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Parametric Amplification in the Context of Vibratory Energy Harvesting

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PARAMETRIC AMPLIFICATION IN THE CONTEXT OF VIBRATORY ENERGY HARVESTING

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Masters of Science
Mechanical Engineering

by
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R. Donovan Bode

(ABSTRACT)

Using a vibratory energy harvester (VEH) to independently power a sensor has become an increasingly popular topic due to the small amount of power current sensors require to operate. This can be achieved by scavenging energy from the ambient environment where the sensor is located. Numerous linear and nonlinear energy harvesters have been proposed in order to deal with various vibratory environments, along with improving the power production and/or bandwidth of the device.

In this Thesis, we propose a technique to harvest energy from excitation sources that possess two frequency components: a fundamental component with large energy content, and a super-harmonic component with smaller energy content at twice the fundamental component. Excitations of this nature are common in the environment due to inherent nonlinearities in the dynamics of the excitation source. Normally, two separate energy harvesters are needed to extract the energy at each frequency; however, this Thesis discusses a single cantilevered piezoelectric VEH that exploits the parametric amplification phenomenon to scavenge energy from both frequencies by varying the tilt angle between the axis of the beam and the direction of the excitation.

To investigate the efficacy of the proposed concept, the equations governing the electromechanical dynamics of the harvester are derived. The resulting partial differential equations and associated boundary conditions are then reduced to a single-mode Galerkin-based reduced-order model. Analytical expressions for the steady-state output power across a purely resistive load are obtained using the method of multiple scales.

Theoretical and experimental results demonstrate that parametric amplification can be used to improve the output power for given excitation parameters, beam tilt angle, and mechanical damping ratio. It is observed that there exists an optimal beam tilt angle at which the flow of energy from the environment to the electric load is maximized. This angle increases as the amplitude of the super-harmonic component of excitation increases and the mechanical damping ratio decreases. Furthermore, the resistive load of the harvesting circuit, which significantly affects the output power, is shown to have little influence on the optimal tilt angle except for very low mechanical damping ratios. Therefore, for a given environment and system parameters, an optimal tilt angle and resistive load combinations should be maintained to maximize the power output of the harvester.

Results indicate that the mechanical damping ratio plays a major role in characterizing performance. Specifically, when the mechanical damping ratio is small, significant enhancement in the output power is attainable even when the magnitude of the super-harmonic is small as compared to the fundamental component. For instance, at a damping ratio of $\zeta = 0.002$, a 20% increase in power is observed at the optimal tilt angle when the super-harmonic component is half that of the fundamental component. However, when the mechanical damping ratio is doubled to $\zeta = 0.004$, while all other design and excitation parameters are kept constant, the enhancement of the output power drops significantly to 4%. Such findings reveal that parametric amplification can be utilized to enhance the output power of a VEH especially for micro-scale applications where the mechanical damping ratio can be easily reduced.

Contents

1	Introduction	1
1.1	Motivations	1
1.2	Current Approaches for Energy Harvesting	2
1.2.1	Mono-Stable Energy Harvesters	4
1.2.2	Bi-Stable Energy Harvesters	12
1.3	Contribution	14
1.4	Thesis Organization	16
2	Problem Formulation	18
2.1	Governing Equations of Motion	19
2.1.1	Potential Energy	20
2.1.2	Inextensibility Constraint	22
2.1.3	Lagrangian	23
2.1.4	Virtual Work	23
2.1.5	Equations of Motion and Boundary Conditions	24
2.2	Reduced-Order Modeling	27

3	Theoretical Analysis	29
3.1	Asymptotic Solutions	31
3.1.1	Validity of The Attained Solutions	34
3.2	Optimizing the Phase Angle	35
3.3	Optimizing the Tilt Angle	37
3.4	Output Power	40
4	Experimental Results	46
5	Conclusions and Future Work	53
A	Location of the Neutral Axis	56

List of Figures

1.1	Hardening frequency-response curves illustrating the extended bandwidth attained as the coefficient of the cubic nonlinearity, β , is increased.	3
1.2	The potential energy function of a bi-stable harvester.	4
1.3	The mono-stable nonlinear energy harvester proposed by Mann and Sims [1]. . .	4
1.4	A schematic of the piezoelectric mono-stable harvester proposed by Stanton et al. [2].	7
1.5	Axially loaded piezoelectric clamped-clamped beam.	10
1.6	A schematic of the piezoelectric bi-stable harvester proposed by Erturk et al. [3].	13
1.7	Frequency spectrum of a rotating machine with small imbalance [4].	15
1.8	Direct and Parametric Excitations	16
2.1	Schematic of a piezoelectric cantilever-type energy harvester.	18
2.2	Differential element of the beam in deformation.	22
3.1	Schematic of a piezoelectric bimorph cantilever-type energy harvester.	30

3.2	Theoretical steady-state voltage-response curves of the cantilever bi-morph for different values of the tilt angle α . ($\alpha = 0[rad]$, solid line, $\alpha = \pi/18[rad]$, dashed lines, and $\alpha = 5\pi/36[rad]$, dotted lines). Numerical integration results are represented by dots. The values used for parametric excitation amplitude, damping, and phase angle are $B = 0.06g \text{ m/sec}^2$, $\zeta = 0.004$, and $\phi = 3\pi/4[rad]$, respectively.	34
3.3	Parametric amplification gain as a function of the phase angle for different tilt angles. The values used for the parametric excitation amplitude and damping are $B = 0.06g \text{ m/sec}^2$ and $\zeta = 0.004$, respectively.	36
3.4	Parametric amplification gain as a function of the super-harmonic excitation amplitude for different tilt angles. The optimal phase angle and $\zeta = 0.004$ are used.	37
3.5	Optimal tilt angle as a function of the super-harmonic excitation amplitude for different values of mechanical damping.	39
3.6	Optimal tilt angle as a function of the load resistance for different values of mechanical damping.	39
3.7	Contours for percentage power enhancement as the super-harmonic excitation amplitude and tilt angle vary. The optimal phase angle and $\zeta = 0.004$ are used.	40
3.8	Contours for percentage power enhancement as the super-harmonic excitation amplitude and tilt angle vary. The optimal phase angle and $\zeta = 0.002$ are used.	41
3.9	(a) Electric damping and (b) Parametric amplification gain as a function of the load resistance. Results are obtained for $B = 0.06g \text{ m/sec}^2$, $\alpha = \pi/6[rad]$, $\phi = 3\pi/4[rad]$, and $\zeta = 0.001$.	43
3.10	Power output as a function of load resistance for mechanical damping values of (a) $\zeta = 0.001$, (b) $\zeta = 0.0012$, and (c) $\zeta = 0.004$. Results are obtained for $B = 0.06g \text{ m/sec}^2$, $\alpha = \pi/6[rad]$, and $\phi = 3\pi/4[rad]$.	44

3.11	Power output as a function of both the load resistance and tilt angle for a mechanical damping of (a) $\zeta = 0.0015$ and (b) $\zeta = 0.002$. Black dots represent the maximum power at each tilt angle.	45
4.1	Experimental Apparatus.	47
4.2	Steady-state voltage output as a function of the phase angle $\Phi[rad]$ for a tilt angle of $\alpha = 5\pi/36[rad]$, super-harmonic excitation of $B = F$, and fundamental excitation frequency of $\omega = 13.5 Hz$. Circles represent experimental data.	47
4.3	Steady-state voltage output as a function of the tilt angle $\alpha [rad]$ ($\alpha = 0$, blue dotted line and empty circles, $\alpha = \pi/18[rad]$, red dashed line and empty squares, and $\alpha = 5\pi/36[rad]$, black solid line and solid circles) for a phase angle of $\Phi = 3\pi/4[rad]$ and super-harmonic excitation of $B = 3F$. Points represent experimental data.	49
4.4	Steady-state voltage output as a function of the super-harmonic excitation amplitude B . ($B = F/2$, blue dotted line and empty squares; $B = F$, red dashed line and empty circles; $B = 2F$, green dash-dot line and solid squares; and $B = 3F$, black solid line and solid circles). Results are obtained for a phase angle of $\Phi = 3\pi/4[rad]$ and a tilt angle of $\alpha = 5\pi/36[rad]$. The long dashed gray line represents the purely direct excitation case. Points represent experimental data.	50
4.5	Maximum voltage output as a function of the tilt angle α for various super-harmonic excitation amplitudes B . ($B = F/2$, blue dotted line and empty circles; $B = F$, red dashed line and empty squares; $B = 2F$, green dash-dot line and solid circles; and $B = 3F$, black solid line and solid squares). Results are obtained for a phase angle of $\Phi = 3\pi/4[rad]$. Points represent experimental data.	51

4.6	Voltage response curves for varying tilt angles and mechanical damping. ($\alpha = 0$ and $\zeta = 0.013$, blue dashed line; $\alpha = 5\pi/36[rad]$ and $\zeta = 0.013$, red solid line; $\alpha = 0$ and $\zeta = 0.012$, green dotted line; and $\alpha = 5\pi/36[rad]$ and $\zeta = 0.012$, black dash-dot line). Results are obtained for a constant super-harmonic excitation amplitude $B = 3F$	52
A.1	Cross-section of uni-morph piezoelectric cantilever-type energy harvester.	56

List of Tables

3.1	Geometric and material properties of the bi-morph cantilever beam. .	30
4.1	Geometric and material properties of the experimental setup.	48

Chapter 1

Introduction

1.1 Motivations

Recent advances in technology have opened new avenues for smaller and more energy efficient sensors requiring minimal power to operate (on the order of μW and mW [5]). Traditionally, these sensors are powered using batteries, which have not kept pace with their demands especially in terms of energy density. Additionally, batteries can only be used for a certain period of time before they must be replaced or recharged, which can be a costly and a cumbersome process especially in situations where the sensors are installed in remote and/or inaccessible locations. To circumvent this critical problem, many micro-power generators have been developed in order to independently power these sensors by transforming ambient vibration energy available in the sensor's environment into electricity.

This process is commonly known as energy harvesting or energy scavenging and can be accomplished by using any of the main four transduction principles: electrostatics, electromagnetism, piezoelectricity, or magnetostriction. In order to transform vibration energy into electricity, electrostatic or capacitive energy harvesters use

variation of the capacitance induced by relative motion of two biased plates, while electromagnetic energy harvesters use the rate of change of flux caused by the motion of a magnetic field. Piezoelectric and magnetostrictive harvesters use active materials that produce a voltage and a magnetic field, respectively, when subjected to external mechanical stresses.

Vibration energy harvesters (VEHs) have recently flourished as a major focus area for micro-power generation due to their ability to autonomously power devices that require small amounts of power to operate. The power density of these VEHs is of utmost importance. The objective of the harvester is to scavenge enough of the available ambient energy to maintain and operate the device being powered. However, due to size limitations on the harvester, research is currently focused on maximizing the energy density of such devices.

