 3.3.4 this implies these edge rings fail Serre's S_2 condition.

Example 4.3.9. Consider the signed graph G with vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{+11, +12, +23, -34, +44, +26, +25 + 56\}$, see Figure 4.7. This graph fails the odd cycle condition since there are loops at 1 and 4 which do not have an alternating path between them. However, it satisfies R_1, in order to find a bipartite subgraph that increases the number of nonbipartite components when deleted, we need to include vertex 2 or 3 in the bipartite component. If vertex 3 is included then, by the sign condition for facets, 2 must be in the bipartite component as well. If vertex 2 is included then vertices 5 and 6 must be inluded as well, otherwise the number of bipartite components increases by two. However, if we include vertices 2,5 and 6 in the component, it is not bipartite. Hence, there is no such bipartite component, and the graph satisfies R_1.

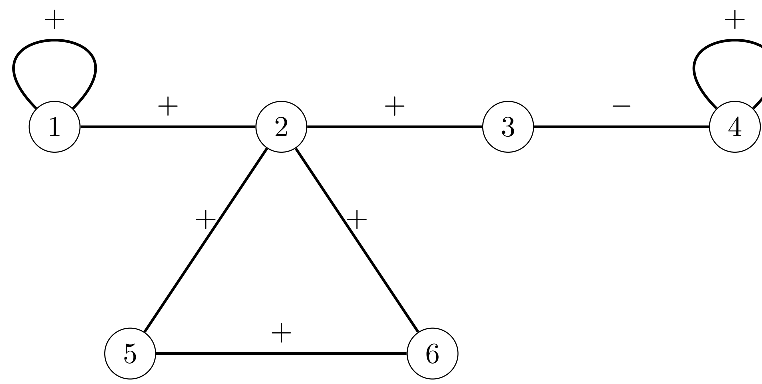


Figure 4.7: The signed graph satisfies R_1 but fails normality. Hence, this is an example of a graph which has an edge ring which is known to be not Cohen-Macaulay, as it fails S_2.

4.4 Future Directions

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_ℓ condition, for $\ell > 1$.
- Determine a combinatorial condition for when an edge ring satisfies Serre's S_ℓ condition, for all ℓ.
- Give a formula or relation determining the face complex of an edge ring, generalizing [23] which counts the number of 1-dimensional faces.

Chapter 5

Mixed Signed Directed Graphs

In this chapter, we extend the results of the previous chapter to determine when general quadratically generated domains are normal. In particular, in the previous sections, the rings are generated by elements of the form $(x_i x_j)^{\pm 1}$. In this section, we allow generators of the rings to be of the form $x_i^{\pm 1} x_j^{\pm 1}$.

Definition 5.0.1. A *mixed signed, directed graph* is a pair $G = (V, E)$ of *vertices*, V, and *edges*, E, where E consists of a set of signed edges, and *directed edges* between distinct vertex pairs. As before, we denote a positive edge between i and j as $+ij$ a negative edge between i and j as $-ij$. A directed edge from i to j is denoted (i, j).

Note that we allow, for any vertex i, positive and negative loops at i denoted $+ii$ and $-ii$ respectively, but not directed loops. Also, given a pair i and j of distinct vertices, we allow any subset of the four possible edges $+ij, -ij, (i, j)$ and (j, i) to be edges in G.

Definition 5.0.2. Let G be a mixed signed, directed graph with d vertices, possibly with loops, and multiple edges. Define $\rho : E(G) \to \mathbb{R}^d$ as $\rho(e) = \text{sgn}(e)(e_i + e_j) \in \mathbb{R}^d$ when $e = \text{sgn}(ij)ij$ is a signed edge of the graph and as $\rho(e) = e_j - e_i$ when $e = (i, j)$ a directed edge of the graph.

Using this definition for ρ, the *edge polytope of G* is defined as in Definition 2.1.12, the semigroups S_1 and S_2 are defined as in Definition 4.2.1, and the Ehrhart and edge rings

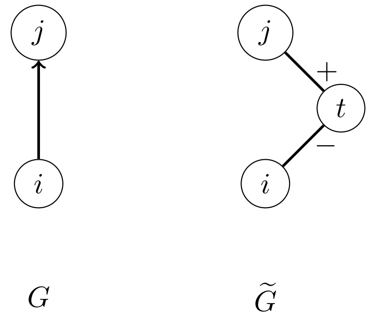


Figure 5.1: The construction of the augmented signed graph $\widetilde{G}$ from the mixed signed, directed graph G replaces directed edges (i,j) with pairs of signed edges $-it$ and $+tj$. Since $\rho((i,j)) = \rho(-it) + \rho(+tj)$, many algebraic properties, such as normality, of $k[G]$ are preserved in $k[\widetilde{G}]$.

are defined as in Definition 4.2.2.

Let G be a mixed signed, directed graph with vertices i, j, and t. Suppose that (i,j) is a directed edge of G and $-it$ and $+tj$ are signed edges of G. Observe that $\rho((i,j)) = \rho(-it) + \rho(+tj)$, see Figure 5.1. We use this equality to construct a signed graph $\widetilde{G}$ from G, on a possibly larger vertex set, such that $k[\widetilde{G}]$ and $k[G]$ have similar algebraic properties, such as normality.

Definition 5.0.3. Let G be a mixed signed, directed graph. The *augmented signed graph* $\widetilde{G}$ of G is a signed graph where each directed edge (i,j) in G is replaced by a vertex $t_{(i,j)}$ and a pair of edges $-it_{(i,j)}$ and $+t_{(i,j)}j$. The new vertex $t_{(i,j)}$, adjacent to only i and j, is called an *artificial vertex*.

By the observation above, any monomial $x^\alpha \in k[G]$ also appears in $k[\widetilde{G}]$ by replacing each use of a directed edge (i,j) with the pair $-it_{(i,j)}$ and $+t_{(i,j)}j$ from $\widetilde{G}$. Therefore, $k[G] \subseteq k[\widetilde{G}]$, and, similarly $\mathcal{A}(\mathcal{P}_G) \subseteq \mathcal{A}(\mathcal{P}_{\widetilde{G}})$. See Figure 5.2 for details on the inclusion relationships.

Lemma 5.0.4. *Let G be a mixed signed, directed graph and $\widetilde{G}$ the augmented signed graph of G. Consider $k[\widetilde{G}]$ and $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$ as subrings of $k[x_1^{\pm1}, \ldots, x_n^{\pm1}, t_1^{\pm1}, \ldots, t_m^{\pm1}]$. Then $k[\widetilde{G}] \cap k[x_1^{\pm1}, \ldots, x_n^{\pm1}] = k[G]$ and $\mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm1}, \ldots, x_n^{\pm1}] = \mathcal{A}(\mathcal{P}_G)$.*

$$\begin{array}{ccc} k[G] & \hookrightarrow & \mathcal{A}(\mathcal{P}_G) \\ \big\downarrow\, \| & & \big\downarrow\, \| \\ k[\widetilde{G}] \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] & \hookrightarrow & \mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \end{array}$$

Figure 5.2: Let G be a mixed signed, directed graph and $\widetilde{G}$ the augmented signed graph for G. The commutative diagram illustrates the inclusion relationship and equalities between $k[G]$, $k[\widetilde{G}]$, $\mathcal{A}(\mathcal{P}_G)$ and $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$. The illustrated equalities are proved in Lemma 5.0.4.

Proof. The inclusion in one direction has been discussed above. Let x^α be a monomial in $k[\widetilde{G}] \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ or $\mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Therefore, α can be represented by positive integral weights if $\alpha \in k[\widetilde{G}]$ and both positive rational weights and positive integral weights if $\alpha \in \mathcal{A}(\mathcal{P}_{\widetilde{G}})$. Let (i, j) be a directed edge of G, then $-it_{(i,j)}$ and $+t_{(i,j)}j$ have the same weight $w_{(i,j)}$ because $e_{t_{(i,j)}}$ has a zero coefficient. Since $\rho((i, j)) = \rho(-it_{(i,j)}) + \rho(+t_{(i,j)}j)$, we can assign weights on the edges of G which generate x^α: in particular, for any signed edge e of G, give e the same weight as the corresponding signed edge in $\widetilde{G}$, and, for any directed edge (i, j) in G, give (i, j) weight $w_{(i,j)}$. If the weights on $\widetilde{G}$ are positive rational, integral, or positive integral, then so are the weights on G. Therefore, α is in S_1 for both rings or S_2 for both rings. □

5.1 Normality

Theorem 5.1.1. Let G be a mixed signed, directed graph and $\widetilde{G}$ the augmented signed graph of G, then $k[G] = \mathcal{A}(\mathcal{P}_G)$ if and only if $\widetilde{G}$ satisfies the odd cycle condition.

Proof. By Theorem 4.2.22, $\widetilde{G}$ satisfies the odd cycle condition if and only if $k[\widetilde{G}]$ is normal, which occurs if and only if $k[\widetilde{G}] = \mathcal{A}(\mathcal{P}_{\widetilde{G}})$. By Lemma 5.0.4,

$$k[G] = k[\widetilde{G}] \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{A}(\mathcal{P}_G).$$

Suppose $\widetilde{G}$ does not satisfy the odd cycle condition. From Theorem 4.2.25, there is a monomial M_Π in $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$, but not in $k[\widetilde{G}]$, where Π is a pair of exceptional cycles.

Observe that if t is an artificial node in $\widetilde{G}$ then the exponent of t in M_Π is zero since in any cycle C containing t, the signature $\mathrm{sig}_C(t) = 0$. Since t is an arbitrary artificial node, $M_\Pi \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Therefore, $M_\Pi \in (\mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \setminus (k[\widetilde{G}] \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. So, by applying Lemma 5.0.4, $k[G] \neq \mathcal{A}(\mathcal{P}_G)$. □

Corollary 5.1.2. Let G be a directed graph, then $k[G]$ is normal.

Proof. Observe that the associated signed graph $\widetilde{G}$ of G is bipartite since every edge of G is replaced with a path of length two. Hence, $\widetilde{G}$ trivially satisfies the odd cycle condition since there are no odd cycles, and, by Theorem 5.1.1, $k[G] = \mathcal{A}(\mathcal{P}_G)$. Since S_1 is a normal semigroup, $\mathcal{A}(\mathcal{P}_G)$, and, hence, $k[G]$ is normal. □

Let G be a mixed signed, directed graph and $\widetilde{G}$ its associated signed graph. Theorem 5.1.1 implies that the normality of $k[G]$ depends only on odd cycles in $k[\widetilde{G}]$. Instead of constructing $\widetilde{G}$, we describe, directly, which subgraphs of G produce exceptional odd cycles in $\widetilde{G}$.

Let C be a cycle in G with k signed edges and ℓ directed edges. The corresponding cycle $\widetilde{C}$ in $\widetilde{G}$ has $k + 2\ell$ signed edges. Observe that there is a natural bijection between cycles of G and cycles of $\widetilde{G}$. Moreover, a cycle in G corresponds to an odd cycle in $\widetilde{G}$ if and only if it has an odd number of signed edges.

We can also describe alternating paths in $\widetilde{G}$. Since a directed edge in G is replaced with an alternating path of length of length two in $\widetilde{G}$, two consecutive directed edges are alternating if and only if they are in the same direction. Moreover, since the directed edge (i,j) is replaced by $-it_{(i,j)}$ and $+t_{(i,j)}j$ in $\widetilde{G}$, a signed edge incident to i in the path must have positive sign, and a signed edge incident to j in the path must have negative sign.

In order to determine which of these cycles in G produce exceptional odd cycles in $\widetilde{G}$, we also determine which subgraphs of G produce alternating paths in $\widetilde{G}$. Recall that each directed edge in G is replaced with an alternating path of length two in $\widetilde{G}$. So, a path of directed edges $(i_0, 1_1), (i_1, i_2) \ldots, (i_{n-1}, i_n)$ gives an alternating path in G, so long as they are directed in the same direction. Similarly, signed edges $\mathrm{sgn}(ai_0)ai_0$ and $\mathrm{sgn}(i_n b)$ on the ends of this directed path then must satisfy $\text{sgn}(ai_0) = +1$ and $\text{sgn}(i_n b) = -1$, as in $\widetilde{G}$ the path corresponds to $+ai_0, -i_0t_1, +t_11_1, \ldots, -i_{n-1}t_n, +t_ni_n, -i_nb$. This gives the definition of a generalized alternating path, and the augmented odd cycle condition.

Definition 5.1.3. Suppose P is a sequence of incident edges in G with vertex set $\{i_0, i_1, \ldots, i_n\}$. We say P is a *generalized alternating path* if:

- the subsequence of signed edges obtained by deleting the directed edges alternates,
- if (i_{a-1}, i_a) is a directed edge, then $\text{sgn}(i_{a-2}i_{a-1}) = +1$ and $\text{sgn}(i_ai_{a+1}) = -1$ for the signed edges $i_{a-2}i_{a-1}$ or i_ai_{a+1} if they exist.
- if (i_{a-1}, i_a) and (i_b, i_{b-1}) are directed edges going in opposite directions in P then there are an odd number of signed edges between them in the sequence,
- if (i_{a-1}, i_a) and (i_{b-1}, i_b) are directed edges going in the same direction in P then there are an even number of signed edges between then in the sequence.

Definition 5.1.4. Let G be a mixed directed signed graph, we say that G satisfies the *generalized odd cycle condition* if for every pair of cycles C and C' of G where C and C' both have an odd number of signed edges, at least one of the following occurs:

1. C and C' are in distinct components,
2. C and C' have a vertex in common,
3. C and C' have a generalized alternating i, j-path in G, where i is a vertex of C and j is a vertex of C'.

Using this definition for the generalized odd cycle condition, we have the following corollary of Theorem 5.1.1:

Corollary 5.1.5. Let G be a mixed signed, directed graph. $k[G]$ is normal if and only if G satisfies the generalized odd cycle condition.

Proof. Since S_1 is the smallest normal semigroup containing S_2, $k[G]$ is normal if and only if $k[G] = \mathcal{A}(\mathcal{P}_G)$. By Theorem 5.1.1, $k[G] = \mathcal{A}(\mathcal{P}_G)$ if and only if $\widetilde{G}$ satisfies the odd cycle condition, which is equivalent to the generalized odd cycle condition on G. □

We can also compute the normalization of $k[G]$ from the normalization of $k[\widetilde{G}]$:

Theorem 5.1.6. Let G be a mixed signed, directed graph, and $\widetilde{G}$ the associated augmented signed graph. If the normalization of $k[\widetilde{G}]$, $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$, is generated by monomials $M_{\Pi_1}, \ldots, M_{\Pi_m}$, over $k[\widetilde{G}]$, then the normalization of $k[G]$, $\mathcal{A}(\mathcal{P}_G)$, is generated by $M_{\Pi_1}, \ldots, M_{\Pi_m}$ over $k[G]$.

Proof. Let (i, j) be a directed edge in G and $t_{(i,j)}$ the corresponding artificial vertex of $\widetilde{G}$. In any cycle C that contains $t_{(i,j)}$, the signature $\operatorname{sig}_C(t_{(i,j)})$ is zero because the two incident edges to $t_{(i,j)}$ have different signs. Therefore, each M_{Π_ℓ} is in $\mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{A}(\mathcal{P}_G)$.

Since $k[\widetilde{G}]$ and $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$ are generated by monomials as an algebra over k and the M_{π_ℓ}'s are also monomials, it follows that (1) it is enough to study the monomials of $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$ and (2) no cancellation is necessary to generate elements of $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$ from elements of $k[\widetilde{G}]$. More precisely, by Theorem 4.2.25, if x^α is a monomial in $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$, then there is some monomial x^β in $k[\widetilde{G}]$ so that $x^\alpha = x^\beta \prod_{p=1}^{r} M_{\Pi_{\ell_p}}$. Moreover, if $x^\alpha \in \mathcal{A}(\mathcal{P}_{\widetilde{G}}) \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then $x^\beta \in k[\widetilde{G}] \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ because each $M_{\Pi_{\ell_p}}$ is in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. This implies that $\mathcal{A}(\mathcal{P}_G)$ is generated by $M_{\Pi_1}, \ldots, M_{\Pi_m}$ as an algebra over $k[G]$. □

5.2 Serre's R_1 Condition for Mixed Signed Directed Graphs

Our study of Serre's R_1 condition for mixed signed directed graphs begins by determining the facets of the cone $\operatorname{cone}(\mathcal{P}_G)$. Similar to Lemma 4.1.5 and Proposition 4.3.1, we determine which dual vectors v^* satisfy $\langle v^*, x\rangle = 0$ for all $x \in \rho(E(G))$.

Lemma 5.2.1. Let G be a mixed signed graph, with bipartite components $\widetilde{G_1}, \ldots, \widetilde{G_r}$ in $\widetilde{G}$ the augmented signed graph. Let $e^*_{L_1} - e^*_{R_1}, \ldots, e^*_{L_r} - e^*_{R_r}$ be the associated dual vectors for $\widetilde{G_1}, \ldots, \widetilde{G_r}$. Suppose $\mathcal{P}_G$ is contained by the hyperplane defined by $\langle v^*, x\rangle = 0$, then v^* is a linear combination of $\pi(e^*_{L_1} - e^*_{R_1}), \ldots, \pi(e^*_{L_k} - e^*_{R_k})$, where $\pi(u^*)$ is the projection of the vector u^* onto the subspace defined by the vertices of G.

Proof. We determine the hyperplanes which contain $\mathcal{P}_G$ by starting with a subgraph of G which is a signed graph, and determine which of these hyperplanes can also contain the directed edges. Using this subgraph and the augmented signed graph we determine which linear combinations of the hyperplanes contain the original graph. We construct the signed graph G' by deleting the directed edges from G. By Lemma 4.1.5 we know that v^* is a linear combination of the bipartite components of G'. Denote the bipartite components of G' by $G'_1, \ldots, G'_s$ and the associated dual vectors $e_{L'_1} - e_{R'_1}, \ldots, e_{L'_s} - e_{R'_s}$. Note that each bipartite component of $\widetilde{G}$ corresponds to at least one bipartite component of G'. Write $v^* = \sum_{i=1}^{s} a_i(e_{L'_i} - e'_{R_i})^*$. Consider a directed edge (u, v) in G, by assumption $\langle v^*, e_v - e_u\rangle = 0$, thus, if u and v are in the same component, say G'_j, in G' then $u, v \in L'_j$ or $u, v \in R'_j$. Similarly, if $u \in G'_j$ and $v \in G'_k$, $i \neq j$, in G' then $u \in L'_j$ and $v \in L'_k$ or $u \in R'_j$ and $v \in R'_k$ as well as $a_j = a_k$.

In either of these cases, we obtain a linear combination of bipartite components in $\widetilde{G}$, as the artificial vertices are in the other part of the bipartition. This gives the desired characterization of the vector. $\square$

Now that the supporting hyperplanes of the form $\{x \in \mathbb{R}^d \ : \ \langle v^*, x\rangle = 0\}$ for the cone $\mathrm{cone}(\mathcal{P}_G)$ have been determined, we use this characterization to find the facets and dimension of $\mathrm{cone}(\mathcal{P}_G)$.

Proposition 5.2.2. Let G be a mixed signed directed graph on n vertices, then

$$\dim \mathrm{cone}(\mathcal{P}_G) = n - \mathrm{BiComp}(\widetilde{G}).$$

Proof. We compute the dimension by finding a maximal linearly independent set of hyperplanes that contain $\mathrm{cone}(\mathcal{P}_G)$. Let $\mathcal{H}$ be the set of hyperplanes given by $\langle e^*_L - e^*_R, x\rangle = 0$

for each bipartite component of $\widetilde{G}$. By Lemma 5.2.1, if the hyperplane $\{x \in \mathbb{R}^d : \langle v^*, x\rangle = 0\}$ contains $\mathcal{P}_G$, then v^* can be written as a linear combination of the vectors which define hyperplanes in $\mathcal{H}$, thus $\mathcal{H}$ is a maximal set of independent hyperplanes. Thus, $\dim(\text{cone}(\mathcal{P}_G)) = n - \#\mathcal{H} = n - \text{BiComp}(\widetilde{G})$. □

We now use the formula for the dimension to determine the facets of cone($\mathcal{P}_G$).

Proposition 5.2.3. For a mixed signed directed graph G, any facet F of cone($\mathcal{P}_G$) is given by a subgraph H of G with one more bipartite component, $L \cup R$ in $\widetilde{H}$, than in $\widetilde{G}$ obtained by deleting positive edges incident to vertices in L and vertices not in R, negative edges incident to vertices in R but not vertices in L, and directed edges (i, j) where $j \in L$, or $i \in R$.

Proof. We know that any facet F of cone($\mathcal{P}_G$) has a support form $\langle v^*, x\rangle$ which does not contain cone($\mathcal{P}_G$). From Lemma 5.2.1, we know that v^* is a linear combination of $e^*_{L_1} - e^*_{R_1}, \dots, e^*_{L_r} - e^*_{R_r}$ where $L_i \cup R_i$ are the bipartite components of $\widetilde{G}$. Thus, any facet corresponds to a subgraph H of G so that $\widetilde{H}$ has one more bipartite component than $\widetilde{G}$.

From Proposition 4.3.1, we know that any deleted positive edge in $\widetilde{G}$ is incident to L, and any negative edge is incident to R. Deleting a directed edge in G is equivalent to deleting a pair of incident edges, one positive one negative in $\widetilde{G}$. Thus, the positive edges incident to a vertex in L and a vertex not in R, and negative edges incident to a vertex in R but not a vertex in L, and directed edges (i, j) where $j \in L$, $i \in R$ or both. □

Note that the support form $\sigma_{L,R}(x) = \langle e^*_L - e^*_R, x\rangle$ is integral on $\rho(E(G))$. If a deleted edge or directed edge e is incident to exactly one vertex in $L \cup R$, then $\sigma_{L,R}(e) = 1$. If the edge or directed edge is incident to two vertices in $L \cup R$ then $\sigma_{L,R}(e) = 2$, thus by using $\sigma_{L,R}$ or $\frac{1}{2}\sigma_{L,R}$, as appropriate, we obtain the integral support form for the facet where $x \in \rho(E(G))$ so that $\sigma_{L,R}(x) = 1$.

