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Computational Bases for Hdiv

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COMPUTATIONAL BASES FOR H_{div}

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ABSTRACT

The H_{div} vector space arises in a number of mixed method formulations, particularly in fluid flow through a porous medium. First we present a Lagrangian computational basis for the Raviert-Thomas (RT) and Brezzi-Douglas-Marini (BDM) approximation subspaces of H_{div} in $\mathbb{R}^3$. Second, we offer three solutions to a numerical problem that arises from the Piola mapping when RT and BDM elements are used in practice.

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Introduction

Introduction

The $H^{div}(\Omega) = \{\mathbf{v} : \mathbf{v} \in L^2(\Omega), \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$ vector space arises in many mixed-method finite element formulations, notably, Darcy and Stokes equations. For the domain Ω partitioned into "elements", $\Omega = \cup_{i=1}^{N} T_i$ (triangles or quadrilaterals in $\mathbb{R}^2$, tetrahedra or hexahedra in $\mathbb{R}^3$) a function lies in $H^{div}(\Omega)$ if it lies in $H^{div}(T_i), i = 1, \cdots N$ and has a continuous normal derivative across the boundary (ie edges in $\mathbb{R}^2$, faces in $\mathbb{R}^3$) of each element. Two popular classes of approximation functions which lie in the H^{div} space are are Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM).
Typically these spaces are characterized as follows. Let $\mathcal{T}_h$ denote a triangulation of $\Omega \subset \mathbb{R}^n$. Further, let e_{ij} denote the edge connecting triangles $T_i, T_j \in \mathcal{T}_h$. Then RT elements on $\mathcal{T}_h$ have the form

$$RT_k(\mathbf{x}) := (P_k(T))^n + \mathbf{x}P_k(T) \tag{1.1}$$

where $\mathbf{x}$ is an n-dimensional vector and $P_k(T)$ denotes the space of order k bivariate polynomials. In 2D $\mathbf{x} = (x, y)'$ and in 3D $\mathbf{x} = (x, y, z)'$. The RT_k approximation space is defined by

$$RT_k(\Omega) := \{\mathbf{u}_h | \mathbf{u}_h \in RT_k(T) \text{ for all } t \in T_h, \mathbf{u}_h \cdot \mathbf{n}_{ij} \text{ is continuous across } e_{ij}, \text{ for all } e_{ij} \in E_h\}.$$

BDM elements have the form

$$BDM_k(\mathbf{x}) := (P_k(T))^n$$

where

$$BDM_k(\Omega) := \{\mathbf{u}_h | \mathbf{u}_h \in BDM_k(T) \text{ for all } t \in T_h, \mathbf{u}_h \cdot \mathbf{n}_{ij} \text{ is continuous across } e_{ij}, \\ \text{for all } e_{ij} \in E_h\}.$$

For computational purposes it is convenient to define the RT space on a reference element, $\hat{T}$, and then use an affine transformation to obtain the RT space on a particular element T in the partition. This results in a slightly different space than that given in 1.1. Specifically, the components of $P_{k+1}(T)$ may differ.

Brezzi and Fortin's Mixed and Hybrid Finite Element methods [1] provides a theoretical discussion of these elements. The authors do not, however, provide an explicit set of computational basis elements. In 2012, Ervin [2] presented 2D computational bases for RT_k and BDM_k on a standard reference triangle.

This work builds on [2] in several ways. First, we construct 3D computational basis for RT_k and BDM_k elements on a standard reference simplex.

Second, we examine a numerical issue - the edge-length restriction - that is only briefly addressed in [2]. When mapping RT_k and BDM_k functions from the reference element to T_h, the Piola Transformation must be used to preserve the orientation of the normal component However, the Piola mapping also introduces a scaling factor to the normal component of the function at element interfaces. Without properly accounting for this, the flux can become discontinuous across the triangle interfaces.

We consider three methods to address the edge length condition. The first is to assign all triangle edge interfaces to an edge of the reference triangle with the same length. This reduces to a coloring problem on the dual graph of the triangulation. The second method is to represent the unknown function in terms of a Lagrangian basis on the physical domain.

Third, we consider using non-standard reference elements for which the edges (faces) of the elements all have the same length (area). This requires redefining the RT_k and BDM_k elements on a new reference element.

This paper proceeds as follows. In Chapter 2, we present an example motivating the H^{div} space. Then in Chapter 3, 3D computational bases on the standard reference simplex are given. In Chapter 4 we introduce the edge length restriction problem and present a graph coloring and basis scaling option to resolving the numerical issue. Finally, in Chapter 5 we construct RT_k and BDM_k bases for equilateral reference elements.

Chapter 2

Motivating Example

2.1 Introduction: Darcy Equation

Darcy's equations describe the relationship between velocity $\mathbf{u}$ and pressure p in a porous medium, Ω,

$$\begin{aligned} \mathbf{u} &= -\frac{\kappa}{\mu}\nabla p, && \text{(velocity } \propto \text{ pressure gradient)} \\ \nabla \cdot \mathbf{u} &= g, && \text{(conservation of mass)} \end{aligned}$$

where κ is porosity and μ represents the effective fluid viscosity.
Equivalently, for a translated velocity, one can express Darcy's equation as

$$\begin{aligned} \mathbf{u} + \nu\nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega. \end{aligned}$$

2.1.1 Weak Formulation

To find the weak formulation, start with the system

$$\mathbf{u} + \nu \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$
$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$
$$\text{OR} \quad p = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

First define

$$X = H_0^{div}(\Omega) = \{\mathbf{v} : \mathbf{v} \in L^2(\Omega), \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$Q = \{q : q \in L^2(\Omega), \int_\Omega q \, d\Omega = 0\} = L_0^2(\Omega),$$

where X is equipped with the norm $\|\mathbf{v}\|_X^2 = (\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2)$.

Lemma 2.1. *For $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, $\|\cdot\|_{H^1}$ is a stronger norm than $\|\cdot\|_X$. That is, there exists $m > 0$ such that for all $\mathbf{v} \in H^{div}(\Omega)$*

$$\|\mathbf{v}\|_X \leq m \|\mathbf{v}\|_{H^1}$$

Proof. Recall

$$\|\mathbf{v}\|_{H_1}^2 = \int_\Omega (\mathbf{v} \cdot \mathbf{v} + \nabla \mathbf{v} : \nabla \mathbf{v}) \, d\Omega,$$
$$\|\mathbf{v}\|_X^2 = \int_\Omega (\mathbf{v} \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^2) \, d\Omega.$$

For $n = 2$,

$$\begin{aligned}(\nabla \cdot \mathbf{v})^2 &= (\mathbf{v}_{1x} + \mathbf{v}_{2y})^2 \\ &\leq 2(\mathbf{v}_{1x}^2 + \mathbf{v}_{2y}^2) \\ &\leq 2(\nabla \mathbf{v} : \nabla \mathbf{v}).\end{aligned}$$

For $n = 3$,

$$
\begin{aligned}
(\nabla \cdot \mathbf{v})^2 &= (\mathbf{v}_{1x} + \mathbf{v}_{2y} + \mathbf{v}_{3z})^2 \\
&\leq 3(\mathbf{v}_{1x}^2 + \mathbf{v}_{2y}^2 + \mathbf{v}_{3z}^2) \\
&\leq 3(\nabla \mathbf{v} : \nabla \mathbf{v}).
\end{aligned}
$$

Therefore, $0 \leq \mathbf{v} \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^2 \leq 3(\mathbf{v} \cdot \mathbf{v} + \nabla \mathbf{v} : \nabla \mathbf{v})$ which implies that

$$
\begin{gathered}
\int_\Omega (\mathbf{v} \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^2)\, d\Omega \leq 3 \int_\Omega (\mathbf{v} \cdot \mathbf{v} + \nabla \mathbf{v} : \nabla \mathbf{v})\, d\Omega \\
\Rightarrow \|\mathbf{v}\|_X^2 \leq 3 \|\mathbf{v}\|_{H^1}^2
\end{gathered}
$$

□

Multiplying equations 2.1, 2.2 with test functions $\mathbf{v} \in H_0^{div}(\Omega)$ and $q \in L_0^2(\Omega)$, and integrating by parts yields

$$
\begin{gathered}
\int_\Omega \mathbf{u} \cdot \mathbf{v}\, d\Omega - \nu \int_\Omega p \nabla \cdot \mathbf{v}\, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega, \\
\int q \nabla \cdot \mathbf{u}\, d\Omega = 0.
\end{gathered}
$$

Define the bilinear forms

$$
\begin{aligned}
a : X \times X \to \mathbb{R}, &\qquad a(\mathbf{w}, \mathbf{u}) = \int_\Omega \mathbf{w} \cdot \mathbf{u}\, d\Omega, \\
b : Q \times X \to \mathbb{R}, &\qquad b(q, \mathbf{v}) = \int_\Omega q \nabla \cdot \mathbf{v}\, d\Omega,
\end{aligned}
$$

and our problem can be recast as find $(\mathbf{u}, p) \in X \times Q$ such that

$$
a(\mathbf{u}, \mathbf{v}) - \nu b(p, \mathbf{v}) = F(\mathbf{v}), \qquad \text{for all } \mathbf{v} \in X, \tag{2.5}
$$

$$
b(q, \mathbf{u}) = 0, \quad \text{for all } q \in Q. \tag{2.6}
$$

Lemma 2.2. *For $F \in X'$, the dual space of X, there exists a unique $(\mathbf{u}, p) \in X \times Q$ satisfying 2.5,2.6.*

Proof. First, introduce $Z = \{\mathbf{v} \in H_0^{div}(\Omega) : b(q, \mathbf{v}) = 0 \text{ for all } q \in Q\}$. Restricted to Z, our problem becomes find $\mathbf{u} \in Z$ such that

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \text{ for all } \mathbf{v} \in Z.$$

From the Lax-Miligram Theorem, this problem has a unique solution provided $a(.,.)$ is continuous and coercive with respect to $\|\cdot\|_X$.
Continuity follows from Cauchy-Schwarz

$$\begin{aligned} a(\mathbf{w}, \mathbf{u}) &= \int_\Omega \mathbf{w} \cdot \mathbf{u} \, d\Omega \\ &\leq \left(\int_\Omega \mathbf{w} \cdot \mathbf{w} \, d\Omega \right)^{\frac{1}{2}} \left(\int_\Omega \mathbf{u} \cdot \mathbf{u} \, d\Omega \right)^{\frac{1}{2}} \\ &\leq \|\mathbf{w}\|_X \, \|\mathbf{u}\|_X \end{aligned}$$

For coercivity, notice that for $\mathbf{v} \in Z$

$$\int_\Omega (\nabla \cdot \mathbf{v}) q \, d\Omega = 0 \text{ for all } q \in Q.$$

Since $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, take $q = \nabla \cdot \mathbf{v} - C \in Q$, where $C = \frac{1}{\Omega} \int_\Omega \nabla \cdot \mathbf{v} \, d\Omega$. Then,

$$\begin{aligned} 0 &= \int_\Omega \nabla \cdot \mathbf{v} \, q \, d\Omega \\ &= \int_\Omega (\nabla \cdot \mathbf{v})^2 d\Omega - C \int_\Omega \nabla \cdot \mathbf{v} d\Omega \\ &= \int_\Omega (\nabla \cdot \mathbf{v})^2 \, d\Omega - C \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, ds + C \int_\Omega \mathbf{v} \cdot \nabla 1 \, d\Omega \\ &= \int_\Omega (\nabla \cdot \mathbf{v})^2 \, d\Omega \end{aligned}$$

Thus, $\nabla \cdot \mathbf{v} = 0$.
As a result,

$$\begin{aligned} a(\mathbf{w}, \mathbf{w}) &= \int_\Omega \mathbf{w} \cdot \mathbf{w} \, d\Omega \\ &= \int_\Omega (\nabla \cdot \mathbf{w})^2 \, d\Omega + \int_\Omega \mathbf{w} \cdot \mathbf{w} \, d\Omega \\ &= \|\mathbf{w}\|_X^2 \text{ for all } w \in Z. \end{aligned}$$

implying that $a(.,.)$ is coercive.
Given that $a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v})$ is well posed, next we consider the pressure. Our problem now is to find $p \in Q$ such that

$$\begin{aligned} b(p, \mathbf{v}) &= a(\mathbf{u}, \mathbf{v}) - F(\mathbf{v}) \quad \text{for all } \mathbf{v} \in Z^{\perp} \\ &= \hat{F}(\mathbf{v}). \end{aligned}$$

To prove that a unique solution exists to this problem, we must show that $\hat{F}$ is a bounded linear functional, $b(p, \mathbf{v})$ is bounded and satisfies the following $\inf - \sup$ condition

$$\alpha \|p\|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in X} \frac{b(p, \mathbf{v})}{\|\mathbf{v}\|_X},$$

where $\alpha > 0$.
Again, continuity follows from Cauchy Schwartz

$$\begin{aligned} b(q, \mathbf{v}) &= \int_{\Omega} q \nabla \cdot \mathbf{v} \, d\Omega \\ &\leq \left(\int_{\Omega} q^2 \, d\Omega \right)^{\frac{1}{2}} \left(\int_{\Omega} (\nabla \cdot \mathbf{v})^2 \, d\Omega \right)^{\frac{1}{2}}. \end{aligned}$$

To show the $\inf - \sup$ condition first note that for all $p \in Q$, there exists $\mathbf{w} \in H^1(\Omega)$ such that $\nabla \cdot \mathbf{w} = p$ and

$$\|\mathbf{w}\|_{H^1} \leq C \|p\|_{L^2(\Omega)}.$$

[3]. Thus, given $p \in Q$, choose, $\mathbf{w} \in H^1 \subset H^{div}$ such that $\nabla \cdot \mathbf{w} = p$. Recalling Lemma 2.1

$$\sup_{v \in X} \frac{b(p, \mathbf{v})}{\|\mathbf{v}\|_X} \geq \frac{b(p, \mathbf{w})}{\|\mathbf{w}\|_X} \geq \frac{\|p\|^2_{L^2(\Omega)}}{m \|\mathbf{w}\|_{H^1}} \geq \frac{\|p\|^2_{L^2(\Omega)}}{mC \|p\|_{L^2(\Omega)}} \geq \frac{1}{mC} \|p\|_{L^2(\Omega)}.$$

Thus, $b(.,.)$ satisfies the $\inf - \sup$ condition ensuring a unique solution exists for the pressure. □

2.1.2 Discrete Approximation

In the discrete approximation, we are interested in finding $(\mathbf{u}_h, p_h) \in X_h \times Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) - \nu\, b(p_h, \nabla \cdot \mathbf{v}_h) = F(\mathbf{v}_h)$$
$$b(q_h, \nabla \cdot \mathbf{u}_h) = 0$$

for all $\mathbf{v}_h \in X_h \subset H_0^{div}$ and $q_h \in Q_h \subset L_0^2(\Omega)$.
To establish existence and uniqueness, we follow a similar pattern as for the continuous problem. Now, however, we let $Z_h = \{\mathbf{z}_h \in X_h : b(q_k, \nabla \cdot \mathbf{z}_h) = 0 \text{ for all } q_h \in Q_h\} \not\subset Z$. Therefore, to show that $a(.,.)$ is coercive as above, it is necessary that $(\nabla \cdot X_h) \subset Q_h$. To establish the $\lim - \sup$ condition for $b(p_h, \mathbf{u_h})$, see [1], [3].

Chapter 3

3D Reference Elements

3.1 Basis Defined on Reference Triangles / Simplex: Introduction

In [2] Ervin, presents a computational basis for RT_k and BDM_k on the reference triangle. Here we extend this and give an explicit RT_k and BDM_k computational basis on a reference simplex.
We start with a definition of the reference simplex and bivariate Lagrangian polynomials. Next construct a 3D basis for RT_k elements on the reference simplex using a generalization of Ervin's method. Finally, we use a novel approach to build a BDM_k basis defined on the reference simplex.

3.1.1 Reference Simplex

In 3D, the reference element is a tetrahedra with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ defined in (ξ, η, ρ) space. The simplex has four faces. Face 1 (F_1) is $\xi + \eta + \rho = 1$ and has outward normal vector $\mathbf{n}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$. Face 2 ($F_2$) is $\xi = 0$ with normal vector $\mathbf{n}_2 = (-1, 0, 0)$. Face 3 ($F_3$) is $\eta = 0$ with normal vector $\mathbf{n}_3 = (0, -1, 0)$ and Face 4 (F_4) is $\rho = 0$ with normal vector $\mathbf{n}_4 = (0, 0, -1)$.

3.1.2 Bivariate Lagrangian Polynomials

Bivariate Lagrangian polynomials are useful for building the 3D basis elements. Unfortunately, bivariate Lagrangian interpolation cannot be directly generalized from one-dimension.

To build a bivariate degree k polynomial, select $i = 1, ..., N = 1/2(k+1)(k+2)$ appropriate points (x_i, y_i, f_i). What constitutes an appropriate set of points is discussed below. Next, let $\mathbf{x}_i = (x_i, y_i)$ and define:

$$\Delta(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n) = \det \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & \cdots & x_1 y_1^{n-1} & y_1^n \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & \cdots & x_2 y_2^{n-1} & y_2^n \\ \vdots & \vdots & & & & & & & \vdots & \vdots \\ 1 & x_N & y_N & x_N^2 & x_N y_N & y_N^2 & x_N^3 & \cdots & x_N y_N^{n-1} & y_N^n \end{pmatrix}$$

Then for $i = 1, \cdots, N$, the following defines a collection of bivariate Lagrangian polynomials.

$$\begin{aligned} \ell_i(\mathbf{x}_1, \cdots, \mathbf{x}, \cdots \mathbf{x}_N) &= \frac{\Delta(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, ..., \mathbf{x}_N)}{\Delta(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)} \\ &= \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{x}_i \\ 0, & \text{if } \mathbf{x} = \mathbf{x}_j, \quad j = 1, 2, ..., N, \quad j \neq i. \end{cases} \end{aligned}$$

In one dimension, a collection of $k+1$ distinct points is enough to build an interpolating polynomial of degree k. In the bivariate case, we require $\Delta(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \neq 0$. Appendix A provides a table with permissible Gaussian quadrature points for constructing these bivariate interpolating polynomials.

3.1.3 Piola Transformation

A key feature of RT_k and BDM_k elements is that they maintain a continuous flux across triangle interfaces. Arbitrary affine transformations do not, however, preserve this property. Instead, an angle preserving mapping, the Piola Transformation, is used to map functions between the reference and target triangles.