1.2 Current Approaches for Energy Harvesting

As of today, most of the energy harvesting research has focused on using linear VEHs that operate at resonance, thereby producing relatively large motions from a small excitation amplitude. These devices are tuned to a certain resonance frequency that will maximize the power output of the system for a specific application and environment. Once the excitation frequency drifts away from the oscillator's resonance frequency, the power output drops significantly. This constitutes a major concern, because in most cases, ambient vibration energy is dispersed over a broad range of frequencies [6]. Also, in situations where these devices are designed to operate on the micro- and nano-levels, fabrication to tune the resonant frequency must be very precise, or else some of the available power is lost.

To remedy this problem which has been hindering the development of efficient VEHs; recent research efforts have focused on purposefully introducing nonlinearities into

that harvester's design. Based on the shape of their associated potential energy function, these nonlinear VEHs can be divided into two major classes: the *mono-stable* harvesters and the *bi-stable* ones. In a mono-stable harvester, the potential energy function has one global minimum representing a stable sink. The nonlinearity comes in the form of a cubic stiffness element, which can either cause a softening or hardening response depending on its sign. Such nonlinearities can be introduced using external design means that will be discussed in the next section. As shown in Fig. 1.1, the nonlinearity can be used to extend the bandwidth of frequencies for which energy can be harvested.

Figure 1.1: Hardening frequency-response curves illustrating the extended bandwidth attained as the coefficient of the cubic nonlinearity, β , is increased.

Recently, power generators with a bi-stable potential, similar to the one shown in Fig. 1.2 were also proposed to enhance the generator's bandwidth and performance in a non-stationary environment. The potential energy function has two local minima (stable sinks) separated by a potential barrier (unstable saddle). Unlike their linear resonant counterparts that require frequency matching for enhanced transduction, it has been shown that generators with a bi-stable potential can provide significant power levels over a wide range of frequencies under steady-state fixed-frequency harmonic excitations. In what follows, we discuss several implementations of nonlinearities into harvesters' design.

Figure 1.2: The potential energy function of a bi-stable harvester.

1.2.1 Mono-Stable Energy Harvesters

Figure 1.3 depicts a schematic of an electromagnetic VEH proposed by Mann and Sims [1] which uses two outer magnets to levitate a fluctuating central magnet. The nonlinearity is introduced in the form of the magnetic restoring force, which also enables the system to be tuned to a specific resonant frequency. This is achieved by varying the distance between the outer magnets and the center one.

Figure 1.3: The mono-stable nonlinear energy harvester proposed by Mann and Sims [1].

For the harvester proposed, the system has an effective cubic hardening nonlinearity. The magnitude of the effective nonlinearity is determined by fitting a cubic power

series to the restoring force as a function of the separation distance between the magnets. While the nonlinearity is not dependent on the separation distance (using the cubic power series approximation), its influence on the VEH's dynamics changes with the separation distance because the linear stiffness is directly affected by it.

In their effort, Mann and Sims [1] derive the equations of motion of the system using system identification in conjunction with basic Newtonian dynamics. The equations of motion, which reduce to the form of a Duffing Oscillator subjected to harmonic excitations, are analyzed using the method of multiple scales [7]. The steady-state nonlinear frequency-response equations are constructed and utilized to study the performance of the harvester.

It is observed that the nonlinearity does not influence the response for small input excitations. However, as the amplitude of the input excitation increases, the nonlinearity becomes apparent, and the maximum amplitude no longer occurs at the linear resonant frequency. The amplitude of the nonlinear response is larger than that of the linear one. Large amplitudes are also realized over a much broader range of frequencies, therefore making the system ideal for applications where there is some variation in the excitation frequency.

For high excitation levels, the system exhibits multiple coexisting stable periodic solutions within a bandwidth of frequencies. Therefore, as the frequency increases or decreases the system exhibits a jump from a high energy level (large amplitude) to a low level (small amplitude) or vice versa. The authors suggest that, in this region, which is very sensitive to initial conditions, a perturbation could be applied to the system to induce a jump to the larger amplitude solution. Another important observation is that the increase in damping greatly reduces the amplitude as well as the range of frequencies over which the larger amplitude motions are present.

In another demonstration, an electromagnetic micro-generator for energy harvesting is proposed by Beeby et al. [8]. Again, the system's response is described using a

nonlinear Duffing oscillator with a base excitation. Four magnets are attached to the end of a cantilever beam, with two on either side of a stationary set of coils. When the beam is excited, the magnets around the coil oscillate to produce a flux gradient in fixed coils, which is extracted as energy.

The paper includes only experimental data that illustrate a similar hardening behavior as shown by Mann and Sims [1]. The jump phenomena as well as the hysteresis of the response is clearly present. However, it is unclear as to whether the amplitude of the nonlinear response is greater than that of the linear response. In order for this micro-power generator to be a more viable option for harvesting ambient vibration energy, theoretical and experimental validation would need to prove that the nonlinear system outperforms a linear one with similar parameters.

Stanton et al. [2] also use a similar setup to that proposed by Beeby et al. [8]. In essence, as shown in Fig. 1.4, a piezoelectric cantilever beam with a tip magnet oscillates in the magnetic field of another fixed magnet. However, the main purpose of Stanton's experiment is to show that the hysteresis resulting from cubic nonlinearity due to magnetic levitation can be either of the hardening or softening type depending on the location of the magnets. The flexibility of the system to undergo either a hardening or softening nonlinearity allows the range of frequencies that can produce ample power to be stretched to either side of the natural frequency.

The mono-stable system proposed by Stanton et al. [2] displays the same characteristics, such as multiple periodic solutions, jump phenomena, etc., that are discussed in previous examples with a cubic nonlinear magnetic restoring force. It is shown again that a nonlinear system can outperform a linear one under similar situations. For the specific setup discussed, the softening response shows a larger bandwidth, but the hardening bandwidth can be increased by placing the attracting magnets closer to the neutral axis of the beam. As expected, the nonlinearities enhance the performance only if the high energy attractor is realized. This can be best achieved

Figure 1.4: A schematic of the piezoelectric mono-stable harvester proposed by Stanton et al. [2].

when the excitation frequency is slowly changed, so that the response can stay at the high energy level. When it is not, then either a mechanical or electrical perturbation can be used to force the oscillator to jump up from the low energy attractor.

Triplett and Quinn [9] study the effect of stiffness nonlinearities as well as the nonlinear coupling of piezoelectric materials on the performance of a VEH. A single-degree-of-freedom base excitation model is used to analyze the system's frequency response near resonance for different values of each nonlinearity.

For the linear system, it is observed that when the electromechanical coupling strength is increased, the frequency range of usable power broadens and the peak value shifts towards a slightly smaller frequency. Typical of the behavior of a linear system when the effective damping is increased. However, the maximum power harvested increases initially until it reaches a maximum value, then begins to decrease as the coupling strength increases further. When the stiffness nonlinearity is introduced, the system initially exhibits hardening characteristics, but these diminish greatly with an increase in the coupling strength. It is also noted that the system's maximum power is only affected by the material nonlinearity and is independent of the stiffness nonlinearity.

When the nonlinear piezoelectric coupling is introduced, the main trends remain similar to the linear one. As the piezoelectric nonlinearity increases, the coupling strength that results in maximum power output decreases and the optimal harvested power increases. However, when the nonlinear coupling becomes very large, the performance of the system can decrease relative to the linear system [9].

A cantilever beam with a magnetoelectric transducer is proposed by Dai et al. [5]. The nonlinearity results again from the magnetic restoring force which is described by a power series as in Mann and Sims [1]. However, the power series representing the nonlinearity is approximated in this case using a fifth-order polynomial.

An optimal initial placement of the magnetoelectric transducer is of utmost importance in this design, and can be determined by the position at which the amplitude of the negative magnetic force exhibits a maximum. This will result in larger variations in the magnetic field for relatively smaller excitations. By inspecting the frequency response of the system versus peak voltage, it is apparent that the nonlinearities created in this device are of the softening type. This results in an increased bandwidth for which energy can be harvested, making it a feasible option for ambient vibrations [5].

Daqaq et al. [10] propose a piezoelectric parametrically-excited (in the direction of the length of the beam) cantilever type harvester. When the excitation frequency is close to twice the first modal frequency of the beam, it undergoes large-amplitude oscillations whose growth is only limited by system's nonlinearities and air-drag. The model presented in the manuscript accounts for nonlinearities arising from the beam's geometry, inertia, and air-drag. The resulting frequency-response curves show that the harvester exhibits a softening-type response.

The paper demonstrates the presence of an optimal coupling coefficient beyond which the harvested energy decreases. Specifically, it is shown that, the voltage measured across an arbitrary resistor increases initially for increasing values of the

coupling coefficient, until it reaches a critical value, beyond which the voltage decreases again. Furthermore, as the coupling coefficient increases, the range of frequencies wherein energy can be harvested decreases. The later is due to an increase in the effective damping which results in the need for a larger amplitude excitation to activate the parametric instability. It is also noted that, for the given experimental design parameters and air-drag, the system does not exhibit the common hysteretic jumps. These findings can be readily used for scavenging energy from ambient vibrations that has a random direction of excitation where the excitation is not necessarily always parallel to the beam deflection as discussed in many other previous studies.

Xue and Hu [11] study the nonlinear behavior of a piezoelectric plate type VEH near resonance. When the plate deflects due to some ambient vibrations, electric energy is harvested via a piezoelectric layer attached to the surface of the beam. Nonlinearities are present in the strain-displacement relations of the von Karman plate theory when considering shear and in-plane deformations. The power-frequency curves demonstrate a hardening nonlinearity with multiple solutions and jump instabilities.

Erturk et al. [12] introduced an L-shaped beam with end masses at the corner and end of the “L-shape”. The goal of the design is to increase the bandwidth of the energy harvester. In such design, a two-to-one internal resonance can be activated that permits energy exchange between the first two vibration modes. However, the authors only discuss the linear behavior of the L-shaped beam and suggest that internal resonances can be used to improve the energy harvesting capability of the system under certain circumstances.

A thin clamped-clamped beam with a proof center mass undergoing large deflections is proposed by Hajati et al. [13] to evaluate the prospect of energy harvesting from the stretching strain as compared to the bending strain. The resulting dynamics can be approximated by a Duffing oscillator exhibiting a hardening-type response.

The beam dimensions, weight of the proof-mass, and residual stresses in the beam are optimized to improve the power density, as well as increase the bandwidth of the harvester. The harmonics of the always-tensile stretching strain occur at twice the frequency of excitation, and improve the voltage output compared to the purely bending strain case. It also doubles the frequency range over which usable power can be harvested.