Proposition 5.2.4. For a connected mixed directed signed graph G, $a \in \mathbb{Z}\rho(E(G))$ if and only if

$$\sum_{i \in V(G)} a_i \in 2\mathbb{Z}$$

if $\widetilde{G}$ is not bipartite and

$$\sum_{i \in L} a_i = \sum_{j \in R} a_j$$

if $\widetilde{G} = L \cup R$ is a bipartite graph with biparition L and R.

Proof. To determine the lattice $\mathbb{Z}\rho(E(G))$, we observe that by Lemma 5.0.4 we know that $\mathbb{Z}\rho(E(G)) = \mathbb{Z}\rho(E(\widetilde{G})) \cap \{t_e = 0 \ : \ e \text{ a directed edge}\}$. Thus, it suffices to determine the lattice of $\widetilde{G}$. By Lemma 4.3.3 and Lemma 4.3.4 we get the desired result. □

By Observation 4.3.5, we know that the lattice of a mixed signed directed graph is the direct sum of the lattices of the components. Similarly, the proofs for Lemma 4.3.6 and Lemma 4.3.7 generalize naturally to the context of mixed signed directed graphs. Thus, we can construct the condition for a mixed signed directed graph to satisfy Serre's R_1 condition.

Theorem 5.2.5. Let G be a graph. $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with a unique new bipartite component $L \cup R$ in $\widetilde{G'}$ obtained from G by deleting positive edges incident to L and not to R, negative edges incident to R and not to L and directed edges (i, j) where $j \in L$ or $i \in R$ satisfies:

$$\text{Comp}(\widetilde{G'}) - \text{BiComp}(\widetilde{G'}) \leq \text{Comp}(\widetilde{G}) - \text{BiComp}(\widetilde{G}).$$

Proof. The intuition behind the proof is based on Lemma 4.3.6 and Lemma 4.3.7, and will have one case related to each of these lemmas.

Case 1: let G' be a subgraph of G that defines a facet of cone$(\mathcal{P}_G)$ with new bipartite component $L \cup R$, so that $\widetilde{G'}$ has more non-bipartite components than $\widetilde{G}$. The only way we can have increased the number of non bipartite components is if there were at least two cycles that were in the same component of G, but are now in distinct components of G', which have odd length in $\widetilde{G'}$. Let i and j be vertices of two such odd cycles. Note that by

Proposition 5.2.4, we know that $e_i + e_j \in \mathbb{Z}\rho(E(G))$. Similarly, i and j are not in $L \cup R$ and, hence, $\langle e_L^* - e_R^*, e_i + e_j \rangle = 0$. However, $e_i + e_j$ is not in $\mathbb{Z}(S_1 \cap F)$ as each is in a distinct component. Hence, G does not satisfy the conditions in Proposition 3.3.7, and thus does not satisfy Serre's R_1 condition.

Case 2: let G' be a subgraph of G that defines a facet of cone($\mathcal{P}_G$) with new bipartite component $L \cup R$, so that $\widetilde{G'}$ has at most as many non-bipartite components as $\widetilde{G}$. Suppose the bipartite component is $L \cup R$ and $\text{Comp}(\widetilde{G'}) = \text{Comp}(\widetilde{G}) + 1$. In this case, the support form $\sigma_{L,R}$ guarantees that $\sum_{i \in L} \alpha_i = \sum_{j \in R} \alpha_j$ for any vector $\alpha \in \mathbb{Z}\rho(E(G)) \cap H$. Thus the bipartite or non-bipartite nature of the other component does not depend on $L \cup R$, and since the other component has even summed degree weights or is bipartite, $\alpha \in \mathbb{Z}(S_1 \cap F)$. Hence, $\mathbb{Z}(S_1 \cap F) = \mathbb{Z}S_1 \cap H$. Similarly, if the component was not bipartite and was made into a bipartite component, then the support form $\sigma_{L,R}$ guarantees that the two vertex sums are equal. □

Example 5.2.6. Consider the signed graph G with vertex set $\{1, 2, 3, 4, 5\}$ and edge set $\{+11, +12, (3, 2), +33, +24, +25, +45\}$, see Figure 5.3. Note that the augmented signed graph of G is given in Figure 4.7, which fails the odd cycle condition. However, it satisfies Serre's R_1 condition as shown in Example 4.3.9. Thus, since G is not normal, by Theorem 4.2.22, and normality is equivalent to satisfying Serre's R_1 and S_2 conditions, we know that G fails S_2. By the definition of Cohen-Macaulay, G fails S_2 implies G is not Cohen-Macaulay.

5.3 Future Directions

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_ℓ condition, for $\ell > 1$.
- Determine a combinatorial condition for when an edge ring satisfies Serre's S_ℓ condition, for all ℓ.
- Characterize the subgraphs which give facets of the edge polytope.

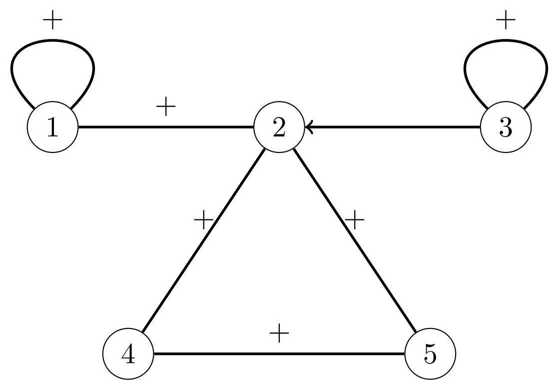


Figure 5.3: The mixed signed directed graph satisfies R_1 but fails normality. Hence, this is an example of a graph which has an edge ring which is known to be not Cohen-Macaulay, as it fails S_2. Note that the augmented signed graph is the graph given in Figure 4.7.

- Find the algorithmic complexity of determining normality, and computing the normalization. Similar problems are NP-hard.

Chapter 6

Homogenized Directed Graphs

6.1 Introduction

In this chapter, we construct normal domains from finite directed graphs, possibly with loops, in a different manner than expressed in Chapter 5. The normal domains that are constructed are graded subalgebras of the Laurent polynomial ring $k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}, s]$, graded with respect to the exponent of s. We do this in several steps: First, we construct an integral polytope $\mathcal{P}_G$ from the directed graph G. We explore the geometry of $\mathcal{P}_G$, in particular characterizing the dimension, hyperplanes that contain $\mathcal{P}_G$, and the facets of $\mathcal{P}_G$. Second, we construct a semigroup from the cone projected through $\mathcal{P}_G$. Using the semigroup and the methodology outlined by Bruns and Herzog in [3], we construct a normal domain $\mathcal{A}(\mathcal{P}_G)$. We characterize a finite generating set of $\mathcal{A}(\mathcal{P}_G)$ in terms of combinatorial behavior on the arcs of G.

6.2 Geometric Results

Recall, we allow directed cycles of length two. That is, (i, j) and (j, i) can both be arcs in G. However, G does not have multiple copies of any arc.

The map, discussed in Chapter 5, from a directed graph G to $\mathbb{R}^d$, is given by $\rho : (i, j) \mapsto e_j - e_i$. This is the map of arcs to their columns in the incident matrix of G.

However, under this map, directed paths are equivalent to arcs. That is, if (i,j), (j,k), and (i,k) are arcs in G then $(e_j - e_i) + (e_k - e_j) = e_k - e_i$, which is the image of the arc (i,k). We avoid the duplication of generators by using the map G to $\mathbb{R}^{d+1}$, where the extra dimension counts the number of arcs in a subgraph. We denote the characteristic vector of this extra dimension as e_s. Using this we now map arc (i,j) to $e_j - e_i + e_s$.

Observe that, geometrically, this map will map the arcs to the hyperplane $\{x \in \mathbb{R}^{d+1} : \langle e_s^*, x\rangle = 1\}$, which is not a subspace. This implies that the facets of the polytope will be facets of the cone projected through the polytope, since no ray intersects the polytope twice. If the polytope is on a subspace we do not have this property.

From an algebraic perspective, the reason that cancellation can occur is that the resulting polynomial ring would not have a grading. The above example illustrates this as, $(x_i^{-1}x_j)(x_j^{-1}x_k) = x_i^{-1}x_k$. That is, the product of two elements in the same level gives a third element in the same level. This can be extended arbitrarily by using longer directed paths starting at i and ending at k. However, adding the extra dimension gives a grading variable, and hence gives a graded algebra.

Definition 6.2.1. Let G be a directed graph with d vertices, possibly with loops, but no multiple arcs. Define the $\rho : E(G) \to \mathbb{R}^{d+1}$ operator as $\rho((i,j)) = e_j - e_i + e_s \in \mathbb{R}^{d+1}$ where (i,j) is an arc of the graph. For simplicity of notation $\rho((ij))$ will be written $\rho(i,j)$. Note that in the case of a loop $i = j$ and $\rho(i,i) = e_s$. Define the *arc polytope of* G as $\mathcal{P}_G := \text{conv}\{\rho(E(G))\}$. Here $\rho(E(G))$ is the image $\rho(E(G)) = \{\rho(i,j) : (i,j) \in E(G)\}$. That is, it is the set of arcs of G with the ρ operator applied to them.

A natural question to ask is "what does such a polytope look like?"

Example 6.2.2. Let G be a directed graph on vertex set $\{1,2,3\}$ and arc set $\{11,12,23,31\}$. Then,

$$\mathcal{P}_G = \text{conv}\{\rho(E(G))\} = \text{conv}\{(0,0,0,1), (-1,1,0,1), (0,-1,1,1), (1,0,-1,1)\}.$$

Observe that $\rho(1,1) = \frac{1}{3}(\rho(1,2) + \rho(2,3) + \rho(3,1))$. This is a convex combination so $\rho(1,1)$

is not an extremal point of $\mathcal{P}_G$. See Figure 6.1.

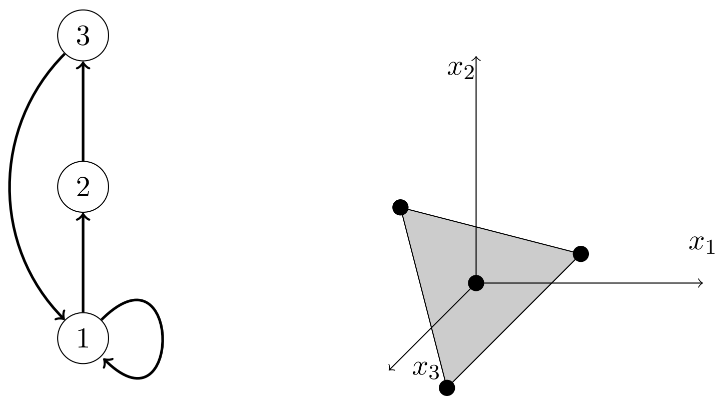


Figure 6.1: The graph G and the polytope $\mathcal{P}_G$. The polytope is a subset of the hyperplane defined by $\langle e_s^*, x\rangle = 1$. Observe that the point $(0,0,0,1)$ is a convex combination of the points $(-1,1,0,1)$,$(0,-1,1,1)$, and $(1,0,-1,1)$.

This example naturally gives rise to the question: "what are the extremal points of $\mathcal{P}_G$?" From Example 6.2.2, we can see that having more than one directed cycle could produce a non-extremal point. We think of loops as directed cycles of length one. Observe that if (i,i) and (j,j) are both loops in G then $\rho(i,i) = \rho(j,j) = e_s$. Thus, for the geometry of $\mathcal{P}_G$, we assume that G has at most one loop.

Definition 6.2.3. We say a directed graph G is *reduced* if G has no loops or G has exactly one directed cycle.

There are three classes of reduced directed graphs. In particular, G is reduced if and only if G satisfies one of:

- G has no directed cycles,
- G has a loop (i,i) such that $G \setminus \{(i,i)\}$ has no directed cycles,
- G has at least one directed cycle, but no loops.

Example 6.2.4. Consider the following directed graphs, see Figure 6.2:

- Let G_1 be the graph on vertex set $\{1, 2, 3\}$ and arc set $\{(1, 2), (2, 3), (1, 3)\}$. This graph does not contain a directed cycle and hence is reduced.

- Let G_2 be the graph on vertex set $\{1, 2, 3\}$ and arc set $\{(1, 2), (2, 3), (1, 3), (1, 1)\}$. This graph contains a loop, $(1, 1)$, however $G_2 \setminus \{(1, 1)\} = G_1$ does not have a directed cycle.

- Let G_3 be the graph on vertex set $\{1, 2, 3\}$ and arc set $\{(1, 2), (2, 3), (3, 1)\}$. This graph has a directed cycle but does not contain a loop.

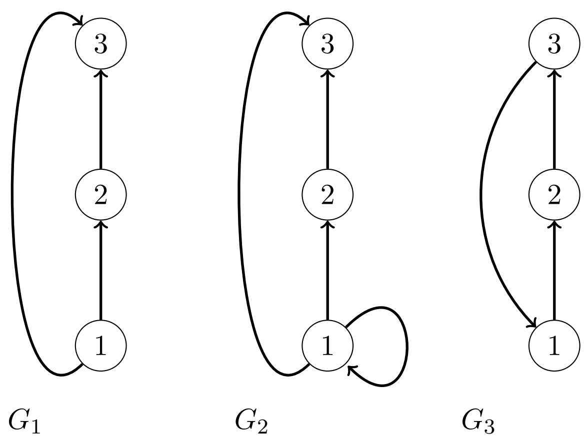


Figure 6.2: Graphs G_1, G_2 and G_3 demonstrate the three forms of reduced graphs. G_1 has no directed cycles, G_2 only has a directed loop, and G_3 has a directed cycle but no directed loop.

Proposition 6.2.5. Every arc of G gives an extremal point if and only if G is a reduced directed graph.

Proof. Suppose (i, j) is an arc in G. Then, the only way to have a convex combination of elements of $\rho(E(G))$ giving $\rho(i, j)$ is if every arc has an end point at i and an end point at j. That is, $\rho(i, j) = \sum_{e \in E(G)} a_e \rho(e)$, where $\sum_{e \in E(G)} a_e = 1$, $a_e \geq 0$ implies $\sum_{k\ :\ (i,k) \in E} a_{ik} = 1$ and $\sum_{k\ :\ (k,j) \in E} a_{kj} = 1$. Thus, if $a_e \neq 0$, then $e = (i, j)$ or $e = (j, i)$. Since $a_{ji}\rho(j, i)$ can not give $\rho(i,j)$, $\rho(i,j)$ is an extremal point for all (i,j). Thus, an arc (i,j) with $i \neq j$ will always be an extremal point.

Now, suppose (i,i) is a loop in G. Suppose $\rho(i,i) = e_s = \sum_{e \in E(G)} a_e \rho(e)$ is a convex combination of elements of $\rho(E(G))$. This equality holds if and only if for every arc (j,k) in G with $a_{jk} \neq 0$, there is an arc of the form (k,l) with $a_{kl} \neq 0$. Since, $(e_s)_k$ is zero, if $a_{jk} > 0$ then there must be at least one more edge incident to k with negative weight. Since there are a finite number of arcs this implies there must be a directed cycle that is not (i,i), and hence G is not reduced. Thus, $\rho(i,i)$ is not an extremal point of $\mathcal{P}_G$ if and only if there is another directed cycle in G. Hence, every arc of G gives an extremal point if and only if G is a reduced directed graph. □

Observe that if G is not reduced, then G has a loop (i,i) and $\mathcal{P}_G = \mathcal{P}_{G \setminus \{(i,i)\}}$. That is, the polytope given by G is equal to the polytope given by $G \setminus \{(i,i)\}$ the graph with the loop at i deleted. Thus, for the geometry of $\mathcal{P}_G$, it suffices to consider reduced graphs, as we may always construct a reduced graph from a graph G.

Assumption 6.2.6. In the rest of this section, we assume that the graphs are reduced.

Now that we have characterized the extremal points of $\mathcal{P}_G$ in terms of the arcs of G, we study the dimension of $\mathcal{P}_G$. We calculate the dimension of $\mathcal{P}_G$ by determining the hyperplanes that contain $\mathcal{P}_G$. Observe that for any graph G, $\mathcal{P}_G \subset \{x \in \mathbb{R}^{d+1} : \langle e_s^*, x \rangle = 1\}$. Thus, we are interested in hyperplanes where the normal vector is not e_s^*. Given a directed graph G and a hyperplane of the form $\mathcal{H} := \{x \in \mathbb{R}^{d+1} : \langle a^*, x \rangle = c\}$ for some vector $a \neq 0$ and $a \neq e_s$, so that $\mathcal{P}_G \subset \mathcal{H}$, there are two cases, $c = 0$ and $c \neq 0$.

Let $c = 0$; since $a \neq 0^*, e_s^*$ there is some coordinate i so that $a_i \neq 0$. Suppose there is a vertex j so that (i,j) is an arc of G, then $a_j - a_i = 0$ as $\rho(i,j) \in \mathcal{H}$ and hence $a_j = a_i$. Similarly, if (j,i) is an arc in G, then $a_j = a_i$. Thus, if $i \neq 0$ then for every j in the same (weakly) connected component as i will have $a_j = a_i$.

Definition 6.2.7. Let G be a finite directed graph. A *component* of G is the subgraph associated with a component of the underlying undirected graph. Let $\mathrm{Comp}(G)$ be the number of components of G.

Let $c \neq 0$ and $a \neq 0$ then there is some coordinate i so that $a_i \neq 0$. Recall that each coordinate, besides s, is associated with a vertex in G. Suppose there is a vertex j so that (i,j) is an arc of G, then $a_j - a_i = c$ as $\rho(i,j) \in \mathcal{H}$ which is true if and only if $\langle a^*, e_j - e_i \rangle = c$ and hence $a_j = a_i + c$. Similarly, if (j,i) is an arc in G, then $a_j = a_i - c$. Now, suppose we have a, possibly undirected, cycle C in G so that i is a vertex of C, then the vertex weights must be consistent. That is, if the cycle has vertex sequence $i_0, i_1, \ldots, i_{n+1} = i_0$ then $a_{i_j} = a_{i_{j+1}} \pm c$ for all $i = 0, \ldots, n$. So, given an orientation of C and a vertex sequence, the number of arcs oriented from vertices with smaller index to larger index is equal to the number oriented from vertices with larger index to smaller index. Notice that, since $c \neq 0$, every arc (i,j) has $a_i \neq 0$, $a_j \neq 0$, or both. Also observe that the vector a is not constant on any of the components of G since the existance of an arc (i,j) implies $a_i \neq a_j$.

Definition 6.2.8. We say an orientation of a cycle, C, with vertex sequence $\{i_0, \ldots, i_{n+1} = i_0\}$ has *signature* n if $n = |\#\{j : (i_j, i_{j+1}) \in E(C)\} - \#\{j : (i_{j+1}, i_j) \in E(C)\}|$. Denote the signature of C as $sig(C)$. We say C is *balanced* if $sig(C) = 0$. We say a graph G is *balanced* if every cycle in G is balanced.

Example 6.2.9. Let C be the cycle on vertex set $V(C) = \{1,2,3,4,5\}$ with arc set $E(C) = \{(1,2),(3,2),(3,4),(5,4),(1,5)\}$. Then $sig(C) = |\#\{j : (i_j, i_{j+1}) \in E(C)\} - \#\{j : (i_{j+1}, i_j) \in E(C)\}| = |\#\{(1,2),(3,4),(1,5)\} - \#\{(3,2),(5,4)\}| = 1$. Thus, C is not balanced and so $\mathcal{P}_C$ is not contained in a hyperplane of the form $\{x \in \mathbb{R}^{d+1} : \langle a^*, x \rangle = c\}$ for $c \neq 0$. See Example 6.3.

There are several ways to view the balanced property. One that seems particularly useful is to view the balanced property in terms of graph distance. Graph distance from a vertex u to a vertex v is the length of the shortest directed path from u to v. If the graph is balanced then all directed paths from u to v have the same length.

Note that, if G is balanced we can explicitly construct a hyperplane by, for each component, choosing an arbitrary vertex i and setting $a_i^* = 0$, and then constructing the

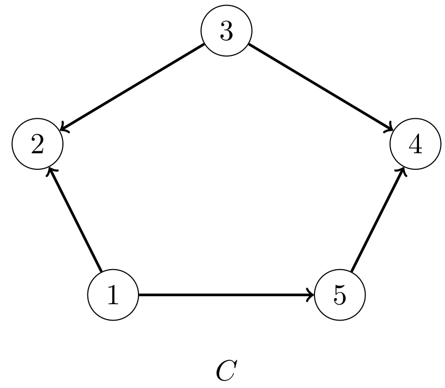


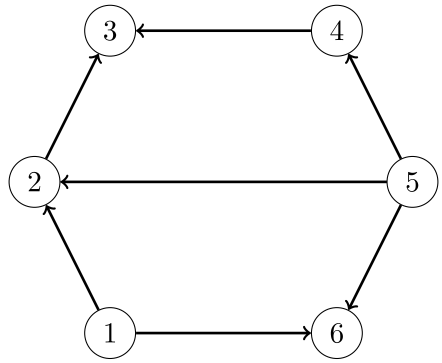
Figure 6.3: Cycle C is an cycle with signature $\text{sig}(C) = 1$. Observe that traversing the vertices in order gives one more backwards arc than forwards arcs. Thus, the signature is 1.

rest of the vertex weights so the graph distance from any vertex u to any vertex v (assuming there is a directed path from u to v) is $a_u^* - a_v^*$. We refer to the resulting vector of vertex weights as a_B^*. Note that if there is another vector $(a'_B)^*$ that also satisfies this property but $(a'_i)^* = c \neq 0$, then since the distances have not been changed, for all u in the same component as i, $(a'_u)^* = a_u^* + c$. Hence, $(a'_B)^* = (a_B + ce_{G_i})$ where G_i is the component of G containing i, and $e_{G_i}^*$ is the dual characteristic vector of G_i. Thus, we can assume a_B is independent of choice of i.

