DETERMINATION OF HIGHER ORDER COEFFICIENTS FOR A CRACK PARALLEL TO AN INTERFACE

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DETERMINATION OF HIGHER ORDER COEFFICIENTS FOR A CRACK PARALLEL TO AN INTERFACE

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mechanical Engineering

by
Venkata Lakshman Kolluru
August 2010

Accepted by:
Dr. Paul F. Joseph, Committee Chair
Dr. Lonny L. Thompson
Dr. Gang Li
ABSTRACT

A general method based on the singular integral equations is developed to computationally determine the higher order coefficients in mixed mode fracture mechanics. These “k” and “T” coefficients are defined with respect to a polar coordinate system centered at a crack tip, and give asymptotic expressions for stresses and displacements according to the William’s eigenfunction expansions,

\[
\sigma_{ij}(r, \theta) = \sum_{n=0}^{\infty} (2r)^{n-1/2} \left[ k_n^i f_0^j (n, \theta) + k_n^r f_0^j (n, \theta) \right] \\
+ T_n^i f_0^{IT} (0, \theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T_n^i f_j^{IT} (n, \theta) + T_n^r f_j^{IT} (n, \theta) \right], \quad i = r, \theta; \quad j = r, \theta,
\]

In the above expression the \( n = 0 \) terms correspond to the modes I and II stress intensity factors and the so called, T-stress. From a method point of view, the higher order k-coefficients are easily obtained, while the T-coefficients require significant post-processing of the singular integral equation solution. A planar crack parallel to an interface between two elastic materials and subjected to far-field tension is considered as an example and extensive results are presented. This example is chosen due to the anomalous behavior of a closing crack tip as the crack approaches the interface for certain material combinations. Such “Comninou contact zones” occur even in a tensile field when the crack is within a critical distance from the interface. Numerous results are provided that compare the asymptotic solutions with that of the full-field. It is shown that up to four k-coefficients and many T-coefficients can be determined for \( h/a = 0.001 \), where \( h \) is the distance of the crack from the interface and \( a \) is the half-crack length. While the application of the method to the case of a crack parallel and very close to an interface focuses on the anomaly of a closing crack tip, in general the ability to determine
higher order coefficients can be used to quantify the size of the zone in which linear fracture mechanics is valid.
DEDICATION

To my parents Mr. Kolluru Sri Rama Sanyasi Setty and Mrs. Kolluru Venkata Ratna
Kumari.
ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Paul F. Joseph, for his guidance and support throughout the thesis. I also thank my committee members Dr. Lonny L. Thompson and Dr. Gang Li, for their patience and understanding.
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CHAPTER I

INTRODUCTION

A crack paralleling an interface of two bonded dissimilar linearly elastic and isotropic materials is considered for discussion in this thesis. Consider the following figure 1.1 which defines the geometry.

Figure 1.1 Two bonded dissimilar materials with a crack parallel to the interface of two materials

In the above figure, ‘a’ is the half-crack length, $\mu_1, \kappa_1$ and $\mu_2, \kappa_2$ are the material properties of the material-1 and material-2 respectively. The material which is above the interface and has the crack is material-1. The one below the interface is material-2. The interface lies parallel to the crack and the crack is along the X-axis. The center of the crack is considered as the origin. ‘h’ is the distance of the crack from the interface and is
very small compared to the crack length ‘2a’. A tensile load of $\sigma_0$ is applied normal to the interface. When the tensile load is applied, shear may develop at the interface. To avoid the development of shear at the interface, loads $\sigma_1$ and $\sigma_2$ are applied parallel to the interface which compensates any shear occurred.

A general and accurate method is used to calculate the coefficients of asymptotic expansion of stresses. Singular integral equation approach is used to calculate the full field stresses and they are compared.

Stress intensity factor, $K$ derived using Linear Elastic Fracture Mechanics (LEFM), can be used to characterize the crack tip conditions. Fracture can be predicted by using stress intensity factor and can be considered a material constant. If any loading and geometry gives the same $K$, the material will respond in the same way with respect to crack growth. But there are many cases, where the stress intensity factor does not account for different geometric non-linearities and materials.

The plastic zone near the crack tip is very small when compared to other dimensions like crack length (2a). Linear Elastic Fracture Mechanics cannot be applied when the plastic zone is more widespread. Elastic-Plastic Fracture Mechanics (EPFM) is applied in such a case which uses J-Integral approach to define the stress field at the crack tip. To account for different materials and geometric non-linearity, the above two approaches might not be sufficient sometimes. In such a case higher order terms can be used.
In the polar co-ordinate system, the stresses and displacements can be expressed as asymptotic series for small distances from the crack tip. The Eigen function expansion near the crack tip according to Williams [1] can be expressed as

\[
\sigma_{ij}(r,\theta) = \sum_{n=0}^{\infty} (2r)^{n+\frac{1}{2}} \left[ k_n f_{ij}^n(n,\theta) + k_n^I g_{ij}^n(n,\theta) \right] + T_0 f_{ij}^I(0,\theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T_n^I f_{ij}^n(n,\theta) + T_n^I f_{ij}^I(n,\theta) \right], \quad i=r,\theta; \quad j=r,\theta,
\]

(1.1)

\[
2\mu u_r(r,\theta) = \sum_{n=0}^{\infty} (2r)^{n+\frac{1}{2}} \left[ k_n^I g_{r}^n(n,\theta) + k_n^I g_{r}^n(n,\theta) \right] + T_0^I (2r) g_{r}^I(0,\theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T_n^I g_{r}^n(n,\theta) + T_n^I g_{r}^n(n,\theta) \right] + E\cos(\theta) + F\sin(\theta),
\]

(1.2)

\[
2\mu u_\theta(r,\theta) = \sum_{n=0}^{\infty} (2r)^{n+\frac{1}{2}} \left[ k_n^I g_{\theta}^n(n,\theta) + k_n^I g_{\theta}^n(n,\theta) \right] + T_0^I (2r) g_{\theta}^I(0,\theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T_n^I g_{\theta}^n(n,\theta) + T_n^I g_{\theta}^n(n,\theta) \right] + G2r \frac{\kappa+1}{8} - E\sin(\theta) + F\cos(\theta).
\]

(1.3)

where, \(r\) is the small distance from the crack tip, \(k\) and \(T\) are coefficients that depend on geometry and loading and they are constant. Even though only a tensile load is applied, the problem has a mixed mode nature induced because of the two materials being bonded. This behaviour can be seen from equations (1.1), (1.2) and (1.3). \(f\) and \(g\) are angular functions. The expansion of the angular functions for stress and displacements for mode I and mode II are detailed in Appendix A.

If LEFM is applied to a problem, it has two length scales namely physical length scale \((r_p)\) and mathematical length scale \((r_m)\). The physical length scale defines the zone in
which LEFM does not consider the phenomena. While mathematical scale involves the truncation of the above series for acceptable level of error.

A singular intergration equation approach has been adopted for the first time for a mixed mode problem to determine the higher order coefficients although it has been applied for mode I and mode II cases separately. The amplitude of crack tip singularity is defined by stress intensity factor. If the stress field around the tip is completely characterized by stress intensity factor, it is called singularity dominated zone.

**Literature Review:**

Larsson and Carlsson [2] first came up with the significance of $T_0$ term in William’s expansion. They studied plastic zone ahead of crack tip using finite element analysis for commonly used fracture specimens. They determined that T-stress is the difference between $\sigma_x$ solutions using finite elements for different specimens and for boundary layer for elements along the crack surface. Leivers and Radon [3] came up with a ratio of stress intensity factor and T-stress as biaxiality ratio $B$, which was non-dimensionalised by geometric parameter like crack length $a$. They came up with the importance of T-stress as a secondary fracture parameter when two specimens are subjected to same stress intensity.

Betegon and Hancock [4] used the modified boundary layer approach and provided an elastic–plastic finite element solution. They showed that a negative T-stress near the crack tip reduces the stresses independent of radial distance from crack tip. Kfouri [5] used Eshelby’s method to evaluate elastic T-term. This involves determining
the T-terms using contour J-integrals along paths close to the crack tip for three different geometries. This method is also suited for finite element analysis.

Sham [6] developed higher order weight functions for calculating power expansion coefficients of a regular elastic field in a 2D body without body forces for both interior points and crack tips. Sham [7] determined the elastic T-term using higher order weight functions. Sham presented values of T-term for single notched specimens subjected tension loading, pure bending and three-point bend. The method to determine T-terms was based on finite element methods. Chidgzey and Deeks [8] determined the coefficients using scaled boundary finite element method. According to Chidgzey and Deeks, if the scaling center is at the crack tip, the scale boundary finite element solution gives the Williams [1] expansion and so stress intensity factor and T-stress can be determined easily. Xiao and Karihaloo [9] used hybrid crack element which allows the calculation of higher order terms directly.

Seed and Nowell [10] determined the T-stress using distributed dislocations method. The singular integral equation approach is used to calculate the stress intensity factors and T-stress by taking an example in which the crack is normal and inclined to a free surface of a half plane loaded by a far field tension. Broberg [11] determined the T-stress using dislocation arrays which gave more accurate results than the finite elements. Chen [12] et al. followed crack front position and crack back position techniques to calculate the T-stress at crack tip using complex variable function. Chen [13] et al. in another study used the perturbation method for a slightly curved crack. All the above work has been done to calculate higher order terms using different methods.
Erdogan [14] was the first to consider a crack parallel to an interface and calculate stress intensity factor based on a set of integral equations. Equations were solved for three adjoining sets of materials with symmetric and anti-symmetric uniform tractions on the crack surfaces. Hutchison et al. [15] derived conditions for a crack to propagate parallel to an interface between two bonded dissimilar materials. The work shows how the stress intensity factors for such a problem can be calculated if the loading and geometry are known.

England [16] considered a crack along the interface of two materials. When equal and opposite normal pressure are applied on a crack there is anomalous behavior at crack tips. According to England, the upper and lower surfaces of the crack wrinkle and overlap near the crack tip which is not physically possible. Comninou and Dundurs [17] came up with a mathematical solution which is a closed crack tip with a small contact zone. Gautesen and Dundurs [18] came up with a solution to how to solve the integral equation exactly. They came up with simple formulae which calculate the length of the contact zones and Mode II stress intensity factor. Rice and Sih[19] found a method to determine Goursat functions for a interface crack problem. It involves eigenfunction expansion and complex function theory.
CHAPTER 2
FREDHOLM KERNELS AND EXPRESSIONS FOR STRESS AROUND THE CRACK TIP

2.1 Fredholm Kernels:

In this chapter the Fredholm kernels for stresses in different material zones are derived.

The eight unknowns in Navier’s equations of elasticity, are reduced to two equations with only two unknowns, horizontal and vertical displacements, \( u(x,y) \) and \( v(x,y) \), respectively.

The reduced equations are

\[
\nabla^2 u + \frac{2}{\kappa-1} \frac{d}{dx} \left[ \frac{du}{dx} + \frac{dv}{dy} \right] = 0, \quad (2.1)
\]

\[
\nabla^2 v + \frac{2}{\kappa-1} \frac{d}{dy} \left[ \frac{du}{dx} + \frac{dv}{dy} \right] = 0. \quad (2.2)
\]

where, \( \kappa = 3 - 4\nu \) for plane strain and \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress. Fourier transforms are applied on equations (2.1) and (2.2). When the transform is in \( x \), the exponential Fourier transform is given by

\[
\tilde{f}(\beta, y) = \int_{-\infty}^{\infty} f(x, y) e^{i\beta x} \, dx, \quad f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\beta, y) e^{-i\beta x} \, d\beta \quad (2.3)
\]

and when the Fourier transform is in \( y \),

\[
\tilde{f}(x, \alpha) = \int_{-\infty}^{\infty} f(x, y) e^{i\alpha y} \, dy, \quad f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(x, \alpha) e^{-i\alpha y} \, d\alpha \quad (2.4)
\]

In this problem, the transform is in \( y \) and, Fourier transforms become
\( \bar{u}(\alpha, y) = \int_{-\infty}^{\infty} u(x, y)e^{i\alpha x}dx, \quad u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(\alpha, y)e^{-i\alpha x}d\alpha, \quad (2.5) \)

\( \bar{v}(\alpha, y) = \int_{-\infty}^{\infty} v(x, y)e^{i\alpha x}dx, \quad v(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{v}(\alpha, y)e^{-i\alpha x}d\alpha \quad (2.6) \)

Applying (2.5) and (2.6) to (2.1) and (2.2) converts the partial differential equations to a constant coefficient ordinary differential equation, and that gives

\( \bar{u}(\alpha, y) = [A_1(\alpha) + yA_2(\alpha)]e^{-j\beta y} + [A_3(\alpha) + yA_4(\alpha)]e^{j\beta y}, \quad (2.7) \)

\( \bar{v}(\alpha, y) = i\frac{\alpha}{|\alpha|}[-A_1(\alpha) - \left(\frac{\kappa}{|\alpha|} + y\right)A_2(\alpha)]e^{-j\beta y} + i\frac{\alpha}{|\alpha|}[-A_3(\alpha) - \left(\frac{\kappa}{|\alpha|} - y\right)A_4(\alpha)]e^{j\beta y}. \quad (2.8) \)

The boundary conditions involve the stresses applied to the surface of the layer, so the Fourier transforms are applied to the stress and strain relations

\( \sigma_{xx}(x, y) = \mu \left(\frac{\kappa - 3}{\kappa - 1}\right) \frac{\partial v}{\partial y} - \mu \left(\frac{\kappa + 1}{\kappa - 1}\right) \frac{\partial u}{\partial x}, \quad (2.9) \)