Masana and Daqaq [14] also utilize an axially-loaded piezoelectric clamped-clamped beam for energy harvesting as shown in Figure 1.5. In their study, the axial preload is compressive and used to tune the natural frequency of the harvester by softening the beam. The authors keep the axial preload below the critical buckling load, therefore the harvester remains in the mono-stable configuration. It is observed that the axial load not only serves to tune the natural frequency of the beam but also enhances the nonlinear characteristics and the electromechanical coupling, allowing for a larger bandwidth of usable energy to be harvested. The response amplitude of the harvester increases with the axial load, as does the effective electric damping of the system allowing for better energy conversion from vibrations to electricity.

Figure 1.5: Axially loaded piezoelectric clamped-clamped beam.

In a recent study, Quinn et al. [15] discuss the implications that the multiple stable solutions present in the response of a mono-stable VEH have. It is well-understood that, over a certain frequency bandwidth, cubic stiffness nonlinearities commonly result in multiple coexisting stable solutions, where one of the stable solutions occurs

at a much high energy level than the other. In general, the higher energy response results from a larger initial condition and vice versa, and in the context of energy harvesting the smaller response is highly undesirable as it reduces the response of the system when compared to the linear one. To remedy this problem, Quinn et al. propose a solution that eliminates the smaller branch of solutions for certain functions describing the nonlinearity, thusly keeping only the large-amplitude solution. If physically realizable, this can enhance the transduction of energy harvesters over a wide range of frequencies.

Daqaq [16] investigated the response of a mono-stable inductive energy harvester to random white and colored excitations. He illustrated that the nonlinearity has no influence on the output power under white excitations, but always decreases the power under colored band-limited noise.

Yang et al. [17] introduces a electromagnetic harvester with two cantilever beams: one beam with a magnetic transducer as a tip mass located slightly above the other beam which consists of two magnetic yokes that fit around the transducer completing a magnetic circuit. The natural frequencies of the two beams are designed to be different but relatively close. A stiffness nonlinearity occurs depending on the initial vertical separation distance between the two beams. As this distance increases from zero, the harvester begins to exhibit local peaks at the natural frequencies of each beam, thus improving the bandwidth of the device. However, for larger separation distances, the resonant frequency of one beam is much smaller than that of the other. This still produces peaks at each frequency, but since they are far apart there is a large gap in frequencies in which usable power can be obtained. Therefore, there is an optimal separation distance that will result in an increased bandwidth for energy harvesting purposes.

1.2.2 Bi-Stable Energy Harvesters

Several energy harvesting devices with bi-stable potential functions were also investigated. In one demonstration, Galchev et al. [18] investigate a VEH that oscillates between two equilibria and scavenges energy using magnetic induction. The proposed design incorporates a frequency up-converter in order to increase the electromechanical coupling, thus increasing the power generated. The device is used to successfully harvest energy from low-frequency and non-periodic excitations. It is further demonstrated that the power harvested as well as bandwidth of the harvester can be optimized by varying the design parameters of the harvester.

Cottone et al. [6] also introduce a bi-stable oscillator using an inverted pendulum with piezoelectric material attached. A tip magnet is placed on the top of the inverted pendulum which, in turn, oscillates in the vicinity of another stationary magnet mounted at a controllable distance from the tip magnet. The system behaves in a linear fashion when the distance between the magnets is large. However, when the distance becomes small, a bi-stable potential is created where the pendulum can oscillate between two stable equilibria.

The potential energy function varies as the distance between the two magnets changes. It goes from having one well (linear) to two wells (nonlinear) as the distance decreases. For very small distances, the “barrier” separating the two wells becomes very large such that escapement of dynamic trajectories from one well to the other requires very large input excitations. As such, in those cases, the pendulum dynamics remain confined to one potential well which severely reduces the efficiency of the harvester. The system is evaluated under random excitation, as in ambient vibrations, and the optimal distance between the two magnets is determined. It is demonstrated that, for some design parameters, the system can outperform the linear design unless the magnets are too close, in which case the linear system produces more power [6].

A device similar to the one devised by Cottone et al. [6] is proposed by Erturk et al. [3] with the only difference being that a ferroelectric steel beam rather than a rigid inverted pendulum is used, see Fig. 1.6. It is demonstrated that, due to the presence of coexisting stable solutions, the initial conditions play a critical role in determining whether the system will oscillate periodically between the two wells (intra-well dynamics) or remain confined in one potential well (inter-well response). For large input harmonic excitations and away from the resonance, the bi-stable system is shown to produce much larger voltages as compared to the linear one. Ferrari et al. [19] fabricated a similar device for applications at the microscale.

Figure 1.6: A schematic of the piezoelectric bi-stable harvester proposed by Erturk et al. [3].

Quinn et al. [20] use magnetic tip forces similar to Cottone et al. [6] to create the bistability, but the model uses two piezoelectric elastic elements (a damper in parallel with a spring and piezoelectric element in series) to support the mass of the beam. In the proposed configuration, the system dynamics result in the common cubic stiffness nonlinearity as well as nonlinear damping and electromechanical coupling. It is known that the cubic stiffness term can increase the efficiency of an energy harvester, but the damping nonlinearity is shown to cause a dynamic instability which can also enhance the ability of the device to harvest energy. Improvement in the power is mostly pronounced when the harvester is subjected to impulsive excitations rather than the common harmonic type.

Daqaq [21] theoretically discusses the prospect of using an inductive energy har-

vester with a symmetric bi-stable potential function to harvest energy from white and exponentially-correlated Gaussian noise. The power output under white Gaussian excitations is shown to be independent of the shape of the potential energy function. As such, a linear; a nonlinear (mono-stable); and a nonlinear (bi-stable) VEHs will harvest similar power levels under white Gaussian noise. However, under exponentially-correlated Gaussian excitations, the shape of the potential energy function; more specifically, the distance between the two potential wells as well as the height of the potential barrier, greatly influence the power output capabilities of the harvester.

1.3 Contribution

Many excitation sources, especially those resulting from vibrating machinery and nonlinear structures, have large energy trapped in their fundamental frequency component and slightly smaller energy components confined to their super- and subharmonics. For instance, consider the frequency spectrum of a rotating machine with small imbalance as shown in Fig. 1.7. The spectrum clearly contains a fundamental frequency component at 25 Hz and a slightly smaller harmonic at 50 Hz resulting from the presence of inherent quadratic nonlinearities in the dynamics. One possible way to maximize the energy capture from this vibration source is to design two VEHs such that each has its fundamental frequency tuned to one of these harmonics. However, this has the adverse effect of reducing the power density of the device and adds to the challenge of conditioning the signals resulting from these two harvesters.

In this effort, we explore a dynamic phenomenon which permits capturing the energy from both harmonics using a single energy harvesting device. This phenomenon, known as *parametric amplification*, utilizes a parametric pump to amplify the influence of an external direct excitation on the response of a certain structure. Using

Figure 1.7: Frequency spectrum of a rotating machine with small imbalance [4].

parametric amplification, the response amplitude near the fundamental frequency of a directly-excited system can be amplified significantly by superimposing a parametric component at twice the fundamental frequency of the system. This concept has been used for a long time in electrical and laser signals amplification and for communication purposes [22, 23] and has been recently explored by various researchers to enhance the sensitivity of microdevices to external resonant excitations [24, 25, 26].

To investigate the prospect of utilizing this phenomenon for enhanced energy harvesting, we study the response behavior of a cantilevered-type VEH to a combination of direct and parametric excitations. As shown in Fig. 1.8, the harvester is tilted such that it makes an angle α with the direction of the input acceleration. This creates a parametric pump resulting from the presence of the super-harmonic component in the frequency spectrum. More specifically, for the purely direct case ($\alpha = 0$), the harvester is only harvesting energy from the fundamental frequency. On the other hand, when the excitation is purely parametric ($\alpha = 90^\circ$), only energy from the super-harmonic is harvested. Therefore, by changing the angle between the excitation and axis of the beam, energy from both frequencies can be harvested. The objective of this Thesis is to theoretically and experimentally investigate the feasibility of this approach and to determine the conditions under which the parametric amplification phenomenon can be utilized for energy harvesting.

Figure 1.8: Direct and Parametric Excitations

1.4 Thesis Organization

The first Chapter provided a brief introduction on vibratory energy harvesting focusing mainly on reviewing the recent literature associated with the purposeful introduction of nonlinearities for enhanced performance. The main motivations and contributions of this Thesis were also summarized. In Chapter 2, an electromechanical model describing the harvester's dynamics is derived using Hamilton's Extended Principle. The model adopts the assumptions of Euler-Bernoulli's thin beam theory in conjunction with the linear constitutive equations for the structural and piezoelectric layers. The resulting partial-differential equations are then discretized into a set of ordinary-differential equations using a Galerkin scheme.

In Chapter 3, theoretical simulations are carried out on the reduced-order model to verify the claim that parametric amplification can, under some conditions, amplify the output power. To achieve this goal, an asymptotic solution governing the response of the system is obtained using the method of multiple scales. Analytical expressions for the output voltage and power of the harvester are derived and ana-

lyzed for different design and excitation parameters. In Chapter 4, an experimental study is performed to validate the theoretical findings under different condition. Finally, in Chapter 5, conclusions and recommendations for future directions that can improve the harvesting capabilities for this VEH are discussed.

Chapter 2

Problem Formulation

In this chapter, we consider the dynamics of a general piezoelectric cantilevered uni-morph harvester similar to that shown in Fig. 2.1. The composite beam has a structural layer (denoted by subscript, b) of width, W , length L , thickness t_b , modulus of elasticity E_b , and mass density ρ_b . The piezoelectric layer (denoted by subscript, p) has the same width with length L_p , thickness t_p , modulus of elasticity E_p , and mass density ρ_p . The piezoelectric layer also has an electric permittivity, ϵ_{33} , and an electromechanical coupling constant, d_{31} .

Figure 2.1: Schematic of a piezoelectric cantilever-type energy harvester.