Example 6.2.10. Let G be the graph on vertex set $\{1, 2, 3, 4, 5, 6\}$ and arc set $\{(1, 2), (2, 3), (4, 3), (5, 4), (5, 6), (1, 6), (5, 2)\}$ see Figure 6.4. This graph has three cycles $\{(1, 2), (5, 2), (5, 6), (1, 6)\}$, $\{(2, 3), (4, 3), (5, 4), (5, 2)\}$ and $\{(1, 2), (2, 3), (4, 3), (5, 4), (5, 6), (1, 6)\}$. Since all three cycles are balanced, G is balanced. The associated hyperplane has $a_B^* = (a_1, a_2, a_3, a_4, a_5, a_6) = (0, 1, 2, 1, 0, 1)$.

Lemma 6.2.11. Let G be a directed graph with k components. Then, there are $k + 1$ orthogonal hyperplanes containing $\mathcal{P}_G$.

Proof. Let $G_1, \ldots, G_k$ be the components of G. Let e_{G_i} represent the characteristic vector of G_i. Then, $\mathcal{P}_G \subseteq \{x \in \mathbb{R}^{d+1} : \langle e_{G_i}^*, x\rangle = 0\}$ for all i. As each of these hyperplanes satisfy



G

Figure 6.4: G is a balanced graph. Observe thaT every cycle is balanced.

$\langle e^*_{G_j}, e_{G_i}\rangle = 0$ for all i and j, they are orthogonal. The final hyperplane is the hyperplane $\{x \in \mathbb{R}^{d+1} : \langle e^*_s, x\rangle = 1\}$. Which also satisfies $\langle e^*_s, e_{G_i}\rangle = 0$. □

This gives that $\dim \mathcal{P}_G \leq d - \mathrm{Comp}(G)$. The only other possible hyperplane that contains the polytope is the hyperplane that comes from a directed graph being balanced.

Definition 6.2.12. Let $\delta_{Bal}(G) = 1$ if G is balanced and $\delta_{Bal}(G) = 0$ otherwise.

Proposition 6.2.13. Let G be a directed graph with $k = \mathrm{Comp}(G)$, on d vertices. Then,

$$\dim \mathcal{P}_G = d + 1 - k - \delta_{Bal}(G).$$

Proof. Let G be a finite directed graph with $k = \mathrm{Comp}(G)$ on d vertices. We embed $\mathcal{P}_G$ in $\mathbb{R}^{d+2}$ by the map $(a_1, \ldots, a_d, 1) \mapsto (a_1, \ldots, a_d, 1, 1)$. Let H be a maximal spanning forest of G. Observe that $\mathcal{P}_H \subseteq \mathcal{P}_G$ and hence $\dim \mathcal{P}_H \leq \dim \mathcal{P}_G$.

Now, let $\mathcal{A}_H$ be the affine spane of $\mathcal{P}_H$ in $\mathbb{R}^{d+2}$, similarly let $\mathcal{A}_G$ be the affine span of $\mathcal{P}_G$ in $\mathbb{R}^{d+2}$. That is, $\mathcal{A}_H$ is the set of all linear combinations of elements of $\mathcal{P}_H$ in $\mathbb{R}^{d+2}$, so that the sum of the coefficients is one. Similarly for $\mathcal{A}_G$. Observe thatdim $\mathcal{P}_G = \dim \mathcal{A}_G - 1$ and $\dim \mathcal{P}_G = \dim \mathcal{A}_H - 1$. Let $\mathcal{A}_H^\perp$ and $\mathcal{A}_G^\perp$ represent the orthogonal complements of $\mathcal{A}_H$ and $\mathcal{A}_G$ respectively. Observe that no arc in H can be written as a linear combination of the other arcs. This is due to the fact that no leaf can be written as a linear combination of the other arcs. If there were such a combination the subgraph induced by the elements of the linear combination would have leaves, and hence a leaf could be written as a linear combination of the other arcs. So, $\mathcal{P}_H$ is a simplex.

From Lemma 6.2.11, we know that $\mathcal{A}_H^\perp$ has the orthogonal basis $\{(e_{G_1}, 0), \ldots, (e_{G_k}, 0), (e_s, 1)\}$ where $(e_{G_i}, 0)$ represents the hyperplane $\{x \in \mathbb{R}^{d+1} : \langle e_{G_i}^*, x\rangle = 0\}$, in addition, since H is a tree it is balanced, and hence $(a_B, 1)$ is also an element of $\mathcal{A}_H^\perp$. Since a_B is not constant on any of the components of G, $(a_B, 1)$ can not be written as a linear combination of $\{(e_{G_1}, 0), \ldots, (e_{G_k}, 0), (e_s, 1)\}$. Thus, $\dim \mathcal{A}_H^\perp \geq k + 2$. Since $\dim \mathcal{A}_H + \dim \mathcal{A}_H^\perp = d + 2$, we know that $\dim \mathcal{A}_H^\perp = k + 2$. We iteratively add edges to H, determine when $\mathcal{A}_H^\perp$ changes, and compute a new basis. Adding arcs to H does not change the dimension of $\mathcal{P}_H$ unless one of the hyperplanes are no longer satisfied. This can only happen if we add an arc that forms a cycle that is not balanced, since we do not decrease the number of components. Hence, $\dim \mathcal{A}_G^\perp = k + 1 + \delta_{Bal}(G)$, which implies $\dim \mathcal{P}_G + 1 = (d + 2) - (k + 1 + \delta_{Bal}(G)) = d + 1 - k - \delta_{Bal}(G)$, as desired. □

As a consequence of having a basis for $\mathcal{A}_G^\perp$, we know that any hyperplane containing $\mathcal{P}_G$ is a linear combination of the hyperplanes found above.

Example 6.2.14. Let G be the graph on vertex set $\{1, 2, 3, 4, 5, 6\}$ and arc set $\{(1, 2), (2, 3), (4, 3), (5, 4), (5, 6), (1, 6), (5, 2)\}$, see Figure 6.4. Observe that G is connected, and hence $\mathrm{Comp}(G) = 1$ moreover G is balanced, see Example 6.2.10. Hence, $\dim \mathcal{P}_G = 6 + 1 - 1 - 1 = 5$.

Having characterized the dimension of $\mathcal{P}_G$ in terms of properties of G, we characterize the facets of $\mathcal{P}_G$ in terms of subgraphs of G. To do this, we analyze when H a subgraph of G has $\dim \mathcal{P}_H = \dim \mathcal{P}_G - 1$. Note that Proposition 6.2.13 gives three ways to produce a subgraph H with smaller dimension: H can have fewer vertices, more components, or if G is not balanced, H can be a balanced subgraph. Note, to get $\dim \mathcal{P}_H = \dim \mathcal{P}_G - 1$, exactly one of these three conditions holds.

First, consider the case where we delete a single vertex u. This subgraph must have $\mathrm{Comp}(G) = \mathrm{Comp}(G \setminus u)$, as well as $\delta_{Bal}(G) = \delta_{Bal}(G \setminus u)$, otherwise $\dim \mathcal{P}_{G\setminus u} < \dim \mathcal{P}_G - 1$. Since $\mathrm{Comp}(G) = \mathrm{Comp}(G\setminus u)$ and $\delta_{Bal}(G) = \delta_{Bal}(G\setminus u)$, the only hyperplane containing $\mathcal{P}_{G\setminus u}$ that does not contain $\mathcal{P}_G$ is the hyperplane associated to the component that contained u in G. Let G_i be the component of G that contains u. The hyperplane $\{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i\setminus u}, x\rangle = 0\}$ is the hyperplane that contains $\mathcal{P}_{G\setminus u}$ and not $\mathcal{P}_G$. Now, suppose we have vertices v and w so that (u, v) and (w, u) are arcs in G. These arcs are not contained in the hyperplane, however, $\langle e^*_{G_i\setminus u}, \rho(u, v),\rangle = 1$ and $\langle e^*_{G_i\setminus u}, \rho(w, u)\rangle = -1$. Since the hyperplane associated to $G_i \setminus u$ is $\{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i\setminus u}, x\rangle = 0\}$ the arcs (u, v) and (w, u) imply that this hyperplane is in fact not a supporting hyperplane. If we choose u so that all the arcs are coming in or all the arcs are going out, then we get a facet.

Definition 6.2.15. Let u be in component G_i, define the hyperplane $\mathcal{H}_u = \{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i} - e^*_u, x\rangle = 0\}$, where e^*_u is the dual characteristic vector for u. The associated half spaces to H_u are $\mathcal{H}^{(+)}_u = \{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i\setminus u}, x\rangle \geq 0\}$, and $\mathcal{H}^{(-)}_u = \{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i\setminus u}, x\rangle \leq 0\}$.

Definition 6.2.16. Let G be a directed graph. Let $\deg^- u = \#\{(u, v) : (u, v) \in E(G)\}$, and $\deg^+ u = \#\{(v, u)0 : (v, u) \in E(G)\}$.

Definition 6.2.17. Let S be a minimal cut set of arcs in G. We say S is *worthwhile* if $\mathrm{Comp}(G \setminus S) = \mathrm{Comp}(G) + 1$, and if $S \subseteq G_i$ is a component of G so that $G_i = G'_i \cup G''_i \cup S$ for components G'_i and G''_i in $G \setminus S$, then every arc in S has head in G'_i and tail in G''_i.

Observation 6.2.18. Let u be a vertex with $\deg^- u = 0$ or $\deg^+ u = 0$. $\mathcal{P}_G \subset \mathcal{H}^{(-)}_u$ if and only if $\deg^- u = 0$. $\mathcal{P}_G \subset \mathcal{H}^{(+)}_u$ if and only if $\deg^+ u = 0$.

Thus,

$$\mathcal{P}_G \subseteq \left(\bigcap_u \mathcal{H}^{(-)}_u\right) \cap \left(\bigcap_v \mathcal{H}^{(+)}_v\right),$$

where, u ranges over all vertices where the incident arcs form a worthwhile set with $\deg^- u = 0$ and v ranges over all worthwhile vertices where the incident arcs form a worthwhile set with $\deg^+ v = 0$.

However, as mentioned above, the facets found by deleting a vertex are not all of the facets of $\mathcal{P}_G$. We still need to characterize the facets found by increasing the number of components and by changing $\delta_{Bal}(G)$.

In particular, we need to increase the number of components of G without deleting a vertex of G. That is, we need a cut set of arcs in G. Note that this is similar to deleting a single vertex in that, in that case, the incident arcs form a cut set. Similar to the case studying vertices, not all cut sets define a facet. If the cut set separates G_i into two new components G_i' and G_i'' and there are arcs (u, v) and (r, s) so that $u, s \in V(G_i')$ and $v, r \in V(G_i'')$, then the new hyperplanes will not define a face of $\mathcal{P}_G$, for the same reason we required $\deg^- u = 0$ or $\deg^+ u = 0$ one arc would give a positive value, and the other a negative value. In particular, letting $G_i' = \{u\}$ and $u = s$ would imply that $\mathcal{P}_{G\setminus u}$ is not a facet of $\mathcal{P}_G$. So, we need a cut set with all the arcs going from one new component to the other new component.

We define the half spaces containing $\mathcal{P}_G$ in terms of the behavior at G_i'.

Definition 6.2.19. Let S be a worthwhile cut set for G_i, a component of G, so that S cuts G_i into components G_i' and G_i'' of $G \setminus S$. Let G_i' be the component with all the tails of S and G_i'' the component with all the heads of S.

So that for every arc in S the head is in G_i'. Then let,

$$\mathcal{H}_S := \{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i'}, x\rangle = 0\}$$

and,

$$\mathcal{H}_S^{(-)} := \{x \in \mathbb{R}^{d+1} : \langle e^*_{G_i'}, x\rangle \leq 0\}.$$

As before, $\mathcal{P}_G \subset \mathcal{H}_S^{(-)}$ for S a worthwhile set.

Example 6.2.20. Let G be the graph on vertex set $\{1, 2, 3, 4, 5, 6\}$ and arc set $\{(1, 2), (2, 3),$ $(4, 3), (5, 4), (5, 6), (1, 6), (5, 2)\}$ see Figure 6.4. Let $S = \{(1, 2), (1, 6)\}$ and $S' = \{(2, 3),$ $(5, 4)\}$. Both of these are worthwhile sets. The associated dual vector for worthwhile set S is $(a_1, a_2, a_3, a_4, a_5, a_6)^* = (1, 0, 0, 0, 0, 0)$. For S' the associated dual vector is

$(a_1, a_2, a_3, a_4, a_5, a_6)^* = (0, 0, 1, 1, 0, 0)$.

Observation 6.2.21. Let G be a finite directed graph then:

$$\mathcal{P}_G \subseteq \bigcup_S \mathcal{H}_S^{(-)},$$

where S runs over all worthwhile arc sets in G.

Note that this generalizes deleting a single vertex where the incident arcs form a worthwhile set. This can be seen by setting $G_i' = \{u\}$ and $G_i'' = G_i \setminus u$ or $G_i'' = \{u\}$ and $G_i'' = G_i \setminus u$, as appropriate. Hence, we only consider cut sets rather than deleting vertices.

The only remaining case is when a graph is not balanced but the subgraph is balanced. Note that given a balanced directed graph G, the associated underlying graph is bipartite. Thus, to construct a balanced subgraph, we begin with a maximal spanning forest.

We want to characterize when a balanced subgraph defines a facet of $\mathcal{P}_G$. The hyperplane for a balanced subgraph is $\mathcal{H} := \{x \in \mathbb{R}^{d+1} : \langle a_B^*, x\rangle = 1\}$ for some vector of vertex weights a_B. Let (i, j) be an arc in G but not in the balancing subgraph. Then, either $\langle a_B^*, \rho(i, j)\rangle > 1$ or $\langle a_B^*, \rho(i, j)\rangle < 1$. In order for us to produce a facet of the polytope $\mathcal{P}_G$, we need every arc not in H to satisfy $\langle a_B^*, \rho(i, j)\rangle > 1$, or every arc not in H satisfies $\langle a_B^*, \rho(i, j)\rangle < 1$.

Definition 6.2.22. Let G be an unbalanced finite, directed graph. We say a subgraph B of G is *balancing* if it is a maximal spanning subgraph that is balanced, and every arc (i, j) not in B satisfies $\langle a_B^*, \rho(i, j)\rangle > 1$ or every arc (i, j) not in H satisfies $\langle a_B^*, \rho(i, j)\rangle < 1$. Here the associated supporting hyperplane is $\mathcal{H}_B := \{x \in \mathbb{R}^{d+1} : \langle a_B^*, x\rangle = 1\}$, for the vector a_B.

Example 6.2.23. Let G be the graph on vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and arc set $E(G) = \{(1, 2), (3, 2), (3, 4), (4, 5), (5, 1), (5, 2)\}$ see Figure 6.5. Observe that G has a three cycle $\{(1, 2), (5, 2), (5, 1)\}$ and hence is not balanced. However, the subgraph H on the same vertex set, with arc set $\{(1,2),(3,2),(3,4),(4,5)\}$, is balanced as it has no cycles. In fact it is balancing. The associated dual vector is $a^* = (a_1, a_2, a_3, a_4, a_5)^* = (0,1,0,1,2)$. The arcs in G but not H are $\{(5,1),(5,2)\}$ which has $\langle a^*, \rho(5,1)\rangle = -2$ and $\langle a^*, \rho(5,2)\rangle = -1$.

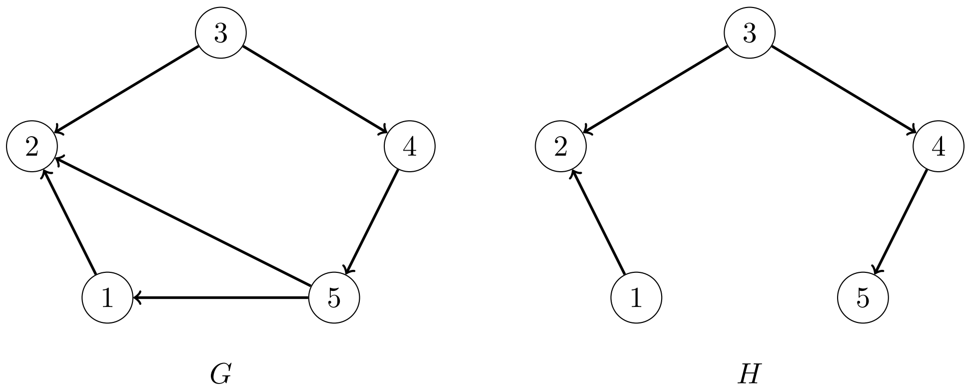


Figure 6.5: G is not a balanced graph since the underlying graph is not bipartite, and the odd cycle $\{1,2,5\}$ is not balanced. H is a maximal balanced subgraph of G. In fact H is a balancing subgraph of G.

Observe that, by Proposition 6.2.13, we have characterized exactly when we have a subgraph that gives a polytope with smaller dimension and, in fact, gives a facet of the polytope.

Theorem 6.2.24. Let G be a finite graph on d vertices and no trivial components, $\mathcal{P}_G$ the associated edge polytope, and $\mathcal{H}$ a hyperplane in $\mathbb{R}^{d+1}$. If $\mathcal{H} \cap \mathcal{P}_G$ is a facet of $\mathcal{P}_G$, then one of the following must be true:

- for some balanced subgraph B, $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_B \cap \mathcal{P}_G$, or
- for some worthwhile set S of arcs, $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_S \cap \mathcal{P}_G$.

Proof. Let H be the subgraph of G induced by $\mathcal{H}$ on vertex set $V(G)$. That is, $(i,j) \in E(H)$ if and only if $\rho(i,j) \in \mathcal{H}$. Note that we can assume $V(H) = V(G)$, since deleting a vertex and deleting all the incident arcs give the same polytope up to embedding in $\mathbb{R}^{d+1}$. Observe that $\mathcal{P}_H = \mathcal{H} \cap \mathcal{P}_G$. Since $\mathcal{P}_H$ is a facet of $\mathcal{P}_G$, $\dim \mathcal{P}_H = \dim \mathcal{P}_G - 1 = d - \text{Comp}(G) - \delta_{Bal}(G)$. Since the number of vertices remains the same, either $\text{Comp}(H) = \text{Comp}(G) + 1$

or $\delta_{Bal}(H) = 1$ and $\delta_{Bal}(G) = 0$. In the former case: after $\mathcal{P}_H$ is given by a worthwhile cut set S hence $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_S \cap \mathcal{P}_G$. In the later case: then H is balanced and G is not, and hence H is a balancing subgraph and $\mathcal{H} \cap \mathcal{P}_G = \mathcal{H}_H \cap \mathcal{P}_G$. □

6.3 Algebraic Results

Let G be a directed graph, possibly with loops, on d vertices. We construct two rings from the edge polytope $\mathcal{P}_G$. To do this we begin by defining the finitely generated lattice $\mathcal{L}_G$, to be the lattice in $\mathbb{Z}^d$ generated by $\rho(E(G))$. The normal domain $\mathcal{A}(\mathcal{P}_G)$ is constructed from cross sections of cone$(\mathcal{P}_G) \cap \mathcal{L}_G$. The second domain, $k[G]$, is constructed from $\rho(E(G))$ directly. We then characterize when these domains are equal, and normal. In the event that $k[G]$ is not normal we provide the normalization.

Similar to the previous chapter, the polytope $n\mathcal{P}_G$ corresponds to the scaling of $\mathcal{P}_G$ on to the hyperplane $\{x \in \mathbb{R}^{d+1} : \langle e_s^*, x\rangle = n\}$ from the origin. Letting n range over $\mathbb{N}$ gives the cross sections of the cone at natural numbers.

We now consider the affine semigroup, $S_1 :=$ cone$(\mathcal{P}_G) \cap \mathcal{L}_G$. Observe that this is, in fact, an affine semigroup, any two points $\alpha, \beta \in$ cone$(\mathcal{P}_G) \cap \mathcal{L}_G$ satisfy $\alpha + \beta \in$ cone$(\mathcal{P}_G) \cap \mathcal{L}_G$ since $\mathcal{L}_G$ is a group and adding two positive linear combinations together gives a positive linear combinations so this is a semigroup. It is an affine semigroup since it contains the origin.

There are several properties of S_1 that we are interested in, for example S_1 is a normal semigroup. To see this observe that S_1 as defined is the intersection between $\mathcal{L}_G$ a finitely generated subgroup of $\mathbb{Q}^{d+1}$, and cone$(\mathcal{P}_G)$ a rational cone defined by a finite number of half space intersections. Hence, by Proposition 2.3.7, S_1 is normal.

Combinatorially, the points in S_1 correspond to integer vertex weights equal to the signed sum of incident arc weights. In particular, for any point in S_1, there is a set of non-negative arc weights and a set of integral arc weights that give the point.

As before, we are interested in characterizing a finite generating set for S_1. We can construct the semigroup using the points in $\rho(E(G))$ as generators. Let,

$$S_2 := \left\{ x \in \mathbb{Z}^{d+1} : \sum_{(i,j)\in E(G)} a_{ij}\rho(i,j),\ a_{ij} \in \mathbb{N}\ \forall (i,j) \in E(G) \right\}.$$

Points in S_2 can be thought of as a set of vertex weights equal to the signed sum of incident arc weights, where the arc weights are non-negative integer weights. Clearly, S_2 is a subsemigroup of S_1 as the generators of S_2 are contained in S_1.

Now, we define the associated integral domains for these two semigroups. We use the notation from Bruns and Herzog [3], where $k[S_i]$ is the polynomial subring of $k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}, s]$ generated by monomials $x_1^{a_1} \cdots x_d^{a_d} s^n$ for each $(a_1, \ldots, a_d, n)$ in S_i. Let $x^a s^n$ represent the monomial $x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} s^n$ for $a = (a_1, \ldots, a_d)$.

Definition 6.3.1. Let $\mathcal{A}(\mathcal{P}_G)_n$ be the vector space over k which is spanned by the monomials of the form $x^a s^n$ such that $(a_1, \ldots, a_d, n) \in n\mathcal{P}_G \cap \mathbb{Z}^d$, where $n\mathcal{P}_G$ is the set of points α in $\mathbb{R}^{d+1}$ such that $\langle e_s^*, \frac{\alpha}{n} \rangle = 1$ and $\frac{\alpha}{n} \in \mathcal{P}_G$.

Definition 6.3.2. Let G be a graph S_1, and S_2 as defined above, then we define:

- the *Ehrhart* Polynomial ring:

$$k[S_1] = \mathcal{A}(\mathcal{P}_G) = \bigoplus_{n=0}^{\infty} \mathcal{A}(\mathcal{P}_G)_n,$$

- the *Edge ring of G*:

$$k[S_2] = k[G] = \langle x^a s^n : (a, n) \in \rho(E(G)) \rangle,$$

Here, $k[G]$ is generated as a subring of $k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}, s]$.