The Piola transformation $\mathcal{P} : L_2(\hat{T}) \to L_2(T)$ is defined as:

$$\mathbf{q}^* \to \mathbf{q} = \mathcal{P}(\mathbf{q}^*)(x) := \frac{1}{|J_T|} J_T \mathbf{q}^*(G_T^{-1}(x))$$

where J_T is the Jacobi of an affine mapping between $\hat{T}$ and T and G_T^{-1} is the inverse of the affine mapping $G_T : \hat{T} \to T$.

3.1.4 RT_k

The dimension of RT_k in 3D is $\frac{1}{2}(k+1)(k+2)(k+4)$ [1]. The following construction follows the method used in Ervin [2]. First, consider the collection of Face Functions

$$\hat{\mathbf{e}}_1(\xi, \eta, \rho) = \sqrt{3} \begin{pmatrix} \xi \\ \eta \\ \rho \end{pmatrix}, \quad \hat{\mathbf{e}}_2(\xi, \eta, \rho) = \begin{pmatrix} \xi - 1 \\ \eta \\ \rho \end{pmatrix},$$
$$\hat{\mathbf{e}}_3(\xi, \eta, \rho) = \begin{pmatrix} \xi \\ \eta - 1 \\ \rho \end{pmatrix}, \quad \hat{\mathbf{e}}_4(\xi, \eta, \rho) = \begin{pmatrix} \xi \\ \eta \\ \rho - 1 \end{pmatrix}.$$

These vectors satisfy:

$$\hat{\mathbf{e}}_i \cdot \mathbf{n_j}|_{F_j} = \begin{cases} 1 & : i = j \text{ and } i = 1,2,3,4 \\ 0 & : i \neq j \end{cases}, \tag{3.1}$$

where F_j is the face corresponding with the unit normal vector $\mathbf{n}_j$.

Lemma 3.1. *The set of Face Functions are linearly independent.*

Proof. This is easily established from the inner-product property with respect to the normal components (3.1). □

Next, define the following Interior Functions.

$$\hat{\mathbf{e}}_5(\xi,\eta,\rho) = \xi \begin{pmatrix} \xi - 1 \\ \eta \\ \rho \end{pmatrix}, \quad \hat{\mathbf{e}}_6(\xi,\eta,\rho) = \eta \begin{pmatrix} \xi \\ \eta - 1 \\ \rho \end{pmatrix},$$
$$\hat{\mathbf{e}}_7(\xi,\eta,\rho) = \rho \begin{pmatrix} \xi \\ \eta \\ \rho - 1 \end{pmatrix},$$

which satisfy $\hat{\mathbf{e}}_i \cdot \mathbf{n_j}|_{F_j} = 0$ for $i = 5,6,7$ and $j = 1,2,3,4$.

Lemma 3.2. *The set of Interior Functions are linearly independent.*

Proof. Consider a linear combination of the elements

$$\alpha_1\hat{\mathbf{e}}_5 + \alpha_2\hat{\mathbf{e}}_6 + \alpha_3\hat{\mathbf{e}}_7 = 0.$$

Since the only ξ^2 term appears in $\hat{\mathbf{e}}_5$, $\alpha_1 = 0$. Similarly, because the only η^2 and ρ^2 terms appear in $\hat{\mathbf{e}}_6$ and $\hat{\mathbf{e}}_7$ respectively, $\alpha_2 = \alpha_3 = 0$. □

RT_0

The dimension of $RT_0(\hat{T})$ is 4. Since the four Face Functions are in $RT_0(\hat{T})$ and are independent (Lemma 3.1), the following forms a basis for RT_0.

$$\Phi_1^{[1]}(\xi,\eta,\rho) = \hat{\mathbf{e_1}}(\xi,\eta,\rho), \quad \Phi_1^{[2]}(\xi,\eta,\rho) = \hat{\mathbf{e_2}}(\xi,\eta,\rho),$$
$$\Phi_1^{[3]}(\xi,\eta,\rho) = \hat{\mathbf{e_3}}(\xi,\eta,\rho), \quad \Phi_1^{[4]}(\xi,\eta,\rho) = \hat{\mathbf{e_4}}(\xi,\eta,\rho).$$

RT_1

The dimension of RT_1 is 15: 12 edge and 3 interior functions.
From Section 3.1.2, a linear bivariate Lagrangian polynomial requires 3 appropriately selected points on each face j, $\mathbf{x}_1^j, \mathbf{x}_2^j, \mathbf{x}_3^j$. Then build the functions:

$$\ell_1^j(\mathbf{x}^j, \mathbf{x}_2^j, \mathbf{x}_3^j), \quad \ell_2^j(\mathbf{x}_1^j, \mathbf{x}^j, \mathbf{x}_3^j), \quad \ell_3^j(\mathbf{x}_1^j, \mathbf{x}_2^j, \mathbf{x}^j).$$

Thus the Face Functions are

$$\begin{aligned}
&\Phi_1^{[1]}(\xi,\eta,\rho) = \ell_1^1(\mathbf{x}^1,\mathbf{x}_2^1,\mathbf{x}_3^1)\hat{\mathbf{e}}_1(\xi,\eta,\rho), \quad \Phi_2^{[1]}(\xi,\eta,\rho) = \ell_2^1(\mathbf{x}_1^1,\mathbf{x}^1,\mathbf{x}_3^1)\hat{\mathbf{e}}_1(\xi,\eta,\rho),\\
&\Phi_3^{[1]}(\xi,\eta,\rho) = \ell_3^1(\mathbf{x}_1^1,\mathbf{x}_2^1,\mathbf{x}^1)\hat{\mathbf{e}}_1(\xi,\eta,\rho),\\
&\Phi_1^{[2]}(\xi,\eta,\rho) = \ell_1^2(\mathbf{x}^2,\mathbf{x}_2^2,\mathbf{x}_3^2)\hat{\mathbf{e}}_2(\xi,\eta,\rho), \quad \Phi_2^{[2]}(\xi,\eta,\rho) = \ell_2^2(\mathbf{x}_1^2,\mathbf{x}^2,\mathbf{x}_3^2)\hat{\mathbf{e}}_2(\xi,\eta,\rho),\\
&\Phi_3^{[2]}(\xi,\eta,\rho) = \ell_3^2(\mathbf{x}_1^2,\mathbf{x}_2^2,\mathbf{x}^2)\hat{\mathbf{e}}_2(\xi,\eta,\rho),\\
&\Phi_1^{[3]}(\xi,\eta,\rho) = \ell_1^3(\mathbf{x}^3,\mathbf{x}_2^3,\mathbf{x}_3^3)\hat{\mathbf{e}}_3(\xi,\eta,\rho), \quad \Phi_2^{[3]}(\xi,\eta,\rho) = \ell_2^3(\mathbf{x}_1^3,\mathbf{x}^3,\mathbf{x}_3^3)\hat{\mathbf{e}}_3(\xi,\eta,\rho),\\
&\Phi_3^{[3]}(\xi,\eta,\rho) = \ell_3^3(\mathbf{x}_1^3,\mathbf{x}_2^3,\mathbf{x}^3)\hat{\mathbf{e}}_3(\xi,\eta,\rho),\\
&\Phi_1^{[4]}(\xi,\eta,\rho) = \ell_1^4(\mathbf{x}^4,\mathbf{x}_2^4,\mathbf{x}_3^4)\hat{\mathbf{e}}_4(\xi,\eta,\rho), \quad \Phi_2^{[4]}(\xi,\eta,\rho) = \ell_2^4(\mathbf{x}_1^4,\mathbf{x}^4,\mathbf{x}_3^4)\hat{\mathbf{e}}_4(\xi,\eta,\rho),\\
&\Phi_3^{[4]}(\xi,\eta,\rho) = \ell_3^4(\mathbf{x}_1^4,\mathbf{x}_2^4,\mathbf{x}^4)\hat{\mathbf{e}}_4(\xi,\eta,\rho).
\end{aligned}$$

The Interior Functions are

$$\Phi_1^{[5]}(\xi,\eta,\rho) = \hat{\mathbf{e}}_5(\xi,\eta,\rho), \quad \Phi_1^{[6]}(\xi,\eta,\rho) = \hat{\mathbf{e}}_6(\xi,\eta,\rho), \quad \Phi_1^{[7]}(\xi,\eta,\rho) = \hat{\mathbf{e}}_7(\xi,\eta,\rho).$$

For the Face Functions, $\Phi_j^{[i]} \cdot \mathbf{n} = 0$ except at one interpolation point. For the Interior Functions, $\Phi_j^{[i]} \cdot \mathbf{n} = 0$ on every face. This implies the Face Functions are linearly independent. Since $\hat{\mathbf{e}}_5, \hat{\mathbf{e}}_6, \hat{\mathbf{e}}_7$ are in $RT_1(\hat{T})$ and independent, the collection of Face and Interior functions form a basis for RT_1.

RT_2

The dimension of RT_2 is 36 and the basis consists of 24 Face Functions and 12 Interior Functions. First choose 6 appropriate points $\mathbf{x}_i^j$ on each face to build six degree two bivariate Lagrangian polynomials. Then the face functions are:

$$\begin{aligned}
&\Phi_i^{[1]}(\xi,\eta,\rho) = \ell_i^1(\mathbf{x}_1^1,\cdots,\mathbf{x}_{i-1}^1,\mathbf{x},\mathbf{x}_{i+1}^1,\cdots,\mathbf{x}_6^1)\hat{\mathbf{e}}_1(\xi,\eta,\rho) \quad \text{for } i = 1,2,3,4,5,6,\\
&\Phi_i^{[2]}(\xi,\eta,\rho) = \ell_i^2(\mathbf{x}_1^2,\cdots,\mathbf{x}_{i-1}^2,\mathbf{x},\mathbf{x}_{i+1}^2,\cdots,\mathbf{x}_6^2)\hat{\mathbf{e}}_2(\xi,\eta,\rho) \quad \text{for } i = 1,2,3,4,5,6,\\
&\Phi_i^{[3]}(\xi,\eta,\rho) = \ell_i^3(\mathbf{x}_1^3,\cdots,\mathbf{x}_{i-1}^3,\mathbf{x},\mathbf{x}_{i+1}^3,\cdots,\mathbf{x}_6^3)\hat{\mathbf{e}}_3(\xi,\eta,\rho) \quad \text{for } i = 1,2,3,4,5,6,\\
&\Phi_i^{[4]}(\xi,\eta,\rho) = \ell_i^4(\mathbf{x}_1^4,\cdots,\mathbf{x}_{i-1}^4,\mathbf{x},\mathbf{x}_{i+1}^4,\cdots,\mathbf{x}_6^4)\hat{\mathbf{e}}_4(\xi,\eta,\rho) \quad \text{for } i = 1,2,3,4,5,6.
\end{aligned}$$

The following gives the set of 12 interior functions:

$$\begin{aligned}
&\Phi_1^{[5]} = (1-\xi-\eta-\rho)\hat{\mathbf{e}}_5, \quad \Phi_2^{[5]} = \xi\hat{\mathbf{e}}_5, \quad \Phi_3^{[5]} = \eta\hat{\mathbf{e}}_5, \quad \Phi_4^{[5]} = \rho\hat{\mathbf{e}}_5,\\
&\Phi_1^{[6]} = (1-\xi-\eta-\rho)\hat{\mathbf{e}}_6, \quad \Phi_2^{[6]} = \xi\hat{\mathbf{e}}_6, \quad \Phi_3^{[6]} = \eta\hat{\mathbf{e}}_6, \quad \Phi_4^{[6]} = \rho\hat{\mathbf{e}}_6,\\
&\Phi_1^{[7]} = (1-\xi-\eta-\rho)\hat{\mathbf{e}}_7, \quad \Phi_2^{[7]} = \xi\hat{\mathbf{e}}_7, \quad \Phi_3^{[7]} = \eta\hat{\mathbf{e}}_7, \quad \Phi_4^{[7]} = \rho\hat{\mathbf{e}}_7.
\end{aligned}$$

These vectors form a basis for $RT_2(\hat{T})$ following an argument analogous to that given for RT_1.

The general case: RT_k

A computational basis for RT_k is obtained in a similar way to RT_2.
First, let $\mathbf{x}_n^j$, $n = 1, \cdots, \frac{1}{2}(k+1)(k+2)$ be appropriately selected points on each face $j = 1, 2, 3, 4$, and construct the bivariate polynomials described in Section 3.1.2 so that

$$\ell_i^j(\mathbf{x}_n^i) \begin{cases} 1 & : \text{if } i = n \\ 0 & : \text{if } i \neq n \end{cases}$$

Also, let $\{b_i(\xi, \eta, \rho) : i = 1, \cdots, \frac{k(k+1)(k+2)}{6}\}$ denote a basis for $P_{k-1}(\hat{T})$.
The following are the Face Functions

$$\begin{aligned}
\Phi_i^{[1]} &= \ell_i^{[1]}(\mathbf{x})\hat{\mathbf{e}}_1(\xi, \eta, \rho) \quad \text{where} \quad i = 1, \cdots, \frac{1}{2}(k+1)(k+2),\\
\Phi_i^{[2]} &= \ell_i^{[2]}(\mathbf{x})\hat{\mathbf{e}}_2(\xi, \eta, \rho) \quad \text{where} \quad i = 1, \cdots, \frac{1}{2}(k+1)(k+2),\\
\Phi_i^{[3]} &= \ell_i^{[3]}(\mathbf{x})\hat{\mathbf{e}}_3(\xi, \eta, \rho) \quad \text{where} \quad i = 1, \cdots, \frac{1}{2}(k+1)(k+2),\\
\Phi_i^{[4]} &= \ell_i^{[4]}(\mathbf{x})\hat{\mathbf{e}}_4(\xi, \eta, \rho) \quad \text{where} \quad i = 1, \cdots, \frac{1}{2}(k+1)(k+2),
\end{aligned}$$

and the following are the Interior Functions:

$$\Phi_i^{[5]} = b_i(\xi,\eta,\rho)\hat{\mathbf{e}}_5(\xi,\eta,\rho) \quad \text{where} \quad i = 1,\cdots,\frac{1}{6}k(k+1)(k+2),$$
$$\Phi_i^{[6]} = b_i(\xi,\eta,\rho)\hat{\mathbf{e}}_6(\xi,\eta,\rho) \quad \text{where} \quad i = 1,\cdots,\frac{1}{6}k(k+1)(k+2),$$
$$\Phi_i^{[7]} = b_i(\xi,\eta,\rho)\hat{\mathbf{e}}_7(\xi,\eta,\rho) \quad \text{where} \quad i = 1,\cdots,\frac{1}{6}k(k+1)(k+2).$$

Together this gives $2(k+1)(k+2)$ Face Functions and $\frac{k(k+1)(k+2)}{2}$ Interior Functions for a total of $\frac{1}{2}(k+1)(k+2)(k+4)$, hence to establish that these functions form a basis, it remains to show independence.
Independence of the Face Functions follows from the Lagrangian property of the Face Functions and the vanishing property of the normal component of the Interior Functions.
To establish independence of the Interior Functions, consider

$$(\alpha_1 b_1 + \cdots + \alpha_n b_n)\hat{\mathbf{e}}_5 + (\beta_1 b_1 + \cdots + \beta_n b_n)\hat{\mathbf{e}}_6 + (\gamma_1 b_1 + \cdots + \gamma_n b_n)\hat{\mathbf{e}}_7 = 0.$$

Since $\hat{\mathbf{e}}_5, \hat{\mathbf{e}}_6, \hat{\mathbf{e}}_7$ are independent (Recall Lemma 3.2)

$$\alpha_1 b_1 + \cdots + \alpha_n b_n = 0$$
$$\beta_1 b_1 + \cdots + \beta_n b_n = 0$$
$$\gamma_1 b_1 + \cdots + \gamma_n b_n = 0$$

Since $b_1, \cdots, b_n$ form a basis, they too are independent and thus $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = \gamma_1 = \cdots = \gamma_n = 0$. Hence, the Interior Functions are independent. Therefore the union of Face and Interior Functions forms a basis.

3.1.5 BDM_k

The dimension of BDM_k in 3D is $\frac{1}{2}(k+1)(k+2)(k+3)$ [1]. Since Ervin's [2] method for building a BDM_k basis does not generalize to 3D, in this section we illustrate a new approach.

Face Functions

Face Functions are built so that they satisfy the following two properties:

1. The normal component of each Face Function $\mathbf{e}_F$, vanishes on all but one reference simplex face. That is, if F_j denotes a simplex face and $\mathbf{n}_j$ the face's outer normal vector, then $\mathbf{e}_{F_j} \cdot n_j|_{F_j} \neq 0$ and $\mathbf{e}_{F_j} \cdot n_i|_{F_i} = 0$ for $i \neq j$.