\( \sigma_{yy}(x, y) = \mu \left(\frac{\kappa + 1}{\kappa - 1}\right) \frac{\partial v}{\partial y} - \mu \left(\frac{\kappa - 3}{\kappa - 1}\right) \frac{\partial u}{\partial x}, \quad (2.10) \)

\( \tau_{xy}(x, y) = \mu \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]. \quad (2.11) \)

When (2.3) is applied to (2.9),(2.10),(2.11),

\( \bar{\sigma}_{xx}(\alpha, y) = \mu a i \left\{ -2A_1(\alpha) + A_2(\alpha) \left(\frac{3 - \kappa}{|\alpha|} - 2y\right) \right\} e^{j\beta y} - \left\{ 2A_3(\alpha) + A_4(\alpha) \left(\frac{3 - \kappa}{|\alpha|} + 2y\right) \right\} e^{j\beta y} \}

\( (2.12) \)
\[ \bar{\sigma}_{yy}(\alpha, y) = \mu \alpha i \left\{ 2A_1(\alpha) + A_2(\alpha) \left( \frac{\kappa + 1}{|\alpha|} + 2y \right) \right\} e^{k |\alpha| y} + \left\{ 2A_4(\alpha) \right\} \left( \frac{\kappa + 1}{|\alpha|} + 2y \right) \right\} e^{k |\alpha| y} \] (2.13)

\[ \bar{\sigma}_{xy}(\alpha, y) = \mu \left\{ -2|\alpha| A_1(\alpha) + A_2(\alpha)(1-\kappa -2|\alpha| y) \right\} e^{k |\alpha| y} + \left\{ 2|\alpha| A_4(\alpha) + A_4(\alpha)(1-\kappa -2|\alpha| y) \right\} e^{k |\alpha| y} \] (2.14)

When the material properties for the three regions are applied, equations (2.7), (2.8), and (2.12), (2.13) and (2.14) can be written as

**Region 1:**

\[ \bar{\sigma}_{xx}(\alpha, y) = \mu \alpha i \left\{ -2A_1(\alpha) + A_2(\alpha) \left( \frac{3-\kappa_i}{|\alpha|} - 2y \right) \right\} e^{k |\alpha| y} - \left\{ 2A_4(\alpha) + A_4(\alpha) \left( \frac{3-\kappa_i}{|\alpha|} + 2y \right) \right\} e^{k |\alpha| y} \] (2.15)

\[ \bar{\sigma}_{yy}(\alpha, y) = \mu \alpha i \left\{ 2A_1(\alpha) - A_2(\alpha) \left( \frac{\kappa_i + 1}{|\alpha|} + 2y \right) \right\} e^{k |\alpha| y} + \left\{ 2A_4(\alpha) - A_4(\alpha) \left( \frac{\kappa_i + 1}{|\alpha|} + 2y \right) \right\} e^{k |\alpha| y} \] (2.16)

\[ \bar{\sigma}_{xy}(\alpha, y) = \mu \left\{ -2|\alpha| A_1(\alpha) + A_2(\alpha)(1-\kappa -2|\alpha| y) \right\} e^{k |\alpha| y} + \left\{ 2|\alpha| A_4(\alpha) + A_4(\alpha)(1-\kappa -2|\alpha| y) \right\} e^{k |\alpha| y} \] (2.17)

\[ \bar{u}(\alpha, y) = \left\{ A_1(\alpha) + yA_2(\alpha) \right\} e^{k |\alpha| y} + \left\{ A_4(\alpha) + yA_4(\alpha) \right\} e^{k |\alpha| y} \] (2.18)

\[ \bar{v}(\alpha, y) = i \alpha \left\{ -A_1(\alpha) - \left( \frac{\kappa_i}{|\alpha|} + y \right) A_2(\alpha) \right\} e^{k |\alpha| y} + \left\{ A_4(\alpha) - \left( \frac{\kappa_i}{|\alpha|} - y \right) A_4(\alpha) \right\} e^{k |\alpha| y} \] (2.19)
Region 2:

\[ \bar{\sigma}_{xx}(\alpha, y) = \mu_2 \alpha i \left[ -2A_y(\alpha) - \left( \frac{3-\kappa_2}{|\alpha|} + 2y \right) A_b(\alpha) \right] e^{k_1 y} \]  
\[ (2.20) \]

\[ \bar{\sigma}_{xy}(\alpha, y) = \mu_2 \left[ 2|\alpha|A_y(\alpha) + (1-\kappa_2 + 2|\alpha|y) A_b(\alpha) \right] e^{k_1 y} \]  
\[ (2.21) \]

\[ \bar{\sigma}_{yy}(\alpha, y) = \mu_2 \alpha i \left[ 2A_y(\alpha) + \left( \frac{\kappa_2 + 1}{|\alpha|} + 2y \right) A_b(\alpha) \right] e^{k_1 y} \]  
\[ (2.22) \]

\[ \bar{u}(\alpha, y) = \left[ A_y(\alpha) + yA_b(\alpha) \right] e^{k_1 y} \]  
\[ (2.23) \]

\[ \bar{v}(\alpha, y) = i \frac{\alpha}{|\alpha|} \left[ A_y(\alpha) - \left( \frac{\kappa_2 - y}{|\alpha|} A_b(\alpha) \right) \right] e^{k_1 y} \]  
\[ (2.24) \]

Region 3:

\[ \bar{\sigma}_{xx}(\alpha, y) = \mu_4 \alpha i \left[ -2A_y(\alpha) + \left( \frac{3-\kappa_1}{|\alpha|} - 2y \right) A_b(\alpha) \right] e^{-k_1 y} \]  
\[ (2.25) \]

\[ \bar{\sigma}_{xy}(\alpha, y) = \mu_4 \left[ -2|\alpha|A_y(\alpha) + (1-\kappa_1 - 2|\alpha|y) A_b(\alpha) \right] e^{-k_1 y} \]  
\[ (2.26) \]

\[ \bar{\sigma}_{yy}(\alpha, y) = \mu_4 \alpha i \left[ 2A_y(\alpha) + \left( \frac{\kappa_1 + 1}{|\alpha|} + 2y \right) A_b(\alpha) \right] e^{-k_1 y} \]  
\[ (2.27) \]

\[ \bar{u}(\alpha, y) = \left[ A_y(\alpha) + yA_b(\alpha) \right] e^{-k_1 y} \]  
\[ (2.28) \]

\[ \bar{v}(\alpha, y) = i \frac{\alpha}{|\alpha|} \left[ -A_y(\alpha) - \left( \frac{\kappa_1 + y}{|\alpha|} A_b(\alpha) \right) \right] e^{-k_1 y} \]  
\[ (2.29) \]
The following boundary conditions are applied to the corresponding equations above to get the arbitrary integral functions $A_1$ through $A_8$ which are cumbersome and are not shown here.

$$u^+_3 - u^-_3 = 0, \ y = 0 \quad (2.30)$$

$$v^+_3 - v^-_1 = 0, \ y = 0 \quad (2.31)$$

$$\sigma^1_{yy} - \sigma^3_{yy} = 0, \ y = 0 \quad (2.32)$$

$$\sigma^1_{xy} - \sigma^3_{xy} = 0, \ y = 0 \quad (2.33)$$

$$\sigma^1_{xy} - \sigma^2_{xy} = 0, \ y = -h \quad (2.34)$$

$$\sigma^1_{yy} - \sigma^2_{yy} = 0, \ y = -h \quad (2.35)$$

$$u_1 - u_2 = 0, \ y = -h \quad (2.36)$$

$$v_1 - v_2 = 0, \ y = -h \quad (2.37)$$

Equations for $\sigma^1_{yy}(x, y)$ and $\sigma^1_{xy}(x, y)$ can now be written as

$$\lim_{y \to 0^+} \frac{\mu i}{2\pi} \int_{-\infty}^{\infty} \left\{ SS1 e^{iky} + SS2 e^{iky} \right\} e^{-i\alpha x} d\alpha = p(x) \quad (2.38)$$

$$\lim_{y \to 0^+} \frac{\mu i}{2\pi} \int_{-\infty}^{\infty} \left\{ SS3 e^{-iky} + SS4 e^{iky} \right\} e^{-i\alpha x} d\alpha = q(x) \quad (2.39)$$

Where,

$$SS1 = \alpha \left[ 2A_1 + A_2 \left( \frac{\kappa_1 + 1}{|\alpha|} + 2y \right) \right] \quad (2.40)$$
\[ SS2 = \alpha \left[ 2A_3 + A_4 \left( -\frac{\kappa_1 + 1}{\alpha} + 2y \right) \right] \]  

(2.41)

\[ SS3 = \left[ -2|\alpha|A_3 + A_4 \left( 1 - \kappa_1 - 2|\alpha|y \right) \right] \]  

(2.42)

\[ SS4 = \left[ 2|\alpha|A_3 + A_4 \left( 1 - \kappa_1 + 2|\alpha|y \right) \right] \]  

(2.43)

and,

\[ Ux = \int_a^\alpha g(t)e^{iat}dt, \quad g(t) = \frac{\partial}{\partial t} \left[ u_3^+ - u_1^- \right] \]  

(2.44)

\[ Vx = \int_a^\alpha f(t)e^{iat}dt, \quad f(t) = \frac{\partial}{\partial t} \left[ v_3^+ - v_1^- \right] \]  

(2.45)

\[ p(x) = \lim_{y \to 0} \sigma_{yy}^1(x, y) \]  

(2.46)

\[ q(x) = \lim_{y \to 0} \sigma_{xy}^1(x, y) \]  

(2.47)

Solving the equations (2.38) and (2.39) and evaluating them at \( y = 0 \) and making use of the following integrals,

\[ \int_0^\infty e^{-\alpha y} \sin(\alpha(t - x))d\alpha = \frac{t - x}{y^2 + (t - x)^2}, \]  

(2.48)

\[ \int_0^\infty \alpha e^{-\alpha y} \sin(\alpha(t - x))d\alpha = \frac{2y(t - x)}{\left[ y^2 + (t - x)^2 \right]^2}, \]  

(2.49)

\[ \int_0^\infty \alpha^2 e^{-\alpha y} \sin(\alpha(t - x))d\alpha = \frac{2(t - x) \left[ 3y^2 - (t - x)^2 \right]}{\left[ y^2 + (t - x)^2 \right]^2}, \]  

(2.50)

\[ \int_0^\infty e^{-\alpha y} \cos(\alpha(t - x))d\alpha = \frac{y}{y^2 + (t - x)^2}, \]  

(2.51)
\[
\int_{0}^{\infty} \alpha e^{-\alpha y} \cos(\alpha(t-x)) d\alpha = \frac{y^2 - (t-x)^2}{y^2 + (t-x)^2}, \quad (2.52)
\]
\[
\int_{0}^{\infty} \alpha^2 e^{-\alpha y} \cos(\alpha(t-x)) d\alpha = \frac{2y[y^2 - 3(t-x)^2]}{y^2 + (t-x)^2}, \quad (2.53)
\]
gives the following singular integral equations,

\[
\int_{a}^{b} \frac{f(t)}{t-x} dt + \int_{a}^{b} f(t) K_{11}(x,t) dt + \int_{a}^{b} g(t) K_{12}(x,t) dt = \pi \frac{1+\kappa_1}{2\mu_1} p(x) \quad a < x < b
\]

\[
\int_{a}^{b} \frac{g(t)}{t-x} dt + \int_{a}^{b} f(t) K_{21}(x,t) dt + \int_{a}^{b} g(t) K_{22}(x,t) dt = \pi \frac{1+\kappa_1}{2\mu_1} q(x) \quad a < x < b
\]

And the kernels are,

\[
K_{11}(x,t) = -\frac{4a_1h^2(t-x)[12h^2-(t-x)^2]}{4h^2+(t-x)^2} - \frac{8a_1h^2(t-x)}{4h^2+(t-x)^2} \left( \frac{a_1-a_2}{2} \right) \frac{(t-x)}{4h^2+(t-x)^2}
\]

\[
K_{22}(x,t) = -\frac{4a_2h^2(t-x)[12h^2-(t-x)^2]}{4h^2+(t-x)^2} + \frac{8a_2h^2(t-x)}{4h^2+(t-x)^2} \left( \frac{a_1-a_2}{2} \right) \frac{(t-x)}{4h^2+(t-x)^2}
\]

\[
K_{12}(x,t) = -K_{21}(x,t) = \frac{8a_1h^2[4h^2-3(t-x)^2]}{4h^2+(t-x)^2} \left( \frac{a_1-a_2}{2} \right) \frac{2h}{4h^2+(t-x)^2}
\]
Where,

\[ a_1 = \frac{\mu_1 - \mu_2}{\mu_1 + \kappa_1 \mu_2}, a_2 = \frac{\kappa_1 \mu_2 - \kappa_2 \mu_1}{2(\mu_2 + \kappa_2 \mu_1)} \]  
(2.59)

A similar method as above is followed to derive \( \sigma_{xx}^3(x, y) \) and the corresponding kernels.