From our work with undirected graphs, we see that Theorem 4.2.13 played an important roll in understanding the structure of graphs that are associated to normal domains. Thus, we construct a similar result for directed graphs.

Theorem 6.3.3. Let G be a directed graph and for each arc (i,j) in G let a_{ij} be an integral arc weight. The following are equivalent:

- $\prod_{(i,j)\in E(G)}(x_i^{-1}x_js)^{a_{ij}} = 1$,
- $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ for all vertices i and $\sum_{e\in E(G)} a_e = 0$,
- There is a multiset of closed, signed, directed walks in G (reversing the negative arcs gives a closed directed walk) so that the signed sum of occurrences of arc (i,j) gives a_{ij} and the total signed sum of the arcs is zero.

Proof. First, we show $\prod_{(i,j)\in E(G)}(x_i^{-1}x_js)^{a_{ij}} = 1$ if and only if $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ for all vertices i and $\sum_{e\in E(G)} a_e = 0$. Second we show $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ for all vertices i and $\sum_{e\in E(G)} a_e = 0$ if and only if there is a multiset of closed, signed, directed walks in G so that the signed sum of occurrences of arc (i,j) gives a_{ij} and the total signed sum of the arcs is zero.

Notice that $\prod_{(i,j)\in E(G)}(x_i^{-1}x_js)^{a_{ij}} = 1$ holds if and only if for each $i \in V(G)$ $\prod_{(i,j)\in E(G)} x_i^{-a_{ij}} \cdot \prod_{(j,i)\in E(G)} x_i^{a_{ji}} = 1$, and $\prod_{e\in E(G)} s^{a_e} = 1$. This is equivalent to saying for all i, $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ and $\sum_{e\in E(G)} a_e = 0$.

Now, assume that the arc weights satisfy the second condition. We construct the multiset of walks iteratively. This is done by constructing a directed, up to reversing for negative weights, walk and decreasing the size of each weight by an arc weight on the walk. This decreases the number of arcs with non-zero weight each iteration. Consider the graph of arcs ij so that $a_{ij} \neq 0$, $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ implies that the non-isolated vertices have two incident arcs, allowing for (i,j) and (j,i). Moreover, if (i,j) has positive weight then there is an arc (j,k) with positive weight or there is an ark (k,j) with negative weight. If (i,j) has negative weight then there is an arc (k,j) with positive weight or there is an arc (j,k) with negative weight. Therefore there is a directed cycle C in this graph, going forwards on the positive arcs and backwards on the negative arcs. Note that since we are allowing multiple edges $V(C) = \{i,j\}$ is possible. Let (u,v) be an arc of C with $\min\{|a_{uv}| \ : \ a_{uv} \in E(C)\}$. Add $|a_{uv}|$ copies of the walk C to the multiset and for every arc (i,j) in C set $a'_{ij} = a_{ij} - |a_{uv}|$ if a_{ij} is positive and in C, $a'_{ij} = a_{ij} + |a_{uv}|$ if a_{ij} is negative and in C, and $a'_{ij} = a_{ij}$ if ij is not in C'. Observe that $a'_{uv} = 0$, and that $\sum_{j:(j,i)\in E(G)} a'_{ji} = \sum_{j:(i,j)\in E(G)} a'_{ij}$, has been maintained. Repeat on the new set of arc weights until all the arc weights are zero. When we terminate we will have the signed sum of the appearances of the arc (i,j) is equal to a_{ij} and the total signed sum of the arcs is zero since $\sum_{(i,j)\in E(G)} a_{ij} = 0$.

Now assume there is a multiset of closed, signed, directed walks in G so that the signed sum of occurrences of arc (i,j) gives a_{ij} and the total signed sum of the arcs is zero. Observe that each walk satisfies $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$, for all i and hence the sum over all the walks satisfies $\sum_{j:(j,i)\in E(G)} a_{ji} = \sum_{j:(i,j)\in E(G)} a_{ij}$ for all i. Moreover, since the total signed sum of the arcs is zero, $\sum_{(i,j)\in E(G)} a_{ij} = 0$. □

Example 6.3.4. Let G be a graph on vertex set $V(G) = \{u,v,w,x,y,z\}$ and arc set $E(G) = \{(u,v),(v,w),(w,x),(y,x),(y,u),(x,z),(y,z)\}$. The closed walk $\{(u,v),(v,w),(w,x),(y,x),(y,u)\}$ has three forwards arcs $\{(u,v),(v,w),(w,x)\}$ and two backwards arcs $\{(y,x),(y,u)\}$ see Figure 6.6. The closed walk $\{(y,x),(x,z),(y,z)\}$ has one forward arc $\{(y,z)\}$ and two backwards arcs $\{(y,x),(x,z)\}$. Together two two walks give a multiset that satisfies the third condition of Theorem 6.3.3. The associated arc weights are:

$(a_{uv}, a_{vw}, a_{wx}, a_{yx}, a_{yu}, a_{xz}, a_{yz}) = (1,1,1,-2,-1,-1,1)$.

We now discuss when monomials of the form s^n appear in $k[G]$, and in $\mathcal{A}(\mathcal{P}_G)$. A monomial $x_i^{-1}x_js^n$ appears in $k[G]$ if and only if there is a set of non-negative, integral, arc weights on $E(G)$ which is a directed weighted walk from i to j of weight one, uses exactly n arcs, and has no other sources and sinks. That is, for all vertices $u \in E(G)$ with $u \neq i, j$, $\sum_{v:(v,u)\in E(G)} a_{vu} = \sum_{v:(u,v)\in E(G)} a_{uv}$. For i we have $\sum_{u:(u,i)\in E(G)} a_{ui}+1 = \sum_{u:(i,u)\in E(G)} a_{iu}$, and for j we have $\sum_{u:(u,j)\in E(G)} a_{uj} = \sum_{u:(j,u)\in E(G)} a_{ju} - 1$. In the language of the previous theorem, there is a path from i to j and a multiset of closed walks so that the number of arcs used in the path and walks sums to n.

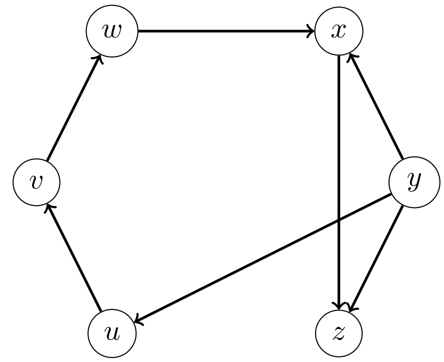


Figure 6.6: The graph G with closed walks $\{(u,v),(v,w),(w,x),(y,x),(y,u)\}$, and $\{(y,x),(x,z),(y,z)\}$ satisfy the conditions in Theorem 6.3.3. Therefore, there are two products of arc weights that give the monomial 1 in the edge ring.

Definition 6.3.5. Let G be a directed graph and a a set of arc weights. The arc weights are a *circulation* if for each vertex u $\sum_{(u,v)} a_{uv} = \sum_{(v,u)} a_{vu}$. That is, if the sum of the weights on the arcs with head at u is equal to the sum of the weights on the arcs with tail at u.

This implies that if a monomial of the form s^n is in $k[G]$, then then every vertex u has $\sum_{v:(v,u)\in E(G)} a_{vu} = \sum_{v:(u,v)\in E(G)} a_{uv}$. That is, the arc weights $\{a_{ij}\}_{(i,j)\in E(G)}$ form a circulation of G. Circulations can be decomposed into directed cycles as follows: take the arc of G with the minimum weight, since every vertex has in and out arcs with weight greater than this value, we can form a directed walk which can be written as a union of directed cycles. Subtract the weight from each of the arcs of the cycle and repeat. That is, s^n is in $k[G]$ implies there is at least one directed cycle in G of length at most n. As a consequence, if s^n is in $\mathcal{A}(\mathcal{P}_G)$, then there is a directed cycle in G. This follows from the definition of $\mathcal{A}(\mathcal{P}_G)$ being the intersection of a lattice and a cone. If there is s^n in $\mathcal{A}(\mathcal{P}_G)$, then $n \cdot e_s$ is in cone$(\mathcal{P}_G)$, which implies that there is a non-negative combination of arc weights that gives $n \cdot e_s$. Using the same argument as above this implies that there is at least one directed cycle. Now, we characterize when s^n is in $\mathcal{A}(\mathcal{P}_G)$ but not in $k[G]$.

Suppose s^n is in $\mathcal{A}(\mathcal{P}_G)$ but not $k[G]$, and s^m is in $k[G]$. Then s^{m-n} will be in the fraction field of $k[G]$. To have a better understanding of when s^n is in $\mathcal{A}(\mathcal{P}_G)$ we, thus, determine the values of n so that s^n is in the fraction field of $k[G]$.

Observe that for any cycle C, we have $s^{sig(C)}$ is in the fraction field of $k[G]$. Suppose, $V(C) = \{u_0, u_1, \ldots, u_n = u_0\}$ is the vertex sequence for C and suppose $sig(C) = \#\{(u_i, u_{i+1}) : (u_i, u_{i+1}) \in C\} - \#\{(u_{i+1}, u_i) : (u_{i+1}, u_i) \in C\}$, note that this always holds up to relabeling of the vertices. Then assign $+1$ to each arc (u_i, u_{i+1}) in C and -1 to each arc (u_{i+1}, u_i) in C. The associated monomial will be $s^{sig(C)}$.

This implies that the smallest n so that s^n is in fraction field of $k[G]$ will be determined by the signatures of the cycles. Let G be a graph that is not balanced and let $g = \gcd\{sig(C) : \text{C is a cycle of G}\}$, then g is the smallest positive integer so that s^g is in the fraction field. Moreover, if $s^g \in k[G]$, then there is a directed cycle in G of length g, otherwise there is an $s^n \in k[G]$ so that n is smaller than g, which is a contradiction to the definition of g. We will first prove a technical lemma.

Lemma 6.3.6. Let $\{a_{ij}\}_{(i,j)\in E(G)}$ be a set of non-negative arc weights so that for each vertex i, $b_i = \sum_{j:(j,i)\in E(G)} a_{ji} - \sum_{j:(i,j)\in E(G)} a_{ij}$ is an integer. Suppose the arcs with positive weights form a forest or a unicyclic graph with a directed cycle, then there exists a set of arc weights which are integral and give the same vertex weights.

Proof. Let H be the subgraph of G induced by the arcs with non-zero weight. Suppose H is a forest. It suffices to consider the subgraph H' of H with non-integral weights on the arcs. Suppose H' has at least one arc. As H' is a forest, there is a vertex i with degree 1. That is, i is incident to only one arc in H'. If b_i is positive then this arc must be (j, i) for some vertex j. Hence, $a_{ji} = b_i$ and so a_{ji} is an integer. Similarly, if b_i is negative then the arc must be (i, j) for some vertex j. Hence, $-a_{ij} = b_i$ and so a_{ij} is an integer. This contradicts the choice of H', thus, H' has no arcs. Since all arcs with positive weights are integral all the weights are integral.

Similarly, if H is a unicyclic graph all the non-cycle arcs have integral weight. All that remains to be shown is that the weights on the cycle arcs are integral. Let the cycle be C with arcs $\{(c_1, c_2), \ldots, (c_{n-1}, c_n), (c_n, c_1)\}$. Consider the arc (c_1, c_2). If this arc has integral weight, then (c_2, c_3) has integral weight as the difference must be integral. Repeating this argument gives all the arcs have integral weight. If the arc (c_1, c_2) has non-integral weight, let $m = a_{c_1c_2} - \lfloor a_{c_1c_2} \rfloor$, then the arc (c_2, c_3) has $m = a_{c_2c_3} - \lfloor a_{c_2c_3} \rfloor$. This is due to the difference of the two arcs gives the value b_{c_2}, which is an integer. Repeating this argument gives every arc in the cycle has m as the non-integral part. Subtracting m from each arc in the cycle gives the same vertex weights but now with integral weights on the all the arcs. □

Now we are ready to characterize when $k[G]$ is equal to $\mathcal{A}(\mathcal{P}_G)$ for graphs G that contain a directed cycle.

Theorem 6.3.7. Let G be a directed graph. Let $g = 0$ if G is balanced and $g = \gcd\{sig(C) : C \text{ is a cycle of } G\}$ otherwise. If $s^g \in k[G]$ then $k[G]$ is normal.

Proof. Let $\{a_{ij}\}_{(i,j)\in E(G)}$ be a set of non-negative arc weights so that for each vertex i, $b_i = \sum_{j:(j,i)\in E(G)} a_{ji} - \sum_{j:(i,j)\in E(G)} a_{ij}$ is an integer. If all the weights are integral we are done. Suppose that the arc weights are not all integral. By Lemma 6.3.6, we know that there is a cycle C in G that has positive weights on each arc. Suppose $V(C) = \{u_0, u_1, \ldots, u_n = u_0\}$ and $sig(C) = c = \#\{(u_i, u_{i+1}) : (u_i, u_{i+1}) \in C\} - \#\{(u_{i+1}, u_i) : (u_{i+1}, u_i) \in C\} \neq 0$. Then, by the definition of g, there is a positive integer h so that $gh = c$. Let C' denote the cycle in G with the minimum number of edges. Since $s^g \in k[G]$ we have $sig(C') = g$. Orient C, with the arcs $\{(u_i, u_{i+1}) : (u_i, u_{i+1}) \in C\}$ positive and $\{(u_{i+1}, u_i) : (u_{i+1}, u_i) \in C\}$ negative. Observe that the multiset of closed walks containing the orientation of C and h copies of C' each arc forwards satisfies the third condition of Theorem 6.3.3. Let m be the minimum weight of the forwards arcs of C. Subtract m from the weight of each forward arc in C, add m to each backwards arc in C, and add $m \cdot h$ to the weight of each arc in C'. While the resulting vertex weights are unchanged, C no longer has all positive arc weights.

Now, suppose $sig(C) = 0$. Then, the cycle C satisfies the third condition of Theorem 6.3.3. Let m be the minimum arc weight of the forward arcs of C. Subtract m from each forward arc and add m to each backwards arc of C. The resulting vertex weights are unchanged, as before, and C no longer has all positive arc weights.

Repeat for all cycles $C \neq C'$ in H. Apply Lemma 6.3.6 to the resulting arc weights to get the arc weights are all non-negative integers. Hence, the monomial associated to vertex weights b_i for all i is in $k[G]$. Thus, $k[G]$ is normal. $\square$

Note that if G is a balanced graph, then $g = 0$ and hence, $s^g = 1$ which is in $k[G]$. Thus, if G is balanced, then $k[G]$ is normal. If G contains a directed cycle then we can prove a characterization of the integral closure of $k[G]$.

Corollary 6.3.8. Let G be a directed graph with a directed cycle, and $g = \gcd\{sig(C) :$ C is a cycle of G$\}$. Then, $k[G]$ is normal if and only if $s^g \in k[G]$.

Proof. From Theorem 6.3.7, we know that if $s^g \in k[G]$, then $k[G]$ is normal. Hence, it suffices to show that if $s^g \notin k[G]$ then $k[G]$ is not normal. Since G contains a directed cycle for some n $s^n \in k[G]$. By the definition of g there is a positive integer h so that $hg = n$. Hence, s^g is a root of the monic polynomial $f(z) = z^h - s^n$. Since s^g is not in $k[G]$, $k[G]$ is not normal. $\square$

Observe that what we have shown is that $k[G]$ is equal to $\mathcal{A}(\mathcal{P}_G)$ when $s^g \in k[G]$ and that $k[G]$ is not normal, and hence not equal to $\mathcal{A}(\mathcal{P}_G)$, when $s^g \notin k[G]$.

Corollary 6.3.9. Suppose G has a directed cycle, and let $g = \gcd\{\text{sig}(C) :$ C is a cycle of $G\}$. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by s^g as an algebra over $k[G]$.

Proof. If $s^g \in k[G]$ then, by Theorem 6.3.7, $k[G]$ is normal, and, hence, $k[G] = \mathcal{A}(\mathcal{P}_G)$. Now, suppose s^g is not in $k[G]$. Let $\{a_{ij}\}_{(i,j)\in E(G)}$ be a set of non-negative arc weights so that for each vertex i, $b_i = \sum_{j:(j,i)\in E(G)} a_{ji} - \sum_{j:(i,j)\in E(G)} a_{ij}$ is an integer. Construct a directed graph G' by taking G and adding a new component that is a directed cycle of length g. By Theorem 6.3.7, $k[G']$ is normal. Thus, there are arc weights a'_{ij} on the arcs of G' that give vertex weights b_i for each vertex i. Let m be the weight on each arc on the cycle of length g added to G. The arc weights on the component correspond to $(s^g)^m$ as a monomial. Hence, the monomial associated to the arc weights is a product of terms of $k[G]$ and s^g. Thus, $\mathcal{A}(\mathcal{P}_G)$ is generated by s^g as an algebra over $k[G]$. □

At this point we observe that there is a directed cycle we can add to any directed graph G with a single generator. Namely, a directed loop at any vertex. Since the length of this loop is 1, it has signature 1.

Corollary 6.3.10. Let R be the ring generated by s as an algebra over $k[G]$. Then, R is a normal domain.

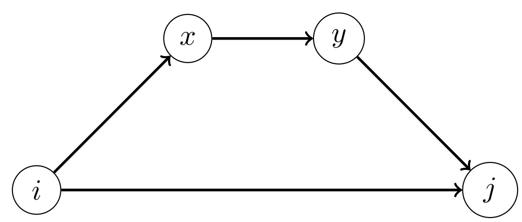
Proof. Let G' be the graph G with a loop L added at some vertex i. Observe that $g' = \gcd\{\text{sig}(C) : C \text{ is a cycle of } G'\} = 1$. Since $\text{sig}(L) = 1$ and $s^1 \in k[G']$, Corollary 6.3.9 gives that $k[G']$ is normal. Since $k[G']$ is generated by s^1 over $k[G]$, we conclude that $R = k[G']$. and hence R is normal. □

At this point, we have characterized when $k[G]$ is equal to $\mathcal{A}(\mathcal{P}_G)$, and hence normal, for G balanced or containing a directed cycle. What remains is to characterize when $k[G]$ is equal to $\mathcal{A}(\mathcal{P}_G)$ when G does not satisfy either of these conditions.

Suppose G is not balanced and does not contain any directed cycles. The monomials in $k[G]$ will, thus, be associated to arc weights with sources and sinks. Let R and T denote disjoint multisets of vertices of G. Let e_R and e_T denote the characteristic vectors for the multisets R and T respectively in $\mathbb{R}^{d+1}$. We characterize for which n is $e_T - e_R + n \cdot e_s$ in S_1 and for what n is it in S_2. Equivalently, when is $x^{e_T - e_R} s^n$ in $k[G]$ and when is it in $\mathcal{A}(\mathcal{P}_G)$?

Let $\{a_{ij}\}_{(i,j)\in E(G)}$ be a set of non-negative, integral, arc weights. Since G does not have a directed cycle, every arc with positive weight is part of at least one path from a vertex in R to a vertex in T. Thus, $e_T - e_R + ne_s$ is in S_2 if and only if there is a set of paths from R to T (with the correct multiplicity of the sources and the sinks) with total path length n.

Example 6.3.11. Let G be a directed graph on vertex set $\{i, j, x, y\}$ and arc set $\{(i, x), (x, y), (y, j), , (i, j)\}$, $R = \{i\}$ and $T = \{j\}$ see Figure 6.7. There are three directed paths from i to j one of length 1,one of length 3, one of length 2. Let each arc have weight $\frac{1}{2}$. The resulting monomial is $x_i^{-1}x_js^2$, as the only source is i, the only sink is j, each of the four arcs has weight $\frac{1}{2}$ for a total of 2. Observe that $(x_i^{-1}x_js^2)^2 = (x_i^{-1}x_js^1)(x_i^{-1}x_js^3)$, thus we have the root of a monic polynomial but the monomial is not in $\mathcal{A}(\mathcal{P}_G)$.



G

Figure 6.7: The graph G with two directed paths between i and j. Attaching a weight of $\frac{1}{2}$ to each arc gives a monomial $x_i^{-1}x_js^2$ which is not possible with integer arc weights in this graph.

This example illustrates why determining if certain monomials are in $\mathcal{A}(\mathcal{P}_G)$, or $k[G]$ is difficult. Thus, we ask the question: given a particular pair of disjoint multisets R and T what possible values of n are there so that $e_T - e_R + ne_s \in S_1$?

As before, if we have n and m so that $e_T - e_R + ne_s \in S_1$ and $e_T - e_R + me_s \in S_1$, then s^{m-n} is in the fraction field of $k[G]$. Thus, as before, let $g = \gcd\{\mathrm{sig}(C) : C \text{ is a cycle of } G\}$, then s^g is the smallest such value. That implies that all the differences in the exponents of the s variable are multiples of g.

Theorem 6.3.12. Let $l_1 = \min\{n : e_T - e_R + ne_s \in S_2\}$, $l_2 = \max\{n : e_T - e_R + ne_s \in S_2\}$ and $g = \gcd\{\mathrm{sig}(C) : C \text{ is a cycle of } G\}$. $e_T - e_R + ne_s \in S_1$ if and only if $n \in \{l_1, l_1 + g, \ldots, l_2\}$.

Proof. Let $n = l_1 + mg$, for $mg \in \{0, \ldots, l_2 - l_1\}$. Observe that $(l_2 - l_1)n = (l_2 - l_1 - mg)l_1 + l_2mg$, since $mg \leq l_2 - l_1$ $(l_2 - l_1 - m)$ is a positive integer. Hence, $(x^{e_T - e_R}s^n)^{l_2 - l_1} = (x^{e_T - e_R}s^{l_1})^{l_2 - l_1 - m}(x^{e_T - e_R}s^{l_2})^m$. Thus $x^{e_T - e_R}s^n$ is the root of a monic polynomial. Moreover, since s^g is in the fraction field of $k[G]$, $x^{e_T - e_R}s^n$ is in the fraction field as well. Hence, $e_T - e_R + ne_s \in S_1$.

