8-2016

A New Look at Matrix Analytic Methods

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A NEW LOOK AT MATRIX ANALYTIC METHODS

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

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by
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August 2016

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Abstract

In the past several decades, matrix analytic methods have proven effective at studying two important sub-classes of block-structured Markov processes: $G/M/1$-type Markov processes and $M/G/1$-type Markov processes. These processes are often used to model many types of random phenomena due to their underlying primitives having phase-type distributions.

When studying block-structured Markov processes and its sub-classes, two key quantities are the "rate matrix" $\mathbf{R}$ and a matrix of probabilities typically denoted $\mathbf{G}$. In [30], Neuts shows that the stationary distribution of a Markov process of $G/M/1$-type, when it exists, possess a matrix-geometric relationship with $\mathbf{R}$. Ramaswami's formula [32] shows that the stationary distribution of an $M/G/1$-type Markov process satisfies a recursion involving a well-defined matrix of probabilities, typically denoted as $\mathbf{G}$.

The first result we present is a new derivation of the stationary distribution for Markov processes of $G/M/1$-type using the random-product theory found in Buckingham and Fralix [9]. This method can also be modified to derive the Laplace transform of each transition function associated with a $G/M/1$-type Markov process.

Next, we study the time-dependent behavior of block-structured Markov processes. In [15], Grassmann and Heyman show that the stationary distribution of block-structured Markov processes can be expressed in terms of infinitely many $\mathbf{R}$ and $\mathbf{G}$ matrices. We show that the Laplace transforms of the transition functions associated with block-structured Markov processes satisfies a recursion involving an infinite collection of $\mathbf{R}$ matrices. The $\mathbf{R}$ matrices are shown to be able to be expressed in terms of an infinite collection of $\mathbf{G}$ matrices, which are solutions to fixed-point equations and can be computed iteratively.

Our final result uses the random-product theory to a study an $M/M/1$ queueing model in a two state random environment. Though such a model is a block-structured Markov process, we

avoid computing any $\mathbf{R}$ or $\mathbf{G}$ matrices and instead show that the stationary distribution can be written exactly as a linear combination of scalars that can be determined recursively.

Acknowledgments

I am grateful to my advisor, Dr. Brian Fralix. His expertise and passion for this material made it possible for me to work in this wonderful and interesting area of mathematics. I thank him for his patience and guidance as he showed me how to navigate the research process from start to finish.

I sincerely thank my committee, Dr. Peter Kiessler, Dr. Xin Liu, and Dr. Robert Lund. Without their willingness to offer guidance, insight, and support throughout my graduate school experience, this work would not be possible.

Without the unconditional support of my family and friends, I do not think I would have sanely made it to the finish line. Their positive words of encouragement made all of the difference.

To my wife, Dongmei, thank you for always believing in me and waiting patiently in Charlotte for the past two years. The real value in this journey was being able to share it with you.

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Chapter 1

Introduction

Many systems and technologies used every day are stochastic in nature. A few examples include the number of users connected to a website, the inventory levels of a warehouse, and the number of customers waiting for service at a bank, airport, or stop-light. The class of stochastic processes known as continuous-time Markov processes [1] (CTMPs) are widely used to model such random phenomena. An important class of CTMPs is referred to in the literature as block structured Markov processes, which we formally define later. The distinguishing feature of a block structured Markov process is that its rate matrix $\mathbf{Q}$ can be partitioned in a way that gives it a repeating block structure. The goal of this dissertation is to present newly discovered techniques – which fall under the category of matrix analytic methods – for studying block structured Markov processes and important sub-classes therein.

The rest of this introductory chapter develops necessary background for the theory to come and is organized as follows. In Section 1.1 and 1.2, we introduce notation, definitions, and some relevant classical results for CTMPs. Section 1.3 formally defines block structured Markov processes as well as two important sub-classes: Markov processes of $G/M/1$-type and Markov processes of $M/G/1$-type. Lastly, Section 1.4 summarizes the main results of the the remaining chapters.

[1] Throughout we use the word 'process' to refer to a countable-state Markov process in continuous-time, while the word 'chain' will be used in the discrete-time setting.

1.1 Continuous-time Markov Processes

We begin by specifying our underlying probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where the set Ω is the sample-space of all outcomes, $\mathcal{F}$ is a σ-field of subsets of Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. A sequence of random variables $X := \{X(t); t \geq 0\}$ is a continuous-time Markov process on a countable state space E if for each $t, s \geq 0$ and $x, y \in E$, the Markov property is satisfied:

$$\mathbb{P}(X(t+s) = y \mid X(t) = x, X(u), u < t) = \mathbb{P}(X(s) = y \mid X(0) = x) =: \mathbb{P}_x(X(s) = y)$$

where, for readability, we are denoting the conditional probability $\mathbb{P}(\cdot \mid X(0) = x)$ as $\mathbb{P}_x(\cdot)$. The transition rate matrix (generator) of X is written as $\mathbf{Q} = [q(x,y)]_{x,y \in E}$, where for $x \neq y$, $q(x,y)$ represents the rate that X transitions from state x to state y, and $q(x,x) = -\sum_{y \neq x} q(x,y)$ is the rate at which X leaves state x. Out of convenience, we define $q(x) := -q(x,x)$.

Associated with X is its set of transition times $\{T_n\}_{n \geq 0}$, where T_n represents the nth transition time of X. The zeroth transition time, T_0, is defined be zero. Embedded at these transition epochs of X is a useful discrete-time Markov chain (DTMC) $\{X_n\}_{n \geq 0}$, where for each $n \geq 0$, $X_n := X(T_n)$ is the state of X immediately after its nth transition. Throughout, we assume that X is a regular CTMP, meaning the number of state transitions made by X in any compact interval is finite with probability one.

The hitting-time random variables associated with X are defined for each $A \subset E$ as $\tau_A := \inf\{t > 0 : X(t-) \neq X(t) \in A\}$, which represents the first time X makes a transition into set A. For notational convenience, we let $\tau_x := \tau_{\{x\}}$ for each state $x \in E$, which represents the first time X makes a transition into state x. Similar hitting times are used for the embedded chain $\{X_n\}_{n \geq 0}$: we let $\eta_A := \inf\{n \geq 1 : X_n \in A\}$ be the first time $\{X_n\}_{n \geq 0}$ visits $A \subset E$, and for convenience, $\eta_x := \eta_{\{x\}}$ is the first time $\{X_n\}_{n \geq 0}$ visits state $x \in E$.

An important feature of τ and η is that they are stopping-times, a fact that many of the forthcoming proofs use. To define the stopping-time $\tau : \Omega \to [0, \infty]$, we recall that a collection of sub-σ-fields $\{\mathcal{F}_t; t \geq 0\}$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$. Typically, we define $\mathcal{F}_t := \sigma(X(s) : 0 \leq s \leq t)$ for each $t \geq 0$. We then say τ is an $\{\mathcal{F}_t\}$-stopping time if

$$\{\tau \leq t\} \in \mathcal{F}_t, \qquad t \geq 0.$$

An analgous definition holds in discrete time for η. The notion of stopping-times is important since it lets us state what is known as the strong Markov property: if τ is finite almost surely, then on the event $\{X(\tau) = i\}$, we have for each $s > 0$ that

$$\mathbb{P}(X(\tau + s) = j \mid \mathcal{F}_\tau) = \mathbb{P}(X(s) = j \mid X(0) = i),$$

where $\mathcal{F}_\tau$ is defined as $\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \geq 0\}$.

1.2 Transition Functions and Stationary Distributions

When modeling random phenomena with X, it may be important to have an understanding of the time-dependent behavior of X. For example, it may be useful to be able to compute $\mathbb{P}_x(X(t) = y)$, for any $x, y \in E$, and any $t \geq 0$. These transient probabilities are conveniently represented with transition functions $p_{x,y} : [0, \infty) \to [0, 1]$, $x, y \in E$, where:

$$p_{x,y}(t) := \mathbb{P}_x(X(t) = y). \tag{1.1}$$

Theoretically, the transition functions satisfy two systems of differential equations known as the Kolmogorov backward and forward equations. The Kolmogorov backward equations state that for each $x, y \in E$,

$$p'_{x,y}(t) = \sum_{k \in E} q(x, k) p_{k,y}(t),$$

and the Kolmogorov forward equations state

$$p'_{x,y}(t) = \sum_{k \in E} p_{x,k}(t) q(k, y).$$

If we view $p_{x,y}(t)$ as the (x, y)th element of the matrix $\mathbf{P}(t)$, and similarly let the (x, y)th element of $\mathbf{P}'(t)$ be $p'_{x,y}(t)$, then the backward equations can be written in matrix form as $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ and the forward equations as $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$. Solving either of these systems, however, can be difficult, or even impossible, analytically. For example, when E is finite, it is well-known that $\mathbf{P}(t) = e^{\mathbf{Q}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{Q}^k$, but numerically computing matrix exponentials can be challenging: see

Moler and Van Loan [31]. Thus, other methods are often required to study the time-dependent behavior of CTMPs.

Even though it can be difficult to compute $p_{x,y}$ directly, it is sometimes possible to derive computable expressions for the Laplace transform $\pi_{x,y}$ of the transient probabilities $p_{x,y}$. These transforms are defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$, the set of complex numbers $\mathbb{C}$ having positive real part, as

$$\pi_{x,y}(\alpha) = \int_0^\infty e^{-\alpha t} p_{x,y}(t)dt, \qquad \alpha \in \mathbb{C}_+. \tag{1.2}$$

Due to the uniqueness of Laplace transforms, knowing the Laplace transform of a continuous function is theoretically equivalent to knowing the function itself. If $\pi_{x,y}(\alpha)$ can be computed for $\alpha \in \mathbb{C}_+$, numerical inversion techniques can be used to numerically evaluate $p_{x,y}(t)$ at various points $t \in [0, \infty)$. In [3], Abate and Whitt present two such inversion algorithms.

In addition to studying the time-dependent behavior of X via its transition functions, we will also be interested in the limiting behavior of X. This information, when it exists, is contained in the stationary distribution $\mathbf{p} := [p(x)]_{x\in E}$, where

$$p_x := \lim_{t\to\infty} \mathbb{P}(X(t) = x \mid X(0) = y) \qquad y \in E. \tag{1.3}$$

We recall that the limit in (1.3) exists when X is ergodic, i.e., irreducible and positive recurrent. We say X is irreducible if $\{X_n\}_{n\geq 0}$ is irreducible, meaning for each $x, y \in E$, there exists integers n_1 and n_2 such that

$$\mathbb{P}_x(X_{n_1} = y) > 0 \text{ and } \mathbb{P}_y(X_{n_2} = x) > 0.$$

Furthermore, we say X is positive recurrent if for any $x \in E$, $\mathbb{E}_x(\tau_x) < \infty$, where $\tau_x := \tau_{\{x\}}$, and $\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot \mid X(0) = x)$.

When the limiting probabilities exist, they can be computed by solving a linear system of equations known as the balance equations, given by

$$\begin{aligned} \mathbf{p}\boldsymbol{Q} &= \mathbf{0} \\ \mathbf{p}\mathbf{e}^T &= 1, \end{aligned}$$

where $\mathbf{e}$ is a row vector of ones of appropriate length, and $\mathbf{A}^T$ represents the transpose of a matrix $\mathbf{A}$. Thus, similar to the transient probabilities, the limiting probabilities of X can be expressed in terms of $\mathbf{Q}$ only. However, solving the balance equations is not always possible, and so other methods are sometimes required to find $\mathbf{p}$.

1.3 Block Structured Markov Processes

Throughout this dissertation, we focus on CTMPs whose state space is given by

$$E = \bigcup_{n \geq 0} L_n, \tag{1.4}$$

where for $n \geq 0$, $L_n := \{(n, 0), (n, 1), \ldots, (n, M)\}$ is referred to as level n, with M being either a finite nonnegative integer or infinite.

We say X is in level n and phase i at time $t \geq 0$ if $X(t) = (n, i)$. These processes can have their the rate matrix $\mathbf{Q}$ partitioned as follows:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \cdots \\ \mathbf{B}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ \mathbf{B}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ \mathbf{B}_{-3} & \mathbf{A}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \cdots \\ \mathbf{B}_{-4} & \mathbf{A}_{-3} & \mathbf{A}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.5}$$

where the $(M+1) \times (M+1)$ matrices $\{\mathbf{B}_n\}_{n \in \mathbb{Z}}$ and $\{\mathbf{A}_n\}_{n \in \mathbb{Z}}$ are easily interpreted: for example, when $n \geq 0$, the (i, j)th element of $\mathbf{B}_n$, denoted $(\mathbf{B}_n)_{ij}$, is $(\mathbf{B}_n)_{ij} = q((0, i), (n, j))$, and for $n \leq 0$, $(\mathbf{B}_n)_{ij} = q((-n, i), (0, j))$. Similarly, the (i, j)th element of $\mathbf{A}_m$ is $(\mathbf{A}_m)_{ij} = q((n, i), (n + m, j))$ whenever $m \geq 0$, and so on. We say X is a level-independent block structured Markov process if its generator has the form given by (1.5).

When X has generator (1.5) with $\mathbf{B}_n = \mathbf{A}_n = \mathbf{0}$ for $n \geq 2$ and $\mathbf{B}_1 = \mathbf{A}_1$, then we say X is a level independent Markov process of $G/M/1$-type. Here, X is said to be skip-free to the right, meaning it can increase in level by at most one at a time.

Where do where Markov processes of $G/M/1$-type get their name? Consider a model that has customers arriving to a single-server system in accordance to a renewal process $\{A(t); t \geq 0\}$. The time between renewals are i.i.d random variables having cdf F. Each arrival brings an exponentially distributed amount of work with rate $\mu > 0$. The server processes work, whenever present, at unit rate. We let $X_n = X(T_n-)$ be the number of customers in the system immediately before the nth arrival, where T_n represents the nth arrival instant of $\{A(t); t \geq 0\}$. The embedded chain $\{X_n\}_{n\geq 0}$ is a DTMC. To compute the transition matrix $\mathbf{P} = [p_{i,j}]_{i,j\geq 0}$, we let the random variable A represent a generic interrenewal time and S an exponential random variable with rate μ. Furthermore, we let $\{N(t); t \geq 0\}$ be a Poisson process have rate μ. Then for each $n \geq 1$ and $i \geq 0$,

$$p_{i,0} = \mathbb{P}(N(A) \geq i+1) = 1 - \mathbb{P}(N(A) \leq i) = 1 - \sum_{k=0}^{i} \mathbb{E}\left[\frac{(\mu A)^k e^{-\mu A}}{k!}\right] =: b_{-i}$$

and for $1 \leq j \leq i+1$,

$$p_{i,j} = \mathbb{P}(N(A) = i+1-j) = \mathbb{E}\left[\frac{(\mu A)^{i+1-j} e^{-\mu A}}{(i+1-j)!}\right] =: a_{j-i}$$

and $p_{i,j} = 0$ otherwise. These observations show $\mathbf{P}$ has the following lower-Hessenberg structure:

$$\mathbf{P} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ b_{-1} & a_0 & a_1 & 0 & 0 & \cdots \\ b_{-2} & a_{-1} & a_0 & a_1 & 0 & \cdots \\ b_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ b_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.6}$$

It is now easy to see that the generator matrix for a Markov process of $G/M/1$-type inherits its block-lower-Hessenberg structure from the lower-Hessenberg structure of $\mathbf{P}$, hence the name.

Block-structured Markov processes contain another important class of CTMPs known as Markov processes of $M/G/1$-type. We say X is an $M/G/1$-type Markov process if it has generator (1.5) with $\mathbf{B}_n = \mathbf{A}_n = \mathbf{0}$ for each $n \leq -2$. Such a process is said to be skip-free to the left since it can transition down at most one level at a time.

Analogous to our study of the $G/M/1$ queueing process, we can study the $M/G/1$ queueing system to discover where the name Markov process of $M/G/1$-type originated. To achieve this, assume customers arrive to a single-server queue according to a Poisson process $\{N(t); t \geq 0\}$ having rate $\lambda > 0$. The amount of work brought by customers are i.i.d random variables having cdf F. The server always processes work at unit rate when available, and we denote a generic service time with the random variable S. Further, we denote the departure time of the nth customer as T_n, $n \geq 1$. Letting $X_n := X(T_n+)$ represent the number of customers in the system immediately after the nth departure, we see that $\{X_n\}_{n\geq 1}$ is an embedded DTMC. Finding the elements of the transition probability matrix $\mathbf{P} := [p_{i,j}]_{i,j\geq 0}$ of X is not difficult: for $i \geq 0$, the zeroth row of $\mathbf{P}$ is given by

$$p_{0,i} = \mathbb{P}(N(S) = i) := b_i, \tag{1.7}$$

and for $k \geq 1$ and $i \geq k-1$, we can write

$$p_{k,i} = \mathbb{P}(N(S) = i - k + 1) := a_{i-k}. \tag{1.8}$$

Using these observations to write $\mathbf{P}$ in matrix form yields

$$\mathbf{P} = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_{-1} & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_{-1} & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.9}$$

While it is true that $b_n = a_{n-1}$ for each $n \geq 0$, we purposefully write $\mathbf{P}$ in the form of (1.9) to illustrate its connection to the generator matrix of $M/G/1$-type Markov processes. By generalizing the upper-Hessenberg structure of $\mathbf{P}$ to a block-upper-Hessenberg generator, it becomes clear why the name Markov processes of $M/G/1$-type was chosen.

1.4 Summary

The rest of this dissertation is organized as follows. Chapter 2 contains a study of the steady-state and time-dependent behavior of Markov processes of $G/M/1$-type. Everything from Chapter 2 can also be found in Joyner and Fralix [21]. In Chapter 3, we present a new viewpoint for studying the steady-state and time-dependent behavior of block-structured Markov processes which yields new insights for such processes. Finally, Chapter 4 gives an exact description of the limiting probabilities of $M/M/1$ queues in a two state random environment.

Chapter 2

An Analysis of $G/M/1$-type Markov Processes

2.1 Introduction

In this chapter, we assume X is a level-independent Markov process of $G/M/1$-type. Recall that such an X has a generator $\mathbf{Q}$ from (1.5) with $\mathbf{B}_n = \mathbf{A}_n = \mathbf{0}$ for each integer $n \geq 2$, and $\mathbf{B}_1 = \mathbf{A}_1$. This gives $\mathbf{Q}$ the following block-lower-Hessenberg structure:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} & \cdots \\ \mathbf{B}_{-3} & \mathbf{A}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \cdots \\ \mathbf{B}_{-4} & \mathbf{A}_{-3} & \mathbf{A}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

A classic result for $G/M/1$-type Markov processes comes from Neuts [30], who shows the stationary distribution of such a process possesses a matrix-geometric structure. Neuts derives the stationary distribution in discrete-time first, by making use of taboo probabilities and sample-path arguments, and then extends to continuous time by applying uniformization to the discrete-time results. A disadvantage of uniformizing from discrete-time, however, is that it requires the diagonal elements of the rate matrix $\mathbf{Q}$ to be bounded.

A derivation of the stationary distribution in continuous-time that does not rely on uniformization can be found in Ramaswami [33] and Latouche and Ramaswami [25], whose argument makes use of Markov renewal theory instead. Their main idea is to 'condition' on a well-defined last jump that occurs before time t in order to derive a formula for the probability of being in a certain state at time t. This approach is extended in [33] to derive analogous results regarding the transient behavior of $G/M/1$-type Markov processes.

Our random-product method, like the approach in [25], is able to derive the stationary distribution directly in continuous-time. Thus, we do not need to place a boundedness assumption on the diagonal elements of $\mathbf{Q}$, and so we can assume M is countably infinite. An additional feature of the random-product theory is its ability to derive the stationary distribution using only basic Markov process concepts—first-step analysis and the strong Markov property—thus removing the reliance on Markov renewal theory and 'last jump' conditioning arguments.

In addition to deriving well-known stationary results using the random-product method, we show how our approach can yield insight into the time-dependent behavior of X. Namely, we are able to show that the Laplace transforms $\{\pi_{x,y}(\alpha)\}_{x\in L_0, y\in E}$ from (1.2) also satisfy a matrix-geometric structure when $X(0) \in L_0$ with probability one. While this result is not entirely new, see [33], we additionally provide an iterative numerical scheme that can be used to compute Laplace transforms when $\alpha \in \mathbb{C}_+$. This is an important result since it allows us to invert the members of $\{\pi_{x,y}(\alpha)\}_{x\in L_0, y\in E}$ in order to compute $p_{x,y}(t)$, $x \in L_0, y \in E$, at any time-point t. As far as we can tell, this is a new contribution.

The remaining two sections of this chapter are as follows: Section 2.2 shows how the recently discovered random-product method from [9] can be used to derive the stationary distribution $\mathbf{p}$ of X. Section 2.3 details how the random-product derivation of $\mathbf{p}$ can be extended to derive computable expressions for the Laplace transforms $\{\pi_{x,y}(\alpha)\}_{x\in L_0, y\in E}$ of X when $X(0) \in L_0$ with probability one.

2.2 Steady-State Behavior

We focus now on deriving the stationary distribution, $\mathbf{p}$, of X using the random-product theory developed in [9]. We partition $\mathbf{p}$ as $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots)$, where for each $n \geq 0$, $\mathbf{p}_n :=$

$(p_{(n,0)}, p_{(n,1)}, \ldots, p_{(n,M)}) \in \mathbb{R}^{(M+1)}$. Element $p_{(n,i)}$ of $\mathbf{p}_n$ represents the long-run proportion of time X spends in state (n, i), $0 \le i \le M$.

To derive $\mathbf{p}$, we follow [9] by selecting a new CTMP $\tilde{X} := \{\tilde{X}(t); t \ge 0\}$ on E with generator $\tilde{\mathbf{Q}} := [\tilde{q}(x, y)]_{x,y \in E}$ satisfying the following two properties:

(i) for each $x \in E$, $\sum_{y \neq x} \tilde{q}(x, y) = \sum_{y \neq x} q(x, y)$, and

(ii) for any two states $x, y \in E$, $x \neq y$, we have $\tilde{q}(x, y) > 0$ if and only if $q(y, x) > 0$.

Though there are infinitely many ways to choose such a $\tilde{\mathbf{Q}}$, a natural and easy choice to work with is simply

$$\tilde{\mathbf{Q}} = \begin{array}{c} \\ L_0 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} L_0 & L_1 & L_2 & L_3 & L_4 & \cdots \end{array} \\ \begin{pmatrix} \tilde{\mathbf{B}}_0 & \tilde{\mathbf{B}}_{-1} & \tilde{\mathbf{B}}_{-2} & \tilde{\mathbf{B}}_{-3} & \tilde{\mathbf{B}}_{-4} & \cdots \\ \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_0 & \tilde{\mathbf{A}}_{-1} & \tilde{\mathbf{A}}_{-2} & \tilde{\mathbf{A}}_{-3} & \cdots \\ \mathbf{0} & \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_0 & \tilde{\mathbf{A}}_{-1} & \tilde{\mathbf{A}}_{-2} & \cdots \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_0 & \tilde{\mathbf{A}}_{-1} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array} \tag{2.2}$$

where the collections of matrices $\{\tilde{\mathbf{B}}_n\}_{n \le 0}$ and $\{\tilde{\mathbf{A}}_n\}_{n \le 1}$ within (2.2) satisfy four properties: (i) for $0 \le j \le M$, $(\tilde{\mathbf{B}}_0)_{j,j} = (\mathbf{B}_0)_{j,j}$ and $(\tilde{\mathbf{A}}_0)_{j,j} = (\mathbf{A}_0)_{j,j}$, (ii) for $0 \le i, j \le M$, $i \neq j$, $(\tilde{\mathbf{B}}_0)_{i,j} > 0$ if and only if $(\mathbf{B}_0)_{j,i} > 0$, and $(\tilde{\mathbf{A}}_0)_{i,j} > 0$ if and only if $(\mathbf{A}_0)_{j,i} > 0$, (iii) $(\tilde{\mathbf{A}}_1)_{i,j} > 0$ if and only if $(\mathbf{A}_1)_{j,i} > 0$, and finally (iv) for each $n \le -1$, and each $0 \le i, j \le M$, $(\tilde{\mathbf{B}}_n)_{i,j} > 0$ if and only if $(\mathbf{B}_n)_{j,i} > 0$, and $(\tilde{\mathbf{A}}_n)_{i,j} > 0$ if and only if $(\mathbf{A}_n)_{j,i} > 0$.

We let $\{\tilde{T}_n\}_{n \ge 0}$ denote the set of transition epochs of $\tilde{X}$. The DTMC embedded at these epochs is denoted $\{\tilde{X}_n\}_{n \ge 0}$, where $\tilde{X}_n := \tilde{X}(\tilde{T}_n)$. Furthermore, for each subset $A \subset E$ we have the hitting times,

$$\tilde{\tau}_A := \inf\{t \ge 0 : \tilde{X}(t) \in A\}, \quad \tilde{\eta}_A := \inf\{n \ge 0 : \tilde{X}_n \in A\},$$

so that $\tilde{\tau}_A$ represents the first time $\tilde{X}$ begins a sojourn in set A (hence $\tilde{\tau}_A = 0$ whenever $\tilde{X}(0) \in A$), and $\tilde{\eta}_A$ is the first time $\{\tilde{X}_n\}_{n \ge 0}$ transitions into A. Note that these are defined slightly differently from their τ_A and η_A counterparts.