2. On the non-vanishing face, the Face Function equals 1 at one of the interpolating points and 0 at the others.

The first step is to find a set of polynomials satisfying Property 2 using the bivariate Lagrangian interpolants discussed in Section 3.1.2. For each simplex face $j = 1, 2, 3, 4$, choose $N = \frac{1}{2}(k+1)(k+2)$ appropriate points $\mathbf{x}_i^j \in F_j, i = 1, \cdots N$, and build the $2(k+1)(k+2)$ polynomials:

$$\ell_1^j = \frac{\Delta(x, x_2^j, ..., x_N^j)}{\Delta(x_1^j, ..., x_N^j)}, \quad \ell_2^j = \frac{\Delta(x_1^j, x, ..., x_N^j)}{\Delta(x_1^j, ..., x_N^j)} \quad, ..., \quad \ell_N^j = \frac{\Delta(x_1^j, x_2^j, ..., x)}{\Delta(x_1^j, ..., x_N^j)}. \tag{3.2}$$

Next, to satisfy Property 2, these polynomials are arranged into vectors where the normal component vanishes on all but one reference simplex face.
To illustrate, consider the Face Functions on Face 1, F_1. Recall that for F_1, $\rho = 1-\xi-\eta$ and $n_1 = \frac{1}{\sqrt{3}}(1,1,1)$. From 3.2, let $\ell_i^1 = \alpha_0^{1,i} + \alpha_1^{1,i}\xi + \alpha_2^{1,i}\eta + \cdots + \alpha_N^{1,i}\eta^k$. Next, arrange ℓ_i^1 so that $\ell_i^1 = \alpha_0^{1,i} + p^{1,i}(\eta) + p^{1,i}(\xi,\eta)$, where $\alpha_0^{1,i}$ is the constant term, $p^{1,i}(\eta)$ is the sum of all terms with only η terms, and $p^{1,i}(\xi,\eta)$ is the sum of all terms with an ξ.
Finally, build the Face Function:

$$\mathbf{e}_i^1(\xi,\eta,\rho) = \sqrt{3}\begin{pmatrix} p^{1,i}(\xi,\eta) + \alpha_0^{1,i}\xi \\ p^{1,i}(\eta) + \alpha_0^{1,i}\eta \\ \alpha_0^{1,i}\rho \end{pmatrix}$$

The normal component of $\mathbf{e}_i^1(\xi, \eta, \rho)$ for $j = 1, 2, 3, 4$ is:

$$\begin{aligned}
\mathbf{e}_i^1 \cdot \mathbf{n}_1|_{\xi+\eta+\rho=1} &= p^{1,i}(\xi,\eta) + \alpha_0^{1,i}\xi + p^{1,i}(\eta) + \alpha_0^{1,i}\eta + \alpha_0^{1,i}(1-\xi-\eta) \\
&= \alpha_0^{1,i} + p^{1,i}(\xi,\eta) + p^{1,i}(\eta), \\
\mathbf{e}_i^1 \cdot \mathbf{n}_2|_{\xi=0} &= -\sqrt{3}\, p^{1,i}(0,\eta) = 0, \\
\mathbf{e}_i^1 \cdot \mathbf{n}_3|_{\eta=0} &= -\sqrt{3}\, (p^{1,i}(0)) = 0, \\
\mathbf{e}_i^1 \cdot \mathbf{n}_4|_{\rho=0} &= -\sqrt{3}\, (\alpha_0^{1,i} 0) = 0.
\end{aligned}$$

As desired, $\mathbf{e}_i^1 \cdot \mathbf{n}_1|_{\xi+\eta+\rho=1} = \alpha_0^{1,i} + p^{1,i}(\xi,\eta) + p^{1,i}(\eta)$, the Lagrangian polynomial built on Face 1, and vanishes on Faces 2, 3 and 4.

Next we illustrate how to find the Face Functions when $j = 2$. A similar idea then works for $j = 3, 4$.

Since $\xi = 0$ on Face 2, consider $\ell_i^2(\eta, \rho)$. Again, partition $\ell_i^2 = \alpha^{2,i} + p^{2,i}(\eta) + p^{2,i}(\eta, \rho)$, where $\alpha^{2,i}$ is the constant term, $p^{2,i}(\eta)$ is a polynomial containing only η terms, and $p^{2,i}(\eta, \rho)$ is a polynomial such that every term contains a ρ.

Next, let $\zeta(\eta)$ and $\omega(\eta, \rho)$ be given by $p^{2,i}(\eta) = \eta\zeta(\eta)$ and $p^{2,i}(\eta, \rho) = \rho\omega(\eta, \rho)$. Finally, define the Face Function:

$$\mathbf{e}_i^2(\xi, \eta, \rho) = \begin{pmatrix} -(a_0^{2,i} + p^{2,i}(\eta) + p^{2,i}(\eta,\rho) - \xi\alpha_0^{2,i}) \\ \eta(\zeta(\eta) + \alpha_0^{2,i}) \\ \rho(\alpha_0^{2,i} + \omega(\eta,\rho)) \end{pmatrix}$$

The normal component of $\mathbf{e}_i^2(\xi, \eta, \rho)$ for $j = 1, 2, 3, 4$ is:

$$
\begin{aligned}
\mathbf{e}_i^2 \cdot \mathbf{n}_1|_{\xi+\eta+\rho=1} &= \frac{1}{\sqrt{3}}(-\alpha_0^{2,i} - p^{2,i}(\eta) - p^{2,i}(\eta,\rho) + \xi a_0^{2,i} + \eta(\zeta(\eta) + a_0^{2,i}) + \rho(\alpha_0^{2,i} + \omega(\eta,\rho))) \\
&= -\frac{1}{\sqrt{3}}(\alpha_0^{2,i} + \eta\zeta(\eta) + (1-\xi-\eta)\omega(\eta,\rho) - \xi\alpha_0^{2,i} - \eta(\zeta(\eta) + \alpha_0^{2,i}) \\
&\quad - (1-\xi-\eta)(\alpha_0^{2,i} + \omega(\eta,\rho))) \\
&= 0, \\
\mathbf{e}_i^2 \cdot \mathbf{n}_2|_{\xi=0} &= -(-\alpha_0^{2,i} - p^{2,i}(\eta) - p^{2,i}(\eta,\rho) + 0\ \alpha_0^{2,i}) \\
&= \alpha_0^{2,i} + p^{2,i}(\eta) + p^{2,i}(\eta,\rho), \\
\mathbf{e}_i^2 \cdot \mathbf{n}_3|_{\eta=0} &= -0\ (\zeta(\eta) + \alpha_0^{2,i}) = 0, \\
\mathbf{e}_i^2 \cdot \mathbf{n}_4|_{z=0} &= -0\ (\alpha_0^{2,i} + \omega(\eta,\rho)) = 0.
\end{aligned}
$$

Again, as desired, $\mathbf{e}_i^2 \cdot \mathbf{n}_2|_{\xi=0} = \alpha_0^{2,i} + p^{2,i}(\eta) + p^{2,i}(\rho,\eta)$, the Lagrangian polynomial built on Face 2, and vanishes on Faces 1, 3 and 4.

Lemma 3.3. *The set of BDM_k Face Functions are linearly independent.*

Proof. The Lagrangian property enforces independence for functions defined on the same face, while the vanishing normal components enforce independence across faces. □

Interior Functions for BDM_k

In 3D, BDM_k has $\dfrac{(k-1)(k+1)(k+2)}{2}$ Interior Functions [1] that we denote C^k. In what follows, we present a set for C^2 (the lowest order for which BDM_k has interior functions) and then show how to modify and extend C^{k-1} to C^k.

First, consider C^2

$$\mathbf{c}_1 = \xi \begin{pmatrix} \xi - 1 \\ \eta \\ \rho \end{pmatrix}, \quad \mathbf{c}_2 = \eta \begin{pmatrix} \xi \\ \eta - 1 \\ \rho \end{pmatrix}, \quad \mathbf{c}_3 = \rho \begin{pmatrix} \xi \\ \eta \\ \rho - 1 \end{pmatrix},$$
$$\mathbf{c}_4 = \xi\eta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_5 = \eta\rho \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{c}_6 = \xi\rho \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

C^2 has the following properties.

Lemma 3.4. $\mathbf{c}_i \cdot \mathbf{n}_j|_{F_j} = 0$ *for* $i = 1, \cdots, 6$ *and* $j = 1, 2, 3, 4$.

Proof. Along Face 1

$$\begin{aligned}
\mathbf{c}_1 \cdot \mathbf{n}_1|_{\xi+\eta+\rho=1} &= \frac{1}{\sqrt{3}}\xi(\xi - 1 + \eta + \rho) = \xi \cdot 0 = 0, \\
\mathbf{c}_1 \cdot \mathbf{n}_2|_{\xi=0} &= -\xi(\xi - 1) = 0 \cdot (-1) = 0, \\
\mathbf{c}_1 \cdot \mathbf{n}_3|_{\eta=0} &= -\xi\eta = 0, \\
\mathbf{c}_1 \cdot \mathbf{n}_4|_{\rho=0} &= -\xi\rho = 0.
\end{aligned}$$

In an analogous fashion it is straight forward to verify $\mathbf{c}_i \cdot \mathbf{n}_j|_{F_j} = 0$ for $i = 2, 3, ..., 6$ and $j = 1, ..., k$. □

Lemma 3.5. *The vectors* $\mathbf{c}_1, \cdots, \mathbf{c}_6$ *are linearly independent.*

Proof. Suppose $\sum_{i=1}^{6} \alpha_i \mathbf{c}_i = \mathbf{0}$. Since the only vectors with ξ^1, η^1 and ρ^1 terms are $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ respectively, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Because $\mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6$ have no common terms, $\alpha_4 = \alpha_5 = \alpha_6 = 0$ also. □

The elements of C^2 are the building blocks for C^k. Indeed, one can partition C^k into sets C_1^k and C_2^k, where C_1^k denotes the elements of C^k derived from $C_1 = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ and C_2^k the elements derived from $C_2 = \{\mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6\}$. As we will see, if $\mathbf{c}^* \in C_1^k$ then $\mathbf{c}^* \in C^{k+i}$ for all $i \geq 0$. In contrast, if $\mathbf{c}^* \in C_2^k$, then $\mathbf{c}^* \notin C^i$ unless $k = i$.
Hence, building C^k from C^{k-1} involves two steps. The first is to extend C_1^{k-1} to C_1^k.

The second is to rebuild C_2^k.

The following result will prove helpful moving forward.

Lemma 3.6. *There are* $\dfrac{(k+N-1)!}{(N-1)!\,k!}$ *distinct solutions to the non-negative integer partition* $x_1 + \cdots + x_N = k$.

Proof. See Appendix B. □

Step 1: C_1^k

First we extend C_1^{k-1} to C_1^k.

Let $C_1^{k,new} = \{\xi^i \eta^j \rho^m \mathbf{c}_l | l = 1,2,3 \text{ and } i+j+m = k-2 \text{ where } i,j,m \geq 0, \in \mathbb{N}\}$. That is, $C_1^{k,new}$ is the product of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ with all degree $k-2$ monomials.

Then $C_1^k = C_1^{k-1} \cup C_1^{k,new}$.

Lemma 3.7. C_1^k *has* $\dfrac{(k-1)k(k+1)}{2}$ *elements.*

Proof. When $k=2$, C_1^k has three elements.

Next, suppose C_1^{k-1} has $\dfrac{(k-2)(k-1)k}{2}$ elements. From Lemma 3.6, $C_1^{k,new}$ has $\dfrac{3(k-1)k}{2}$ elements. Thus

$$\frac{(k-2)(k-1)k}{2} + \frac{3k(k-1)}{2} = \frac{(k-1)k(k+1)}{2}$$

□

Lemma 3.8. C_1^k *are Interior Functions.*

Proof. All elements of C_1^k are polynomial multiples of $\mathbf{c}_i$ for $i = 1,2,3$. Let $p(\xi,\eta,\rho)$ be an arbitrary polynomial, then from Lemma 3.1.5 $p(\xi,\eta,\rho)\mathbf{c}_i \cdot \mathbf{n}_j = p(\xi,\eta,\rho)(\mathbf{c}_i \cdot \mathbf{n}_j) = 0$ for $i = 1,2,3$ and $j = 1,2,3,4$. □

Lemma 3.9. C_1^k *is linearly independent.*

Proof. In Lemma 3.5 we established independence for $k = 2$.
Now suppose C_1^{k-1} is independent. For $k > 2$, we can characterize $C_1^{k-1} = C_1^{k-2} \cup C_1^{k-1,new}$ where the degree of the polynomial terms making up these elements satisfy $k-2 \leq C_1^{k-1,new} \leq k-1$ and $1 \leq C_1^{k-2} \leq k-2$.
The elements of $C_1^{k,new}$ have the form

$$(\xi^i\eta^j\rho^m)\mathbf{c}_1 = (\xi^{i+1}\eta^j\rho^m)\begin{pmatrix}\xi-1\\ \eta\\ \rho\end{pmatrix}, \quad (\xi^i\eta^j\rho^m)\mathbf{c}_2 = (\xi^i\eta^{j+1}\rho^m)\begin{pmatrix}\xi\\ \eta-1\\ \rho\end{pmatrix},$$

$$(\xi^i\eta^j\rho^m)\mathbf{c}_3 = (\xi^i\eta^j\rho^{m+1})\begin{pmatrix}\xi\\ \eta\\ \rho-1\end{pmatrix},$$

for $i+j+m = (k-2)$. Thus all terms in $C_1^{k,new}$ have degree $k-1$ or k.
Consider

$$\sum_{\mathbf{d}_i \in C_1^k} \alpha_i \mathbf{d}_i = 0 \tag{3.3}$$

where α_i is a scalar. For all $\mathbf{d}_i \in C_1^{k-2}$, $\alpha_i = 0$ because C_1^{k-1} is independent and all elements in $C^{k,new}$ have lowest order terms of $k-1$.
Hence, the only possible non-zero coefficients α_i's must be associated with $\mathbf{d}_i \in C_1^{k-1,new}$ or $C_1^{k,new}$. Since $C_1^{k-1,new}$ is independent and every element has a polynomial term of degree $k-2$ (recall $\mathbf{d}_i \in C_1^{k,new}$ does not have any $k-2$ terms), $\alpha_i = 0$ for all elements in $C_1^{k-1,new}$.
This leaves $C^{k,new}$ as the only set with potential non-zero coefficients α_i. Notice that the only order $k-1$ terms in row one are in the set $(\xi^{i+1}\eta^j\rho^m)\mathbf{c}_1$. Thus, these elements are independent from $(\xi^i\eta^{j+1}\rho^m)\mathbf{c}_2$ and $(\xi^i\eta^j\rho^{m+1})\mathbf{c}_3$. Hence, all three groups are independent of each other. Since every combination $\xi^{i+1}\eta^j\rho^m$ is distinct, vectors within each group are independent.
Therefore, $\alpha_i = 0$ for all $\alpha_i \in 3.3$. □

Step 2: C_2^k

To build C_2^k, start with the functions

$$\mathbf{f}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{f}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Then, $C_2^k = \{(\xi^{i+1}\eta^{j+1}\rho^m)f_1, (\xi^i\eta^{j+1}\rho^{m+1})f_2, (\xi^{r+1}\rho^{s+1})f_3\}$ where $\{i, j, m\}$ is the set of integers such that $i + j + m = k - 2$ and $\{r, s\}$ are such that $r + s = k - 2$.

Lemma 3.10. *C_2^k has (k-1)(k+1) elements.*

Proof. Follows from Lemma 3.6. □

Lemma 3.11. *C_2^k are Interior Functions.*

Proof. This follows from a similar argument given in Lemma 3.8. □

Lemma 3.12. *The elements of C_2^k are independent.*

Proof. First, define the polynomial sets $A = \{\xi^{i+1}\eta^{j+1}\rho^m : i + j + m = k - 2,\ i, j, m \in \mathbb{N} \cup \{0\}\}$, $B = \{\xi^i\eta^{j+1}\rho^{m+1} : i + j + m = k - 2,\ i, j, m \in \mathbb{N} \cup \{0\}\}$ and $C = \{\xi^{i+1}\rho^m : i + m = k - 2,\ i, m \in \mathbb{N} \cup \{0\}\}$.
Notice that $C \cap A = \emptyset$ and $C \cap B = \emptyset$. (Suppose this were not the case. Then $\xi^{i+1}\eta^{j+1}\rho^m = \xi^{r+1}\rho^{s+1}$ which implies $j = -1$, a contradiction.)
Next, suppose

$$\sum_{p \in A} \alpha_p p \mathbf{f}_1 + \sum_{p \in B} \beta_p p \mathbf{f}_2 + \sum_{p \in C} \gamma_p p \mathbf{f}_3 = 0$$

where α_p, β_p and γ_p are scalar coefficients. Since C is disjoint from A and B, and every $p \in C$ is distinct, $\gamma_p = 0$ for all $p \in C$. Next, because $\mathbf{f}_1$ and $\mathbf{f}_2$ are independent, it must be that $\sum_{p \in A} \alpha_p p = 0$ and $\sum_{p \in B} \beta_p p = 0$. Since the every element of A is a distinct non-zero polynomial, it must be that $\alpha_p = 0$. Similarly, since every element of B is a distinct non-zero polynomial, $\beta_p = 0$. □

Finally,

Lemma 3.13. *The set $C = C_1^k \cup C_2^k$ is independent.*

Proof. Consider $\sum_{d_i \in C} \alpha_i \mathbf{d}_i = 0$ where α_i is a scalar.
For $\mathbf{d}_i \in C_1^k$, every polynomial term in $\mathbf{d}_i$ has degree, $k-i$ or $k-i+1$ for $i \in k-1, \cdots 1$.
For $\mathbf{d}_i \in C_2^k$, every polynomial term has degree k. Because C_1^k is independent, and no $\mathbf{d}_i \in C_2^k$ has a term of degree less than k, $\alpha_i = 0$ for all $\mathbf{d}_i \in C_1^k$ of degree is $k-1$.
Thus, the only possible dependences exist between elements of C_1^k with degree k terms and elements of C_2^k. Yet $\mathbf{d}_i \in C_1^k$ with a degree k term also has a degree $k-1$ term. Since $\alpha_i = 0$ for all other $\mathbf{d}_i \in C$ with degree $k-1$, it must be that $\alpha_i = 0$ for all $\mathbf{d}_i \in C_1^k$. This leaves $\sum_{d_i \in C_2^k} \alpha_i \mathbf{d}_i = 0$ and since C_2^k is independent, it follows that $\alpha_i = 0$ for all $c \in C$. □

Since the elements of $C = C_1^k \cup C_2^k$ are interior, it follows from Lemma's 3.7, 3.10 and 3.13 that C has $\dfrac{(k-1)(k+1)(k+2)}{2}$ interior functions as required.

BDM_1

The dimension of BDM_1 is 12 and the basis consists of three Face Functions on each face. An explicit basis using the 2D Gauss quadrature points is given in Appendix A.

BDM_2

The dimension of BDM_2 is 30, consisting of 24 Face Functions (six per face) and 6 interior functions. The 2D Gauss quadrature points for constructing the Face Functions are given in Appendix A.

The general case: *BDM_k*

Building a general basis for BDM_k is a two step process. First we build $2(k+1)(k+2)$ independent Face functions and then $\dfrac{1}{2}(k-1)(k+1)(k+2)$ independent Interior functions for a combined $\frac{1}{2}(k+1)(k+2)(k+3)$ linearly independent vectors.

Theorem 3.14. *BDM_k is a basis.*

Proof. As mentioned at the beginning of Section 3.1.5, the dimension of BDM_k is $\frac{1}{2}(k+1)(k+2)(k+3)$.

Previously we've seen that all Face and Interior functions built in Section 3.1.5 belong to BDM_k and are independent. Therefore, since

$$\frac{1}{2}(k-1)(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)(k+3)=(k+1)(k+2)(k+3)$$

BDM_k forms a basis. □

Chapter 4

Edge Length Restriction

4.1 Introduction

In Chapter 3, we presented RT_k and BDM_k bases for the standard reference triangle and simplex. Next we address a numerical issue - the edge length restriction - that arises from the Piola Transformation and can lead to an approximating function $\mathbf{u}_h$ where $\mathbf{u}_h \cdot \mathbf{n}$ (ie the "flux") is discontinuous across triangle interfaces. To see how this issue arises we begin with a review of the finite element representation of a scalar function on Ω.