Equation for \( \sigma_{xx}^3(x, y) \) can be written as,

\[
\lim_{y \to 0} \frac{\mu_i i}{2\pi} \int_{-\infty}^{\infty} SS e^{-h_b^i} e^{-ia_y} d\alpha = \lim_{y \to 0} \sigma_{xx}^3(x, y)
\]  
(2.60)

where,

\[
SS = \alpha \left[ -A_r + A_s \left( \frac{3 - \kappa_1}{|\alpha|} - 2y \right) \right]
\]  
(2.61)

Equation (2.60) solved as above to get the following singular equation,

\[
\int_{a}^{b} \frac{f(t)}{t-x} \, dt + \int_{a}^{b} f(t) L_1(x,t) \, dt + \int_{a}^{b} g(t) L_2(x,t) \, dt + 2\pi g(x) = \pi \frac{1 + \kappa_i}{2\mu_1} \sigma_{xx}^3(x, 0^+)
\]  
(2.62)

And the kernels are,

\[
L_1(x,t) = \frac{4a_i h^2 (t-x) \left[ 12h^2 - (t-x)^2 \right]}{4h^2 + (t-x)^2} - \frac{8a_i h^2 (t-x)}{4h^2 + (t-x)^2} \left( \frac{3}{2} a_1 + a_2 \right) \frac{(t-x)}{4h^2 + (t-x)^2}
\]  
(2.63)

\[
L_2(x,t) = -\frac{8a_i h^2 \left[ 4h^2 - 3(t-x)^2 \right]}{4h^2 + (t-x)^2} + \frac{4a_i h \left[ 4h^2 - (t-x)^2 \right]}{4h^2 + (t-x)^2} - \left( \frac{3}{2} a_1 - a_2 \right) \frac{2h}{4h^2 + (t-x)^2}
\]  
(2.64)
To derive the kernels of $u_\phi^3(x, y)$

$$\lim_{y \to 0} \frac{i}{2\pi} \int_{-\pi}^{\pi} SS6 e^{-i\beta y} e^{-j_0 x} d\alpha = \lim_{y \to 0} u_\phi^3(x, y)$$  \hspace{1cm} (2.65)$$

where,

$$SS6 = \frac{\alpha}{|\alpha|} \left[ -A_\gamma - A_\delta \left( \frac{\kappa_1}{|\alpha|} + y \right) \right]$$  \hspace{1cm} (2.66)$$

Taking a derivative with respect to ‘x’, and evaluating at y=0, (2.65) can be written as

$$\int_a^b \frac{g(t)}{t-x} dt + \int_a^b f(t) M_1(x,t) dt + \int_a^b g(t) M_2(x,t) dt - \pi \frac{\kappa_1 + 1}{\kappa_1 - 1} f(x) = \left[ \frac{-4\mu_1}{\kappa_1 - 1} \right] \frac{1 + \kappa_1}{2\mu_1} \frac{\partial \nu^3}{\partial x}(x, 0^-)$$

where the kernels are

$$M_1(x,t) = \frac{16h^2 a_8}{(\kappa_1 - 1)} \left( \frac{4h^2 - 3(t-x)^2}{4h^2 + (t-x)^2} \right)^3 + 2h a_8 \left( \frac{\kappa_1 + 1}{(4h^2 + (t-x)^2)} \right)^3 \left( \frac{4h^2 - (t-x)^2}{4h^2 + (t-x)^2} \right)$$

$$M_2(x,t) = \frac{8h^2 a_8}{(\kappa_1 - 1)} \left( \frac{12h^2 - (t-x)^2}{4h^2 + (t-x)^2} \right)^3 \left( \frac{4h^2 + (t-x)^2}{4h^2 + (t-x)^2} \right)^3 \left( \frac{-\mu_1^2 \kappa_2 - \mu_1 \kappa_1 - 2\mu_1 \kappa_2 \kappa_1 + 2\mu_1 \kappa_2 - 2\mu_1 \kappa_1 + 2\mu_1 \kappa_1 \kappa_2 + 2\mu_1 \kappa_1}{(\kappa_1 - 1)(\mu_1 + \kappa_1 \mu_2)(\mu_1 + \kappa_1 \mu_2)} \right)$$

$$\hspace{1cm} \left( \frac{(t-x)^2}{(4h^2 + (t-x)^2)} \right)^3$$

(2.67)

(2.68)

(2.69)

### 2.2 Expressions for stresses in all the three regions around the crack tip

The equation set from (2.15) through (2.29) are used to derive the kernels for different stresses in different regions. The boundary conditions (2.30) to (2.37) are applied to get the integral functions as done earlier.
Region 1:

Equations for $\sigma_{xx}^1(x, y)$, $\sigma_{yy}^1(x, y)$ and $\sigma_{xy}^1(x, y)$ can be written as,

$$\frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} \{SS9 + SS10\} e^{-iax} \, d\alpha = \sigma_{xx}^1(x, y) \quad (2.70)$$

$$\frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} \{SS17 + SS18\} e^{-iax} \, d\alpha = \sigma_{yy}^1(x, y) \quad (2.71)$$

$$\frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} \{SS14 + SS15\} e^{-iax} \, d\alpha = \sigma_{xy}^1(x, y) \quad (2.72)$$

Where,

$$SS9 = \alpha \left[ -2A_1(\alpha) + A_2(\alpha) \left( 3 - \frac{\kappa_1}{|\alpha|} - 2y \right) \right] \quad (2.73)$$

$$SS10 = \alpha \left[ -2A_1(\alpha) - A_4(\alpha) \left( 3 - \frac{\kappa_1}{|\alpha|} + 2y \right) \right] \quad (2.74)$$

$$SS17 = \alpha \left[ 2A_1(\alpha) + A_2(\alpha) \left( \frac{\kappa_1 + 1}{|\alpha|} + 2y \right) \right] \quad (2.75)$$

$$SS18 = \alpha \left[ 2A_3(\alpha) + A_4(\alpha) \left( -\frac{\kappa_1 + 1}{|\alpha|} + 2y \right) \right] \quad (2.76)$$

$$SS14 = -2|\alpha| A_1(\alpha) + A_4(\alpha) \left( 1 - \kappa_1 - 2|\alpha|y \right) \quad (2.77)$$

$$SS15 = 2|\alpha| A_3(\alpha) + A_4(\alpha) \left( 1 - \kappa_1 - 2|\alpha|y \right) \quad (2.78)$$

Solving equations (2.70), (2.71) and (2.72) and making use of the integrals from (2.48) to (2.53), we get the following integral equations
\[ \int_{a}^{b} f(t)XX^1_f(x,t)dt + \int_{a}^{b} g(t)XX^1_g(x,t)dt = \pi \frac{1 + \kappa_1}{2\mu_1} \sigma^1_{xx}(x,y) \] (2.79)

\[ \int_{a}^{b} f(t)YY^1_f(x,t)dt + \int_{a}^{b} g(t)YY^1_g(x,t)dt = \pi \frac{1 + \kappa_1}{2\mu_1} \sigma^1_{yy}(x,y) \] (2.80)

\[ \int_{a}^{b} f(t)XY^1_f(x,t)dt + \int_{a}^{b} g(t)XY^1_g(x,t)dt = \pi \frac{1 + \kappa_1}{2\mu_1} \sigma^1_{xy}(x,y) \] (2.81)

and the kernels for region 1 are

\[ XX^1_f(x,t) = \frac{\mu_1}{a_1} \left( \frac{a_1(t-x)}{(2h+y)^2+(t-x)^2} - \frac{2a_4(2h-y)(2h+y)(t-x)}{(2h+y)^2+(t-x)^2} \right) + \frac{4a_4h(h+y)\left[3(2h+y)^2-(t-x)^2\right]}{(2h+y)^3+(t-x)^3} \]

\[ + \frac{2\mu_1}{a_2} \left( \frac{(t-x)}{y^2+(t-x)^2} - \frac{2y^2(t-x)}{(y^2+(t-x)^2)^2} \right) \] (2.82)

\[ XX^1_g(x,t) = \frac{\mu_1}{a_1} \left( \frac{a_1(2h+y)}{(2h+y)^2+(t-x)^2} + \frac{a_4(4h+y)(2h+y)^2-(t-x)^2)}{(2h+y)^2+(t-x)^2} - \frac{4a_4h(h+y)(2h+y)(2h+y)^2-3(t-x)^2)}{(2h+y)^3+(t-x)^3} \]

\[ - \frac{2\mu_1}{a_2} \left( \frac{-2y}{y^2+(t-x)^2} + \frac{y(y^2-(t-x)^2)}{(y^2+(t-x)^2)^2} \right) \] (2.83)

\[ YY^1_f(x,t) = \frac{\mu_1}{a_1} \left( \frac{a_1(t-x)}{(2h+y)^2+(t-x)^2} + \frac{2a_4(2h+y)^2(t-x)}{(2h+y)^2+(t-x)^2} \right) + \frac{4a_4h(h+y)\left[3(2h+y)^2-(t-x)^2\right]}{(2h+y)^3+(t-x)^3} \]

\[ + \frac{2\mu_1}{a_2} \left( \frac{(t-x)}{y^2+(t-x)^2} + \frac{2y^2(t-x)}{(y^2+(t-x)^2)^2} \right) \] (2.84)

\[ YY^1_g(x,t) = \frac{\mu_1}{a_1} \left( \frac{a_1(2h+y)}{(2h+y)^2+(t-x)^2} + \frac{a_4y(2h+y)^2-(t-x)^2)}{(2h+y)^2+(t-x)^2} \right) - \frac{4a_4h(h+y)(2h+y)(2h+y)^2-3(t-x)^2)}{(2h+y)^3+(t-x)^3} \]

\[ - \frac{\mu_1}{a_2} \frac{2y(y^2-(t-x)^2)}{(y^2+(t-x)^2)^2} \] (2.85)
\( XY_j(x,t) = \frac{\mu_i}{a_i} \left( \frac{a_x(2h+y)}{(2h+y)^2+(t-x)^2} - \frac{a_y \left( (2h+y)^2-(t-x)^2 \right)}{\left( (2h+y)^2+(t-x)^2 \right)^2} - \frac{4a_i h(h+y)(2h+y) \left( (2h+y)^2-3(t-x)^2 \right)}{\left( (2h+y)^2+(t-x)^2 \right)^3} \right) \\
+ \frac{\mu_i}{a_i} \frac{2y(y^2-(t-x)^2)}{(y^2+(t-x)^2)^2} \)  \hspace{1cm} (2.86)

\( XY^i(x,t) = -\frac{\mu_i}{a_i} \left( \frac{a_x(t-x)}{(2h+y)^2+(t-x)^2} - \frac{2a_y(2h+y) \left( (2h+y)^2-3(t-x)^2 \right)(t-x)}{\left( (2h+y)^2+(t-x)^2 \right)^3} \right) \\
+ \frac{2\mu_i}{a_i} \left( \frac{(t-x)}{y^2+(t-x)^2} - \frac{2y^2(t-x)}{(y^2+(t-x)^2)^2} \right) \)  \hspace{1cm} (2.87)

The above equations can be normalized using

\[ t = \frac{b-a}{2} r + \frac{b+a}{2}, \quad x = \frac{b-a}{2} s + \frac{b+a}{2} \] \hspace{1cm} (2.88)

\[ f(t) = \frac{1+\kappa_i}{2\mu_i} \sigma_0 \bar{f}(r), \quad g(t) = \pi \frac{1+\kappa_i}{2\mu_i} \sigma_0 \bar{g}(r) \] \hspace{1cm} (2.89)

\( XX_j(s,r) = \frac{b-a}{2} XX^i(x,t) \) \hspace{1cm} (2.90)

\( YY_j(s,r) = \frac{b-a}{2} YY^i(x,t) \) \hspace{1cm} (2.91)

\( XY_j(s,r) = \frac{b-a}{2} XY^i(x,t) \) \hspace{1cm} (2.92)

**Region 3:**

The method in the derivation of kernels in region 1 is followed here.

Equations for \( \sigma^3_{xx}(x,y) \), \( \sigma^3_{yy}(x,y) \) and \( \sigma^3_{xy}(x,y) \) can be written as,

\[ \frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} SS16 e^{-i\alpha} e^{-iax} d\alpha = \sigma^3_{xx}(x,y) \] \hspace{1cm} (2.93)
\[
\frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} SS7 e^{-|\alpha|} e^{-i\alpha x} d\alpha = \sigma_{x_3}^3(x, y)
\]  
(2.94)

\[
\frac{\mu_i}{2\pi} \int_{-\infty}^{\infty} SS8 e^{-|\alpha|} e^{-i\alpha x} d\alpha = \sigma_{x_3}^3(x, y)
\]  
(2.95)

Where,

\[
SS16 = \alpha \left[ -2A_y(\alpha) + A_k(\alpha) \left( \frac{3 - \kappa_i}{|\alpha|} - 2y \right) \right]
\]  
(2.96)

\[
SS7 = \alpha \left[ 2A_y(\alpha) + A_k(\alpha) \left( \frac{\kappa_i + 1}{|\alpha|} + 2y \right) \right]
\]  
(2.97)

\[
SS8 = -2|\alpha| A_y(\alpha) + A_k(\alpha) \left( 1 - \kappa_i - 2|\alpha| y \right)
\]  
(2.98)

Solving the equations (2.104), (2.105) and (2.106) and using the integrals from (2.48) to (2.53), we get the following integral equations

\[
\int_a^b f(t)XX_f^3(x,t)dt + \int_a^b g(t)XX_g^3(x,t)dt = \pi \frac{1 + \kappa_i}{2\mu_i} \sigma_{x_3}^3(x, y)
\]  
(2.99)

\[
\int_a^b f(t)YY_f^3(x,t)dt + \int_a^b g(t)YY_g^3(x,t)dt = \pi \frac{1 + \kappa_i}{2\mu_i} \sigma_{y_3}^3(x, y)
\]  
(2.100)

\[
\int_a^b f(t)XY_f^3(x,t)dt + \int_a^b g(t)XY_g^3(x,t)dt = \pi \frac{1 + \kappa_i}{2\mu_i} \sigma_{x_y}^3(x, y)
\]  
(2.101)

and the kernels for region 3 are

\[
XX_f^3(x,t) = \frac{\mu_i}{a_1} \left( \frac{a_y(t-x)}{(2h+y)^2+(t-x)^2} + \frac{2a_y(2h-y)(2h+y)(t-x)}{(2h+y)^2+(t-x)^2} \right)
\]

\[
- \frac{4a_y h(h+y) \left[ 3(2h+y)^2-(t-x)^2 \right] (t-x)}{(2h+y)^2+(t-x)^2} \right)
\]

\[
- \frac{\mu_i}{a_1} \left( \frac{a_y(t-x)}{y^2+(t-x)^2} + \frac{2a_y y^2(t-x)}{(y^2+(t-x)^2)^2} \right)
\]  
(2.102)
\[ XX^3(x,t) = -\mu_t \frac{a_{1}(2h+y)}{(2h+y)^2+(t-x)^2} - \frac{a_{4}(4h+y)((2h+y)^2-(t-x)^2)}{(2h+y)^2+(t-x)^2} + \frac{4a_{4}h(y)(2h+y)((2h+y)^2-3(t-x)^2)}{(2h+y)^2+(t-x)^2} + \mu_t \left( -\frac{2a_{10}y}{y^2+q^2} + \frac{a_{10}y(y^2-(t-x)^2)}{(y^2+(t-x)^2)^2} \right) \] 