Assume $e_T - e_R + ne_s \in S_1$. We know that $n = l_1 + mg$ for some integer m, as the difference is a multiple of g. Thus, all we need to prove is that $l_1 \leq n \leq l_2$. Let $m = \min\{n \ : \ e_T - e_R + ne_s \in S_1\}$. For this set of weights, we decompose the arcs with positive weight into directed paths going from R to T of a common weight, q, allowing repetition of paths. The resulting graph each vertex has degree q times the number of times it appears in R or T, denote this value as qb_v for each vertex v. By Theorem 2.1.10 we can find a subgraph where each vertex v has has degree b_v. Delete this subgraph and repeat applying Theorem 2.1.10 $q-1$ times. This decomposes the paths into q sets of paths with the vector of that set of paths $q(e_T - e_R + m_i e_s)$, for $i = 1, \ldots, q$ for each set. For each of these clearing the common fraction weight gives an non-negative integer weight $e_T - e_R + m_i e_s$ and since, by assumption $m_i \geq l_1$ we know that $m \geq l_1$ and thus, by choice of m, $m = l_1$. Similarly $m = \max\{n \ : \ e_T - e_R + ne_s \in S_1\} = l_2$. Hence, $e_T - e_R + ne_s \in S_1$ if and only if $n \in \{l_1, l_1 + g, \ldots, l_2\}$.

□

Now that we have a characterization of the element of $\mathcal{A}(\mathcal{P}_G)$ in terms of subgraphs of G, we construct a finite generating set for $\mathcal{A}(\mathcal{P}_G)$.

Definition 6.3.13. We say a subgraph H of G is *max-min* for disjoint multisets R and T if H is the union of subgraphs H_1 and H_2. Here, H_1 is the subgraph of G that has the minimum number of arcs with source set R and sink set T, and H_2 is the subgraph of G that has the maximum number of arcs with source set R and sink set T. Denote the max-min subgraph for R and T as $max - min(R, T)$.

Theorem 6.3.14. Let G be a directed graph without a directed cycle Let $\{R_j\}_{j=1}^m$, and $\{T_j\}_{j=1}^m$ denote the pairs of sets so that $R_j \cap T_j = \emptyset$ and $max - min(R_j, T_j)$ is a closed walk of G. Let $\mathcal{M} = \{x^{e_{T_j} - e_{R_j}} s^{n_j}\}_{j, n_j}$ be the set of monomials where $j = 1, \ldots, m$ and, for each j, if $l_1 = \min\{n_j \ : \ x^{e_{T_j} - e_{R_j}} s^{n_j} \in k[G]\}$ and $l_2 = \max\{n_j \ : \ x^{e_{T_j} - e_{R_j}} s^{n_j} \in k[G]\}$, then n_j ranges over $\{l_1, l_1 + g, \ldots, l_2\}$. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by $\mathcal{M}$ as an algebra over $k[G]$.

Proof. Let R and T be a pair of disjoint multisets of G so that $x^{e_T - e_R} s^n \in \mathcal{A}(\mathcal{P}_G)$. From the characterization in Theorem 6.3.12, we know that $n \in \{l_1, l_1 + g, \ldots, l_2\}$ where $l_1 = \min\{n : x^{e_T - e_R} s^n \in k[G]\}$ and $l_2 = \max\{n : x^{e_T - e_R} s^n \in k[G]\}$.

Consider $G' = max - min(R, T)$. Observe that G' decomposes into closed walks where each source and sink shows up at most once. This is done by alternatively taking forwards paths from the maximum subgraph and backwards paths from the minimum subgraph. These walks are max-min as the minimum paths can not be replaced with a smaller set of arcs and the maximum paths can not be replaced with a larger set of arcs. Denote the sets that give the max-min closed walks in the decomposition by $l_{i,1} = \min\{n : x^{e_{T_i} - e_{R_i}} s^n \in k[G]\}$ and $l_{i,2} = \max\{n : x^{e_{T_i} - e_{R_i}} s^n \in k[G]\}$ over R_i and T_i. Hence, n can be written as $n = \sum_{i=1}^{k} n_i$ where $n_i \in \{l_{i,1}, l_{i,1} + g, \ldots, l_{i,2}\}$. Thus, $x^{e_T - e_R} s^n = \prod_{i=1}^{k} x^{e_{T_i} - e_{R_i}} s^{n_i}$. Since each of the monomials $x^{e_{T_i} - e_{R_i}} s^{n_i}$ is in $\mathcal{M}$, $\mathcal{A}(\mathcal{P}_G)$ is generated as an algebra by $\mathcal{M}$ over $k[G]$. □

Observe that checking if a cycle, where each source and sink appears once, is max-min may be complicated. However, an easier set to build is $\mathcal{M}' = \{x^{e_{T_j} - e_{R_j}} s^{n_j}\}_{j, n_j}$ where j ranges over all closed walks in G where each source and sink appears once in the walk and $n_j = \{l_{j,1}, l_{j,1} + g, \ldots, l_{j,2}\}$ over the appropriate values of n. Since $\mathcal{M} \subseteq \mathcal{M}'$ this set of monomials is a generating set for $\mathcal{A}(\mathcal{P}_G)$.

6.4 Future Directions

- Improve the normalization condition for Homogenized Directed Graphs. In the current form it is not difficult to tell if it is or is not normal, is there a condition that does not require checking to see if it has a directed cycle or is balanced?

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_1 condition, for $\ell > 1$.

- Generalize results to when the homogenizing variable is raised to different powers than just 1.

Chapter 7

Hypergraphs

We begin this chapter by describing the hypergraphs we are studying. We study *signed hypergraphs* (G, sgn) where, analogous to signed graphs, $\text{sgn}(e) = \{-1, +1\}$ for all $e \in E$. In Section 7.1, we present basic algebraic results about edge rings of hypergraphs, with some specific results for hypergraphs with only 2-vertex and 3-vertex hyperedges. In Section 7.3, we study *completely separable* hypergraphs, where the hyperedges can take on any number of vertices, that is, for each hyperedge e with more than two vertices, $G \setminus e$ has each of the vertices of e in a separate component. In Sections 7.2 and 7.4, a hypergraph $G = (V, E_2 \cup E_3)$ will have a set of vertices V, and a set of hyperedges $E_2 \cup E_3$ where E_2 are 2-vertex hyperedges, referred to as *edges* and E_3 the 3-vertex hyperedges, referred to as *hyperedges*. Note that we refer to the set of all edges and hyperedges of a graph as the set of hyperedges, unless we refer to specifically the 2-vertex hyperedges. We also assume that the hypergraph is *separable*, that is, for each hyperedge e with three vertices, $G \setminus e$ will have more components than G.

7.1 Algebraic Results

We consider the relation between hyperedge weights and vertex weights as a map, $f : \mathbb{Z}^E \to \mathbb{Z}^V$ where the vertex weights are the signed sum of the hyperedge weights.

For signed graphs, Theorem 4.2.13 presents a characterization of the kernel of the map $f : \mathbb{Z}^E \to \mathbb{Z}^V$ in terms of a basis, where the basis elements are closed even walks with alternating signs. In the case where we are considering a signed hypergraph instead, the combinatorial structure that is associated with a kernel element is not as well-studied. Thus, instead of using a combinatorial description of the elements, we refer to them as *kernel elements* and give a more combinatorial description of their behavior. Note that the kernel elements are primarily used as a computational tool by, potentially, adding edges to the graph, and then show that there is a kernel element with certain properties as a way to adjust the edge weights.

Definition 7.1.1. A *kernel element* is a connected subhypergraph of G, with non-zero integral edge and hyperedge weights and, for each vertex, the sum of the incident edge and hyperedge weights is zero.

Note that we assume the hyperedges which are not contained in a kernel element to have weight zero. We now describe the basis elements for the kernel. The given construction generalizes the construction of the basis elements given in Theorem 4.2.13. In particular, we study kernel elements which have a restricted structure.

Definition 7.1.2. Let $G = (V, E)$ be a hypergraph, where for every hyperedge $e \in E$, at most two vertices in e are contained in the same component of $G \setminus e$. Given a set of weights w_e for each $e \in E$, add a signed edge $\text{sgn}(e)ij$ of weight w_e for each pair of vertices i, j of e in the same component of $G \setminus e$ to the graph and call the resulting graph G'. Note that, in some sense, G' maintains most of the structure of G for the components of $G \setminus e$. That is, if e contains two vertices in the same component of $G \setminus e$, then G' can model how they interact in G. A kernel element H of G is said to be a *basis element* if the edges with non-zero weight in G' gives a set of alternating walks in G' where the vertices of odd degree are contained in hyperedges of G.

Note that this definition is the generalization of the definition given in Theorem 4.2.13 in that locally the kernel elements are a set of sign alternating closed walks that contain the hyperedges, or sign alternating paths connecting vertices in different hyperedges, and each vertex has even degree.

Lemma 7.1.3. A kernel element H of G with hyperedge weight w_e, for $e \in E$, can be decomposed into a set of basis kernel elements $H_1, \ldots, H_r$ with hyperedge weights $w_{e,1}, \ldots, w_{e,r}$ for each hyperedge $e \in E$ where $w_e = w_{e,1} + \ldots + w_{e_r}$ and $w_{e,i} > 0$ for any $i = 1, \ldots, r$ if and only if $w_e > 0$.

Proof. Let H be a kernel element with hyperedge weights w_e for each $e \in E$. Suppose the hyperedge e has weight $w_e > 0$ and contains the vertex i, then as H is a kernel element, there is a hyperedge e', containing i so that $w_{e'} < 0$. Similarly if $w_e < 0$, then there is an e' with $w_{e'} > 0$.

Let $\mathcal{E}$ be the multiset of pairs of signed hyperedges of G and weights; in particular, for each hyperedge e in G, the edge-weight pair $(e, \text{sgn}(a_e))$ occurs $|a_e|$ times. Partition the resulting graph into minimal connected sign alternating subgraphs. Note that each subgraph contains either a closed even walk or contains at least one hyperedge. If the subgraph contains at least one hyperedge, then the edges form walks connecting vertices in hyperedges. If the walk connects two vertices in the same hyperedge then, since it is alternating, the walk has odd length. Similarly, if a hyperedge e has two vertices, i and j, in the same component after deleting e, then the replacing e with ij the resulting subgraph on that component has to be alternating, and hence be a closed alternating walk. Thus, the kernel element can be decomposed into basis kernel elements. $\square$

The method used in Lemma 7.1.3 of replacing a hyperedge with a subgraph is one of the main methods used throughout the remainder of this chapter. There are four hypergraphs we generally use for each hyperedge.

Definition 7.1.4. For a hypergraph G and hyperedge $e = \text{sgn}(ijk)ijk$, define the following graphs:

- $G_i := (G \setminus e) \cup \{\text{sgn}(ijk)jk, \text{sgn}(ijk)it_i\}$ where t_i is a new vertex added to the graph,

- $G_j := (G \setminus e) \cup \{\mathrm{sgn}(ijk)ik, \mathrm{sgn}(ijk)jt_j\}$ where t_j is a new vertex added to the graph,
- $G_k := (G \setminus e) \cup \{\mathrm{sgn}(ijk)ij, \mathrm{sgn}(ijk)kt_k\}$ where t_k is a new vertex added to the graph,

Note that this can be generalized to arbitrary hypergraphs.

Definition 7.1.5. For a hypergraph G, hyperedge e, and partition of the vertices in e, $p = p_1 \cup \ldots \cup p_k$ where $p_j, p_{j+1}, \ldots, p_r$ are the elements of the partition with a single vertex, define the graph:

$$G_p := (G \setminus e) \cup \left(\bigcup_{i=1}^{j-1} \mathrm{sgn}(e) p_i \right) \cup \left(\bigcup_{i=j}^{r} \mathrm{sgn}(e) t_i \cup p_i \right)$$

where $t_i \cup p_i$ is a regular edge for $i = j, \ldots, r$, and p_i is a hyperedge for $i = 1, \ldots, j-1$.

Note that the trivial partition $p = e$ of one set gives the original hypergraph, and the partition into isolated elements replaces the hyperedge with a collection of edges running between vertices of the hyperedge and new vertices. The graphs given in Definition 7.1.4 are associated with the three partitions of three elements into a pair and an isolated element.

In order to discuss the normality of $k[G]$ we need to define the subgroup containing $\mathcal{P}_G$ and the cone containing $\mathcal{P}_G$.

Definition 7.1.6. Let $G = (V, E)$ be a hypergraph on d vertices, then the *lattice of* G is the sublattice of $\mathbb{Z}^d$ defined by,

$$\mathcal{L}_G := \mathbb{Z}\rho(E) = \left\{ \sum_{e \in E} z_e \rho(e) \ : \ z_e \in \mathbb{Z} \right\}.$$

and the *cone of* G is the cone in $\mathbb{R}^d$ defined by,

$$\mathrm{cone}(\mathcal{P}_G) := \mathbb{R}_+ \rho(E) = \left\{ \sum_{e \in E} a_e \rho(e) \ : \ a_e \geq 0 \right\}.$$

Note that using this definition, the kernel elements are the preimage of the zero vector in $\mathcal{L}_G$. We use the definitions of G_i, G_j, and G_k, as well as the kernel elements to study the structure of $\mathcal{L}_G$.

Theorem 7.1.7. Let G be a hypergraph and $e = \text{sgn}(ijk)ijk$ a hyperedge of G, then:

1. $\mathcal{L}_G \subseteq \pi(\mathcal{L}_{G_i}) \cap \pi(\mathcal{L}_{G_j}) \cap \pi(\mathcal{L}_{G_k})$,

2. $\text{cone}(\mathcal{P}_G) \subseteq \pi(\text{cone}(\mathcal{P}_{G_i})) \cap \pi(\text{cone}(\mathcal{P}_{G_i})) \cap \pi(\text{cone}(\mathcal{P}_{G_i}))$,

3. $\mathbb{N}\rho(\mathcal{P}_G) \subseteq \pi(\mathbb{N}\rho(\mathcal{P}_{G_i})) \cap \pi(\mathbb{N}\rho(\mathcal{P}_{G_i})) \cap \pi(\mathbb{N}\rho(\mathcal{P}_{G_i}))$,

where π represents the projection onto the coordinates associated with the vertices of G.

Proof. Note that the differences between $\mathcal{L}_G$, $\text{cone}(\mathcal{P}_G)$ and $\mathbb{N}\rho(\mathcal{P}_G)$ can be thought of as what hyperedge weights we use to produce the vertex weights, integer, non-negative, and non-negative integer respectively. Starting with a set of hyperedge weights, and weight w_e on edge $e = \text{sgn}(e)ijk$, we can obtain a set of hyperedge weights on G_i, G_j and G_k, with the same properties, by using the same weights on all hyperedges except e. For G_i, the edges $\text{sgn}(e)it_i$ and $\text{sgn}(e)jk$ both get weight w_e. Note that the resulting weights are integral or non-negative if and only if the original weights were, and that the resulting vertex weights match the vertex weights of G for all vertices except t_i, which is not in G. Thus, $\mathcal{L}_G \subseteq \pi(\mathcal{L}_{G_i})$, $\text{cone}(\mathcal{P}_G) \subseteq \pi(\text{cone}(\mathcal{P}_{G_i}))$ and $\mathbb{N}\rho(\mathcal{P}_G) \subseteq \pi(\mathbb{N}\rho(\mathcal{P}_{G_i}))$. Applying the same logic to G_j and G_k gives the desired subset relations. $\square$

Note that this relation is not an equality in general, see Example 7.1.8. However, as a computational tool it is very useful, starting with a set of vertex weights, in $\mathcal{L}_G$ and $\text{cone}(\mathcal{P}_G)$, if we can show that they are not in $\pi(\mathbb{N}\rho(\mathcal{P}_{G_i})) \cap \pi(\mathbb{N}\rho(\mathcal{P}_{G_i})) \cap \pi(\mathbb{N}\rho(\mathcal{P}_{G_i}))$, then we have shown that the edge ring $k[G]$ is not normal. Similarly, if we can iteratively determine the hyperedge weights in a component of G_i, and computationally force the edge weight of $\text{sgn}(e)it_i$ and $\text{sgn}(e)jk$ to be equal, then we have produced a set of edge weights that we can use on the hyperedges of G.

Example 7.1.8. Consider the hypergraph $G = (V, E)$ with vertex set $V = \{i, j, k, \ell\}$ and hyperedge set $E = \{ijk, j\ell, k\ell\}$, see Figure 7.1. The vertex weights $(i, j, k, \ell) = (0, 1, 1, 0)$

are not in $\mathcal{L}_G$, as ijk has weight zero, the weight of $j\ell$ is the negative of the weight of $k\ell$ for ℓ to have vertex weight zero but both of these can not happen at the same time so it is not in the lattice.

However, consider G_i, $V(G_i) = \{i, j, k, \ell, t_i\}$, $E(G_i) = \{it_i, jk, j\ell, k\ell\}$. Letting the edge weights be one on jk and zero on the other edges gives $(i, j, k, \ell) = (0, 1, 1, 0)$. Thus, $(0, 1, 1, 0)$ is in $\pi(\mathcal{L}_{G_i})$. Similarly, for G_j, $V(G_j) = \{i, j, k, \ell, t_j\}$, $E(G_j) = \{ik, k\ell, j\ell, jt_j\}$, edge weight one on $k\ell$, negative one on $j\ell$ and two on jt_j gives $(i, j, k, \ell) = (0, 1, 1, 0)$. Hence, $(0, 1, 1, 0)$ is in $\pi(\mathcal{L}_{G_j})$. By symmetry $(i, j, k, \ell) = (0, 1, 1, 0)$ is in $\pi(\mathcal{L}_{G_k})$. Thus, we have $\mathcal{L}_G \neq \pi(\mathcal{L}_{G_i}) \cap \pi(\mathcal{L}_{G_j}) \cap \pi(\mathcal{L}_{G_k})$, and hence is not tight in general.

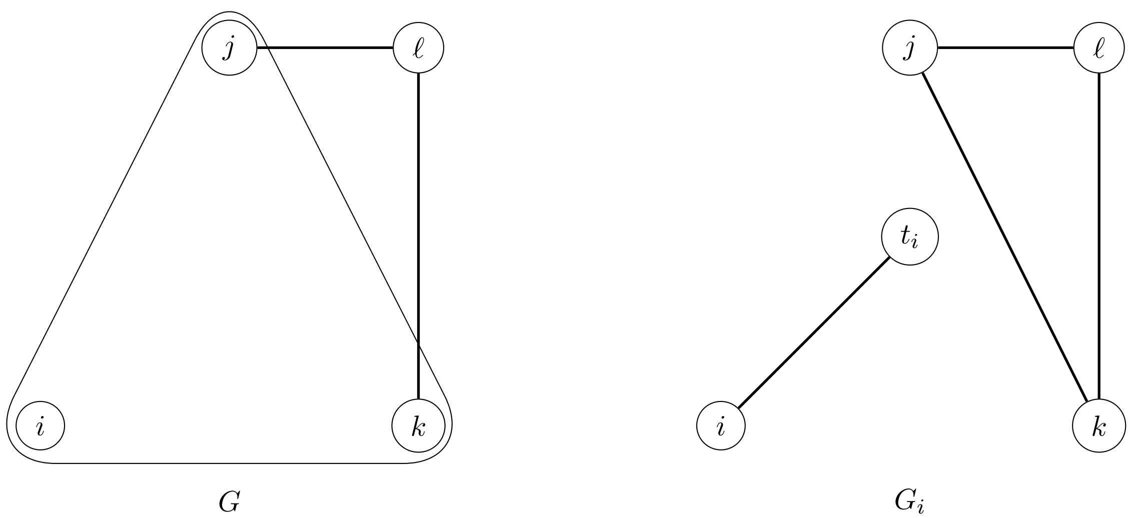


Figure 7.1: The hypergraph G with vertex set $\{i, j, k, \ell\}$ and hyperedge set $\{ijk, j\ell, k\ell\}$ and the graph G_i with vertex set $\{i, j, k, \ell, t_i\}$ and edge set $\{jk, j\ell, k\ell, it_i\}$. Note that $\mathcal{L}_G \subset \pi(\mathcal{L}_{G_i})$, but are not equal.

We can generalize Theorem 7.1.7 by replacing the three hypergraphs G_i, G_j, and G_k with any set of hypergraphs given by a partition of the vertices of a hyperedge G_p as given in Definition 7.1.5. The proof follows exactly as stated.

Theorem 7.1.9. Let G be a hypergraph, $e =$ a hyperedge of G, and a p partition of the vertices of e into at least two parts then:

1. $\mathcal{L}_G \subseteq \pi(\mathcal{L}_{G_p})$,

2. $\text{cone}(\mathcal{P}_G) \subseteq \pi(\text{cone}\,\mathcal{P}_{G_p})$,

3. $\mathbb{N}\rho(\mathcal{P}_G) \subseteq \pi(\mathbb{N}\rho(\mathcal{P}_{G_p}))$,

where π represents the projection onto the coordinates associated with the vertices of G.

Proof. The intuition behind this proof is exactly the same as the intuition behind the proof of Theorem 7.1.7. Note that the differences between $\mathcal{L}_G$, $\text{cone}(\mathcal{P}_G)$ and $\mathbb{N}\rho(\mathcal{P}_G)$ can be thought of as what hyperedge weights we use to produce the vertex weights, integer, non-negative, and non-negative integer respectively. Starting with a set of hyperedge weights, and weight w_e on edge e we can obtain a set of hyperedge weights on G_p, with the same properties, by using the same weights on all hyperedges except e. The edges in $E(G_p) \setminus E(G \setminus e)$ are assigned weight w_e. Note that the resulting weights are integral or non-negative if and only if the original weights were, and that the resulting vertex weights will match the vertex weights of G for all vertices except the vertices which are not in G. Thus, $\mathcal{L}_G \subseteq \pi(\mathcal{L}_{G_p})$, $\text{cone}(\mathcal{P}_G) \subseteq \pi(\text{cone}(\mathcal{P}_{G_p}))$ and $\mathbb{N}\rho(\mathcal{P}_G) \subseteq \pi(\mathbb{N}\rho(\mathcal{P}_{G_p}))$. □

Definition 7.1.10. Let G be a hypergraph with p 3-vertex hyperedges, let $\mathcal{G}(G)$ denote the set of 3^p graphs produced by replacing all p hyperedges in one of the three ways discussed above in Definition 7.1.4.