First, we introduce some notation. Let $\mathcal{T}_h$ be a triangulation of the domain and $\hat{T}$ be the standard reference triangle. Next, suppose $T_i, T_j \in \mathcal{T}_h$ share a common edge. If T_i is the active triangle, denote the edge e_{ij}. If T_j is the active triangle, denote the edge e_{ji}. Let $\hat{e}_i = F_T^{-1}(e_{ij})$ be the corresponding mapping edge from $\hat{T}$ to $T \in \mathcal{T}_h$.

Now, suppose $\hat{\phi}(\boldsymbol{\xi})$ is a scalar Lagrangian function defined on $\hat{T}$, and $\hat{\mathbf{q}}_k$, $k = 1, \cdots, r$ are points in $\hat{T}$, which satisfies

$$\hat{\phi}(\hat{\mathbf{q}}_k) = \begin{cases} 1 & : k = i \\ 0 & : k = 1, \cdots, r,\ k \neq i \end{cases} .$$

For $F_T : \hat{T} \to T$ denoting the affine transformation from $\hat{T}$ to T described in Chapter 3 let $\mathbf{q}_k = F_T(\hat{\mathbf{q}}_k)$ and $\phi(\boldsymbol{x}) = \hat{\phi}(F_T^{-1}(\boldsymbol{x}))$.

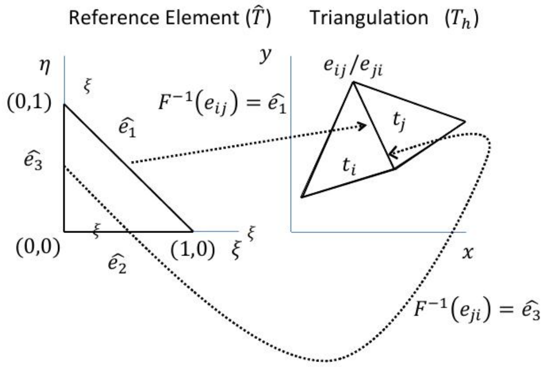

FIGURE 4.1: Example of a problem mapping

Then $\phi(\boldsymbol{x})$ is a Lagrangian function on T satisfying

$$\phi(\mathbf{q}_k) = \begin{cases} 1 & : k = i \\ 0 & : k = 1, \cdots, r, \; k \neq i \end{cases} .$$

In finite element computations, for $\{\phi_j\}$ a basis defined on Ω, we represent a finite element function as

$$u(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \phi_j(\boldsymbol{x}) \tag{4.1}$$

$$= \sum_{j=1}^{N} c_j \hat{\phi}_{\rho(T,j)}(F_T(\boldsymbol{x})) \tag{4.2}$$

where $\rho(T, j)$ is an index function for the mapping $\hat{\phi}_{1,2,3} \to \phi_j$.
The representation (4.2) is particularly useful as all finite element computations are performed on the reference triangle $\hat{T}$.
For vector valued functions defined on $\hat{T}$ the affine transformation F_T does not preserve vector orientations relative to triangle edges. For example, with

$$\hat{\boldsymbol{\phi}}(\boldsymbol{\xi}) = \begin{pmatrix} \hat{\sigma}(\boldsymbol{\xi}) \\ \hat{\tau}(\boldsymbol{\xi}) \end{pmatrix} \qquad \boldsymbol{\phi}(\boldsymbol{\xi}) = \begin{pmatrix} \sigma(\boldsymbol{\xi}) \\ \tau(\boldsymbol{\xi}) \end{pmatrix}$$

where $\sigma(\mathbf{x}) = \hat{\sigma}(F_T^{-1}(\mathbf{x}))$, $\tau(\mathbf{x}) = \hat{\tau}(F_T^{-1}(\mathbf{x}))$, $\hat{\mathbf{n}}_i$, the normal to edge $\hat{e}_i$ on $\hat{T}$, $\mathbf{n}_i$ the normal to $e_i = F(\hat{e}_i)$ on T, in general

$$\hat{\phi}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i|_{\xi \in \hat{e}_i} = 0 \not\Rightarrow \phi(\boldsymbol{x}) \cdot \mathbf{n}_i|_{x=e_i} = 0.$$

Thus to preserve the vector orientations to triangle edges we use the Piola transformation as described Chapter 3.

Under the Piola transformation, with $\tilde{\phi}(\boldsymbol{x}) = \mathcal{P}_T\, \hat{\phi}(F_T^{-1}(\boldsymbol{x}))$ we have

$$\hat{\phi}(\boldsymbol{\xi}) \cdot \hat{\boldsymbol{n}}_i|_{\boldsymbol{\xi} \in \hat{e}_i} = \tilde{\phi}(x) \cdot \mathbf{n}_i|_{x=F_T(\boldsymbol{\xi}) \in e_i} \frac{length(\hat{e_i})}{length(e_i)} \tag{4.3}$$

Note that if $\hat{\phi}(\boldsymbol{\xi}) \cdot \hat{\mathbf{n}}_i$ has a Lagrangian property, in general, $\hat{\phi}(\mathbf{x}) \cdot \mathbf{n}_i$ does not. In particular, note the dependence on the right hand side on the edge length of e_i and $\hat{e}_i$.

In a similar manner as 4.2, writing

$$u(\mathbf{x}) = \sum_{j=1}^{N} c_j \mathcal{P}_T \hat{\phi}_{\rho(T,j)}(F_T^{-1}(\boldsymbol{\xi}))$$

results in a function $\mathbf{u}$ which, because of 4.3, may not have a continuous normal derivative across each edge, e, in the triangulation. In order to guarantee the continuity of $\mathbf{u} \cdot \mathbf{n}$ across e for each triangle T having e as an edge, the pre-image of e under F_T on $\hat{T}$, $\hat{e}$ must have the same length. For example, if e is the common edge between T_1 and T_2 and F_{T_1} maps $\hat{e}_1$ to e, then F_{T_2} also maps $\hat{e}_1$ to e_1. However, if F_{T_1} maps $\hat{e}_2$ to e, then F_{T_2} may map either $\hat{e}_2$ or $\hat{e}_3$ to e as $length(\hat{e}_2) = length(\hat{e}_3) = 1$. See Figure 4-1 for an illustration.

In this Section we present two ways to satisfy this edge length restriction. The first is to align the triangulation of the domain so that triangles sharing an edge map from an edge with the same length on the reference element. The second is to normalize the columns of the finite element coefficient matrix by the multiplicative factor introduced by the Piola transformation.

4.2 Method 1: Edge Coloring

The explicit bases given in [2] and Chapter 3 are defined on reference elements with a non-conforming edge length (or face area) that creates a scaling issue when used in practice. One solution is to arrange the mappings $F_T : \hat{T} \to T$ where $length(F_T^{-1}(e_{ij})) = length(F_T^{-1}(e_{ji}))$ for all T and e_{ij} pairs in $\mathcal{T}_h$.
Finding such a configuration of F_T is a coloring problem. Suppose every edge in $\mathcal{T}_h$ is assigned the color red or black in a way that every triangle has exactly one red edge. The mappings $F_T : \hat{T} \to T$ can then be arranged so that the non-conforming edge of $\hat{T}$ maps to T's red edge, while the conforming edges of $\hat{T}$ map to T's black edges. Formally,

Definition 1. *A conforming coloring assigns every edge in $\mathcal{T}_h$ the color red or black such that every triangle in $\mathcal{T}_h$ has exactly one red edge.*

Lemma 4.1. *A conforming coloring provides an arrangement of the mappings F_T, $T \in \mathcal{T}_h$ that satisfy the edge length restriction.*

Proof. Let $\mathcal{C}$ be a conforming coloring of $\mathcal{T}_h$, and let $\hat{e}^*$ be $\hat{T}$'s non-conforming edge. Arrange $\{F_T : T \in \mathcal{T}_h\}$ (the collection of affine mappings between $\hat{T}$ and $T \in \mathcal{T}_h$) so that $F_T^{-1}(e_{ij}) = \hat{e}^*$ if e_{ij} is colored red in $\mathcal{C}$.
This arrangement is well posed because every triangle has one red edge, and the reference element has one non-conforming coloring. The pre-image of the black edges can be assigned to the reference triangle in any way that is consistent with this.
Now, suppose there exists an F_T and $e_{ij} \in \mathcal{T}_h$ such that $length(F^{-1}(e_{ij})) \neq length(F^{-1}(e_{ji}))$. This implies that either $F^{-1}(e_{ij}) = \hat{e}^*$ and $F^{-1}(e_{ji}) \neq \hat{e}^*$ or $F^{-1}(e_{ji}) = \hat{e}^*$ and $F^{-1}(e_{ij}) \neq \hat{e}^*$. If $F^{-1}(e_{ij}) = \hat{e}^*$, then it implies that e_{ij} is colored red in $\mathcal{C}$, but this creates a contradiction because $F^{-1}(e_{ji}) \neq \hat{e}^*$ yet e_{ij} and e_{ji} refer to the same edge in $\mathcal{T}_h$. □

Now our interest turns towards identifying conforming mappings of $\mathcal{T}_h$. In the following sections, we rely heavily on graph theory.

4.2.1 Graph Theory Definitions

The following definitions come from [4].

Definition 2. *A graph is an ordered pair $G = (V, E)$ of vertices (V) and edges (E). V is a finite, nonempty set of objects and E is a collection of 2-element subsets of V.*

Definition 3. *A graph $G = (V, E)$ is called a directed graph or a digraph if E is a collection of ordered pairs. Edges in a digraph are often referred to as arcs and vertices are often referred to as nodes.*

Definition 4. *A graph G is planar if it can be embedded in the plane. That is, if G can be drawn in such a way that edges only intersect at their endpoints.*

Definition 5. *A $u - v$ walk is a sequence of adjacent vertices that starts at u and ends at v. A walk that does not traverse an edge more than once is called a trail. A walk in which no vertex is repeated is a path.*

Definition 6. *A circuit is a trail of length 3 or more that begins and ends on the same vertex. A circuit in which no vertex, except the starting one, is repeated is called a cycle. A graph is acyclic if it has no cycles.*

Definition 7. *Let $G = (V, E)$ be a graph. Two vertices a, b are connected if there exists an $a - b$ walk. A graph G is connected if all vertex pairs in V are connected.*

Definition 8. *If removing a single edge $e \in E$ causes a connected graph to become disconnected, e is called a bridge. A connected graph with no bridges is called bridgeless.*

Definition 9. *A graph $G = (V, E)$ is a tree if it is both acyclic and connected.*

Definition 10. *For a graph $G = (V, E)$, the degree of a vertex $v \in V$ is number of edges containing v. A graph is called k-regular if all vertices in G have degree k.*

Note: Of particular importance here are 3 and 4-regular graphs. For these graphs, every vertex is incident to exactly 3 and 4 edges, respectively.

Definition 11. *Given a graph G, a matching is a collection of edges for which no vertex appears in more than one edge. A perfect matching is a matching which contains every vertex.*

Definition 12. *A graph G is bipartiate if the set of vertices can be partitioned into disjoint sets U, V such that every edge connects a vertex in U with a vertex in V.*

4.2.2 The Dual Graph

Next we introduce the dual graph, $G_{\mathcal{T}_h}$, of an arbitrary finite element triangulation $\mathcal{T}_h$. As we will see, a perfect matching of $G_{\mathcal{T}_h}$ gives a conforming coloring of $\mathcal{T}_h$.

Let $\mathcal{T}_h$ be a bounded finite element triangulation with no hanging nodes. We can view $\mathcal{T}_h$ as a graph, $\mathcal{T}_h = (T, E)$, where $T = \{t_1, \cdots, t_n\}$ is a collection of triangles and $E = \{e_1, \cdots e_m\}$ is a collection of edges. For a k-dimensional triangulation, every $e \in E$ is a hyperplane connecting k-coordinates in $\mathbb{R}^k$, and every $t \in T$ is a $k+1$-tuple of edges. For example, in Figure 4.2 $t_4 = \{e_7, e_2, e_1\}$. The hanging node restriction implies that each edge belongs to at most two triangles.

Two triangles are *adjacent* if they share a common edge. If all of a triangle's edges belong to two triangles, the triangle is *interior*. Triangles that are not interior are *exterior*. An edge is a *boundary edge* if it is incident to a single triangle, and we denote the set of boundary edges in $\mathcal{T}_h$ as $E_{\partial\Omega}$.

Given $\mathcal{T}_h = (T, E)$, define the dual graph $G_{T_h} = (V_{T_h}, E_{T_h})$ where

$$V_{T_h} = \{i : t_i \in T\}$$
$$E_{T_h} = \{(i, j) : \text{if triangles } i, j \text{ share a common edge in } \mathcal{T}_h\}$$

and $(i, j), (j, i)$ are identified.

First note that $\mathcal{T}_h$ boundary edges are not included in the dual graph. If a vertex in V_{T_h} has fewer than $k + 1$ edges, we call it an exterior vertex.

Figure 4.3 provides an illustration of the dual graph for a 2D triangulation. The dots are elements of V_{T_h} and the double lines are elements of E_{T_h}. Notice that exterior triangles in T_h correspond to exterior vertices in V_{T_h}.

The functions, $\kappa : V_{T_h} \to T$ and $\phi : E_{T_h} \to E \setminus E_{\partial\Omega}$ link the G_{T_h} and $\mathcal{T}_h = (T, E)$, where

$$\kappa(i) = t_i, \text{ and } \quad \phi(i, j) = e_i, \text{ where } e_i \text{ is the edge shared by triangles } t_i \text{ and } t_j.$$

Lemma 4.2. *$\kappa : V_{T_h} \to T$ and $\phi : E_{T_h} \to E \setminus E_{\partial\Omega}$ are bijections.*

Proof. First consider κ. Suppose $\kappa(i) = \kappa(j)$, then $i = j$, so κ is one-to-one. Let $t_i \in T$, then by the definition of V_{T_h}, there exists an $i \in V_{T_h}$ such that $\kappa(i) = t_i$. Therefore, κ is onto.

Next, consider ϕ. Let $e_i \in E \setminus E_{\partial\Omega}$. Then there exist $t_i, t_j \in T$ such that e_i is an

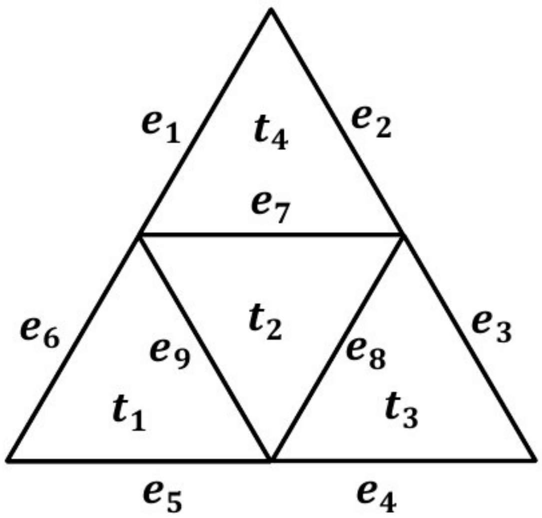

FIGURE 4.2: A simple 2D triangulation

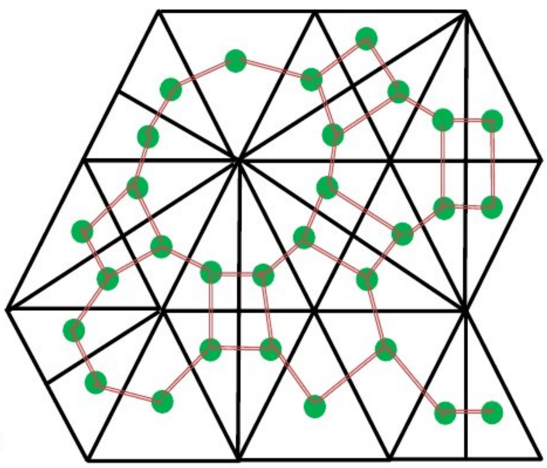

FIGURE 4.3: The Dual Graph of a 2D triangulation

edge of t_i and t_j. Since $t_i, t_j \in T$ share a common edge, there exists $i, j \in V_{T_h}$ and $(i,j) = (j,i) \in E_{T_h}$ such that $\phi(i,j) = e_i$. Therefore, ϕ is onto.

Next, suppose $\phi(i,j) = \phi(s,t)$. If $\phi(i,j) = e_i$, then the edge e_i connects triangles i and j. Since every edge in $E \setminus \partial E_{\partial E}$ belongs exactly two triangles, it follows that $(i,j) = (s,t)$. Therefore, ϕ is one-to-one. □

Lemma 4.3. *A perfect matching of G_{T_h} corresponds to a $\mathcal{T}_h$ conforming coloring.*

Proof. Let $P_e = \{e_1, \cdots, e_n\} \subset E_{T_h}$ be a perfect matching of G_{T_h}. This implies that every $v \in V_{T_h}$, is uniquely associated with an element of P_e.

Now let every edge in E be black and then color every element in $\{\phi(e) : e \in P_e\}$ red. Because $\kappa(V_{T_h}) = T$, it follows that every $t_i \in T$ now has exactly one red edge. □

Lemma 4.4. *In 2D, G_{T_h} is a planar graph.*

Proof. This follows from the fact $\mathcal{T}_h$ is a valid triangulation and can be drawn in the plane. See Figure 4.3. □

4.2.3 2D Solutions

Next we turn our attention to finding a perfect matching for G_{T_h}, the dual graph of a bounded 2D triangulation T_h with no hanging nodes.
The follow result helps prove the existence of such a matching.