The key quantities used to derive $\mathbf{p}$ are referred to in the literature as the rate matrices $\{\mathbf{R}_{n,n+k}\}_{n\geq 0,k\geq 1}$: for each $n \geq 0, k \geq 1$, the (i,j)th element of $\mathbf{R}_{n,n+k}$ is

$$(\mathbf{R}_{n,n+k})_{i,j} = q((n,i))\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{C_n}} \mathbf{1}(X(t) = (n+k,j))dt\right], \qquad 0 \leq i,j \leq M, \tag{2.3}$$

where for $x \in E$, $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot \mid X(0) = x)$ is a probability conditional on $X(0) = x$, and $\mathbb{E}_x(\cdot)$ its corresponding expectation, $C_n := \cup_{k=0}^{n} L_k$ denotes the collection of all states residing within or below level n, and $\mathbf{1}(\cdot)$ represents an indicator function, equal to one if $(\cdot)$ is true, and zero otherwise. Each term $(\mathbf{R}_{n,n+k})_{i,j}$, $0 \leq i,j \leq M$, represents $q((n,i))$ times the expected amount of time spent by X in state $(n+k,j)$ before returning to a level at or lower than level n, given X starts in state (n,i).

Our first result shows that for $n \geq 0, k \geq 1$, we can interpret each element of $\mathbf{R}_{n,n+k}$ as the expected value of a random product governed by the embedded DTMC $\{\tilde{X}_n\}_{n\geq 0}$.

Lemma 2.2.1 *For each $0 \leq i,j \leq M$, $n \geq 0$, $k \geq 1$, we have*

$$(\mathbf{R}_{n,n+k})_{i,j} = \mathbb{E}_{(n+k,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right]. \tag{2.4}$$

Note that in (2.4), $\mathbb{E}_{(n+k,j)}(\cdot)$ represents conditional expectation, given $\tilde{X}_0 = (n+k,j)$. Throughout we will let $\mathbb{E}_x$ represent conditional expectation, given either $X_0 = x$ or $\tilde{X}_0 = x$, since it should always be clear from the context which one is being used. Representation (2.4) for $(\mathbf{R}_{n,n+k})_{i,j}$ appears to be much harder to understand intuitively than (2.3), but it will allow us to derive many well-known facts about the stationary distribution of X using only first-step analysis, and the strong Markov property.

Proof It helps to first notice that, by definition of $\tilde{\mathbf{Q}}$, a feasible path $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ from x to y under $\tilde{\mathbf{Q}}$ yields a feasible path $y = x_n, x_{n-1}, \ldots, x_1, x_0 = x$ from y to x under $\mathbf{Q}$, and vice versa. Proceeding then as in the proof of Theorem 1.1 of [9], we find after summing over all finite feasible paths from state $(n+k,j)$ to state (n,i) under $\tilde{\mathbf{Q}}$ that

$$\begin{aligned}
& \mathbb{E}_{(n+k,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
= & \sum_{m=1}^{\infty} \sum_{\substack{x_0,\ldots,x_m: \\ x_0=(n+k,j), x_m=(n,i), x_1,\ldots,x_{m-1}\in C_n^c}} \left[\prod_{\ell=1}^{m} \frac{q(x_\ell, x_{\ell-1})}{\tilde{q}(x_{\ell-1}, x_\ell)}\right]\left[\prod_{\ell=1}^{m} \frac{\tilde{q}(x_{\ell-1}, x_\ell)}{\tilde{q}(x_{\ell-1})}\right] \\
= & \sum_{m=1}^{\infty} \sum_{\substack{x_0,\ldots,x_m: \\ x_0=(n+k,j), x_m=(n,i), x_1,\ldots,x_{m-1}\in C_n^c}} \left[\prod_{\ell=1}^{m} \frac{q(x_\ell, x_{\ell-1})}{\tilde{q}(x_{\ell-1})}\right] \\
= & \frac{q((n,i))}{q((n+k,j))}\sum_{m=1}^{\infty} \sum_{\substack{x_0,\ldots,x_m: \\ x_0=(n+k,j), x_m=(n,i), x_1,\ldots,x_{m-1}\in C_n^c}} \left[\prod_{\ell=1}^{n_1} \frac{q(x_\ell, x_{\ell-1})}{q(x_\ell)}\right] \\
= & \frac{q((n,i))}{q((n+k,j))}\sum_{m=1}^{\infty} \mathbb{P}_{(n,i)}(X_m = (n+k,j), \eta_{C_n} > m) \\
= & \frac{q((n,i))}{q((n+k,j))}\mathbb{E}_{(n,i)}\left[\sum_{m=0}^{\eta_{C_n}-1} \mathbf{1}(X_m = (n+k,j))\right] \\
= & q((n,i))\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{C_n}} \mathbf{1}(X(t) = (n+k,j))dt\right]
\end{aligned}$$

where the last equality follows from an argument similar to that used to establish Lemma 31 on pg. 259 of Serfozo [37]. $\diamondsuit$

Representation (2.4) can be used to derive other well-known properties of these rate matrices: one such property is given in Lemma 2.2.2. This lemma is well-known, but our proof makes use of (2.4).

Lemma 2.2.2 *For each $n = 0, 1, 2, \ldots$, $k = 1, 2, 3, \ldots$, we have*

$$\mathbf{R}_{n,n+k} = \prod_{\ell=1}^{k} \mathbf{R}_{n+\ell-1,n+\ell}. \tag{2.5}$$

Proof To establish (2.5), we fix an integer $n \geq 0$ and apply an induction argument on k. Clearly (2.5) holds when $k = 1$. Next, assume (2.5) holds for a fixed integer $k \geq 1$. Then for $0 \leq i, j \leq M$, we have after starting in state $(n+k+1, j)$ and summing over all possible values for $\tilde{X}_{\tilde{\eta}_{C_{n+k}}}$, the first state visited in C_{n+k}, that

$$
\begin{aligned}
(\mathbf{R}_{n,n+k+1})_{i,j} = \mathbb{E}_{(n+k+1,j)} &\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i)) \prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
= \sum_{\nu=0}^{M} \mathbb{E}_{(n+k+1,j)} &\Bigg[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n+k}}} = (n+k,\nu)) \\
&\times \left(\prod_{\ell=1}^{\tilde{\eta}_{C_{n+k}}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right) \prod_{\ell=\tilde{\eta}_{C_{n+k}}+1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \Bigg] \\
= \sum_{\nu=0}^{M} \mathbb{E}_{(n+k+1,j)} &\left[\mathbf{1}(\tilde{\eta}_{C_{n+k}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n+k}}} = (n+k,\nu)) \prod_{\ell=1}^{\tilde{\eta}_{C_{n+k}}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
&\times \mathbb{E}_{(n+k,\nu)} \left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i)) \prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
= \sum_{\nu=0}^{M} &(\mathbf{R}_{n,n+k})_{i,\nu}(\mathbf{R}_{n+k,n+k+1})_{\nu,j}
\end{aligned}
$$

where the third equality follows from applying the strong Markov property at the stopping time $\tilde{\eta}_{C_{n+k}}$, the first time $\{\tilde{X}_n\}_{n\geq 0}$ reaches the set C_{n+k}. Using this observation with our induction hypothesis then gives

$$
\mathbf{R}_{n,n+k+1} = \mathbf{R}_{n,n+k}\mathbf{R}_{n+k,n+k+1} = \prod_{\ell=1}^{k+1} \mathbf{R}_{n+\ell-1,n+\ell}
$$

proving the claim. $\diamondsuit$

The following corollary follows immediately from the level-independent structure of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$.

Corollary 2.2.1 *For each $n \geq 0$, we have $\mathbf{R}_{n,n+k} = \mathbf{R}^k$, where $\mathbf{R} := \mathbf{R}_{0,1}$.*

We now state and prove the main result of this section. While the theorem is well-known, our proof makes use of random-products. In Section 2.3, we will see the random-product methodology can be extended to give an analogous result for the Laplace transform of each transition function associated with X.

Theorem 2.2.1 *For each $n \geq 0$, we have*

$$
\mathbf{p}_{n+1} = \mathbf{p}_n\mathbf{R}_{n,n+1} = \mathbf{p}_0\mathbf{R}^{n+1} \tag{2.6}
$$

where the rate matrix $\mathbf{R}$ *is the minimal nonnegative solution to the equation*

$$\sum_{k=0}^{\infty} \mathbf{X}^k \mathbf{A}_{1-k} = \mathbf{0}. \tag{2.7}$$

Furthermore, $\mathbf{R}$ *also satisfies*

$$\mathbf{R} = \lim_{N \to \infty} \mathbf{Y}(N) \tag{2.8}$$

where $\{\mathbf{Y}(N)\}_{N \geq 0}$ *is a sequence of matrices in* $\mathbb{R}^{(M+1)\times(M+1)}$ *that are pointwise nondecreasing in* N *and satisfy the recursion*

$$\mathbf{Y}(N+1) = (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(N)^k \mathbf{A}_{1-k})(-\mathbf{A}_0^{-1}), \qquad N \geq 0 \tag{2.9}$$

having initial condition $\mathbf{Y}(0) = \mathbf{0}$*.*

Statement (2.6) shows $\mathbf{p}$ has a matrix-geometric form. In most cases the rate matrix $\mathbf{R}$ cannot be computed explicitly, but Recursion (2.9) can be used to approximate $\mathbf{R}$ with $\mathbf{Y}(N)$ for large N.

Proof Our proof of Theorem 2.2.1 consists of three steps. In Step 1, we use Theorem 1.1 of [9], combined with Lemmas 2.2.1 and 2.2.2, and Corollary 2.2.1 to establish (2.6). Next, in Step 2 we show $\mathbf{R}$ is a solution to (2.7), and in Step 3 we establish $\mathbf{R}$ as the minimal nonnegative solution of (2.7), where $\lim_{N\to\infty} \mathbf{Y}(N) = \mathbf{R}$.

Step 1: Fix $(0, l_0)$ as a reference point, $0 \leq l_0 \leq M$, and recall from Theorem 1.1 of [9] that since state $(0, l_0)$ is recurrent, an invariant measure of X is given by $\mathbf{w} = \{w(x)\}_{x \in E}$, where $w((0, l_0)) = 1$, and for $(n, i) \neq (0, l_0)$,

$$w((n,i)) = \mathbb{E}_{(n,i)} \left[\mathbf{1}(\tilde{\eta}_{(0,l_0)} < \infty) \prod_{\ell=1}^{\tilde{\eta}_{(0,l_0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right].$$

For $n \geq 0$, $0 \leq j \leq M$, we have after summing over all possible ways $\{\tilde{X}_n\}_{n \geq 0}$ can first reach the set C_n that

$$\begin{aligned} w((n+1,j)) &= \mathbb{E}_{(n+1,j)}\left[\mathbf{1}(\tilde{\eta}_{(0,l_0)}<\infty)\prod_{\ell=1}^{\tilde{\eta}_{(0,l_0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\ &= \sum_{\nu=0}^{M}\mathbb{E}_{(n+1,j)}\Big[\mathbf{1}(\tilde{\eta}_{(0,l_0)}<\infty)\mathbf{1}(\tilde{\eta}_{C_n}<\infty)\mathbf{1}(X_{\tilde{\eta}_{C_n}}=(n,\nu)) \qquad (2.10) \\ &\quad \times \prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\prod_{\ell=\tilde{\eta}_{C_n}+1}^{\tilde{\eta}_{(0,l_0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\Big] \\ &= \sum_{\nu=0}^{M}w((n,\nu))\mathbb{E}_{(n+1,j)}\Big[\mathbf{1}(\tilde{\eta}_{C_n}<\infty)\mathbf{1}(X_{\tilde{\eta}_{C_n}}=(n,\nu)) \qquad (2.11) \\ &\quad \times \prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\Big] \\ &= \sum_{\nu=0}^{M}w((n,\nu))(\mathbf{R}_{n,n+1})_{\nu,j} \qquad (2.12) \end{aligned}$$

where the third equality follows from applying the strong Markov property at time $\tilde{\eta}_{C_n}$, and the last equality from Lemma 2.2.1. Dividing both sides of (2.10) by the total mass of $\mathbf{w}$ and applying Lemma 2.2.1 further yields

$$\mathbf{p}_{n+1}=\mathbf{p}_n\mathbf{R}_{n,n+1}=\mathbf{p}_n\mathbf{R}=\mathbf{p}_0\mathbf{R}^{n+1}$$

which establishes (2.6). Note that w is still an invariant measure that has a matrix-geometric form when $(0,l_0)$ is only assumed to be null recurrent: these types of questions are studied in Latouche et al. [24].

Step 2: We next show $\mathbf{R}$ is a solution to (2.7). Fix $0\le i,j\le M$: starting with the representation (2.4) for $(\mathbf{R}_{0,1})_{i,j}$ and conditioning on $\tilde{X}_1$, we see after simplifying that

$$\begin{aligned} (\mathbf{R}_{0,1})_{i,j} &= \mathbb{E}_{(1,j)}\left[\mathbf{1}(\tilde{\eta}_{C_0}<\infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}}=(0,i))\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\ &= \frac{q((0,i),(1,j))}{q((1,j))}+\sum_{\substack{\nu=0\\ \nu\neq j}}^{M}\left(\frac{q((1,\nu),(1,j))}{q((1,j))}\right)\mathbb{E}_{(1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0}<\infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}}=(0,i))\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\ &+ \sum_{n=1}^{\infty}\sum_{\nu=0}^{M}\left(\frac{q((n+1,\nu),(1,j))}{q((1,j))}\right)\mathbb{E}_{(n+1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0}<\infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}}=(0,i))\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \end{aligned}$$

or

$$-(\mathbf{A}_0)_{j,j}(\mathbf{R}_{0,1})_{i,j} = (\mathbf{A}_1)_{i,j} + \sum_{\substack{\nu=0 \\ \nu\neq j}}^{M}(\mathbf{A}_0)_{\nu,j}(\mathbf{R}_{0,1})_{i,\nu} + \sum_{n=2}^{\infty}\sum_{\nu=0}^{M}(\mathbf{A}_{1-n})_{\nu,j}(\mathbf{R}_{0,n})_{i,\nu}$$

which, upon using matrix notation, shows $\mathbf{R}_{0,1}$ satisfies (2.7) since

$$\mathbf{0} \;\; = \;\; \mathbf{A}_1 + \mathbf{R}_{0,1}\mathbf{A}_0 + \sum_{n=2}^{\infty}\mathbf{R}_{0,n}\mathbf{A}_{1-n} = \sum_{n=0}^{\infty}\mathbf{R}^n\mathbf{A}_{1-n}.$$

Step 3: It remains to prove (2.8): doing so also establishes $\mathbf{R}$ as the minimal nonnegative solution to (2.7). To prove $\lim_{N\to\infty}\mathbf{Y}(N) \leq \mathbf{R}$, it suffices to show $\mathbf{Y}(N)$ is pointwise nondecreasing in N, and $\mathbf{Y}(N) \leq \mathbf{R}$ for each $N \geq 0$. We omit the proof, as it is analogous to steps used in the proof of Lemma 1.2.3 in [30].

Next, we prove $\lim_{n\to\infty}\mathbf{Y}(N) \geq \mathbf{R}$. Fix an arbitrary state $x \in E$: under the measure $\mathbb{P}_x$, let $\tilde{\gamma}_{C_n}$ represent the number of level transitions made by $\{\tilde{X}_n\}_{n\geq 0}$ as it moves from state x to the set C_n when $x \in C_n^c$. Likewise, for each $n \geq 0$, $N \geq 0$, $k \geq 1$, define the matrix $\mathbf{R}_{n,n+k}(N)$ whose (i,j)th element is given by

$$(\mathbf{R}_{n,n+k}(N))_{i,j} = \mathbb{E}_{(n+k,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\mathbf{1}(\tilde{\gamma}_{C_n} \leq N)\prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right].$$

By the monotone convergence theorem, we have

$$\lim_{N\to\infty}\mathbf{R}_{0,1}(N) = \mathbf{R}$$

and so to prove $\lim_{N\to\infty}\mathbf{Y}(N) \geq \mathbf{R}$, it suffices instead to show that for each integer $N \geq 1$, $\mathbf{R}_{0,1}(N) \leq \mathbf{Y}(N)$. Starting with the (i,j)th element from $\mathbf{R}_{0,1}(N)$, we find after applying first-step analysis and using matrix notation that

$$\mathbf{R}_{0,1}(N) = (\mathbf{A}_1 + \sum_{n=2}^{\infty}\mathbf{R}_{0,n}(N-1)\mathbf{A}_{1-n})(-\mathbf{A}_0^{-1}). \tag{2.13}$$

The final step of the proof involves establishing the inequality

$$\mathbf{R}_{0,n}(N-1) \leq \mathbf{R}_{0,1}(N-1)^n \tag{2.14}$$

for each $n \geq 1$, $N \geq 1$. Once this has been shown, another induction argument can be used to prove $\mathbf{R}_{0,1}(N) \leq \mathbf{Y}(N)$ for each $N \geq 0$, completing the proof.

To establish $\mathbf{R}_{0,n}(N-1) \leq \mathbf{R}_{0,1}(N-1)^n$, we first fix $i, j \in \{0, 1, \ldots, M\}$. For $n \geq 2$, we find after summing over both the number of level transitions it takes to get to L_{n-1} and the first state visited in L_{n-1} that

$$\begin{aligned}
& (\mathbf{R}_{0,n}(N-1))_{i,j} \\
= \; & \mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1)\prod_{\ell=1}^{\tilde{\eta}_{C_0}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
= \; & \sum_{k=1}^{N-1}\sum_{\nu=0}^{M} \mathbb{E}_{(n,j)}\Big[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu)) \\
& \times \mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1)\prod_{\ell=1}^{\tilde{\eta}_{C_0}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\Bigg] \\
= \; & \sum_{k=1}^{N-1}\sum_{\nu=0}^{M} \mathbb{E}_{(n-1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1-k)\prod_{\ell=1}^{\tilde{\eta}_{C_0}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
\times \; & \mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{C_{n-1}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k)\prod_{\ell=1}^{\tilde{\eta}_{C_{n-1}}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
\leq \; & \sum_{\nu=0}^{M}\sum_{k=1}^{N-1} (\mathbf{R}_{0,n-1}(N-1))_{i,\nu}\mathbb{E}_{(n,j)}\Big[\mathbf{1}(\tilde{\eta}_{C_{n-1}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k) \\
& \times \prod_{\ell=1}^{\tilde{\eta}_{C_{n-1}}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\Bigg] \\
= \; & \sum_{\nu=0}^{M} (\mathbf{R}_{0,n-1}(N-1))_{i,\nu}(\mathbf{R}_{n-1,n}(N-1))_{\nu,j}
\end{aligned}$$

where the third equality follows from again applying the strong Markov property at the stopping time $\tilde{\eta}_{C_{n-1}}$, with the following inequality being a consequence of the monotonicity property of expectation. Hence, $\mathbf{R}_{0,n}(N-1) \leq \mathbf{R}_{0,n-1}(N-1)\mathbf{R}_{n-1,n}(N-1)$, and further iterations of this inequality yield

$$\mathbf{R}_{0,n}(N-1) \leq \prod_{k=1}^{n} \mathbf{R}_{k-1,k}(N-1) = \mathbf{R}_{0,1}(N-1)^n$$

proving Theorem 2.2.1. $\diamondsuit$

We close this section by stating the well-known fact that once $\mathbf{R}$ has been either computed or approximated, $\mathbf{p}_0$ can be computed by solving

$$\mathbf{p}_0 \sum_{n=0}^{\infty} \mathbf{R}^n \mathbf{B}_{-n} = \mathbf{0}, \qquad 1 = \mathbf{p}_0 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{e}$$

where $\mathbf{e}$ is a $(M+1) \times 1$ column vector of ones, since

$$\mathbf{0} = \sum_{n=0}^{\infty} \mathbf{p}_n \mathbf{B}_{-n} = \sum_{n=0}^{\infty} \mathbf{p}_0 \mathbf{R}^n \mathbf{B}_{-n} = \mathbf{p}_0 \sum_{n=0}^{\infty} \mathbf{R}^n \mathbf{B}_{-n}, \qquad 1 = \sum_{n=0}^{\infty} \mathbf{p}_n \mathbf{e} = \mathbf{p}_0 \sum_{n=0}^{\infty} \mathbf{R}^n \mathbf{e} = \mathbf{p}_0 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{e}.$$

2.3 Time-Dependent Behavior

Our goal now is to derive computable expressions for the Laplace transform $\pi_{(0,i_0),(n,j)}$ when $n \geq 0$ and $0 \leq i_0, j \leq M$. Once the transform is known for $\alpha \in \mathbb{C}_+$, numerical transform inversion techniques can be used to compute $p_{(0,i_0),(n,j)}$; see Abate and Whitt [3] and den Iseger [12] for more on this idea.

The approach presented in this section to derive $\pi_{(0,i_0),(n,j)}$ greatly mirrors that used in Section 2.2 to derive $\mathbf{p}$. An important collection of matrices appearing within these Laplace transforms are the matrices $\{\mathbf{R}_{n,n+k}(\alpha)\}_{n \geq 0, k \geq 1}$ for $\alpha \in \mathbb{C}_+$, which are defined as follows: for $n \geq 0, k \geq 1$ and $0 \leq i, j \leq M$, the (i,j)th element of $R_{n,n+k}(\alpha)$ is given by

$$(\mathbf{R}_{n,n+k}(\alpha))_{i,j} = (q((n,i)) + \alpha) \mathbb{E}_{(n,i)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+k, j)) dt \right], \qquad \alpha \in \mathbb{C}_+, \qquad (2.15)$$

where again, τ_{C_n} represents the first time X makes a transition to a state within or below level n.

Our first lemma shows that each element of $\mathbf{R}_{n,n+k}(\alpha)$ represents the Laplace transform of a nonnegative function.

Lemma 2.3.1 *The (i,j)th element of the matrix $\mathbf{R}_{n,n+k}(\alpha)$ is the Laplace transform of a nonnegative function, where the transform is defined on $\alpha \in \mathbb{C}_+$.*

Proof Fix $\alpha \in \mathbb{C}_+$. Conditioning on $X(T_1)$ yields

$$\begin{aligned}
& \mathbb{E}_{(n,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+k,j))dt\right] \\
= & \sum_{\nu=0}^{M}\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+k,j))dt \mid X(T_1)=(n+1,\nu)\right]\frac{(\mathbf{A}_1)_{i,\nu}}{-(\mathbf{A}_0)_{i,i}} \\
= & \sum_{\nu=0}^{M}\mathbb{E}_{(n,i)}\left[e^{-\alpha T_1}\int_{T_1}^{\tau_{C_n}} e^{-\alpha (t-T_1)}\mathbf{1}(X(t)=(n+k,j))dt \mid X(T_1)=(n+1,\nu)\right]\frac{(\mathbf{A}_1)_{i,\nu}}{-(\mathbf{A}_0)_{i,i}} \\
= & \sum_{\nu=0}^{M}\frac{-(\mathbf{A}_0)_{i,i}}{\alpha-(\mathbf{A}_0)_{i,i}}\mathbb{E}_{(n+1,\nu)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+k,j))dt\right]\frac{(\mathbf{A}_1)_{i,\nu}}{-(\mathbf{A}_0)_{i,i}} \\
= & \frac{1}{\alpha-(\mathbf{A}_0)_{i,i}}\sum_{\nu=0}^{M}(\mathbf{A}_1)_{i,\nu}\mathbb{E}_{(n+1,\nu)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+k,j))dt\right].
\end{aligned}$$

To establish the third equality, use a change of variable, along with the fact that conditional on $X(T_1)$, T_1 is exponential with rate $q((n,i))$, and independent of the future behavior of X beyond time T_1. Noting further that $\alpha-(\mathbf{A}_0)_{i,i}=\alpha+q((n,i))$, we find after multiplying both sides of the previous equality by $\alpha+q((n,i))$ that

$$\begin{aligned}
(\mathbf{R}_{n,n+k}(\alpha))_{i,j} &= \sum_{\nu=0}^{M}(\mathbf{A}_1)_{i,\nu}\mathbb{E}_{(n+1,\nu)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+k,j))dt\right] \\
&= \int_0^{\infty} e^{-\alpha t}\left[\sum_{\nu=0}^{M}(\mathbf{A}_1)_{i,\nu}\mathbb{P}_{(n+1,\nu)}(X(t)=(n+k,j),\tau_{C_n}>t)\right]dt
\end{aligned}$$

where the last line follows from an application of Fubini's theorem (two applications if $M=\infty$). Hence, each element of $\mathbf{R}_{n,n+k}(\alpha)$ can be interpreted as a Laplace transform of a nonnegative function. Since this function is also bounded, its Laplace transform is well-defined on $\mathbb{C}_+$, and this proves our claim. $\diamondsuit$

Our next lemma shows that for $\alpha \in \mathbb{C}_+$, each element of $\mathbf{R}_{n,n+k}(\alpha)$ exhibits a random-product representation analogous to the representation given in Lemma 2.2.1 for each element of $\mathbf{R}_{n,n+k}$.