Example 7.1.11. Suppose the graph G has two hyperedges $e = ijk$ and $e' = sro$ then $\mathcal{G}(G)$ has 9 graphs, $\mathcal{G}(G) = \{(G_i)_o, (G_i)_r, (G_i)_s, (G_j)_o, (G_j)_r, (G_j)_s, (G_k)_o, (G_k)_r, (G_k)_s\}$.

Corollary 7.1.12. Let G be a hypergraph then,

1. $\mathcal{L}_G \subseteq \bigcap_{H \in \mathcal{G}(G)} \pi(\mathcal{L}_H)$,

2. $\text{cone}(\mathcal{P}_G) \subseteq \bigcap_{H \in \mathcal{G}(G)} \pi(\text{cone}(\mathcal{P}_H))$,

3. $\mathbb{N}\rho(E(G)) \subseteq \bigcap_{H \in \mathcal{G}(G)} \pi(\mathbb{N}\rho(E(H))$,

Proof. Iteratively replace the hyperedges, as described in Definition 7.1.4. At each step apply Theorem 7.1.7 to maintain the subset inequality. □

As before, we can construct a similar result to Corollary 7.1.12 to Theorem 7.1.9.

7.2 Non-Normality

In this section, we determine conditions for non-normality of hypergraphs. In particular, we study separable hypergraphs with only 2-vertex and 3-vertex edges.

Proposition 7.2.1. Let $G = (V, E_2 \cup E_3)$ be a hypergraph with E_3 the set of all 3-vertex hyperedges, and let C_1 and C_2 be two odd cycles in $G \setminus E_3$ that violate the odd cycle condition in $G \setminus E_3$. If there exists an $H \in \mathcal{G}(G)$ where C_1 and C_2 violate the odd cycle condition and the component containing C_1 and C_2 does not have any vertices that are not in G, then G is not normal.

Proof. Suppose G is a hypergraph, C_1, C_2 odd cycles and H a graph as stated in the assumptions. By Theorem 4.2.9, we know that the monomial $\prod_{\ell \in C} x_\ell^{\text{sig}_C(\ell)} \prod_{\ell \in C'} x_\ell^{\text{sig}_{C'}(\ell)}$ can not be obtained in the ring $k[H]$. Note that in the projection, π, the component of H that contains C_1 and C_2 is an isomorphism. That is, the vertices in the component are all in G. So, there is no set of non-negative edge weights, on the component, that gives this set of vertex weights. Hence, on these vertices, the projections $\pi\mathbb{N}\rho(E(G))$ and $\pi\mathbb{N}\rho(E(H))$ are equal. Hence, this set of vertex weights are not in the projection, and hence $k[G]$ is not normal. $\square$

Proposition 7.2.1 gives some insight into the methods that we use to find other non-normality conditions. We first find a set of vertex weights in G, and then choose the correct graph $H \in \mathcal{G}(G)$ so that the vertex weights are not in the projection $\pi(\mathbb{N}\rho(E(H)))$. Note that to use the condition, as stated, we need to use the constructed graph. For an example, see Example 7.2.2. Moreover, Example 7.2.2 and Example 7.2.3 show that we can have normality or non-normality of the edge ring if we do not satisfy all the conditions of Proposition 7.2.1

Example 7.2.2. Consider the hypergraph G with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and hyperedge set $\{+12, +13, +22, +33, +234, -456, +57, -78, +68\}$, see Figure 7.2. The pair of odd cycles $+22$ and $+33$ have the path $+12$, $+13$ connecting them. Assign vertex weights $(0,1,1,0,0,0,0,0)$ to the vertices. Consider the graph $(G_2)_4 = (G \setminus \{234, 456\}) \cup \{34, 56, 2t_2, 4t_4\}$, in this graph $+22$ and $+33$ violate the odd cycle condition, however, putting edge weight 1 on $+2t_2$, $+34$ and $-4t_4$, and zero on all the other edges produces the desired vertex weights.

In the original hypergraph, putting weight 1 on $+234$, -456, $+57$, -78, and $+68$, and putting weight zero on all the other hyperedges produces this set of vertex weights.

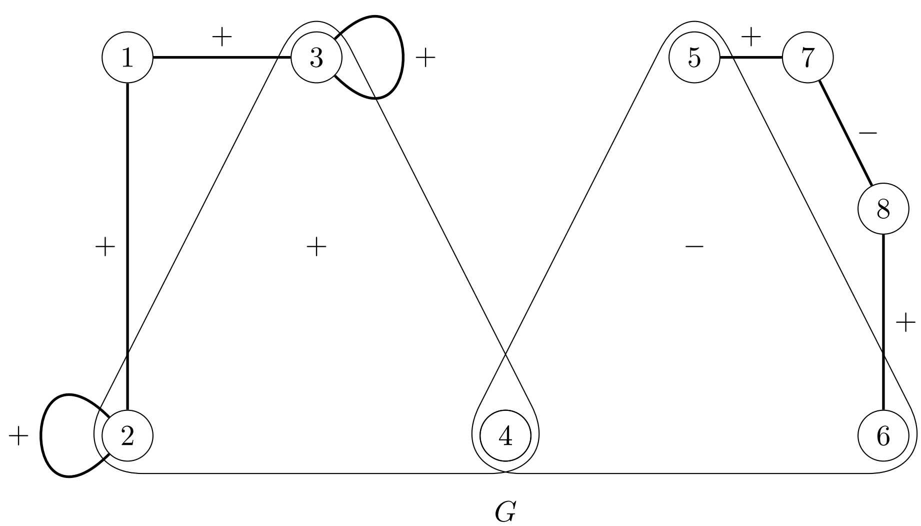


Figure 7.2: The hypergraph G with vertex set $\{1,2,3,4,5,6,7,8\}$ and hyperedge set $\{+12, +13, +22, +33, +234, -456, +57, -78, +68\}$. In $(G_2)_4$, $+22$ and $+33$ violate the odd cycle condition. The subhypergraph given by hyperedges $\{+12, +13, +234, \}$ is an odd bud. The subhypergraph given by hyperedges $\{-456, +57, -78, +68\}$ is an even bud, and the hyperwalk given by $\{-456, +57, -78, +68\}$ is an example of an alternating walk with one leaf, 4.

Example 7.2.3. Consider the hypergraph G with vertex set $\{1,2,3,4\}$ and hyperedge set $\{+12, +13, +22, +33, +234\}$, see Figure 7.3. The pair of odd cycles $+22$ and $+33$ have the path $+12$, $+13$ connecting them. Consider the vertex weights $(0,1,1,0)$. Note that the graphs G_2, G_3, G_4 and $G \setminus \{+234\}$ fail the odd cycle condition. There are no non-negative integer hyperedge weights in the hypergraph that gives $(0,1,1,0)$ as a set of vertex weights. Thus, the edge ring of this graph is not normal.

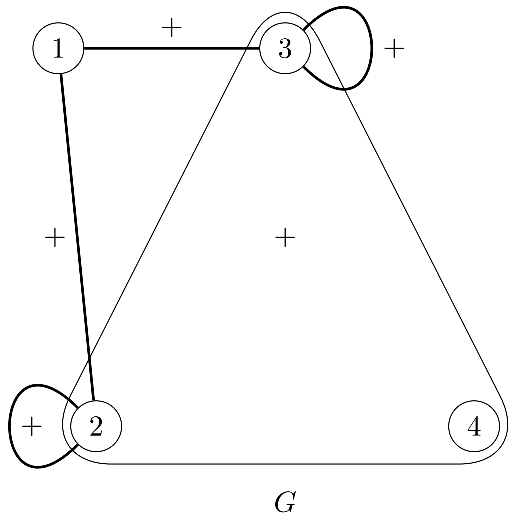


Figure 7.3: The hypergraph G with vertex set $V(G) = \{1, 2, 3, 4\}$ and hyperedge set $E(G) = \{+12, +13, +22, +33, +234\}$. In $(G_2)_4$, $+22$ and $+33$ violate the odd cycle condition.

Note that Proposition 7.2.1 gives a characterization of when an hyperedge ring is not normal. The behavior of the 3-vertex hyperedges can be thought of as not influencing the odd cycles. That is, they can not be used to produce the monomials that guarantee non normality. The remaining conditions, thus, depend on 3-vertex hyperedges interacting with the odd cycles and each other. First, we give some motivating examples of graphs that produce non-normal edge rings.

Example 7.2.4. Let G be the hypergraph with vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{+11, +12, +34, +44, +56, +66, +235\}$, see Figure 7.4. Consider the vertex weights $(1, 0, 0, 1, 0, 1)$, that is, weight one on 1, 4 and 6, and weight zero on 2, 3 and 5. This set of vertex weights is in cone$(\mathcal{P}_G)$, by placing weight $\frac{1}{2}$ on +11,+44 and +66, and in $\mathcal{L}_G$ by placing weight one on +12, +34, +56 and weight -1 on +235. However, it is not in $\mathbb{N}\rho(E(G))$ as it is not in $\mathbb{N}(\rho(E(G_2))$.

Note that in Example 7.2.4, the odd cycles appear to violate the odd cycle condition, and in fact do violate the odd cycle condition in G_2, G_3 and G_5. In this case, a generalization of the odd cycle condition guarantees the non-normality of the edge ring, and is shown in

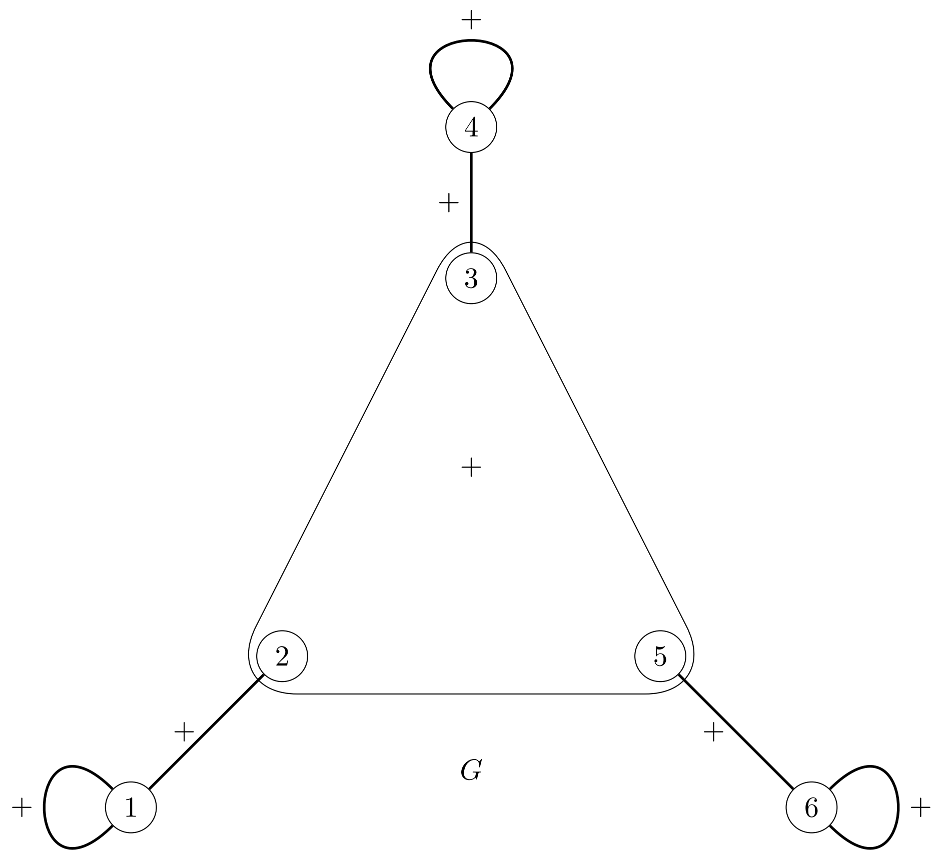


Figure 7.4: The hypergraph G with vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and hyperedge set $E(G) = \{+11, +12, +34, +44, +56, +66, +235\}$. The edge ring of G is not normal, as demonstrated by putting vertex weight of 1 on 1,4 and 6. The hypergraph G is an example of a kernel element from Theorem 7.3.3, and the hypergraph formed by taking $\{+12, +34, +56, +235\}$ is an example of a non-alternating hyperwalk with leaves $\{1, 4, 6\}$.

Theorem 7.3.3.

Example 7.2.5. Let G be the hypergraph with vertex set $\{1, 2, 3, 4\}$ and hyperedge set $\{+11, +123, +234, +44\}$, see Figure 7.5. Consider the vertex weights $(1, 0, 0, 1)$, that is, weight one on 1 and 4 and weight zero on 2 and 3. This is in cone($\mathcal{P}_G$), and can be obtained by placing weight $\frac{1}{2}$ on +11 and +44. This set of vertex weights is also in $\mathcal{L}_G$, and can be obtained by placing weight 1 on +123, -1 on +234 and weight 2 on +44. However, it is not in $\mathbb{N}\rho(E(G))$ as $(G_2)_2$ fails the odd cycle condition. Intuitively, the two hyperedges +123 and +234 behave like edges, in that vertices 2 and 3 are identical and changing the weight of one of them changes the other by the same amount.

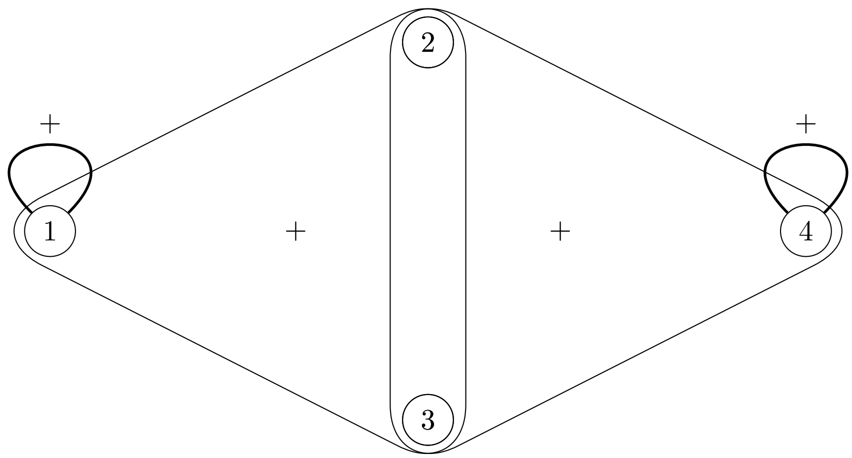


Figure 7.5: The hypergraph G with vertex set $V(G) = \{1, 2, 3, 4\}$ and hyperedge set $E(G) = \{+11, +123, +234, +44\}$. A vertex weight of 1 on vertices 1 and 4 is shows $k[G]$ is not normal. The hyperedges +123 and +234 is an example of a hyperwalk with leaves 1 and 4.

The following example shows that the property controlling normality is not as simple as having odd cycles separated by hyperedges, and paths. In particular, the lattice of the hypergraphs needs to be taken into consideration.

Example 7.2.6. Let G be the hypergraph with vertex set $\{1, 2, 3, 4, 5\}$, and hyperedge set $\{+11, +123, +24, +345, +55\}$, see Figure 7.6. Consider the vertex weights $(1, 0, 0, 0, 1)$, that is, weight one on 1 and 5, weight zero on 2,3, and 4.

While this set of vertex weights violate the odd cycle condition in $(G_3)_3$, it is not in the lattice of $(G_1)_3$ and hence not in $\mathcal{L}_G$.

Definition 7.2.7. A connected hypergraph P is a *hyperwalk* if every graph in $\mathcal{G}(P)$ is a set of vertex disjoint walks with distinct endpoints, or closed even walks. The set of vertices which are endpoints in every graph in $\mathcal{G}(P)$, if they exist, are called the *leaves* of P. If P does not have a leaf, then we say P is *closed*, and if P has exactly one leaf then we say

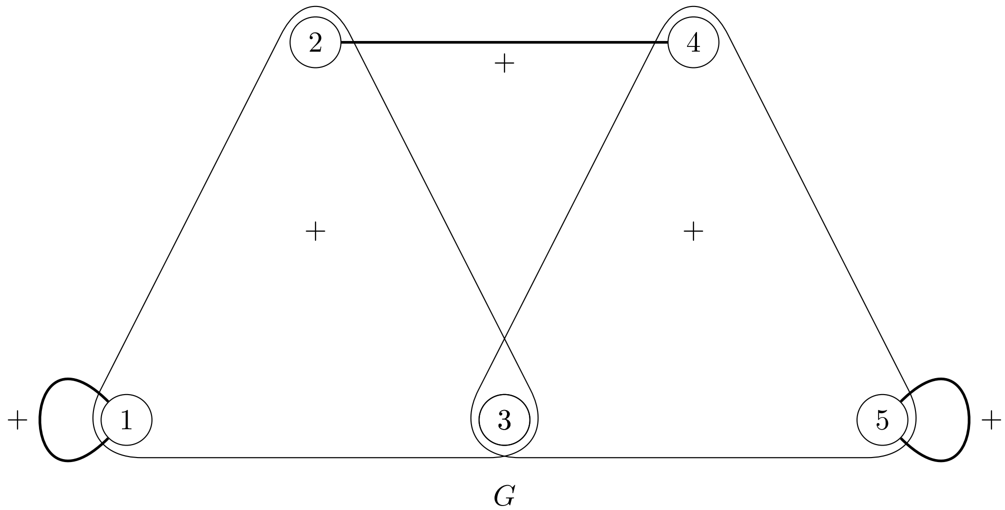


Figure 7.6: The hypergraph G with vertex set $V(G) = \{1, 2, 3, 4, 5\}$, and hyperedge set $E(G) = \{+11, +123, +24, +345, +55\}$. Notice that the vertex weights 1 on vertex 1 and 5 can not be obtained using this graph. In particular, the graph $(G_3)_3$ violates the odd cycle condition, but the weights do not appear in the lattice of $(G_1)_3$. This is an example of a subgraph that is not a hyperwalk, as the graph $(G_1)_5$ produces an odd cycle $\{2, 3, 4\}$.

P is *degenerate*. A hyperwalk which does not contain any hyperedges with more than two vertices is said to be *trivial*.

A hyperwalk P is said to be *alternating* if an assignment of non-negative hyperedge weights results in weight zero for the non-leaves of P and weight ± 1 on the leaves of P.

Note that, unlike a walk, the number of leaves of a hyperwalk can be any non-negative integer. For example the hyperwalk with hyperedges $\{+123, +12\}$ has 3 as the only leaf, In general, taking a path and replacing edges with hyperedges produces any number greater than or equal to two leaves in a hyperwalk.

Proposition 7.2.8. Let $G = (V, E)$ be a completely separable hypergraph, $\Pi = \{C_1, \ldots, C_r\}$ a set of odd cycles and P a hyperwalk where the leaves of P are $\{v_1, \ldots, v_r\}$ for v_i a vertex in C_i. Suppose there does not exist alternating hyperwalks so that each cycle has exactly one leaf, then the monomial:

$$M_\Pi = \prod_{i=1}^{r} \prod_{\ell \in C_i} x_\ell^{\mathrm{sig}_{C_i}(\ell)} \tag{7.1}$$

is contained in $\mathcal{A}(\mathcal{P}_G)$ but not $k[G]$.

Proof. We begin by showing that M_Π is in $\mathcal{A}(\mathcal{P}_G)$ by showing $\sum_{i=1}^{r} \sum_{\ell \in C_i} \mathrm{sig}_{C_i}(e_\ell)$, which is the vector of exponents for M_Π, is in $\mathrm{cone}(\mathcal{P}_G)$ and then showing it is in $\mathcal{L}_G$. To represent the vector of exponents as an element of $\mathrm{cone}(\mathcal{P}_G)$, we put weight $\frac{1}{2}$ on each edge in Π. The resulting vertex weight for each vertex $\ell \in C_i$ is $\mathrm{sig}_{C_i}(\ell)$. To represent the vector of exponents as an element of $\mathcal{L}_G$, we first observe that for some $Q \in \mathcal{G}(P)$ we can algorithmically assign ± 1 to the edges in Q by starting at a leaf v of P, assigning the incident edge weight 1, and alternate for the path containing v. Every other edge of this path is either an edge in P or is one of two edges associated with a hyperedge in P. If it is associated with a hyperedge, assign the other edges in Q the same weight, and then assign alternating ± 1 on the other edges in that component of Q. Note that the assumption of G being completely separable guarantees that the components are uniquely assigned weights by this process, and that every edge in Q gets an assignment. In fact, this produces an assignment of hyperedge weights in P, as each of the edges associated with the hyperedge have the same weight, where the leaves have weight ± 1 and the non-leaf vertices in P have weight zero. Using the same method as in Observation 4.2.8 we can assign additional weights to the edges of the cycles in Π so that each vertex $\ell \in C_i$ has weight $\mathrm{sig}_{C_i}(\ell)$ as desired.

Now that we have shown that the vector of exponents is an element of $\mathrm{cone}(\mathcal{P}_G) \cap \mathcal{L}_G$, we will show that it is not in $\mathbb{N}\rho(E(G))$. Let C_1 be an odd cycle which does not have an alternating hyperwalk connecting it to a subset of the other cycles. Note that this means that in some of the graphs of $\mathcal{G}(G)$, this cycle and one of the other cycles violate the odd cycle condition. We choose a graph H from $\mathcal{G}(G)$ to be one in which there is not an alternating path connecting C_1 to a vertex in $V(H) \setminus V(G)$. Since G is completely separable there always exists such a choice. Hence, this set of vertex weights is not in $\pi(\mathbb{N}\rho(E(H)))$

and hence not in $\mathbb{N}\rho(E(G))$. □

Note that, as a consequence of Proposition 7.2.8, we have that the graphs, and vertex weights, in Example 7.2.4 are not normal, as the non-zero edges are a set of odd cycles connected with a hyperwalk, where one of the odd cycles does not have an alternating path to one of the other cycles.

Proposition 7.2.9. Let G be a hypergraph with integer hyperedge weights, where the induced vertex weights are, for each vertex, ± 1 or 0. Then, G can be decomposed into a set of hyperwalks and a kernel element.