(2.103)

\[ YY^3(x,t) = -\mu_t \frac{a_{12}(t-x)}{(2h+y)^2+(t-x)^2} + \frac{2a_{4}(2h+y)^3(t-x)}{(2h+y)^2+(t-x)^2} + \frac{4a_{4}h(y)(3(2h+y)^2-(t-x)^2)(t-x)}{(2h+y)^2+(t-x)^2} + \mu_t \left( a_{10}(t-x) \right) + \mu_t \left( \frac{2a_{10}y(2h+y)}{(y^2+(t-x)^2)^2} \right) \] 

(2.104)

\[ YY^3(x,t) = -\mu_t \frac{a_{13}(2h+y)}{(2h+y)^2+(t-x)^2} - \frac{a_{3}y((2h+y)^2-(t-x)^2)}{(2h+y)^2+(t-x)^2} + \frac{4a_{4}h(y)(2h+y)((2h+y)^2-3(t-x)^2)}{(2h+y)^2+(t-x)^2) \] 

+ \mu_t \left( \frac{a_{10}y(y^2-(t-x)^2)}{(y^2+(t-x)^2)^2} \right) \] 

(2.105)

\[ XY^3(x,t) = -\mu_t \frac{a_{12}(t-x)}{(2h+y)^2+(t-x)^2} - \frac{a_{13}(2h+y)}{(2h+y)^2+(t-x)^2} + \frac{2a_{4}(2h+y)^3(t-x)}{(2h+y)^2+(t-x)^2} + \frac{4a_{4}h(y)(3(2h+y)^2-(t-x)^2)(t-x)}{(2h+y)^2+(t-x)^2) \] 

- \mu_t \left( -\frac{a_{10}(t-x)}{y^2+(t-x)^2} + \frac{2a_{10}y(2h+y)}{(y^2+(t-x)^2)^2} \right) \] 

(2.106)

\[ XY^3(x,t) = -\mu_t \frac{a_{12}(t-x)}{(2h+y)^2+(t-x)^2} - \frac{a_{13}(2h+y)}{(2h+y)^2+(t-x)^2} + \frac{2a_{4}(2h+y)^3(t-x)}{(2h+y)^2+(t-x)^2} + \frac{4a_{4}h(y)(3(2h+y)^2-(t-x)^2)(t-x)}{(2h+y)^2+(t-x)^2) \] 

- \mu_t \left( -\frac{a_{10}(t-x)}{y^2+(t-x)^2} + \frac{2a_{10}y(2h+y)}{(y^2+(t-x)^2)^2} \right) \] 

(2.107)

Region 2:

Equations for \( \sigma_{xx}^2(x,y) \), \( \sigma_{yy}^2(x,y) \) and \( \sigma_{xy}^2(x,y) \) can be written as,

\[ \frac{\mu_{t}i}{2\pi} \int_{-\infty}^{\infty} SS1e^{iyx} e^{-i\alpha x} d\alpha = \sigma_{xx}^2(x,y) \] 

(2.108)
\[
\frac{\mu}{2\pi} \int_{-\infty}^{\infty} SS12 e^{i|\alpha|} e^{-iax} d\alpha = \sigma_{xx}^2(x, y) \tag{2.109}
\]

\[
\frac{\mu}{2\pi} \int_{-\infty}^{\infty} SS13 e^{i|\alpha|} e^{-iax} d\alpha = \sigma_{yy}^2(x, y) \tag{2.110}
\]

Where,

\[
SS11 = \alpha \left[ -2A_2(\alpha) - A_b(\alpha) \left( \frac{3 - \kappa_2}{|\alpha|} + 2y \right) \right] \tag{2.111}
\]

\[
SS12 = \alpha \left[ 2A_2(\alpha) + A_b(\alpha) \left( -\frac{\kappa_2 + 1}{|\alpha|} + 2y \right) \right] \tag{2.112}
\]

\[
SS13 = 2|\alpha|A_2(\alpha) + A_b(\alpha) \left( 1 - \kappa_2 + 2|\alpha|y \right) \tag{2.113}
\]

Solving the equations (2.128), (2.129) and (2.130) and using the integrals from (2.48) to (2.53), we get the following integral equations

\[
\int_{a}^{b} \frac{f(t)XX_f(x, t)dt}{a} + \int_{a}^{b} \frac{g(t)XX_g(x, t)dt}{a} = \frac{1 + \kappa_2}{2\mu_2} \sigma_{xx}^2(x, y) \tag{2.114}
\]

\[
\int_{a}^{b} \frac{f(t)YY_f(x, t)dt}{a} + \int_{a}^{b} \frac{g(t)YY_g(x, t)dt}{a} = \frac{1 + \kappa_2}{2\mu_2} \sigma_{yy}^2(x, y) \tag{2.115}
\]

\[
\int_{a}^{b} \frac{f(t)XY_f(x, t)dt}{a} + \int_{a}^{b} \frac{g(t)XY_g(x, t)dt}{a} = \frac{1 + \kappa_2}{2\mu_2} \sigma_{xy}^2(x, y) \tag{2.116}
\]

and the kernels for region 2 are

\[
XX_f^2(x, t) = \frac{\mu_1 \mu_2}{a_{14}} \left( \frac{a_{15}(t-x)}{y^2 + (t-x)^2} - \frac{2(a_{16}h + a_{17}y)y(t-x)}{(y^2 + (t-x)^2)^2} \right) \tag{2.117}
\]
\[ XX^2_{g}(x,t) = \frac{\mu_1 \mu_2}{a_{14}} \left( \frac{a_{18} y}{y^2 + (t-x)^2} - \frac{(a_{16} h + a_{17} y)(y^2 - (t-x)^2)}{(y^2 + (t-x)^2)^2} \right) \]  
\hspace{2cm} (2.118)

\[ YY^2_{f}(x,t) = -\frac{\mu_1 \mu_2}{a_{14}} \left( \frac{a_{19} (t-x)}{y^2 + (t-x)^2} - \frac{2(a_{16} h + a_{17} y)y(t-x)}{(y^2 + (t-x)^2)^2} \right) \]  
\hspace{2cm} (2.119)

\[ YY^2_{g}(x,t) = -\frac{\mu_1 \mu_2}{a_{14}} \left( -\frac{a_{20} y}{y^2 + (t-x)^2} - \frac{(a_{16} h + a_{17} y)(y^2 - (t-x)^2)}{(y^2 + (t-x)^2)^2} \right) \]  
\hspace{2cm} (2.120)

\[ XY^2_{f}(x,t) = \frac{\mu_1 \mu_2}{a_{14}} \left( -\frac{a_{20} y}{y^2 + (t-x)^2} + \frac{(a_{16} h + a_{17} y)(y^2 - (t-x)^2)}{(y^2 + (t-x)^2)^2} \right) \]  
\hspace{2cm} (2.121)

\[ XY^2_{g}(x,t) = \frac{\mu_1 \mu_2}{a_{14}} \left( \frac{a_{19} (t-x)}{y^2 + (t-x)^2} - \frac{2(a_{16} h + a_{17} y)y(t-x)}{(y^2 + (t-x)^2)^2} \right) \]  
\hspace{2cm} (2.122)

The constants from \( a_1 \) to \( a_{20} \) are listed in the Appendix B.
CHAPTER 3
KERNELS FOR A CRACK PARALLEL TO AN INTERFACE AND THE
COMPUTATIONAL DETERMINATION OF HIGHER ORDER TERMS

Following Erdogan (1971) and Achenbach (1980) the integral equations for a crack parallel to an interface are:

\[ \int_{a}^{b} f(t) \frac{d}{dt} \left[ v_{i}(t,0^{+}) - v_{i}(t,0^{-}) \right] dt + \int_{a}^{b} g(t) K_{11}(x,t) dt = \pi \frac{1 + \kappa_{i}}{2\mu_{i}} p(x) \quad a < x < b, \quad (3-1) \]

\[ \int_{a}^{b} g(t) \frac{d}{dt} \left[ u_{j}(t,0^{+}) - u_{j}(t,0^{-}) \right] dt + \int_{a}^{b} f(t) K_{21}(x,t) dt = \pi \frac{1 + \kappa_{i}}{2\mu_{i}} q(x) \quad a < x < b, \quad (3-2) \]

where:

\[ f(t) = \frac{d}{dt} \left[ v_{j}(t,0^{+}) - v_{j}(t,0^{-}) \right] = \frac{dV}{dt}, \quad (3-3) \]

\[ g(t) = \frac{d}{dt} \left[ u_{i}(t,0^{+}) - u_{i}(t,0^{-}) \right] = \frac{dU}{dt}, \quad (3-4) \]

\[ p(x) = \lim_{y \to 0^{+}} \sigma_{yy}^{1}(x,y) \quad (3-5) \]

\[ q(x) = \lim_{y \to 0^{+}} \tau_{xy}^{1}(x,y). \quad (3-6) \]

The kernels in (3-1) and (3-2) are:

\[ K_{11}(x,t) = -\frac{4a_{i}h^{2}(t-x)\left[12h^{2}-(t-x)^{2}\right]}{4h^{2}+(t-x)^{2}} + \frac{8a_{i}h^{2}(t-x)}{4h^{2}+(t-x)^{2}} \left( \frac{a_{1}}{2} - a_{2} \right) \frac{t-x}{4h^{2}+(t-x)^{2}} \]

\[ K_{22}(x,t) = -\frac{4a_{i}h^{2}(t-x)\left[12h^{2}-(t-x)^{2}\right]}{4h^{2}+(t-x)^{2}} + \frac{8a_{i}h^{2}(t-x)}{4h^{2}+(t-x)^{2}} \left( \frac{a_{1}}{2} - a_{2} \right) \frac{t-x}{4h^{2}+(t-x)^{2}}. \]
\[ K_{12}(x,t) = -K_{21}(x,t) = \frac{8a_i h^3}{4h^2 - 3(t-x)^2} - \left( \frac{a_i}{2} + a_2 \right) \frac{2h}{4h^2 + (t-x)^2} \quad (3-7) \]

The constants are given by:

\[ a_i = \frac{\mu_i - \mu_2}{\mu_1 + \kappa_1 \mu_2} = \frac{1}{\mu_1 + \mu_2} \left( \mu_2 / \mu_1 \right) - \beta - \alpha, \quad a_2 = \frac{\kappa_1 \mu_2 - \kappa_2 \mu_1}{2 \left( \mu_2 + \kappa_1 \mu_1 \right)} = \frac{\beta + \alpha}{2(1 - \beta)}, \quad (3-8) \]

where the Dundurs’ (1969) parameters are:

\[ \beta = \frac{\mu_2 (\kappa_1 - 1) - \mu_1 (\kappa_2 - 1)}{\mu_2 (\kappa_1 + 1) + \mu_1 (\kappa_2 + 1)} = \frac{2a_2 + a_1}{2a_2 + 2 - a_1}, \quad \alpha = \frac{\mu_2 (\kappa_1 + 1) - \mu_1 (\kappa_2 + 1)}{\mu_2 (\kappa_1 + 1) + \mu_1 (\kappa_2 + 1)} = -\frac{4a_2 - a_1 + 2a_2}{2a_2 + 2 - a_1}. \quad (3-9) \]

The solution of these integral equations allows for the determination of the “K” coefficients using:

\[ \text{COD} = u_y^+ - u_y^- = -u_y^+ + u_y^- = \frac{\kappa + 1}{2\mu} \sum_{n=0}^{\infty} (-1)^n \frac{k_n^I}{2n+1} (2\rho)^{n+\frac{1}{2}}, \quad \rho = b - x, \quad (3-10) \]

\[ \text{CSD} = u_x^+ - u_x^- = -u_x^+ + u_x^- = \frac{\kappa + 1}{2\mu} \sum_{n=0}^{\infty} (-1)^n \frac{k_n^{II}}{2n+1} (2\rho)^{n+\frac{1}{2}}, \quad \rho = b - x, \quad (3-11) \]

where the polar coordinate variable in (3-10) and (3-11), \( \rho = b - x \) for the right crack tip at \( x = b \). Following the Appendix A, the mode I “T” coefficients are determined using either

\[ \sigma_{rr}(\rho,0) = \sum_{n=0}^{\infty} k_n^I (2\rho)^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} T_n^I (2\rho)^n, \quad \rho = x - b, \quad (3-12) \]

or

\[ \sigma_{rr}(\rho,\pm\pi) = \sum_{n=0}^{\infty} (-1)^n T_n^I (2\rho)^n \mp 2 \sum_{n=0}^{\infty} (-1)^n k_n^{II} (2\rho)^{n+\frac{1}{2}}, \quad \rho = b - x \quad (3-13) \]