Theorem 4.5 (Petersen's Theorem)**.** *Every bridgeless 3-regular graph has a perfect matching.*

Proof. See [4]. □

As Figure 4.3 shows, G_{T_h} is not cubic (ie, a 3-regular graph). While every interior vertex in G_{T_h} has three incident edges, the exterior vertices do not. Thus, to apply Petersen's Theorem, we must extend G_{T_h} to a cubic graph $G_3 = (V_3, E_3)$. This is done by adding a set of artificial nodes V_s and edges E_s (see Figure 4.5). We will see that a perfect matching for G_3 still provides a valid edge assignment for T_h.
Recall $E_{\partial\Omega} \subset E$ is the set of all boundary edges in $T_h = (T, E)$. Since T_h is bounded, $|E_{\partial\Omega}| = p < \infty$. Next, re-index T_h so that $E_{\partial\Omega} = \{m-p, m-p+1, \cdots, m\}$ form a clockwise trail circumscribing T_h.
Assume T_h has n triangles, and define $V_s = \{n+1, \cdots, n+p\}$ where $V_s[i]$ is sequentially assigned to $E_{\partial\Omega}[i]$. This ensures that $v_i \in V_s$ is adjacent to v_{i-1} and v_{i+1} (if $i = n+1$, then v_i is adjacent to v_{i+2} and v_{n+p}). Finally, define $V_3 = V_{T_h} \cup V_s = \{1, \cdots, n, n+1, \cdots, n+p\}$.
To complete the G_3 extension, a set of artificial edges E_s are created. First, let $E_{s_1} = \{e_{m+1}, \cdots, e_{m+p}\}$ denote a set of edges connecting each node in V_s with its corresponding exterior vertex in V_{T_h}. Second, $E_{s_2} = \{e_{m+p+1}, \cdots, e_{m+2p}\}$ connects each artificial vertex with its adjacent artificial vertices. Then $E_s = E_{s_1} \cup E_{s_2}$ and $E_3 = E_{T_h} \cup E_s$.

Lemma 4.6. *G_3 has no bridges.*

Proof. From [4], an edge e of a graph is a bridge if and only if e does not lie on a cycle of G_3. Thus, if e is a bridge, then $e \notin E_s$ because the edges in E_s circumvent the graph, forming a cycle.

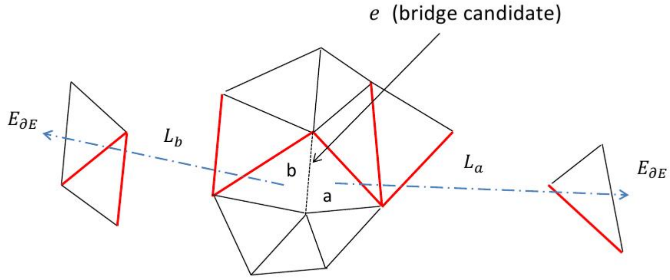

FIGURE 4.4: Example of building a path to the boundary

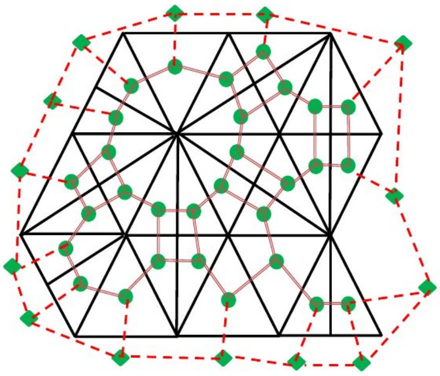

FIGURE 4.5: Extension to a 3-regular graph

Instead, suppose $e \in E_{T_h}$, the edge incident to nodes a and b, is a bridge. Note that if we can construct paths from a and b to E_s, then e would be part of a cycle and thus not a bridge.

To find such a path, let $\kappa(V_{T_h})$ and $\phi(E_{T_h} \setminus e)$ represent a modified form of the triangulation $\mathcal{T}_h$. With $\phi(e)$ removed, T_a and T_b still both have two remaining edges. Since the triangulation is finite, there exist lines L_a and L_b from T_a and T_b to ∂E_{T_h} such that the line does not cross the intersection of two edges. Let $\{e_{a1}, e_{a2}, \cdots e_{an}\}$ and $\{e_{b1}, e_{b2}, \cdots, e_{bn}\}$ denote the set of edges intersected by L_a and L_b respectively. See Figure 4.4.

It follows then that $\{\phi^{-1}(e_{a1}), \phi^{-1}(e_{a2}), \cdots, \phi^{-1}(e_{an})\}$ and $\{\phi^{-1}(e_{b1}), \phi^{-1}(e_{b2}), \cdots,$ $\phi^{-1}(e_{bn})\}$ form a path from node a and b respectively to E_s. Therefore, e lies on a cycle which contradicts that it is a bridge. □

Thus, since G_3 is a 3-regular, bridgeless graph, Petersen's theorem implies a perfect matching exists.

Lemma 4.7. *A perfect matching for G_3 gives a solution to G_{T_h}.*

Proof. If the perfect matching includes an edge connecting two artificial vertices, this edge is ignored in the assignment. If the perfect matching includes an edge connect an exterior vertex with an artificial vertex, then the corresponding boundary edge is assigned to the non-conforming edge of the $\hat{T}$. □

The final consideration is to find a perfect matching for G_3. Here we rely on [5] who provide an $O(n)$-time algorithm for finding a perfect matching of a cubic planar graph.

4.2.4 3D Coloring Problems

In the 2D case, a solution to the coloring problem exists, and there is a linear time algorithm for its discovery.

Unfortunately, the results for 3D triangulations are not as nice. Since each simplex in T_h has four faces, the dual graph G_{T_h} resembles a 4-regular graph. However, not all 4-regular graphs have perfect matchings [6]. Hence it remains an open question whether a coloring exists for all 3D triangulations.

3D Triangulation Method

While a general solution may not exist, here we present one way a 3D triangulation with the desired coloring can be built.

Step 1 Create a cube that encompasses the entire approximation domain Ω.

Step 2 Subdivide the cube into $\frac{1}{h}$ equally sized cubes. The faces of these cubes are the faces that will be mapped to the non-conforming face of the reference simplex (ie the faces are colored red).

Step 3 Partition each cube into six pyramid shaped prisms (Figure 4.6).

Step 4 Slice each pyramid along the diagonal edges of the cube face (Figure 4.7).

Step 5 Remove all tetrahedra that do not intersect with Ω.

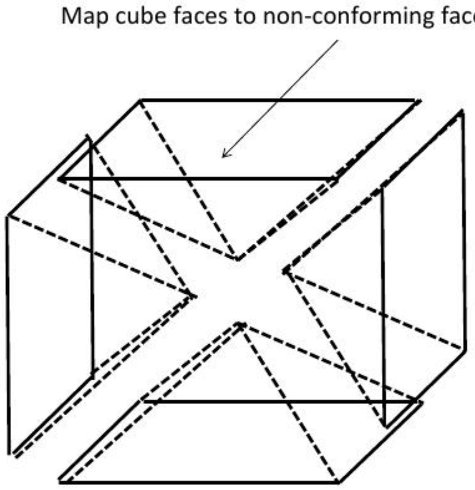

FIGURE 4.6: Partition cube into six pyramids

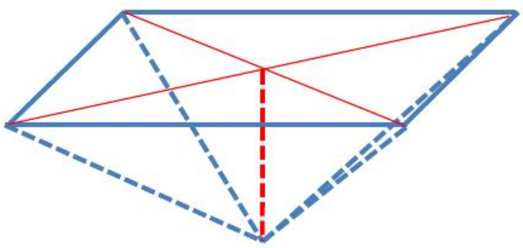

FIGURE 4.7: Slice pyramids to obtain tetrahedra

4.3 Method 2: Multiplicative Factor

Another method to resolve the edge length restriction is to multiply the Finite Element inner products with a scaling factor that offsets the one introduced by the Piola Transformation.

Let A be the coefficient matrix and b the right hand side vector from the standard finite element procedure. The elements of A and b are calculated from the discrete approximation inner products. For example

$$A_{ij} = \sum_{t \in T_h} a(x_i, w_j) \quad b_i = \sum_{t \in T_h} F(x_i)$$

where $a(\mathbf{w}, \mathbf{u})$ is a bilinear form, $F(x_i)$ is a linear functional and $x_i = \{\mathbf{u}_i, p_i\}$ and $w_j = \{\mathbf{v}_j, q_j\}$ are the test and trial spaces for the velocity and pressure.

To account for the edge length restriction, we introduce a scaling function $\delta(t, i)$. Also let e_{ij} denote the edges of $t_i \in T_h$. Let $F(e_{ij}) = \hat{e}_k$ for $k = 1, 2, 3$ denote the edge on the reference triangle corresponding to the edge e_{ij}. Next, every test function $\mathbf{u}_i$ is either an edge or interior function. If it is an edge function, we associate $\mathbf{u}_i$ with the edge e_{ij} where its normal component does not vanish. If $\mathbf{u}_i$ is interior, we associate the test function with edge e_{ij}. Now define $\delta(t, i) = \dfrac{length(e_{ij})}{length(F(e_{ij}))}$.

Using $\delta(t,i)$, we scale the inner product calculations so that

$$A_{ij} = \sum_{t \in T_h} a(\{\delta(t,i)\mathbf{u}_i, p_i\}, w_j) \quad b_i = \sum_{t \in T_h} F(\{\delta(t,i)\mathbf{u}_i, p_i\}, w_j)$$

This scales the test functions corresponding to the velocity by the reciprocal of the scaling factor introduced by the Piola transformation. As a result, when calculating $\mathbf{u}_h$, $\delta(t,i)$ is included with the basis elements, that is

$$\begin{aligned}\mathbf{u}_h(x,y) &= \sum_k c_k \phi_k(x,y) \\ &= \sum_{t \in T_h} [\sum_{i=1} \sum_j c_{\kappa(i,j,T)} nsgn(i,T) \mathcal{P}(\delta(t,i)\phi_j^{[i]}(F^{-1}(x,y))]\end{aligned}$$

Chapter 5

Equilateral Reference Element

5.1 Basis Defined on Equilateral Triangles / Simplex : Introduction

To use RT_k and BDM_k elements on standard reference elements, one must account for the edge length restriction (see Chapter 4) in the numerical representation. This can be avoided if the RT_k and BDM_k bases are defined on a reference element where the edges (or faces) all have the same length (or area).
This gives rise to using an equilateral triangle (tetrahedra) as a reference element. In this section, RT_k and BDM_k bases are given for these elements.

5.2 2D-Equilateral Triangle

This section presents 2D RT_k and BDM_k bases on an equilateral reference triangle.

5.2.1 Equilateral Triangle

First we introduce a new reference element T^*, defined on γ, ψ space.
T^* is the equilateral triangle with vertices $S_1 = (\frac{-\sqrt{3}}{2}, -\frac{1}{2}), S_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and $S_3 = (0, 1)$ with edges $E_1 := \psi = -\sqrt{3}\gamma + 1, E_2 := \psi = \sqrt{3}\gamma + 1$, and $E_3 := \psi = -\frac{1}{2}$.
The outward unit normal vectors are $\mathbf{n}_1 = \frac{1}{2}(\sqrt{3}, 1)'$, $\mathbf{n}_2 = \frac{1}{2}(-\sqrt{3}, 1)'$, and $\mathbf{n}_3 = (0, -1)'$.

5.2.2 Piola Transformation

A key feature of RT_k and BDM_k elements is that they maintain a continuous flux across triangle interfaces. Arbitrary affine transformations do not, however, preserve this property. Instead, an angle preserving mapping, the Piola Transformation, is used to map functions between the reference and target triangles.
The Piola transformation $\mathcal{P} : L_2(T^*) \to L_2(T)$ is defined as:

$$\mathbf{q}^* \to \mathbf{q} = \mathcal{P}(\mathbf{q}^*)(x) := \frac{1}{|J_T|} J_T \mathbf{q}^*(G_T^{-1}(x))$$

where J_T is the Jacobi of an affine mapping between T^* and T and and G_T^{-1} is the inverse of the affine mapping $G_T : T^* \to T$.
To map with T^* to a triangle T with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, use the affine transformation

$$G_T(\gamma, \psi) = \begin{cases} x &= \dfrac{x_2 + x_1}{2} + \dfrac{1}{\sqrt{3}}(x_2 - x_1)\gamma + \dfrac{2}{3}(x_3 - \dfrac{x_2 + x_1}{2})(\psi + \frac{1}{2}) \\ y &= \dfrac{y_2 + y_1}{2} + \dfrac{1}{\sqrt{3}}(y_2 - y_1)\gamma + \dfrac{2}{3}(y_3 - \dfrac{y_2 + y_1}{2})(\psi + \frac{1}{2}) \end{cases}.$$

Hence,

$$J_T = \begin{pmatrix} \dfrac{\partial x}{\partial \gamma} & \dfrac{\partial x}{\partial \psi} \\ \dfrac{\partial y}{\partial \gamma} & \dfrac{\partial y}{\partial \psi} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{3}}(x_2 - x_1) & \dfrac{2}{3}\left(x_3 - \dfrac{x_2 + x_1}{2}\right) \\ \dfrac{1}{\sqrt{3}}(y_2 - y_1) & \dfrac{2}{3}\left(y_3 - \dfrac{y_2 + y_1}{2}\right) \end{pmatrix}.$$

The Piola transformation can map established RT_k and BDM_k bases on $\hat{T}$ (the standard reference element) to T^*. The affine mapping from $\hat{T}$ to T^* is

$$F_T(\xi, \eta) = \begin{cases} \gamma = -\dfrac{\sqrt{3}}{2} + \sqrt{3}\xi + \dfrac{\sqrt{3}}{2}\eta \\ \psi = -\dfrac{1}{2} + \dfrac{3}{2}\eta \end{cases}, \quad F_T^{-1}(\gamma, \psi) = \begin{cases} \xi = \dfrac{1}{3} + \dfrac{\sqrt{3}}{3}\gamma - \dfrac{1}{3}\psi \\ \eta = \dfrac{1}{3} + \dfrac{2}{3}\psi \end{cases}.$$

Therefore,

$$\begin{aligned} q(\gamma, \psi) &= \frac{1}{|J_T|} J_T \hat{q}(\hat{x}) \\ &= \frac{2}{3\sqrt{3}} \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ 0 & \frac{3}{2} \end{pmatrix} \hat{q}(F_T^{-1}(x)) \end{aligned}$$

where $\hat{q}(\xi)$ is a function defined on the standard reference triangle.
The following is helpful to establish independence when basis elements are derived from the reference triangle or simplex.

Lemma 5.1. *The Piola Transformation preserves independence.*

Proof. Suppose $\alpha_1 f_1^* + \cdots + \alpha_n f_n^* = 0$ if and only if $\alpha_1 = \cdots = \alpha_n = 0$. Let

$$f_1 = \mathcal{P}(f_1^*(x^*)) = \frac{1}{|J_T|} J_T f_1^*(x^*),$$
$$\vdots$$
$$f_n = \mathcal{P}(f_n^*(x^*)) = \frac{1}{|J_T|} J_T f_n^*(x^*).$$

Then,

$$\begin{aligned}\beta_1 f_1 + \cdots + \beta_n f_n &= \beta_1 \frac{1}{|J_T|} J_T f_1^*(x^*) + \cdots + \beta_n \frac{1}{|J_T|} J_T f_n^*(x^*) \\ &= \frac{1}{|J_T|} J_T(\beta_1 f_1^* + \cdots + \beta_n f_n^*).\end{aligned}$$

Since $|J_T|$ is non-zero for all T in a valid finite element triangulation $\mathcal{T}_h$, $\beta_1 f_1^* + \cdots + \beta_n f_n^* = 0$ if and only if $\beta_1 = \cdots = \beta_n = 0$. □

5.2.3 *RT* Basis Elements

Next we present an RT_k basis on T^*. First define

$$\mathbf{e}_1^*(\gamma,\psi) = \frac{1}{3}\begin{pmatrix}\sqrt{3}+2\gamma \\ 1+2\psi\end{pmatrix}, \quad \mathbf{e}_2^*(\gamma,\psi) = \frac{1}{3}\begin{pmatrix}-\sqrt{3}+2\gamma \\ 1+2\psi\end{pmatrix}, \mathbf{e}_3^*(\gamma,\psi) = \frac{2}{3}\begin{pmatrix}\gamma \\ \psi-1\end{pmatrix},$$
$$\mathbf{e}_4^*(\gamma,\psi) = (\psi+\sqrt{3}\gamma-1)\begin{pmatrix}\sqrt{3}+2\gamma \\ 1+2\psi\end{pmatrix}, \quad \mathbf{e}_5^*(\gamma,\psi) = (\psi-\sqrt{3}\gamma-1)\begin{pmatrix}-\sqrt{3}+2\gamma \\ 1+2\psi\end{pmatrix}.$$

Notice that

$$\mathbf{e}_i^* \cdot \mathbf{n}_j|_{E_j} = \begin{cases} 1 & : i=j \text{ and } i=1,2,3 \\ 0 & : i\neq j \text{ or } i=4,5\end{cases} \tag{5.1}$$

where E_j is the edge corresponding to $\mathbf{n}_j$.

Basis for $RT_0(T^*)$

The dimension of $RT_0(T^*)$ is 3. Let,

$$\Phi_1^{[1]}(\xi,\eta) = \mathbf{e}_1^*, \quad \Phi_1^{[2]}(\xi,\eta) = \mathbf{e}_2^*, \quad \Phi_1^{[3]}(\xi,\eta) = \mathbf{e}_3^*.$$

Linear independence follows from the Lagrangian property 5.1. Thus, since $\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*$ are in $RT_0(T^*)$, they form a basis of $RT_0(T^*)$.

Basis for $RT_1(T^*)$

The 8 basis elements of RT_1 are made up of 6 Edge and 2 Interior Functions.
Let $g_1^\gamma, g_2^\gamma, g_1^\psi$ and g_2^ψ be Gaussian quadrature points along the γ and ψ axes (see Appendix A for values). Define the Lagrangian polynomials:

$$\ell_1^{[\psi]}(t) = \frac{(t-g_2^\psi)}{(g_1^\psi - g_2^\psi)}, \qquad \ell_2^{[\psi]}(t) = \frac{(t-g_1^\psi)}{(g_2^\psi - g_1^\psi)},$$
$$\ell_1^{[\gamma]}(t) = \frac{(t-g_2^\gamma)}{(g_1^\gamma - g_2^\gamma)}, \qquad \ell_2^{[\gamma]}(t) = \frac{(t-g_1^\gamma)}{(g_2^\gamma - g_1^\gamma)}.$$

Then let

$$\Phi_1^{[1]} = \ell_1^{[\psi]}(\psi)\mathbf{e}_1^*(\gamma,\psi), \qquad \Phi_2^{[1]} = \ell_2^{[\psi]}(\psi)\mathbf{e}_1^*(\gamma,\psi),$$
$$\Phi_1^{[2]} = \ell_2^{[\psi]}(\psi)\mathbf{e}_2^*(\gamma,\psi), \qquad \Phi_1^{[2]} = \ell_1^{[\psi]}(\psi)\mathbf{e}_2^*(\gamma,\psi),$$
$$\Phi_1^{[3]} = \ell_1^{[\gamma]}(\gamma)\mathbf{e}_3^*(\gamma,\psi), \qquad \Phi_2^{[3]} = \ell_2^{[\gamma]}(\gamma)\mathbf{e}_3^*(\gamma,\psi),$$
$$\Phi_1^{[4]} = \mathbf{e}_4^*(\gamma,\psi), \qquad \Phi_1^{[5]} = \mathbf{e}_5^*(\gamma,\psi).$$

The functions are independent (following from the familiar arguments given previously) and in $RT_1(T^*)$. Therefore, the collection above forms a basis for $RT_2(T^*)$.