Proof. The proof is given algorithmically. Start at a vertex, v with non-zero vertex weight. There is an edge or hyperedge with non-zero weight incident to v, choose an edge e so that $\operatorname{sgn}(e)w_e$ has the same sign as the vertex weight v. Decrease the weight of e by one. Continue this process for the other vertex or vertices in e. Since the original weights had the other vertex or vertices in e with weight zero, they now have a non-zero weight. Note that if the process encounters another of the original non-zero weight vertices, the process may stop if the vertex being considered now has weight zero, otherwise we ignore this vertex's weight and continue. If there are no more vertices with non-zero vertex weight produced by a sequence of hyperedges starting from v, then we have a hyperwalk. Repeat this process for all vertices with non-zero weight. If there are no remaining vertices of non-zero weight then the remaining edge weights form a kernel element. □

7.3 The Completely Separable Case

In this section, we study the case of a particularly nice set of hypergraphs, hypergraphs which are completely separable. Recall that a hypergraph is *completely separable* for each hyperedge e with more than two vertices, $G \setminus e$ has each of the vertices of e in a separate component.

Definition 7.3.1. Let $G = (V, E_2 \cup E_3)$ be a signed hypergraph and let H be a subhypergraph in G. Define the *signature* of vertex u, where $\deg_H(u) = 2$, of H as the average of the signs of the hyperedges incident to u in H, i.e.,

$$\operatorname{sig}_H(u) = \frac{1}{2}\left(\sum_{uv \in E_2} \operatorname{sgn}(uv) + \sum_{uvw \in E_3} \operatorname{sgn}(uvw)\right).$$

Corollary 7.3.2. Let $G = (V, E)$ be a hypergraph where E' is the set of hyperedges with more than two vertices, and for each hyperedge $e \in E$, a_e is a non-negative weight. Let $G' = G \setminus E'$ be the graph produced by deleting the hyperedges from G. Then there is a subgraph H of G' where each component of H is a tree, or a unicyclic graph with an odd cycle. Moreover, there are positive edge weights b_e on H so that $\sum_{e \in G'} a_e \rho(e) = \sum_{e \in H} b_e \rho(e)$.

Proof. Proof follows from Corollary 4.2.21 applied to G'. □

Theorem 7.3.3. Let $G = (V, E)$ be a completely separable hypergraph. Then, G is normal if and only if for every set of vertex disjoint odd cycles connected by a set of hyperwalks, the set is connected by a set of alternating hyperwalks.

Proof. Note that Proposition 7.2.8 shows the contrapositive that any hypergraph G which fails this condition is not normal. Thus, it suffices to prove that any hypergraph which does satisfy this condition is normal.

Let $G = (V, E)$ be a separable hypergraph with E' the set of hyperedges with more than two vertices. Let $\alpha \in \mathbb{Z}^V$ be a set of vertex weights in $\mathcal{L}_G \cap \operatorname{cone}(\mathcal{P}_G)$, and let $a \in \mathbb{R}_+^E$ be a set of non-negative hyperedge weights that induce vertex weights α. Consider the graph $G' = G \setminus E'$, that is the graph produced by deleting the hyperedges, but not the vertices contained in the hyperedges. By Corollary 7.3.2, we may assume that the edges with non-zero edge weights in G' form trees and unicyclic graphs with odd cycles, we call this subgraph H. Moreover, since it suffices to give non-negative integral hyperedge weights for the given vertex weights, we may assume that each of the non-zero hyperedge weights are in the interval $(0, 1)$, as we can subtract the non-negative integer $\lfloor a_e \rfloor$ from each edge weight. Note that this condition implies that the vertices of degree 1 in H are in hyperedges of G.

We give a construction so that when it terminates, we have a set of odd cycles, connected by paths to hyperedges, and vertices in hyperedges connected to other vertices in hyperedges by paths. Moreover, except for a unique vertex in each odd cycle, each vertex has degree two: Suppose $e \in E'$ has non-zero weight. Then for each i a vertex of e, there exists another hyperedge with non-zero weight. Iterating this process, we construct a path from i to either, an odd cycle, or to another edge in E'. If the path reaches an edge in E', we repeat the process for the other vertices.

By assigning weight ϵ to e and $\pm\epsilon$ to each edge and hyperedge in the paths, we obtain a hyperwalk that has a leaf in each odd cycle. Assigning weights $\pm\frac{\epsilon}{2}$ to the edges of the odd cycles together with the earlier assignments produces a kernel element because each vertex now has weight zero. Choose ϵ so that the resulting hyperedge weights are non-negative, and at least one of them is integral, this can be done because, by assumption, the weights are non-negative. Remove the integral hyperedge weights and repeat this process for another hyperedge $e \in E'$ with positive weight.

This process depends only on there being a hyperedge in E' with non-zero weight. Thus, we can no longer find construct such a kernel elements when all the hyperedges in E' have weight zero. By assumption, this implies that the remaining edges with non-zero edge weight form odd cycles. For an odd cycle C with non-zero edge weights, the vertex weights are $\text{sig}_C(v)$ for each vertex $v \in C$.

We now show that there must be an alternating hyperwalk in order for the vertex weights to be in the lattice $\mathcal{L}_G$. By assigning weight 1 or 0 to each edge in the odd cycles found above we produce vertex weights $\text{sig}_C(v)$ for all but one vertex in each odd cycle C, the remaining vertex has weight 0 when $\text{sig}_C(v) \neq 0$. By assumption, there was a set of integral hyperedge weights that produced the given vertex weights, we subtracted non-negative integers from each hyperedge weight, thus we have a set of hyperedge weights that give weight ± 1 on distinct vertices.

By Proposition 7.2.9, we know that the hyperedge weights can be decomposed into a set of hyperwalks and a kernel element. By the assumption, this implies there are a set of alternating hyperwalks with a leaf in each odd cycle. Using these hyperwalks, and the assignments of weight 1 and 0 to the edges of the odd cycles, we produce a set of non-negative integer edge weights that give $\mathrm{sig}_C(v)$ on each of the odd cycles. □

Definition 7.3.4. Let G be a completely separable hypergraph. A set of odd cycles is said to be *hyperwalk connected* if there is a set of hyperwalks so that each odd cycle contains exactly one leaf, if there are alternating hyperwalks with this property they are said to be *alternating hyperwalk connected.*

Definition 7.3.5. Let $G = (V, E)$ be a completely separable hypergraph. A set of odd cycles $\{C_1, \dots, C_r\}$ is said to be *exceptional* if, they are hyperwalk connected but not alternating hyperwalk connected.

For a set of exceptional odd cycles $\Pi = \{C_1, \dots, C_r\}$, define the monomial

$$M_\Pi = \prod_{i=1}^{r} \prod_{\ell \in C_i} x_\ell^{\mathrm{sig}_{C_i}(\ell)} \tag{7.2}$$

Theorem 7.3.6. Let $G = (V, E)$ be a separable hypergraph and $k[G]$ the edge ring of G. Let $\Pi_1 = \{C_1^1, \dots, C_{r_1}^1\}, \dots, \Pi_q = \{C_1^q, \dots, C_{r_q}^q\}$ denote the exceptional sets of odd cycles in G. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by the monomials $M_{\Pi_1} \dots, M_{\Pi_q}$ as an algebra over $k[G]$.

Proof. In the proof of Theorem 7.3.3, proving that some $\alpha \in S_1$ is in S_2 amounted to proving that $\sum_{i=1}^{r} \alpha'_{C_i}$ is in S_2 as well for a set of odd cycles $\{C_1, \dots, C_r\}$ which are hyperwalk connected (where α' are the reduced weights determined in Theorem 7.3.3). If the odd cycles are not an exceptional set, then the proof of Theorem 7.3.3 shows that $\sum_{i=1}^{r} \alpha'_{C_i}$ is in S_2. Therefore, extending S_2 by $\sum_{i=1}^{r} \alpha'_{C_i}$ when $\{C_1, \dots, C_r\}$ are an exceptional set extends S_2 to S_1. Since $x^{\sum_{i=1}^{r} \alpha'_{C_i}} = M_\Pi$ for $\Pi = \{C_1, \dots, C_r\}$, this extension extends $k[G]$ by M_Π where Π is exceptional. □

7.4 The Separable Case

Note that the special case studied in Section 7.3 illustrates that the normality behavior of hypergraphs is similar to that of graphs when the hyperedges with more than two verties have the completely separable condition. However, this is not the case when there are hyperedges in E so that $\text{Comp}(G \setminus e) = \text{Comp}(G) + 1$, even when the hypergraph is separable and has only 2-vertex and 3-vertex hyperedges. In order to see this, consider Example 7.4.1.

Example 7.4.1. Let G be the hypergraph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and hyperedge set $\{+11, +12, +234, +35, +45, +56, +67, +77\}$, see Figure 7.7. Consider the vertex weights $(1, 2, 3, 4, 5, 6, 7) = (1, 1, 1, 1, 1, 0, 1)$. This set of vertex weights are given by non-negative hyperedge weights $\frac{1}{4}$ on $+11$, $\frac{1}{2}$ on $+23$, $+234$, $+35$, $+45$ and $+77$. The vertex weights are also given by integer hyperedge weights 1 on $+12$, $+35$, $+45$, $+67$, and weight -1 on $+56$. Moreover, this set of vertex weights are not in $\mathbb{N}\rho(E(G))$. To see this, observe that the vertex weights are not in $\mathbb{N}\rho(E(G_2))$, as the graph violates the odd cycle condition.

Note that in Example 7.4.1 the subgraph given by the hyperedges $+234$ $+35$ and $+45$ behaves like an odd cycle. By that we mean that it does not have a perfect matching. As in the unsigned case, normality is equivalent to showing that certain sets of vertices have a perfect matching between them, we use terminology inspired by Edmonds' Blossom Algorithm [9].

Definition 7.4.2. A subgraph H of a hypergraph G given by an edge $e = \text{sgn}(e)ijk$ and a path $P = v_1 v_2 \cdots v_\ell$, where v_1 and v_ℓ are distinct vertices in $\{i, j, k\}$, is a *bud*, the vertex not equal to v_1 or v_ℓ is called the *stem* of the bud. If ℓ is even P and the hyperedge is referred to as an *even bud* and if ℓ is odd then it is referred to as an *odd bud*. Note that if the path P contains a hyperedge, then the bud can have more than one stem.

Note that in Example 7.4.1, there is an odd bud $+234, +35, +45$. For a graph G, an odd bud given by hyperedge $\text{sgn}(e)ijk$ and path P with leaves i and j, has an odd cycle

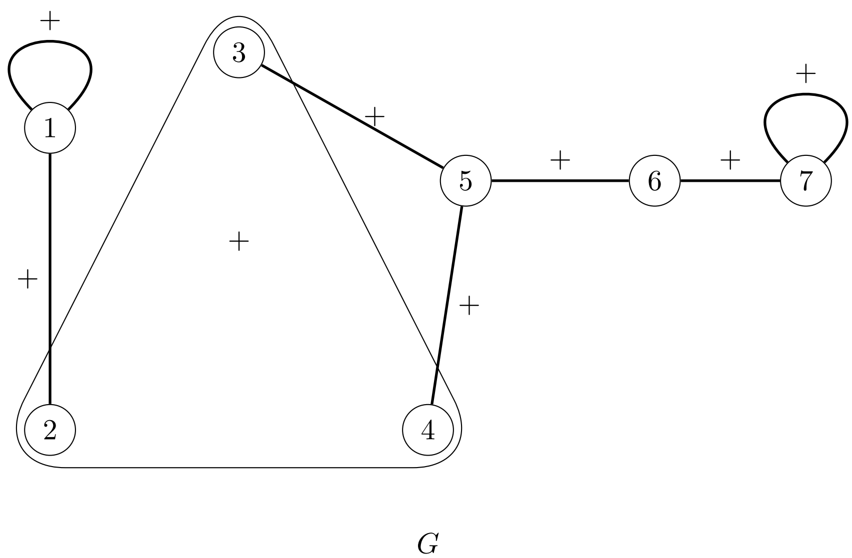


Figure 7.7: The hypergraph G with vertex set $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and hyperedge set $E(G) = \{+11, +12, +234, +35, +45, +56, +67, +77\}$. The subhypergraph given by $\{+11, +12, +234, +35, +45\}$ is a bush with odd bud given by $\{+234, +35, +45\}$. The edge ring of G is not normal by Theorem 7.4.10.

in G_k given by P and the edge $\text{sgn}(e)ij$. However, the interaction between odd buds is complex, as seen from the following examples.

Example 7.4.3. Let G be the hypergraph on vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d\}$ and hyperedge set $\{11, 12, 23, 34, 35, 456, 67, 78, 89a, 9b, ab, bc, cd, dd\}$. Note that this hypergraph has two odd buds, given by $\{34, 35, 456\}$ and $\{89a, 9b, ab\}$. The set of vertex weights $(1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1)$, that is weight 1 on each vertex except 0 on 2 and c, can be obtained by placing non-negative weights $\frac{1}{2}$ on 11, 34, 35, 456, 67, 78, $89a$, $9b$, ab and dd. The vertex weights $(1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1)$ can also be obtained by integer weights, by placeing weight 1 on 12 34, 35, 67, $89a$, bc and dd, and weight -1 on 23 and cd. Moreover, we know that this set of vertex weights are not in $\mathbb{N}\rho(E(G))$ as it is not in $\mathbb{N}\rho(E(G_6))$.

Example 7.4.4. Let G be the hypergraph on vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b\}$ and hyperedge set $\{11, 12, 23, 24, 345, 56, 67, 789, 8a, 9a, ab, bb\}$. Consider the vertex weights $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. This set of vertex weights can be obtained by placing a non-

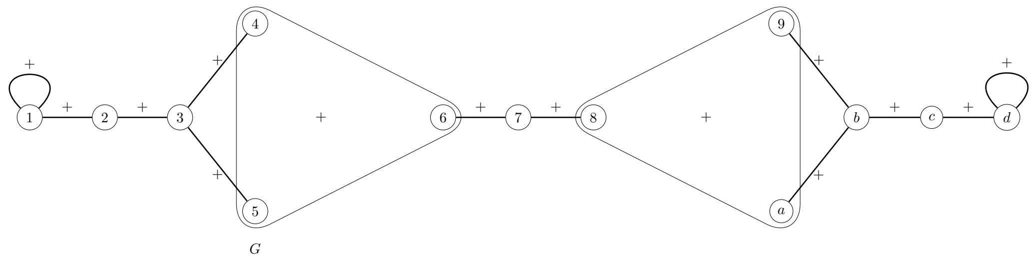


Figure 7.8: G, the hypergraph on vertex set $\{1,2,3,4,5,6,7,8,9,a,b,c,d\}$ and hyperedge set $\{11,12,23,34,35,456,67,78,89a,9b,ab,bc,cd,dd\}$. With a irreducible bush on vertices $\{3,4,5,6,7,8,9,a,b\}$, odd buds on $\{\{3,4,5,6\}$ and $\{8,9,a,b\}$. The edge ring of G is not normal by Theorem 7.4.10.

negative weight of $\frac{1}{2}$ on all the hyperedges, except 12 and ab, which have a weight of zero. This set of vertex weights can be obtained by placing integer weights of 1 on 12, 345, 67, $8a$, $9a$ and bb, and weight -1 on ab. Note that in G_5 and G_7, the odd cycles produced from the odd buds do not violate the odd cycle condition. However, this set of vertex weights can not be produced in $\mathbb{N}\rho(E(G))$. To see this, note that the weights of 11 and bb must be zero, and hence 12, 345, 789 and ab must have weight 1. However, this implies that 6 has weight zero as 56 and 67 must both have weight 0. Thus, this is an example of when $\mathbb{N}\rho(E(G)) \neq \bigcap_{H\in\mathcal{G}(G)} \pi(\mathbb{N}\rho(E(H)))$.

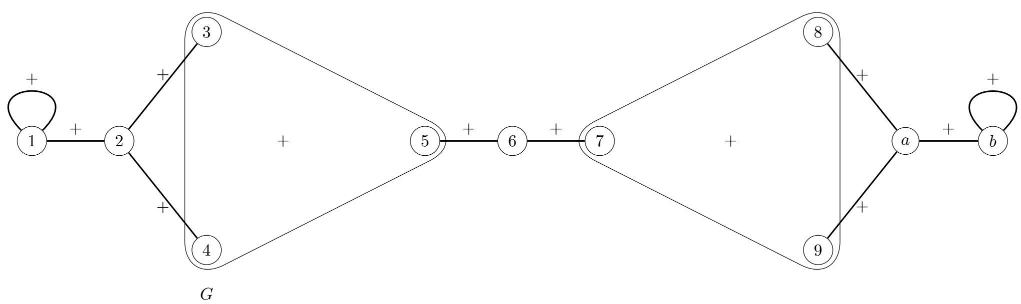


Figure 7.9: G, the hypergraph on vertex set $\{1,2,3,4,5,6,7,8,9,a,b\}$ and hyperedge set $\{11,12,23,24,345,56,67,789,8a,9a,ab,bb\}$. The graph does not satisfy the conditions for Theorem 7.4.10, and is not normal. This is also an example of when $\mathbb{N}\rho(E(G)) \neq \bigcap_{H\in\mathcal{G}(G)} \pi(\mathbb{N}\rho(E(H)))$.

In the case where the hypergraph is completely separable, we found a kernel element, which was, essentially, a hyperwalk and a collection of odd cycles. If G is separable but not completely separable hypergraph, suppose i and j are in the same component of $G \setminus \{ijk\}$, then we can consider G_k. After performing this replacement on all the hyperedges $e \in E_3$ where $\mathrm{Comp}(G \backslash e) = \mathrm{Comp}(G)+1$, we have produced a hypergraph, G', which is completely separable. Applying the proof of Theorem 7.3.3, we can find a subgraph of G' that is a hyperwalk and a collection of odd cycles, one for each leaf of the hyperwalk. However, if $e = ijk$ is a hyperedge that was replaced to form G', and ij is used in the subgraph, then the subgraph can not be used to produce a kernel element.

If the kernel element of G' changes ij by ϵ, we need to change kt_k by ϵ as well. While kt_k has another hyperedge of non-negative weight incident to it, it is possible that the path produced by iteratively choosing incident hyperedges with non-negative weights will lead to a bud. If all of these paths terminate at an odd cycle, or an even bud, then we can assign $\pm\epsilon$ to the hyperedges in the hyperwalk and the even buds, and $\pm\frac{\epsilon}{2}$ to the edges of the odd cycles to obtain a kernel element of G. However, if the path terminates at an odd bud, we can not construct such a kernel element. We thus, need to describe what these subgraphs look like.

Definition 7.4.5. A *bush* is a non-cycle connected subgraph of G which can be decomposed into a collection of buds, odd cycles and walks running between the stems of the buds vertices in the odd cycles, and vertices in hyperedges. A bush is called *trivial* if it does not contain any odd buds. and it is called *irreducible* if it does not contain a trivial bush as a subgraph.

Corollary 7.4.6. Let G be a hypergraph with non-negative hyperedge weights a. If the positive hyperedge weights form a non-irreducible bush, then there are non-negative hyperedge weights, b, where the non-integral edges form a collection of odd cycles, and irreducible bushes.

Proof. It suffices to show that there is an assignment of weights to the hyperedges of a trivial bush that produces a kernel element. Note that, for a path, if a positive edge is given weight ϵ, then we can assign weight $-\epsilon$ to an incident edge e if $\mathrm{sgn}(e) = 1$ and ϵ if $\mathrm{sgn}(e) = -1$. Note that this guarantees that the common vertex has weight zero. For every vertex of even degree this process produces a vertex of weight zero. A vertex of odd degree occurs only when a walk has a leaf at a vertex of an odd cycle. If the path vertex is labeled ϵ, assign $\pm\frac{\epsilon}{2}$ to the two edges of the odd cycle so that the sum is zero, and repeat for the remaining hyperedge weights.

This gives a kernel element that contains the hyperedges of the trivial bush. Choose ϵ so that when the assigned weights get added to the weights in w, they remain non-negative, and at least one of them is integral. Repeat this process on the non-integral edge weights. When we can no longer find a trivial bush, we have a set of integral hyperedge weights, and a set of non-integral weights which form a collection of irreducible bushes and odd cycles. □

The following observation follows from Theorem 7.1.7.

Observation 7.4.7. Let G be a hypergraph with non-negative hyperedge weights taken from the interval $[0, 1)$ that induce a set of vertex weights in $\mathcal{L}_G$. Suppose there is a bud and after replacing the stems of the bud with two edges and an artificial vertex we get a cycle, C in a hypergraph G'. If all the vertices of C have weight $\mathrm{sig}_C(v)$, then there is a vertex, not in C, with non-zero weight in the component containing C.

At this point, to summarize, given a hypergraph G, and a set of non-negative hyperedge weights that induce a set of vertex weights in $\mathcal{L}_G$, Corollary 7.3.2 tells us that the positive non-integer weights may be assumed to be on odd cycles, hyperedges, and paths connecting vertices in odd cycles, to hyperedges, and hyperedges to other hyperedges. The definition of a bush tells us that this is a collection of bushes and odd cycles. By Corollary 7.4.6, the bushes may be rewritten as a collection of irreducible bushes and odd cycles. Finally by Observation 7.4.7, we know that each odd cycle and odd bud with vertex weights $\mathrm{sig}_C(v)$ can not be in a component where all the other vertices have weight 0, after applying the correct replacements.

Definition 7.4.8. A set of odd cycles and odd buds are said to be *hyperwalk connected*, and respectively *alternating hyperwalk connected* if, after replacing the stems and turning the buds into cycles they are hyperwalk connected, respectively, alternating hyperwalk connected.

Theorem 7.4.9. Let $G = (V, E_2 \cup E_3)$ be a hypergraph and let w be a set of non-negative hyperedge weights. Suppose the hyperedges with non-zero weight in w form a set of pairwise disjoint and irreducible bushes and odd cycles, G' so that the weights on the odd cycles are $\mathrm{sig}_{G'}(u)$, and the weights on the odd cycles and buds are the canonical weights. If the vertex weights are in $\mathcal{L}_G$, then they are hyperwalk connected

Proof. Suppose that the vertex weights are in $\mathcal{L}_G$. Replace the stems of the odd buds so that the odd buds become odd cycles, call the resulting hypergraph H. By Theorem 7.1.7 we know that the vertex weights given by w are in $\pi(\mathcal{L}_H)$.