Lemma 2.3.2 *Suppose $\alpha \in \mathbb{C}_+$. Then for each $0 \le i,j \le M, n \ge 0, k \ge 1$, we have*

$$(\boldsymbol{R}_{n,n+k}(\alpha))_{i,j} = \mathbb{E}_{(n+k,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n}<\infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}}=(n,i))e^{-\alpha\tilde{\tau}_{C_n}}\prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]. \qquad (2.16)$$

Proof Expression (2.16) can be derived by adjusting the proof of Lemma 2.2 found in [13] so that it is similar to the proof of Lemma 2.2.1 of this dissertation: we omit the details. ♢

The next lemma is analogous to Lemma 2.2.2 but pertains to the matrices $\{\mathbf{R}_{n,n+k}(\alpha)\}_{n\geq 0,k\geq 1}$. The result is also stated in Lemma 3.1 of [33]

Lemma 2.3.3 *Let $n \geq 0$ and $k \geq 1$ be two integers. Then for $\alpha \in \mathbb{C}_+$,*

$$\mathbf{R}_{n,n+k}(\alpha) = \prod_{\ell=1}^{k} \mathbf{R}_{n+\ell-1,n+\ell}(\alpha). \tag{2.17}$$

Proof We use an induction argument that is very much analogous to the argument used to prove Lemma 2.2.2. Let $\alpha \in \mathbb{C}_+$ and fix an integer $n \geq 0$. Note that (2.17) is trivially true when $k = 1$. Next, assume (2.17) holds for a fixed positive k. Then for $0 \leq i, j \leq M$, we have

$$(\mathbf{R}_{n,n+k+1}(\alpha))_{i,j} = \mathbb{E}_{(n+k+1,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))e^{-\alpha\tilde{\tau}_{C_n}}\prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]$$

$$\begin{aligned}
&= \sum_{\nu=0}^{M}\mathbb{E}_{(n+k+1,j)}\Big[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n+k}}} = (n+k,\nu))e^{-\alpha\tilde{\tau}_{C_{n+k}}} \\
&\quad \times \left(\prod_{\ell=1}^{\tilde{\eta}_{C_{n+k}}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right)e^{-\alpha(\tilde{\tau}_{C_n}-\tilde{\tau}_{C_{n+k}})}\prod_{\ell=\tilde{\eta}_{C_{n+k}}+1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\Big] \\
&= \sum_{\nu=0}^{M}\mathbb{E}_{(n+k+1,j)}\left[\mathbf{1}(\tilde{\eta}_{C_{n+k}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n+k}}} = (n+k,\nu))e^{-\alpha\tilde{\tau}_{C_{n+k}}}\prod_{\ell=1}^{\tilde{\eta}_{C_{n+k}}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\
&\quad \times \quad \mathbb{E}_{(n+k,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))e^{-\alpha\tilde{\tau}_{C_n}}\prod_{\ell=1}^{\tilde{\eta}_{C_n}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\
&= \sum_{\nu=0}^{M}(\mathbf{R}_{n,n+k}(\alpha))_{i,\nu}(\mathbf{R}_{n+k,n+k+1}(\alpha))_{\nu,j}
\end{aligned}$$

which, coupled with our induction hypothesis, gives

$$\mathbf{R}_{n,n+k+1}(\alpha) = \mathbf{R}_{n,n+k}(\alpha)\mathbf{R}_{n+k,n+k+1}(\alpha) = \prod_{\ell=1}^{k+1}\mathbf{R}_{n+\ell-1,n+\ell}(\alpha),$$

thus proving the claim. ♢

Corollary 2.3.1 *For each $n \geq 0$, we have $\mathbf{R}_{n,n+k}(\alpha) = \mathbf{R}(\alpha)^k$, where $\mathbf{R}(\alpha) := \mathbf{R}_{0,1}(\alpha)$.*

Before stating and proving the main result of this section, we introduce additional notation. For each fixed state $(m, i) \in E$ and each integer $n \geq 0$, it is convenient to place the transforms $\pi_{(m,i),(n,j)}(\alpha)$ within the row vector $\boldsymbol{\pi}_{(m,i),n}(\alpha)$, which is defined as

$$\boldsymbol{\pi}_{(m,i),n}(\alpha) := [\pi_{(m,i),(n,0)}(\alpha) \ , \ \ldots \ , \ \pi_{(m,i),(n,M)}(\alpha)], \qquad \alpha \in \mathbb{C}_+.$$

Furthermore, given levels $m, n \geq 0$ we define $\mathbf{\Pi}_{m,n}(\alpha) = \left[\pi_{(m,i),(n,j)}(\alpha)\right]_{0 \leq i,j \leq M} \in \mathbb{C}^{(M+1)\times(M+1)}$, the set of $(M+1) \times (M+1)$ matrices having complex-valued elements, and we further define $\mathbf{\Pi}(\alpha)$, where

$$\mathbf{\Pi}(\alpha) = \begin{array}{c} \\ L_0 \\ L_1 \\ L_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} L_0 & L_1 & L_2 & \cdots \end{array} \\ \begin{pmatrix} \mathbf{\Pi}_{0,0}(\alpha) & \mathbf{\Pi}_{0,1}(\alpha) & \mathbf{\Pi}_{0,2}(\alpha) & \cdots \\ \mathbf{\Pi}_{1,0}(\alpha) & \mathbf{\Pi}_{1,1}(\alpha) & \mathbf{\Pi}_{1,2}(\alpha) & \cdots \\ \mathbf{\Pi}_{2,0}(\alpha) & \mathbf{\Pi}_{2,1}(\alpha) & \mathbf{\Pi}_{2,2}(\alpha) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

contains all of the Laplace transforms associated with each transition function of X.

We now present the main result of this section.

Theorem 2.3.1 *For each $n \geq 0$ and $Re(\alpha) > 0$, we have*

$$\boldsymbol{\pi}_{(0,l_0),n+1}(\alpha) = \boldsymbol{\pi}_{(0,l_0),n}(\alpha)\mathbf{R}_{n,n+1}(\alpha) = \boldsymbol{\pi}_{(0,l_0),n}(\alpha)\mathbf{R}(\alpha) \tag{2.18}$$

where the matrix $\mathbf{R}(\alpha)$ satisfies

$$\mathbf{R}(\alpha) = \lim_{N\to\infty} \mathbf{Y}(N, \alpha) \tag{2.19}$$

with $\{\mathbf{Y}(N, \alpha)\}_{N \geq 0}$ satisfying the recursion

$$\mathbf{Y}(N+1, \alpha) = (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(N, \alpha)^k \mathbf{A}_{1-k})(\alpha \mathbf{I} - \mathbf{A}_0)^{-1}, \qquad N \geq 0 \tag{2.20}$$

having initial condition $\mathbf{Y}(0,\alpha) = \mathbf{0}$*. Moreover, for* $\alpha \in \mathbb{C}_+$*,* $\mathbf{R}(\alpha)$ *is a solution to the equation*

$$\mathbf{A}_1 + \mathbf{X}(\mathbf{A}_0 - \alpha\mathbf{I}) + \sum_{k=2}^{\infty} \mathbf{X}^k \mathbf{A}_{1-k} = \mathbf{0} \tag{2.21}$$

and when $\alpha > 0$*,* $\mathbf{R}(\alpha)$ *is also the minimal solution to (2.21).*

Most of Theorem 2.3.1 has been discovered before, and has been derived with a number of different methods. Limit (2.19), Recursion (2.20) and Equation (2.21) all appear in the work of Hsu and He [20] for the case where α is a nonnegative real number. There, $\mathbf{R}(\alpha)$ appears as the rate matrix of a modified Markov process of $G/M/1$-type, where $\mathbf{A}_0$ is changed to $\mathbf{A}_0 - \alpha\mathbf{I}$ and each $\mathbf{B}_n$ is changed to $\mathbf{B}_n + \alpha\mathbf{I}$, for $n \leq -1$. This process represents a clearing model, but there is no immediate connection between the stationary distribution of this model and the transition functions of the model corresponding to the case where $\alpha = 0$. Equations (2.18) and (2.21) were also derived using Green's function methods in Keilson and Masuda [22] for the special case where X is a QBD process. Almost all of Theorem 2.3.1 can be found in Ramaswami [33], who derives the results by making use of the Markov renewal approach discussed in detail in [25]. Finally, in the work of Bean et al [6], a recursion and limiting result very similar to (2.20) and (2.19) are also derived for QBD chains in discrete-time, but there the emphasis is more on studying convergence rates associated with the chain. Other approaches used to study the time-dependent behavior of QBD processes exist as well, see e.g. Van Velthoven et al [38], as well as Zhang and Coyle [42].

While our main contribution here is in the approach we use to derive these results, we have also shown that the limiting result (2.19) is still valid for the case where $\alpha \in \mathbb{C}_+$: to the best of our knowledge, such a limit result has not been stated previously. This extension is important since many numerical transform inversion algorithms [3, 12] require evaluating Laplace transforms at points outside of the real line. Similar issues are discussed in Abate and Whitt [1] within the context of studying the busy period distribution of the $M/G/1$ queue.

Proof We prove this result in four steps. Step 1 consists in verifying (2.18) for $\alpha \in \mathbb{C}_+$, and each integer $n \geq 0$. In Step 2, we establish $\mathbf{R}(\alpha)$ as a solution to equation (2.21), and as the minimal nonnegative solution when $\alpha > 0$. Next, in Step 3 we establish that $\lim_{N\to\infty} \mathbf{Y}(N,\alpha) = \mathbf{R}(\alpha)$ for $\alpha > 0$, and finally in Step 4 we show that each element of $\lim_{N\to\infty} \mathbf{Y}(N,\alpha)$ is a Laplace transform for $\alpha \in \mathbb{C}_+$, and so from Step 3 it must be the case that this limit is $\mathbf{R}(\alpha)$.

Step 1: We first derive (2.18) by making use of Lemma 2.2 of [13]: for $n \geq 0$, $0 \leq j \leq M$, and a fixed state $(0,l_0)$, $0 \le l_0 \le M$, we have

$$\pi_{(0,l_0),(n+1,j)}(\alpha) = \pi_{(0,l_0),(0,l_0)}(\alpha) w_{(0,l_0),(n+1,j)}(\alpha), \qquad \alpha \in \mathbb{C}_+, \tag{2.22}$$

where for $x, y \in E$, $w_{x,y}(\alpha)$ is defined on $\mathbb{C}_+$ as

$$w_{x,y}(\alpha) = \mathbb{E}_y \left[\mathbf{1}(\tilde{\eta}_x < \infty) e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right].$$

Working further with $w_{(0,l_0),(n+1,j)}(\alpha)$, we see that

$$\begin{aligned}
w_{(0,l_0),(n+1,j)}(\alpha) &= \mathbb{E}_{(n+1,j)} \left[\mathbf{1}(\tilde{\eta}_{(0,l_0)} < \infty) e^{-\alpha \tilde{\tau}_{(0,l_0)}} \prod_{\ell=1}^{\tilde{\eta}_{(0,l_0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
= \quad & \sum_{\nu=0}^{M} \mathbb{E}_{(n+1,j)} \Big[\mathbf{1}(\tilde{\eta}_{(0,l_0)} < \infty) e^{-\alpha(\tilde{\tau}_{(0,l_0)} - \tilde{\tau}_{C_n})} \mathbf{1}(\tilde{\eta}_{C_n} < \infty) e^{-\alpha \tilde{\tau}_{C_n}} \mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,\nu)) \\
& \times \prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \prod_{\ell=\tilde{\eta}_{C_n}+1}^{\tilde{\eta}_{(0,l_0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \Bigg] \\
= \quad & \sum_{\nu=0}^{M} w_{(0,l_0),(n,\nu)}(\alpha) \mathbb{E}_{(n+1,j)} \left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty) e^{-\alpha \tilde{\tau}_{C_n}} \mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,\nu)) \prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
= \quad & \sum_{\nu=0}^{M} w_{(0,l_0),(n,\nu)}(\alpha) (\mathbf{R}_{n,n+1}(\alpha))_{i,j}.
\end{aligned}$$

Multiplying both sides of this expression by $\pi_{(0,l_0),(0,l_0)}(\alpha)$ and applying (2.22) yields, for $\alpha \in \mathbb{C}_+$,

$$\pi_{(0,l_0),(n+1,j)}(\alpha) = \sum_{\nu=0}^{M} \pi_{(0,l_0),(n,\nu)}(\alpha) (\mathbf{R}_{n,n+1}(\alpha))_{i,j}. \tag{2.23}$$

Writing (2.23) in matrix form establishes (2.18).

Step 2: Next, we show that $\mathbf{R}(\alpha)$ is a solution to the matrix equation (2.21) for $\alpha \in \mathbb{C}_+$. For

$0 \le i, j \le M$, we find after conditioning on $\tilde{X}_1$ that

$$
\begin{aligned}
&(\mathbf{R}_{0,1}(\alpha))_{i,j} = \mathbb{E}_{(1,j)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
= \quad & \left(\frac{q((1,j))}{q((1,j))+\alpha)}\right)\left(\frac{q((0,i),(1,j)))}{\tilde{q}((1,j),(0,i))}\right)\left(\frac{\tilde{q}((1,j),(0,i))}{\tilde{q}((1,j))}\right) \\
+ \quad & \sum_{\nu\neq j}\left(\frac{q((1,j))}{q((1,j))+\alpha)}\right)\left(\frac{q((1,\nu),(1,j)))}{\tilde{q}((1,j),(1,\nu))}\right)\left(\frac{\tilde{q}((1,j),(1,\nu))}{\tilde{q}((1,j))}\right) \\
\times \quad & \mathbb{E}_{(1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
+ \quad & \sum_{n=1}^{\infty}\sum_{\nu=0}^{M}\left(\frac{q((1,j))}{q((1,j))+\alpha)}\right)\left(\frac{q((n+1,\nu),(1,j)))}{\tilde{q}((1,j),(n+1,\nu))}\right)\left(\frac{\tilde{q}((1,j),(n+1,\nu))}{\tilde{q}((1,j))}\right) \\
\times \quad & \mathbb{E}_{(n+1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
= \quad & \frac{q((0,i),(1,j))}{q((1,j))+\alpha} \\
+ \quad & \sum_{\nu\neq j}\left(\frac{q((1,\nu),(1,j))}{q((1,j))+\alpha}\right)\mathbb{E}_{(1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
+ \quad & \sum_{n=1}^{\infty}\sum_{\nu=0}^{M}\left(\frac{q((n+1,\nu),(1,j))}{q((1,j))+\alpha}\right)\mathbb{E}_{(n+1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right]
\end{aligned}
$$

which simplifies to

$$
(-(\mathbf{A}_0)_{j,j} + \alpha)(\mathbf{R}_{0,1}(\alpha))_{i,j} = (\mathbf{A}_1)_{i,j} + \sum_{\nu\neq j}(\mathbf{A}_0)_{\nu,j}(\mathbf{R}_{0,1}(\alpha))_{i,\nu} + \sum_{n=2}^{\infty}\sum_{\nu=0}^{M}(\mathbf{A}_{1-n})_{\nu,j}(\mathbf{R}_{0,n}(\alpha))_{i,\nu}
$$

or, in matrix notation,

$$
\begin{aligned}
\mathbf{0} &= \mathbf{A}_1 + \mathbf{R}_{0,1}(\alpha)(\mathbf{A}_0 - \alpha\mathbf{I}) + \sum_{n=2}^{\infty}\mathbf{R}_{0,n}(\alpha)\mathbf{A}_{1-n} \\
&= \mathbf{A}_1 + \mathbf{R}(\alpha)(\mathbf{A}_0 - \alpha\mathbf{I}) + \sum_{n=2}^{\infty}\mathbf{R}(\alpha)^n\mathbf{A}_{1-n} \qquad (2.24)
\end{aligned}
$$

thus establishing $\mathbf{R}(\alpha)$ as a solution to Equation (2.21) when $\alpha \in \mathbb{C}_+$.

Step 3: Now we work toward showing $\mathbf{R}(\alpha) = \lim_{N\to\infty}\mathbf{Y}(N,\alpha)$ for $\alpha > 0$. To do this, it suffices to show that $\mathbf{Y}(N,\alpha)$ is pointwise nondecreasing in N, that $\lim_{N\to\infty}\mathbf{Y}(N,\alpha) \le \mathbf{R}(\alpha)$, and finally that $\lim_{N\to\infty} \mathbf{Y}(N,\alpha) \geq \mathbf{R}(\alpha)$.

Observe first that $\mathbf{Y}(0,\alpha) = \mathbf{0} \leq \mathbf{R}(\alpha)$. Next,

$$\mathbf{Y}(1,\alpha) = (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(0,\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} = \mathbf{A}_1(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \geq \mathbf{0} = \mathbf{Y}(0,\alpha)$$

and also

$$\mathbf{Y}(1,\alpha) = (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(0,\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \leq (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{R}(\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \leq \mathbf{R}(\alpha).$$

Proceeding by induction, assume that $\mathbf{Y}(0,\alpha) \leq \mathbf{Y}(1,\alpha) \leq \cdots \leq \mathbf{Y}(N,\alpha) \leq \mathbf{R}(\alpha)$. Then we have by nonnegativity of $\mathbf{Y}(N,\alpha)$, $\mathbf{A}_1$, $(\alpha\mathbf{I} - \mathbf{A}_0)^{-1}$, and $\mathbf{A}_k$ for $k \leq -1$ that

$$\begin{aligned} \mathbf{Y}(N+1,\alpha) &= (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(N,\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \\ &\geq (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(N-1,\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} = \mathbf{Y}(N,\alpha) \end{aligned}$$

which shows that sequence of matrices $\{\mathbf{Y}(N,\alpha)\}$ is pointwise nondecreasing. In addition,

$$\begin{aligned} \mathbf{Y}(N+1,\alpha) &= (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{Y}(N,\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \\ &\leq (\mathbf{A}_1 + \sum_{k=2}^{\infty} \mathbf{R}(\alpha)^k \mathbf{A}_{1-k})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} = \mathbf{R}(\alpha) \end{aligned}$$

and so $\lim_{N\to\infty} \mathbf{Y}(N,\alpha) \leq \mathbf{R}(\alpha)$.

The next step is to show that $\lim_{N\to\infty} \mathbf{Y}(N,\alpha) \geq \mathbf{R}(\alpha)$. For each $n \geq 0$, $k \geq 1$, $N \geq 0$, we define the matrix $\mathbf{R}_{n,n+k}(N,\alpha)$, whose (i,j)th element is given by

$$(\mathbf{R}_{n,n+k}(N,\alpha))_{i,j} = \mathbb{E}_{(n+k,j)}\left[\mathbf{1}(\tilde{\eta}_{C_n} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_n}} = (n,i))\mathbf{1}(\tilde{\gamma}_{C_n} \leq N)e^{-\alpha\tilde{\tau}_{C_n}} \prod_{\ell=1}^{\tilde{\eta}_{C_n}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right],$$

where under $\mathbb{P}_x$, we recall $\tilde{\gamma}_{C_n}$ represents the number of level transitions made by $\{\tilde{X}\}_{n\geq 0}$ as it moves from state $x \in C_n^c$ to the set C_n. An application of the monotone convergence theorem yields

$$\lim_{N\to\infty} \mathbf{R}_{0,1}(N,\alpha) = \mathbf{R}(\alpha)$$

and so to finish the proof, it suffices to show that for each integer $N \geq 1$, $\mathbf{R}_{0,1}(N,\alpha) \leq \mathbf{Y}(N,\alpha)$. Starting with the (i,j)th element from the matrix $\mathbf{R}_{0,1}(N,\alpha)$, we observe through conditioning on $\tilde{X}_1$ that

$$(\mathbf{R}_{0,1}(N,\alpha))_{i,j} = \mathbb{E}_{(1,j)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N)e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]$$

$$= \frac{q((0,i),(1,j))}{q((1,j))+\alpha}$$

$$+ \sum_{\nu\neq j}\left(\frac{q((1,\nu),(1,j))}{q((1,j))+\alpha}\right)\mathbb{E}_{(1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N)e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]$$

$$+ \sum_{n=1}^{\infty}\sum_{\nu=0}^{M}\left(\frac{q((n+1,\nu),(1,j))}{q((1,j))+\alpha}\right)\mathbb{E}_{(n+1,\nu)}\Big[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(X_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1)e^{-\alpha\tilde{\tau}_{C_0}}$$

$$\times\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\Bigg]$$

which, in matrix notation, is equivalent to

$$\mathbf{0} = \mathbf{A}_1 + \mathbf{R}_{0,1}(N,\alpha)(\mathbf{A}_0 - \alpha\mathbf{I}) + \sum_{n=2}^{\infty}\mathbf{R}_{0,n}(N-1,\alpha)\mathbf{A}_{1-n}$$

or, equivalently,

$$\mathbf{R}_{0,1}(N,\alpha) = \left(\mathbf{A}_1 + \sum_{n=2}^{\infty}\mathbf{R}_{0,n}(N-1,\alpha)\mathbf{A}_{1-n}\right)(\alpha\mathbf{I} - \mathbf{A}_0)^{-1}. \tag{2.25}$$

Next, note that for $i, j \in \{0, 1, \ldots, M\}$, we have for $n \geq 2$

$$\begin{aligned}
& (\mathbf{R}_{0,n}(N-1,\alpha))_{i,j} \\
= \; & \mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1)e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
= \; & \sum_{k=1}^{N-1}\sum_{\nu=0}^{M}\mathbb{E}_{(n,j)}\Bigg[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1) \\
& \times e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\Bigg] \\
= \; & \sum_{k=1}^{N-1}\sum_{\nu=0}^{M}\mathbb{E}_{(n-1,\nu)}\left[\mathbf{1}(\tilde{\eta}_{C_0} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_0}} = (0,i))\mathbf{1}(\tilde{\gamma}_{C_0} \leq N-1-k)e^{-\alpha\tilde{\tau}_{C_0}}\prod_{\ell=1}^{\tilde{\eta}_{C_0}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
\times \; & \mathbb{E}_{(n,j)}\left[\mathbf{1}(\tilde{\eta}_{C_{n-1}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k)e^{-\alpha\tilde{\tau}_{C_{n-1}}}\prod_{\ell=1}^{\tilde{\eta}_{C_{n-1}}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\
\leq \; & \sum_{\nu=0}^{M}\sum_{k=1}^{N-1}(\mathbf{R}_{0,n-1}(N-1,\alpha))_{i,\nu}\mathbb{E}_{(n,j)}\Bigg[\mathbf{1}(\tilde{\eta}_{C_{n-1}} < \infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_{C_{n-1}}} = (n-1,\nu))\mathbf{1}(\tilde{\gamma}_{C_{n-1}} = k)e^{-\alpha\tilde{\tau}_{C_{n-1}}} \\
& \times \prod_{\ell=1}^{\tilde{\eta}_{C_{n-1}}}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\Bigg] \\
= \; & \sum_{\nu=0}^{M}(\mathbf{R}_{0,n-1}(N-1,\alpha))_{i,\nu}(\mathbf{R}_{n-1,n}(N-1,\alpha))_{\nu,j}
\end{aligned}$$

implying

$$\mathbf{R}_{0,n}(N-1,\alpha) \leq \mathbf{R}_{0,n-1}(N-1,\alpha)\mathbf{R}_{n-1,n}(N-1,\alpha)$$

and further iterating gives

$$\mathbf{R}_{0,n}(N-1,\alpha) \leq \prod_{k=1}^{n}\mathbf{R}_{k-1,k}(N-1,\alpha) = \mathbf{R}_{0,1}(N-1,\alpha)^n. \tag{2.26}$$

Equation (2.26) can be used to show $\mathbf{R}_{0,1}(N,\alpha) \leq \mathbf{Y}(N,\alpha)$ for each $N \geq 1$. Observe that $\mathbf{R}_{0,1}(0,\alpha) = \mathbf{Y}(0,\alpha)$,

$$\mathbf{R}_{0,1}(1,\alpha) = \mathbf{A}_1(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} = \mathbf{Y}(1,\alpha)$$

and for each $N \geq 2$, we note that if $\mathbf{R}_{0,1}(N-1,\alpha) \leq \mathbf{Y}(N-1,\alpha)$, then by Equation (2.26) we have

$$\begin{aligned}
\mathbf{R}_{0,1}(N,\alpha) &= (\mathbf{A}_1 + \sum_{n=2}^{\infty} \mathbf{R}_{0,n}(N-1,\alpha)\mathbf{A}_{1-n})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \\
&\leq (\mathbf{A}_1 + \sum_{n=2}^{\infty} \mathbf{R}_{0,1}(N-1,\alpha)^n\mathbf{A}_{1-n})(\alpha\mathbf{I} - \mathbf{A}_0)^{-1} \\
&\leq (\mathbf{A}_1 + \sum_{n=2}^{\infty} \mathbf{Y}(N-1,\alpha)^n\mathbf{A}_{1-n})(\alpha\mathbf{I} - \mathbf{A}_1)^{-1} = \mathbf{Y}(N,\alpha)
\end{aligned}$$

which, inductively, proves $\mathbf{R}_{0,1}(N,\alpha) \leq \mathbf{Y}(N,\alpha)$ for each $N \geq 0$, $\alpha > 0$, and so $\lim_{N\to\infty}\mathbf{Y}(N,\alpha) \geq \mathbf{R}(\alpha)$.

