5-2012

Enhanced Physics Schemes for the 2D NS-alpha Models of Incompressible Flow

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A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master's
Mathematical Science

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by
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May 2012

Accepted by:
Dr. Leo Rebholz, Committee Chair
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Abstract

In this thesis, we study algorithms for the 2D NS-α model of incompressible flow. These schemes conserve both discrete energy and discrete enstrophy in the absence of viscous and external forces, and otherwise admit exact balances for them analogous to those of true fluid flow. This model belongs to a very small group that conserves both of these quantities in the continuous case, and in this work, we develop finite element algorithms for the vorticity-stream formulation of this model that will preserve numerical energy and enstrophy in the computed solutions.

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Chapter 1

Introduction

This thesis studies a numerical scheme for approximating solutions to the vorticity-stream formulation of the Navier-Stokes Equations (NSE). In this work, we show that the proposed scheme conserves model energy and enstrophy and has optimal convergence for higher-order polynomials. Because the models we study have the same structure as the Barotropic Vorticity Equations (BVE), we anticipate the results shown in this work to aid our future study of the BVE. Currently, the BVE are an area of active research, and it is hoped they will provide insight into the complex behavior of oceanic and atmospheric currents. This work is a test of deconvolution modeling techniques on these equations that is conducted in the simplified setting of periodic boundary conditions.

For 2D flows, energy and enstrophy are believed to be fundamental to the organization of large structures, and thus predicting these conserved quantities correctly is paramount for many problems. The first schemes for the NSE to enforce this dual conservation were designed by Arakawa [1] for finite difference methods, and by G. Fix [6] for finite element methods. It was found that the enhanced physical accuracy of these methods (compared to energy-only conserving schemes) led to longer time stability and accuracy. However, many flow schemes do not conserve energy (they dissipate it, even in the absence of viscosity), and almost all do not conserve enstrophy. Thus, any such method could be flawed due to a lack of *physical* accuracy. Herein, we study a numerical scheme for the NS-α deconvolution model of the BVE, which conserves both energy and enstrophy.

1.1 The NS-α model

NS-α belongs to a special class of fluid flow models that preserve both energy and enstrophy [7, 10]. In addition, NS-α is well-posed [8, 12] and cascades energy at the same rate as the NSE up to a filtering-radius dependent lengthscale after which it dissipates energy more quickly [7]. The NS-α model can be derived in several ways, the simplest of which is regularizing part of the NSE nonlinearity with the Helmholtz filter. The NS-α model is given by

$$v_t + (\nabla \times v) \times \overline{v} + \nabla P - \nu \Delta v = f, \tag{1.1.1}$$

$$\nabla \cdot \overline{v} = 0, \tag{1.1.2}$$

$$-\alpha^2 \Delta \overline{v} + \overline{v} - v = 0. \tag{1.1.3}$$

Using notation consistent with the literature, v denotes the velocity, $\overline{v}$ denotes the filtered velocity, f denotes the body force, and ν denotes the kinematic viscosity.

1.2 2D Vorticity-stream NS-α deconvolution formulation

In this work, we examine the vorticity-stream formulation of the NS-α model and prove that it preserves model energy and enstrophy in the continuous case and when discretized using Crank-Nicolson. We then apply Van Cittert deconvolution to the filtered vorticity and show that model energy and enstrophy are still conserved both continuously and discretely. To obtain the vorticity-stream deconvolution formulation of NS-α, we curl (1.1.1), which eliminates the pressure term and reduces (1.1.1)-(1.1.3) to the following system of *scalar* equations, where $\hat{z}$ denotes the vector $[0, 0, 1]^T$:

$$w_t - (\hat{z} \times \nabla\phi) \cdot \nabla w - \nu \Delta w = (\nabla \times f) \cdot \hat{z}, \tag{1.2.1}$$

$$-D_N \overline{w} - \Delta\phi = 0, \tag{1.2.2}$$

$$-\alpha^2 \Delta \overline{w} + \overline{w} - w = 0. \tag{1.2.3}$$

We define the streamfunction ϕ by the solution to $D_N \overline{v} = (\phi_y, -\phi_x)$ and the vorticity as $w := \nabla \times v$. Note $D_N \overline{v} = -(\hat{z} \times \nabla\phi)$. We define the filtered value $\overline{\phi}$ by $(-\alpha^2\Delta + I)^{-1}\phi$, and for ease of notation, we define $F := (-\alpha^2\Delta + I)^{-1}$. $D_N := \sum_{n=0}^{N} (I - F)^n$ denotes the Nth-order Van Cittert approximate deconvolution operator.

This thesis is arranged as follows. Chapter two provides the notation and preliminary results used throughout this paper. In chapter three, we state and prove the conservation laws mentioned earlier in this introduction. Chapter four gives the statement and proof of our convergence theorem, and in chapter five, we provide numerical experiments lending support to these results. We conclude the thesis with chapter six.

Chapter 2

Notation and Preliminaries

Throughout this paper, the domain Ω will represent the periodic box $(0, L)^2$. While periodic boundary conditions are typically unrealistic, studying conservation laws in this setting is still relevant because the laws must hold even with the nonlinearity present. In addition, the periodic case provides us with a starting point from which we can expand into more complicated settings.

Definition 2.0.1. *Define the H^1 and L^2 spaces in the periodic case as follows:*

$$H^1_{\#}(\Omega) = \left(\phi \in H^1(\Omega) \mid \phi(x + Le_i) = \phi(x), \int_\Omega \phi = 0\right),$$

$$L^2_{\#}(\Omega) = \left(q \in L^2(\Omega) \mid q(x + Le_i) = q(x), \int_\Omega q = 0\right).$$

Definition 2.0.2. *Define the discrete velocity space as below, given a regular mesh $\tau_h(\Omega)$:*

$$X^h_{\#}(\Omega) = \left(\phi_h \in H^1_{\#}(\Omega) \mid \phi_h \in P_k(e) \cap C^0(\Omega) \quad \forall e \in \tau_h\right).$$

Definition 2.0.3. *Given $\phi \in L^2_{\#}(\Omega)$ with filtering radius α, define $\overline{\phi}$ to be the unique L-periodic solution of*

$$-\alpha^2 \Delta \overline{\phi} + \overline{\phi} = \phi.$$

To simplify later notation, we also denote filtering by $F := (-\alpha^2 \Delta + I)^{-1}$. Observe that F is self-adjoint in the L^2 inner product [5].

Definition 2.0.4. *Define the Van Cittert approximate deconvolution operator D_N for a fixed finite N by*

$$D_N\phi = \sum_{n=0}^{N}(I-F)^n\phi.$$

Note that D_N is self-adjoint since F is. Moreover, D_N is positive and $1 \le \|D_N\| \le N+1$ [5].

Definition 2.0.5. *Define the energy norm (which is natural for the models we study herein) to be*

$$\|v\|_{E,N}^2 := (v, D_N\overline{v}).$$

When, $N = 0$, we will denote

$$\|v\|_{E,\alpha}^2 := (v, \overline{v}) = \|\overline{v}\|^2 + \alpha^2\|\nabla\overline{v}\|^2.$$

Lemma 2.0.1. *Define the discrete Laplacian $\Delta_h : H^1_\# \to X^h_\#$ as follows: Given $\phi \in H^1_\#, \Delta_h\phi$ is the unique solution in $X^h_\#$ to $(\Delta_h\phi, v_h) = -(\nabla\phi, \nabla v_h)$ for all $v_h \in X^h_\#$.*

Proof. It is easy to verify that the discrete Laplacian is well-defined. Also, when we restrict the domain to $X^h_\#$, Δ_h is self-adjoint on $X^h_\#$ since for $\phi_h, v_h \in X^h_\#$,

$$(\Delta_h\phi_h, v_h) = -(\nabla\phi_h, \nabla v_h) = (\phi_h, \Delta_h v_h).$$

Finally, Δ_h is negative definite, and hence Δ_h has an inverse. □

Definition 2.0.6. *Define the discrete filtering operator as follows: Given $\phi \in L^2_\#(\Omega)$ and $\alpha > 0, \overline{\phi}^h$ is the unique solution in $X^h_\#$ to*

$$(\phi, v) = (\overline{\phi}^h, v) - \alpha^2(\Delta_h\overline{\phi}^h, v) \quad \forall v \in X^h_\#.$$

It is easy to check that the discrete filtering operation is well defined. Moreover, the discrete filter is self-adjoint under periodic boundary conditions since the discrete Laplacian is self-adjoint under periodic boundary conditions. To simplify later notation, we denote discrete filtering by the operator F^h. Again, observe that F^h is self-adjoint in the L^2 inner product.