2.1 Governing Equations of Motion

The electromechanical response of the system can be described using the function, $w(s, t)$, which represents the spatiotemporal deflection of the beam; the function, $u(s, t)$, representing the spatiotemporal elongation of the beam; and the voltage, $V(t)$, dissipated in an electric load, R , which is assumed to be purely resistive in this study. The equations governing the evolution of the system dynamics can be obtained using Hamilton's extended principle

$$\delta\mathcal{H} = \int_{t_1}^{t_2} (\delta\mathcal{L} + \delta\mathcal{W})dt = 0, \quad (2.1)$$

where δ is the variational operator, and the time interval t_1 to t_2 is arbitrary. Here, the Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = \mathcal{T} - \mathcal{U}$, where $\mathcal{T}$ and $\mathcal{U}$ are the kinetic and potential energy of the system, respectively, and $\mathcal{W}$ accounts for the work done by non-conservative forces. The kinetic energy is defined as

$$\mathcal{T} = \frac{1}{2} \int_0^L M(s)(\dot{u}^2 + \dot{w}^2)ds, \quad (2.2)$$

where the overdot represents the derivative with respect to time, and the mass per unit length, $M(s)$, is given by

$$M(s) = W\rho_b t_b + W\rho_p t_p [H(s) - H(s - L_p)]. \quad (2.3)$$

Here, $H(s)$ is a smooth representation of the Heaviside function that can be expressed as

$$H(s) = \lim_{k \rightarrow \infty} \frac{1}{1 + e^{-2ks}}, \quad (2.4)$$

to account for the fact that the piezoelectric layer does not cover the entire length of the structural layer.

2.1.1 Potential Energy

The potential energy of the system is composed of the strain energy of the composite beam, $\mathcal{U}_s$, as well as the electric potential stored in the piezoelectric layer, $\mathcal{U}_e$. The strain energy can be expressed as

$$\mathcal{U}_s = \frac{1}{2} \iiint_{\mathcal{V}} (\sigma_b \epsilon_b + \sigma_p \epsilon_p) d\mathcal{V}, \quad (2.5)$$

where $\mathcal{V}$ is the domain, and σ and ϵ represent, respectively, the axial stress and strain. Using linear Euler-Bernoulli beam theory, the strain developed in both layers can be expressed in terms of the transverse deflection using

$$\epsilon = -zw'', \quad (2.6)$$

where the prime denotes a derivative with respect to the arc length, s , and z represents the distance measured from the neutral axis of the beam (see Figure 2.1). Using the linear constitutive relations, the axial stresses in each layer can be expressed as

$$\begin{aligned} \sigma_b &= E_b \epsilon_b, \\ \sigma_p &= E_p (\epsilon_p - d_{31} E_3), \end{aligned} \quad (2.7)$$

where E_3 represents the electric field developed in the piezoelectric layer. Assuming homogeneous distribution of charges across the piezoelectric layer, the electric field can be further related to the voltage developed across the load and the thickness of the piezoelectric layer using $E_3 = -V(t)/t_p$. Also, the voltage can be related to the current $\dot{Q}_R(t)$ via $V(t) = R\dot{Q}_R(t)$. Substituting these relations back into Equation (2.7), we obtain

$$\sigma_p = E_p \left[\epsilon_p + \frac{d_{31}}{t_p} R\dot{Q}_R(t) \right]. \quad (2.8)$$

Upon substitution of Equations (2.6), (2.7), and (2.8) back into Equation (2.5), the strain energy of the system becomes

$$\mathcal{U}_s = \frac{1}{2} \iiint_{\mathcal{V}} \left\{ (E_b + E_p) z^2 w''^2 - z \frac{E_p d_{31}}{t_p} R\dot{Q}_R(t) w'' \right\} d\mathcal{V}. \quad (2.9)$$

which upon integration over the area becomes

$$\mathcal{U}_s = \frac{1}{2} \int_0^L [EI(s)w''^2 - \theta(s)R\dot{Q}_R(t)w''] ds. \quad (2.10)$$

Here, $EI(s)$ is the bending stiffness given by

$$\begin{aligned} EI(s) = & \frac{1}{3}[WE_b(h_b^3 - h_a^3) + WE_p(h_c^3 - h_b^3)][H(s) - H(s - L_p)] \\ & + \frac{WE_bt_b^3}{12}[H(s - L_p) - H(s - L)], \end{aligned} \quad (2.11)$$

and

$$\theta(s) = \frac{E_p W d_{31}}{2t_p}(h_c^2 - h_b^2)[H(s) - H(s - L_p)], \quad (2.12)$$

is an electromechanical coupling term. As shown in Fig. 2.1, h_a represents the distance from the neutral axis to the bottom surface of the structural layer, h_b denotes the distance between the neutral axis and the upper surface of the structural layer, and, h_c represents the distance between the neutral axis and upper surface of the piezoelectric layer; see Appendix A for details.

The electric potential energy stored in the capacitive piezoelectric layer, $\mathcal{U}_e$, can be expressed as

$$\mathcal{U}_e = -\frac{1}{2} \iiint_{\mathcal{V}} E_3 D_3 d\mathcal{V}. \quad (2.13)$$

where the electric displacement, D_3 , is well approximated by the linear piezoelectric constitutive relation

$$D_3 = d_{31}E_p\epsilon_p - e_{33}E_3. \quad (2.14)$$

Substituting Equation (2.14) into Equation (2.13) and integrating over the area yields

$$\mathcal{U}_e = -\frac{1}{2} \int_0^L \theta(s)R\dot{Q}_R(t)w'' ds - \frac{1}{2}C_p[R\dot{Q}_R(t)]^2, \quad (2.15)$$

where $C_p = e_{33}WL_p/t_p$ is the piezoelectric capacitance. Combining the strain and electric potential energy given in Equations (2.10) and (2.15), respectively, the total

potential energy becomes

$$\mathcal{U} = \frac{1}{2} \int_0^L [EI(s)w''^2 - 2\theta(s)R\dot{Q}_R(t)w'']ds + \frac{1}{2}C_p[R\dot{Q}_R(t)]^2. \quad (2.16)$$

2.1.2 Inextensibility Constraint

Figure 2.2: Differential element of the beam in deformation.

The composite beam is assumed to be inextensional. This implies that there is no elongation along the neutral axis of the beam. Using a differential beam element as shown in Figure (2.2), the longitudinal displacement, $u(s, t)$, can then be related to the transversal displacement, $w(s, t)$. To achieve this goal, the elongation, e , along the neutral axis of the beam can be set equal to zero. That is

$$e = \sqrt{(ds + du)^2 + dw^2} - ds = 0. \quad (2.17)$$

Dividing Equation (2.17) by ds yields

$$(1 + u')^2 + w'^2 = 1. \quad (2.18)$$

Solving Equation (2.18) for u' and using a first order Taylor-series approximation gives

$$u' = \sqrt{1 - w'^2} - 1 \approx -\frac{1}{2}w'^2. \quad (2.19)$$

It is worth noting that the first term expansion is an acceptable approximation for the cantilever beam case, because midplane stretching and coupling between stretching and other motions can be neglected [27].

2.1.3 Lagrangian

Using the kinetic energy, potential energy, and the inextensibility constraint, defined in Equations (2.2), (2.16), and (2.18) respectively, the Lagrangian of the system can be written as

$$\begin{aligned} \mathcal{L} = \frac{1}{2} \int_0^L \left\{ M(s)(\dot{u}^2 + \dot{w}^2) - EI(s)w''^2 + 2\theta(s)w''R\dot{Q}_R(t) \right. \\ \left. + \lambda(s, t)[1 - (1 + u')^2 - w'^2] \right\} ds - \frac{1}{2}C_p[R\dot{Q}_R(t)]^2, \end{aligned} \quad (2.20)$$

where the Lagrange multiplier $\lambda(s, t)$ is introduced to enforce the inextensibility constraint.

2.1.4 Virtual Work

The virtual work term, $\mathcal{W}$, accounts for the work done by non-conservative forces. In this case, we have two non-conservative energy fields: the environmental base excitation which pumps energy into the system; and the internal damping which is assumed to be a linear viscous damping in the transversal direction only. The composite beam is assumed to be subjected to a base excitation consisting of two acceleration components, with one having twice the frequency of the other. Such excitation sources are common and can result from the nonlinear response of structures having quadratic nonlinearities. The excitations are also offset by a phase angle, Φ , and can be assumed to take the form

$$\ddot{x}_b = F \cos(\omega t + \Phi) + B \cos(2\omega t), \quad (2.21)$$

where F and B are the acceleration magnitudes corresponding to the fundamental and super-harmonic components, respectively, and ω is the excitation frequency. In order to exploit the parametric amplification phenomenon associated with the super-harmonic, the composite beam is tilted by an angle α about the y -axis. This results in two base acceleration components, one forcing the beam directly, $\ddot{w}_b = \ddot{x}_b \cos \alpha$, and the other forcing the beam parametrically, $\ddot{u}_b = \ddot{x}_b \sin \alpha$. With that, the virtual work term becomes

$$\delta\mathcal{W} = \int_0^L \left[Q_u \delta u + (Q_w - c_w \dot{w}) \delta w - R \dot{Q}_R(t) \delta Q_R \right] ds, \quad (2.22)$$

where $Q_u = \ddot{u}_b$, $Q_w = \ddot{w}_b$, and $c_w \dot{w}$ represents the linear viscous damping term.

2.1.5 Equations of Motion and Boundary Conditions

The function inside the integral of the Lagrangian, Equation (2.20), is also known as the Lagrangian density, $l(s, t, r', r'', \dot{r})$, where $r(s, t)$ is a general function that can represent the elongation, $u(s, t)$, the transversal deflection, $w(s, t)$, or the current passing through the resistive load, $\dot{Q}_R(t)$. Substituting Equations (2.20) and (2.22) back into Equation (2.1) yields

$$\delta\mathcal{H} = \int_{t_1}^{t_2} \int_0^L \left\{ \delta l + Q_u \delta u + (Q_w - c_w \dot{w}) \delta w - R \dot{Q}_R(t) \delta Q_R \right\} ds dt, \quad (2.23)$$

where δu , δw , and δQ_R are zero at both t_1 and t_2 .