Step 4: It remains to show $\lim_{n\to\infty}\mathbf{Y}(N,\alpha) = \mathbf{R}(\alpha)$ for $\alpha \in \mathbb{C}_+$. Since this has already been established for $\alpha > 0$ in Step 3, and each element of $\mathbf{R}(\alpha)$ is a well-defined Laplace transform on $\mathbb{C}_+$, it suffices to show that each element of $\lim_{N\to\infty}\mathbf{Y}(N,\alpha)$ exists on $\mathbb{C}_+$, and is a Laplace transform of a nonnegative function.

For each $k \leq 1$, we define $a_{k,(i,j)} : [0,\infty) \to \mathbb{R}$ to be the function whose Laplace transform, defined on $\mathbb{C}_+$, is given by the (i,j)th element of the matrix $\mathbf{A}_h(\alpha\mathbf{I} - \mathbf{A}_0)^{-1}$. Next, we define, for integers $N \geq 1$, $k \geq 1$, $0 \leq i,j \leq M$, $y_{i,j}^{(N,k)} : [0,\infty) \to \mathbb{R}$ as the function whose Laplace transform is given by the (i,j)th element of $\mathbf{Y}(N,\alpha)^k$: when $k=1$, we write $y_{i,j}^{(N,1)}$ as $y_{i,j}^{(N)}$.

The key to showing the (i,j)th element of $\lim_{N\to\infty}\mathbf{Y}(N,\alpha)$ is a Laplace transform of a nonnegative function is to instead show that $y_{i,j}^{(N)}$ is pointwise nondecreasing in N on $[0,\infty)$, which we establish via induction. Clearly $y_{i,j}^{(1)}$ is nonnegative on $[0,\infty)$ since its Laplace transform is given by the (i,j)th element of $\mathbf{A}_1(\alpha\mathbf{I} - \mathbf{A}_0)^{-1}$, and also $y_{i,j}^{(1)} \geq 0 = y_{i,j}^{(0)}$.

From here we proceed by induction: suppose $y_{i,j}^{(N)} \geq y_{i,j}^{(N-1)}$ for some $N \geq 1$. We first make use of this hypothesis to show that $y_{i,j}^{(N,k)} \geq y_{i,j}^{(N-1,k)}$ for each $k \geq 2$, using another induction argument. For $k=2$, we have for each $t \geq 0$

$$\begin{aligned}
y_{i,j}^{(N,2)}(t) &= \sum_{\nu=0}^{M}\int_0^t y_{i,\nu}^{(N)}(s)y_{\nu,j}^{(N)}(t-s)ds \\
&\geq \sum_{\nu=0}^{M}\int_0^t y_{i,\nu}^{(N-1)}(s)y_{\nu,j}^{(N-1)}(t-s)ds \\
&= y_{i,j}^{(N-1,2)}(t).
\end{aligned}$$

Next, assume $y_{i,j}^{(N,k)} \geq y_{i,j}^{(N-1,k)}$ on $[0,\infty)$ for some $k \geq 2$. Then for each $t \geq 0$,

$$\begin{aligned} y_{i,j}^{(N,k+1)}(t) &= \sum_{\nu=0}^{M} \int_0^t y_{i,\nu}^{(N,k)}(s) y_{\nu,j}^{(N)}(t-s) ds \\ &\geq \sum_{\nu=0}^{M} \int_0^t y_{i,\nu}^{(N-1,k)}(s) y_{\nu,j}^{(N-1)}(t-s) ds \\ &= y_{i,j}^{(N-1,k+1)}(t) \end{aligned}$$

Now we are ready to use the induction hypothesis $y_{i,j}^{(N)} \geq y_{i,j}^{(N-1)}$ for some $N \geq 1$ to show $y_{i,j}^{(N+1)} \geq y_{i,j}^{(N)}$. Since $y_{i,j}^{(N+1)}$ has as its Laplace transform the (i,j)th element of $\mathbf{Y}(N+1)$, we find from inverting (2.20) that

$$\begin{aligned} y_{i,j}^{(N+1)}(t) &= a_{1,(i,j)}(t) + \sum_{k=2}^{\infty} \int_0^t y_{i,\nu}^{(N,k)}(s) a_{1-k,(\nu,j)}(t-s) ds \\ &\geq a_{1,(i,j)}(t) + \sum_{k=2}^{\infty} \int_0^t y_{i,\nu}^{(N-1,k)}(s) a_{1-k,(\nu,j)}(t-s) ds \\ &= y_{i,j}^{(N)}(t) \end{aligned}$$

Thus, by Theorem A.1 from the Appendix, the (i,j)th element of $\lim_{N\to\infty} \mathbf{Y}(N,\alpha)$ is a Laplace transform, defined on $\mathbb{C}_+$, of a nonnegative function, namely the pointwise limit $\lim_{N\to\infty} y_{i,j}^{(N)}$. This proves $\lim_{N\to\infty} \mathbf{Y}(N,\alpha) = \mathbf{R}(\alpha)$ for $\alpha \in \mathbb{C}_+$. $\diamondsuit$

Next we show how to compute the row vector $\boldsymbol{\pi}_{(0,l_0),0}(\alpha)$ using the Kolmogorov forward equations associated with X, which are as follows: for $x, y \in E$, where possibly $x = y$, we have

$$p'_{x,y}(t) = \sum_{h \in E} p_{x,h}(t) q(h,y). \tag{2.27}$$

Set $x = (0,l_0), y = (0,k)$ for $0 \leq k \leq M$: integrating both sides of (2.27) with respect to $e^{-\alpha t}dt$, $\alpha \in \mathbb{C}_+$, yields

$$(\alpha + q((0,k)))\pi_{(0,l_0),(0,k)}(\alpha) = \mathbf{1}(l_0 = k) + \sum_{\substack{h \in E \\ h \neq (0,k)}} \pi_{(0,l_0),h}(\alpha) q(h,(0,k)), \tag{2.28}$$

which in turn can be written as

$$
\begin{aligned}
(\alpha - (\mathbf{B}_0)_{k,k})\pi_{(0,l_0),(0,k)}(\alpha) = \mathbf{1}(l_0 = k) \quad &+ \quad \sum_{\substack{\nu=0 \\ \nu\neq k}}^{M} (\mathbf{B}_0)_{\nu,k}\pi_{(0,l_0),(0,\nu)}(\alpha) \\
&+ \quad \sum_{n=1}^{\infty}\sum_{\nu=0}^{M} (\mathbf{B}_{-n})_{\nu,k}\pi_{(0,\ell),(n,\nu)}(\alpha) \qquad (2.29)
\end{aligned}
$$

or, equivalently,

$$
\mathbf{0} = \mathbf{e}_{l_0} + \boldsymbol{\pi}_{(0,l_0),0}(\alpha)\left[\mathbf{B}_0 - \alpha\mathbf{I} + \sum_{n=1}^{\infty} \mathbf{R}^n(\alpha)\mathbf{B}_{-n}\right],
$$

where $\mathbf{e}_{l_0} \in \mathbb{R}^M$ is a unit row vector, with its l_0th component equal to one and all other components equal to zero. Thus, if the matrix $\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty} \mathbf{R}^n(\alpha)\mathbf{B}_n$ is invertible for $\alpha \in \mathbb{C}_+$, then

$$
\boldsymbol{\pi}_{(0,l_0),0}(\alpha) = \mathbf{e}_\ell \left[\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty} \mathbf{R}^n(\alpha)\mathbf{B}_n\right]^{-1}. \qquad (2.30)
$$

To see why the inverse in (2.30) exists, recall that stating the Kolmogorov forward equations associated with X using Laplace transforms yields, for $\alpha \in \mathbb{C}_+$,

$$
\mathbf{\Pi}(\alpha)(\alpha\mathbf{I} - \mathbf{Q}) = \mathbf{I}. \qquad (2.31)
$$

After taking the dot product of the zeroth row of $\mathbf{\Pi}(\alpha)$ with the zeroth column of $(\alpha\mathbf{I} - \mathbf{Q})$, we get

$$
\begin{aligned}
\mathbf{I} &= \mathbf{\Pi}_{0,0}(\alpha)(\alpha\mathbf{I} - \mathbf{B}_0) - \sum_{n=1}^{\infty} \mathbf{\Pi}_{0,n}(\alpha)\mathbf{B}_{-n} \\
&= \mathbf{\Pi}_{0,0}(\alpha)\left[\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty} \mathbf{R}^n(\alpha)\mathbf{B}_{-n}\right]
\end{aligned}
$$

which proves $\left[\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty} \mathbf{R}^n(\alpha)\mathbf{B}_n\right]$ is invertible when $M < \infty$.

Extra care must be taken to show invertibility if $M = \infty$. Starting instead with the Kolmogorov backward equations associated with X, we find that for $\alpha \in \mathbb{C}_+$,

$$
(\alpha\mathbf{I} - \mathbf{Q})\mathbf{\Pi}(\alpha) = \mathbf{I}. \qquad (2.32)
$$

Writing down the system of equations given by the dot product of the zeroth column of $\mathbf{\Pi}(\alpha)$ with the ith row of $(\alpha\mathbf{I} - \mathbf{Q})$ for each $i \geq 0$, we get

$$\begin{aligned}
\mathbf{I} &= (\alpha\mathbf{I} - \mathbf{B}_0)\mathbf{\Pi}_{0,0}(\alpha) - \mathbf{A}_1\mathbf{\Pi}_{1,0}(\alpha) \\
\mathbf{0} &= -\mathbf{B}_{-1}\mathbf{\Pi}_{0,0}(\alpha) + (\alpha\mathbf{I} - \mathbf{A}_0)\mathbf{\Pi}_{1,0}(\alpha) - \mathbf{A}_1\mathbf{\Pi}_{2,0}(\alpha) \\
\mathbf{0} &= -\mathbf{B}_{-2}\mathbf{\Pi}_{0,0}(\alpha) - \mathbf{A}_{-1}\mathbf{\Pi}_{1,0}(\alpha) + (\alpha\mathbf{I} - \mathbf{A}_0)\mathbf{\Pi}_{2,0}(\alpha) - \mathbf{A}_1\mathbf{\Pi}_{3,0}(\alpha) \\
&\vdots
\end{aligned}$$

Multiplying both sides of each of these equations by an appropriate power of $\mathbf{R}(\alpha)$ further yields

$$\begin{aligned}
\mathbf{I} &= (\alpha\mathbf{I} - \mathbf{B}_0)\mathbf{\Pi}_{0,0}(\alpha) - \mathbf{A}_1\mathbf{\Pi}_{1,0}(\alpha) \\
\mathbf{0} &= -\mathbf{R}(\alpha)\mathbf{B}_{-1}\mathbf{\Pi}_{0,0}(\alpha) + \mathbf{R}(\alpha)(\alpha\mathbf{I} - \mathbf{A}_0)\mathbf{\Pi}_{1,0}(\alpha) - \mathbf{R}(\alpha)\mathbf{A}_1\mathbf{\Pi}_{2,0}(\alpha) \\
\mathbf{0} &= -\mathbf{R}(\alpha)^2\mathbf{B}_{-2}\mathbf{\Pi}_{0,0}(\alpha) - \mathbf{R}(\alpha)^2\mathbf{A}_{-1}\mathbf{\Pi}_{1,0}(\alpha) + \mathbf{R}(\alpha)^2(\alpha\mathbf{I} - \mathbf{A}_0)\mathbf{\Pi}_{2,0}(\alpha) - \mathbf{R}(\alpha)^2\mathbf{A}_1\mathbf{\Pi}_{3,0}(\alpha) \\
&\vdots
\end{aligned}$$

and upon summing together all of these equations, applying (2.21), and simplifying, we get

$$\mathbf{I} = \left[\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty}\mathbf{R}(\alpha)^n\mathbf{B}_{-n}\right]\mathbf{\Pi}_{0,0}(\alpha).$$

Hence, $\Pi_{0,0}(\alpha)$ is both a left and right inverse of $\left[\alpha\mathbf{I} - \mathbf{B}_0 - \sum_{n=1}^{\infty}\mathbf{R}(\alpha)^n\mathbf{B}_n\right]$, even when $M = \infty$.

Chapter 3

On the Behavior of Block-Structured Markov Processes

3.1 Introduction

In this chapter, we study both the steady-state and time-dependent behavior of block-structured Markov processes. To the best of our knowledge, the first attempt at studying block-structured Markov processes was made in Grassmann and Heyman [15], where the stationary distribution of block-structured Markov processes with repeating rows is studied by making use of a technique from Markov process theory known as censoring. There they show that block-structured Markov processes have infinitely many **R** matrices and infinitely many **G** matrices, and they also illustrate how their theory can be used to rederive the stationary distributions of Markov processes of $M/G/1$-type and $G/M/1$-type, thus providing some unification to the theory. Readers should also see Zhao [39] for a further discussion of block-structured Markov processes, as well as Grassmann and Heyman [16] for a practical algorithm designed to compute the stationary distribution of a block-structured Markov process when the matrices are banded.

The methodology we develop to study both the steady-state and time-dependent behavior of block-structured Markov processes does not rely on classical censoring techniques and yields results that are more analogous to corresponding results for Markov processes of $M/G/1$-type, as well as Markov processes of $G/M/1$-type. More specifically, to derive the Laplace transforms of the transition functions associated with a block-structured Markov process, we will first use a known result from the theory of Markov processes of $G/M/1$-type to rederive an important recursion satisfied by these Laplace transforms that features an infinite collection of $\mathbf{R}$ matrices: these are very similar to the ones discussed in [15] and the work of Zhao, Li, and Braun [39, 40, 41], and they are continuous-time analogues of those found in the works of Li and Zhao [27, 28], and the book of Li [26], where β-invariant measures of transient, discrete-time versions of these processes are studied in detail. Once we have this recursion, we then rewrite the $\mathbf{R}$ matrices appearing in this recursion in terms of an infinite collection of $\mathbf{G}$ matrices: interestingly, we do not seem to use the same collection of $\mathbf{G}$ matrices used in [15, 16, 26, 27, 28, 39, 40, 41]. Next, we show that the $\mathbf{G}$ matrices we use are also solutions to a fixed-point equation, and we provide an iterative procedure for computing these matrices. An analogous procedure can be used to compute the Laplace transforms of the transition functions of a block-structured Markov process as well.

In short, we feel our contribution (i) sheds more light on the connection between Markov processes of $M/G/1$-type and those of $G/M/1$-type, and (ii) our iterative scheme for computing the $\mathbf{G}$ matrices could perhaps be improved upon considerably at some point, given the amount of past work devoted towards computing the $\mathbf{R}$ and $\mathbf{G}$ matrices of a quasi-birth-death process in the most efficient manner possible. We focus throughout only on deriving Laplace transforms of transition functions on the set $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$, the set of complex numbers having positive real part, but readers will see that very analogous results can be derived for stationary distributions, or even when $Re(\alpha) \leq 0$—which is analogous to what is studied in [26, 27, 28] in a discrete-time setting—if one assumes various expected values encountered in the analysis are finite.

It also worth mentioning here that while we do not use the technique of censoring, the algebra used to simplify some of the expectations we will encounter is very much reminiscent of techniques used to study quantities associated with censored Markov processs. The papers [15, 16, 26, 27, 28, 39, 40, 41] cited all frequently use some form of the censoring technique, but our avoidance of censoring allows us to discover new equations that block-structured Markov processes satisfy.

3.2 A Key Result

While the main results of this chapter regard block-structured Markov processs, our first result is most easily stated in terms of an arbitrary CTMP X with countable state-space E and transition rate matrix (generator) $\mathbf{Q} := [q(x,y)]_{x,y\in E}$. For simplicity, we assume X is regular, meaning $\lim_{n\to\infty} T_n = \infty$. Furthermore, recall that associated with X are its collection of transition functions $\{p_{x,y}\}_{x,y\in E}$, where for each $x, y \in E$, $p_{x,y} : [0,\infty) \to [0,1]$ is defined as $p_{x,y}(t) := \mathbb{P}(X(t) = y \mid X(0) = x)$ for each $t \geq 0$. The Laplace transform $\pi_{x,y}$ of $p_{x,y}$ is defined on the subset $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$ of complex numbers $\mathbb{C}$ as

$$\pi_{x,y}(\alpha) = \int_0^\infty e^{-\alpha t} p_{x,y}(t)dt, \qquad \alpha \in \mathbb{C}_+.$$

Theorem 3.2.1 *Suppose T and D are two disjoint subsets of E. Then for each $x \in T$, $y \in D$,*

$$\pi_{x,y}(\alpha) = \sum_{z\in T} \pi_{x,z}(\alpha)(q(z)+\alpha)\mathbb{E}_z\left[\int_0^{\tau_T} e^{-\alpha t}\mathbf{1}(X(t)=y)dt\right], \qquad \alpha \in \mathbb{C}_+. \tag{3.1}$$

Theorem 3.2.1 is not new. Equation (3.1) is a Laplace transform version of the equation found at the top of page 124 of Latouche and Ramaswami [25], which was derived using Markov renewal theory. In Equation (3.1), the symbol $\mathbb{E}_z$ represents a conditional expectation, where we condition on $X(0) = z$.

Another way to derive (3.1) involves using the random-product representations featured in [9, 13, 21], as well as Chapter 2 of this dissertation. Recall this approach has us select another CTMP $\tilde{X} := \{\tilde{X}(t); t \geq 0\}$ on E having a rate matrix $\tilde{\mathbf{Q}} := [\tilde{q}(x,y)]_{x,y\in E}$ that satisfies two properties: (i) $\tilde{q}(x,x) = q(x,x)$ for each $x \in E$; (ii) for any two states $x, y \in E$, $\tilde{q}(x,y) > 0$ if and only if $q(y,x) > 0$. Once $\tilde{X}$ has been specified, we let $\{\tilde{X}_n\}_{n\geq 0}$ represent the DTMC formed by the transition epochs of $\tilde{X}$, and we further define, for each subset $T \subset E$, the random variables

$$\tilde{\tau}_T := \inf\{t \geq 0 : \tilde{X}(t) \in T\}, \qquad \tilde{\eta}_T := \inf\{n \geq 0 : \tilde{X}_n \in T\}$$

where $\tilde{\tau}_T$ represents the first time the CTMP $\tilde{X}$ visits the set T (so $\tilde{\tau}_T = 0$ when $\tilde{X}(0) \in T$) and $\tilde{\eta}_T$ the first time the embedded DTMC $\{\tilde{X}_n\}_{n\geq 0}$ makes a transition into T. Given $\tilde{X}$, one can show that for each $x, y \in E$ (see Theorem 2.1 of [13])

$$\pi_{x,y}(\alpha) = \pi_{x,x}(\alpha) w_{x,y}(\alpha), \qquad \alpha \in \mathbb{C}_+ \tag{3.2}$$

where $w_{x,y}(\alpha)$ is defined on $\mathbb{C}_+$ as

$$w_{x,y}(\alpha) = \mathbb{E}_y \left[e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \tag{3.3}$$

meaning $w_{x,y}(\alpha) = 1$ when $x = y$. Equation (3.1) can be derived by starting with $w_{x,y}(\alpha)$, summing over all possible values of $\tilde{X}(\tilde{\tau}_T)$ and applying the strong Markov property at the stopping time $\tilde{\tau}_T$, while noting that for $z \in T, y \in D$,

$$\mathbb{E}_y \left[\mathbf{1}(\tilde{X}(\tilde{\tau}_T) = z) e^{-\alpha \tilde{\tau}_T} \prod_{\ell=1}^{\tilde{\eta}_T} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] = (q(z) + \alpha) \mathbb{E}_z \left[\int_0^{\tau_T} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right]. \tag{3.4}$$

Readers interested in seeing the full details of this procedure should consult Sections 3 and 4 of [21], or Chapter 2 of this work.

3.3 Main Results

We assume throughout the rest of this chapter that $X := \{X(t); t \geq 0\}$ is a block-structured Markov process, meaning its state space is given by $E = \bigcup_{n \geq 0} L_n$ from (1.4) and its transition rate matrix $\mathbf{Q}$ is defined as in (1.5). Furthermore, we assume $X(0) \in L_0$ with probability one.

3.3.1 The R matrices

Our first result shows the row vectors $\boldsymbol{\pi}_{(0,i_0),n}(\alpha)$, $n \geq 0$, satisfy the following recursion.

Theorem 3.3.1 *For each integer $n \geq 0$, $\alpha \in \mathbb{C}$,*

$$\boldsymbol{\pi}_{(0,i_0),n+1}(\alpha) = \sum_{k=0}^{n} \boldsymbol{\pi}_{(0,i_0),k}(\alpha) \mathbf{R}_{k,n+1}(\alpha) \tag{3.5}$$

where for $0 \le k \le n$, $\mathbf{R}_{k,n+1}(\alpha)$ *is an* $(M+1) \times (M+1)$ *matrix whose* (i,j)*th element is given by*

$$(\mathbf{R}_{k,n+1}(\alpha))_{i,j} = (q((k,i)) + \alpha)\mathbb{E}_{(k,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t) = (n+1,j))dt\right]. \tag{3.6}$$

Proof For each integer $n \ge 0$, choose $T = C_n := \cup_{k=0}^{n} L_k$, and $D = L_{n+1}$. Then by Theorem 3.2.1,

$$\pi_{(0,i_0),(n+1,j)}(\alpha) = \sum_{k=0}^{n}\sum_{\ell=0}^{M}\pi_{(0,i_0),(k,\ell)}(\alpha)(q((k,\ell)) + \alpha)\mathbb{E}_{(k,\ell)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t) = (n+1,j))dt\right]$$

for each phase $j \ge 0$, proving (3.5). $\diamondsuit$

The stationary version of Theorem 3.3.1 appears in [15], and this result also appears in Li and Zhao [27] for Markov processs of $M/G/1$ type, as well as in Li [26] for block-structured Markov processs: in all of these references, the result is derived by making use of censoring, but here we show it also follows from ideas from the theory of Markov processes of $G/M/1$-type. Ramaswami also notices the stationary version of (3.3.1) in [33], at least for the case where $M = 0$, and when X is a Markov process of $M/G/1$-type.

More can be said about each $\mathbf{R}_{k,n+1}(\alpha)$ matrix, as shown in the following result.

Theorem 3.3.2 *For each* $n \ge 0$*, we have*

$$\mathbf{R}_{k,n+1}(\alpha) = \begin{cases} \displaystyle\sum_{m=n+1-k}^{\infty} \mathbf{B}_m\mathbf{G}_{k+m,n+1}(\alpha)\mathbf{N}_{n+1}(\alpha), & k = 0 \\ \displaystyle\sum_{m=n+1-k}^{\infty} \mathbf{A}_m\mathbf{G}_{k+m,n+1}(\alpha)\mathbf{N}_{n+1}(\alpha), & k \ge 1 \end{cases} \tag{3.7}$$

where for integers $n_1 \ge 1$ *and* $n_2 \ge 0$ *with* $n_1 \ge n_2$*, the* (i,j)*th element of* $\mathbf{G}_{n_1,n_2}(\alpha)$ *is given by*

$$(\mathbf{G}_{n_1,n_2}(\alpha))_{i,j} = \begin{cases} \mathbb{E}_{(n_1,i)}\left[e^{-\alpha\tau_{C_{n_2}}}\mathbf{1}(X(\tau_{C_{n_2}}) = (n_2,j))\right], & n_1 > n_2 \\ \mathbf{1}(i = j), & n_1 = n_2 \end{cases} \tag{3.8}$$

and the (i,j)*th element of the matrix* $\mathbf{N}_{n+1}(\alpha)$*,* $n \ge 0$*, is given by*

$$(\mathbf{N}_{n+1}(\alpha))_{i,j} = \mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t) = (n+1,j))dt\right]. \tag{3.9}$$

Proof It suffices to establish (3.7) for each matrix $\mathbf{R}_{k,n+1}(\alpha)$ when $1 \le k \le n$, $n \ge 1$, as an analogous argument can be used to derive $\mathbf{R}_{0,n+1}(\alpha)$ for each integer $n \ge 0$.