The required expression to make use of (3-12) or (3-13) is
\[ \int_{a}^{b} f(t) \frac{dt}{t-x} + \int_{a}^{b} f(t) L_{1}(x,t) dt + \int_{a}^{b} g(t) L_{2}(x,t) dt \pm 2\pi g(x) = \pi \frac{1+\kappa_{1}}{2\mu_{1}} \sigma_{e}^{\prime}(x,0^{+}) , \] (3-14)

where the kernels are:

\[ L_{1}(x,t) = \frac{4a_{i}h^{2}(t-x)[12h^{2}-(t-x)^{2}]}{[4h^{2}+(t-x)^{2}]^{3}} - \frac{8a_{i}h^{2}(t-x)}{[4h^{2}+(t-x)^{2}]^{2}} - \left( \frac{3}{2} a_{1} + a_{2} \right) \frac{(t-x)}{4h^{2}+(t-x)^{2}} , \]

\[ L_{2}(x,t) = - \frac{8a_{i}h^{2}[4h^{2}-3(t-x)^{2}]}{[4h^{2}+(t-x)^{2}]^{3}} + \frac{4a_{i}h[4h^{2}-(t-x)^{2}]}{[4h^{2}+(t-x)^{2}]^{2}} - \left( \frac{3}{2} a_{1} - a_{2} \right) \frac{2h}{4h^{2}+(t-x)^{2}} , \] (3-15)

and the plus sign for the delta function term is for the upper crack surface \((i = 3)\) while the negative sign is for the lower crack surface \((i = 1)\). The mode II “T” coefficients are determined using either

\[ 2\mu \frac{\partial u_{\theta}(\rho,0)}{\partial \rho} = -\frac{\kappa-1}{2} \sum_{n=0}^{\infty} k_{n}^{II}(2\rho)^{n-1} + \frac{\kappa+1}{4} \sum_{n=1}^{\infty} T_{n}^{II}(2\rho)^{n} , \] (3-16)

or

\[ 2\mu \frac{\partial u_{\theta}(\rho,\pm \pi)}{\partial \rho} = \mp \frac{\kappa+1}{2} \sum_{n=0}^{\infty} (-1)^{n} k_{n}^{I}(2\rho)^{n-1} + \frac{\kappa+1}{4} \sum_{n=1}^{\infty} (-1)^{n} T_{n}^{II}(2\rho)^{n} . \] (3-17)

The required expression in this case is

\[ \int_{a}^{b} g(t) \frac{dt}{t-x} + \int_{a}^{b} f(t) M_{1}(x,t) dt + \int_{a}^{b} g(t) M_{2}(x,t) dt - \pi \frac{\kappa_{1}+1}{\kappa_{1}-1} f(x) = \left( \frac{-4\mu_{1}}{\kappa_{1}-1} \right) \pi \frac{1+\kappa_{1}}{2\mu_{1}} \frac{\partial v^{3}(x,0^{+})}{\partial x} , \] (3-18)

where the kernels are
These functions cannot be expressed in terms of $a_1$ and $a_2$ alone, although as will be shown numerically, the mode II $T$-coefficients determined using (3-18) are only functions of these two constants. Next consider the numerical work.

**Numerical solution.**

Equations (3-1) and (3-2) are normalized using

\[
t = \frac{b-a}{2} r + \frac{b+a}{2}, \quad x = \frac{b-a}{2} s + \frac{b+a}{2},
\]

(3-20)

\[
f(t) = \frac{1+\kappa_1}{2\mu_1} \sigma_0 \bar{f}(r), \quad g(t) = \frac{1+\kappa_1}{2\mu_1} \sigma_0 \bar{g}(r),
\]

(3-21)

\[
\bar{K}_0(s,r) = \frac{b-a}{2} K_0(x,t)
\]

(3-22)

which gives

\[
\int_{-1}^{1} \bar{f}(r) \, dr + \int_{-1}^{1} \bar{f}(r) \bar{K}_{11}(s,r) \, dr + \int_{-1}^{1} \bar{g}(r) \bar{K}_{12}(s,r) \, dr = \pi \frac{p(s)}{\sigma_0} \quad -1 < s < 1,
\]

(3-23)

\[
\int_{-1}^{1} \bar{f}(r) \, dr + \int_{-1}^{1} \bar{f}(r) \bar{K}_{21}(s,r) \, dr + \int_{-1}^{1} \bar{g}(r) \bar{K}_{22}(s,r) \, dr = \pi \frac{q(s)}{\sigma_0} \quad -1 < s < 1.
\]

(3-24)

Defining the non-dimensional length parameter,

\[
\varepsilon = \frac{2h}{b-a},
\]

(3-25)

which is simply $h$ divided by the half-crack length, the normalized kernels (3-22) become
\[ \bar{K}_{11}(s,r) = \frac{b-a}{2} K_{11}(x,t) \]

\[ = -\frac{4a_i \varepsilon^2 (r-s) \left[ 12 \varepsilon^2 - (r-s)^2 \right]}{4 \varepsilon^2 + (r-s)^2} - \frac{8a_i \varepsilon^2 (r-s)}{4 \varepsilon^2 + (r-s)^2} - \left( \frac{a_i}{2} - a_s \right) \frac{(r-s)}{4 \varepsilon^2 + (r-s)^2}, \quad (3-26) \]

\[ \bar{K}_{22}(s,r) = \frac{b-a}{2} K_{22}(x,t) \]

\[ = -\frac{4a_i \varepsilon^2 (r-s) \left[ 12 \varepsilon^2 - (r-s)^2 \right]}{4 \varepsilon^2 + (r-s)^2} + \frac{8a_i \varepsilon^2 (r-s)}{4 \varepsilon^2 + (r-s)^2} - \left( \frac{a_i}{2} - a_s \right) \frac{(r-s)}{4 \varepsilon^2 + (r-s)^2}, \quad (3-27) \]

\[ \bar{K}_{12}(s,r) = -\bar{K}_{21}(s,r) = \frac{8a_i \varepsilon^3 \left[ 4 \varepsilon^2 - 3(r-s)^2 \right]}{4 \varepsilon^2 + (r-s)^2} - \left( \frac{a_i}{2} + a_s \right) \frac{2 \varepsilon}{4 \varepsilon^2 + (r-s)^2}. \quad (3-28) \]

Taking into account that both crack tips are closed and the stress is singular at both tips, the solution of (3-23) and (3-24) is obtained by using

\[ \bar{f}(r) = \sum_{i=1}^{N} a_i T_i(r) \sqrt{1-r^2} \rightarrow \frac{\sum_{i=1}^{N} a_i T_{i-1}(r)}{\sqrt{1-r^2}}, \]

\[ \bar{g}(r) = \sum_{i=1}^{N} b_i T_i(r) \sqrt{1-r^2} \rightarrow \frac{\sum_{i=1}^{N} b_i T_{i+1}(r)}{\sqrt{1-r^2}}, \quad (3-29) \]

where the \( T_i \) functions are Chebyshev polynomials of the first kind. The expressions to the right take advantage of symmetry. Using (3-20) and (3-21), (3-29) can be integrated to obtain,

\[ V(s) = \sigma_0 \frac{b-a}{2} \frac{1 + K_i}{2 \mu_1} \sum_{i=1}^{N} \frac{-a_i}{i} U_{i-1}(s) \sqrt{1-s^2} \rightarrow \sigma_0 \frac{b-a}{2} \frac{1 + K_i}{2 \mu_1} \sum_{i=1}^{N} \frac{-a_i}{2i-1} U_{2i-2}(s) \sqrt{1-s^2} \]
A useful expression to normalize displacement in terms of material 2 instead of material 1, which can be used to correspond to a negative $h/a$, is

$$\frac{\mu_2}{\mu_1} \frac{1 + \kappa_2}{1 + \kappa_1} = \frac{1 - \alpha}{1 + \alpha}$$  \hspace{1cm} (3-31)

Starting the sum at one in (3-30) instead of zero automatically satisfies the requirements,

$$\int_{-1}^{1} f(t) dt = \int_{-1}^{1} \left[ v_i(t,0^+) - v_i(t,0^-) \right] dt = \left[ v_i(t,0^+) - v_i(t,0^-) \right]_{-1}^{1} = 0,$$  \hspace{1cm} (3-32)

$$\int_{-1}^{1} g(t) dt = \int_{-1}^{1} \left[ u_i(t,0^+) - u_i(t,0^-) \right] dt = \left[ u_i(t,0^+) - u_i(t,0^-) \right]_{-1}^{1} = 0,$$

due to the orthogonality condition,

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} dt = \begin{cases} 0, & i \neq j \\ \pi/2, & i = j \neq 0. \\ \pi, & i = j = 0 \end{cases}$$  \hspace{1cm} (3-33)

Substitution of (3-29) into (3-23) and (3-24) leads to the following system of linear equations:

$$\sum_{i=1}^{N} a_i \int_{-1}^{1} \frac{1}{r-s} + \mathbf{K}_{11}(s,r) \frac{T_i(r)}{\sqrt{1-r^2}} dr = \frac{\pi p(s)}{\sigma_0},$$  \hspace{1cm} (3-34)

$$\sum_{i=1}^{N} a_i \int_{-1}^{1} T_i(r) \mathbf{K}_{21}(s,r) dr + \sum_{i=1}^{N} b_i \int_{-1}^{1} \frac{1}{r-s} + \mathbf{K}_{22}(s,r) \frac{T_i(r)}{\sqrt{1-r^2}} dr = \frac{\pi q(s)}{\sigma_0}.$$  \hspace{1cm} (3-35)

The singular integral is given by
\[
\int_{-1}^{1} \frac{T_i(r)}{(r-s)\sqrt{1-r^2}} dr = \pi U_{i-1}(s), \tag{3-36}
\]

where \( U_{i-1} \) are the Chebyshev polynomials of the second kind.

**Determination of the “K” coefficients.**

Given the numerical solution of (3-29) using (3-34) and (3-35), (3-3) and (3-4) together with (3-10) and (3-11) can be used to determine the \( K \)-coefficients. First (3-29) is integrated to obtain,

\[
f(x) = \frac{d}{dx} \left[ v_j(x,0^+) - v_i(x,0^-) \right] = \frac{2}{b-a} d \frac{d}{ds} \left[ v_j(x,0^+) - v_i(x,0^-) \right] = \frac{1+\kappa_1}{2\mu_1} \frac{1}{\sigma_0} \sum_{i=1}^{N} a_i T_i(s) \frac{1}{\sqrt{1-s^2}}
\]

\[
\frac{2}{b-a} \frac{2\mu_1}{1+\kappa_1} \frac{1}{\sigma_0} \sum_{i=1}^{N} a_i T_i(q) \frac{1}{\sqrt{1-q^2}} dq = -\sqrt{1-s^2} \sum_{i=1}^{N} a_i U_{i-1}(s)
\]

(3-37)

Following Ananthasayanam, et al (2007), (3-37) can be expressed mathematically in terms of \((1-s)\) as follows,

\[
\frac{2}{b-a} \frac{2\mu_1}{1+\kappa_1} \frac{1}{\sigma_0} \sum_{i=1}^{N} a_i T_i(q) \frac{1}{\sqrt{1-q^2}} dq = -\sqrt{2(1-s)} \sum_{i=1}^{N} a_i \sum_{n=0}^{\infty} d_n^i (1-s)^n,
\]

(3-38)

where

\[
d_0^i = 1, \quad d_n^i = \frac{d_{n-1}^i}{n(2n+1)} \left[ \left( \frac{2n-1}{2} \right)^2 - i^2 \right].
\]

(3-39)

Taking into account that the radial coordinate, “\( \rho \)” at the right crack tip \((x = b)\) in (3-10) is

\[
\rho = b-x = \frac{b-a}{2} (1-s),
\]

(3-40)
the left side of (3-38) is now expressed using (3-10) as follows:

\[
\frac{2}{b-a} \frac{2\mu}{\kappa_1+1} \frac{1}{\sigma_0} \left[ u_y^+ - u_y^- \right] = \frac{2}{b-a} \frac{1}{\sigma_0} \sum_{n=0}^{\infty} (-1)^n \frac{k_n^I}{2n+1} (b-a)^{n+\frac{1}{2}} (1-s)^{n+\frac{1}{2}}. \tag{3-41}
\]

Equating (3-38) and (3-41) term by term gives the normalized mode I “k coefficients” as follows:

\[
-\sqrt{2} \sum_{i=1}^{N} \sum_{n=0}^{\infty} a_n^i (1-s)^n = \frac{2}{b-a} \frac{1}{\sigma_0} \sum_{n=0}^{\infty} (-1)^n \frac{k_n^I}{2n+1} (b-a)^{n+\frac{1}{2}} (1-s)^n
\]

\[
\bar{k}_n^I (b) = \frac{k_n^I (b)(b-a)^n}{\sigma_0 \sqrt{\frac{b-a}{2}}} = (-1)^{n+1} (2n+1) \sum_{i=1}^{N} a_n^i d_n^i,
\]

\[
\bar{k}_n^I (a) = \frac{k_n^I (a)(b-a)^n}{\sigma_0 \sqrt{\frac{b-a}{2}}} = (-1)^n (2n+1) \sum_{i=1}^{N} (-1)^i a_n^i d_n^i. \tag{3-42}
\]

It is observed that for \( n = 0 \) the normalized stress intensity factor is obtained. In the same manner the mode II coefficients at both crack tips are

\[
\bar{k}_n^{II} (b) = \frac{k_n^{II} (b)(b-a)^n}{\sigma_0 \sqrt{\frac{b-a}{2}}} = (-1)^{n+1} (2n+1) \sum_{i=1}^{N} b_n^i d_n^i,
\]

\[
\bar{k}_n^{II} (a) = \frac{k_n^{II} (a)(b-a)^n}{\sigma_0 \sqrt{\frac{b-a}{2}}} = (-1)^n (2n+1) \sum_{i=1}^{N} (-1)^i b_n^i d_n^i. \tag{3-43}
\]

In order to understand how the asymptotic coefficients can represent the actual stress and displacement fields around the crack tip, it is necessary to develop expressions for the stresses in from of the crack tip. The logical and simple starting point are the stresses
which are easily obtained from (3-1) and (3-2), or more conveniently, (3-23) and (3-24).