The Gâteaux, or first variational, is defined as [28]

$$\delta\mathcal{H}[h] = \lim_{\kappa \rightarrow 0} \frac{\mathcal{H}[r + \kappa h] - \mathcal{H}[r]}{\kappa} = 0, \quad (2.24)$$

where $h(s, t)$ is a test function and κ is real. Applying Equation (2.24) to Equation

(2.23) yields

$$\delta\mathcal{H}[h] = \lim_{\kappa \rightarrow 0} \frac{1}{\kappa} \int_{t_1}^{t_2} \int_0^L \left\{ l(s, t, r' + \kappa h', r'' + \kappa h'', \dot{r} + \kappa \dot{h}) - l(s, t, r', r'', \dot{r}) \right\} ds dt + Q^* = 0, \quad (2.25)$$

where Q^* represents the non-conservative forces that relate to the corresponding r (i.e. $Q^* = Q_u$ when $r = u$). Taking the Taylor expansion up to the first order of κ and simplifying gives

$$\delta\mathcal{H}[h] = \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial l}{\partial r'} h' + \frac{\partial l}{\partial r''} h'' + \frac{\partial l}{\partial \dot{r}} \dot{h} \right) ds dt + Q^* = 0. \quad (2.26)$$

Integrating Equation (2.26) and simplifying yields

$$\begin{aligned} \delta\mathcal{H}[h] = & \int_{t_1}^{t_2} \int_0^L \left\{ -\frac{\partial}{\partial s} \left(\frac{\partial l}{\partial r'} \right) + \frac{\partial^2}{\partial s^2} \left(\frac{\partial l}{\partial r''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial l}{\partial \dot{r}} \right) \right\} h ds dt \\ & + \int_{t_1}^{t_2} \left\{ \frac{\partial l}{\partial r'} - \frac{\partial}{\partial s} \left(\frac{\partial l}{\partial r''} \right) \right\} h \Big|_{s=0}^L dt + \int_{t_1}^{t_2} \frac{\partial l}{\partial r''} h' \Big|_{s=0}^L dt + Q^* = 0. \end{aligned} \quad (2.27)$$

Since Equation (2.27) must be true for all $h(s, t)$, each part can be set equal to zero independently with Q^* being included in the first. This yields the equations of motion and boundary conditions that govern this system dynamics as

$$-\frac{\partial}{\partial s} \left(\frac{\partial l}{\partial r'} \right) + \frac{\partial^2}{\partial s^2} \left(\frac{\partial l}{\partial r''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial l}{\partial \dot{r}} \right) = -Q^* \quad (2.28)$$

$$\frac{\partial l}{\partial r'} - \frac{\partial}{\partial s} \left(\frac{\partial l}{\partial r''} \right) \Big|_{s=0}^L = 0; \quad \frac{\partial l}{\partial r''} \Big|_{s=0}^L = 0. \quad (2.29)$$

Now, setting $r(s, t) \equiv u(s, t)$ in Equations (2.28) and (2.29) gives

$$\{\lambda(1 + u')\}' - M(s)\ddot{u} = -Q_u, \quad (2.30)$$

$$\lambda(1 + u') \Big|_{s=0}^L = 0, \quad (2.31)$$

where only the upper limit is considered for the boundary condition, because at the lower limit $u = 0$. Using the inextensibility constraint given by Equation (2.19), and substituting into Equations (2.30) and (2.31) yields

$$\left\{ \lambda \left(1 - \frac{w'^2}{2} \right) \right\}' + \frac{1}{2} M(s) \int_0^L \ddot{w}'^2 ds = -Q_u, \quad (2.32)$$

$$\lambda \left(1 - \frac{w'^2}{2} \right) \Big|_{s=L} = 0. \quad (2.33)$$

Solving these equations together for the Lagrange multiplier $\lambda(s, t)$, we obtain

$$\lambda(s, t) = -\frac{1}{2} \int_L^s M(s) \int_0^L \ddot{w}'^2 ds ds - \int_L^s Q_u ds. \quad (2.34)$$

Now, setting $r(s, t) \equiv w(s, t)$ in Equations (2.28) and (2.29), and using Equation (2.34) while retaining only linear terms gives

$$M(s)\ddot{w} + [EI(s)w'']'' + \theta''(s)R\dot{Q}_R(t) = Q_w - \left(w' \int_L^s Q_u ds \right)', \quad (2.35)$$

and the boundary conditions

$$w(0, t) = w'(0, t) = 0 \quad w''(L, t) = w'''(L, t) = 0. \quad (2.36)$$

Note that the boundary conditions are equivalent to the standards for a fixed-free beam [29].

Setting $r(s, t) \equiv Q_R(t)$ in Equation (2.28) yields

$$\frac{\partial}{\partial t} \int_0^L R\theta(s)w'' ds - C_p R^2 \ddot{Q}_R(t) = R\dot{Q}_R(t). \quad (2.37)$$

Substituting the non-conservative forces expressions back into Equations (2.35) and simplifying yields the following system's model

$$\begin{aligned} M(s)\ddot{w} + c_w \dot{w} + [EI(s)w'']'' + \theta''(s)V(t) &= M(s)\ddot{w}_b - M(s)\ddot{u}_b \{w''(s-L) + w'\}, \\ -\frac{\partial}{\partial t} \int_0^L \theta(s)w'' ds + C_p \dot{V}(t) + \frac{V(t)}{R} &= 0. \end{aligned} \quad (2.38)$$

$$w(0, t) = w'(0, t) = 0; \quad w''(L, t) = w'''(L, t) = 0$$

2.2 Reduced-Order Modeling

A single-mode Galerkin scheme is used to discretize the equations of motion that govern the system by setting the spatiotemporal function to $w(s, t) = \phi(s)q(t)$, where $\phi(s)$ is the first modal shape of the cantilever beam dynamics and $q(t)$ is a generalized coordinate. The first modal shape can be expressed as [30]

$$\begin{aligned} \phi(s) = C \left[\left(\sin \frac{\lambda}{L} - \sinh \frac{\lambda}{L} \right) \left(\sin \frac{\lambda}{L} s - \sinh \frac{\lambda}{L} s \right) \right. \\ \left. + \left(\cos \frac{\lambda}{L} - \cosh \frac{\lambda}{L} \right) \left(\cos \frac{\lambda}{L} s - \cosh \frac{\lambda}{L} s \right) \right], \end{aligned} \quad (2.39)$$

where the constants λ and C can be obtained using [30]

$$\cosh \lambda \cos \lambda = -1; \quad \int_0^L M(s) \phi^2(s) ds = 1. \quad (2.40)$$

Substituting the Galerkin expansion for $w(s, t)$ into Equation (2.38), multiplying by $\phi(s)$, and integrating over the domain yields

$$\begin{aligned} \ddot{q}(t) + \bar{c}\dot{q}(t) + \omega_n^2 q(t) + \alpha_1 V(t) &= \gamma_2 \ddot{u}_b - \gamma_1 \ddot{u}_b q(t), \\ -\alpha_2 \dot{q}(t) + C_p \dot{V}(t) + \frac{V(t)}{R} &= 0, \end{aligned} \quad (2.41)$$

where

$$\begin{aligned} \bar{c} &= c_w \int_0^L \phi^2(s) ds; \quad \omega_n^2 = \int_0^L \phi(s) [EI(s) \phi''(s)]'' ds; \\ \alpha_1 &= \int_0^L \phi(s) \theta''(s) ds; \quad \alpha_2 = \int_0^L \theta(s) \phi''(s) ds; \\ \gamma_1 &= \int_0^L M(s) [\phi(s) \phi''(s)(s - L_p) + \phi(s) \phi'(s)] ds; \quad \gamma_2 = \int_0^L M(s) \phi(s) ds. \end{aligned}$$

Introducing the time scale $t \equiv \tau/\omega_n$, and using the expressions for $\ddot{u}_b$ and $\ddot{w}_b$ described earlier yields the final discretized equations of motion governing the system

as

$$\begin{aligned}
 \ddot{q} + 2\zeta\dot{q} + q + \theta_1 V &= -4q[\lambda_1 \cos(\Omega\tau + \Phi) + \lambda_2 \cos(2\Omega\tau)] \\
 &\quad + 2[\eta_1 \cos(\Omega\tau + \Phi) + \eta_2 \cos(2\Omega\tau)], \\
 -\alpha_2\dot{q} + C_p\dot{V} + \theta_2 V &= 0,
 \end{aligned} \tag{2.42}$$

where the overdot now corresponds to a derivative with respect to τ and

$$\begin{aligned}
 \Omega &= \frac{\omega}{\omega_n}; \quad \zeta = \frac{\bar{c}}{2\omega_n}; \quad \theta_1 = \frac{\alpha_1}{\omega_n^2}; \quad \theta_2 = \frac{1}{R\omega_n} \\
 \lambda_1 &= \gamma_1 \frac{F}{4\omega_n^2} \sin \alpha; \quad \lambda_2 = \gamma_1 \frac{B}{4\omega_n^2} \sin \alpha \quad \eta_1 = \gamma_2 \frac{F}{2\omega_n^2} \cos \alpha; \quad \eta_2 = \gamma_2 \frac{B}{2\omega_n^2} \cos \alpha.
 \end{aligned} \tag{2.43}$$

Chapter 3

Theoretical Analysis

The previous chapter discussed the derivation of a general linear model representing the dynamics of a cantilevered unimorph harvester. In this chapter, the special case of the piezoelectric bi-morph, shown in Figure 3.1, is considered where two piezoelectric layers cover both sides of the beam entirely. As such, there is no longer a shift in the neutral axis of the composite beam, and hence, the mass per unit length $M(s)$, the effective modulus of elasticity $EI(s)$, and the electromechanical coupling $\theta(s)$ are not functions of the arc length s , but are constant. The properties of the piezoelectric bi-morph that will be used throughout this chapter can be seen in Table 3.1, where it must be noted that the length of the piezoelectric layer, L_p , is equivalent to that of the beam, L_b . Theoretical simulations are carried out in order to validate the claim that parametric amplification can, under some conditions, amplify the power output. To achieve this goal, an asymptotic solution governing the response of Equation (2.42) is obtained using the method of multiple scales [31] and analyzed for different design parameters.

Figure 3.1: Schematic of a piezoelectric bimorph cantilever-type energy harvester.

Table 3.1: Geometric and material properties of the bi-morph cantilever beam.