Fix two levels k, n satisfying $1 \leq k \leq n$, and fix two phases $i, j \geq 0$. Applying first-step analysis to the expected value in (3.6) yields

$$\begin{aligned}
& \mathbb{E}_{(k,i)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \right] \\
= & \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} \mathbb{E}_{(k,i)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \mid X(T_1) = (m+k, x) \right] \\
& \qquad \times \ \mathbb{P}_{(k,i)}(X(T_1) = (m+k, x)) \\
= & \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} \mathbb{E}_{(k,i)} \left[\int_{T_1}^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \mid X(T_1) = (m+k, x) \right] \\
& \qquad \times \ \mathbb{P}_{(k,i)}(X(T_1) = (m+k, x)) \\
= & \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} \mathbb{E}_{(k,i)} \left[e^{-\alpha T_1} \int_{T_1}^{\tau_{C_n}} e^{-\alpha (t - T_1)} \mathbf{1}(X(t) = (n+1, j)) dt \mid X(T_1) = (m+k, x) \right] \\
& \qquad \times \ \mathbb{P}_{(k,i)}(X(T_1) = (m+k, x)) \\
= & \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} \mathbb{E}_{(k,i)}[e^{-\alpha T_1}] \mathbb{E}_{(m+k,x)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \right] \\
& \qquad \times \ \mathbb{P}_{(k,i)}(X(T_1) = (m+k, x)) \\
= & \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} \frac{(\mathbf{A}_m)_{i,x}}{q((k,i)) + \alpha} \mathbb{E}_{(m+k,x)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \right]
\end{aligned}$$

implying

$$(\mathbf{R}_{k,n+1}(\alpha))_{i,j} = \sum_{m=n+1-k}^{\infty} \sum_{x=0}^{M} (\mathbf{A}_m)_{i,x} \mathbb{E}_{(m+k,x)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \right].$$

The next step is to simplify, for each level $m > n + 1$, the expectation

$$\mathbb{E}_{(m,x)} \left[\int_0^{\tau_{C_n}} e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j)) dt \right]. \tag{3.10}$$

Observe that when $X(0) \in L_m$, the quantity $e^{-\alpha t} \mathbf{1}(X(t) = (n+1, j))$ is zero for each $t < \tau_{C_{n+1}}$. Summing over every way we can first reach C_{n+1}, performing a change-of-variable, and applying the strong Markov property at time $\tau_{C_{n+1}}$ yields

$$
\begin{aligned}
& \mathbb{E}_{(m,x)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= & \sum_{y=0}^{M}\mathbb{E}_{(m,x)}\left[\mathbf{1}(X(\tau_{C_{n+1}})=(n+1,y))\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= & \sum_{y=0}^{M}\mathbb{E}_{(m,x)}\left[\mathbf{1}(X(\tau_{C_{n+1}})=(n+1,y))e^{-\alpha\tau_{C_{n+1}}}\int_{\tau_{C_{n+1}}}^{\tau_{C_n}} e^{-\alpha(t-\tau_{C_{n+1}})}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= & \sum_{y=0}^{M}\mathbb{E}_{(m,x)}\left[\mathbf{1}(X(\tau_{C_{n+1}})=(n+1,y))e^{-\alpha\tau_{C_{n+1}}}\right]\mathbb{E}_{(n+1,y)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= & \sum_{y=0}^{M}(\mathbf{G}_{m,n+1}(\alpha))_{x,y}(\mathbf{N}_{n+1}(\alpha))_{y,j}.
\end{aligned}
$$

Thus, we conclude that the expectation in (3.10) is simply the (x,j)th element of the matrix $\mathbf{G}_{m,n+1}(\alpha)\mathbf{N}_{n+1}(\alpha)$. Hence,

$$
\mathbf{R}_{k,n+1}(\alpha)=\sum_{m=n+1-k}^{\infty}\mathbf{A}_m\mathbf{G}_{m+k,n+1}(\alpha)\mathbf{N}_{n+1}(\alpha) \tag{3.11}
$$

$\diamondsuit$

The next result shows that each matrix $\mathbf{N}_{n+1}(\alpha)$, $n\geq 0$, can be written in terms of $\mathbf{G}$ matrices.

Theorem 3.3.3 *For each $n\geq 0$ and $\alpha\in\mathbb{C}_+$,*

$$
\mathbf{N}_{n+1}(\alpha)=\left[\alpha\mathbf{I}-\mathbf{A}_0-\sum_{m=1}^{\infty}\mathbf{A}_m\mathbf{G}_{n+1+m,n+1}(\alpha)\right]^{-1}. \tag{3.12}
$$

Proof Starting with the (i,j) element of $\mathbf{N}_{n+1}(\alpha)$, we see that for $\alpha \in \mathbb{C}_+$,

$$\begin{aligned}
& \mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= \; & \mathbb{E}_{(n+1,i)}\left[\int_0^{T_1} e^{-\alpha t}dt\right]\mathbf{1}(i=j) + \mathbb{E}_{(n+1,i)}\left[\int_{T_1}^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= \; & \left(\frac{\mathbf{1}(i=j)}{q((n+1,i))+\alpha}\right) \\
+ \; & \sum_{x\neq i}\mathbb{E}_{(n+1,i)}\left[\int_{T_1}^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt \mid X(T_1)=(n+1,x)\right] \\
& \quad \times \; \mathbb{P}_{(n+1,i)}(X(T_1)=(n+1,x)) \\
+ \; & \sum_{m=1}^{\infty}\sum_{x=0}^{M}\mathbb{E}_{(n+1,i)}\left[\int_{T_1}^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt \mid X(T_1)=(m+n+1,x)\right] \\
& \qquad \times \; \mathbb{P}_{(n+1,i)}(X(T_1)=(m+n+1,x)) \\
= \; & \left(\frac{\mathbf{1}(i=j)}{-(\mathbf{A}_0)_{i,i}+\alpha}\right) + \sum_{x\neq i}\mathbb{E}_{(n+1,x)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right]\frac{(\mathbf{A}_0)_{i,x}}{-(\mathbf{A}_0)_{i,i}+\alpha} \\
+ \; & \sum_{m=1}^{\infty}\sum_{x=0}^{M}\mathbb{E}_{(m+n+1,x)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right]\frac{(\mathbf{A}_m)_{i,x}}{-(\mathbf{A}_0)_{i,i}+\alpha}.
\end{aligned}$$

This implies

$$\begin{aligned}
& (-(\mathbf{A}_0)_{i,i}+\alpha)\mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right] \\
= \; & \mathbf{1}(i=j) + \sum_{x\neq i}\mathbb{E}_{(n+1,x)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right](\mathbf{A}_0)_{i,x} \\
+ \; & \sum_{m=1}^{\infty}\sum_{x=0}^{M}\mathbb{E}_{(m+n+1,x)}\left[\int_0^{\tau_{C_n}} e^{-\alpha t}\mathbf{1}(X(t)=(n+1,j))dt\right](\mathbf{A}_m)_{i,x},
\end{aligned}$$

which in matrix form gives

$$\alpha\mathbf{N}_{n+1}(\alpha) = \mathbf{I} + \mathbf{A}_0\mathbf{N}_{n+1}(\alpha) + \sum_{m=1}^{\infty}\mathbf{A}_m\mathbf{G}_{n+1+m,n+1}(\alpha)\mathbf{N}_{n+1}(\alpha)$$

and solving for $\mathbf{N}_{n+1}(\alpha)$ yields (3.12). $\diamondsuit$

3.3.2 Computing the G matrices

At this point we see that each $\mathbf{R}$ matrix can be expressed in terms of $\mathbf{G}$ matrices, but to make this observation useful we need some way of computing the $\mathbf{G}$ matrices. The homogeneous nature of the transition rate matrix $\mathbf{Q}$ implies that for integers $n \geq 1$, $k \geq 1$,

$$\mathbf{G}_{k+n,n}(\alpha) = \mathbf{G}_{k+1,1}(\alpha), \qquad \alpha \in \mathbb{C}_+.$$

This simplification introduces the definition of the matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$, where for each integer $k \geq 1$, $\mathbf{G}_k(\alpha) := \mathbf{G}_{k+1,1}(\alpha)$.

Our next result provides an iterative procedure for computing the matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$, as well as the matrices $\{\mathbf{G}_{k,0}\}_{k\geq 1}$. This result is still of limited practical use since there are infinitely many $\mathbf{G}$ matrices to compute, but later we will briefly discuss practical methods for computing these matrices when additional assumptions are made about the transition structure of X.

Theorem 3.3.4 *For $\alpha > 0$, the matrices $\{\mathbf{G}_{k,0}(\alpha)\}_{k\geq 1}$ form the minimal nonnegative solution set to the following equations, defined for each $k \geq 1$ as*

$$\mathbf{Y}_{k,0}(\alpha) \quad = \quad (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_{-k} + \sum_{\substack{m=1 \\ m\neq k}}^{\infty} (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_{m,0}(\alpha). \tag{3.13}$$

Similarly, the matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$ form the minimal nonnegative set of solutions to the system of equations

$$\mathbf{Y}_k(\alpha) \quad = \quad (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{-k} + \sum_{\substack{m=1 \\ m\neq k}}^{\infty} (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_m(\alpha). \tag{3.14}$$

Furthermore, each matrix $\mathbf{G}_{k,0}(\alpha)$, $k \geq 1$, satisfies, for $\alpha \in \mathbb{C}_+$,

$$\mathbf{G}_{k,0}(\alpha) = \lim_{N\to\infty} \mathbf{Y}_{k,0}(N, \alpha), \tag{3.15}$$

where $\mathbf{Y}_{k,0}(0, \alpha) = \mathbf{0}$ and for each $N \geq 0$,

$$\mathbf{Y}_{k,0}(N+1, \alpha) \quad = \quad (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_{-k} + \sum_{\substack{m=1 \\ m\neq k}}^{\infty} (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_{m,0}(N, \alpha).$$

Similarly, the matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$ *each satisfy, for* $\alpha \in \mathbb{C}_+$

$$\mathbf{G}_k(\alpha) = \lim_{N\to\infty} \mathbf{Y}_k(N,\alpha), \tag{3.16}$$

where $\mathbf{Y}_k(0,\alpha) = \mathbf{0}$ *and for each* $N \geq 0$,

$$\mathbf{Y}_k(N+1,\alpha) \;=\; (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{-k} + \sum_{\substack{m=1\\ m\neq k}}^{\infty} (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_m(N,\alpha).$$

Proof We prove (3.13) and (3.15), which concern the $\{\mathbf{G}_{k,0}(\alpha)\}_{k\geq 1}$ matrices. Analogous arguments can be used to obtain (3.14) and (3.16).

First, we show $\mathbf{G}_{k,0}(\alpha)$ is a solution to (3.13) for each $k \geq 1$. After conditioning on X_1 and simplifying, we obtain, for each $0 \leq i, j \leq M$,

$$(\mathbf{G}_{k,0}(\alpha))_{i,j} = \mathbb{E}_{(k,i)}\left[e^{-\alpha\tau_{C_0}}\mathbf{1}(X(\tau_{C_0}) = (0,j))\right]$$

$$= \frac{q((k,i),(0,j))}{q((k,i))+\alpha} + \sum_{m=1}^{k-1}\sum_{y=0}^{M}\left(\frac{q((k,i),(m,y))}{q((k,i))+\alpha}\right)\mathbb{E}_{(m,y)}\left[e^{-\alpha\tau_{C_0}}\mathbf{1}(X(\tau_{C_0}) = (0,j))\right] \tag{3.17}$$

$$+ \sum_{\substack{y=0\\ y\neq i}}^{M}\left(\frac{q((k,i),(k,y))}{q((k,i))+\alpha}\right)\mathbb{E}_{(k,y)}\left[e^{-\alpha\tau_{C_0}}\mathbf{1}(X(\tau_{C_0}) = (0,j))\right] \tag{3.18}$$

$$+ \sum_{m=k+1}^{\infty}\sum_{y=0}^{M}\left(\frac{q((k,i),(m,y))}{q((k,i))+\alpha}\right)\mathbb{E}_{(m,y)}\left[e^{-\alpha\tau_{C_0}}\mathbf{1}(X(\tau_{C_0}) = (0,j))\right], \tag{3.19}$$

which, after writing in matrix form, establishes $\mathbf{G}_{k,0}(\alpha)$ as a solution to (3.13) for each $k \geq 1$.

Next, we prove (3.15) for $\alpha > 0$. To do so, we first show $\lim_{N\to\infty}\mathbf{Y}_{k,0}(N,\alpha) \leq \mathbf{G}_{k,0}(\alpha)$ by arguing (i) $\mathbf{Y}_{k,0}(N,\alpha)$ is (componentwise) monotone increasing in N and (ii) $\mathbf{Y}_{k,0}(N,\alpha) \leq \mathbf{G}_{k,0}(\alpha)$ for each $N \geq 0$ and $k \geq 1$. Clearly $\mathbf{Y}_{k,0}(N,\alpha)$ is monotone increasing in N:

$$\mathbf{Y}_{k,0}(0,\alpha) = \mathbf{0} \leq (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_{-k} = \mathbf{Y}_{k,0}(1,\alpha),$$

establishing the base case, and the induction step is shown by

$$\begin{aligned}\mathbf{Y}_{k,0}(N+1,\alpha) &= (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_{-k} + \sum_{\substack{m=1\\ m\neq k}}^{\infty} (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_{k,0}(N,\alpha)\\ &\geq (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_{-k} + \sum_{\substack{m=1\\ m\neq k}}^{\infty} (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{Y}_{k,0}(N-1,\alpha)\\ &= \mathbf{Y}_{k,0}(N,\alpha).\end{aligned}$$

A similar induction argument can be used to show $\mathbf{Y}_{k,0}(N,\alpha) \leq \mathbf{G}_{k,0}(\alpha)$ for each $N \geq 0$ and $k \geq 1$. These observations yield $\lim_{N\to\infty} \mathbf{Y}_{k,0}(N,\alpha) \leq \mathbf{G}_{k,0}(\alpha)$.

Next, we prove $\mathbf{G}_{k,0}(\alpha) \leq \lim_{N\to\infty} \mathbf{Y}_{k,0}(N,\alpha)$. Fix an arbitrary state $x \in E$ and under the measure $\mathbb{P}_x$, let γ_{C_n} represent the number of level transitions made by $\{X_n\}_{n\geq 0}$ as it travels from state x to the set C_n when $x \in C_n^c$. Then we can define, for each $k \geq 1$, $N \geq 0$ the matrix $\mathbf{G}_{k,0}(N,\alpha)$ whose (i,j)th element is given by

$$(\mathbf{G}_{k,0}(N,\alpha))_{i,j} = \mathbb{E}_{(k,i)}\left[e^{-\alpha\tau_{C_0}}\mathbf{1}(X(\tau_{C_0}) = (0,j))\mathbf{1}(\gamma_{C_0} \leq N)\right].$$

By the monotone convergence theorem, we have $\lim_{N\to\infty} \mathbf{G}_{k,0}(N,\alpha) = \mathbf{G}_{k,0}(\alpha)$: hence, to show $\mathbf{G}_{k,0}(\alpha) \leq \lim_{N\to\infty} \mathbf{Y}_{k,0}(N,\alpha)$, it suffices to prove $\mathbf{G}_{k,0}(N,\alpha) \leq \mathbf{Y}_{k,0}(N,\alpha)$ for each $N \geq 0$. Starting with the (i,j)th element of $\mathbf{G}_{k,0}(N,\alpha)$, we find that conditioning on X_1, and simplifying yields an equation that, in matrix form, is given by

$$\mathbf{G}_{k,0}(N+1,\alpha) = (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_{-k} + \sum_{\substack{m=1\\ m\neq k}}^{\infty} (\alpha\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{A}_{m-k}\mathbf{G}_{k,0}(N,\alpha).$$

From here, a simple induction argument can be used to show $\mathbf{G}_{k,0}(N,\alpha) \leq \mathbf{Y}_{k,0}(N,\alpha)$ for each $N \geq 0$. This proves (3.15) holds when $\alpha > 0$.

To extend this result to the case where $\alpha \in \mathbb{C}_+$, first notice that each element of $\mathbf{G}_{k,0}(\alpha)$ represents a Laplace transform of a nonnegative Lebesgue-integrable function (this can be seen by conditioning on τ_{C_0}). Once this has been observed, we can use an argument analogous to that given in Part 4 of the proof of Theorem 4.1 in [21] to show that (3.15) still holds for $\alpha \in \mathbb{C}_+$: we omit the details. $\diamondsuit$

It remains to address the problem of computing the matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$ and $\{\mathbf{G}_{k,0}(\alpha)\}_{k\geq 1}$. Suppose we assume the transition rate matrix $\mathbf{Q}$ of X is banded, meaning there exists a positive integer u and a negative integer l satisfying $\mathbf{A}_k = \mathbf{B}_k = 0$ for $k < l$, $k > u$. Under this assumption, one naive approach is to make use of both matrix generating functions, and numerical transform inversion.

Define, for $\alpha \in \mathbb{C}_+$ and $z \in \mathbb{C}$ satisfying $|z| < 1$, the matrix generating functions

$$\mathbf{G}(z,\alpha) := \sum_{k=1}^{\infty} z^k \mathbf{G}_k(\alpha), \qquad \mathbf{G}_0(z,\alpha) := \sum_{k=1}^{\infty} z^k \mathbf{G}_{k,0}(\alpha)$$

and for each integer $N \geq 0$, define

$$\mathbf{Y}(z,N,\alpha) := \sum_{k=1}^{\infty} z^k \mathbf{Y}_k(N,\alpha), \qquad \mathbf{Y}_0(z,N,\alpha) := \sum_{k=1}^{\infty} z^k \mathbf{Y}_{k,0}(N,\alpha).$$

Clearly

$$\lim_{N\to\infty} \mathbf{Y}(z,N,\alpha) = \mathbf{G}(z,\alpha), \qquad \lim_{N\to\infty} \mathbf{Y}_0(z,N,\alpha) = \mathbf{G}_0(z,\alpha)$$

and so the goal is to write down one recursion satisfied by the $\mathbf{Y}(z,N,\alpha)$ matrices, and another by the $\mathbf{Y}_0(z,N,\alpha)$ matrices. Doing this requires us to define the matrix generating functions

$$\mathbf{A}_+(z) := \sum_{k=1}^{u} z^k \mathbf{A}_k, \qquad \mathbf{A}_-(z) := \sum_{k=l}^{-1} z^k \mathbf{A}_k$$

associated with $\{\mathbf{A}_k\}_{k=l}^{u}$, as well as

$$\mathbf{B}_+(z) := \sum_{k=1}^{u} z^k \mathbf{A}_k, \qquad \mathbf{B}_-(z) := \sum_{k=l}^{-1} z^k \mathbf{B}_k.$$

Theorem 3.3.5 *For each integer $N \geq 0$, we have*

$$\begin{aligned}\mathbf{Y}(z,N+1,\alpha) &= (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_-(1/z) \\ &\quad + (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\left[\mathbf{A}_-(1/z) + \mathbf{A}_+(1/z)\right]\left[\mathbf{Y}(z,N,\alpha) - \sum_{m=1}^{u} z^m \mathbf{Y}_m(N,\alpha)\right] \\ &\quad + (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\sum_{m=1}^{u}\left[\mathbf{A}_-(1/z) + \sum_{k=1}^{m-1} z^{k-m}\mathbf{A}_{m-k}\right] z^m \mathbf{Y}_m(N,\alpha).\end{aligned}$$

Furthermore, for each integer $N \geq 0$, we have

$$\begin{aligned}
\mathbf{Y}_0(z, N+1, \alpha) &= (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_-(1/z) \\
&+ (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\left[\mathbf{A}_-(1/z) + \mathbf{A}_+(1/z)\right]\left[\mathbf{Y}_0(z, N, \alpha) - \sum_{m=1}^{u} z^m \mathbf{Y}_{m,0}(N, \alpha)\right] \\
&+ (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\sum_{m=1}^{u}\left[\mathbf{A}_-(1/z) + \sum_{k=1}^{m-1} z^{k-m}\mathbf{A}_{m-k}\right] z^m \mathbf{Y}_{m,0}(N, \alpha).
\end{aligned}$$

Proof This result can be proven with simple algebra, so we omit the details. ♢

What the recursions from Theorem 3.3.5 suggest is the following two-step procedure: given $\mathbf{Y}(z, N, \alpha)$ evaluated at a finite number of values $z_1, z_2, \ldots, z_m$, first (i) use the discrete numerical transform inversion algorithm of Abate and Whitt [1] to compute each matrix $\mathbf{Y}_1(N, \alpha), \mathbf{Y}_2(N, \alpha), \ldots, \mathbf{Y}_u(N, \alpha)$ from $\mathbf{Y}(z_1, N, \alpha), \mathbf{Y}(z_2, N, \alpha), \ldots, \mathbf{Y}(z_m, N, \alpha)$—after modifying Theorem 1 of [1] in an obvious way so that complex numbers can be handled—and then (ii) use these matrices to compute $\mathbf{Y}(z_k, N + 1, \alpha)$, $1 \leq k \leq m$. The values $z_1, z_2, \ldots, z_m$ are chosen based on Theorem 1 of [1]. A similar naive technique for computing the generating functions $\{\mathbf{Y}_0(z, N, \alpha)\}_{N \geq 0}$ should work as well: we plan to investigate possible methods for computing these $\mathbf{G}$ matrices in future work.

3.3.3 Computing the boundary vector $\pi_{(0,\ell_0),0}(\alpha)$

We conclude this section by showing how to compute the row vector $\boldsymbol{\pi}_{(0,\ell_0),0}(\alpha)$ once both sets of $\mathbf{G}$ matrices have been computed. For the upcoming derivations, it is useful to define the following notation: let $\mathbf{A}^{(*r,*c)}$ represent the matrix $\mathbf{A}$ with row r and column c removed. Additionally, $\mathbf{A}^{(*r,\cdot)}$ denotes $\mathbf{A}$ with row r removed, and similarly, $\mathbf{A}^{(\cdot,*c)}$ is $\mathbf{A}$ with column c removed. With this notation, we introduce the following convention: for the matrix $\mathbf{A}^{(*r,*c)}$, we no longer let $(\mathbf{A}^{(*r,*c)})_{i,j}$ represent the ith row and jth column of $\mathbf{A}^{(*r,*c)}$. Instead, the notation $(\mathbf{A}^{(*r,*c)})_{i,j}$ represents the element of $\mathbf{A}^{(*r,*c)}$ which previously was the (i, j) element of $\mathbf{A}$.

Furthermore, we introduce the $(M + 1) \times (M + 1)$ matrix $\mathbf{N}_0(\alpha)$, where, for $0 \leq i, j \leq M$, the (i, j)th element of $\mathbf{N}_0(\alpha)$ is given by

$$(\mathbf{N}_0(\alpha))_{i,j} = \mathbb{E}_{(0,i)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0, j))dt\right].$$

Theorem 3.3.6 *For* $\alpha \in \mathbb{C}_+$,

$$\mathbf{N}_0^{(*\ell_0,*\ell_0)}(\alpha) = \left[\alpha \mathbf{I}^{(*\ell_0,*\ell_0)} - \mathbf{B}_0^{(*\ell_0,*\ell_0)} - \sum_{m=1}^{\infty} \mathbf{B}_m^{(*\ell_0,\cdot)} \mathbf{G}_{m,0}^{(\cdot,*\ell_0)}(\alpha)\right]^{-1}. \tag{3.20}$$

Remark Observe that the $\{\mathbf{G}_{m,0}(\alpha)\}$ matrices are present in (3.20), but they do not appear in Theorem 3.3.2.

Proof We begin with $(\mathbf{N}_0^{(*\ell_0,*\ell_0)}(\alpha))_{i,j}$ for $i, j \neq \ell_0$ and proceed in a manner analogous to the proof of Theorem 3.3.3: applying first-step analysis yields

$$\begin{aligned}
(\mathbf{N}_0^{(*\ell_0,*\ell_0)}(\alpha))_{i,j} &= \mathbb{E}_{(0,i)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \\
&= \left(\frac{\mathbf{1}(i = j)}{-(\mathbf{B}_0)_{i,i} + \alpha}\right) + \sum_{\substack{x=0 \\ x\neq i,\ell_0}}^{M} \mathbb{E}_{(0,x)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \frac{(\mathbf{B}_0)_{i,x}}{-(\mathbf{B}_0)_{i,i} + \alpha} \\
&+ \sum_{m=1}^{\infty}\sum_{x=0}^{M} \mathbb{E}_{(m,x)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \frac{(\mathbf{B}_m)_{i,x}}{-(\mathbf{B}_0)_{i,i} + \alpha}.
\end{aligned} \tag{3.21}$$

Working further with the expectation in (3.21), we see that for $m \geq 1$,

$$\begin{aligned}
&\mathbb{E}_{(m,x)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \\
= &\sum_{\substack{y=0 \\ y\neq \ell_0}}^{M} \mathbb{E}_{(m,x)}\left[\mathbf{1}(X(\tau_{C_0}) = (0,y))e^{-\alpha \tau_{C_0}}\right] \mathbb{E}_{(0,y)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right].
\end{aligned} \tag{3.22}$$

Plugging (3.22) into (3.21) and writing in matrix form yields (3.20). ♢

To compute $\pi_{(0,\ell_0),(0,j)}(\alpha)$ whenever $j \neq \ell_0$, we make use of Theorem 3.2.1 with $T = \{(0,\ell_0)\}$ and $D = \{(0,j)\}$ to get

$$\pi_{(0,\ell_0),(0,j)}(\alpha) = \pi_{(0,\ell_0),(0,\ell_0)}(\alpha)((q(0,\ell_0)) + \alpha)\mathbb{E}_{(0,\ell_0)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right], \tag{3.23}$$

where, in a calculation very similar to the proof of Theorem 3.3.6, we see

$$\begin{aligned}
&((q(0,\ell_0)) + \alpha)\mathbb{E}_{(0,\ell_0)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \\
= \quad & \sum_{\substack{x=0 \\ x\neq \ell_0}}^{M} (\mathbf{B}_0)_{\ell_0,x}\mathbb{E}_{(0,x)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right] \\
+ \quad & \sum_{m=1}^{\infty}\sum_{x=0}^{M}\sum_{\substack{y=0 \\ y\neq \ell_0}}^{M} (\mathbf{B}_m)_{\ell_0,x}\mathbb{E}_{(m,x)}\left[\mathbf{1}(X(\tau_{C_0}) = (0,y))e^{-\alpha\tau_{C_0}}\right]\mathbb{E}_{(0,y)}\left[\int_0^{\tau_{(0,\ell_0)}} e^{-\alpha t}\mathbf{1}(X(t) = (0,j))dt\right].
\end{aligned}$$

Hence, each transform $\pi_{(0,\ell_0),(0,j)}(\alpha)$ can be expressed in terms of $\pi_{(0,\ell_0),(0,\ell_0)}(\alpha)$. In fact,

$$\begin{aligned}
\boldsymbol{\pi}_{(0,\ell_0),0}^{(\cdot,*\ell_0)}(\alpha) \quad = \quad & \pi_{(0,\ell_0),(0,\ell_0)}(\alpha)\left[\left(\mathbf{B}_0^{(\cdot,*\ell_0)}\right)_{(\ell_0,\cdot)}\left(\mathbf{N}_0^{(*\ell_0,*\ell_0)}(\alpha)\right)\right. \\
& \left. + \sum_{m=1}^{\infty}\left((\mathbf{B}_m)_{(\ell_0,\cdot)}\right)\left(\mathbf{G}_{m,0}^{(\cdot,*\ell_0)}(\alpha)\right)\left(\mathbf{N}_0^{(*\ell_0,*\ell_0)}(\alpha)\right)\right]. \qquad (3.24)
\end{aligned}$$

It remains to compute the remaining transform $\pi_{(0,\ell_0),(0,\ell_0)}(\alpha)$. One way to do this is to first set $\pi_{(0,\ell_0),(0,\ell_0)}(\alpha) = 1$, use (3.24) to compute the rest of the $\pi_{(0,\ell_0),(0,j)}(\alpha)$ transforms (after computing the $\{\mathbf{G}_{m,0}(\alpha)\}_{m\geq 1}$ matrices), then use the recursion from Theorem 3.3.1 to compute the $\pi_{(0,\ell_0),k}(\alpha)$ vectors (after computing the $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$ matrices). Once the $\pi_{(0,\ell_0),k}(\alpha)$ vectors have been found for sufficiently large k, normalization can then be used to find the correct value for $\pi_{(0,\ell_0),(0,\ell_0)}(\alpha)$.