Basis for $RT_2(T^*)$

The 15 basis elements of RT_2 include nine Edge and six Interior Functions.
Let $g_1^\psi, g_2^\psi, g_3^\psi, g_1^\gamma, g_2^\gamma$ and g_3^γ be Gaussian quadrature points along the ψ and γ axes (see

Appendix A for values). Define the Lagrangian polynomials:

$$\ell_1^{[\psi]}(t) = \frac{(t-g_2^{\psi})(t-g_3^{\psi})}{(g_1^{\psi}-g_2^{\psi})(g_1^{\psi}-g_3^{\psi})}, \qquad \ell_2^{[\psi]}(t) = \frac{(t-g_1^{\psi})(t-g_3^{\psi})}{(g_2^{\psi}-g_1^{\psi})(g_2^{\psi}-g_3^{\psi})},$$
$$\ell_3^{[\psi]}(t) = \frac{(t-g_1^{\psi})(t-g_2^{\psi})}{(g_3^{\psi}-g_1^{\psi})(g_3^{\psi}-g_2^{\psi})}, \qquad \ell_1^{[\gamma]}(t) = \frac{(t-g_2^{\gamma})(t-g_3^{\gamma})}{(g_1^{\gamma}-g_2^{\gamma})(g_1^{\gamma}-g_3^{\gamma})},$$
$$\ell_2^{[\gamma]}(t) = \frac{(t-g_1^{\gamma})(t-g_3^{\gamma})}{(g_2^{\gamma}-g_1^{\gamma})(g_2^{\gamma}-g_3^{\gamma})}, \qquad \ell_3^{[\gamma]}(t) = \frac{(t-g_1^{\gamma})(t-g_2^{\gamma})}{(g_3^{\gamma}-g_1^{\gamma})(g_3^{\gamma}-g_2^{\gamma})}.$$

Then

$$\begin{aligned}
\Phi_1^{[1]} &= \ell_1^{[\psi]}(\psi)\mathbf{e}_1^*(\gamma,\psi), \\
\Phi_3^{[1]} &= \ell_3^{[\psi]}(\psi)\mathbf{e}_1^*(\gamma,\psi), \qquad \Phi_2^{[1]} = \ell_2^{[\psi]}(\psi)\mathbf{e}_1^*(\gamma,\psi), \\
\Phi_1^{[2]} &= \ell_3^{[\psi]}(\psi)\mathbf{e}_2^*(\gamma,\psi), \\
\Phi_3^{[2]} &= \ell_1^{[\psi]}(\psi)\mathbf{e}_2^*(\gamma,\psi), \qquad \Phi_2^{[2]} = \ell_2^{[\psi]}(\psi)\mathbf{e}_2^*(\gamma,\psi), \\
\Phi_1^{[3]} &= \ell_1^{[\gamma]}(\gamma)\mathbf{e}_3^*(\gamma,\psi), \\
\Phi_3^{[3]} &= \ell_2^{[\psi]}(\gamma)\mathbf{e}_3^*(\gamma,\psi), \qquad \Phi_2^{[3]} = \ell_2^{[\gamma]}(\gamma)\mathbf{e}_3^*(\gamma,\psi), \\
\Phi_1^{[4]} &= (\psi+\sqrt{3}\gamma-1)\mathbf{e}_4^*(\gamma,\psi), \\
\Phi_3^{[4]} &= (\psi-\frac{1}{2})\mathbf{e}_4^*(\gamma,\psi), \qquad \Phi_2^{[4]} = (\psi-\sqrt{3}\gamma-1)\mathbf{e}_4^*(\gamma,\psi), \\
\Phi_1^{[5]} &= (\psi+\sqrt{3}\gamma-1)\mathbf{e}_5^*(\gamma,\psi), \\
\Phi_3^{[5]} &= (\psi-\frac{1}{2})\mathbf{e}_5^*(\gamma,\psi), \qquad \Phi_2^{[5]} = (\psi-\sqrt{3}\gamma-1)\mathbf{e}_5^*(\gamma,\psi),
\end{aligned}$$

forms a basis for $RT_2(T^*)$.

Basis for $RT_k(T^*)$

In 2D, the dimension of RT_k is $(k+1)(k+3)$.
Let g_n^{ψ} denote the $k+1$ Gaussian quadrature points on $[-\frac{1}{2},1]$ and g_n^{γ} the $k+1$ Gaussian quadrature points on $[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]$. Using these points, build two sets of Lagrangian

polynomials

$$\ell_i^{\psi}(t) = \frac{(t-g_1^{\psi})(t-g_2^{\psi})\cdots(t-g_{i-1}^{\psi})(t-g_{i+1}^{\psi})\cdots(t-g_n^{\psi})}{(g_i^{\psi}-g_1^{\psi})(g_i^{\psi}-g_2^{\psi})\cdots(g_i^{\psi}-g_{i-1}^{\psi})(g_i^{\psi}-g_{i+1}^{\psi})\cdots(g_i^{\psi}-g_n^{\psi})}$$
$$\ell_i^{\gamma}(t) = \frac{(t-g_1^{\gamma})(t-g_2^{\gamma})\cdots(t-g_{i-1}^{\gamma})(t-g_{i+1}^{\gamma})\cdots(t-g_n^{\gamma})}{(g_i^{\gamma}-g_1^{\gamma})(g_i^{\gamma}-g_2^{\gamma})\cdots(g_i^{\gamma}-g_{i-1}^{\gamma})(g_i^{\gamma}-g_{i+1}^{\gamma})\cdots(g_i^{\gamma}-g_n^{\gamma})}$$

for $i = 1, \cdots, k+1$.

Next with these functions build the edge functions

$$\begin{aligned}
\phi_i^{[1]}(\gamma,\psi) &= \ell_i^{\psi}(t)\mathbf{e}_1^*(\gamma,\psi) \quad i = 1,\cdots,k+1,\\
\phi_i^{[2]}(\gamma,\psi) &= \ell_{k+2-i}^{\psi}\mathbf{e}_2^*(\gamma,\psi) \quad i = 1,\cdots,k+1,\\
\phi_i^{[3]}(\gamma,\psi) &= \ell_i^{\gamma}\mathbf{e}_3^*(\gamma,\psi) \quad i = 1,\cdots,k+1.
\end{aligned}$$

For the interior functions, let $\{b_i(\gamma,\psi), i = 1,2,\cdots,k(k+1)/2\}$ denote a basis for $P_{k-1}(T^*)$. Then the Interior Functions are

$$\begin{aligned}
\Phi_i^{[4]}(\gamma,\psi) &= b_i(\gamma,\psi)\mathbf{e}_4^*(\gamma,\psi) \quad i = 1,2,\cdots,k(k+1)/2,\\
\Phi_i^{[5]}(\gamma,\psi) &= b_i(\gamma,\psi)\mathbf{e}_5^*(\gamma,\psi) \quad i = 1,2,\cdots,k(k+1)/2.
\end{aligned}$$

This gives $3(k+1)$ Edge Functions and $k(k+1)$ Interior Functions for a total of $(k+1)(k+3)$.

Independence

Arguments analogous to those in Section 3.1.4 establish the independence of these functions.

Since every function is an element of $RT_k(T^*)$, they form a basis for $RT_k(T^*)$.

5.2.4 BDM Elements

For BDM_k on T^*, define the edge functions

$$\mathbf{e}_1^*(s_1^\psi, s_2^\psi) = \frac{1}{3\sqrt{3}(s_2^\psi - s_1^\psi)} \begin{pmatrix} 2(s_2^\psi + \frac{1}{2})(1+\sqrt{3}\gamma - \psi) + (s_2^\psi - 1)(2\psi + 1) \\ \sqrt{3}(s_2^\psi - 1)(2\psi+1) \end{pmatrix},$$

$$\mathbf{e}_2^*(s_1^\psi, s_2^\psi) = \frac{1}{3\sqrt{3}(s_2^\psi - s_1^\psi)} \begin{pmatrix} -2(s_2^\psi + \frac{1}{2})(1-\sqrt{3}\gamma - \psi) - (s_2^\psi - 1)(2\psi + 1) \\ \sqrt{3}(s_2^\psi - 1)(2\psi+1) \end{pmatrix},$$

$$\mathbf{e}_3^*(s_1^\gamma, s_2^\gamma) = \frac{1}{3(s_2^\gamma - s_1^\gamma)} \begin{pmatrix} 2s_2^\gamma(1+\sqrt{3}\gamma - \psi) + (2s_2^\gamma + \sqrt{3})(\psi + \frac{1}{2}) - 3(s_2^\gamma + \frac{\sqrt{3}}{2}) \\ 3(1+\sqrt{3}\gamma-\psi) + \sqrt{3}(2s_2^\gamma + \sqrt{3})(\psi + \frac{1}{2}) - 3\sqrt{3}(s_2^\gamma + \frac{\sqrt{3}}{2}) \end{pmatrix},$$

where $s_1^\psi, s_2^\psi \in (-\frac{1}{2}, 1)$ and $s_1^\gamma, s_2^\gamma \in (-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ are distinct.
These Edge Functions satisfy:

$$\begin{aligned} &\text{for } \psi = s_1^\psi, \quad \mathbf{e}_1^* \cdot \mathbf{n}_1|_{E_1} = 1, \quad \text{for } \psi = s_1^\psi, \quad \mathbf{e}_2^* \cdot \mathbf{n}_2|_{E_2} = 1, \text{ for } \gamma = s_1^\gamma, \quad \mathbf{e}_3^* \cdot \mathbf{n}_3|_{E_3} = 1, \\ &\text{for } \psi = s_2^\psi, \quad \mathbf{e}_1^* \cdot \mathbf{n}_1|_{E_1} = 0, \quad \text{for } \psi = s_2^\psi, \quad \mathbf{e}_2^* \cdot \mathbf{n}_2|_{E_2} = 0, \text{ for } \gamma = s_2^\gamma, \quad \mathbf{e}_3^* \cdot \mathbf{n}_3|_{E_3} = 0, \end{aligned}$$

where E_1, E_2, E_3 are the edges of the equilateral triangle T^*. Also $\mathbf{e}_i^* \cdot \mathbf{n}_j = 0$ for $i \neq j$.
The tangent functions are derived from the edge functions:

$$\begin{aligned} \mathbf{e}_4^*(\gamma, \psi) &= (\psi + \sqrt{3}\gamma - 1)\mathbf{e}_1^* \\ \mathbf{e}_5^*(\gamma, \psi) &= (\psi - \sqrt{3}\gamma - 1)\mathbf{e}_2^* \\ \mathbf{e}_6^*(\gamma, \psi) &= (\psi + \frac{1}{2})\mathbf{e}_3^* \end{aligned}$$

where $\mathbf{e}_j^*(\gamma, \psi) \cdot \mathbf{n}_i|_{E_i} = 0$ for $j = 4, 5, 6$ and $i = 1, 2, 3$.
Finally, the interior bubble functions are

$$\mathbf{e}_7^*(\gamma, \psi) = (\psi + \sqrt{3}\gamma - 1)(\psi - \sqrt{3}\gamma - 1)(\psi + \frac{1}{2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\mathbf{e}_8^*(\gamma, \psi) = (\psi + \sqrt{3}\gamma - 1)(\psi - \sqrt{3}\gamma - 1)(\psi + \frac{1}{2}) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\mathbf{e}_j^*(\gamma, \psi) \cdot \mathbf{n}_i|_{E_i} = 0$ for $j = 7, 8$ and $i = 1, 2, 3$.

Basis for BDM_1

The dimension of $BDM_1(T^*)$ is 6. Let $s_1^\psi = g_1^\psi, s_2^\psi = g_2^\psi, s_1^\gamma = g_1^\gamma, s_2^\gamma = g_2^\gamma$, be the Gaussian quadrature points for T^* (see Appendix A). Then the Edge Functions are

$$\begin{aligned} \Phi_1^{[1]}(\gamma,\psi) &= \mathbf{e}_1^*(g_1^\psi, g_2^\psi), & \Phi_2^{[1]}(\gamma,\psi) &= \mathbf{e}_1^*(g_2^\psi, g_1^\psi), \\ \Phi_1^{[2]}(\gamma,\psi) &= \mathbf{e}_2^*(g_1^\psi, g_2^\psi), & \Phi_2^{[2]}(\gamma,\psi) &= \mathbf{e}_2^*(g_2^\psi, g_1^\psi), \\ \Phi_1^{[3]}(\gamma,\psi) &= \mathbf{e}_3^*(g_1^\gamma, g_2^\gamma), & \Phi_2^{[3]}(\gamma,\psi) &= \mathbf{e}_3^*(g_2^\gamma, g_1^\gamma). \end{aligned}$$

The Lagrangian property of the normal components can be used to show the $\Phi_j^{[i]}$s are independent. Therefore, since each element belongs to $BDM_1(T^*)$ they form a basis for $BDM_1(T^*)$.

Basis for BDM_2

The dimension of BDM_2 is 12 and the basis includes nine Edge Functions and three Tangent Functions.
Define the Lagrangian polynomial $\ell(g_1, g_2, t) := \dfrac{(t - g_1)}{(g_1 - g_2)}$ and use the Gaussian quadrature points $g_1^\psi, g_2^\psi, g_3^\psi, g_1^\gamma, g_2^\gamma, g_3^\gamma$, to build the edge functions

$$\begin{aligned} \Phi_1^{[1]}(\gamma,\psi) &= \ell(g_1^\psi, g_3^\psi; \psi)\mathbf{e}_1^*(g_1^\psi, g_2^\psi), \\ \Phi_3^{[1]}(\gamma,\psi) &= \ell(g_3^\psi, g_2^\psi; \psi)\mathbf{e}_1^*(g_3^\psi, g_1^\psi), \\ \Phi_1^{[2]}(\gamma,\psi) &= \ell(g_3^\psi, g_2^\psi; \psi)\mathbf{e}_2^*(g_3^\psi, g_1^\psi), \\ \Phi_3^{[2]}(\gamma,\psi) &= \ell(g_1^\psi, g_3^\psi; \psi)\mathbf{e}_2^*(g_2^\psi, g_3^\psi), \\ \Phi_1^{[3]}(\gamma,\psi) &= \ell(g_1^\gamma, g_3^\gamma; \gamma)\mathbf{e}_3^*(g_1^\gamma, g_2^\gamma), \\ \Phi_3^{[3]}(\gamma,\psi) &= \ell(g_3^\gamma, g_2^\gamma; \gamma)\mathbf{e}_3^*(g_3^\gamma, g_1^\gamma). \end{aligned}$$

$$\begin{aligned} \Phi_2^{[1]}(\gamma,\psi) &= \ell(g_2^\psi, g_1^\psi; \psi)\mathbf{e}_1^*(g_2^\psi, g_3^\psi), \\ \Phi_2^{[2]}(\gamma,\psi) &= \ell(g_2^\psi, g_1^\psi; \psi)\mathbf{e}_2^*(g_2^\psi, g_3^\psi), \\ \Phi_2^{[3]}(\gamma,\psi) &= \ell(g_2^\gamma, g_1^\gamma; \gamma)\mathbf{e}_3^*(g_2^\gamma, g_3^\gamma), \end{aligned}$$

The tangent functions are:

$$\Phi_1^{[4]}(\gamma,\psi) = \mathbf{e}_4^*(g_1^\psi, g_2^\psi), \quad \Phi_1^{[5]}(\gamma,\psi) = \mathbf{e}_5^*(g_1^\psi, g_2^\psi), \quad \Phi_1^{[6]}(\gamma,\psi) = \mathbf{e}_6^*(g_1^\gamma, g_2^\gamma).$$

Finally, these functions form a basis following from an analogous argument as given in [2].

Basis from BDM_k

In 2D, the dimension of BDM_k is $(k+1)(k+2)$.
Let $\mathbf{g}_n^\psi$ denote the $k+1$ Gaussian quadrature points on $[-\frac{1}{2}, 1]$ and $\mathbf{g}_n^\gamma$ the $k+1$ Gaussian quadrature points on $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. Using these points, build the Lagrangian polynomials

$$\ell_i^\psi(t) = \frac{(t-g_1^\psi)(t-g_2^\psi)\cdots(t-g_{i-1}^\psi)(t-g_{i+1}^\psi)\cdots(t-g_n^\psi)}{(g_i^\psi-g_1^\psi)(g_i^\psi-g_2^\psi)\cdots(g_i^\psi-g_{i-1}^\psi)(g_i^\psi-g_{i+1}^\psi)\cdots(g_i^\psi-g_n^\psi)},$$
$$\ell_i^\gamma(t) = \frac{(t-g_1^\gamma)(t-g_2^\gamma)\cdots(t-g_{i-1}^\gamma)(t-g_{i+1}^\gamma)\cdots(t-g_n^\gamma)}{(g_i^\gamma-g_1^\gamma)(g_i^\gamma-g_2^\gamma)\cdots(g_i^\gamma-g_{i-1}^\gamma)(g_i^\gamma-g_{i+1}^\gamma)\cdots(g_i^\gamma-g_n^\gamma)},$$

for $i = 1, \cdots, k+1$.
Next, let $\{\delta_i(\gamma), i = 0, 1, \cdots, (k-2)\}$ denote a basis for $P_{k-2}(T^*)$ and $\{v_i(\gamma,\psi), i = 1, 2, \cdots, (k-2)(k-1)/2\}$ denote a basis for $P_{k-3}(T^*)$.
The Edge Functions are

$$\Phi_j^{[1]}(g_1^\psi, \cdots, g_j^\psi, \cdots, g_{k+1}^\psi, \gamma, \psi) = \ell_j(\psi)\mathbf{e}_1^*(g_j^\psi, g_{j+1}^\psi), \quad j = 1, 2, \cdots, k+1,$$
$$\Phi_j^{[2]}(g_1^\psi, \cdots, g_j^\psi, \cdots, g_{k+1}^\psi, \gamma, \psi) = \ell_j(\psi)\mathbf{e}_2^*(g_j^\psi, g_{j+1}^\psi), \quad j = 1, 2, \cdots, k+1,$$
$$\Phi_j^{[3]}(g_1^\gamma, \cdots, g_j^\gamma, \cdots, g_{k+1}^\gamma, \gamma, \psi) = \ell_j(\gamma)\mathbf{e}_3^*(g_j^\gamma, g_{j+1}^\gamma), \quad j = 1, 2, \cdots, k+1.$$

The Tangent Functions are

$$\Phi_j^{[4]}(\gamma,\psi) = \rho_j(\psi)\mathbf{e}_4^*(g_1^\psi, g_2^\psi), \quad \Phi_j^{[5]}(\gamma,\psi) = \rho_j(\psi)\mathbf{e}_5^*(g_1^\psi, g_2^\psi), \quad \Phi_j^{[6]}(\gamma,\psi) = \rho_j(\gamma)\mathbf{e}_6^*(g_1^\gamma, g_2^\gamma),$$

for $j = 0, 1, \cdots, k-2$.
Finally, the Interior Functions are

$$\Phi_j^{[7]}(\gamma,\psi) = v_j(\gamma,\psi)\mathbf{e}_7^*(\gamma,\psi), \quad \Phi_j^{[8]}(\gamma,\psi) = v_j(\gamma,\psi)\mathbf{e}_8^*(\gamma,\psi),$$

for $j = 0, 1, \cdots, (k-2)(k-1)/2$.
Altogether, this gives $3(k+1)$ Edge Functions, $3(k-1)$ Tangent Functions and $(k-2)(k-1)$ Interior Functions. Following the argument in [2] these $(k+1)(k+2)$ functions form a basis.