Definition 2.0.7. *Define the discrete approximate deconvolution operator D_N^h for a fixed finite N by*

$$D_N^h \phi = \sum_{n=0}^{N} (I - F^h)^n \phi.$$

Note that D_N^h is self-adjoint and positive [10].

Lemma 2.0.2. *$\| \cdot \|_{E_h,N}$ defined by $\|v\|_{E_h,N} = \sqrt{(v, D_N^h \overline{v}^h)}$ is a norm on Ω as is $\| \cdot \|_{E_h,\alpha}$ defined by $\|v\|_{E_h,\alpha} = \sqrt{(v, \overline{v}^h)}$.*

Proof. See [13]. □

Chapter 3

Conservation Laws and Stability

In this chapter, we prove conservation laws for the vorticity-stream formulation of NS-α and then prove analogous conservation laws for the schemes we propose. Stability readily follows.

Lemma 3.0.3. *Provided sufficiently smooth data and under periodic boundary conditions, solutions to the vorticity stream formulation of NS-*α (1.2.1)-(1.2.3) *satisfy the following conservation laws:*

NS-α energy:

$$\frac{1}{2}(\alpha^2\|\overline{w}(t)\|^2+\|\nabla\phi(t)\|^2)+\nu\int_0^t \|w(t')\|_{E,\alpha}^2\,dt' = \int_0^t ((\nabla\times f(t'))\cdot\hat{z},\phi(t'))\,dt'+\frac{1}{2}\left(\alpha^2\|\overline{w_0}\|^2+\|\nabla\phi_0\|^2\right)$$

NS-α enstrophy:

$$\frac{1}{2}\|w(t)\|^2+\nu\int_0^t \|\nabla w(t')\|\,dt' = \frac{1}{2}\|w_0\|^2+\int_0^t ((\nabla\times f(t'))\cdot\hat{z},w(t'))\,dt'.$$

Proof. For NS-α energy, multiply (1.2.1) by ϕ and integrate over the domain. Note the nonlinear term vanishes since $\nabla\cdot(\hat{z}\times\nabla\phi)=0$ in each element, and thus

$$((\nabla\phi\times\hat{z})\cdot\nabla w,\phi) = -((\nabla\phi\times\hat{z})\cdot\nabla\phi,w) = 0$$

since $\nabla\phi$ is perpendicular to $(\nabla\phi\times\hat{z})$. Thus, we have

$$(w_t,\phi)-\nu(\Delta w,\phi) = ((\nabla\times f)\cdot\hat{z},\phi). \tag{3.0.1}$$

For the first term, we use (1.2.2), (1.2.3), and integration by parts to get

$$\begin{aligned}(w_t, \phi) &= -\alpha^2 \left(\frac{d}{dt}\Delta\overline{w}, \phi\right) + \left(\frac{d}{dt}\overline{w}, \phi\right) \\ &= \alpha^2 \left(\frac{d}{dt}\Delta\Delta\phi, \phi\right) - \left(\frac{d}{dt}\Delta\phi, \phi\right) \\ &= \frac{\alpha^2}{2}\frac{d}{dt}\|\Delta\phi\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\phi\|^2. \end{aligned} \tag{3.0.2}$$

For the viscous term in (3.0.1), we use (1.2.2) to obtain

$$-\nu(\Delta w, \phi) = -\nu(w, \Delta\phi) = \nu(w, \overline{w}) = \nu\|w\|_{E,\alpha}^2. \tag{3.0.3}$$

Now, combining (3.0.1)-(3.0.3) gives

$$\frac{1}{2}\frac{d}{dt}(\alpha^2\|\overline{w}(t)\|^2 + \|\nabla\phi(t)\|^2) + \nu\|w\|_{E,\alpha} = ((\nabla \times f) \cdot \hat{z}, \phi).$$

Integrating over time gives the result.

To prove the NS-α enstrophy result, multiplying (1.2.1) by w, integrating over space, applying Green's theorem to $\nu\Delta w$, then integrating over time immediately gives the result. Note that because we are in the periodic case, all boundary integrals vanish. □

3.1 Background for the NS-α deconvolution model and derivation of vorticity-stream formulations in the deconvolution case.

We now define the NS-α deconvolution model, give a brief background, and derive its vorticity-stream formulations; the enhanced-physics based numerical schemes for the model will be based off of this formulation. As given by Rebholz [14], the deconvolution NS-α model is given by

$$v_t + (\nabla \times v) \times D_N\overline{v} + \nabla P - \nu\Delta v = f, \tag{3.1.1}$$

$$\nabla \cdot D_N\overline{v} = 0, \tag{3.1.2}$$

$$-\alpha^2\Delta\overline{v} + \overline{v} - v = 0. \tag{3.1.3}$$

This model is part of a class of models known as the deconvolution α-models, which can be derived by regularizing all or part of the NSE nonlinearity with the Helmholtz filter, as we did for NS-α, and then applying the Van Cittert deconvolution operator D_N to the filtered value. By doing so, it was shown in [14] that significantly better accuracy could be obtained compared to NS-α. Moreover, these models are known to be well-posed under periodic boundary conditions [15], obey fundamental conservation laws (such as energy and enstrophy) [14], cascade energy at the same rate as the NSE up to a filtering-radius dependent lengthscale [15], produce accurate energy dissipation rates [11], and have many other desirable properties such as frame invariance [15].

We now show that model energy and enstrophy are conserved for vorticity-stream formulations in the NS-α deconvolution case.

3.1.1 Vorticity-stream formulations in the deconvolution case

Applying the same procedure used to obtain the vorticity-stream equation for NS-α, we obtain the NS-α deconvolution model.

We again define $w := \nabla \times v$ and define a stream function ϕ as the solution to the equation $D_N\overline{v} = (\phi_y, -\phi_x)$. Note $D_N\overline{v} = -(\hat{z} \times \nabla\phi)$ and $-D_N\overline{w} = \Delta\phi$. We now have the vorticity stream formulation of the NS-α deconvolution model:

$$w_t - (\hat{z} \times \nabla\phi) \cdot \nabla w - \nu\Delta w = (\nabla \times f) \cdot \hat{z}, \tag{3.1.4}$$

$$-D_N\overline{w} - \Delta\phi = 0, \tag{3.1.5}$$

$$-\alpha^2\Delta\overline{w} + \overline{w} - w = 0. \tag{3.1.6}$$

Next, we derive the energy and enstrophy balances for the NS-α deconvolution model, again assuming sufficiently smooth data and periodic boundary conditions on a rectangular domain. As expected, we find that energy and enstrophy are conserved in terms of the stream and vorticity functions in the deconvolution case.

Lemma 3.1.1. *Provided sufficiently smooth data and under periodic boundary conditions, solutions to the vorticity stream formulations of NS-α deconvolution* (3.1.4)-(3.1.6) *satisfy the following conservation laws:*

NS-α deconvolution energy:

$$\frac{1}{2}\left(\alpha^2\|D_N^{1/2}\overline{w}(t)\|^2 + \|D_N^{-1/2}\nabla\phi\|^2\right) + \nu\int_0^t \|w(t')\|_{E,N}\ dt' = \int_0^t ((\nabla\times f(t'))\cdot\hat{z}, \phi(t'))\ dt' + \frac{1}{2}\left(\alpha^2\|D_N^{1/2}\overline{w_0}\|^2 + \|D_N^{-1/2}\nabla\phi_0\|^2\right),$$

NS-α deconvolution enstrophy (identical to NS-α enstrophy):

$$\frac{1}{2}\|w(t)\|^2 + \nu\int_0^t \|\nabla w(t')\|\ dt' = \frac{1}{2}\|w_0\|^2 + \int_0^t ((\nabla\times f(t'))\cdot\hat{z}, w(t'))\ dt'.$$

Proof. For NS-α deconvolution energy, multiply (3.1.4) by ϕ as before and integrate over the domain. Again, the nonlinear term vanishes, and we have

$$(w_t,\phi) - \nu(\Delta w,\phi) = ((\nabla\times f)\cdot\hat{z},\phi). \tag{3.1.7}$$