Properties/Beam Material	
Modulus of elasticity, $E_b[GPa]$	200
Density, $\rho_b[kg/m^3]$	7800
Length, $L_b[mm]$	150
Width, $w[mm]$	20
Thickness, $t_b[mm]$	0.13
Piezoelectric layer	
Density, $\rho_p[kg/m^3]$	7850
Modulus of elasticity, $E_p[GPa]$	66
Thickness, $t_p[mm]$	0.03
Electromechanical coupling constant, $d_{31}[pC/N]$	-212
Permittivity, $\epsilon_{33}[nF/m]$	13.28
Other	
Acceleration at fundamental frequency, $F[m^2/sec]$	0.065 g
Load Resistance, $R[kOhm]$	300

3.1 Asymptotic Solutions

The steady-state solutions for the deflection of the beam and voltage are needed to assess the effect that the super-harmonic has on the amount of power the energy harvester can produce. This is done by utilizing the method of multiple scales [31], which expands the time dependence in terms of difference time scales as

$$T_n = \varepsilon^n \tau, \quad \text{where } n = 0, 1, \dots \quad (3.1)$$

where ε is a bookkeeping parameter. The time derivatives are given by

$$\begin{aligned} \frac{d}{d\tau} &= D_0 + \varepsilon D_1 + O(\varepsilon^2), \\ \frac{d^2}{d\tau^2} &= D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2), \end{aligned} \quad (3.2)$$

where D_0 and D_1 represent the partial derivatives with respect to T_0 and T_1 , respectively. The deflection and voltage must also be expanded as

$$q(\tau) = q_0(T_0, T_1) + \varepsilon q_1(T_0, T_1) + O(\varepsilon^2), \quad (3.3a)$$

$$V(\tau) = V_0(T_0, T_1) + \varepsilon V_1(T_0, T_1) + O(\varepsilon^2). \quad (3.3b)$$

The effect of damping, forcing functions, and the electromechanical coupling is scaled to be on the same order of the perturbation problem; therefore, we let

$$\zeta = \varepsilon \zeta, \quad \eta_1 = \varepsilon \eta_1, \quad \eta_2 = \varepsilon \eta_2, \quad \lambda_1 = \varepsilon \lambda_1, \quad \lambda_2 = \varepsilon \lambda_2, \quad \theta_1 = \varepsilon \theta_1. \quad (3.4)$$

Since large amplitude responses will appear near the mechanical natural frequency, we choose the excitation frequency, Ω , to be close to the natural frequency (normalized to one) by introducing the detuning parameter, σ , such that

$$\Omega = 1 + \varepsilon \sigma, \quad (3.5)$$

Substituting Equations (3.2) - (3.5) into Equation (2.43), and collecting terms in equal powers of ε gives

$O(1)$:

$$D_0^2 q_0 + q_0 = 0, \quad (3.6a)$$

$$C_p D_0 V_0 + \theta_2 V_0 = \alpha_2 D_0 q_0, \quad (3.6b)$$

$O(\varepsilon)$:

$$\begin{aligned} D_0^2 q_1 + q_1 = & -2D_0 D_1 q_0 - 2\zeta D_0 q_0 - \theta_1 V_0 - 4q_0[\lambda_1 \cos(T_0 + \sigma T_1 + \Phi) \\ & + \lambda_2 \cos(2T_0 + 2\sigma T_1)] + 2[\eta_1 \cos(T_0 + \sigma T_1 + \Phi) + \eta_2 \cos(2T_0 + 2\sigma T_1)], \end{aligned} \quad (3.7a)$$

$$C_p D_0 V_1 + \theta_2 V_1 = \alpha_2 (D_0 q_1 + D_1 q_0) - C_p D_1 V_0, \quad (3.7b)$$

The solution to Equations (3.6a) and (3.6b) is given by

$$q_0 = A(T_1)e^{iT_0} + cc, \quad (3.8a)$$

$$V_0 = ZA(T_1)e^{iT_0} + cc, \quad (3.8b)$$

where cc is the complex conjugate of the preceding term, $A(T_1)$ is a complex valued function, and $Z = \frac{i\alpha_2}{C_p i + \theta_2}$. Substituting the zeroth-order perturbation solution, Equations (3.8a) and (3.8b), into Equation (3.7a), and setting the secular terms with coefficients of $e^{\pm iT_0}$ to zero yields

$$-2\zeta i A(T_1) - 2i D_1 A(T_1) - i \frac{\alpha_2 \theta_1 (\theta_2 - C_p i)}{C_p^2 + \theta_2^2} A(T_1) - 2\lambda_2 \bar{A}(T_1) e^{2i\sigma T_1} + \eta_1 e^{i\Phi} e^{i\sigma T_1} = 0. \quad (3.9)$$

To solve Equation (3.9), we express the complex-valued function $A(T_1)$ in the polar form, $A(T_1) = 1/2a(T_1)e^{i\beta(T_1)}$; where $a(T_1)$ and $\beta(T_1)$ are real-valued functions representing the amplitude and phase of the response, respectively. We then substitute the polar expansion into Equation (3.9) and separate the real and imaginary terms to obtain

$$D_1 a = -(\zeta + \zeta_e)a - \lambda_2 a \sin 2\delta + \eta_1 \sin(\delta + \Phi), \quad (3.10a)$$

$$a D_1 \delta = (\sigma + \omega_e)a - \lambda_2 a \cos 2\delta + \eta_1 \cos(\delta + \Phi), \quad (3.10b)$$

where

$$\delta = \sigma T_1 - \beta, \quad \zeta_e = \frac{1}{2} \frac{\alpha_2 \theta_1 \theta_2}{(\theta_2^2 + C_p^2)}, \quad \omega_e = -\frac{1}{2} \frac{\alpha_2 \theta_1 C_p}{(\theta_2^2 + C_p^2)}.$$

Here, ζ_e corresponds to the electric damping in the system as a result of the energy harvesting process, and ω_e represents a shift in the natural frequency due to the electric coupling. To obtain the steady-state amplitude and phase of the response, the time derivatives are set to zero in Equations (3.10a) and (3.10b). The resulting algebraic equations are subsequently solved for a_0 and δ_0 . This yields

$$a_0 = \eta_1 \sqrt{\frac{\lambda_2^2 + \zeta_{eff}^2 + \omega_{eff}^2 + 2\lambda_2[\omega_{eff} \cos(2\Phi) - \zeta_{eff} \sin(2\Phi)]}{(\lambda_2^2 - \zeta_{eff}^2 - \omega_{eff}^2)^2}}, \quad (3.11)$$

$$\delta_0 = \arctan \left(\frac{(\lambda_2 - \omega_{eff}) \sin \Phi - \zeta_{eff} \cos \Phi}{(\lambda_2 + \omega_{eff}) \cos \Phi - \zeta_{eff} \sin \Phi} \right), \quad (3.12)$$

where $\zeta_{eff} = \zeta + \zeta_e$ and $\omega_{eff} = \sigma + \omega_e$. With that and using Equations (3.8a) and (3.8b), the steady-state response amplitude and output voltage can be written as

$$q_{ss}(t) = a_0 \cos(\Omega t - \delta_0), \quad V_{ss}(t) = \frac{\alpha_2 a_0}{\sqrt{(C_p^2 + \theta_2^2)}} \cos(\Omega t - \delta_0 + \psi), \quad (3.13)$$

where $\psi = \arctan \theta_2$. The steady-state power dissipated in the load can be further expressed as

$$P_{ss} = \frac{\alpha_2^2 a_0^2}{R(C_p^2 + \theta_2^2)}. \quad (3.14)$$

To confirm these steady-state analytical solutions are accurate, the steady-state voltage given by Equation (3.13) is compared to the voltage obtained by numerically integrating the discretized equations of motion, Equation (2.42). Steady-state frequency-response curves for three beam tilt angles are depicted in Fig. 3.2. Simulations obtained using the numerical values listed in Table 3.1, demonstrate excellent agreement for the different values of the beam tilt angle. The figure also illustrates the amplified response near resonance due to the change in the tilt angle of the beam.

Figure 3.2: Theoretical steady-state voltage-response curves of the cantilever bi-morph for different values of the tilt angle α . ($\alpha = 0[\text{rad}]$, solid line, $\alpha = \pi/18[\text{rad}]$, dashed lines, and $\alpha = 5\pi/36[\text{rad}]$, dotted lines). Numerical integration results are represented by dots. The values used for parametric excitation amplitude, damping, and phase angle are $B = 0.06g \text{ m/sec}^2$, $\zeta = 0.004$, and $\phi = 3\pi/4[\text{rad}]$, respectively.

3.1.1 Validity of The Attained Solutions

By inspecting Equation (3.11), it becomes evident that the steady-state amplitude of the response can be amplified when the denominator approaches zero regardless of the amplitude of the direct excitation η_1 . This occurs when $\lambda_2^2 \rightarrow \zeta_{eff}^2 + \omega_{eff}^2$. This represents the same excitation threshold above which the well-known *parametric instability* is activated [31]. When $\lambda_2^2 > \zeta_{eff}^2 + \omega_{eff}^2$, the beam can undergo large amplitude oscillations due to the principle parametric resonance. In such a case, the amplitude of the system's response can be predicted only when including the limiting beam geometric nonlinearities which were neglected in this study. As such, the solutions presented herein can predict the actual response behavior of the VEH only when $\lambda_2^2 < \zeta_{eff}^2 + \omega_{eff}^2$. This, however, does not impose any additional constraints on the scope of this study, as the amplification occurs below this threshold.

3.2 Optimizing the Phase Angle

To assess the influence of the super-harmonic on the response amplitude near the primary resonance, we define the following measure of the parametric gain:

$$\chi = \zeta_{eff} \frac{\sqrt{\lambda_2^2 + \zeta_{eff}^2 + \omega_{eff}^2 - 2\lambda_2\zeta_{eff}\sin(2\Phi)}}{\lambda_2^2 - \zeta_{eff}^2}. \quad (3.15)$$

Equation (3.15) represents the ratio between the steady-state response amplitude at resonance ($\sigma = \omega_e$) for any beam tilt angle α , and the steady-state response amplitude at resonance for the purely direct excitation case ($\sigma = \omega_e$, $\alpha = 0$). This gain reveals that the potential of utilizing the parametric amplification phenomenon to enhance the output power of a VEH depends on the effective damping ζ_{eff} , the design parameters of the beam embedded within λ_2 , and the excitation parameters which include the acceleration magnitude of the super-harmonic and the phase angle between the primary and super-harmonic components, Φ .

In order to determine the phase angle that maximizes the parametric amplification gain, the partial derivative of Equation (3.15) with respect to the phase angle is set equal to zero; this yields

$$\frac{\partial \chi}{\partial \Phi} = -2 \frac{\zeta_{eff}}{\lambda_2^2 - \zeta_{eff}^2} [\lambda_2^2 + \zeta_{eff}^2 - 2\lambda_2\zeta_{eff}\sin 2\Phi_{opt}] [\lambda_2\zeta_{eff}\cos 2\Phi_{opt}] = 0. \quad (3.16)$$

For nontrivial solutions, the second bracketed item is set to zero. Thus,

$$\cos 2\Phi_{opt} = 0, \Rightarrow \Phi_{opt} = \frac{n\pi}{4}, \text{ where } n \text{ is odd.} \quad (3.17)$$

It is important to note that the optimal phase angle is not dependent on α . As such, it will remain constant for all tilt angles. Since we are dealing with trigonometric functions, the values of Φ_{opt} will result in alternating maxima and minima for χ . When the second derivative of χ with respect to Φ is negative (concave down), the amplification gain will observe a maximum and vice versa.

$$\frac{\partial^2 \chi}{\partial \Phi^2} = 4 \frac{\lambda_2 \zeta_{eff}^2}{\lambda_2^2 - \zeta_{eff}^2} \{ [\lambda_2^2 + \zeta_{eff}^2] \sin 2\Phi_{opt} - 2\lambda_2\zeta_{eff} \sin^2 2\Phi_{opt} \}. \quad (3.18)$$

The concavity of the first two n values ($\pi/4$ and $3\pi/4$) is tested to determine which corresponds to a maximum in χ :

$$4 \frac{\lambda_2 \zeta_{eff}^2}{\lambda_2^2 - \zeta_{eff}^2} \{(\lambda_2 - \zeta_{eff})^2\} \text{ for } \Phi_{opt} = \pi/4;$$

$$4 \frac{\lambda_2 \zeta_{eff}^2}{\lambda_2^2 - \zeta_{eff}^2} \{-(\lambda_2 + \zeta_{eff})^2\} \text{ for } \Phi_{opt} = 3\pi/4. \quad (3.19)$$

By inspecting Equations (3.19), it becomes apparent that, in order to maximize the parametric amplification gain, the phase angle between the two excitations must be $3\pi/4$ [rad]. This can also be observed in Fig. 3.3, which verifies that the optimal phase angle is not affected by the tilt angle. It must be noted that there is a small window of phase angles for which the amplification gain is greater than one. Therefore, analysis of the characteristics of the excitation source is essential prior to exploring the parametric amplification for energy harvesting.