5.2.5 Computational Example

Here we consider a numerical approximation to the Darcy Flow equation

$$\begin{aligned} \mathbf{u} + \nu \nabla p = \mathbf{f}, &\quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, &\quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, &\quad \text{on } \partial\Omega, \end{aligned}$$

using the RT and BDM elements built on the reference equilateral triangle T^* described in Section 5.2.1.

For this example, we let $\Omega = (-1, 1) \times (0, 1)$, $\nu = 1$ and use the known solution

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x)\cos(\pi x) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix},$$

$$p = xy.$$

Below are the convergence results for $(RT_0, discP_0)$, $(RT_1, discP_1)$, $(RT_2, discP_2)$, $(BDM_1, discP_0)$ and $(BDM_2, discP_1)$.

$1/h$	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. Rate	$\|div(u_h)\|_{L^2(\Omega)}$	$\|p - p_h\|_{L^2(\Omega)}$	Cvg. Rate
4	3.74E-01		1.00E-15	7.90E-02	
6	2.48E-01	1	1.00E-15	5.15E-02	1
8	1.86E-01	1	2.00E-15	3.84E-02	1
10	1.49E-01	1	2.00E-15	3.06E-02	1
12	1.24E-01	1	2.00E-15	2.55E-02	1
Exp. rate		1			1

TABLE 5.1: Convergence rates for $RT_0, discP_0$ approximation pair on T^*.

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. Rate	$\|div(u_h)\|_{L^2(\Omega)}$	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. Rate
4	6.20E-02		3.00E-15	3.50E-03	
6	2.83E-02	2	4.00E-15	1.42E-03	2
8	1.61E-02	2	6.00E-15	7.73E-04	2
10	1.04E-02	2	6.00E-15	4.87E-04	2
12	7.21E-03	2	7.00E-15	3.35E-04	2
Exp. rate		2			2

TABLE 5.2: Convergence rates for $RT_1, discP_1$ approximation pair on T^*.

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. Rate	$\|div(u_h)\|_{L^2(\Omega)}$	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. Rate
4	4.80E-03		7.00E-15	1.35E-04	
6	1.40E-03	3	8.00E-15	2.56E-05	4
8	5.88E-04	3	1.30E-14	7.93E-06	4
10	3.00E-04	3	1.20E-14	3.22E-06	4
12	1.73E-04	3	2.90E-14	1.54E-06	4
Exp. rate		3			3

TABLE 5.3: Convergence rates for $RT_2, discP_2$ approximation pair on T^*.

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. Rate	$\|div(u_h)\|_{L^2(\Omega)}$	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. Rate
4	6.20E-02		3.00E-15	7.66E-02	
6	2.83E-02	2	4.00E-15	5.09E-02	1
8	1.61E-02	2	6.00E-15	3.81E-02	1
10	1.04E-02	2	6.00E-15	3.05E-02	1
12	7.21E-03	2	9.00E-15	2.54E-02	1
Exp. rate		2			1

TABLE 5.4: Convergence rates for $BDM_1, discP_0$ approximation pair on T^*.

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. Rate	$\|div(u_h)\|_{L^2(\Omega)}$	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. Rate
4	4.80E-03		7.00E-15	3.51E-03	
6	1.40E-03	3	8.00E-15	1.43E-03	2
8	5.88E-04	3	1.30E-14	7.73E-04	2
10	3.00E-04	3	1.20E-14	4.87E-04	2
12	1.73E-04	3	2.90E-14	3.35E-04	2
Exp. rate		3			2

TABLE 5.5: Convergence rates for $BDM_2, discP_1$ approximation pair on T^*.

5.3 3D- Equilateral Tetrahedra

This section presents 3D RT_k and BDM_k bases on an equilateral reference triangle.

5.3.1 Equilateral Simplex

Here we introduce a 3D reference element T^* on (γ, ψ, β) space.
T^* is an equilateral simplex with vertices $S_1 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{2}}{4})$, $S_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{2}}{4})$, $S_3 = (0, 1, -\frac{\sqrt{2}}{4})$ and $S_4 = (0, 0, \frac{3\sqrt{2}}{4})$. The faces are $F_1 := \sqrt{3}\gamma + \psi + \frac{1}{\sqrt{2}}\beta = \frac{3}{4}$, $F_2 := -\sqrt{3}\gamma + \psi + \frac{1}{\sqrt{2}}\beta = \frac{3}{4}$, $F_3 := -2\psi + \frac{1}{\sqrt{2}}\beta = \frac{3}{4}$ and $F_4 := \beta = -\frac{\sqrt{2}}{4}$.
The outward normal vectors corresponding to these faces are $\mathbf{n}_1 = \frac{\sqrt{2}}{3}(\sqrt{3}, 1, \frac{1}{\sqrt{2}})'$, $\mathbf{n}_2 = \frac{\sqrt{2}}{3}(-\sqrt{3}, 1, \frac{1}{\sqrt{2}})'$, $\mathbf{n}_3 = \frac{\sqrt{2}}{3}(0, -2, \frac{1}{\sqrt{2}})'$ and $\mathbf{n}_4 = (0, 0, -1)'$.

5.3.2 3D Piola Transformation

The Piola mapping $\mathcal{P} : L^2(T^*) \to L^2(T)$ is defined in Chapter 5.2.2.
The affine transformation from T^* to a tetrahedra T with coordinates (x_1, y_1, z_1),

(x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$G(\gamma,\psi,\beta) = \begin{cases} x &= \dfrac{x_1+x_2+x_3+x_4}{4} + \dfrac{\sqrt{3}}{3}(x_2-x_1)\gamma + \dfrac{2x_3-x_1-x_2}{3}\psi + \\ & \dfrac{\sqrt{2}}{6}(3x_4-x_1-x_2-x_3)\beta \\ y &= \dfrac{y_1+y_2+y_3+y_4}{4} + \dfrac{\sqrt{3}}{3}(y_2-y_1)\gamma + \dfrac{2y_3-y_1-y_2}{3}\psi + \\ & \dfrac{\sqrt{2}}{6}(3y_4-y_1-y_2-y_3)\beta \\ z &= \dfrac{z_1+z_2+z_3+z_4}{4} + \dfrac{\sqrt{3}}{3}(z_2-z_1)\gamma + \dfrac{2z_3-z_1-z_2}{3}\psi + \\ & \dfrac{\sqrt{2}}{6}(3z_4-z_1-z_2-z_3)\beta. \end{cases} \tag{5.2}$$

Thus,

$$J_T = \begin{pmatrix} \dfrac{\partial x}{\partial \gamma} & \dfrac{\partial x}{\partial \psi} & \dfrac{\partial x}{\partial \beta} \\ \dfrac{\partial y}{\partial \gamma} & \dfrac{\partial y}{\partial \psi} & \dfrac{\partial y}{\partial \beta} \\ \dfrac{\partial z}{\partial \gamma} & \dfrac{\partial z}{\partial \psi} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{3}}{3}(x_2-x_1) & \dfrac{2x_3-x_1-x_2}{3} & \dfrac{\sqrt{2}}{6}(3x_4-x_1-x_2-x_3) \\ \dfrac{\sqrt{3}}{3}(y_2-y_1) & \dfrac{2y_3-y_1-y_2}{3} & \dfrac{\sqrt{2}}{6}(3y_4-y_1-y_2-y_3) \\ \dfrac{\sqrt{3}}{3}(z_2-z_1) & \dfrac{2z_3-z_1-z_2}{3} & \dfrac{\sqrt{2}}{6}(3z_4-z_1-z_2-z_3) \end{pmatrix}.$$

Also of interest is mapping a basis defined on the reference simplex $\hat{T}$ to the equilateral tetrahedra T^* using $\mathcal{P} : L^2(\hat{T}) \to L^2(T^*)$.
An affine mapping from $\hat{T}$ to T^* is

$$G(\xi,\eta,\rho) = \begin{cases} \gamma = -\dfrac{\sqrt{3}}{2} + \sqrt{3}\xi + \dfrac{\sqrt{3}}{2}\eta + \dfrac{\sqrt{3}}{2}\rho \\ \psi = -\dfrac{1}{2} + \dfrac{3}{2}\eta + \dfrac{1}{2}\rho \\ \beta = -\dfrac{\sqrt{2}}{4} + \sqrt{2}\rho \end{cases} \tag{5.3}$$

Thus

$$J_T = \begin{pmatrix} \dfrac{\partial \gamma}{\partial \xi} & \dfrac{\partial \gamma}{\partial \eta} & \dfrac{\partial \gamma}{\partial \rho} \\ \dfrac{\partial \psi}{\partial \xi} & \dfrac{\partial \psi}{\partial \eta} & \dfrac{\partial \psi}{\partial \rho} \\ \dfrac{\partial \beta}{\partial \xi} & \dfrac{\partial \beta}{\partial \eta} & \dfrac{\partial \beta}{\partial \rho} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \dfrac{\sqrt{3}}{2} & \dfrac{\sqrt{3}}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

The inverse affine mapping is

$$G_T^{-1}(\gamma,\psi,\beta) = \begin{cases} \xi = \dfrac{1}{4} + \dfrac{\sqrt{3}}{3}\gamma - \dfrac{1}{3}\psi - \dfrac{\sqrt{2}}{6}\rho \\ \eta = \dfrac{1}{4} + \dfrac{2}{3}\psi - \dfrac{\sqrt{2}}{6}\beta \\ \rho = \dfrac{1}{4} + \dfrac{3\sqrt{2}}{6}\beta \end{cases}.$$

5.3.3 RT_k

The dimension of RT_k in 3D is $\frac{1}{2}(k+1)(k+2)(k+4)$ [1]. The following construction follows Ervin's method in 2D [2].

First consider the collection of Face Functions

$$\mathbf{e}_1^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}\begin{pmatrix} \gamma + \frac{\sqrt{3}}{2} \\ \psi + \frac{1}{2} \\ \beta + \frac{\sqrt{2}}{4} \end{pmatrix}, \qquad \mathbf{e}_2^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}\begin{pmatrix} \gamma - \frac{\sqrt{3}}{2} \\ \psi + \frac{1}{2} \\ \beta + \frac{\sqrt{2}}{4} \end{pmatrix},$$

$$\mathbf{e}_3^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}\begin{pmatrix} \gamma \\ \psi - 1 \\ \beta + \frac{\sqrt{2}}{4} \end{pmatrix}, \qquad \mathbf{e}_4^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}\begin{pmatrix} \gamma \\ \psi \\ \beta - \frac{3}{2\sqrt{2}} \end{pmatrix}.$$

These functions satisfy

$$\mathbf{e}_i^* \cdot \mathbf{n}_j|_{F_j} = \begin{cases} 1 & : i = j \text{ and } i = 1,2,3,4 \\ 0 & : i \neq j \end{cases} \tag{5.4}$$

where F_j is the face corresponding with the unit normal vector $\mathbf{n}_j$.

Lemma 5.2. *The set of Face Functions is linearly independent.*

Proof. This follows from the Lagrangian property 5.4. □

Next, define the Interior Functions

$$\mathbf{e}_5^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}(\sqrt{3}\gamma + \psi + \frac{1}{\sqrt{2}}\beta - \frac{3}{4})\begin{pmatrix}\gamma + \frac{\sqrt{3}}{2}\\ \psi + \frac{1}{2}\\ \beta + \frac{\sqrt{2}}{4}\end{pmatrix},$$

$$\mathbf{e}_6^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}(-\sqrt{3}\gamma + \psi + \frac{1}{\sqrt{2}}\beta - \frac{3}{4})\begin{pmatrix}\gamma - \frac{\sqrt{3}}{2}\\ \psi + \frac{1}{2}\\ \beta + \frac{\sqrt{2}}{4}\end{pmatrix},$$

$$\mathbf{e}_7^*(\gamma,\psi,\beta) = \frac{\sqrt{2}}{2}(-2\psi + \frac{1}{\sqrt{2}}\beta - \frac{3}{4})\begin{pmatrix}\gamma\\ \psi - 1\\ \beta + \frac{\sqrt{2}}{4}\end{pmatrix}.$$

These functions satisfy $\mathbf{e}_i^* \cdot \mathbf{n}_j|_{F_j} = 0$ for $i = 5, 6, 7$ and $j = 1, 2, 3, 4$.

Lemma 5.3. *The set of Interior Functions is linearly independent.*

Proof. Consider

$$\alpha_1\mathbf{e}_5^* + \alpha_2\mathbf{e}_6^* + \alpha_3\mathbf{e}_7^* = 0.$$

Since the only γ^2 appears in $\mathbf{e}_5^*$, $\alpha_1 = 0$. Similarly, because the only ψ^2 and β^2 terms appear in $\mathbf{e}_6^*$ and $\mathbf{e}_7^*$ respectively, $\alpha_2 = \alpha_3 = 0$. □

Basis for RT_0

The dimension of RT_0 is 4. Let

$$\Phi_1^{[1]}(\gamma,\psi,\beta) = \mathbf{e}_1^*(\gamma,\psi,\beta), \qquad \Phi_1^{[2]}(\gamma,\psi,\beta) = \mathbf{e}_2^*(\gamma,\psi,\beta),$$
$$\Phi_1^{[3]}(\gamma,\psi,\beta) = \mathbf{e}_3^*(\gamma,\psi,\beta), \qquad \Phi_1^{[4]}(\gamma,\psi,\beta) = \mathbf{e}_4^*(\gamma,\psi,\beta).$$

These functions are all elements of RT_0 and independent following from the Lagrangian property 5.4. Therefore, they form a basis.

Basis for RT_1

The dimension of RT_1 is 15, and the basis is made up of 12 Edge and 3 Interior Functions. From Chapter 3, a linear bivariate Lagrangian polynomial requires 3 appropriately selected points on each face $j, \mathbf{x}_1^j, \mathbf{x}_2^j, \mathbf{x}_3^j$. Let

$$\ell_1^j(\mathbf{x}^j, \mathbf{x}_2^j, \mathbf{x}_3^j), \quad \ell_2^j(\mathbf{x}_1^j, \mathbf{x}^j, \mathbf{x}_3^j), \quad \ell_3^j(\mathbf{x}_1^j, \mathbf{x}_2^j, \mathbf{x}^j),$$

be the bivariate Lagrangian polynomials built with these points.
This Face Functions are

$$\Phi_1^{[1]}(\gamma,\psi,\beta) = \ell_1^1(\mathbf{x}^1, \mathbf{x}_2^1, \mathbf{x}_3^1)\mathbf{e}_1^*(\gamma,\psi\beta), \qquad \Phi_2^{[1]}(\gamma,\psi,\beta) = \ell_2^1(\mathbf{x}_1^1, \mathbf{x}^1, \mathbf{x}_3^1)\mathbf{e}_1^*(\gamma,\psi\beta),$$
$$\Phi_3^{[1]}(\gamma,\psi,\beta) = \ell_3^1(\mathbf{x}_1^1, \mathbf{x}_2^1, \mathbf{x}^1)\mathbf{e}_1^*(\gamma,\psi\beta),$$
$$\Phi_1^{[2]}(\gamma,\psi,\beta) = \ell_1^2(\mathbf{x}^2, \mathbf{x}_2^2, \mathbf{x}_3^2)\mathbf{e}_2^*(\gamma,\psi\beta), \qquad \Phi_2^{[2]}(\gamma,\psi,\beta) = \ell_2^2(\mathbf{x}_1^2, \mathbf{x}^2, \mathbf{x}_3^2)\mathbf{e}_2^*(\gamma,\psi\beta),$$
$$\Phi_3^{[2]}(\gamma,\psi,\beta) = \ell_3^2(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}^2)\mathbf{e}_2^*(\gamma,\psi\beta),$$
$$\Phi_1^{[3]}(\gamma,\psi,\beta) = \ell_1^3(\mathbf{x}^3, \mathbf{x}_2^3, \mathbf{x}_3^3)\mathbf{e}_3^*(\gamma,\psi\beta), \qquad \Phi_2^{[3]}(\gamma,\psi,\beta) = \ell_2^3(\mathbf{x}_1^3, \mathbf{x}^3, \mathbf{x}_3^3)\mathbf{e}_3^*(\gamma,\psi\beta),$$
$$\Phi_3^{[3]}(\gamma,\psi,\beta) = \ell_3^3(\mathbf{x}_1^3, \mathbf{x}_2^3, \mathbf{x}^3)\mathbf{e}_3^*(\gamma,\psi\beta),$$
$$\Phi_1^{[4]}(\gamma,\psi,\beta) = \ell_1^4(\mathbf{x}^4, \mathbf{x}_2^4, \mathbf{x}_3^4)\mathbf{e}_4^*(\gamma,\psi\beta), \qquad \Phi_2^{[4]}(\gamma,\psi,\beta) = \ell_2^4(\mathbf{x}_1^4, \mathbf{x}^4, \mathbf{x}_3^4)\mathbf{e}_4^*(\gamma,\psi\beta)$$
$$\Phi_3^{[4]}(\gamma,\psi,\beta) = \ell_3^4(\mathbf{x}_1^4, \mathbf{x}_2^4, \mathbf{x}^4)\mathbf{e}_4^*(\gamma,\psi\beta).$$

The Interior Functions are

$$\Phi_1^{[5]}(\gamma,\psi,\beta) = \mathbf{e}_5^*(\gamma,\psi,\beta), \quad \Phi_1^{[6]}(\gamma,\psi,\beta) = \mathbf{e}_6^*(\gamma,\psi,\beta), \quad \Phi_1^{[7]}(\gamma,\psi,\beta) = \mathbf{e}_7^*(\gamma,\psi,\beta).$$

For the Faces Functions, $\Phi_j^{[i]} \cdot \mathbf{n} = 0$ except at one interpolation point. For the Interior Functions, $\Phi_j^{[i]} \cdot \mathbf{n} = 0$ on every face. This implies the Face Functions are linearly independent. Since $\mathbf{e}_5^*, \mathbf{e}_6^*, \mathbf{e}_7^*$ are in RT_1 and independent, these vectors form a basis for RT_1.