For the first term, we use (3.1.5), (3.1.6), integration by parts, and the fact that differential operators commute under periodic boundary conditions to get

$$\begin{aligned}(w_t,\phi) &= -\alpha^2\left(\frac{d}{dt}\Delta\overline{w},\phi\right) + \left(\frac{d}{dt}\overline{w},\phi\right)\\ &= -\alpha^2\left(\frac{d}{dt}\overline{w},\Delta\phi\right) - \left(\frac{d}{dt}D_N^{-1}\Delta\phi,\phi\right)\\ &= \alpha^2\left(\frac{d}{dt}\overline{w},D_N\overline{w}\right) + \left(\frac{d}{dt}\nabla D_N^{-1/2}\phi,\nabla D_N^{-1/2}\phi\right)\\ &= \frac{\alpha^2}{2}\frac{d}{dt}\|D_N^{1/2}\overline{w}\|^2 + \frac{1}{2}\frac{d}{dt}\|D_N^{-1/2}\nabla\phi\|^2. \end{aligned}\tag{3.1.8}$$

For the viscous term in (3.1.7), we can use (3.1.5) to get

$$-\nu(\Delta w,\phi) = -\nu(w,\Delta\phi) = \nu(w,D_N\overline{w}) = \nu\|w\|_{E,N}^2. \tag{3.1.9}$$

Combining (3.1.7)-(3.1.9) gives

$$\frac{\alpha^2}{2}\frac{d}{dt}\|D_N^{1/2}\overline{w}\|^2 + \frac{1}{2}\frac{d}{dt}\|D_N^{-1/2}\nabla\phi\|^2 + \nu\|w\|_{E,N}^2 = ((\nabla\times f)\cdot\hat{z},\phi).$$

Integrating over time gives the result.

The proof of the NS-α deconvolution enstrophy result is identical to the proof of NS-α enstrophy.

$\square$

3.2 Conservation laws for NS-α in the discrete case

After discretizing with Crank-Nicolson, we obtain the following discrete scheme for NS-α. Let Δt denote the timestep, $t^n = n\Delta t$, $t^{n+\frac{1}{2}} = (n+\frac{1}{2})\Delta t$, and u_h^n represent the approximation to $u(x,t^n)$ on the appropriate finite element space. We will denote $\frac{1}{2}(u_h^{n+1}+u_h^n)$ by $u_h^{n+\frac{1}{2}}$ and $f(t^{n+\frac{1}{2}},x)$ by $f^{n+\frac{1}{2}}$.

NS-α : Find $(\phi_h^n, w_h^n, \overline{w_h^n}^h) \in (X_\#^h, X_\#^h, X_\#^h)$ such that for all $(v_h, \chi_h, \psi_h) \in (X_\#^h, X_\#^h, X_\#^h)$, the following equations hold:

$$\begin{aligned} \frac{1}{\Delta t}(w_h^{n+1} - w_h^n, v_h) - ((\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}})\cdot\nabla w_h^{n+\frac{1}{2}}, v_h) & \\ +\ \nu(\nabla w_h^{n+\frac{1}{2}}, \nabla v_h) &= ((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z}, v_h), \qquad (3.2.1) \\ -(\overline{w_h^{n+1}}^h, \chi_h) + (\nabla\phi_h^{n+1}, \nabla\chi_h) &= 0, \qquad (3.2.2) \\ \alpha^2(\nabla\overline{w_h^{n+1}}^h, \nabla\psi_h) + (\overline{w_h^{n+1}}^h, \psi_h) - (w_h^{n+1}, \psi_h) &= 0. \qquad (3.2.3) \end{aligned}$$

Given $w_h^0 \in X_\#^h$, define $\phi_h^0 \in X_\#^h$ to be the unique (after enforcing mean zero) solution of (3.2.2). Since w_h^n is given at a particular time step n, we iterate through the de-coupled system by first filtering using (3.2.3), then solving for ϕ_h^n using (3.2.2) to obtain an initial guess for (3.2.1). Then, we solve (3.2.1) to obtain an approximation for w_h^{n+1}. We repeat this process using a linear iteration scheme for the non-linearity until the solutions are within a 10^{-10} tolerance of each other.

Lemma 3.2.1. *Provided sufficiently smooth data and under periodic boundary conditions, solutions to the vorticity stream formulations of NS-α* (3.2.1)-(3.2.3) *satisfy the following conservation laws: NS-α energy:*

$$\frac{1}{2}\left(\alpha^2\|\overline{w_h^M}^h\|^2 + \|\nabla\phi_h^M\|^2\right) + \Delta t\sum_{n=0}^{M-1}\nu\|w_h^{n+\frac{1}{2}}\|_{E_h,\alpha}^2 =$$
$$\frac{1}{2}\left(\alpha^2\|\overline{w_h^0}^h\|^2 + \|\nabla\phi_h^0\|^2\right) + \Delta t\sum_{n=0}^{M-1}((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z}, \phi_h^{n+\frac{1}{2}}),$$

NS-α enstrophy:

$$\frac{1}{2}\|w_h^M\|^2 + \Delta t \sum_{n=0}^{M-1} \nu \|\nabla w_h^{n+\frac{1}{2}}\|^2 = \frac{1}{2}\|w_h^0\|^2 + \Delta t \sum_{n=0}^{M-1} ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, w_h^{n+\frac{1}{2}}).$$

Proof. To prove NS-α energy, choose v_h in (3.2.1) to be $\phi_h^{n+\frac{1}{2}}$. Since $\nabla \cdot (\hat{z} \times \nabla \phi_h^n) = 0$ on each triangle, and ϕ_h^n is comprised of continuous piecewise polynomials, $\nabla \cdot (\hat{z} \times \nabla \phi_h^n) = 0$ on Ω. So,

$$((\hat{z} \times \nabla \phi_h^{n+\frac{1}{2}}) \cdot \nabla w_h^{n+\frac{1}{2}}, \phi_h^{n+\frac{1}{2}}) \quad = \quad ((\hat{z} \times \nabla \phi_h^{n+\frac{1}{2}}) \cdot \nabla \phi_h^{n+\frac{1}{2}}, w_h^{n+\frac{1}{2}}) = 0. \tag{3.2.4}$$

Hence, the non-linear term vanishes, and we have

$$\frac{1}{\Delta t}(w_h^{n+1} - w_h^n, \frac{\phi_h^{n+1} + \phi_h^n}{2}) + \nu(\nabla w_h^{n+\frac{1}{2}}, \nabla \phi_h^{n+\frac{1}{2}}) \quad = \quad ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, \phi_h^{n+\frac{1}{2}}). \tag{3.2.5}$$

Coupling the definition of Δ_h with (3.2.2), we see that $(\Delta_h \phi_h^{n+1}, v_h) = -(\overline{w_h^{n+1}}^h, v_h) \forall v_h \in X_h^{\#}$. So, $\Delta_h \phi_h^{n+1} = -\overline{w_h^{n+1}}^h$. For the first term, we use (3.2.2) and (3.2.3) to obtain:

$$\begin{aligned}
\frac{1}{\Delta t}\left(w_h^{n+1} - w_h^n, \frac{\phi_h^{n+1} + \phi_h^n}{2}\right) &= \frac{1}{2\Delta t}(w_h^{n+1}, \phi_h^{n+1}) - \frac{1}{2\Delta t}(w_h^n, \phi_h^n) \\
&= \frac{1}{2\Delta t}\left(\alpha^2(\nabla \overline{w_h^{n+1}}^h, \nabla \phi_h^{n+1}) + (\overline{w_h^{n+1}}^h, \phi_h^{n+1})\right) - \frac{1}{2\Delta t}\left(\alpha^2(\nabla \overline{w_h^n}^h, \nabla \phi_h^n) + (\overline{w_h^n}^h, \phi_h^n)\right) \\
&= \frac{1}{2\Delta t}\left(-\alpha^2(\overline{w_h^{n+1}}^h, \Delta_h \phi_h^{n+1}) + (\overline{w_h^{n+1}}^h, \phi_h^{n+1})\right) - \frac{1}{2\Delta t}\left(-\alpha^2(\overline{w_h^n}^h, \Delta_h \phi_h^n) + (\overline{w_h^n}^h, \phi_h^n)\right) \\
&= \frac{1}{2\Delta t}\left(\alpha^2(\overline{w_h^{n+1}}^h, \overline{w_h^{n+1}}^h) + (\nabla \phi_h^{n+1}, \nabla \phi_h^{n+1})\right) - \frac{1}{2\Delta t}\left(\alpha^2(\overline{w_h^n}^h, \overline{w_h^n}^h) + (\nabla \phi_h^n, \nabla \phi_h^n)\right) \\
&= \frac{1}{2\Delta t}\left(\alpha^2\|\overline{w_h^{n+1}}^h\|^2 + \|\nabla \phi_h^{n+1}\|^2\right) - \frac{1}{2\Delta t}\left(\alpha^2\|\overline{w_h^n}^h\|^2 + \|\nabla \phi_h^n\|^2\right).
\end{aligned}$$