Figure 3.3: Parametric amplification gain as a function of the phase angle for different tilt angles. The values used for the parametric excitation amplitude and damping are $B = 0.06g \text{ m/sec}^2$ and $\zeta = 0.004$, respectively.

3.3 Optimizing the Tilt Angle

Another important factor influencing the parametric amplification gain is the amplitude of the super-harmonic excitation, B , which is embedded in λ_2 of Equation (3.15). Referring back to Equation (2.43), λ_2 is directly related to B and $\sin \alpha$. Therefore as the excitation amplitude is increased, the amplification gain also increases due to the larger energy component at the super-harmonic frequency. Figure 3.4 shows this increase for various tilt angles. When the tilt angle is small, even a relatively small super-harmonic excitation amplitude will result in a parametric amplification gain that is greater than one. On the other hand, for larger tilt angles, the super-harmonic component of acceleration must be larger than the direct excitation component in order to observe any benefit. This is expected due to the decrease in energy channeled from the direct excitation as the tilt angle increases.

Figure 3.4: Parametric amplification gain as a function of the super-harmonic excitation amplitude for different tilt angles. The optimal phase angle and $\zeta = 0.004$ are used.

It is evident that for a given super-harmonic excitation, there exists a tilt angle that optimizes the amplification gain, and thus the energy the device can harvest. To

obtain the optimal tilt angle, we differentiate Equation (3.15) with respect to α and solve the resulting equation for α_{opt} . This yields

$$\alpha_{opt} = \arcsin \frac{\gamma_1}{4L\omega_n^2} \frac{B}{\zeta_{eff}}, \quad (3.20)$$

where B and ζ_{eff} can be varied, while the remainder of the parameters stay constant. Equation (3.20) reveals that the optimal tilt angle is directly proportional to the magnitude of the super-harmonic and inversely proportional to the effective damping. Figure 3.5 presents the change in the optimal tilt angle as the super-harmonic amplitude and mechanical damping vary. For larger values of damping (i.e. $\zeta = 0.004$) the relationship between the optimal tilt angle and the amplitude of the super-harmonic excitation is almost linear. When the damping decreases, the optimal tilt angle increases as expected, and for low damping (i.e. $\zeta = 0.001$) very large tilt angles maximize the amplification gain even for small super-harmonic components of acceleration. As the effective damping decreases, larger optimal tilt angles are necessary to maximize the parametric gain. This is expected because the parametric component of acceleration tends to have more influence for smaller damping ratios. As such, in order to maximize the gain, the beam should be tilted towards the parametric component of acceleration when its magnitude increases or when the effective damping decreases.

The optimal tilt angle also depends on the electric damping, which, in turn, is a function of the electric load resistance. Figure 3.6 depicts variation of the optimal tilt angle with the load resistance for different values of the mechanical damping. It is evident that the optimal tilt angle is not very sensitive to variations in the load resistance when the mechanical damping ratio is large. This stems from the smaller influence of the electric damping on the total effective damping. As such, even when R and hence ζ_e vary significantly, their influence on the effective damping, and, hence, on the optimal angle is negligible. However, when the mechanical damping ratio is very low, small variations in R can yield significant change in the optimal

Figure 3.5: Optimal tilt angle as a function of the super-harmonic excitation amplitude for different values of mechanical damping.

tilt angle.

Figure 3.6: Optimal tilt angle as a function of the load resistance for different values of mechanical damping.

3.4 Output Power

In this section, we investigate how enhancement in the parametric gain influences the output power. Figure 3.7 depicts percentage power enhancement contours as function of the magnitude of the super-harmonic acceleration and the tilt angle α . It is evident that significant power enhancement can be achieved at resonance especially as the super-harmonic acceleration component increases and becomes comparable in magnitude to the fundamental component. For instance, when $B = F = 0.06g \text{ m/sec}^2$, a 21% improvement in the output power is attained when the beam is tilted by $\alpha = 0.4[\text{rad}]$. Even when the acceleration magnitude drops to $0.03g$, power enhancement of around 4% is still attainable.

Figure 3.7: Contours for percentage power enhancement as the super-harmonic excitation amplitude and tilt angle vary. The optimal phase angle and $\zeta = 0.004$ are used.

Such numbers can be vastly improved when the damping ratio is decreased to $\zeta = 0.002$ as shown in Figure 3.8. Significant enhancement in the harvester's output power becomes possible even when the magnitude of acceleration associated with the super-harmonic component decreases to very small values when compared to the direct excitation component. This can prove very beneficial when designing harvesters at the microscale. In such applications, the harvester can be packaged under almost vacuum conditions, which permits achieving extremely high quality factors.

Figure 3.8: Contours for percentage power enhancement as the super-harmonic excitation amplitude and tilt angle vary. The optimal phase angle and $\zeta = 0.002$ are used.

While the mechanical damping ratio can be altered to enhance the amplification gain, the electric damping can only be altered by changing the load resistance (as-

suming the electromechanical coupling is constant). As such, it places a limit on how small the effective damping of the system can be. Furthermore, reducing the electrical damping to enhance the parametric gain could have an adverse influence on the output power. Figure 3.9 demonstrates variation of the electric damping, and amplification gain with the load resistance. As the load resistance increases from short to open circuit conditions, the electric damping increases initially, exhibits a peak, then decreases again. As a result, the parametric gain, which is expected to follow opposite trends, decreases initially, exhibits a minimum, then increases again.

At one point, as the electric damping increases and the parametric amplification gain decreases for small electric loads, these opposite effects balance each other yielding a peak in the output power, Figure 3.10 (a). Similarly, for large electric loads, as the electric damping decreases and the parametric amplification gain increases, their effects balance each other resulting in a second peak in the output power. At the value of the load resistance where the electric damping exhibits a peak and the parametric gain exhibits a minimum, the output power exhibits a local minimum.

When the damping ratio increases, the maximum power decreases and the optimal load resistances come closer together merging into one global optimal value, Figure 3.10(c). This is generally in agreement to what has been already observed in the optimization of the electric load for the common directly-excited energy harvester [32]. In that case, two optimal load resistances, one corresponding to the resonance frequency and the other to the anti-resonance frequency exist for small damping ratios, while only one optimal electric load exists beyond a threshold damping.

Since maintaining an optimal load resistance is essential toward maximizing the output power, it is critical to investigate the effect of the tilt angle on the optimal electric load. Towards that end, we study variation of the output power with the load resistance and the tilt angle. As depicted in Fig. 3.11 (a), for a small mechanical damping ratio and small values of the tilt angle, only one optimal load exists. This

Figure 3.9: (a) Electric damping and (b) Parametric amplification gain as a function of the load resistance. Results are obtained for $B = 0.06g \text{ m/sec}^2$, $\alpha = \pi/6[\text{rad}]$, $\phi = 3\pi/4[\text{rad}]$, and $\zeta = 0.001$.

global maximum is insensitive to variations in α . However, as the tilt angle exceeds a threshold value, α_{cr} , this same optimal load resistance produces a minimum in the output power, and two new optimal loads corresponding to two equal maxima are born. The smaller of the two optimal loads decreases with the tilt angle while the larger increases. If we were to consider α as a bifurcation parameter, then one can loosely portray this behavior as a pitchfork bifurcation occurring in the $R_{opt} - \alpha$ space at some α_{cr} . As shown in Fig. 3.11 (b), for a larger mechanical damping ratio, this behavior ceases to occur and only one optimal load, which is insensitive to variations in α , exists.

(a)

(b)

(c)

Figure 3.10: Power output as a function of load resistance for mechanical damping values of (a) $\zeta = 0.001$, (b) $\zeta = 0.0012$, and (c) $\zeta = 0.004$. Results are obtained for $B = 0.06g \text{ m/sec}^2$, $\alpha = \pi/6[\text{rad}]$, and $\phi = 3\pi/4[\text{rad}]$.

(a)

(b)

Figure 3.11: Power output as a function of both the load resistance and tilt angle for a mechanical damping of (a) $\zeta = 0.0015$ and (b) $\zeta = 0.002$. Black dots represent the maximum power at each tilt angle.

Chapter 4

Experimental Results

Experimental testing is performed to verify the theoretical claims made in the previous chapters. The test apparatus shown in Fig. 4.1 is designed to mimic the theory presented in Chapter 2, with the piezoelectric patch partially covering one side of the cantilever beam. A Macro Fiber Composite (MFC) patch from *Smart Material* [33] was used as the active energy harvesting element. The composite beam is securely fastened to a base to create perfect clamping conditions. The base is attached to an adjustable setup that permits tilting the beam through different angles with respect to the direction of input acceleration provided by the shaker. The properties of the harvesting beam are listed in Table 4.1, where the properties of the piezoelectric layer are obtained from *Smart Material* [33]. The fundamental frequency of acceleration is measured via an accelerometer and the voltage is obtained across an open circuit.

We start by verifying the optimal phase angle, Φ , between the two excitation components which was theoretically determined to be constant at $\Phi_{opt} = 3\pi/4$ [rad] for any design parameters. Figure 4.2 shows that the experimental results closely agree with the theoretical findings. They both show that maximum output voltage occurs at a phase angle of $\Phi_{opt} = 3\pi/4$ [rad]. The experiment is repeated for varying tilt

Figure 4.1: Experimental Apparatus.

Figure 4.2: Steady-state voltage output as a function of the phase angle $\Phi[rad]$ for a tilt angle of $\alpha = 5\pi/36[rad]$, super-harmonic excitation of $B = F$, and fundamental excitation frequency of $\omega = 13.5 Hz$. Circles represent experimental data.

angles and different super-harmonic excitation amplitudes. The maximum voltage output always corresponds to a phase angle which is very close to $3\pi/4 [rad]$. Therefore, this optimal phase angle will be used for the remainder of the experimental

Table 4.1: Geometric and material properties of the experimental setup.