3.4 $M/G/1$-type Markov processes

In this section we assume X is a Markov process of $M/G/1$-type, meaning $\mathbf{B}_n = \mathbf{A}_n = \mathbf{0}$ for each integer $n \leq -2$. Such a process is said to have level transitions that are skip-free to the left (i.e., skip-free with respect to state transitions to lower levels).

A classic result in the theory of $M/G/1$-type Markov processes is known as Ramaswami's formula [32] which provides a numerically stable recursion for computing the stationary distribution of an ergodic Markov process of $M/G/1$-type. The main result of this section (Theorem 3.4.2) shows that the Laplace transforms of the transition functions of X satisfy a formula analogous to Ramaswami's formula. A discrete time version of this formula can be found in [27].

Since X is skip-free to the left, we no longer need the full set of matrices $\{\mathbf{G}_k(\alpha)\}_{k\geq 1}$ introduced in Equation 3.8, nor will we need the matrices $\{\mathbf{G}_{k,0}(\alpha)\}_{k\geq 1}$. Instead, we will only need the matrix $\mathbf{G}(\alpha)$ defined on $\mathbb{C}_+$ as follows: for $i, j \in \{0, 1, \ldots, M\}$, we have

$$(\mathbf{G}(\alpha))_{i,j} = \mathbb{E}_{(2,i)}\left[\mathbf{1}(X(\tau_{L_1}) = (1, j))e^{-\alpha\tau_{L_1}}\right].$$

Under the skip-free-to-the-left assumption, one can easily use both induction and the strong Markov property to show that for each integer $k \geq 1$,

$$\mathbf{G}_k(\alpha) = \mathbf{G}(\alpha)^k. \tag{3.25}$$

The next result is a special case of Theorem 3.3.4: most of this result can also be found in Theorem 4.4 of [33].

Theorem 3.4.1 *The matrix $\mathbf{G}(\alpha)$ is, for $\alpha \geq 0$, the minimal nonnegative solution to the matrix equation*

$$\mathbf{0} = \mathbf{A}_{-1} + (\mathbf{A}_0 - \alpha\mathbf{I})\mathbf{Y}(\alpha) + \sum_{k=1}^{\infty} \mathbf{A}_k \mathbf{Y}(\alpha)^k. \tag{3.26}$$

Furthermore, for each $\alpha \in \mathbb{C}_+$, we have

$$\mathbf{G}(\alpha) = \lim_{n\to\infty} \mathbf{Y}(N, \alpha) \tag{3.27}$$

pointwise, where $\{\mathbf{Y}(N, \alpha)\}_{N\geq 0}$ is a sequence of matrices satisfying $\mathbf{Y}(0, \alpha) = \mathbf{0}$, and for each integer $N \geq 0$,

$$\mathbf{Y}(N + 1, \alpha) = (\alpha\mathbf{I} - \mathbf{A}_0)^{-1}\left[\mathbf{A}_{-1} + \sum_{n=1}^{\infty} \mathbf{A}_n \mathbf{Y}(N, \alpha)^n\right]. \tag{3.28}$$

We are now ready to state Theorem 3.4.2. Here, we define $\mathbf{e}_{i_0}$ as the column vector whose i_0th element is equal to one, with all other elements equal to zero.

Theorem 3.4.2 *Suppose X is a Markov process of $M/G/1$-type. Then for each integer $n \geq 0$, we have for $\alpha \in \mathbb{C}_+$ that*

$$\boldsymbol{\pi}_{(0,i_0),n+1}(\alpha) = \boldsymbol{\pi}_{(0,i_0),0}(\alpha)\overline{\mathbf{B}}_{n+1}(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1} + \sum_{k=1}^{n} \boldsymbol{\pi}_{(0,i_0),k}(\alpha)\overline{\mathbf{A}}_{n+1-k}(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1} \tag{3.29}$$

and

$$\boldsymbol{\pi}_{(0,i_0),0}(\alpha)[\alpha\mathbf{I} - \mathbf{B}_0 - \overline{\mathbf{B}}_1(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1}\mathbf{B}_{-1}] = \mathbf{e}_{i_0} \tag{3.30}$$

where for each integer $n \geq 0$,

$$\overline{\mathbf{B}}_n(\alpha) = \sum_{m=n}^{\infty} \mathbf{B}_m \mathbf{G}(\alpha)^{m-n}, \qquad \overline{\mathbf{A}}_n(\alpha) = \sum_{m=n}^{\infty} \mathbf{A}_m \mathbf{G}(\alpha)^{m-n}.$$

Remark Ramaswami's formula can be derived from (3.29) by first multiplying both sides of (3.29) by α, then letting $\alpha \downarrow 0$. Typically Ramaswami's formula is derived with the censoring technique (see e.g. [32, 33] as well as the tutorial of Riska and Smirni [35]). ♣

Remark The linear system (3.30) has a unique solution when $\alpha \in \mathbb{C}_+$: uniqueness follows from both the Kolmogorov forward equations—which have a unique solution when the process is regular—and the recursion given by Theorem 3.4.2. Another way to compute $\pi_{(0,\ell_0),0}(\alpha)$ is to use both (3.24) and normalization.

Solving for $\boldsymbol{p}_0 := \lim_{\alpha \downarrow 0} \alpha \boldsymbol{\pi}_{(0,i_0),0}(\alpha)$ using (3.30) is less straightforward, as one still needs to argue that the solution set of the limiting linear system is of dimension one. This step can be avoided if one instead uses (3.24) and the monotone convergence theorem. Neuts [29], Ramaswami [32], and Helmut [19] also discuss ways to solve for $\boldsymbol{p}_0$. ♣

Proof In light of Theorem 3.3.1, it suffices to show for $\alpha \in \mathbb{C}_+$ and each integer $n \geq 0$ that

$$\mathbf{R}_{0,n+1}(\alpha) = \overline{\mathbf{B}}_{n+1}(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1},$$

and for each integer $1 \leq k \leq n$,

$$\mathbf{R}_{k,n+1}(\alpha) = \overline{\mathbf{A}}_{n+1-k}(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1}.$$

The steps needed to show this are extremely similar to those used in the proof of Theorem 3.3.2 and Theorem 3.3.3, so we omit the details. ♢

3.4.1 Starting Outside of Level Zero

We conclude this chapter by deriving the Laplace transforms found in $\boldsymbol{\pi}_{(n_0,i_0),n}(\alpha)$, for each integer $n_0 \geq 1$, $i_0 \geq 0$, when X is a Markov process of $M/G/1$-type. First notice that for each integer $n \geq n_0$, we have for each $\alpha \in \mathbb{C}_+$ that

$$\boldsymbol{\pi}_{(n_0,i_0),n+1}(\alpha) = \boldsymbol{\pi}_{(n_0,i_0),0}(\alpha)\overline{\mathbf{B}}_{n+1}(\alpha)(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1} + \sum_{k=1}^{n} \boldsymbol{\pi}_{(n_0,i_0),k}(\alpha)\overline{\mathbf{A}}_{n+1-k}(\alpha\mathbf{I} - \overline{\mathbf{A}}_0(\alpha))^{-1} \quad (3.31)$$

Equation (3.31) shows that once we are able to compute the vectors $\boldsymbol{\pi}_{(n_0,i_0),n}(\alpha)$, $0 \leq n \leq n_0$, all other vectors $\boldsymbol{\pi}_{(n_0,i_0),n}(\alpha)$ for $n > n_0$ can be computed recursively. An important set of matrices appearing in the expressions of these vectors is $\{\mathbf{R}_{n+1,k}(\alpha)\}_{0 \leq n < n_0; 0 \leq k \leq n}$, where the (i,j)th element of $\mathbf{R}_{n+1,k}(\alpha)$ is defined for $\alpha \in \mathbb{C}_+$ as

$$(\mathbf{R}_{n+1,k}(\alpha))_{i,j} = (q((n+1,i)) + \alpha)\mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{L_{n+1}}} e^{-\alpha t}\mathbf{1}(X(t) = (k,j))dt\right].$$

The elements of these matrices can also be expressed as expected values of random products governed by another CTMP: choosing a CTMP $\tilde{X}$ as we did in our sketched proof of Theorem 3.2.1, we can also show that for $i, j \geq 0$,

$$(\mathbf{R}_{n+1,k}(\alpha))_{i,j} = \mathbb{E}_{(k,j)}\left[e^{-\alpha\tilde{\tau}_{n+1}}\mathbf{1}(\tilde{X}_{\tilde{\eta}_{n+1}} = (n+1,i))\prod_{\ell=1}^{\tilde{\eta}_{n+1}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right]. \quad (3.32)$$

Theorem 3.4.3 *For $0 \leq n < n_0$,*

$$\boldsymbol{\pi}_{(n_0,i_0),n}(\alpha) = \boldsymbol{\pi}_{(n_0,i_0),n+1}(\alpha)\mathbf{R}_{n+1,n}(\alpha) \quad (3.33)$$

where the matrices $\{\mathbf{R}_{n+1,k}(\alpha)\}_{0 \leq n < n_0; 0 \leq k \leq n}$ satisfy the equations $\mathbf{R}_{1,0}(\alpha) = \mathbf{B}_{-1}(\alpha\mathbf{I} - \mathbf{B}_0)^{-1}$ and for $1 \leq n < n_0$

$$\boldsymbol{R}_{n+1,n}(\alpha) = \left[\mathbf{B}_{-1}\mathbf{1}(n=1) + \mathbf{A}_{-1}\mathbf{1}(n \geq 2)\right]\left[\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1}\mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right]^{-1} \quad (3.34)$$

and for $n \geq 1$, $0 \leq k \leq n$,

$$\mathbf{R}_{n+1,k}(\alpha) = \mathbf{R}_{n+1,n}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\cdots\mathbf{R}_{k+1,k}(\alpha). \quad (3.35)$$

Finally, the vector $\boldsymbol{\pi}_{(n_0,i_0),n_0}(\alpha)$ *satisfies*

$$\begin{aligned} \boldsymbol{e}_{i_0} &= \boldsymbol{\pi}_{(n_0,i_0),n_0}(\alpha)\Bigg[\alpha \boldsymbol{I} - \boldsymbol{A}_0 - \sum_{k=1}^{n_0-1}\left(\prod_{h=0}^{n_0-k-1}\mathbf{R}_{n_0-h,n_0-1-h}(\alpha)\right)\Big(\mathbf{A}_{n_0-k} + \mathbf{R}_{k,n_0+1}(\alpha)\boldsymbol{A}_{-1}\Big) \\ &\qquad - \mathbf{R}_{n_0,n_0+1}(\alpha)\boldsymbol{A}_{-1} - \left(\prod_{h=0}^{n_0-1}\mathbf{R}_{n_0-h,n_0-1-h}(\alpha)\right)\mathbf{B}_{n_0}\Bigg] \end{aligned} \tag{3.36}$$

This theorem provides a procedure for computing each of the matrices $\mathbf{R}_{n+1,n}(\alpha)$, $0 \le n < n_0$. Starting with the boundary condition $\mathbf{R}_{1,0}(\alpha) = \mathbf{B}_{-1}\left(\alpha\mathbf{I} - \mathbf{B}_0\right)^{-1}$, we can successively compute the matrices $\mathbf{R}_{2,1}(\alpha), \mathbf{R}_{3,2}(\alpha), \ldots, \mathbf{R}_{n_0,n_0-1}(\alpha)$, by alternating between Equations (3.34) and (3.35).

Proof Equation (3.33) follows from Theorem 3.2.1: choose $T = C_n^c$ and $D = L_n$.

Next we show $\mathbf{R}_{1,0}(\alpha) = \mathbf{B}_{-1}\left(\alpha\mathbf{I} - \mathbf{B}_0\right)^{-1}$. Starting with representation (3.32) of the (i,j)th element of $\mathbf{R}_{1,0}(\alpha)$, we find after conditioning on $\tilde{X}_1$ and simplifying that

$$\begin{aligned} (\mathbf{R}_{1,0}(\alpha))_{i,j} &= \mathbb{E}_{(0,j)}\left[e^{-\alpha\tilde{\tau}_1}\mathbf{1}(\tilde{X}_{\tilde{\eta}_1} = (1,i))\prod_{\ell=1}^{\tilde{\eta}_1}\frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right] \\ &= \frac{(\mathbf{B}_{-1})_{i,j}}{(-(\mathbf{B}_0)_{j,j} + \alpha)} + \sum_{\nu \neq j}\left(\frac{(\mathbf{B}_0)_{\nu,j}}{-(\mathbf{B}_0)_{j,j} + \alpha}\right)(\mathbf{R}_{1,0}(\alpha))_{i,\nu} \end{aligned}$$

or, equivalently,

$$\alpha(\mathbf{R}_{1,0}(\alpha))_{i,j} = (\mathbf{B}_{-1})_{i,j} + \sum_{\nu \geq 0}(\mathbf{R}_{1,0}(\alpha))_{i,\nu}(\mathbf{B}_0)_{\nu,j}$$

which proves $\alpha\mathbf{R}_{1,0}(\alpha) = \mathbf{B}_{-1} + \mathbf{R}_{1,0}(\alpha)\mathbf{B}_0$, so $\mathbf{R}_{1,0}(\alpha) = \mathbf{B}_{-1}\left(\alpha\mathbf{I} - \mathbf{B}_0\right)^{-1}$. To establish (3.35), it suffices to show

$$\mathbf{R}_{n+1,k}(\alpha) = \mathbf{R}_{n+1,k+1}(\alpha)\mathbf{R}_{k+1,k}(\alpha)$$

which can be proven from (3.32) by summing over all possible ways $\tilde{X}$ can enter the set C_k^c, and applying the strong Markov property at this entrance time: since $\tilde{X}$ is skip-free to the right, the only way C_k^c can be entered from C_k is through L_{k+1}.

We next simplify $\mathbf{R}_{n+1,n}(\alpha)$. Conditioning on the value of $\tilde{X}_1$ in $(\mathbf{R}_{n+1),n}(\alpha))_{i,j}$ gives, in matrix notation,

$$\mathbf{R}_{n+1,n}(\alpha) = \left[\mathbf{B}_{-1}\mathbf{1}(n=1) + \mathbf{A}_{-1}\mathbf{1}(n \geq 2) + \sum_{k=1}^{n-1} \mathbf{R}_{n+1,k}(\alpha)\mathbf{A}_{n-k} + \mathbf{R}_{n+1,0}(\alpha)\mathbf{B}_n\right](\alpha\mathbf{I} - \mathbf{A}_0)^{-1}$$

which, after making use of (3.35), proves (3.34).

Lastly, (3.36) is obtained from the Laplace transform version of the Kolmogorov forward equations associated with X, while also making use of (3.31) and (3.33). $\diamondsuit$

To show the matrix $\left(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1}\mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right)$ is invertible (this is used to derive (3.34)) for $1 \leq n < n_0$, consider a new CTMP $X^{(n)} := \{X^{(n)}(t); t \geq 0\}$ on E having generator $\mathbf{Q}^{(n)}$, which is equivalent to $\mathbf{Q}$ except that the row corresponding to level L_{n+1} is instead

$$\begin{array}{c} \\ L_{n+1} \end{array} \begin{array}{c} \begin{array}{ccccccccc} L_0 & L_1 & L_2 & \cdots & L_n & L_{n+1} & L_{n+2} & L_{n+3} & L_{n+4} & \cdots \end{array} \\ \left[\begin{array}{cccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_0 & \mathbf{A}_{-1} & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \end{array}\right]. \end{array}$$

Furthermore, for each subset $A \subset E$, we let $\tau_A^{(n)} := \inf\{t \geq 0 : X^{(n)}(t-) \neq X^{(n)}(t) \in A\}$ represent the first time $X^{(n)}$ makes a transition into the set A. Then since the rows corresponding to $L_0, L_1, \ldots, L_n$ in $\mathbf{Q}^{(n)}$ are identical to those in $\mathbf{Q}$, we have that $X^{(n)}$ behaves probabilistically as X when in levels zero through n. Hence, we obtain, for each $0 \leq k \leq n$,

$$\begin{aligned} (\mathbf{R}_{n,k}(\alpha))_{i,j} &= (q((n,i)) + \alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{L_n}} e^{-\alpha t}\mathbf{1}(X(t) = (k,j))dt\right] && (3.37) \\ &= (q^{(n)}((n,i)) + \alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{L_n}^{(n)}} e^{-\alpha t}\mathbf{1}(X^{(n)}(t) = (k,j))dt\right]. && (3.38) \end{aligned}$$

Thus, the matrix $\left(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1}\mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right)$ is unchanged when working with the newly introduced $X^{(n)}$ process instead of the X process. Having this in mind, we define the matrix of matrices $\mathbf{\Pi}^{(n)}(\alpha)$ as

$$\mathbf{\Pi}^{(n)}(\alpha) = \begin{array}{c} \\ L_0 \\ L_1 \\ L_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} L_0 & L_1 & L_2 & \cdots \end{array} \\ \begin{pmatrix} \mathbf{\Pi}^{(n)}_{0,0}(\alpha) & \mathbf{\Pi}^{(n)}_{0,1}(\alpha) & \mathbf{\Pi}^{(n)}_{0,2}(\alpha) & \cdots \\ \mathbf{\Pi}^{(n)}_{1,0}(\alpha) & \mathbf{\Pi}^{(n)}_{1,1}(\alpha) & \mathbf{\Pi}^{(n)}_{1,2}(\alpha) & \cdots \\ \mathbf{\Pi}^{(n)}_{2,0}(\alpha) & \mathbf{\Pi}^{(n)}_{2,1}(\alpha) & \mathbf{\Pi}^{(n)}_{2,2}(\alpha) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}, \qquad \alpha \in \mathbb{C}_+$$

where each block element $\mathbf{\Pi}^{(n)}_{m,v}(\alpha)$ is an $(M+1) \times (M+1)$ matrix whose $(i,j)th$ element is the Laplace transform

$$\pi^{(n)}_{(m,i),(v,j)}(\alpha) := \int_0^\infty e^{-\alpha t} \mathbb{P}_{(m,i)}(X^{(n)}(t) = (v,j))dt, \qquad \alpha \in \mathbb{C}_+.$$

We are now ready to prove $\left(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1} \mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right)$ is invertible. From the forward equations of $X^{(n)}$, we have $\mathbf{\Pi}^{(n)}(\alpha)\left(\alpha\mathbf{I} - \mathbf{Q}^{(n)}\right) = \mathbf{I}$, for $\alpha \in \mathbb{C}_+$. Taking the dot product of the column in $\left(\alpha\mathbf{I} - \mathbf{Q}^{(n)}\right)$ corresponding to L_n (i.e., the $(n+1)$st column) with the $(n+1)$st row of $\mathbf{\Pi}^{(n)}(\alpha)$ gives

$$-\mathbf{\Pi}^{(n)}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1} \mathbf{\Pi}^{(n)}_{n,k}(\alpha)\mathbf{A}_{n-k} + \mathbf{\Pi}^{(n)}_{n,n}(\alpha)\left(\alpha\mathbf{I} - \mathbf{A}_1\right) = \mathbf{I}$$

which, after repeatedly applying (3.33), reduces to

$$\mathbf{\Pi}^{(n)}_{n,n}(\alpha)\left(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1} \mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right) = \mathbf{I},$$

thus proving $\left(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}_{n,0}(\alpha)\mathbf{B}_n - \sum_{k=1}^{n-1} \mathbf{R}_{n,k}(\alpha)\mathbf{A}_{n-k}\right)$ is invertible when $M < \infty$. Unfortunately, this argument does not establish invertibility when $M = \infty$.

Chapter 4

$M/M/1$ in a Random Environment

4.1 Introduction

In this chapter, we are interested in the stationary behavior of an $M/M/1$ queueing system whose arrival and service rates are goverened by an external Markovian environment $Y := \{Y(t); t \geq 0\}$ having state space $E' := \{0, 1\}$ and generator $\mathbf{Q}' := [q'(x, y)]_{x,y \in E'}$. In the literature, this is referred to as an $M/M/1$ queue in a two state random environment. The dynamics of such a queue are as follows: while the Y process is in phase i, $i \in \{0, 1\}$, customers arrive to a single server queueing system in accordance to a Poisson process with rate $\lambda_i > 0$. Each arrival brings an exponentially distributed amount of work having unit rate and the server processes work in a first-come-first-serve manner with exponential rate $\mu_i > 0$. For each $t \geq 0$ we let $Q(t)$ represent the number of customers present in the system at time t and we let $X(t) := (Q(t), Y(t))$. Then $\{X(t); t \geq 0\}$ is a CTMP on state space $E = \{0, 1, 2, \ldots\} \times \{0, 1\}$ with generator $\mathbf{Q}$ satisfying

$$q((j,i),(\ell,k)) = \begin{cases} \lambda_i, & \ell = j+1,\ k = i; \\ \mu_i, & \ell = j-1,\ n \geq 1,\ k = i; \\ q'(i,k), & \ell = j,\ i \neq k; \\ 0, & \text{otherwise.} \end{cases}$$

As in previous chapters, we denote the stationary distribution of X as $\mathbf{p} := [p(x)]_{x \in E}$. Our goal is to compute $\mathbf{p}$. Before proceeding, it is worth noting that the generator of X can be written in the block-matrix form of (1.5) with $M = 1$ and $\mathbf{A}_n = \mathbf{B}_n = \mathbf{0}$ for $n \in \{\dots, -3, -2\} \cup \{2, 3, \dots\}$. The process X is known as a quasi-birth-death process, which is a special case of the $G/M/1$-type Markov processes studied in Chapter 2. Hence, to derive the stationary distribution of X, one could start with (4.1) and apply the matrix-geometric theory of Chapter 2 to compute $\mathbf{p}$. Recall, however, that matrix-geometric methods require computing the elements of the rate matrix $\mathbf{R}$, which in most cases cannot be done in closed form and thus are numerically approximated instead.

In this chapter, we bypass the matrix-geometric approach of previous chapters, and instead show that the random-product invariant measure $\mathbf{w} := [w(x)]_{x \in E}$ found in [9], which is given by $w((0,0)) = 1$ and for $(n,i) \neq (0,0)$,

$$w((n,i)) = \mathbb{E}_{(n,i)} \left[\mathbf{1}(\eta_{(0,0)} < \infty) \prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)} \right] \tag{4.1}$$

can be written in closed form as a linear combination of scalars whose coefficients can be determined by solving a recursion. Note that in (4.1), we have made use of the fact that we can choose $\tilde{X} = X$ and thus $\tilde{\mathbf{Q}} = \mathbf{Q}$. Once $\mathbf{w}$ has been computed, we obtain $\mathbf{p}$ by normalizing $\mathbf{w}$.

4.2 Main Results

To work further with each $w((n,i))$ term, we let T count the number of environment transitions made by X on its journey from state (n,i) to state $(0,0)$. Using the monotone convergence theorem, we can say that for each $n \geq 0$ and $i \in \{0,1\}$, with $(n,i) \neq (0,0)$,

$$w((n,i)) = \sum_{t=0}^{\infty} \mathbb{E}_{(n,i)} \left[\mathbf{1}(\eta_{(0,0)} < \infty) \mathbf{1}(T = t) \prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)} \right] = \sum_{t=0}^{\infty} w_t((n,i)), \tag{4.2}$$

where for each integer $t \geq 0$, and each state (n,i),

$$w_t((n,i)) := \mathbb{E}_{(n,i)} \left[\mathbf{1}(\eta_{(0,0)} < \infty) \mathbf{1}(T = t) \prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)} \right].$$

Furthermore, it is not difficult to see that when $i = 0$, (4.2) can be written as

$$w((n,0)) = \sum_{t=0}^{\infty} w_t((n,0)) = \sum_{\substack{t=0 \\ t \text{ even}}}^{\infty} w_t((n,0)), \tag{4.3}$$

and when $i = 1$,

$$w((n,1)) = \sum_{t=0}^{\infty} w_t((n,1)) = \sum_{\substack{t=0 \\ t \text{ odd}}}^{\infty} w_t((n,1)). \tag{4.4}$$

Hence, one way of computing each $w((n,0))$ term is to devise a method for computing each $w_t((n,0))$ term when t is even. Similarly, computing each $w((n,1))$ term can be done by developing a way to compute each $w_t((n,1))$ term for odd values of t.