Note that the cross terms in the first term of the above string of inequalities disappear by the following argument:

$$
\begin{aligned}
(w_h^n, \phi_h^{n+1}) &= (w_h^n, \Delta_h^{-1}\Delta_h \phi_h^{n+1}) \\
&= -(w_h^n, \Delta_h^{-1}\overline{w_h^{n+1}}^h) \\
&= -(\Delta_h^{-1} w_h^n, \overline{w_h^{n+1}}^h) \\
&= -(\Delta_h^{-1}\overline{w_h^n}^h, w_h^{n+1}) \\
&= (\Delta_h^{-1}\Delta_h \phi_h^n, w_h^{n+1}) \\
&= (\phi_h^n, w_h^{n+1}).
\end{aligned}
\tag{3.2.6}
$$

We are guaranteed the existence of Δ_h^{-1} since Δ_h is self-adjoint and negative definite. So, $(w_h^n, \phi_h^{n+1}) - (w_h^{n+1}, \phi_h^n) = 0$. Now, for the viscous term, we use (3.2.2) to obtain

$$
-\nu(\nabla w_h^{n+\frac{1}{2}}, \nabla \phi_h^{n+\frac{1}{2}}) = \nu(w_h^{n+\frac{1}{2}}, \overline{w_h^{n+\frac{1}{2}}}^h) = \nu \|w_h^{n+\frac{1}{2}}\|_{E_h,\alpha}^2. \tag{3.2.7}
$$

Combining (3.2.5)-(3.2.7) gives

$$
\frac{1}{2\Delta t}\left(\alpha^2\|\overline{w_h^{n+1}}^h\|^2 + \|\nabla\phi_h^{n+1}\|^2\right) - \frac{1}{2\Delta t}\left(\alpha^2\|\overline{w_h^n}^h\|^2 + \|\nabla\phi_h^n\|^2\right) + \nu\|w_h^{n+\frac{1}{2}}\|_{E_h,\alpha}^2 = ((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z}, \phi_h^{n+\frac{1}{2}}).
$$

Summing both sides from $n = 0$ to $n = M - 1$ yields (by telescoping sums)

$$
\frac{1}{2}\left(\alpha^2\|\overline{w_h^M}^h\|^2 + \|\nabla\phi_h^M\|^2\right) + \Delta t\sum_{n=0}^{M-1}\nu\|w_h^{n+\frac{1}{2}}\|_{E_h,\alpha}^2 = \frac{1}{2}\left(\alpha^2\|\overline{w_h^0}^h\|^2 + \|\nabla\phi_h^0\|^2\right) + \Delta t\sum_{n=0}^{M-1}((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z}, \phi_h^{n+\frac{1}{2}}).
$$

For enstrophy in the NS-α discrete case, we choose v_h to be $w_h^{n+\frac{1}{2}}$ in (3.2.1). The nonlinear term immediately disappears by the same reasoning used in the proof for NS-α energy. The cross terms in the expression $\frac{1}{2\Delta t}(w_h^{n+1} - w_h^n, w_h^{n+1} + w_h^n)$ clearly vanish, and we have

$$
\frac{1}{\Delta t}(\|w_h^{n+1}\|^2 - \|w_h^n\|^2) + \nu\|\nabla w_h^{n+\frac{1}{2}}\|^2 = ((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z}, w_h^{n+\frac{1}{2}}).
$$

Summing both sides from $n = 0$ to $n = M - 1$ yields

$$\frac{1}{2}\|w_h^M\|^2 + \Delta t \sum_{n=0}^{M-1} \nu \|\nabla w_h^{n+\frac{1}{2}}\|^2 = \frac{1}{2}\|w_h^0\|^2 + \Delta t \sum_{n=0}^{M-1} ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, w_h^{n+\frac{1}{2}}).$$

□

3.3 Conservation laws for the NS-α in the discrete case with deconvolution

Discretizing using Crank-Nicolson for the deconvolution model yields the following discrete formulation for NS-α in the deconvolution case.

NS-α deconvolution: Again, let Δt denote the timestep, $t^n = n\Delta t$, $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$, and u_h^n represent the approximation to $u(x, t^n)$. We will denote $\frac{1}{2}(u_h^{n+1} + u_h^n)$ by $u_h^{n+\frac{1}{2}}$ and will denote $f(t^{n+\frac{1}{2}}, x)$ by $f^{n+\frac{1}{2}}$. Then, we find $(\phi_h^n, w_h^n, \overline{w_h^n}^h) \in (X_\#^h, X_\#^h, X_\#^h)$ such that $\forall (v_h, \chi_h, \psi_h) \in (X_\#^h, X_\#^h, X_\#^h)$,

$$\begin{aligned} \frac{1}{\Delta t}(w_h^{n+1} - w_h^n, v_h) - ((\hat{z} \times \nabla \phi_h^{n+\frac{1}{2}}) \cdot \nabla w_h^{n+\frac{1}{2}}, v_h) & \\ + \ \nu(\nabla w_h^{n+\frac{1}{2}}, \nabla v_h) &= ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, v_h), \quad (3.3.1) \\ -(D_N^h \overline{w_h^{n+1}}^h, \chi_h) + (\nabla \phi_h^{n+1}, \nabla \chi_h) &= 0, \quad (3.3.2) \\ \alpha^2(\nabla \overline{w_h^{n+1}}^h, \nabla \psi_h) + (\overline{w_h^{n+1}}^h, \psi_h) - (w_h^{n+1}, \psi_h) &= 0. \quad (3.3.3) \end{aligned}$$

As before, given $w_h^0 \in X_\#^h$, define $\phi_h^0 \in X_\#^h$ to be the unique (after enforcing mean zero) solution of (3.3.2). Since we have w_h^n at a given time step n, we iterate through the de-coupled system by first filtering w_h^n using (3.3.3), applying deconvolution, then solving for ϕ_h^n using (3.3.2) to obtain an initial guess for (3.3.1). Then, we solve (3.3.1) to obtain an approximation for w_h^{n+1}. We repeat this process using a linear iteration scheme for the non-linearity until the solutions are within 10^{-10} of each other.

Lemma 3.3.1. *Provided sufficiently smooth data and under periodic boundary conditions, solutions to the vorticity stream formulations of NS-α deconvolution* (3.3.1)-(3.3.3) *satisfy the following conservation laws: NS-α energy:*

$$\frac{1}{2}\left(\alpha^2\|D_N^{h^{1/2}}w_h^M\| + \|D_N^{h^{-1/2}}\nabla\phi_h^M\|^2\right) + \Delta t\sum_{n=0}^{M-1}\nu\|w_h^{n+\frac{1}{2}}\|_{E_h,N}^2 = \frac{1}{2}\left(\alpha^2\|D_N^{h^{1/2}}w_h^0\| + \|D_N^{h^{-1/2}}\nabla\phi_h^0\|^2\right) + \Delta t\sum_{n=0}^{M-1}((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z},\phi_h^{n+\frac{1}{2}}),$$

NS-α enstrophy (identical to discrete case):

$$\frac{1}{2}\|w_h^M\|^2 + \Delta t\sum_{n=0}^{M-1}\nu\|\nabla w_h^{n+\frac{1}{2}}\|^2 = \frac{1}{2}\|w_h^0\|^2 + \Delta t\sum_{n=0}^{M-1}((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z},w_h^{n+\frac{1}{2}}).$$