Properties/Beam Material	
Modulus of elasticity, $E_b[GPa]$	200
Density, $\rho_b[kg/m^3]$	7800
Length, $L_b[mm]$	167.5
Width, $w[mm]$	14.6
Thickness, $t_b[mm]$	0.508
Piezoelectric layer	
Density, $\rho_p[kg/m^3]$	4750
Modulus of elasticity, $E_p[GPa]$	15.86
Length, $L_p[mm]$	108
Thickness, $t_p[mm]$	0.3
Electromechanical coupling constant, $d_{31}[pC/N]$	-470
Permittivity, $e_{33}[nC/m]$	13.28
Other	
Fundamental frequency, $\omega[Hz]$	13.77
Acceleration at fundamental frequency, $F[m^2/sec]$	5.9
Mechanical damping, ζ	0.013

study.

The tilt angle of the beam is adjusted in increments of 5° ($\pi/36$ [rad]), starting at zero (purely direct excitation) up to 40° ($2\pi/9$ [rad]). For each tilt angle, the experiments were repeated using four values of the super-harmonic excitation amplitude, while the amplitude of the fundamental component is kept constant. First, the effect of the tilt angle on the steady-state response is studied while keeping the amplitude of the super-harmonic excitation constant. Figure 4.3 depicts the frequency response of

Figure 4.3: Steady-state voltage output as a function of the tilt angle α [rad] ($\alpha = 0$, blue dotted line and empty circles, $\alpha = \pi/18$ [rad], red dashed line and empty squares, and $\alpha = 5\pi/36$ [rad], black solid line and solid circles) for a phase angle of $\Phi = 3\pi/4$ [rad] and super-harmonic excitation of $B = 3F$. Points represent experimental data.

the purely direct case ($\alpha = 0$) as well as two other tilt angles. Both the theoretical and experimental results demonstrate an increase in the voltage output near the fundamental frequency when the beam is tilted. While the experimental data does not show as much of an increase as the theoretical results, the increase in output voltage is still close to 10% at $\alpha = 5\pi/36$ [rad] when compared to the purely direct excitation case.

The steady-state voltage response is also influenced by the super-harmonic excitation amplitude as seen in Fig. 4.4. As expected larger excitations result in larger voltage outputs. Again, the experimental data does not show as much of an increase as the theoretical results, but the increase is still significant and follows similar trends. For this experimental scenario, and since the damping is significantly larger than the theoretical values used in the previous chapters, the super-harmonic excitation needs to be greater than the fundamental component to observe any benefit from

tilting the angle of the beam.

Figure 4.4: Steady-state voltage output as a function of the super-harmonic excitation amplitude B . ($B = F/2$, blue dotted line and empty squares; $B = F$, red dashed line and empty circles; $B = 2F$, green dash-dot line and solid squares; and $B = 3F$, black solid line and solid circles). Results are obtained for a phase angle of $\Phi = 3\pi/4[rad]$ and a tilt angle of $\alpha = 5\pi/36[rad]$. The long dashed gray line represents the purely direct excitation case. Points represent experimental data.

To obtain a better grasp of how both the super-harmonic excitation and tilt angle influence the steady-state response, Fig. 4.5 depicts the maximum voltage output as a function of the tilt angle for four excitation levels. The experimental data follows similar trends as the theoretical results, both showing an increase in the optimal phase angle as the super-harmonic excitation increases. This can be clearly seen when inspecting Equation (3.20). For $B = F/2$ and $B = F$, there is a small increase in the output voltage for small tilt angles followed by a decrease in the output voltage at larger tilt angles. For $B = 2F$ and $B = 3F$, the voltage increase is much more significant, and there exists an optimal tilt angle that maximizes the voltage output for a given super-harmonic excitation.

As demonstrated theoretically in the previous chapter, the mechanical damping has

Figure 4.5: Maximum voltage output as a function of the tilt angle α for various super-harmonic excitation amplitudes B . ($B = F/2$, blue dotted line and empty circles; $B = F$, red dashed line and empty squares; $B = 2F$, green dash-dot line and solid circles; and $B = 3F$, black solid line and solid squares). Results are obtained for a phase angle of $\Phi = 3\pi/4[\text{rad}]$. Points represent experimental data.

a considerable influence on the performance of the device. The mechanical damping ratio also affects how the tilt angle and super-harmonic excitation amplitude influence the voltage output. The mechanical damping was obtained experimentally to be $\zeta = 0.013$. Figure 4.6 demonstrates how the voltage response varies significantly when the damping is just slightly reduced to $\zeta = 0.012$. As predicted, the smaller mechanical damping yields an increase in the response amplitude. For similar values of the amplitude of the super-harmonic component, the increase in the voltage output is much larger for a smaller damping ratio demonstrating again the importance of achieving the smallest mechanical damping for enhanced performance.

Figure 4.6: Voltage response curves for varying tilt angles and mechanical damping. ($\alpha = 0$ and $\zeta = 0.013$, blue dashed line; $\alpha = 5\pi/36[rad]$ and $\zeta = 0.013$, red solid line; $\alpha = 0$ and $\zeta = 0.012$, green dotted line; and $\alpha = 5\pi/36[rad]$ and $\zeta = 0.012$, black dash-dot line). Results are obtained for a constant super-harmonic excitation amplitude $B = 3F$.

Chapter 5

Conclusions and Future Work

This Thesis provides an initial theoretical and experimental investigation into the prospect of utilizing a parametric pump to enhance the energy transduction of a VEH subjected to an excitation having a combination of a fundamental harmonic and a super-harmonic at twice the fundamental frequency. Due to inherent nonlinearities in the dynamics of the excitation source, such excitations can be commonly found in the environment. Traditionally, two linear energy harvesters, each tuned to one of these frequency components are necessary to efficiently harvest energy from such excitations. However, this has adverse influence of reducing the power density of the harvester.

To that end, we study the response behavior of a piezoelectric cantilevered-type harvester whose axis is tilted with an angle with respect to the direction of the input acceleration. This creates two acceleration components; one is perpendicular and the other is parallel to the axis of the beam. Because of the presence of the super-harmonic component in the frequency spectrum, the component parallel to the axis of the beam can act as a parametric pump which amplifies the response amplitude near the fundamental frequency of the harvester. To study the response characteristics of the harvester, an electromechanical model governing the system

dynamics is obtained using Hamilton's extended principle. The model adopts Euler-Bernoulli's beam theory for thin beams combined with linear constitutive relations for the structural and active piezoelectric layers. The resulting partial-differential equations and associated boundary conditions are subsequently discretized using a single-mode Galerkin expansion.

The method of multiple scales is used to obtain an analytical solution of the resulting reduced-order model. Steady-state analytical expressions for the beam deflection, output voltage, and power are derived and validated against a numerical integration of the original equations of motion. The resulting expressions are then used to investigate how the phase angle between the two excitation components, the tilt angle of the beam, the amplitude of super-harmonic component, the mechanical and electrical damping influence the steady-state response of the system.

Results reveal that, while possible, power enhancement near the fundamental frequency depends largely on the excitation parameters and the amount of mechanical damping present in the system. Specifically, it is observed that, for the parametric pump to amplify the response near the fundamental frequency, the super-harmonic component should be delayed by a range of phase angles, or otherwise the response amplitude can be deamplified (reverse energy pumping). The optimal phase angle was found to correspond to $3\pi/4$ [rad] delay between the super-harmonic and direct components. As such, for enhanced performance, significant analysis of the environmental excitation is necessary prior to designing of the harvester.

Also it is shown that there exists an optimal tilt angle that maximizes the power at resonance. The optimal tilt angle is a function of the properties of the beam, the amplitude of the super-harmonic excitation, and the effective damping of the system, which is the sum of mechanical and electrical damping. As the super-harmonic excitation increases or the damping decreases, a larger tilt angle is necessary to maximize the voltage output. Furthermore, it is shown that, for a large mechan-

ical damping ratio, improvement in the output power is mostly pronounced when the magnitude of the super-harmonic is comparable to that of the fundamental frequency. On the other hand, when the mechanical damping ratio is very small, significant enhancement in the output power is attainable even when the magnitude of the super-harmonic is very small when compared to the fundamental frequency. Such findings reveal that, under certain conditions, parametric amplification can be utilized to enhance the output power of a VEH especially at the microscale where the damping ratio can be easily controlled.

The Thesis also involved an experimental component to validate the theoretical claims. The experimental data are shown to have good agreement with the theoretical trends. However, the enhanced performance was not as pronounced. The main reason being the presence of larger damping in the experiments. As such, the super-harmonic excitation had to be larger than the fundamental component in order to observe an increase in the output power. The concept of this harvester is more targeted towards microscale applications where the damping ratio can be very small.

Including the inherent beam nonlinearities in the Euler-Bernoulli beam theory would increase the accuracy of the theoretical results. Also, introducing some of the nonlinearities detailed in Section 1.2, and observing their response to the direct and parametric excitations described is an area that could be looked into further. Future work would need to be done to ensure that this device would operate as expected on the microscale.

Appendix A

Location of the Neutral Axis

Figure A.1: Cross-section of uni-morph piezoelectric cantilever-type energy harvester.

As shown in Figure A.1, the location of the neutral axis relative to the bottom surface of the beam, h_a , occurs where the bending stress, σ , is equal to zero. Therefore summing the forces in the x -direction gives us

$$\sum F_x = 0 = \int \sigma dA = \int_{A_b} \sigma_b dA + \int_{A_p} \sigma_p dA. \quad (\text{A.1})$$

Since this is a composite beam, the radius of curvature is the same for both materials

and Equation (A.1) becomes

$$\sum F_x = 0 = E_b \int_{A_b} z_b dA + E_p \int_{A_p} z_p dA, \quad (\text{A.2})$$

where the integrals are the first moment of each material area, and z_b and z_p are the distance from the neutral axis to the centroid of the structural and piezoelectric layer, respectively. Since the location of the centroids of each material is known, the first moment can be found by multiplying the distance the centroid is from the neutral axis by the area. Therefore Equation (A.2) becomes

$$\sum F_x = 0 = E_b z_b A_b + E_p z_p A_p, \quad (\text{A.3})$$

where

$$z_b = t_b - h_a; \quad A_b = W t_b; \quad z_p = t_b + t_p - h_a; \quad A_p = W t_p. \quad (\text{A.4})$$

Solving Equation (A.3) for h_a and using Figure 2.1 gives

$$h_a = \frac{1}{2} \frac{E_b t_b^2 + 2E_p t_b t_p + E_p t_p^2}{E_b t_b + E_p t_p}; \quad h_b = t_b - h_a; \quad h_c = t_p + h_b. \quad (\text{A.5})$$

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