Computing $w_0((n,0))$ for each $n \geq 1$ is straightforward. Letting ϕ_i represent the Laplace-Stieltjes transform of the busy period of an $M/M/1$ queueing system having arrival rate λ_i and service rate μ_i, and defining $r_i := \rho_i \phi_i(\alpha_i)$ for each $i \in \{0,1\}$, we find that

$$\begin{aligned} w_0((n,0)) &= \mathbb{E}_{(n,0)}\left[\mathbf{1}(\eta_{(0,0)} < \infty)\mathbf{1}(T = 0) \prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)}\right] \\ &= \rho_0^n \mathbb{E}_{(n,0)}[\mathbf{1}(\tau_{(0,0)} < \tau_{(\cdot,1)})] = (\rho_0 \phi_0(\alpha_0))^n = r_0^n. \end{aligned} \tag{4.5}$$

Furthermore, we write

$$1 = w((0,0)) = \sum_{t=0}^{\infty} w_t((0,0))$$

with $w_0((0,0)) = 1$, and $w_t((0,0)) = 0$ for each $t \geq 1$.

Our main result (Theorem 4.2.1) shows how to compute $\{w_t((n,i))\}_{t \geq 1, n \geq 1}$. For convenience, we borrow notation from [10] and define, for each i,

$$\Omega_i := \frac{\rho_i \phi_i(\alpha_i)}{\lambda_i(1 - \rho_i \phi_i(\alpha_i)^2)}, \qquad \Gamma_i := \frac{1 - \rho_i \phi_i(\alpha_i)}{\alpha_i}.$$

The following proposition provides a key recursion.

Proposition 4.2.1 *For $t \geq 1$, $n \geq 1$,*

$$\begin{aligned} w_{2t}((n,0)) = \quad & \alpha_1 \Omega_0 \sum_{\ell=1}^{n-1} r_0^{n-\ell}(1 - (r_0 \phi_0(\alpha_0))^\ell) w_{2t-1}((\ell,1)) \\ + \quad & \alpha_1 \Omega_0 \sum_{\ell=n}^{\infty} \phi_0(\alpha_0)^{\ell-n}(1 - (r_0 \phi_0(\alpha_0))^n) w_{2t-1}((\ell,1)). \end{aligned}$$

Furthermore, for $t \geq 0$ and $n \geq 0$,

$$\begin{aligned} w_{2t+1}((n,1)) = \quad & \alpha_0 \Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n w_{2t}((\ell,0)) \\ + \quad & \alpha_0 \Omega_1 \sum_{\ell=1}^{n-1} r_1^{n-\ell}(1-(r_1\phi_1(\alpha_1))^{\ell}) w_{2t}((\ell,0)) \\ + \quad & \alpha_0 \Omega_1 \sum_{\ell=n}^{\infty} \phi_1(\alpha_1)^{\ell-n}(1-(r_1\phi_1(\alpha_1))^{n}) w_{2t}((\ell,0)). \end{aligned}$$

When constructing this recursion, we follow the convention that $w_0((0,0)) = 1$, and $w_t((0,0)) = 0$ for $t \geq 1$.

Proof Fix an integer $t \geq 1$, and let $\tau_{(\cdot,1)} := \inf\{s \geq 0 : X(s) \in \{(0,1),(1,1),(2,1),\ldots\}\}$. Then we have

$$\begin{aligned} & w_{2t}((n,0)) = \mathbb{E}_{(n,0)}\left[\mathbf{1}(\eta_{(0,0)} < \infty)\mathbf{1}(T = 2t)\prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)}\right] \\ = \quad & \sum_{\ell=1}^{\infty} \mathbb{E}_{(n,0)}\left[\mathbf{1}(\eta_{(0,0)} < \infty)\mathbf{1}(\tau_{(\cdot,1)} < \tau_{(0,0)})\mathbf{1}(X(\tau_{(\cdot,1)}) = (\ell,1))\mathbf{1}(T = 2t)\prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)}\right] \\ = \quad & \frac{\alpha_1}{\alpha_0}\sum_{\ell=1}^{\infty} \rho_0^{n-\ell}\mathbb{E}_{(n,0)}\left[\mathbf{1}(\tau_{(\cdot,1)} < \tau_{(0,0)})\mathbf{1}(X(\tau_{(\cdot,1)}) = (\ell,1))\right] w_{2t-1}((\ell,1)) \\ = \quad & \frac{\alpha_1}{\alpha_0}\sum_{\ell=1}^{n-1} \rho_0^{n-\ell}\alpha_0\pi_{n,\ell}^{(0)}(\alpha_0) w_{2t-1}((\ell,1)) + \frac{\alpha_1}{\alpha_0}\sum_{\ell=n}^{\infty}\left(\frac{1}{\rho_0}\right)^{\ell-n}\alpha_0\pi_{n,\ell}^{(0)}(\alpha_0) w_{2t-1}((\ell,1)) \\ = \quad & \alpha_1\Omega_0\sum_{\ell=1}^{n-1} r_0^{n-\ell}(1-(r_0\phi_0(\alpha_0))^{\ell}) w_{2t-1}((\ell,1)) + \alpha_1\Omega_0\sum_{\ell=n}^{\infty}\phi_0(\alpha_0)^{\ell-n}(1-(r_0\phi_0(\alpha_0))^{n}) w_{2t-1}((\ell,1)). \end{aligned}$$

Here the second equality follows from summing over all possible values of $X(\tau_{(\cdot,1)})$ on the set $\{T = 2t\}$, the third equality follows from applying the Strong Markov Property at the stopping time $\tau_{(\cdot,1)}$, the fourth simply follows from

$$\mathbb{E}_{(n,0)}\left[\mathbf{1}(\tau_{(\cdot,1)} < \tau_{(0,0)})\mathbf{1}(X(\tau_{(\cdot,1)}) = (\ell,1))\right] = \alpha_0\pi_{n,\ell}^{(0)}(\alpha_0),$$

and the final equality follows from using Appendix B to rewrite each $\pi_{n,\ell}^{(0)}(\alpha_0)$ term.

Furthermore, for $t \geq 0$, $n \geq 0$, we also have

$$w_{2t+1}((n,1)) = \mathbb{E}_{(n,1)}\left[\mathbf{1}(\eta_{(0,0)} < \infty)\mathbf{1}(T = 2t+1)\prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)}\right]$$

$$= \sum_{\ell=0}^{\infty}\mathbb{E}_{(n,1)}\left[\mathbf{1}(\eta_{0,0} < \infty)\mathbf{1}(T = 2t+1)\mathbf{1}(X(\tau_{(\cdot,0)}) = (\ell,0))\prod_{l=1}^{\eta_{(0,0)}} \frac{q(X_l, X_{l-1})}{q(X_{l-1}, X_l)}\right]$$

$$= \frac{\alpha_0}{\alpha_1}\sum_{\ell=0}^{\infty}\rho_1^{n-\ell}\mathbb{E}_{(n,1)}\left[\mathbf{1}(X(\tau_{(\cdot,0)}) = (\ell,0))\right] w_{2t}((\ell,0))$$

$$= \frac{\alpha_0}{\alpha_1}\sum_{\ell=0}^{n-1}\rho_1^{n-\ell}\alpha_1\pi_{n,\ell}^{(1)}(\alpha_1)w_{2t}((\ell,0)) + \frac{\alpha_0}{\alpha_1}\sum_{\ell=n}^{\infty}\left(\frac{1}{\rho_1}\right)^{\ell-n}\alpha_1\pi_{n,\ell}^{(1)}(\alpha_1)w_{2t}((\ell,0))$$

which, after plugging in the correct expressions for each $\pi_{n,\ell}^{(1)}(\alpha_1)$ term from Appendix B, gives the recursion. $\diamondsuit$

We next give two lemmas which are very helpful in simplifying algebra needed in the proof of the main result. These follow from the identities found in Appendix C of [10].

Lemma 4.2.1 *For each integer $n \geq 0$, and any two environment phases i, k, the following identities are true.*

$$\sum_{\ell=1}^{n-1}\phi_i(\alpha_i)^\ell r_i^n\binom{\ell-1+u}{u}r_i^\ell = \left[\frac{r_i\phi_i(\alpha_i) - (r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1}}\right]r_i^n - \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right]\frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1-x}}.$$

$$\sum_{\ell=1}^{n-1}\phi_i(\alpha_i)^\ell r_i^n\binom{\ell-1+u}{u}r_k^\ell = \left[\frac{r_k\phi_i(\alpha_i) - (r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1}}\right]r_i^n - \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right]\frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1-x}}$$

$$\sum_{\ell=n}^{\infty}\phi_i(\alpha_i)^\ell r_i^n\binom{\ell-1+u}{u}r_i^\ell = \frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1}} + \sum_{x=1}^{u}\left[\binom{n-1+x}{x}-\binom{n-1+x-1}{x-1}\right]\frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1-x}}.$$

$$\sum_{\ell=n}^{\infty}\phi_i(\alpha_i)^\ell r_i^n\binom{\ell-1+u}{u}r_k^\ell = \frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1}} + \sum_{x=1}^{u}\left[\binom{n-1+x}{x}-\binom{n-1+x-1}{x-1}\right] \times\frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1-x}}.$$

$$\sum_{\ell=1}^{n-1}r_i^{n-\ell}(1-(r_i\phi_i(\alpha_i))^\ell)\binom{\ell-1+u}{u}r_i^\ell = \left[\binom{n-1+u+1}{u+1}-\binom{n-1+u}{u}\right]r_i^n - \left[\frac{r_i\phi_i(\alpha_i)}{(1-r_i\phi_i(\alpha_i))^{u+1}}\right]r_i^n + \frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1}} + \sum_{x=1}^{u}\left[\binom{n-1+x}{x}-\binom{n-1+x-1}{x-1}\right] \times\frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1-x}}.$$

$$\sum_{\ell=1}^{n-1}r_i^{n-\ell}(1-(r_i\phi_i(\alpha_i))^\ell)\binom{\ell-1+u}{u}r_k^\ell = \left[\frac{r_i^u r_k}{(r_i-r_k)^{u+1}}\right]r_i^n - \left[\frac{r_i^{u+1}}{(r_i-r_k)^{u+1}}\right]r_k^n - \sum_{x=1}^{u}\left[\binom{n-1+x}{x}-\binom{n-1+x-1}{x-1}\right] \times\left[\frac{r_i}{r_i-r_k}\right]^{u+1-x} r_k^n - \left[\frac{r_k\phi_i(\alpha_i)}{(1-r_k\phi_i(\alpha_i))^{u+1}}\right]r_i^n + \frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1}} + \sum_{x=1}^{u}\left[\binom{n-1+x}{x}-\binom{n-1+x-1}{x-1}\right] \times\frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1-x}}.$$

$$
\begin{aligned}
\sum_{\ell=n}^{\infty} \phi_i(\alpha_i)^{\ell-n}(1-(r_i\phi_i(\alpha_i))^n)\binom{\ell-1+u}{u} r_i^{\ell} \;&=\; \frac{r_i^n}{(1-r_i\phi_i(\alpha_i))^{u+1}} - \frac{(r_i r_i \phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1}} \\
&+ \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right] \\
&\quad \times \frac{r_i^n}{(1-r_i\phi_i(\alpha_i))^{u+1-x}} \\
&- \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right] \\
&\quad \times \frac{(r_i r_i\phi_i(\alpha_i))^n}{(1-r_i\phi_i(\alpha_i))^{u+1-x}}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\ell=n}^{\infty} \phi_i(\alpha_i)^{\ell-n}(1-(r_i\phi_i(\alpha_i))^n)\binom{\ell-1+u}{u} r_k^{\ell} \;&=\; \frac{r_k^n}{(1-r_k\phi_i(\alpha_i))^{u+1}} - \frac{(r_i r_k \phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1}} \\
&+ \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right] \\
&\quad \times \frac{r_k^n}{(1-r_k\phi_i(\alpha_i))^{u+1-x}} \\
&- \sum_{x=1}^{u}\left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right] \\
&\quad \times \frac{(r_i r_k\phi_i(\alpha_i))^n}{(1-r_k\phi_i(\alpha_i))^{u+1-x}}.
\end{aligned}
$$

Proof The proof follows directly from Lemmas 6, 7, and 8 from Appendix C of [10]. $\diamondsuit$

The next lemma makes use of Lemma 4.2.1; it will be used to define our recursions. It is a special case of Lemma 1 and Lemma 2 from [10].

Lemma 4.2.2 *Define, for each $i \in \{0,1\}$, and each $n, \ell \geq 0$,*

$$
\beta_{n,\ell}^{(i)} = \begin{cases} r_i^{n-\ell}(1-(r_i\phi_i(\alpha_i))^{\ell}), & \ell \leq n; \\ \phi_i(\alpha_i)^{\ell-n}(1-(r_i\phi_i(\alpha_i))^n), & \ell \geq n. \end{cases}
$$

Then

$$
\sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(i)}\binom{\ell-1+u}{u} r_i^{\ell} = \binom{n-1+u+1}{u+1} r_i^n + \sum_{x=1}^{u}\left[\frac{r_i\phi_i(\alpha_i)}{(1-r_i\phi_i(\alpha_i))^{u+1-x}}\right]\binom{n-1+x}{x} r_i^n.
$$

Furthermore, for $k \neq i$,

$$\begin{aligned}\sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(i)} \binom{\ell-1+u}{u} r_k^{\ell} &= \left[\frac{r_k r_i^u}{(r_i - r_k)^{u+1}} - \frac{r_k \phi_i(\alpha_i)}{(1 - r_k \phi_i(\alpha_i))^{u+1}}\right] r_i^n \\ &\quad + \sum_{x=0}^{u} \left[\frac{r_k \phi_i(\alpha_i)}{(1 - r_k \phi_i(\alpha_i))^{u+1-x}} - \frac{r_k r_i^{u-x}}{(r_i - r_k)^{u+1-x}}\right] \binom{n-1+x}{x} r_k^n.\end{aligned}$$

Proof When $k = i$,

$$\begin{aligned}\sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(i)} \binom{\ell-1+u}{u} r_i^{\ell} &= \sum_{\ell=1}^{n-1} r_i^{n-\ell}(1 - (r_i \phi_i(\alpha_i))^{\ell}) \binom{\ell-1+u}{u} r_i^{\ell} \\ &+ \sum_{\ell=n}^{\infty} \phi_i(\alpha_i)^{\ell-n}(1 - (r_i \phi_i(\alpha_i))^{n}) \binom{\ell-1+u}{u} r_i^{\ell} \\ &= \left[\frac{1}{(1 - r_i \phi_i(\alpha_i))^{u}}\right] r_i^n \\ &+ \sum_{x=1}^{u+1} \left[\binom{n-1+x}{x} - \binom{n-1+x-1}{x-1}\right] \frac{1}{(1 - r_i \phi_i(\alpha_i))^{u+1-x}} r_i^n \\ &= \binom{n-1+u+1}{u+1} r_i^n + \sum_{x=1}^{u} \left[\frac{r_i \phi_i(\alpha_i)}{(1 - r_i \phi_i(\alpha_i))^{u+1-x}}\right] \binom{n-1+x}{x} r_i^n,\end{aligned}$$

where the second equality follows from Lemma 4.2.1. The case where $k \neq i$ follows similarly. $\diamondsuit$

We are now ready to state our main result.

Theorem 4.2.1 *For each $n \geq 1$ and $t \geq 0$, we have*

$$w_{2t}((n,0)) = \sum_{u=0}^{t} c_{2t,u}^{(0)} \binom{n-1+u}{u} r_0^n + \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \binom{n-1+u}{u} r_1^n$$

where the elements of $\{c_{2t,u}^{(0)}\}_{0 \leq u \leq t}$ and $\{c_{2t,u}^{(1)}\}_{0 \leq u \leq t-1}$ are given, for $t \geq 1$ as

$$\begin{aligned}c_{2t,0}^{(0)} &= \alpha_1 \Omega_0 \sum_{k=0}^{t-1} c_{2t-1,k}^{(1)} \left[\frac{r_1 r_0^k}{(r_0 - r_1)^{k+1}} - \frac{r_1 \phi_0(\alpha_0)}{(1 - r_1 \phi_0(\alpha_0))^{k+1}}\right] \\ c_{2t,u}^{(0)} &= \alpha_1 \Omega_0 \left(c_{2t-1,u-1}^{(0)} + \sum_{k=u}^{t-1} c_{2t-1,k}^{(0)} \frac{r_0 \phi_0(\alpha_0)}{(1 - r_0 \phi_0(\alpha_0))^{k+1-u}}\right) \qquad (1 \leq u \leq t) \\ c_{2t,u}^{(1)} &= \alpha_1 \Omega_0 \sum_{k=u}^{t-1} c_{2t-1,k}^{(1)} \left[\frac{r_1 \phi_0(\alpha_0)}{(1 - r_1 \phi_0(\alpha_0))^{k+1-u}} - \frac{r_1 r_0^{k-u}}{(r_0 - r_1)^{k+1-u}}\right] \qquad (0 \leq u \leq t-1),\end{aligned}$$

and $c_{0,0}^{(0)} = 1$.

Alternatively, for each $n \geq 1$ and $t \geq 0$, we have

$$w_{2t+1}((n,1)) \quad = \quad \sum_{u=0}^{t} c_{2t+1,u}^{(0)} \binom{n-1+u}{u} r_0^n + \sum_{u=0}^{t} c_{2t+1,u}^{(1)} \binom{n-1+u}{u} r_1^n$$

where the elements of $\{c_{2t+1,u}^{(0)}\}_{0 \leq u \leq t}$ and $\{c_{2t+1,u}^{(1)}\}_{0 \leq u \leq t}$ are given by

$$\begin{aligned}
c_{2t+1,u}^{(0)} = &\quad \alpha_0 \Omega_1 \sum_{k=u}^{t} c_{2t,k}^{(0)} \left[\frac{r_0 \phi_1(\alpha_1)}{(1 - r_0 \phi_1(\alpha_1))^{k+1-u}} - \frac{r_0 r_1^{k-u}}{(r_1 - r_0)^{k+1-u}} \right] \qquad (0 \leq u \leq t) \\
c_{2t+1,0}^{(1)} = &\quad \alpha_0 \Gamma_1 \sum_{k=0}^{t} c_{2t,k}^{(0)} \left[\frac{r_0 \phi_1(\alpha_1)}{(1 - r_0 \phi_1(\alpha_1))^{k+1}} \right] \\
&+ \alpha_0 \Gamma_1 \sum_{k=0}^{t-1} c_{2t,k}^{(1)} \left[\frac{r_1 \phi_1(\alpha_1)}{(1 - r_1 \phi_1(\alpha_1))^{k+1}} \right] \\
&+ \alpha_0 \Omega_1 \sum_{k=0}^{t} c_{2t,k}^{(0)} \left[\frac{r_0 r_1^k}{(r_1 - r_0)^{k+1}} - \frac{r_0 \phi_1(\alpha_1)}{(1 - r_0 \phi_1(\alpha_1))^{k+1}} \right] \\
c_{2t+1,u}^{(1)} = &\quad \alpha_0 \Omega_1 \left(c_{2t,u-1}^{(1)} + \sum_{k=u}^{t-1} c_{2t,k}^{(1)} \frac{r_1 \phi_1(\alpha_1)}{(1 - r_1 \phi_1(\alpha_1)^{k+1-u}} \right) \qquad (1 \leq u \leq t),
\end{aligned}$$

where $c_{1,0}^{(0)} = \alpha_0 \Omega_1 \left(\frac{1}{1 - r_0 \phi_1(\alpha_1)} - \frac{r_1}{r_1 - r_0} \right)$
and $c_{1,0}^{(1)} = \left[\alpha_0 \Gamma_1 \left(\frac{1}{1 - r_0 \phi_1(\alpha_1)} \right) + \alpha_0 \Omega_1 \left(\frac{r_0}{r_1 - r_0} - \frac{r_0 \phi_1(\alpha_1)}{1 - r_0 \phi_1(\alpha_1)} \right) \right]$

Proof We prove Theorem 4.2.1 by strong induction on t.

Base Cases We have two base cases: $w_0((n,0))$ and $w_1((n,1))$. From (4.5), we know

$$w_0((n,0)) = r_0^n = c_{0,0}^{(0)} r_0^n$$

by recalling that $c_{0,0}^{(0)} = 1$, thus establishing the first base case.

Regarding $w_1((n,1))$, we see for each $n \geq 1$,

$$\begin{aligned}
w_1((n,1)) = \quad & \alpha_0\Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n w_0(\ell,0) \\
+ \quad & \alpha_0\Omega_1 \sum_{\ell=n}^{\infty} \phi_1(\alpha_1)^{\ell-n}(1-(r_1\phi_1(\alpha_1))^n) w_0(\ell,0) \\
= \quad & \alpha_0\Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n r_0^{\ell} \\
+ \quad & \alpha_0\Omega_1 \sum_{\ell=1}^{n-1} r_1^{n-\ell}(1-(r_1\phi_1(\alpha_1))^{\ell}) r_0^{\ell} \\
+ \quad & \alpha_0\Omega_1 \sum_{\ell=n}^{\infty} \phi_1(\alpha_1)^{\ell-n}(1-(r_1\phi_1(\alpha_1))^n) r_0^{\ell} \\
= \quad & \left[\frac{\alpha_0\Gamma_1}{1-r_0\phi_1(\alpha_1)}\right] r_1^n + r_1\left[\frac{\alpha_0\Omega_1}{r_0-r_1}\right] r_0^n - r_0\left[\frac{\alpha_0\Omega_1}{r_0-r_1}\right] r_1^n \\
& -r_0\phi_1(\alpha_1)\left[\frac{\alpha_0\Omega_1}{1-r_0\phi_1(\alpha_1)}\right] r_1^n + \left[\frac{\alpha_0\Omega_1}{1-r_0\phi_1(\alpha_1)}\right](r_0 r_1\phi_1(\alpha_1))^n \\
+ \quad & \left[\frac{\alpha_0\Omega_1}{1-r_0\phi_1(\alpha_1)}\right](1-(r_1\phi_1(\alpha_1))^n) r_0^n
\end{aligned}$$

where the first equality is due to Proposition 4.2.1 and the last equality is due to Lemma 4.2.1. After simplifying, we conclude that for $n \geq 1$,

$$\begin{aligned}
w_1((n,1)) = \quad & \alpha_0\Omega_1\left(\frac{1}{1-r_0\phi_1(\alpha_1)} - \frac{r_1}{r_1-r_0}\right) r_0^n \\
+ \quad & \left[\alpha_0\Gamma_1\left(1+\frac{r_0\phi_1(\alpha_1)}{1-r_0\phi_1(\alpha_1)}\right) + \alpha_0\Omega_1\left(\frac{r_0}{r_1-r_0} - \frac{r_0\phi_1(\alpha_1)}{1-r_0\phi_1(\alpha_1)}\right)\right] r_1^n \\
= \quad & c_{1,0}^{(0)} r_0^n + c_{1,0}^{(1)} r_1^n,
\end{aligned}$$

which shows the final base case.