Proof. To prove NS-α energy, choose v_h in (3.2.1) to be $\phi_h^{n+\frac{1}{2}}$. Again, the nonlinear term vanishes, and we have

$$\frac{1}{\Delta t}(w_h^{n+1} - w_h^n, \frac{\phi_h^{n+1}+\phi_h^n}{2}) + \nu(\nabla w_h^{n+\frac{1}{2}}, \nabla\phi_h^{n+\frac{1}{2}}) \quad = \quad ((\nabla\times f^{n+\frac{1}{2}})\cdot\hat{z},\phi_h^{n+1/2}). \qquad (3.3.4)$$

Coupling the definition of Δ_h with (3.3.2), we see that $(\Delta_h\phi_h^{n+1}, v_h) = -(D_N^h\overline{w_h^{n+1}}^h, v_h)\forall v_h \in X_h$. So, $\Delta_h\phi_h^{n+1} = -D_N^h\overline{w_h^{n+1}}^h$. For the first term, we have

$$\frac{1}{\Delta t}(w_h^{n+1} - w_h^n, \frac{\phi_h^{n+1}+\phi_h^n}{2}) = \frac{1}{2\Delta t}(w_h^{n+1},\phi_h^{n+1}) - \frac{1}{2\Delta t}(w_h^n,\phi_h^n).$$

The cross terms again disappear using the same argument as in the discrete case. By (3.3.3), we see that

$$\frac{1}{2\Delta t}(w_h^{n+1},\phi_h^{n+1}) - \frac{1}{2\Delta t}(w_h^n,\phi_h^n) = \frac{1}{2\Delta t}\left(\alpha^2(\nabla\overline{w_h^{n+1}}^h, \nabla\phi_h^{n+1}) + (\overline{w_h^{n+1}}^h, \phi_h^{n+1})\right) - \frac{1}{2\Delta t}\left(\alpha^2(\nabla\overline{w_h^n}^h, \nabla\phi_h^n) + (\overline{w_h^n}^h, \phi_h^n)\right).$$

Using the definition of Δ_h, (3.3.2), the fact that $\overline{w_h^n}^h = D_N^h D_N^{h^{-1}}\overline{w_h^n}^h$, and that differential operators commute under periodic boundary conditions, we obtain the equivalent expression

$$\frac{1}{2\Delta t}\left(-\alpha^2(\overline{w_h^{n+1}}^h, \Delta_h \phi_h^{n+1}) + (\nabla \phi_h^{n+1}, D_N^{h^{-1}} \nabla \phi_h^{n+1})\right) \\ - \frac{1}{2\Delta t}\left(-\alpha^2(\overline{w_h^{n}}^h, \Delta_h \phi_h^{n}) + (\nabla \phi_h^{n}, D_N^{h^{-1}} \nabla \phi_h^{n})\right).$$

Substitution and noting that $D_N^{h^{-1}} = D_N^{h^{-1/2}} D_N^{h^{-1/2}}$ since D_N^h is self-adjoint and positive, we get the equivalent expression

$$\frac{1}{2\Delta t}\left(\alpha^2(\overline{w_h^{n+1}}^h, D_N^h \overline{w_h^{n+1}}^h) + (D_N^{h^{-1/2}} \nabla \phi_h^{n+1}, D_N^{h^{-1/2}} \nabla \phi_h^{n+1})\right) \\ - \frac{1}{2\Delta t}\left(\alpha^2(\overline{w_h^{n}}^h, D_N^h \overline{w_h^{n}}^h) + (D_N^{h^{-1/2}} \nabla \phi_h^{n}, D_N^{h^{-1/2}} \nabla \phi_h^{n})\right)$$

which, since D_N^h is positive and self-adjoint, can be re-written in terms of the L^2 norm as

$$\frac{1}{2\Delta t}\left(\alpha^2\|D_N^{h^{1/2}} \overline{w_h^{n+1}}^h\|^2 + \|D_N^{h^{-1/2}} \nabla \phi_h^{n+1}\|^2\right) - \frac{1}{2\Delta t}\left(\alpha^2\|D_N^{h^{1/2}} \overline{w_h^{n}}^h\|^2 + \|D_N^{h^{-1/2}} \nabla \phi_h^{n}\|^2\right).$$

Now, for the viscous term, we use (3.3.2) to obtain

$$-\nu(\nabla w_h^{n+\frac{1}{2}}, \nabla \phi_h^{n+\frac{1}{2}}) = \nu(w_h^{n+\frac{1}{2}}, D_N^h \overline{w_h^{n+\frac{1}{2}}}^h) = \nu\|w_h^{n+\frac{1}{2}}\|_{E_h,N}^2. \tag{3.3.5}$$

Combining (3.3.4)-(3.3.5) gives

$$\frac{1}{2\Delta t}\left(\alpha^2\|D_N^{h^{1/2}} \overline{w_h^{n+1}}^h\|^2 + \|D_N^{h^{-1/2}} \nabla \phi_h^{n+1}\|^2\right) - \frac{1}{2\Delta t}\left(\alpha^2\|D_N^{h^{1/2}} \overline{w_h^{n}}^h\|^2 + \|D_N^{h^{-1/2}} \nabla \phi_h^{n}\|^2\right) + \\ \nu\|w_h^{n+\frac{1}{2}}\|_{E_h,N}^2 = ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, \phi_h^{n+1/2}).$$

Summing both sides from $n = 0$ to $n = M - 1$ yields (by telescoping sums)

$$\frac{1}{2}\left(\alpha^2 \|D_N^{h^{1/2}} \overline{w_h^M}^h\|^2 + \|D_N^{h^{-1/2}} \nabla \phi_h^M\|^2\right) + \Delta t \sum_{n=0}^{M-1} \nu \|w_h^{n+\frac{1}{2}}\|_{E_h,N}^2$$
$$= \frac{1}{2}\left(\alpha^2 \|D_N^{h^{1/2}} \overline{w_h^0}^h\| + \|D_N^{h^{-1/2}} \nabla \phi_h^0\|^2\right) + \Delta t \sum_{n=0}^{M-1} ((\nabla \times f^{n+\frac{1}{2}}) \cdot \hat{z}, \phi_h^{n+\frac{1}{2}}).$$

The proof of enstrophy in the NS-α deconvolution discrete case is identical to the proof of enstrophy in the NS-α discrete case. $\square$

We now prove that our model is unconditionally stable. While proving unconditional stability is often difficult, we see that in this case, stability follows quite directly from our enstrophy conservation result.

3.4 Stability Analysis

Lemma 3.4.1. *We obtain the following bound on w_h:*

$$\|w_h^M\|^2 + \Delta t \left(\nu \sum_{n=0}^{M-1} \|\nabla w_h^{n+\frac{1}{2}}\| \right) \leq C(data),$$

and the following bound on ϕ_h :

$$\|\Delta_h \phi_h^n\| \leq \|D_N^h \overline{w_h^n}^h\| \leq C\|w_h^n\| \leq C(data),$$

which implies that

$$\|\phi_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C(data).$$

Proof. Setting $v_h = w_h^{n+\frac{1}{2}}$ in (3.3.1) yields (by using Cauchy-Schwarz and Young's inequalities)

$$\frac{1}{2\Delta t}(\|w_h^{n+1}\|^2 - \|w_h^n\|^2) + \nu\|\nabla w_h^{n+\frac{1}{2}}\|^2 = (\nabla \times f^{n+\frac{1}{2}}, w_h^{n+\frac{1}{2}}) \leq \frac{\nu^{-1}}{2}\|\nabla \times f^{n+\frac{1}{2}}\|_{-1}^2 + \frac{\nu}{2}\|\nabla w_h^{n+\frac{1}{2}}\|^2.$$

So,

$$\frac{1}{\Delta t}(\|w_h^{n+1}\|^2 - \|w_h^n\|^2) + \frac{\nu}{2}\|\nabla w_h^{n+\frac{1}{2}}\| \leq \frac{\nu^{-1}}{2}\|\nabla \times f^{n+\frac{1}{2}}\|_{-1}^2.$$

Summing from n=0 to $M-1$, we get

$$\|w_h^M\|^2 + \Delta t \left(\nu \sum_{n=0}^{M-1} \|\nabla w_h^{n+\frac{1}{2}}\| \right) \leq C(data)$$

as desired. Next, setting $\chi_h = \Delta_h \phi_h^n$ in (3.3.2) and applying Cauchy-Schwarz, we see that

$$\|\Delta_h \phi_h^n\|^2 \leq \|D_N^h \overline{w_h^n}^h\| \|\Delta_h \phi_h^n\|.$$