Assume a set of hyperedge weights that give these vertex weights is given by integer hyperedge weights a. Since the sum of the vertex weights on each odd cycle, with non-zero edge weights in w, is odd, we can assign integer edge weights to the odd cycles with vertex weights $\mathrm{sig}_C(u)$ for each odd cycle C and vertex u in C, except for one vertex say u_C for each in each odd cycle C. This vertex has $\mathrm{sig}_C(u) \neq 0$, but the edge weights have weight zero. Subtracting these new weights from the weights a gives a set of weights that produce vertex weight zero on all vertices except u_C, which has weight ± 1. By Proposition 7.2.9, the resulting set of integer hyperedge weights can be decomposed into a set of hyperwalks and a kernel element. Thus, the odd cycles are hyperwalk connected in H, and hence the odd buds and odd cycles are hyperwalk connected in G. □

Theorem 7.4.10. Let $G = (V, E_2 \cup E_3)$ be a hypergraph and let w be a set of non-negative hyperedge weights that induce a set of vertex weights in $\mathcal{L}_G$. Suppose the hyperedges with non-zero weight in w form a set of irreducible bushes and odd cycles that are not alternating hyperwalk connected. Then $k[G]$ is not normal.

Proof. By Theorem 7.4.9 we know that the odd cycles and odd buds with non-zero hyperedge weights in w are hyperwalk connected. Let H be the hypergraph formed by turning the buds into cycles. Note that H is a completely separable hypergraph. By Theorem 7.3.3,

$k[H]$ is not a normal. By assumption, the vertex weights are in cone$(\mathcal{P}_G) \cap \mathcal{L}_G$ and hence in $\mathcal{A}(\mathcal{P}_G)$ but not in $k[G]$, and thus, $k[G]$ is not normal. □

7.5 Future Directions

- Completely characterize the bushes.
- Determine a combinatorial characterization for normality when Theorem 7.4.10 is not satisfied.
- Determine the lattice for completely separable hypergraphs, and the condition to be a facet of $\mathcal{P}_G$.

Chapter 8

Summary

In this chapter we summarize the main results of Chapter 3 through Chapter 7 grouping them by the nature of the theorem. This, we hope, allows the reader to highlight the similarities and the differences between the edge rings and polytopes for various combinatorial structures.

8.1 Dimension Formulas

In this section we list the results relating to the dimension of the edge polytope, or the cone of the edge polytope for various combinatorial structures.

Propsition 3.1.9. Let G be a graph on d vertices with $\mathrm{BiComp}(G) = k$. Then,

$$\dim \mathcal{P}_G = d - k - 1.$$

Theorem 4.1.19. Let G be a signed graph on n vertices, then

$$\dim \mathcal{P}_G = n - \mathrm{BiComp}(G) - \mathrm{Strat}(G).$$

Proposition 5.2.2. Let G be a mixed signed directed graph on n vertices, then

$$\dim \text{cone}(\mathcal{P}_G) = n - \text{BiComp}(\widetilde{G}).$$

Theorem 6.2.13. Let G be a directed graph with $k = \text{Comp}(G)$, on d vertices. Then,

$$\dim \mathcal{P}_G = d + 1 - k - \delta_{Bal}(G).$$

8.2 Normality Condition

In this section we list the combinatorial characterization determining normality of edge rings for various combinatorial structures.

Theorem 3.2.18. Given a graph G, $k[G]$ is normal if and only if G satisfies the odd cycle condition.

Theorem 4.2.22. Let G be a signed graph, then $k[G]$ is normal if and only if G satisfies the odd cycle condition.

Theorem 5.1.1. Let G be a mixed signed, directed graph and $\widetilde{G}$ the augmented signed graph of G, then $k[G] = \mathcal{A}(\mathcal{P}_G)$ if and only if $\widetilde{G}$ satisfies the odd cycle condition.

Corollary 6.3.8. Let G be a directed graph with a directed cycle, and $g = \gcd\{sig(C) :$ C is a cycle of G$\}$. Then, $k[G]$ is normal if and only if $s^g \in k[G]$.

Theorem 7.3.3. Let $G = (V, E)$ be a completely separable hypergraph. Then, G is normal if and only if for every set of vertex disjoint odd cycles connected by a set of hyperwalks, the set is connected by a set of alternating hyperwalks.

8.3 Normalization

In this section we list the monomials that give the normalization of edge rings for various combinatorial structures.

Definition 3.2.20. We say a pair $\Pi = \{C, C'\}$ of minimal odd cycles in a component of G that are vertex disjoint are *exceptional* if there exists no edge connecting C and C' in G. Given an exceptional pair $\Pi = \{C, C'\}$ we write

$$\frac{1}{2}\rho(\Pi) = \frac{1}{2}\sum_{ij \in C} \rho(ij) + \frac{1}{2}\sum_{ij \in C'} \rho(ij),$$

and $M_\Pi = (\prod_{i \in V(C)} t_i)(\prod_{j \in V(C')} t_j) \in k[t_1, \dots, t_d]$.

Theorem 3.2.21. Let G be a finite graph possibly with loops and $k[G]$ be the edge ring. Let $\Pi_1 = \{C_1, C_1'\}, \dots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs of minimal odd cycles in G. Then the normalization of $k[G]$ is generated by the monomials $M_{\Pi_1} \dots, M_{\Pi_q}$ as an algebra over $k[G]$

Definition 4.2.24. Let G be a signed graph. We say a pair $\Pi = \{C, C'\}$ of odd cycles in a component of G that are vertex disjoint are *exceptional* if there does not exist an alternating path connecting C and C' in G. Given an exceptional pair $\Pi = \{C, C'\}$, let

$$\frac{1}{2}\rho(\Pi) = \frac{1}{2}\sum_{\pm ij \in C} \rho(ij) + \frac{1}{2}\sum_{\pm ij \in C'} \rho(ij),$$

and

$$M_\Pi = x^{\frac{1}{2}\rho(\Pi)} = \prod_{\ell \in C} x_\ell^{\mathrm{sig}_C(x_\ell)} \prod_{\ell \in C'} x_\ell^{\mathrm{sig}_{C'}(x_\ell)}$$

in $k[x_1, \dots, x_n]$.

Corollary 4.2.25. Let G be a signed graph and $k[G]$ the edge ring of G. Let $\Pi_1 = \{C_1, C_1'\}, \dots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs of odd cycles in G. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by the monomials $M_{\Pi_1} \dots, M_{\Pi_q}$ as an algebra over $k[G]$.

Theorem 5.1.6. Let G be a mixed signed, directed graph, and $\widetilde{G}$ the associated augmented signed graph. If the normalization of $k[\widetilde{G}]$, $\mathcal{A}(\mathcal{P}_{\widetilde{G}})$, is generated by monomials $M_{\Pi_1}, \dots, M_{\Pi_m}$, over $k[\widetilde{G}]$, then the normalization of $k[G]$, $\mathcal{A}(\mathcal{P}_G)$, is generated by $M_{\Pi_1}, \dots, M_{\Pi_m}$ over $k[G]$.

Corollary 6.3.9. Suppose G has a directed cycle, and let $g = \gcd\{\mathrm{sig}(C) : C \text{ is a cycle of } G\}$. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by s^g as an algebra over $k[G]$.

Theorem 6.3.14. Let G be a directed graph without a directed cycle Let $\{R_j\}_{j=1}^m$, and $\{T_j\}_{j=1}^m$ denote the pairs of sets so that $R_j \cap T_j = \emptyset$ and $max - min(R_j, T_j)$ is a closed walk of G. Let $\mathcal{M} = \{x^{e_{T_j} - e_{R_j}} s^{n_j}\}_{j,n_j}$ be the set of monomials where $j = 1, \ldots, m$ and, for each j, if $l_1 = \min\{n_j \ : \ x^{e_{T_j} - e_{R_j}} s^{n_j} \in k[G]\}$ and $l_2 = \max\{n_j \ : \ x^{e_{T_j} - e_{R_j}} s^{n_j} \in k[G]\}$, then n_j ranges over $\{l_1, l_1 + g, \ldots, l_2\}$. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by $\mathcal{M}$ as an algebra over $k[G]$.

Theorem 7.3.6. Let $G = (V, E)$ be a separable hypergraph and $k[G]$ the edge ring of G. Let $\Pi_1 = \{C_1^1, \ldots, C_{r_1}^1\}, \ldots, \Pi_q = \{C_1^q, \ldots, C_{r_q}^q\}$ denote the exceptional sets of odd cycles in G. Then, $\mathcal{A}(\mathcal{P}_G)$ is generated by the monomials $M_{\Pi_1} \ldots, M_{\Pi_q}$ as an algebra over $k[G]$. Here, $M_\Pi = \prod_{i=1}^r \prod_{\ell \in C_i} x_\ell^{\mathrm{sig}_{C_i}(\ell)}$.

8.4 Serre's R_1 Condition

In this section we list the combinatorial condition that determines when edge rings for various combinatorial structures satisfy Serre's R_1 condition.

Proposition 3.3.8. Let G be a finite, non-bipartite graph. $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with exactly one bipartite component $L \cup R$ that is not bipartite in G, obtained from G be deleting edges incident to L and not R and satisfies $\mathrm{BiComp}(G') = \mathrm{BiComp}(G)$, has $\mathrm{Comp}(G') \leq \mathrm{Comp}(G) + 1$.

Theorem 4.3.8. Let G be a graph, then $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with a unique new bipartite component $L \cup R$ obtained from G by deleting positive edges incident to L and not to R, and negative edges incident to R and not to L satisfies: $\mathrm{Comp}(G') - \mathrm{BiComp}(G') \leq \mathrm{Comp}(G) - \mathrm{BiComp}(G)$.

Theorem 5.2.5. Let G be a graph. $k[G]$ satisfies Serre's R_1 condition if and only if for every subgraph G' with a unique new bipartite component $L \cup R$ in $\widetilde{G'}$ obtained from G by deleting positive edges incident to L and not to R, negative edges incident to R and not to L and directed edges (i,j) where $j \in L$ or $i \in R$ satisfies: $\text{Comp}(\widetilde{G'}) - \text{BiComp}(\widetilde{G'}) \leq \text{Comp}(\widetilde{G}) - \text{BiComp}(\widetilde{G})$.

8.5 Future Directions

In this section we list, again, the future directions for the relevant chapters:

For signed graphs:

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_ℓ condition, for $\ell > 1$.
- Determine a combinatorial condition for when an edge ring satisfies Serre's S_ℓ condition, for all ℓ.
- Give a formula or relation determining the face complex of an edge ring, generalizing [23] which counts the number of 1-dimensional faces.

For mixed signed directed graphs:

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_ℓ condition, for $\ell > 1$.
- Determine a combinatorial condition for when an edge ring satisfies Serre's S_ℓ condition, for all ℓ.
- Characterize the subgraphs which give facets of the edge polytope.
- Find the algorithmic complexity of determining normality, and computing the normalization. Similar problems are NP-hard.

For homogenized edge rings of directed graphs:

- Improve the normalization condition for Homogenized Directed Graphs. In the current form it is not difficult to tell if it is or is not normal, is there a condition that does not require checking to see if it has a directed cycle or is balanced?

- Determine a combinatorial condition for when an edge ring satisfies Serre's R_1 condition, for $\ell > 1$.
- Generalize results to when the homogenizing variable is raised to different powers than just 1.

For signed hypergraphs:

- Completely characterize the bushes.
- Determine a combinatorial characterization for normality when Theorem 7.4.10 is not satisfied.
- Determine the lattice for completely separable hypergraphs, and the condition to be a facet of $\mathcal{P}_G$.

Bibliography

[1] Adrián Alcántar. Noether normalizations of some subrings of graphs. *Communications in Algebra*, 29(12):5525–5534, 2001.

[2] I. Bermejo, I. Garcí a Marco, and E. Reyes. Graphs and complete intersection toric ideals. *Journal of Algebra and its Applications*, 14(9):1540011, 37, 2015.

[3] W. Bruns and J. Herzog. *Cohen-Macaulay Rings, Revised Ed.* Cambridge University Press, 1996.

[4] Winfried Bruns, Wolmer V. Vasconcelos, and Rafael H. Villarreal. Degree bounds in monomial subrings. *Illinois Journal of Mathematics*, 41(3):341–353, 09 1997.

[5] Giuseppa Carra' Ferro and Daniela Ferrarello. Ideals and graphs, Gröbner bases and decision procedures in graphs. *Discrete Mathematics*, 308(2-3):287–298, 2008.

[6] G. Chartrand and P. Zhang. *Introduction to Graph Theory.* McGraw-Hill, 2005.

[7] D. Cox, J. Little, and H. Schenck. *Toric Varieties.* American Mathematical Society, 2011.

[8] Alessio D'Alì. Toric ideals associated with gap-free graphs. *Journal of Pure and Applied Algebra*, 219(9):3862–3872, 2015.

[9] Jack Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449 – 467, 1965.

[10] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, 2004.

[11] Oscar Fernández-Ramos and Philippe Gimenez. First nonlinear syzygies of ideals associated to graphs. *Communications in Algebra*, 37(6):1921–1933, 2009.

[12] Isidoro Gitler and Carlos E. Valencia. Bounds for invariants of edge-rings. *Communications in Algebra*, 33(5):1603–1616, 2005.

[13] Isidoro Gitler and Carlos E. Valencia. Multiplicities of edge subrings. *Discrete Mathematics*, 302(1-3):107–123, 2005.

[14] Huy Tài Hà and Kuei-Nuan Lin. Normal 0-1 polytopes. *SIAM Journal on Discrete Mathematics*, 29(1):210 – 223, 2015.

[15] Jürgen Herzog, Takayuki Hibi, and Ngô Viêt Trung. Symbolic powers of monomial ideals and vertex cover algebras. *Advances in Mathematics*, 210(1):304–322, 2007.

[16] Jürgen Herzog, Takayuki Hibi, and Xinxian Zheng. Monomial ideals whose powers have a linear resolution. *Mathematica Scandinavica*, 95(1):23–32, 2004.

[17] Takayuki Hibi, Akihiro Higashitani, Lukas Katthän, and Ryota Okazaki. Normal cyclic polytopes and cyclic polytopes that are not very ample. *Journal of the Australian Mathematical Society*, 96(1):61–77, 2014.

[18] Takayuki Hibi, Akihiro Higashitani, Kyouko Kimura, and Augustine B. O'Keefe. Depth of edge rings arising from finite graphs. *Proceedings of the American Mathematical Society*, 139(11):3807–3813, 2011.

[19] Takayuki Hibi, Akihiro Higashitani, Kyouko Kimura, and Augustine B. O'Keefe. Depth of initial ideals of normal edge rings. *Communications in Algebra*, 42(7):2908–2922, 2014.

[20] Takayuki Hibi and Lukas Katthän. Edge rings satisfying Serre's condition (R1). *Proceedings of the American Mathematical Society*, 142(7):2537–2541, 2014.

[21] Takayuki Hibi, Nan Li, and Yan X. Zhang. Separating hyperplanes of edge polytopes. *Journal of Combinatorial Theory. Series A*, 120(1):218–231, 2013.

[22] Takayuki Hibi, Kazunori Matsuda, and Hidefumi Ohsugi. Strongly Koszul edge rings. *Acta Mathematica Vietnamica*, 41(1):69–76, 2016.

[23] Takayuki Hibi, Aki Mori, Hidefumi Ohsugi, and Akihiro Shikama. The number of edges of the edge polytope of a finite simple graph. *Ars Mathematica Contemporanea*, 10(2):323–332, 2016.

[24] Akihiro Higashitani. Non-normal very ample polytopes and their holes. *Electronic Journal of Combinatorics*, 21(1):Paper 1.53, 12, 2014.

[25] Akihiro Higashitani. Smooth Fano polytopes arising from finite directed graphs. *Kyoto Journal of Mathematics*, 55(3):579–592, 2015.

[26] Akihiro Higashitani. Minkowski sum of polytopes and its normality. *Journal of the Mathematical Society of Japan*, 68(3):1147–1159, 2016.

[27] Lukas Katthän. Non-normal affine monoid algebras. *Manuscripta Mathematica*, 146(1-2):223–233, 2015.

[28] Mordechai Katzman. Bipartite graphs whose edge algebras are complete intersections. *Journal of Algebra*, 220(2):519–530, 1999.

[29] Kyouko Kimura, Naoki Terai, and Ken-ichi Yoshida. Arithmetical rank of squarefree monomial ideals of small arithmetic degree. *Journal of Algebraic Combinatorics*, 29(3):389–404, 2009.

[30] Kuei-Nuan Lin and Jason McCullough. Hypergraphs and regularity of square-free monomial ideals. *International Journal of Algebra and Computation*, 23(07):1573–1590, 2013.

[31] Florian Luca and Rita Zuazua. Standard Noether normalizations of the graph subring. *Journal of Algebraic Combinatorics. An International Journal*, 24(4):415–436, 2006.

[32] José Martínez-Bernal, Edwin O'Shea, and Rafael Villarreal. Ehrhart clutters: Regularity and max-flow min-cut. *the Electronic Journal of Combinatorics*, 17:# R52, 2010.

[33] Tetsushi Matsui. Ehrhart series for connected simple graphs. *Graphs and Combinatorics*, 29(3):617–635, 2013.

[34] Tetsushi Matsui, Akihiro Higashitani, Yuuki Nagazawa, Hidefumi Ohsugi, and Takayuki Hibi. Roots of Ehrhart polynomials arising from graphs. *Journal of Algebraic Combinatorics. An International Journal*, 34(4):721–749, 2011.

[35] Susan Morey. Depths of powers of the edge ideal of a tree. *Communications in Algebra*, 38(11):4042–4055, 2010.

[36] Hidefumi Ohsugi. Toric ideals and an infinite family of normal $(0,1)$-polytopes without unimodular regular triangulations. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 27(4):551–565, 2002.

[37] Hidefumi Ohsugi, Jürgen Herzog, and Takayuki Hibi. Combinatorial pure subrings. *Osaka Journal of Mathematics*, 37(3):745–757, 2000.

[38] Hidefumi Ohsugi and Takayuki Hibi. Normal polytopes arising from finite graphs. *Journal of Algebra*, 207(2):409 – 426, 1998.

[39] Hidefumi Ohsugi and Takayuki Hibi. Toric ideals generated by quadratic binomials. *Journal of Algebra*, 218(2):509–527, 1999.

[40] Hidefumi Ohsugi and Takayuki Hibi. Hamiltonian tournaments and Gorenstein rings. *European Journal of Combinatorics*, 23(4):463–470, 2002.

[41] Hidefumi Ohsugi and Takayuki Hibi. Special simplices and Gorenstein toric rings. *Journal of Combinatorial Theory. Series A*, 113(4):718–725, 2006.

[42] Hidefumi Ohsugi and Takayuki Hibi. Toric ideals and their circuits. *Journal of Commutative Algebra*, 5(2):309–322, 2013.

[43] Hidefumi Ohsugi and Takayuki Hibi. A Gröbner basis characterization for chordal comparability graphs. *European Journal of Combinatorics*, 59:122–128, 2017.

[44] William R. Pulleyblank. Edmonds, matching and the birth of polyhedral combinatorics. *Documenta Mathematica*, Extra Volume ISMP:181 – 197, 2012.

[45] Enrique Reyes, Christos Tatakis, and Apostolos Thoma. Minimal generators of toric ideals of graphs. *Advances in Applied Mathematics*, 48(1):64–78, 2012.

[46] Enrique Reyes, Rafael H. Villarreal, and Leticia Zárate. A note on affine toric varieties. *Linear Algebra and its Applications*, 318(1-3):173–179, 2000.

[47] Eliseo Sarmiento, Maria Vaz Pinto, and Rafael H. Villarreal. On the vanishing ideal of an algebraic toric set and its parametrized linear codes. *Journal of Algebra and its Applications*, 11(4):1250072, 16, 2012.

[48] A. Simis, W.V. Vasconcelos, and R.H. Villarreal. On the ideal theory of graphs. *Journal of Algebra*, 167(2):389 – 416, 1994.

[49] Aron Simis, Wolmer V Vasconcelos, and Rafael H Villarreal. The integral closure of subrings associated to graphs. *Journal of Algebra*, 199(1):281 – 289, 1998.

[50] Aron Simis and Rafael H. Villarreal. Constraints for the normality of monomial subrings and birationality. *Proceedings of the American Mathematical Society*, 131(7):2043–2048, 2003.

[51] R. Stanley. *Enumerative Combinatorics, Volume 1, Second Ed.* Cambridge University Press, 2012.

[52] Bernd Sturmfels and Volkmar Welker. Commutative algebra of statistical ranking. *Journal of Algebra*, 361:264–286, 2012.

[53] Seth Sullivant. Combinatorial symbolic powers. *Journal of Algebra*, 319(1):115–142, 2008.

[54] Yukihide Takayama. Normality of affine semigroup rings generated by 2-dimensional cone type simplicial complexes. *Communications in Algebra*, 29(4):1499–1512, 2001.

[55] Christos Tatakis and Apostolos Thoma. On complete intersection toric ideals of graphs. *Journal of Algebraic Combinatorics. An International Journal*, 38(2):351–370, 2013.

[56] Tuan Tran and Günter M. Ziegler. Extremal edge polytopes. *Electronic Journal of Combinatorics*, 21(2):Paper 2.57, 16, 2014.

[57] Tran Nam Trung. Stability of depths of powers of edge ideals. *Journal of Algebra*, 452:157 – 187, 2016.

[58] Rafael H. Villarreal. Rees algebras of edge ideals. *Communications in Algebra*, 23(9):3513–3524, 1995.

[59] Rafael H. Villarreal. Normality of subrings generated by square free monomials. *Journal of Pure and Applied Algebra*, 113(1):91 – 106, 1996.

[60] Rafael H. Villarreal. Normality of semigroups with some links to graph theory. *Discrete Mathematics*, 302(1-3):267–284, 2005.

[61] Marie A. Vitulli. Serre's condition Rℓ for affine semigroup rings. *Communications in Algebra*, 37(3):743–756, 2009.