These expressions give,

\[
\frac{\sigma_{\theta\theta}(x)}{\sigma_0} = \frac{1}{\pi} \int_{-1}^{1} \bar{f}(r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{f}(r) \bar{K}_{11}(s, r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{g}(r) \bar{K}_{12}(s, r) \, dr ,
\]

\[
\frac{\sigma_{\varphi\varphi}(x)}{\sigma_0} = \frac{1}{\pi} \int_{-1}^{1} \bar{g}(r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{f}(r) \bar{K}_{21}(s, r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{g}(r) \bar{K}_{22}(s, r) \, dr .
\]

It is noted that the above two expressions are valid on and off the crack surfaces, although when on the crack this is simply an expression of the boundary conditions.

**Determination of the “T” coefficients.**

The next step is to obtain an expression for the “T-coefficients,” which must include the radial stress component from (3-14), which is repeated below,

\[
\pi \frac{1 + \kappa}{2\mu} \sigma_{ss}^l(x, 0^+) = \int_{a}^{b} \frac{f(t)}{t-x} \, dt + \int_{a}^{b} f(t) L_t(x, t) \, dt + \int_{a}^{b} g(t) L_s(x, t) \, dt + 2\pi g(x) .
\]

The first step is to express (3-46) in terms of normalized quantities using (3-20) and (3-21) as follows:

\[
\frac{\sigma_{ss}^3(x, 0^+)}{\sigma_0} = \frac{1}{\pi} \int_{-1}^{1} \bar{f}(r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{f}(r) \bar{L}_t(s, r) \, dr + \frac{1}{\pi} \int_{-1}^{1} \bar{g}(r) \bar{L}_s(s, r) \, dr + 2\bar{g}(s) ,
\]

where (3-47) is valid for all \( s \) and

\[
L_t(s, r) = \frac{4a_t \varepsilon^2 (r-s) \left[ 12 \varepsilon^2 - (r-s)^2 \right]}{4 \varepsilon^2 + (r-s)^2} - \frac{8a_t \varepsilon^2 (r-s)}{4 \varepsilon^2 + (r-s)^2} - \left( \frac{3}{2} a_1 + a_2 \right) \frac{(r-s)}{4 \varepsilon^2 + (r-s)^2} .
\]

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$$L_z(s,r) = -\frac{8a_1\varepsilon^3}{4\varepsilon^2+(r-s)^2} + \frac{4a_2\varepsilon^2-(r-s)^2}{4\varepsilon^2+(r-s)^2} = \left(\frac{3}{2}a_1-a_2\right)\frac{2\varepsilon}{4\varepsilon^2+(r-s)^2}. \\
(3-48)$$

Ananthasayanam et. al (2007) obtained the T-coefficients were obtained very accurately by combining (3-46) with $\sigma_{ys}$, the key point being the elimination of the singular integral.

Following this approach, (3-44) and (3-47) give:

$$\frac{\sigma_{ys}(x)}{\sigma_0} - \frac{\sigma_{as}(x,0^+)}{\sigma_0} = \frac{1}{\pi} \int_{-1}^{1} \tilde{f}(r)[\bar{K}_{11}(s,r) - \bar{L}_z(s,r)] dr + \frac{1}{\pi} \int_{-1}^{1} \tilde{g}(r)[\bar{K}_{13}(s,r) - \bar{L}_z(s,r)] dr - 2\bar{g}(s) \quad (3-49)$$

Off of the crack, the right side of (3-49) can be expanded in terms of small values of $(s-1)$ using the notation,

$$\sigma_{ys}(x) - \sigma_{as}(x,0^+) = \sigma_0 \sum_{n=0}^{M} h_n (s-1)^n + O(s-1)^{M+1}, \quad (3-50)$$

where the “$h$ constants” are obtained from integrals of known functions using (3-49).

This same combination of stresses on the upper crack surface can be expressed asymptotically as follows:

$$\sigma_{00}(r,0) - \sigma_{rr}(r,0) = -\sum_{n=0}^{\infty} T_n'(2r)^n. \quad (3-51)$$

Comparing (3-50) and (3-51) gives the normalized coefficients,

$$\frac{T_n'(b-a)^n}{\sigma_0} = -h_n', \quad (3-52)$$
Now consider the case when \( x \) is on the crack and the last term of (3-47) plays a role.

First from (3-37 to 3-39),

\[
\frac{T_i(s)}{\sqrt{1-s}} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} d_n^i (2n+1)(1-s)^{n-\frac{1}{2}}. \tag{3-53}
\]

When \( x \) is on the crack, the right side of (3-49) can be expanded in terms of small values of \((1 - s)\) using the notation,

\[
\sigma_{yy}(x,0^+) - \sigma_{xx}^3(x,0^+) = \sigma_0 \sum_{n=0}^{M} h_n^2 (1-s)^n + O(1-s)^{M+1} - \sqrt{2}\sigma_0 \sum_{i=1}^{N} b_i \sum_{n=0}^{\infty} d_n^i (2n+1)(1-s)^{n-\frac{1}{2}}, \tag{3-54}
\]

where the “\( h \) constants” are obtained from integrals of known functions using (3-49). This same combination of stresses on the upper crack surface can be expressed asymptotically as follows:

\[
\sigma_{yy}(\rho, \pm\pi) - \sigma_{rr}(\rho, \pm\pi) = -\sigma_0 - \sum_{n=0}^{\infty} (-1)^n T_n^I (2\rho)^n \pm 2 \sum_{n=0}^{\infty} (-1)^n k_n^{II} (2\rho)^{n\frac{1}{2}}. \tag{3-55}
\]

Comparing (3-54) and (3-55) gives the normalized T coefficients as follows

\[
\frac{T_n^I}{\sigma_0} = -1 - h_0^2,
\]

\[
\frac{T_n^I (b-a)^n}{\sigma_0} = (-1)^n h_n^2, \quad n > 0 \tag{3-56}
\]

and the identical result (3-41) for the K coefficients. In order to determine the T-coefficients for mode II, (3-18) is used in the normalized form,
\[
\left(-4\mu_1\right) \frac{1}{\kappa_1-1} \frac{\partial v^y(x,0^+)}{\partial x} = \frac{1}{\pi} \int_{-1}^{+1} \frac{g(r)}{r-s} dr \\
+ \frac{1}{\pi} \int_{-1}^{+1} f(r) \tilde{M}_1(s,r) dr + \frac{1}{\pi} \int_{-1}^{+1} g(r) \tilde{M}_2(s,r) dr - \frac{\kappa_1+1}{\kappa_1-1} \tilde{f}(s), 
\]

(3-57)

where

\[
\tilde{M}_1(s,r) = \frac{b-a}{2} M_1(x,t).
\]

\[
\tilde{M}_1(s,r) = \frac{16\varepsilon^3 a_i}{(\kappa_1-1)} \left(\frac{4\varepsilon^2 - 3(r-s)^2}{4\varepsilon^2 + (r-s)^2}\right)^3 + 2\varepsilon a_i \left(\frac{\kappa_1+1}{\kappa_1-1}\right) \left(\frac{4\varepsilon^2 - (r-s)^2}{4\varepsilon^2 + (r-s)^2}\right)^2 + \frac{b_1}{2\varepsilon},
\]

\[
\tilde{M}_2(s,r) = \frac{8\varepsilon^2 a_i (r-s)}{(\kappa_1-1)} \left(\frac{12\varepsilon^2 - (r-s)^2}{4\varepsilon^2 + (r-s)^2}\right)^3 + \frac{8\varepsilon^2 a_i (r-s)}{(4\varepsilon^2 + (r-s)^2)^2} - \frac{b_2}{(4\varepsilon^2 + (r-s)^2)}.
\]

\[
b_1 = \frac{(\kappa_1+1)(\mu_1^2 \kappa_2 - \mu_2^2 \kappa_1)}{(\kappa_1-1)(\mu_1 + \kappa_1 \mu_2)(\mu_2 + \kappa_2 \mu_1)},
\]

\[
b_2 = \frac{-\mu_1^2 \kappa_2 - \mu_2^2 \kappa_1 + \mu_1^2 \kappa_2 \kappa_1 + \mu_2^2 \kappa_2 \kappa_1 - 2\mu_1 \mu_2 \kappa_1 \kappa_2 + 2\mu_1 \mu_2 \kappa_1}{(\kappa_1-1)(\mu_1 + \kappa_1 \mu_2)(\mu_2 + \kappa_2 \mu_1)}.
\]

(3-58)

Once again eliminating the singular integral by combining (3-54) with (3-45) as follows:

\[
\frac{\sigma_{xx}(x)}{\sigma_0} = -\frac{4\mu_1}{\kappa_1-1} \frac{1}{\sigma_0} \frac{\partial v^y(x,0^+)}{\partial x} \\
= \frac{1}{\pi} \int_{-1}^{+1} f(r) \left[ \tilde{K}_{21}(s,r) - \tilde{M}_1(s,r) \right] dr + \frac{1}{\pi} \int_{-1}^{+1} g(r) \left[ \tilde{K}_{22}(s,r) - \tilde{M}_2(s,r) \right] dr + \frac{\kappa_1+1}{\kappa_1-1} \tilde{f}(s).
\]

(3-59)

When \( s \) is off the crack, the \( f \)-bar function is zero and the right side of (3-59) can be expanded in terms of small values of \( (s - 1) \) using the notation.
\[
\sigma_{xy}(x) - \frac{-4\mu_i}{\kappa_i - 1} \frac{\partial v^3(x,0^+)}{\partial x} = \sigma_0 \sum_{n=0}^{M} h_n^3(s-1)^n + O(s-1)^{M+1},
\]  

(3-60)

where the “\( h \) constants” are obtained from integrals of known functions using (3-59).

This same combination of quantities in front of the crack tip can be expressed asymptotically as follows:

\[
\sigma_{r\theta}(\rho,0) = \frac{-4\mu_i}{\kappa_i - 1} \frac{\partial u_{\theta\theta}(\rho,0)}{\partial \rho} = \frac{1}{2} \frac{\kappa_i + 1}{\kappa_i - 1} \sum_{n=1}^{\infty} T_n^n (2\rho)^n + \frac{1}{\kappa_i - 1} G .
\]

(3-61)

Comparing (3-60) and (3-59) gives the normalized T-coefficients,

\[
\frac{T_n^n(b-a)^n}{\sigma_0} = 2 \frac{\kappa_i - 1}{\kappa_i + 1} h_3^n, \ n \geq 0,
\]

(3-62)

where for \( n = 0 \) the rigid body rotation term, \( G = T_0^n \).

When \( s \) is on the crack, the right side of (3-59) can be expanded in terms of small values of \( (1-s) \) using the notation,

\[
\sigma_{xy}(x) - \frac{-4\mu_i}{\kappa_i - 1} \frac{\partial v^3(x,0^+)}{\partial x} = \sigma_0 \sum_{n=0}^{M} h_n^3(1-s)^n + O(1-s)^{M+1} + \frac{\kappa_i + 1}{\kappa_i - 1} \frac{\sigma_0}{2} \sum_{i=1}^{N_i} d_n^i (2n+1)(1-s)^{-\frac{n-1}{2}},
\]

(3-63)

where the “\( h \) constants” are obtained from integrals of known functions using (3-59).

This same combination of stresses on the upper crack surface can be expressed asymptotically as follows:

\[
\sigma_{r\theta}(\rho,+\tau) = \frac{-4\mu_i}{\kappa_i - 1} \frac{\partial u_{\theta\theta}}{\partial \rho}
\]
\[-\tau_0 = \frac{-2}{\kappa_1 - 1} \left\{ -\frac{\kappa_1 + 1}{2} \sum_{n=0}^{\infty} (-1)^n \kappa_n^l (2\rho)^{n-\frac{1}{2}} + \frac{\kappa_1 + 1}{4} \sum_{n=1}^{\infty} (-1)^n T_n^H (2\rho)^n + \frac{\kappa_1 + 1}{4} G \right\} \right].

(3-64)

Comparing (3-61) and (3-62) gives the normalized T coefficients as follows

\[ \frac{T_n^H (b-a)^n}{\sigma_o} = (-1)^n \frac{2}{\kappa_1} \frac{\kappa_1 - 1}{\kappa_1 + 1} h_n^4, \quad n \geq 0, \]

(3-65)

where for \( n = 0 \) the rigid body rotation term, \( G = T_0^H \), and the identical result (3-42) for the k coefficients. A very interesting point is that the T coefficients for mode II should only be dependent on the bi-material constants introduced in (3-8). We have not been able to write the kernels in (3-19) as a function of these constants, and therefore the expressions (3-59), (3-60) and (3-63) must be such that when the evaluation in (3-62) and (3-65) is made, all dependence on constants other than (3-8) disappears.