Recalling that $\|D_N^h\| \leq N+1$, we get

$$\|\Delta_h \phi_h^n\| \leq \|D_N^h \overline{w_h^n}^h\| \leq C\|w_h^n\| \leq C(data)$$

for n = 1,2, $\ldots$, M. By Poincaré's inequality, we also have that

$$\max_{1\leq n\leq M} \|\phi_n^h\| \leq C(data),$$

and

$$\max_{1\leq n\leq M} \|\nabla \phi_n^h\| \leq C(data),$$

so we have unconditional stability as desired. We note that well-posedness of the discrete system can be proven in the usual way, as a result of the stability bounds. $\square$

Chapter 4

Convergence Analysis

For simplicity in stating the convergence theorem, we now summarize the regularity assumptions for $\phi(x,t)$ and $w(x,t)$, assuming P_k elements to approximate w and ϕ.

$$w, \phi \in L^2(0,T; H^{k+1}(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \tag{4.0.1}$$

$$w_t, \phi_t \in L^2(0,T; H^{k+1}(\Omega)) \cap L^\infty(0,T; H^{k+1}(\Omega)) \tag{4.0.2}$$

$$w_{tt}, \phi_{tt} \in L^2(0,T; H^{k+1}(\Omega)) \tag{4.0.3}$$

$$w_{ttt}, \phi_{ttt} \in L^2(0,T; L^2(\Omega)) \tag{4.0.4}$$

Theorem 4.0.1. *If w and ϕ are solutions of the vorticity-stream deconvolution formulation of NSE with w, ϕ satisfying* (4.0.1)-(4.0.4) *with $f \in L^2(0,T; L^2(\Omega))$, and Δt is sufficiently small, we have that*

$$\|w(T) - w_h^M\| + \left(\nu \Delta t \sum_{n=0}^{M-1} \|\nabla(w^{n+\frac{1}{2}} - w_h^{n+\frac{1}{2}})\|^2\right)^{\frac{1}{2}} + \left(\Delta t \sum_{n=0}^{M-1} \|\nabla(\phi^{n+\frac{1}{2}} - \phi_h^{n+\frac{1}{2}})\|^2\right)^{\frac{1}{2}}$$
$$\leq C(T,\nu,w)(h^k + (\Delta t)^2 + \alpha^{2N+2}).$$

Proof. Since w solves the vorticity-stream formulation of the NSE, we have for all $v_h \in X_{\#}^h$ that

$$(w_t(t^{n+\frac{1}{2}}), v_h) - (\hat{z} \times \nabla\phi(t^{n+\frac{1}{2}}) \cdot \nabla w(t^{n+\frac{1}{2}}), v_h) + \nu(\nabla w(t^{n+\frac{1}{2}}), \nabla v_h) = (\nabla \times f(t^{n+\frac{1}{2}}), v_h). \tag{4.0.5}$$

Adding $\left(\frac{w^{n+1}-w^n}{\Delta t}, v_h\right)$ and $\nu(\nabla w^{n+\frac{1}{2}}, \nabla v_h)$ to both sides, we obtain

$$\begin{aligned}&\frac{1}{\Delta t}(w^{n+1}-w^n, v_h) - (\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}), v_h) + \nu(\nabla w^{n+\frac{1}{2}}, \nabla v_h)\\ &= (\nabla\times f(t^{n+\frac{1}{2}}), v_h) + \left(\frac{w^{n+1}-w^n}{\Delta t} - w_t(t^{n+\frac{1}{2}}), v_h\right) + \nu(\nabla w^{n+\frac{1}{2}} - \nabla w(t^{n+\frac{1}{2}}), \nabla v_h). \end{aligned} \tag{4.0.6}$$

Next, subtracting (3.2.1) from (4.0.6) and labeling $e^n := w^n - w_h^n$ gives

$$\begin{aligned}\frac{1}{\Delta t}(e^{n+1}-e^n, v_h) + \nu(\nabla e^{n+\frac{1}{2}}, \nabla v_h) &= (\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}), v_h) - (\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla w_h^{n+\frac{1}{2}}, v_h)\\ &+ \left(\frac{w^{n+1}-w^n}{\Delta t} - w_t(t^{n+\frac{1}{2}}), v_h\right) + \nu\left(\nabla w^{n+\frac{1}{2}} - \nabla w(t^{n+\frac{1}{2}}), \nabla v_h\right).\end{aligned}$$

We split the vorticity error into two pieces Φ_w and η_w: $e^n = (w^n - \xi_h^n) + (\xi_h^n - w_h^n) := \eta_w + \Phi_w$ where ξ_h^n is an arbitrary element of $X_{\#}^h$. With this notation, rearranging gives

$$\begin{aligned}\frac{1}{\Delta t}(\Phi_w^{n+1}-\Phi_w^n, v_h) + \nu(\nabla\Phi_w^{n+\frac{1}{2}}, \nabla v_h) &= -\frac{1}{\Delta t}(\eta_w^{n+1}-\eta_w^n, v_h) - \nu(\nabla\eta_w^{n+\frac{1}{2}}, \nabla v_h)\\ &+ (\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}), v_h) - (\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla w_h^{n+\frac{1}{2}}, v_h)\\ &+ \left(\frac{w^{n+1}-w^n}{\Delta t} - w_t(t^{n+\frac{1}{2}}), v_h\right) + \nu\left(\nabla w^{n+\frac{1}{2}} - \nabla w(t^{n+\frac{1}{2}}), \nabla v_h\right).\end{aligned}$$

Choosing $v_h = \Phi_w^{n+\frac{1}{2}}$ yields

$$\begin{aligned}\frac{1}{2\Delta t}(\|\Phi_w^{n+1}\|^2 - \|\Phi_w^n)\|^2) + \nu\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2 &= -\frac{1}{\Delta t}(\eta_w^{n+1}-\eta_w^n, \Phi_w^{n+\frac{1}{2}}) - \nu(\nabla\eta_w^{n+\frac{1}{2}}, \nabla\Phi_w^{n+\frac{1}{2}})\\ &+ (\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}), \Phi_w^{n+\frac{1}{2}}) - (\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla w_h^{n+\frac{1}{2}}, \Phi_w^{n+\frac{1}{2}})\\ &+ \left(\frac{w^{n+1}-w^n}{\Delta t} - w_t(t^{n+\frac{1}{2}}), \Phi_w^{n+\frac{1}{2}}\right) + \nu\left(\nabla w^{n+\frac{1}{2}} - \nabla w(t^{n+\frac{1}{2}}), \nabla\Phi_w^{n+\frac{1}{2}}\right).\end{aligned}$$

We have the following bounds for the terms on the RHS (see [4]).

$$-\nu(\nabla\eta_w^{n+\frac{1}{2}}, \nabla\Phi_w^{n+\frac{1}{2}}) \le \frac{\nu}{12}\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2 + 3\nu\|\nabla\eta_w^{n+\frac{1}{2}}\|^2, \tag{4.0.7}$$

and

$$
\begin{aligned}
-\frac{1}{\Delta t}(\eta_w^{n+1}-\eta_w^n),\Phi_w^{n+\frac{1}{2}}) &\leq \frac{1}{2}\left\|\frac{\eta_w^{n+1}-\eta_w^n}{\Delta t}\right\|^2+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2 \\
&= \frac{1}{2}\int_\Omega\left(\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\frac{d}{dt}\eta_w dt\right)^2 d\Omega+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2 \\
&\leq \frac{1}{2}\int_\Omega\left(2|\frac{d}{dt}\eta_w(t^{n+1})|^2+2\int_{t^n}^{t^{n+1}}|\frac{d^2}{dt^2}\eta_w|dt\right)d\Omega+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2 \\
&= \|\frac{d}{dt}\eta_w(t^{n+1})\|^2+\int_{t^n}^{t^{n+1}}\|\frac{d^2}{dt^2}\eta_w\|^2dt+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2. \qquad (4.0.8)
\end{aligned}
$$