Basis for RT_2

The dimension of RT_2 is 36, and the basis is made up for 24 Face and 12 Interior Functions. First, select six appropriate points $\mathbf{x}_i^j$, $j = 1, \cdots, 6$ on each face and build six degree two bivariate Lagrangian polynomials ℓ_j^i, $i = 1, \cdots, 4$ on each face.

The Face Functions are

$$\begin{aligned}
\Phi_i^{[1]}(\gamma,\psi,\beta) &= \ell_i^1(\mathbf{x}_1^1,\cdots,\mathbf{x}_{i-1}^1,\mathbf{x},\mathbf{x}_{i+1}^1,\cdots,\mathbf{x}_6^1)\mathbf{e}_1^*(\gamma,\psi,\beta) \quad \text{for } i=1,2,3,4,5,6,\\
\Phi_i^{[2]}(\gamma,\psi,\beta) &= \ell_i^2(\mathbf{x}_1^2,\cdots,\mathbf{x}_{i-1}^2,\mathbf{x},\mathbf{x}_{i+1}^2,\cdots,\mathbf{x}_6^2)\mathbf{e}_2^*(\gamma,\psi,\beta) \quad \text{for } i=1,2,3,4,5,6,\\
\Phi_i^{[3]}(\gamma,\psi,\beta) &= \ell_i^3(\mathbf{x}_1^3,\cdots,\mathbf{x}_{i-1}^3,\mathbf{x},\mathbf{x}_{i+1}^3,\cdots,\mathbf{x}_6^3)\mathbf{e}_3^*(\gamma,\psi,\beta) \quad \text{for } i=1,2,3,4,5,6,\\
\Phi_i^{[4]}(\gamma,\psi,\beta) &= \ell_i^4(\mathbf{x}_1^4,\cdots,\mathbf{x}_{i-1}^4,\mathbf{x},\mathbf{x}_{i+1}^4,\cdots,\mathbf{x}_6^4)\mathbf{e}_4^*(\gamma,\psi,\beta) \quad \text{for } i=1,2,3,4,5,6.
\end{aligned}$$

The Interior Functions are

$$\begin{aligned}
\Phi_1^{[5]} &= (\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_5^*, & \Phi_2^{[5]} &= (-\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_5^*,\\
\Phi_3^{[5]} &= (-2\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_5^*, & \Phi_4^{[5]} &= (\beta+\frac{\sqrt{2}}{4})\mathbf{e}_5^*,\\
\Phi_1^{[6]} &= (\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_6^*, & \Phi_2^{[6]} &= (-\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_6^*,\\
\Phi_3^{[6]} &= (-2\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_6^*, & \Phi_4^{[6]} &= (\beta+\frac{\sqrt{2}}{4})\mathbf{e}_6^*,\\
\Phi_1^{[7]} &= (\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_7^*, & \Phi_2^{[7]} &= (-\sqrt{3}\gamma+\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_7^*,\\
\Phi_3^{[7]} &= (-2\psi+\frac{1}{\sqrt{2}}\beta-\frac{3}{4})\mathbf{e}_7^*, & \Phi_4^{[7]} &= (\beta+\frac{\sqrt{2}}{4})\mathbf{e}_7^*.
\end{aligned}$$

This collection of functions forms a basis for $RT_2(T^*)$ using analogous arguments used in Section 3.1.4.

The General Case: RT_k

A computational basis for RT_k is obtained in a similar manner to that of RT_2.

First, let $\mathbf{x}_n^j$, $n=1,\cdots,\frac{1}{2}(k+1)(k+2)$ be appropriately selected points on each face $j=1,2,3,4$ and construct the bivariate polynomials described in Chapter 3 so that

$$\ell_i^j(\mathbf{x}_n^i) = \begin{cases} 1 & : \text{ if } i=n \\ 0 & : \text{ if } i\neq n \end{cases}$$

Also, let $\{b_i(\gamma, \psi, \beta) : i = 1, \cdots, \frac{k(k+1)(k+2)}{6}\}$ denote a basis for $P_{k-1}(T^*)$.
The Face Functions are

$$\Phi_i^{[1]} = \ell_i^{[1]}(\mathbf{x})\mathbf{e}_1^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{2}(k+1)(k+2),$$
$$\Phi_i^{[2]} = \ell_i^{[2]}(\mathbf{x})\mathbf{e}_2^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{2}(k+1)(k+2),$$
$$\Phi_i^{[3]} = \ell_i^{[3]}(\mathbf{x})\mathbf{e}_3^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{2}(k+1)(k+2),$$
$$\Phi_i^{[4]} = \ell_i^{[4]}(\mathbf{x})\mathbf{e}_4^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{2}(k+1)(k+2).$$

The Interior Functions are

$$\Phi_i^{[5]} = b_i(\gamma, \psi, \beta)\mathbf{e}_5^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{6}k(k+1)(k+2),$$
$$\Phi_i^{[6]} = b_i(\gamma, \psi, \beta)\mathbf{e}_6^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{6}k(k+1)(k+2),$$
$$\Phi_i^{[7]} = b_i(\gamma, \psi, \beta)\mathbf{e}_7^*(\gamma, \psi, \beta) \text{ where } i = 1, \cdots, \frac{1}{6}k(k+1)(k+2).$$

Together this gives $2(k+1)(k+2)$ Face Functions and $\frac{k(k+1)(k+2)}{2}$ Interior Functions for a total of $\frac{1}{2}(k+1)(k+2)(k+4)$.

Establishing these functions form a basis for $RT_k(T^*)$ follows a similar argument presented in Section 3.1.4.

5.3.4 BDM_k

To build a basis for BDM_k, the Piola transformation in Section 5.3.2 can be applied to the basis defined in on $\hat{T}$ in Chapter 3.
An additional consideration is that the points Lagrangian Interpolating points used must also be scaled using the affine transformation 5.3 in Section 5.3.2.

Appendix A

Gaussian Quadrature Points for Simplex

The following Gaussian Quadrature points are used for building bivariate Lagrangian polynomials on the reference and equilateral simplex [7].

Reference Simplex $k = 1$

For the simplex, when $k = 1$, there are three quadrature points per face.

Face	Quad Point 1	Quad Point 2	Quad Point 3
Face 1	$g_1^{xy} = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$	$g_2^{xy} = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	$g_3^{xy} = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$
Face 2	$g_1^{yz} = (0, \frac{1}{6}, \frac{1}{6})$	$g_2^{yz} = (0, \frac{2}{3}, \frac{1}{6})$	$g_3^{yz} = (0, \frac{1}{6}, \frac{2}{3})$
Face 3	$g_1^{xz} = (\frac{1}{6}, 0, \frac{1}{6})$	$g_2^{xz} = (\frac{2}{3}, 0, \frac{1}{6})$	$g_3^{xz} = (\frac{1}{6}, 0, \frac{2}{3})$
Face 4	$g_1^{xy} = (\frac{1}{6}, \frac{1}{6}, 0)$	$g_2^{xy} = (\frac{2}{3}, \frac{1}{6}, 0)$	$g_3^{xy} = (\frac{1}{6}, \frac{2}{3}, 0)$

3D BDM_1 Basis

Next we illustrate the construction of the 3D BDM_1 Face Functions.
The bivariate Lagrangian polynomials are

$$\begin{aligned}
\ell_1^1 &= -2\xi - 2\eta + \frac{5}{3}, & \ell_2^1 &= 2\xi - \frac{1}{3}, & \ell_3^1 &= 2\eta - \frac{1}{3},\\
\ell_1^2 &= -2\eta - 2\rho + \frac{5}{3}, & \ell_2^2 &= 2\eta - \frac{1}{3}, & \ell_3^2 &= 2\rho - \frac{1}{3},\\
\ell_1^3 &= -2\xi - 2\rho + \frac{5}{3}, & \ell_2^3 &= 2\xi - \frac{1}{3}, & \ell_3^3 &= 2\eta - \frac{1}{3},\\
\ell_1^4 &= -2\xi - 2\eta + \frac{5}{3}, & \ell_2^4 &= 2\xi - \frac{1}{3}, & \ell_3^4 &= 2\eta - \frac{1}{3}.
\end{aligned}$$

Following the method in Section 3.1.5, this yields the following set of face functions

$$\mathbf{e}_1^1 = \sqrt{3}\begin{pmatrix} -2\xi + \frac{5}{3}\xi \\ -2\eta + \frac{5}{3}\eta \\ \frac{5}{3}\rho \end{pmatrix}, \quad \mathbf{e}_2^1 = \sqrt{3}\begin{pmatrix} 2\xi - \frac{1}{3}\xi \\ -\frac{1}{3}\eta \\ -\frac{1}{3}\rho \end{pmatrix}, \quad \mathbf{e}_3^1 = \sqrt{3}\begin{pmatrix} -\frac{1}{3}\xi \\ 2\eta - \frac{1}{3}\eta \\ -\frac{1}{3}\rho \end{pmatrix},$$

$$\mathbf{e}_1^2 = \begin{pmatrix} -(\frac{5}{3} - 2\eta - 2\rho - \frac{5}{3}\xi) \\ \eta(-2 + \frac{5}{3}) \\ \rho(-2 + \frac{5}{3}) \end{pmatrix}, \quad \mathbf{e}_1^2 = \begin{pmatrix} -(-\frac{1}{3} + 2\eta + \frac{1}{3}\xi) \\ \eta(2 - \frac{1}{3}) \\ \rho(-\frac{1}{3}) \end{pmatrix}, \mathbf{e}_3^2 = \begin{pmatrix} -(-\frac{1}{3} + 2\rho + \frac{1}{3}\xi) \\ \eta(-1/3) \\ \rho(2 - \frac{1}{3}) \end{pmatrix},$$

$$\mathbf{e}_1^3 = \begin{pmatrix} \xi(\frac{5}{3} - 2) \\ -(\frac{5}{3} - 2\xi - 2\rho - \frac{5}{3}\eta) \\ \rho(\frac{5}{3} - 2) \end{pmatrix}, \quad \mathbf{e}_1^3 = \begin{pmatrix} \xi(2 - \frac{1}{3}) \\ -(-\frac{1}{3} + 2\xi + \frac{1}{3}\eta) \\ \rho(-\frac{1}{3}) \end{pmatrix}, \mathbf{e}_3^3 = \begin{pmatrix} \xi(-1/3) \\ -(-\frac{1}{3} + 2\rho + \frac{1}{3}\eta) \\ \rho(2 - \frac{1}{3}) \end{pmatrix},$$

$$\mathbf{e}_1^4 = \begin{pmatrix} \xi(\frac{5}{3} - 2) \\ \eta(\frac{5}{3} - 2) \\ -(\frac{5}{3} - 2\xi - 2\eta - \frac{5}{3}\rho) \end{pmatrix}, \quad \mathbf{e}_1^4 = \begin{pmatrix} \xi(2 - \frac{1}{3}) \\ \eta(-\frac{1}{3}) \\ -(-\frac{1}{3} + 2\xi + \frac{1}{3}\rho) \end{pmatrix}, \mathbf{e}_3^4 = \begin{pmatrix} \xi(-1/3) \\ \eta(2 - \frac{1}{3}) \\ -(-\frac{1}{3} + 2\eta + \frac{1}{3}\rho) \end{pmatrix}.$$

Equilateral Simplex $k = 1$

Applying the standard affine transformation, the Gaussian quadrature points for T^* are.

Face	Quad Point 1	Quad Point 2	Quad Point 3
Face 1	$g_1^{xy} = (\frac{-\sqrt{3}}{4}, -\frac{1}{4}, 0)$	$g_2^{xy} = (\frac{\sqrt{3}}{4}, -\frac{1}{4}, 0)$	$g_3^{xy} = (0, \frac{1}{2}, 0)$
Face 2	$g_1^{yz} = (0, \frac{-\sqrt{3}}{4}, -\frac{1}{4})$	$g_2^{yz} = (0, \frac{\sqrt{3}}{4}, -\frac{1}{4})$	$g_3^{yz} = (0, 0, \frac{1}{2})$
Face 3	$g_1^{xz} = (\frac{-\sqrt{3}}{4}, 0, -\frac{1}{4})$	$g_2^{xz} = (\frac{\sqrt{3}}{4}, 0, -\frac{1}{4})$	$g_3^{xz} = (0, 0, \frac{1}{2})$
Face 4	$g_1^{xy} = (\frac{-\sqrt{3}}{4}, -\frac{1}{4}, 0)$	$g_2^{xy} = (\frac{\sqrt{3}}{4}, -\frac{1}{4}, 0)$	$g_3^{xy} = (0, \frac{1}{2}, 0)$

Reference Simplex k=2

An exact representation of six Gaussian quadrature on a 2D triangle is non-trivial, hence the values are given for five decimal points of precision.

Face	Quad Point 1	Quad Point 2
Face 1	$g_1^{xy} = (0.44594, 0.44594, 0.10810)$	$g_2^{xy} = (0.44594, 0.10810, 0.44594)$
Face 2	$g_1^{yz} = (0, 0.44594, 0.44594)$	$g_2^{yz} = (0, 0.44594, 0.10810)$
Face 3	$g_1^{xz} = (0.44594, 0, 0.44594)$	$g_2^{xz} = (0.44594, 0, 0.10810)$
Face 4	$g_1^{xy} = (0.44594, 0.44594, 0)$	$g_2^{xy} = (0.44594, 0.10810, 0)$

Face	Quad Point 3	Quad Point 4
Face 1	$g_3^{xy} = (0.10810, 0.44594, 0.44594)$	$g_4^{xy} = (0.09158, 0.09158, 0.81685)$
Face 2	$g_3^{yz} = (0, 0.10810, 0.44594)$	$g_4^{yz} = (0, 0.09158, 0.09158)$
Face 3	$g_3^{xz} = (0.10810, 0, 0.44594)$	$g_4^{xz} = (0.09158, 0, 0.09158)$
Face 4	$g_3^{xy} = (0.10810, 0.44594, 0)$	$g_4^{xy} = (0.09158, 0.09158, 0)$

Face	Quad Point 5	Quad Point 6
Face 1	$g_5^{xy} = (0.09158, 0.81685, 0.09158)$	$g_6^{xy} = (0.81685, 0.09158, 0.09158)$
Face 2	$g_5^{yz} = (0, 0.09158, 0.81685)$	$g_6^{yz} = (0, 0.81685, 0.9158)$
Face 3	$g_5^{xz} = (0.09158, 0, 0.81685)$	$g_6^{xz} = (0.09158, 0, 0.9158)$
Face 4	$g_5^{xy} = (0.09158, 0.81685, 0)$	$g_6^{xy} = (0.81685, 0.9158, 0)$

Appendix B

Proof of Some Results

Proposition B.1. *For any positive integer N,*

$$\sum_{j=0}^{k} \frac{(j+N-1)!}{(N-1)!\ \ j!} = \frac{(k+N)!}{N!\,k!} \tag{B.1}$$

Proof. Using induction, suppose $k = 1$. For

$$j = 0, \qquad \frac{(N-1)!}{(N-1)!} = 1,$$

$$\text{and for } j = 1, \qquad \frac{N!}{(N-1)!} = N.$$

Therefore $\sum_{j=0}^{1} \frac{(j+N-1)!}{(N-1)!j!} = N + 1 = \frac{(1+N)!}{N!\,1!}$.

Next, assume

$$\sum_{j=0}^{k-1} \frac{(j+N-1)!}{(N-1)!\,j!} = \frac{(k-1+N)!}{N!\,(k-1)!}.$$

Then,

$$\sum_{j=0}^{k} \frac{(j+N-1)!}{(N-1)!\, j!} = \sum_{j=0}^{k-1} \frac{(j+N-1)!}{(N-1)!\, j!} + \frac{(k-1+N)!}{(N-1)!\, k!}$$
$$= \frac{(k-1+N)!}{N!\,(k-1)!} + \frac{(k-1+N)!}{(N-1)!\, k!} = \frac{(k+N)!}{N!\, k!}.$$

□

Lemma B.2. *There are* $\frac{(k+N-1)!}{(N-1)!k!}$ *distinct solutions to the non-negative integer partition* $x_1 + \cdots + x_N = k$*.*

Proof. Using induction, suppose $N = 1$. Then $x_1 = k$ only has a single solution.
Next, suppose $x_1 + \cdots + x_{N-1} = k$ has $\frac{(k+N-2)!}{(N-2)!\, k!}$ solutions.
Consider $x_1 + \cdots + x_N = k$. Since x_N can assume any value $j = 0, \cdots, k$, it follows $x_1 + \cdots x_{N-1} = k - j$. Equivalently $x_1 + \cdots + x_{N-1} = j$ for $j = 0, \cdots k$, and $x_N = k - j$.
From the induction hypothesis, this has $\frac{(j+N-2)!}{(N-2)!\ \ j!}$ solutions. Hence from B.1:

$$\sum_{j=0}^{k} \frac{(j+N-2)!}{(N-2)!j!} = \frac{(k+N-1)!}{(N-1)!\, k!}$$

□

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