A critical expression used to obtain the T coefficients is obtained by using partial fractions and complex variables,

\[ \frac{1}{4\varepsilon^2 + (r-s)^2} = -\frac{1}{2\varepsilon} \text{Im} \left[ \frac{1}{s-r+2i\varepsilon} \right] = -\frac{1}{2\varepsilon} \text{Im} \sum_{n=0}^{\infty} \left[ \frac{(1-s)^n}{(1-r)^{n+1} + 2i\varepsilon} \right], \]

\[ = \frac{1}{2\varepsilon} \sum_{n=0}^{\infty} \frac{\sin[(n+1)\theta]}{(1-r)^2 + 4\varepsilon^2} (1-s)^n, \quad \theta = \tan^{-1} \frac{2\varepsilon}{1-r}. \]

(3-66)

This gives

\[ \frac{1}{4\varepsilon^2 + (r-s)^2} = \sum_{n=0}^{\infty} c_n (1-s)^n, \]

(3-67)

where
\[ c_n^1 = \frac{1}{2\epsilon} \frac{\sin[(n+1)\theta]}{[(1-r)^2 + 4\epsilon^2]^{n+1/2}}. \]  

(3-68)

The other required expressions are

\[ \frac{1}{[4\epsilon^2 + (r-s)^2]} = \sum_{n=0}^{\infty} c_n^2 (1-s)^n, \quad c_n^2 = \sum_{i=0}^{n} c_i^1 c_{n-i}^1, \]  

(3-69)

\[ \frac{1}{[4\epsilon^2 + (r-s)^2]} = \sum_{n=0}^{\infty} c_n^3 (1-s)^n, \quad c_n^3 = \sum_{i=0}^{n} c_i^1 c_{n-i}^2. \]  

(3-70)

Using (3-49) and (3-50) for the mode I T coefficients and (3-59) and (3-60) for mode II, requires that the following be expressed in terms of small distances from the right crack tip, i.e., \( s \) near \( I \):

\[ [\tilde{K}_{11}(s,r) - \tilde{L}_1(s,r)] = \frac{(r-s)(a_1 + 2a_2)}{[4\epsilon^2 + (r-s)^2]} - \frac{8a_1\epsilon^2(r-s)[12\epsilon^2 - (r-s)^2]}{[4\epsilon^2 + (r-s)^2]^3}. \]  

(3-71)

\[ [\tilde{K}_{12}(s,r) - \tilde{L}_2(s,r)] = \frac{2\epsilon(a_1 - 2a_2)}{[4\epsilon^2 + (r-s)^2]} - \frac{4a_1\epsilon[4\epsilon^2 - (r-s)^2]}{[4\epsilon^2 + (r-s)^2]^2} + \frac{16a_1\epsilon^3[4\epsilon^2 - 3(r-s)^2]}{[4\epsilon^2 + (r-s)^2]^3}. \]  

(3-72)

\[ [\tilde{K}_{21}(s,r) - \tilde{M}_1(s,r)] = \frac{\epsilon[a_1 + 2a_2 - 2b_1]}{[4\epsilon^2 + (r-s)^2]} - \frac{2a_1\epsilon[k_1 + 1][4\epsilon^2 - (r-s)^2]}{(k_1-1)[4\epsilon^2 + (r-s)^2]^2} - \frac{8a_1\epsilon^3(k_1 + 1)[4\epsilon^2 - 3(r-s)^2]}{(k_1-1)[4\epsilon^2 + (r-s)^2]^3}. \]  

(3-73)

\[ [\tilde{K}_{22}(s,r) - \tilde{M}_2(s,r)] = -\frac{1}{2} \frac{(r-s)(a_1 - 2a_2 - 2b_2)}{[4\epsilon^2 + (r-s)^2]^2} \]
\[-\frac{4a_1\varepsilon^2(r-s)(\kappa_1+1)[12\varepsilon^2-(r-s)^2]}{(\kappa_1-1)[4\varepsilon^2+(r-s)^2]^3}\]  \hspace{1cm} (3-74) 

The next step is to write the above four expressions in terms of one series as follows:

\[
\left[\bar{K}_{11}(s,r) - \bar{L}_1(s,r)\right] = \sum_{n=0}^{\infty} hf_n^I (1-s)^n,
\]

\[hf_n^I = -(a_1 + 2a_2)\left[(1-r)c_n^1 - c_{n-1}^1\right] + 8a_1\varepsilon^2 \left\{(1-r)[12\varepsilon^2 - (1-r)^2]\right\}c_n^3 + \left\{3(1-r)^2 - 12\varepsilon^2\right\}c_{n-1}^3 - 3(1-r)c_{n-2}^3 + c_{n-3}^3 \]  \hspace{1cm} (3-75) 

\[
\left[\bar{K}_{12}(s,r) - \bar{L}_2(s,r)\right] = \sum_{n=0}^{\infty} hg_n^I (1-s)^n,
\]

\[hg_n^I = 2\varepsilon(a_1 - 2a_2)c_n^1 - 4a_1\varepsilon \left\{4\varepsilon^2 - (1-r)^2\right\}c_n^2 + 2(1-r)c_{n-1}^2 - c_{n-2}^2 \] 

\[+16a_1\varepsilon^3 \left\{4\varepsilon^2 - 3(1-r)^2\right\}c_n^3 + 6(1-r)c_{n-1}^3 - 3c_{n-2}^3 \]  \hspace{1cm} (3-76) 

\[
\left[\bar{K}_{21}(s,r) - \bar{M}_1(s,r)\right] = \sum_{n=0}^{\infty} hf_n^{II} (1-s)^n,
\]

\[hf_n^{II} = \varepsilon(a_1 + 2a_2 - 2b_1)c_n^1 - 2a_1\varepsilon \frac{\kappa_1+1}{\kappa_1-1} \left\{4\varepsilon^2 - (1-r)^2\right\}c_n^2 + 2(1-r)c_{n-1}^2 - c_{n-2}^2 \] 

\[+8a_1\varepsilon^3 \frac{\kappa_1+1}{\kappa_1-1} \left\{4\varepsilon^2 - 3(1-r)^2\right\}c_n^3 + 6(1-r)c_{n-1}^3 - 3c_{n-2}^3 \]  \hspace{1cm} (3-77) 

\[
\left[\bar{K}_{22}(s,r) - \bar{M}_2(s,r)\right] = \sum_{n=0}^{\infty} hg_n^{II} (1-s)^n,
\]

\[hg_n^{II} = \frac{1}{2} (a_1 - 2a_2 - 2b_2) \left[(1-r)c_n^1 - c_{n-1}^1\right] \]
\[ +4a e^2 \kappa_1 +1 \left\{ (1-r) \left[ 12e^2 - (1-r)^2 \right] c_n^3 + \left[ 3(1-r)^2 - 12e^2 \right] c_n^3 - 3(1-r)c_{n-2}^3 + c_{n-3}^3 \right\} \]

(3-78)

In the above it is understood that

\[ c_n^I = 0, n < 0. \]  

(3-79)

Comparing (3-54) and (3-56), and using (3-75) and (3-76) for the case on the crack,

\[ \bar{T}_n^I = \frac{T_n^I (b-a)^n}{\sigma_0} = (-1)^{n+1} \frac{1}{\pi} \int_{-1}^{1} \left[ h f_n^I \bar{f}(r) + h g_n^I \bar{g}(r) \right] dr, \]

(3-80)

and similarly for mode II, using (3-63) leads to

\[ \bar{T}_n^{II} = \frac{T_n^{II} (b-a)^n}{\sigma_0} = (-1)^n \frac{\kappa_1 -1}{2} \frac{1}{\kappa_1 +1} \int_{-1}^{1} \left[ h f_n^{II} \bar{f}(r) + h g_n^{II} \bar{g}(r) \right] dr. \]

(3-81)

In order to compare full field results to the asymptotic expressions obtained using the stress intensity factors and T-stress coefficients, the following results apply off the crack:

\[ \frac{\sigma_{xx}(r,0)}{\sigma_0} = \frac{1}{\sigma_0} \left\{ \sum_{n=0}^{\infty} k_n^I (2r)^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} T_n^I (2r)^n \right\} = \sum_{n=0}^{\infty} \left[ \frac{\bar{k}_n^I}{\sqrt{2(s-1)}} + \bar{T}_n^I \right] (s-1)^n \]

(3-82)

\[ \frac{\sigma_{yy}(r,0)}{\sigma_0} = \frac{1}{\sigma_0} \sum_{n=0}^{\infty} k_n^I (2r)^{n-\frac{1}{2}} = \sum_{n=0}^{\infty} \bar{k}_n^I \frac{(s-1)^n}{\sqrt{2(s-1)}} \]

(3-83)

\[ \frac{\sigma_{xy}(r,0)}{\sigma_0} = \frac{1}{\sigma_0} \sum_{n=0}^{\infty} k_n^{II} (2r)^{n-\frac{1}{2}} = \sum_{n=0}^{\infty} \bar{k}_n^{II} \frac{(s-1)^n}{\sqrt{2(s-1)}} \]

(3-84)

\[ \frac{1}{\sigma_0} \frac{1}{\kappa_1 + 1} \frac{e^2(\bar{v}(r,0))}{dr} = \frac{1}{\sigma_0} \left\{ \sum_{n=0}^{\infty} k_n^{II} (2r)^{n-\frac{1}{2}} - \frac{1}{2} \frac{\kappa_1 +1}{\kappa_1 -1} \sum_{n=0}^{\infty} T_n^{II} (2r)^n \right\} \]

\[ = \sum_{n=0}^{\infty} \left[ \frac{\bar{k}_n^{II}}{\sqrt{2(s-1)}} - \frac{1}{2} \frac{\kappa_1 +1}{\kappa_1 -1} \bar{T}_n^{II} \right] (s-1)^n \]

(3-85)
Similarly, along the crack flanks where $\theta = \pm \pi$, the pertinent expressions are:

$$\frac{\sigma_{xx}(r, \pm \pi)}{\sigma_0} = \frac{1}{\sigma_0} \left[ \sum_{n=0}^{\infty} (-1)^n T_n^I (2r)^n + 2 \sum_{n=0}^{\infty} (-1)^n k_n^H (2r)^{n-\frac{1}{2}} \right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[ \frac{-2k_n^H}{\sqrt{2(1-s)}} \right] (1-s)^n$$

(3-86)

$$-\frac{1}{\sigma_0} \frac{4\mu}{\kappa - 1} \frac{\partial v(r, \pm \pi)}{\partial r} = \frac{1}{\sigma_0} \left[ \pm \frac{\kappa + 1}{\kappa - 1} \sum_{n=0}^{\infty} (-1)^n k_n^I (2r)^{n-\frac{1}{2}} - \frac{1}{2} \frac{\kappa + 1}{\kappa - 1} \sum_{n=0}^{\infty} (-1)^n T_n^H (2r)^n \right]$$

$$= \frac{\kappa + 1}{\kappa - 1} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{\pm k_n^I}{\sqrt{2(1-s)}} - \frac{1}{2} T_n^H \right] (1-s)^n.$$

(3-87)
CHAPTER 4

STRESS INTENSITY FACTORS, T-STRESS COEFFICIENTS AND THE
STRESS FIELD AROUND THE TIP OF A CRACK THAT IS PARALLEL TO AN INTERFACE

In this Chapter all the results are presented starting with a convergence study for the modes I and II, k- and T-coefficients. This is followed by a set of contour plots of the coefficients for the full range of $\alpha$ and $\beta$. After this the focus is on details of the stress field for the case when the crack tip closes for $\alpha = 0.98$ and $\beta = 0.495$. Several plots of displacement and stress along the line of the crack and for the stress field around the crack tip are presented.

4.1 Convergence Study

The numerical approach detailed in the previous chapter is used to determine the $K$ and $T$-coefficients using a double precision computer program written in Fortran. Unfortunately there are no results for validation from the literature for the higher order coefficients for a crack parallel to an interface. However, the approach has been validated for the pure mode I case by Ananthasayanam, et. al (2007) and for the pure mode II case by Ananthasayanam (2008). Furthermore, as will be seen throughout this Chapter, the asymptotic results are consistent with the full field stress field.

In Table 4.1, for $a/h = 0.001$, converged values of the four different coefficients are presented to show how convergence behaves with respect to the coefficient order, $n$. This
is a difficult case for convergence since the length parameter, \( h/a \), is small. As shown by Ananthasayanam, et. al (2007), it is easier to obtain the T-coefficients than the K-coefficients, the reason being that the former are the result of an integration of a function, while the latter are obtained by evaluating the derivative of the function at the endpoint, \( s = -1 \).