Induction Step Now assume the theorem holds for nonnegative integers less than or equal to $2t$, $t \geq 0$. Then for $2t+1$, we obtain

$$
\begin{aligned}
w_{2t+1}((n,1)) &= \alpha_0\Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n w_{2t}((\ell,0)) + \alpha_0\Omega_1 \sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(1)} w_{2t}((\ell,0)) \\
&= \alpha_0\Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n \sum_{u=0}^{t} c_{2t,u}^{(0)} \binom{\ell-1+u}{u} r_0^{\ell} \\
&+ \alpha_0\Gamma_1 \sum_{\ell=0}^{\infty} \phi_1(\alpha_1)^{\ell} r_1^n \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \binom{\ell-1+u}{u} r_1^{\ell} \\
&+ \alpha_0\Omega_1 \sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(1)} \sum_{u=0}^{t} c_{2t,u}^{(0)} \binom{\ell-1+u}{u} r_0^{\ell} \\
&+ \alpha_0\Omega_1 \sum_{\ell=1}^{\infty} \beta_{n,\ell}^{(1)} \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \binom{\ell-1+u}{u} r_1^{\ell},
\end{aligned}
$$

where the first equality follows from Proposition 4.2.1 and the second equality is our induction hypothesis. After applying Lemmas 4.2.1 and 4.2.2, we get

$$
\begin{aligned}
w_{2t+1}((n,1)) &= \alpha_0\Gamma_1 \sum_{u=0}^{t} c_{2t,u}^{(0)} \left[\frac{r_0\phi_1(\alpha_1)}{(1-r_0\phi_1(\alpha_1))^{u+1}}\right] r_1^n \\
&+ \alpha_0\Gamma_1 \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \left[\frac{r_1\phi_1(\alpha_1)}{(1-r_1\phi_1(\alpha_1))^{u+1}}\right] r_1^n \\
&+ \alpha_0\Omega_1 \sum_{u=0}^{t} c_{2t,u}^{(0)} \left[\frac{r_0 r_1^u}{(r_1-r_0)^{u+1}} - \frac{r_0\phi_1(\alpha_1)}{(1-r_0\phi_1(\alpha_1))^{u+1}}\right] r_1^n \\
&+ \alpha_0\Omega_1 \sum_{u=0}^{t} c_{2t,u}^{(0)} \left[\frac{1}{1-r_0\phi_1(\alpha_1)} - \frac{r_1}{r_1-r_0}\right] \binom{n-1+u}{u} r_0^n \\
&+ \alpha_0\Omega_1 \sum_{u=0}^{t} c_{2t,u}^{(0)} \sum_{x=0}^{u-1} \left[\frac{r_0\phi_1(\alpha_1)}{(1-r_0\phi_1(\alpha_1))^{u+1-x}} - \frac{r_0 r_1^{u-x}}{(r_1-r_0)^{u+1-x}}\right] \binom{n-1+x}{x} r_0^n \\
&+ \alpha_0\Omega_1 \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \binom{n-1+u+1}{u+1} r_1^n \\
&+ \alpha_0\Omega_1 \sum_{u=0}^{t-1} c_{2t,u}^{(1)} \sum_{x=1}^{u} \left[\frac{r_1\phi_1(\alpha_1)}{(1-r_1\phi_1(\alpha_1))^{u+1-x}}\right] \binom{n-1+x}{x} r_1^n \\
&= \sum_{u=0}^{t} c_{2t+1,u}^{(0)} \binom{n-1+u}{u} r_0^n + \sum_{u=0}^{t} c_{2t+1,u}^{(1)} \binom{n-1+u}{u} r_1^n,
\end{aligned}
$$

where the last equality follows from collecting like terms and equating coefficients appropriately. The algebra for $w_{2t+2}((n,0))$ follows analogously, and hence is omitted. $\diamondsuit$

4.3 The Normalization Constant

Now that Theorem 4.2.1 has provided a way to compute the invariant measure $\mathbf{w}$, we focus on computing the total mass of $\mathbf{w}$. This allows us normalize $\mathbf{w}$, thus giving the stationary distribution of X. The mass of $\mathbf{w}$ is given by

$$\sum_{n=0}^{\infty} w((n,0)) + \sum_{n=0}^{\infty} w((n,1)),$$

which can be written as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} w((n,0)) + \sum_{n=0}^{\infty} w((n,1)) \quad &= \quad w((0,0)) + \sum_{n=1}^{\infty}\sum_{t=0}^{\infty} w_{2t}((n,0)) \\ &\quad + \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} w_{2t+1}((n,1)). \end{aligned} \tag{4.6}$$

Working closer with (4.6), we see

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{t=0}^{\infty} w_{2t}((n,0)) \quad &= \quad \sum_{n=1}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t,u}^{(0)} \binom{n-1+u}{u} r_0^n + \sum_{n=1}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \binom{n-1+u}{u} r_1^n \\ &= \quad \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t,u}^{(0)} \sum_{n=1}^{\infty} \binom{n-1+u}{u} r_0^n + \sum_{t=0}^{\infty}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \sum_{n=1}^{\infty} \binom{n-1+u}{u} r_1^n \\ &= \quad \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t,u}^{(0)} \frac{r_0}{(1-r_0)^{u+1}} + \sum_{t=0}^{\infty}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \frac{r_1}{(1-r_1)^{u+1}} \end{aligned} \tag{4.7}$$

where the first equality follows from Theorem 4.2.1 and the third equality follows from Lemma 4.2.1. Similarly,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{t=0}^{\infty} w_{2t+1}((n,1)) \quad &= \quad \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(0)} \sum_{n=0}^{\infty} \binom{n-1+u}{u} r_0^n + \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(1)} \sum_{n=0}^{\infty} \binom{n-1+u}{u} r_1^n \\ &= \quad \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(0)} \frac{1}{(1-r_0)^{u+1}} + \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(1)} \frac{1}{(1-r_1)^{u+1}}, \end{aligned} \tag{4.8}$$

where the first equality again is due to Theorem 4.2.1 and the second equality follows from Lemma 4.2.1. Combining (4.7), (4.8), and the fact that $w((0,0)) = 1$, (4.6) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} w((n,0)) + \sum_{n=0}^{\infty} w((n,1)) \quad = \quad & 1 + \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t,u}^{(0)} \frac{r_0}{(1-r_0)^{u+1}} \\ & + \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(0)} \frac{1}{(1-r_0)^{u+1}} \\ & + \sum_{t=0}^{\infty}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \frac{r_1}{(1-r_1)^{u+1}} \\ & + \sum_{t=0}^{\infty}\sum_{u=0}^{t} c_{2t+1,u}^{(1)} \frac{1}{(1-r_1)^{u+1}}. \qquad (4.9)\end{aligned}$$

Finally, dividing $\mathbf{w}$ by (4.9) yields $\mathbf{p}$. Another use of (4.9) is as a convergence criterion when implementing Theorem 4.2.1 numerically. That is, we truncate the sums in (4.9) at $t = M$, where M is smallest positive integer such that

$$\begin{aligned}& \sum_{t=0}^{M}\sum_{u=0}^{t} c_{2t,u}^{(0)} \frac{r_0}{(1-r_0)^{u+1}} + \sum_{t=0}^{M}\sum_{u=0}^{t} c_{2t+1,u}^{(0)} \frac{1}{(1-r_0)^{u+1}} \\ + & \sum_{t=0}^{M}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \frac{r_1}{(1-r_1)^{u+1}} + \sum_{t=0}^{M}\sum_{u=0}^{t} c_{2t+1,u}^{(1)} \frac{1}{(1-r_1)^{u+1}}\end{aligned}$$

differs from

$$\begin{aligned}& \sum_{t=0}^{M-1}\sum_{u=0}^{t} c_{2t,u}^{(0)} \frac{r_0}{(1-r_0)^{u+1}} + \sum_{t=0}^{M-1}\sum_{u=0}^{t} c_{2t+1,u}^{(0)} \frac{1}{(1-r_0)^{u+1}} \\ + & \sum_{t=0}^{M-1}\sum_{u=0}^{t-1} c_{2t,u}^{(1)} \frac{r_1}{(1-r_1)^{u+1}} + \sum_{t=0}^{M-1}\sum_{u=0}^{t} c_{2t+1,u}^{(1)} \frac{1}{(1-r_1)^{u+1}}\end{aligned}$$

by less than some prescribed tolerance. The resulting value of M can be used heuristically as a truncation rule for the infinite sums $w((n,i)) = \sum_{t=0}^{\infty} w_t((n,i))$, i.e., heuristically we compute $w((n,i)) \approx \sum_{t=0}^{M} w_t((n,i))$. Unfortunately, numerical implementation of Theorem 4.2.1 has not yet proven to be effective: see Section 4.4 for a brief discussion on implementation challenges.

4.4 Conclusion

This chapter has provided a new way to compute the stationary distribution of an $M/M/1$ queue in a random environment when the random environment has two states. The main idea is to use the random-product representation of the invariant measure, $\mathbf{w}$, from [9] and condition on the number of environment transitions made as the process journeys from its starting state to state $(0,0)$. By doing so, it is possible to write each element of $\{w_t((n,i))\}_{t\geq 0, n\geq 1, i\in E'}$ as a linear combination of scalars whose coefficients can be determined recursively.

The scalar equations derived here provide an exact representation of $\mathbf{p}$, once the exact representation of $\mathbf{w}$ has been normalized. This is in contrast to the well-known matrix-geometric theory which requires computing the elements of the rate matrix $\mathbf{R}$ which are not known to have a closed form solution in general; instead, $\mathbf{R}$ is represented as the minimal nonnegative solution to a matrix equation. Numerically, however, the method provided by Theorem 4.2.1 has not outperformed the matrix-geometric approach. This is due to the numerical issues that arise when computing the coefficients $\{c^{(0)}_{2t+1,u}\}_{0\leq u\leq t}$ and $\{c^{(1)}_{2t+1,u}\}_{0\leq u\leq t}$ as well as the coefficients $\{c^{(0)}_{2t,u}\}_{0\leq u\leq t}$ and $\{c^{(1)}_{2t,u}\}_{0\leq u\leq t-1}$. As Theorem 4.2.1 shows, the recursion used to compute these coefficients involves subtractions, often of terms that differ greatly in magnitude due to either extremely large or extremely small denominators. This can cause a loss of accuracy which grows more pronounced as t grows larger. We plan to investigate these numerical issues as well as look for other ways to avoid such issues in future work.

Lastly, we comment that while the main result is for environment processes having two states, the same ideas generalize very nicely to environment processes $\{Y(t); t\geq 0\}$ having state space $E = \{0, 1, \ldots, N\}$ for $N \geq 1$, whose embedded DTMC $\{Y_n\}_{n\geq 0}$ satisfies

$$\mathbb{P}_i(Y_1 = j) = \begin{cases} 1, & j = i+1,\ 0 \leq i \leq N-1; \\ 1, & i = N,\ j = 0; \\ 0, & \text{otherwise} \end{cases}, \tag{4.10}$$

Thus, using the ideas of Theorem 4.2.1 to derive $\mathbf{p}$ for $M/M/1$ queues in a random environment that satisfy (4.10) is another avenue for future work. We also plan to adapt the ideas of Theorem 4.2.1 in order to compute $\mathbf{p}$ for other $M/M/1$ queues in a random environment whose environment processes are not forced to satisfy (4.10).

Appendices

Appendix A Limit of Laplace Transforms

Here we give a rather obvious closure property of Laplace transforms under limits, but due to its importance in our arguments in Chapter 2, we formally state the result here, and provide a proof.

Theorem A.1 *Suppose $f_1, f_2, f_3, \dots$ is a collection of nonnegative functions on $[0,\infty)$, where $f_n \leq f_{n+1}$ pointwise on $[0,\infty)$ for each $n \geq 1$. Furthermore, assume the function $f := \lim_{n\to\infty} f_n$ satisfies*

$$\int_0^\infty e^{-\alpha t} f(t)dt < \infty$$

for each $\alpha > 0$. Then the Laplace transform of f on $\mathbb{C}_+$ is given by

$$\int_0^\infty e^{-\alpha t} f(t)dt = \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} f_n(t)dt.$$

Proof The result is trivial for $\alpha > 0$, since in this case it follows as an immediate consequence of the monotone convergence theorem. Suppose now that $\alpha \in \mathbb{C}_+$, and write $\alpha = \alpha_0 + i\alpha_1$, where $\alpha_0 = Re(\alpha)$, and $\alpha_1 = Im(\alpha)$. Then for each $n \geq 1$,

$$\int_0^\infty e^{-\alpha t} f_n(t)dt = \int_0^\infty e^{-\alpha_0 t} \cos(\alpha_1 t) f_n(t)dt + i \int_0^\infty e^{-\alpha_0 t} \sin(\alpha_1 t) f_n(t)dt.$$

Next, define $I_1 = \{t \geq 0 : \cos(\alpha_1 t) \geq 0\}$, and $I_2 = \{t \geq 0 : \sin(\alpha_1 t) \geq 0\}$. Then by the monotone convergence theorem,

$$\lim_{n\to\infty} \int_{I_1} e^{-\alpha_0 t} \cos(\alpha_1 t) f_n(t)dt = \int_{I_1} e^{-\alpha_0 t} \cos(\alpha_1 t) f(t)dt$$

and similarly, by the monotone convergence theorem

$$\lim_{n\to\infty} \int_{I_1^c} e^{-\alpha_0 t} (-\cos(\alpha_1 t)) f_n(t)dt = \int_{I_1^c} e^{-\alpha_0 t} (-\cos(\alpha_1 t)) f(t)dt$$

which further implies

$$\lim_{n\to\infty} \int_{I_1^c} e^{-\alpha_0 t} \cos(\alpha_1 t) f_n(t)dt = \int_{I_1^c} e^{-\alpha_0 t} \cos(\alpha_1 t) f(t)dt.$$

Putting these facts together gives, since the Laplace transform of f is finite for each $\alpha > 0$,

$$\lim_{n\to\infty} \int_0^{\infty} e^{-\alpha_0 t} \cos(\alpha_1 t) f_n(t) dt = \int_0^{\infty} e^{-\alpha_0 t} \cos(\alpha_1 t) f(t) dt$$

and the same logic can be used to prove convergence of the analogous integral containing the sine term: adding these two limits of integrals together verifies our claim. $\diamondsuit$

Appendix B Time-dependent results for the $M/M/1$ queue

This appendix contains a study of the time-dependent behavior of both an ordinary $M/M/1$ queueing model, as well as a modified $M/M/1$ model where state 0 serves as an absorbing state. Namely, let $\{B_1(t); t \geq 0\}$ and $\{B_0(t); t \geq 0\}$ be two CTMPs on a state space $E = \{0, 1, 2, \ldots\}$ having generators $\mathbf{B}_1 := [b_1(j,\ell)]_{j,\ell \geq 0}$ and $\mathbf{B}_0 := [b(j,\ell)]_{j,\ell \geq 0}$, respectively, where for $j, \ell \in E$, $j \neq \ell$,

$$b_1(j,\ell) = \begin{cases} \lambda, & j \geq 0,\ \ell = j+1; \\ \mu, & j \geq 1,\ \ell = j-1; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$b_0(j,\ell) = \begin{cases} \lambda, & j \geq 1,\ \ell = j+1; \\ \mu, & j \geq 1,\ \ell = j-1; \\ 0, & \text{otherwise.} \end{cases}$$

Both of these CTMPs can be interpreted as follows: for each $t \geq 0$, $B(t)$ represents the number of customers present at time t in an ordinary $M/M/1$ queue having arrival rate λ, service rate μ, while $B_0(t)$ instead represents the number of customers present at time t in a modified $M/M/1$ queue with arrival rate λ and service rate μ, in that whenever the queue empties, it stays empty (meaning that if it is empty at time zero, then it remains empty for all time).

Let $\{p^{(1)}_{j,\ell}\}_{j,\ell \in E}$ and $\{p^{(0)}_{j,\ell}\}_{j,\ell \in E}$ represent the collections of transition functions associated with $\{B_1(t); t \geq 0\}$ and $\{B_0(t); t \geq 0\}$, respectively. Our goal in this section is to compute the Laplace transforms $\{\pi^{(1)}_{j,\ell}\}_{j,\ell \in E}$ and $\{\pi^{(0)}_{j,\ell}\}_{j,\ell \in E}$ associated with these transitions functions, which are defined as follows: for $j, \ell \in E$, $\alpha \in \mathbb{C}_+$,

$$\pi^{(1)}_{j,\ell}(\alpha) = \int_0^\infty e^{-\alpha t} p^{(1)}_{j,\ell}(t) dt, \qquad \pi^{(0)}_{j,\ell}(\alpha) = \int_0^\infty e^{-\alpha t} p^{(0)}_{j,\ell}(t) dt.$$

Our next theorem gives explicit expressions for each of these transforms: these expressions are known, but to make the presentation easier to follow we will both formally state and derive them. We will find that these expressions can be expressed solely in terms of the traffic intensity

$\rho := \lambda/\mu$, and the Laplace-Stieltjes transform ϕ of the busy period of B_1, where

$$\phi(\alpha) = \frac{\alpha + \lambda + \mu - \sqrt{(\alpha + \lambda + \mu)^2 - 4\lambda\mu}}{2\lambda}.$$

Theorem B.1 *The Laplace transforms* $\{\pi_{j,\ell}^{(1)}\}_{j,\ell \geq 0}$ *and* $\{\pi_{j,\ell}^{(1)}\}_{j \geq 1, \ell \geq 0}$ *are as follows:*

(1) For each $j \geq 0$*, we have for* $0 \leq \ell \leq j$*,*

$$\pi_{j,\ell}^{(1)}(\alpha) = \Gamma(\alpha)(\rho\phi(\alpha))^\ell \phi(\alpha)^j + \Omega(\alpha)\phi(\alpha)^{j-\ell}(1 - (\rho\phi(\alpha)^2)^\ell)$$

and for $\ell \geq j$*,*

$$\pi_{j,\ell}^{(1)}(\alpha) = \Gamma(\alpha)(\rho\phi(\alpha))^\ell \phi(\alpha)^j + \Omega(\alpha)(\rho\phi(\alpha))^{\ell-j}(1 - (\rho\phi(\alpha)^2)^j).$$

(2) For each $j \geq 0$*, we have* $\pi_{j,0}^{(0)}(\alpha) = \phi(\alpha)^j/\alpha$*: furthermore, for* $1 \leq \ell \leq j$*,*

$$\pi_{j,\ell}^{(0)}(\alpha) = \Omega(\alpha)\phi(\alpha)^{j-\ell}(1 - (\rho\phi(\alpha)^2)^\ell)$$

and for $\ell \geq j$*,*

$$\pi_{j,\ell}^{(0)}(\alpha) = \Omega(\alpha)(\rho\phi(\alpha))^{\ell-j}(1 - (\rho\phi(\alpha)^2)^j).$$

Proof We first begin by deriving all Laplace transforms $\pi_{0,\ell}^{(1)}$, for $\ell \geq 0$. Using Theorem 2.1 of [13], it is easy to see that for $Re(\alpha) > 0$,

$$\pi_{0,\ell}^{(1)}(\alpha) = \pi_{0,0}(\alpha)\mathbb{E}_\ell \left[e^{-\alpha\tau_0} \prod_{l=1}^{\eta_0} \frac{b(B_l^{(1)}, B_{l-1}^{(1)})}{b(B_{l-1}^{(1)}, B_l^{(1)})} \right] = \pi_{0,0}(\alpha)(\rho\phi(\alpha))^\ell.$$

Summing over $\ell \geq 0$ then gives

$$\frac{1}{\alpha} = \sum_{\ell=0}^{\infty} \pi_{0,0}^{(1)}(\alpha) = \pi_{0,0}^{(1)}(\alpha) \sum_{\ell=0}^{\infty} (\rho\phi(\alpha))^\ell = \frac{\pi_{0,0}^{(1)}(\alpha)}{1 - \rho\phi(\alpha)}$$

or equivalently,

$$\pi_{0,0}^{(1)}(\alpha) = \frac{1 - \rho\phi(\alpha)}{\alpha}$$

so for each $\ell \geq 0$,

$$\pi^{(1)}_{0,\ell}(\alpha) = \frac{(1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell}}{\alpha}. \tag{11}$$

The next step is to derive $\pi^{(1)}_{j,\ell}(\alpha)$, $\alpha > 0$, for each $j \geq 1$, $\ell \geq 0$. Letting e_α denote an exponential random variable with rate α, independent of $\{B^1(t); t \geq 0\}$, we easily see that for $j \geq 0$, $\ell \geq 0$,

$$\alpha\pi_{j,\ell}(\alpha) = \mathbb{P}_j(B_1(e_\alpha) = \ell).$$

One way to simplify this probability is to make use of the factorization results discussed in [14]: there it was shown that under each probability measure $\mathbb{P}_j$, the random variables

$$\inf_{0\leq s\leq e_\alpha} B_1(s), \qquad B_1(e_\alpha) - \inf_{0\leq s\leq e_\alpha} B_1(s)$$

are independent, and furthermore $B_1(e_\alpha) - \inf_{0\leq s\leq e_\alpha} B_1(s)$ has the same law as the conditional law of $B_1(e_\alpha)$ given $B_1(0) = 0$.

For $j \geq 1$, $0 \leq \ell \leq j$, we have

$$\begin{aligned}
\mathbb{P}_j(B_1(e_\alpha) = \ell) &= \mathbb{P}_j(B_1(e_\alpha) = \ell \mid \inf_{0\leq s\leq e_\alpha} B(s) = 0)\mathbb{P}_j(\inf_{0\leq s\leq e_\alpha} B(s) = 0) \\
&+ \sum_{k=1}^{\ell} \mathbb{P}_j(B_1(e_\alpha) = \ell \mid \inf_{0\leq s\leq e_\alpha} B(s) = k)\mathbb{P}_j(\inf_{0\leq s\leq e_\alpha} B_1(s) = k) \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell}\phi(\alpha)^j + \sum_{k=1}^{\ell}(1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell-k}(1-\phi(\alpha))\phi(\alpha)^{j-k} \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell}\phi(\alpha)^j + (1-\rho\phi(\alpha))(1-\phi(\alpha))\phi(\alpha)^{j-\ell}\sum_{k=1}^{\ell}(\rho\phi(\alpha)^2)^{\ell-k} \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell}\phi(\alpha)^j + \left[\frac{(1-\rho\phi(\alpha))(1-\phi(\alpha))}{1-\rho\phi(\alpha)^2}\right]\phi(\alpha)^{j-\ell}(1-(\rho\phi(\alpha)^2)^{\ell}).
\end{aligned}$$

Observe further that since the Laplace-Stieltjes transform ϕ satisfies

$$\lambda\phi(\alpha)^2 - (\lambda+\mu+\alpha)\phi(\alpha) + \mu = 0$$

we can also state that

$$\frac{\alpha}{\mu}\phi(\alpha) = 1-(1+\rho)\phi(\alpha)+\rho\phi(\alpha)^2 = (1-\rho\phi(\alpha))(1-\phi(\alpha))$$

meaning

$$\frac{(1-\rho\phi(\alpha))(1-\phi(\alpha))}{1-\rho\phi(\alpha)^2} = \frac{\alpha\phi(\alpha)}{\mu(1-\rho\phi(\alpha)^2)} = \frac{\alpha\rho\phi(\alpha)}{\lambda(1-\rho\phi(\alpha)^2)}$$

so that if we define

$$\Omega(\alpha) := \frac{\rho\phi(\alpha)}{\lambda(1-\rho\phi(\alpha)^2)}, \qquad \Gamma(\alpha) := \frac{1-\rho\phi(\alpha)}{\alpha}$$

then

$$\mathbb{P}_j(B_1(e_\alpha)=\ell) = \alpha\Gamma(\alpha)(\rho\phi(\alpha))^\ell\phi(\alpha)^j + \alpha\Omega(\alpha)\phi(\alpha)^{j-\ell}(1-(\rho\phi(\alpha)^2)^\ell).$$

or, equivalently,

$$\pi^{(1)}_{j,\ell}(\alpha) = \Gamma(\alpha)(\rho\phi(\alpha))^\ell\phi(\alpha)^j + \Omega(\alpha)\phi(\alpha)^{j-\ell}(1-(\rho\phi(\alpha)^2)^\ell).$$

A similar formula can be derived for the case where $j \geq 1$, $\ell \geq j$: here

$$\begin{aligned}
\mathbb{P}_j(B_1(e_\alpha)=\ell) &= \mathbb{P}_j(B_1(e_\alpha)=\ell \mid \inf_{0\leq s\leq e_\alpha} B(s)=0)\mathbb{P}_j(\inf_{0\leq s\leq e_\alpha} B(s)=0) \\
&+ \sum_{k=1}^{j}\mathbb{P}_i(B_1(e_\alpha)=\ell \mid \inf_{0\leq s\leq e_\alpha} B(s)=k)\mathbb{P}_j(\inf_{0\leq s\leq e_\alpha} B_1(s)=k) \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^\ell\phi(\alpha)^j + \sum_{k=1}^{j}(1-\rho\phi(\alpha))(\rho\phi(\alpha))^{\ell-k}(1-\phi(\alpha))\phi(\alpha)^{j-k} \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^\ell\phi(\alpha)^j + (1-\rho\phi(\alpha))(1-\phi(\alpha))(\rho\phi(\alpha))^{\ell-j}\sum_{k=1}^{j}(\rho\phi(\alpha)^2)^{j-k} \\
&= (1-\rho\phi(\alpha))(\rho\phi(\alpha))^\ell\phi(\alpha)^j + \left[\frac{(1-\rho\phi(\alpha))(1-\phi(\alpha))}{1-\rho\phi(\alpha)^2}\right](\rho\phi(\alpha))^{\ell-j}(1-(\rho\phi(\alpha)^2)^j).
\end{aligned}$$

meaning

$$\pi^{(1)}_{j,\ell}(\alpha) = \Gamma(\alpha)(\rho\phi(\alpha))^{\ell}\phi(\alpha)^j + \Omega(\alpha)(\rho\phi(\alpha))^{\ell-j}(1-(\rho\phi(\alpha)^2)^j).$$

The same technique can be used to compute each $\pi^{(0)}_{j,\ell}$ transform: first notice that

$$\mathbb{P}_j(B_0(e_\alpha)=0)=\phi(\alpha)^j$$

since once B_0 reaches state 0, it stays there forever. Hence,

$$\pi^{(0)}_{j,0}(\alpha)=\frac{\phi(\alpha)^j}{\alpha}.$$

Furthermore, for $j \geq 1$, $1 \leq \ell \leq j$,

$$\pi^{(0)}_{j,\ell}(\alpha) = \Omega(\alpha)\phi(\alpha)^{j-\ell}(1-(\rho\phi(\alpha)^2)^{\ell})$$

and for $j \geq 1$, $\ell \geq j$,

$$\pi^{(0)}_{j,\ell}(\alpha) = \Omega(\alpha)(\rho\phi(\alpha))^{\ell-j}(1-(\rho\phi(\alpha)^2)^j).$$

$\diamondsuit$

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