Utilizing Taylor-series expansions as in [4], we see that

$$
\begin{aligned}
\left(\frac{w^{n+1}-w^n}{\Delta t}-w_t(t^{n+\frac{1}{2}}),\Phi_w^{n+\frac{1}{2}}\right) &\leq \frac{1}{2}\left\|\frac{w^{n+1}-w^n}{\Delta t}-w_t(t^{n+\frac{1}{2}})\right\|^2+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2 \\
&= \frac{\Delta t^3}{2560}\int_{t^n}^{t^{n+1}}\|w_{ttt}\|^2dt+\frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|^2, \qquad (4.0.9)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu(\nabla(w^{n+\frac{1}{2}}-w(t^{n+\frac{1}{2}})),\nabla\Phi_w^{n+\frac{1}{2}}) &\leq \frac{\nu}{12}\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2+3\nu\|\nabla(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}})\|^2 \\
&= \frac{\nu}{12}\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2+\frac{\nu}{16}(\Delta t)^3\int_{t^n}^{t^{n+1}}\|\nabla w_{tt}\|^2dt. \qquad (4.0.10)
\end{aligned}
$$

Combining these results, we now have

$$
\begin{aligned}
\frac{1}{2\Delta t}\left(\|\Phi_w^{n+1}\|^2-\|\Phi_w^n\|^2\right)+\frac{5\nu}{6}\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2 &\leq 3\nu\|\nabla\eta_w^{n+\frac{1}{2}}\|^2+\frac{\Delta t^3}{2560}\int_{t^n}^{t^{n+1}}\|w_{ttt}\|^2dt \\
+\frac{\nu}{16}(\Delta t)^3\int_{t^n}^{t^{n+1}}\|\nabla w_{tt}\|^2dt &+ \|\Phi_w^{n+\frac{1}{2}}\|^2+\|\frac{d}{dt}\eta_w(t^{n+1})\|^2+\int_{t^n}^{t^{n+1}}\|\frac{d^2}{dt^2}\eta_w\|^2dt \\
+(\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}),\Phi_w^{n+\frac{1}{2}}) &- (\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla w_h^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}). \qquad (4.0.11)
\end{aligned}
$$

We isolate the nonlinear term, add and subtract $(\hat{z}\times\nabla\phi^{n+\frac{1}{2}}\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}})$, and obtain

$$
\begin{aligned}
(\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla w(t^{n+\frac{1}{2}}),\Phi_w^{n+\frac{1}{2}}) &- (\hat{z}\times\nabla\phi^{n+\frac{1}{2}}\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) \\
+(\hat{z}\times\nabla\phi^{n+\frac{1}{2}}\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) &- (\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla w_h^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) \\
=(\hat{z}\times\nabla[\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}}]\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) &+ (\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}}),\Phi_w^{n+\frac{1}{2}}) \\
+(\hat{z}\times\nabla\phi_h^{n+\frac{1}{2}}\cdot\nabla e^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) &+ (\hat{z}\times\nabla(\phi^{n+\frac{1}{2}}-\phi_h^{n+\frac{1}{2}})\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}). \qquad (4.0.12)
\end{aligned}
$$

Using our stability results and then applying the extension of the Poincaré inequality applied to the discrete Laplacian as in [9], we can bound the third term in (4.0.12) as follows. We first use Hölder's inequality to obtain

$$(\hat{z} \times \nabla \phi_h^{n+\frac{1}{2}} \cdot \nabla e^{n+\frac{1}{2}}, \Phi_w^{n+\frac{1}{2}}) \leq \|\nabla \phi_h^{n+\frac{1}{2}}\|_{L^4}(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}).$$

Now, by Ladyzhenskaya's inequality, we have

$$\begin{aligned}\|\nabla \phi_h^{n+\frac{1}{2}}\|_{L^4}&(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}) \\ &\leq C\|\nabla \phi_h^{n+\frac{1}{2}}\|^{\frac{1}{2}}\|\Delta_h \phi_h^{n+\frac{1}{2}}\|^{\frac{1}{2}}(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}).\end{aligned}$$

Using the Poincaré inequality to the discrete Laplacian as in [9], we see that

$$\begin{aligned}C\|\nabla \phi_h^{n+\frac{1}{2}}\|^{\frac{1}{2}}\|\Delta_h \phi_h^{n+\frac{1}{2}}\|^{\frac{1}{2}}&(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}) \\ &\leq C\|\Delta_h \phi_h^{n+\frac{1}{2}}\|(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}).\end{aligned}$$

Applying our stability bound and Young's inequality, we have

$$\begin{aligned}C\|\Delta_h \phi_h^{n+\frac{1}{2}}\|(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}) &\leq C(\|\nabla \eta_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4} + \|\nabla \Phi_w^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}) \\ &\leq C_1\|\nabla \eta_w^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}^2 + \frac{\nu}{12}\|\nabla \Phi_w^{n+\frac{1}{2}}\|^2 + C_2\nu^{-1}\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}^2.\end{aligned}$$

Applying Ladyzhenskaya's inequality again, we obtain our result that

$$\begin{aligned}C_1\|\nabla \eta_w^{n+\frac{1}{2}}\|^2 &+ \frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}^2 + \frac{\nu}{12}\|\nabla \Phi_w^{n+\frac{1}{2}}\|^2 + C_2\nu^{-1}\|\Phi_w^{n+\frac{1}{2}}\|_{L^4}^2 \\ &\leq C_1\|\nabla \eta_w^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|\Phi_w^{n+\frac{1}{2}}\|\|\nabla \Phi_w^{n+\frac{1}{2}}\| + C_2\nu^{-1}\|\Phi_w^{n+\frac{1}{2}}\|\|\nabla \Phi_w^{n+\frac{1}{2}}\|^2 \\ &\leq C_1\|\nabla \eta_w^{n+\frac{1}{2}}\|^2 + \frac{\nu}{12}\|\nabla \Phi_w^{n+\frac{1}{2}}\|^2 + C_2\nu^{-1}\|\Phi_w^{n+\frac{1}{2}}\|^2 \\ &\quad + \frac{\nu}{12}\|\nabla \Phi_w^{n+\frac{1}{2}}\|^2 + C_3\nu^{-1}\|\Phi_w^{n+\frac{1}{2}}\|^2.\end{aligned}$$

To bound the fourth term in (4.0.12), we denote $\phi^{n+\frac{1}{2}} - \phi_h^{n+\frac{1}{2}} = \eta_\phi^{n+\frac{1}{2}} + \Phi_\phi^{n+\frac{1}{2}}$ where $\eta_\phi = \phi - \xi_h, \Phi_\phi = \xi_h - \phi_h$ and ξ_h is an arbitrary element of $X_\#^h$. We now use straightforward techniques and obtain

$$
\begin{aligned}
(\hat{z}\times\nabla(\phi^{n+\frac{1}{2}}-\phi_h^{n+\frac{1}{2}})\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}}) &\leq \|\nabla w^{n+\frac{1}{2}}\|_{L^\infty}\|\nabla(\phi^{n+\frac{1}{2}}-\phi_h^{n+\frac{1}{2}})\|\|\Phi_w^{n+\frac{1}{2}}\| \\
&\leq C(\|\nabla\eta_\phi\|\|\Phi_w^{n+\frac{1}{2}}\|+\|\nabla\Phi_\phi^{n+\frac{1}{2}}\|\|\Phi_w^{n+\frac{1}{2}}\|) \\
&\leq \frac{C_1}{2}\|\nabla\eta_\phi\|^2+C_2\|\Phi_w^{n+\frac{1}{2}}\|+\frac{1}{8}\|\nabla\Phi_\phi^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

We now use Taylor series to bound the first two terms in (4.0.12).

$$
\begin{aligned}
&(\hat{z}\times\nabla(\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}})\cdot\nabla w^{n+\frac{1}{2}},\Phi_w^{n+\frac{1}{2}})+(\hat{z}\times\nabla\phi(t^{n+\frac{1}{2}})\cdot\nabla(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}}),\Phi_w^{n+\frac{1}{2}}) \\
&\leq \|\nabla(\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}})\|\|\nabla w^{n+\frac{1}{2}}\|_{L^\infty}\|\Phi_w^{n+\frac{1}{2}}\|+\|\nabla\phi(t^{n+\frac{1}{2}})\|_{L^\infty}\|\nabla(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}})\|\|\Phi_w^{n+\frac{1}{2}}\| \\
&\qquad\leq C_1\|\nabla(\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}})\|^2+C_2\|\Phi_w^{n+\frac{1}{2}}\|^2+ \\
&\qquad\quad C_3\|\nabla(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}})\|^2+C_4\|\Phi_w^{n+\frac{1}{2}}\|^2 \\
&\qquad\leq C_1(\Delta t)^3\int_{t^n}^{t^{n+1}}\|\nabla\phi_{tt}\|^2dt+C_2(\Delta t)^3\int_{t^n}^{t^{n+1}}\|\nabla w_{tt}\|^2dt+C_3\|\Phi_w^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

From the second set of equations in (3.2.2), we have

$$
(\nabla\phi(t^{n+\frac{1}{2}}),\nabla\chi_h)=(w(t^{n+\frac{1}{2}}),\chi_h).
$$

We now add $\pm(\nabla\phi^{n+\frac{1}{2}},\nabla\chi_h)$ to the LHS and $\pm(w^{n+\frac{1}{2}},\chi_h)$ to the RHS and subtract $(\nabla\phi_h^{n+\frac{1}{2}},\nabla\chi_h)=(D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\chi_h)$ from the equation and obtain

$$
\begin{aligned}
(\nabla\Phi_\phi^{n+\frac{1}{2}},\chi_h)&=(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}},\chi_h)+(w^{n+\frac{1}{2}}-D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\chi_h) \\
&\quad-(\nabla[\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}}],\nabla\chi_h)-(\nabla\eta_\phi^{n+\frac{1}{2}},\nabla\chi_h).
\end{aligned}
$$

Setting $\chi_h=\Phi_\phi^{n+\frac{1}{2}}$ yields

$$
\begin{aligned}
\|\nabla\Phi_\phi^{n+\frac{1}{2}}\|^2&=(w(t^{n+\frac{1}{2}})-w^{n+\frac{1}{2}},\Phi_\phi^{n+\frac{1}{2}})+(w^{n+\frac{1}{2}}-D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\Phi_\phi^{n+\frac{1}{2}})- \\
&\quad(\nabla[\phi(t^{n+\frac{1}{2}})-\phi^{n+\frac{1}{2}}],\nabla\Phi_\phi^{n+\frac{1}{2}})-(\nabla\eta_\phi^{n+\frac{1}{2}},\nabla\Phi_\phi^{n+\frac{1}{2}}).
\end{aligned}
$$

To obtain a bound for $(w^{n+\frac{1}{2}}-D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\Phi_\phi^{n+\frac{1}{2}})$, we note that

$$
(w^{n+\frac{1}{2}}-D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\Phi_\phi^{n+\frac{1}{2}})=(w^{n+\frac{1}{2}}-D_N^h\overline{w^{n+\frac{1}{2}}}^h,\Phi_\phi^{n+\frac{1}{2}})+(D_N^h\overline{w^{n+\frac{1}{2}}}^h-D_N^h\overline{w_h^{n+\frac{1}{2}}}^h,\Phi_\phi^{n+\frac{1}{2}}).
$$

We now recall that by [2], $D_N^h\overline{e^{n+\frac{1}{2}}}^h \leq C(N)\|e^{n+\frac{1}{2}}\|^2 \leq \frac{1}{2}\|\Phi_w\|^2 + \frac{1}{2}\|\eta_w\|^2$. We also recall that by Lemma 2.9 in [3], $\|w - D_N^h\overline{w}^h\| \leq C\alpha^{2N+2}\|w\| + C(\alpha h^k + h^{k+1})\left(\sum_{n=1}^{N+1}|F^n w|_{k+1}\right)$, where for N=1, $\sum_{n=1}^{N+1}|F^n w|_{k+1}$ is finite. The rest of the terms are handled in a straightforward way with Cauchy-Schwarz, Taylor's Series, Young's inequalities, and Poincaré's inequality. Combining our results, we have

$$\begin{aligned}
\frac{1}{2\Delta t}(\|\Phi_w^{n+1}\|^2 - \|\Phi_w^n\|^2) + \frac{\nu}{4}\|\nabla\Phi_w^{n+\frac{1}{2}}\|^2 + \frac{1}{8}\|\nabla\Phi_\phi\|^2 \leq \\
C_1(\nu)\|\nabla\eta_w^{n+\frac{1}{2}}\|^2 + C_2\|\nabla\eta_\phi^{n+\frac{1}{2}}\|^2 + C_3\|\nabla\eta_\phi^{n+\frac{1}{2}}\|^2 \\
+ C_4(\nu)(\Delta t)^3 + C_5\|\frac{d}{dt}\eta_w(t^{n+1})\|^2 + C_6\int_{t^n}^{t^{n+1}}\|\frac{d^2}{dt^2}\eta_w\|^2 dt \\
+ C_7(\nu)\|\Phi_w^{n+\frac{1}{2}}\|^2 + C_8(N)\alpha^{4N+4} + C_9\alpha^{4N+4}\|u^{n+\frac{1}{2}}\|_{H^{2N+2}}.
\end{aligned}$$

After summing over time steps, multiplying through by $2\Delta t$, and applying the discrete form of Gronwall's inequality, we obtain the desired convergence bound of $C(T,\nu,w)(\Delta t^2 + h^k + \alpha^{2N+2})$. □

Chapter 5

Numerical Experiments

In this chapter, we perform numerical experiments verifying the results of our analysis. All computations were carried out using FreeFem++.

The tables below contain errors and rates for the schemes' approximation to the chosen solution

$$u = [\cos(8\pi y), \sin(8\pi x)]^T (1 + 0.01t), p = 0.$$

Note that p is arbitrary since $\nabla \times (\nabla p) = 0$. For our calculations, we let $\alpha = h, \nu = 1, T = 0.01$. We obtained $\nabla \times f$ by plugging our chosen solution (u, p) into the NSE and taking the curl.

From this experiment, we see that higher accuracy can be realized from the use of deconvolution. The results for (3.3.1)-(3.3.3) using P_3 are shown in tables 5.1-5.3 on the next page. In the first table, we see the expected cubic rates for the vorticity-stream formulation of the NSE with no filtering or deconvolution applied. In the second table where we apply filtering without deconvolution, we observe sub-optimal rates as expected. Finally, in the third table where we apply deconvolution to the filtered term, we recover the cubic rates as desired.

h	Δt	$\|w - w_h\|_{L^2(0,T;H^1)}$	Rate	$\|\phi - \phi_h\|_{L^2(0,T;H^1)}$	Rate
1/4	0.01	42.5783	-	0.084373	-
1/8	0.01/3	9.0403	2.236	0.013962	2.595
1/16	0.01/9	1.1018	3.036	0.0017382	3.005
1/32	0.01/27	0.1417	2.958	0.000224	2.955
1/64	0.01/81	0.0180	2.981	2.8×10^{-5}	2.979

Table 5.1: Error rates on the vorticity-stream formulation of the NSE (no filtering) using P_3 elements

h	Δt	$\|w - w_h\|_{L^2(0,T;H^1)}$	Rate	$\|\phi - \phi_h\|_{L^2(0,T;H^1)}$	Rate
1/4	0.01	42.5442	-	0.083751	-
1/8	0.01/3	9.0385	2.235	0.075069	0.158
1/16	0.01/9	1.1054	3.032	0.063541	0.241
1/32	0.01/27	0.1504	2.878	0.034824	0.868
1/64	0.01/81	0.0269	2.482	0.012408	1.489

Table 5.2: Error rates on the vorticity-stream formulation of NS-α with N=0 using P_3 elements

h	Δt	$\|w - w_h\|_{L^2(0,T;H^1)}$	Rate	$\|\phi - \phi_h\|_{L^2(0,T;H^1)}$	Rate
1/4	0.01	42.5479	-	0.083799	-
1/8	0.01/3	9.0378	2.235	0.067560	0.311
1/16	0.01/9	1.1010	3.037	0.04470	0.596
1/32	0.01/27	0.1435	2.939	0.013426	1.735
1/64	0.01/81	0.0189	2.921	0.001817	2.885

Table 5.3: Error rates on the vorticity-stream formulation of NS-α with N=1 using P_3 elements

Chapter 6

Conclusions

We have proved both discrete and continuous model energy and enstrophy conservation for the NS-α vorticity-streamfunction scheme with and without deconvolution. As an easy consequence of these conservation laws, we proved unconditional stability for this system. Moreover, we proved convergence results which demonstrated the ability to use higher-order polynomials with the NS-α deconvolution scheme without losing accuracy. Finally, we performed numerical tests which demonstrated the strengths of the deconvolution approach in the vorticity-streamfunction setting.

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