Table 4.1: Converged asymptotic coefficients defined by Equations (3-42, 3-43, 3-56, 3-65) for \( a/h = 0.01 \), \( \alpha = 0.48 \), \( \beta = 0.495 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{k_1 a^n}{\sigma_0 \sqrt{a}} )</th>
<th>( \frac{k_2 a^n}{\sigma_0 \sqrt{a}} )</th>
<th>( \frac{T_1 (2a)^n}{\sigma_0} )</th>
<th>( \frac{T_2 (2a)^n}{\sigma_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.23097979E+00</td>
<td>-0.61204009E+00</td>
<td>0.54422967E+01</td>
<td>-0.18432797E+02</td>
</tr>
<tr>
<td>1</td>
<td>0.11775661E+03</td>
<td>0.21589759E+03</td>
<td>0.60417942E+04</td>
<td>-0.29049912E+04</td>
</tr>
<tr>
<td>2</td>
<td>0.12780295E+06</td>
<td>0.3176309E+05</td>
<td>-0.18137165E+07</td>
<td>0.44182013E+07</td>
</tr>
<tr>
<td>3</td>
<td>-0.300965E+08</td>
<td>0.641008E+08</td>
<td>0.21210725E+10</td>
<td>0.12629598E+10</td>
</tr>
<tr>
<td>4</td>
<td>-0.5034E+11</td>
<td>-0.1373E+11</td>
<td>0.63203819E+12</td>
<td>-0.17231618E+13</td>
</tr>
<tr>
<td>5</td>
<td>0.1175E+14</td>
<td>-0.244E+14</td>
<td>-0.82948942E+15</td>
<td>-0.47021641E+15</td>
</tr>
<tr>
<td>6</td>
<td>0.19E+17</td>
<td>0.55E+16</td>
<td>-0.22692141E+18</td>
<td>0.63102698E+18</td>
</tr>
<tr>
<td>7</td>
<td>?</td>
<td>?</td>
<td>0.30340792E+21</td>
<td>0.16283599E+21</td>
</tr>
<tr>
<td>11</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
<td>0.30340792E+21</td>
<td>0.16283599E+21</td>
</tr>
</tbody>
</table>

In Table 4.2 convergence of the 4\textsuperscript{th} \( K \)- and 12\textsuperscript{th} \( T \)-coefficients, with respect to the parameter, \( N \) defined in (3-29), are presented for \( h/a = 0.001 \), \( \alpha = 0.98 \) and \( \beta = 0.495 \). Convergence data for the case of \( h/a = 0.1 \) are presented in Table 3, which shows an increasing level of difficulty as the crack gets closer to the interfa...
Table 4.2: Convergence study of asymptotic coefficients with respect to $N$ defined in Equation (3-29) for $a/h = 0.001$, $\alpha = 0.98$, $\beta = 0.495$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_1 a^3 \over (\sigma_0 \sqrt{a})$</th>
<th>$k_1^{\prime} a^3 \over (\sigma_0 \sqrt{a})$</th>
<th>$T_{ii}(2a)^{11} \over \sigma_0$</th>
<th>$T_{ii}^{\prime}(2a)^{11} \over \sigma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>From (40)</td>
<td>From (41)</td>
<td>From (54)</td>
<td>From (60)</td>
</tr>
<tr>
<td>40</td>
<td>0.33226405E+08</td>
<td>0.28236903E+08</td>
<td>0.33557133E+32</td>
<td>0.16786781E+32</td>
</tr>
<tr>
<td>80</td>
<td>0.11670084E+08</td>
<td>0.20226995E+09</td>
<td>0.33864010E+32</td>
<td>0.17079087E+32</td>
</tr>
<tr>
<td>120</td>
<td>-0.22580818E+08</td>
<td>0.14926987E+08</td>
<td>0.33879512E+32</td>
<td>0.17055089E+32</td>
</tr>
<tr>
<td>160</td>
<td>-0.34109549E+08</td>
<td>0.71972925E+08</td>
<td>0.33879156E+32</td>
<td>0.17054409E+32</td>
</tr>
<tr>
<td>200</td>
<td>-0.29327462E+08</td>
<td>0.63345405E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>240</td>
<td>-0.30187094E+08</td>
<td>0.64142307E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>280</td>
<td>-0.30089157E+08</td>
<td>0.64100422E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>320</td>
<td>-0.30096989E+08</td>
<td>0.64100693E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>360</td>
<td>-0.30096533E+08</td>
<td>0.64100880E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>400</td>
<td>-0.30096543E+08</td>
<td>0.64100847E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
<tr>
<td>440</td>
<td>-0.30096554E+08</td>
<td>0.64100865E+08</td>
<td>0.33879146E+32</td>
<td>0.17054406E+32</td>
</tr>
</tbody>
</table>

Table 4.3: Same as Table 4.2 for $a/h = 0.1$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_1 a^3 \over (\sigma_0 \sqrt{a})$</th>
<th>$k_1^{\prime} a^3 \over (\sigma_0 \sqrt{a})$</th>
<th>$T_{ii}(2a)^{11} \over \sigma_0$</th>
<th>$T_{ii}^{\prime}(2a)^{11} \over \sigma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>From (40)</td>
<td>From (41)</td>
<td>From (54)</td>
<td>From (60)</td>
</tr>
<tr>
<td>8</td>
<td>-0.21353093E+03</td>
<td>0.17581891E+03</td>
<td>0.40374055E+09</td>
<td>-0.17708400E+09</td>
</tr>
<tr>
<td>16</td>
<td>-0.10869035E+03</td>
<td>0.44302018E+02</td>
<td>0.40364999E+09</td>
<td>-0.17739224E+09</td>
</tr>
<tr>
<td>32</td>
<td>-0.10102608E+03</td>
<td>0.40568528E+02</td>
<td>0.40364989E+09</td>
<td>-0.17739218E+09</td>
</tr>
<tr>
<td>64</td>
<td>-0.10102582E+03</td>
<td>0.40568770E+02</td>
<td>0.40364989E+09</td>
<td>-0.17739218E+09</td>
</tr>
<tr>
<td>128</td>
<td>-0.10102575E+03</td>
<td>0.40568768E+02</td>
<td>0.40364989E+09</td>
<td>-0.17739218E+09</td>
</tr>
</tbody>
</table>

In the last table of this section it is demonstrated that as the number of terms is increased for small $r/a$, the asymptotic solutions converge to the full-field solution. To do this the following percent error definition is made for the $\sigma_{xx}$ stress in front of the crack tip:

\[
\text{Percent Error} = \frac{\text{Full Field Solution} - \text{Asymptotic Solution}}{\text{Full Field Solution}} \times 100
\]
\[ E_M = 100 \times \frac{\frac{\sigma_{xx}(r,0)}{\sigma_0} - A^M_{xx}}{\frac{\sigma_{xx}(r,0)}{\sigma_0}} , \] (4-1)

where from (3-82) the asymptotic series involves the mode I \( k \)- and \( T \)-coefficients as follows,

\[ \frac{\sigma_{xx}(r,0)}{\sigma_0} \approx A^M_{xx} = \sum_{n=0}^{\infty} \frac{k_n^I}{\sqrt{2(s-1)}} + T_n^I (s-1)^n . \] (4-2)

Values of the error measure from (4-1) are presented in Table 4.4.

Table 4.4: Convergence of \( \sigma_{xx} \) in front of the crack tip with respect to the number of terms as defined in (4-2) for \( a/h = 0.1, \alpha = 0.98, \beta = 0.495 \). The percent error measure, \( E_M \), is defined in (4-1).

<table>
<thead>
<tr>
<th>( r/a ) ( \theta = 0 )</th>
<th>( \frac{\sigma_{xx}(r,0)}{\sigma_0} )</th>
<th>( E_0 )</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th>( E_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.697E+00</td>
<td>0.440E+02</td>
<td>0.368E+02</td>
<td>-0.764E+01</td>
<td>-0.155E+02</td>
</tr>
<tr>
<td>0.01</td>
<td>0.361E+01</td>
<td>0.510E+01</td>
<td>0.190E+00</td>
<td>-0.128E-01</td>
<td>-0.727E-03</td>
</tr>
<tr>
<td>0.001</td>
<td>0.131E+02</td>
<td>0.553E+00</td>
<td>0.156E-02</td>
<td>-0.152E-04</td>
<td>-0.577E-07</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.434E+02</td>
<td>0.563E-01</td>
<td>0.145E-04</td>
<td>-0.158E-07</td>
<td>-0.102E-10</td>
</tr>
</tbody>
</table>

In Table 4.5 the error of the \( \sigma_{xx} \) component of stress on the upper crack surface is presented using the following asymptotic expression from (3-86),
\[
\frac{\sigma_{xx}(r, \pi)}{\sigma_0} \approx A_{xx}^M = \sum_{n=0}^{\infty} (-1)^n \left( \frac{\tilde{T}_n - \frac{2k_n^II}{\sqrt{2(1-s)}}}{1-s} \right)^n,
\]

where now the mode II \( k \)-coefficients are involved.

Table 4.5: Convergence of \( \sigma_{xx} \) on the upper crack surface with respect to the number of terms as defined in (4-3) for \( a/h = 0.1 \), \( \alpha = 0.98 \), \( \beta = 0.495 \). The percent error measure, \( E_M \), is defined in (4-1).

<table>
<thead>
<tr>
<th>( r/a ) ( \theta = +\pi )</th>
<th>( \sigma_{xx}(r, 0)/\sigma_0 )</th>
<th>( E_0 )</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th>( E_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.726E-01</td>
<td>-0.265E+03</td>
<td>-0.474E+03</td>
<td>0.538E+02</td>
<td>0.190E+03</td>
</tr>
<tr>
<td>0.01</td>
<td>0.290E+01</td>
<td>-0.440E+01</td>
<td>-0.491E+00</td>
<td>0.121E-01</td>
<td>0.200E-02</td>
</tr>
<tr>
<td>0.001</td>
<td>0.117E+02</td>
<td>-0.448E+00</td>
<td>-0.414E-02</td>
<td>0.133E-04</td>
<td>0.170E-06</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.393E+02</td>
<td>-0.459E-01</td>
<td>-0.397E-04</td>
<td>0.139E-07</td>
<td>0.292E-10</td>
</tr>
</tbody>
</table>

Two things are clear from these tables: 1) the value of \( r/a \) must be “small enough” and 2) if it is, as the number of terms increases, the error drops, i.e., the asymptotic solution quantified by a few constants that can be determined from an analysis such as this, define the conditions of stress around the crack tip.

In the next section a study of the effect of the bi-material constants, \( \alpha \) and \( \beta \), is made by using contour plots.

4.2. Asymptotic coefficients as a function of the bi-material constants

The stress intensity factor results have been checked with values reported in the literature; however, there are no reported values for the higher order terms. One of the
most convincing arguments that the results are correct is that the mode II, T-coefficients are only a function of \( \alpha \) and \( \beta \). This was not proven analytically; rather the numerical solutions presented in this section confirm it. For a given set of \( \alpha \) and \( \beta \), multiple values of \( \mu_1, \mu_2, \kappa_1, \) and \( \kappa_2 \) that correspond to these values give the same results.

In the previous section it was demonstrated that several higher order coefficients can be determined very accurately using the singular integral equation approach. In this section the versatility of the formulation is used to generate contour plots for the full range of material pair possibilities and for non-dimensional crack lengths of \( a/h = 1, 0.1, 0.01, 0.001 \) and \( 0.0001 \). Results are presented for the first eight coefficients, which corresponds to the first two in each of the four categories: modes \( I \) and \( II, K- \) and \( T- \) coefficients. It is recalled from the discussion involving Equations (3-61) through (3-65) that the first mode \( II T \)-coefficient is actually the rigid body rotation term, \( G \), first introduced in Equation (A-2). For each crack length through \( h/a = 0.001 \) there are a series of eight plots in the order:

\[
\frac{k_0^I}{\sigma_0 \sqrt{a}}, \frac{k_0^{II}}{\sigma_0 \sqrt{a}}, \frac{T_0^I}{\sigma_0}, \frac{T_0^{II}}{\sigma_0} = G, \frac{k_1^I a}{\sigma_0 \sqrt{a}}, \frac{k_1^{II} a}{\sigma_0 \sqrt{a}}, \frac{T_1^I 2a}{\sigma_0}, \frac{T_1^{II} 2a}{\sigma_0}. \tag{4-1}
\]

For the case of \( h/a = 0.0001 \), only the first coefficients in (4-1) are presented due to numerical difficulty. There are a total of forty-four figures that follow in this section. The first eight figures, Figures 4.1-4.8, are for \( a/h = 1 \). The remaining figures correspond to \( a/h = 0.1, 0.01, 0.001 \) and \( 0.0001 \), respectively.
Figure 4.1. Contour plot for Mode-I $k$ coefficient when $n=0$ and $h/a=0.5$

Figure 4.2. Contour plot for Mode-II $k$ coefficient when $n=0$ and $h/a=0.5
Figure 4.3 Contour plot for Mode-I T coefficient when \( n=0 \) and \( h/a=0.5 \)

Figure 4.4. Contour plot for Mode-II T coefficient when \( n=0 \) and \( h/a=0.5 \)
Figure 4.5 Contour plot for Mode-I $k$ coefficient when $n=1$ and $h/a=0.5$

Figure 4.6 Contour plot for Mode-II $k$ coefficient when $n=1$ and $h/a=0.5
Figure 4.7 Contour plot for Mode-I T coefficient when n=1 and h/a=0.5

Figure 4.8 Contour plot for Mode-II T coefficient when n=1 and h/a=0.5
Figure 4.9 Contour plot for Mode-I k coefficient when n=0 and h/a=0.1

Figure 4.10 Contour plot for Mode-II k coefficient when n=0 and h/a=0.1
Figure 4.11 Contour plot for Mode-I T coefficient when n=0 and h/a=0.1

Figure 4.12 Contour plot for Mode-II T coefficient when n=0 and h/a=0.1
Figure 4.13 Contour plot for Mode-I k coefficient when n=1 and h/a=0.1

Figure 4.14 Contour plot for Mode-II k coefficient when n=1 and h/a=0.1
Figure 4.15 Contour plot for Mode-I $T$ coefficient when $n=1$ and $h/a=0.1$

Figure 4.16 Contour plot for Mode-II $T$ coefficient when $n=1$ and $h/a=0.1
Figure 4.17 Contour plot for Mode-I $k$ coefficient when $n=0$ and $h/a=0.01$

Figure 4.18 Contour plot for Mode-II $k$ coefficient when $n=0$ and $h/a=0.01$
Figure 4.19 Contour plot for Mode-I $T$ coefficient when $n=0$ and $h/a=0.01$

Figure 4.20 Contour plot for Mode-II $T$ coefficient when $n=0$ and $h/a=0.01$
Figure 4.21 Contour plot for Mode-I $k$ coefficient when $n=1$ and $h/a=0.01$

Figure 4.22 Contour plot for Mode-II $k$ coefficient when $n=1$ and $h/a=0.01$
Figure 4.23 Contour plot for Mode-I (T coefficient)/100 when n=1 and h/a=0.01

Figure 4.24 Contour plot for Mode-II (T coefficient)/100 when n=1 and h/a=0.01
Figure 4.25 Contour plot for Mode-I k coefficient when n=0 and h/a=0.001

Figure 4.26 Contour plot for Mode-II k coefficient when n=0 and h/a=0.001
Figure 4.27 Contour plot for Mode-I T coefficient when n=0 and h/a=0.001

Figure 4.28 Contour plot for Mode-II T coefficient when n=0 and h/a=0.001
Figure 4.29 Contour plot for Mode-I (k coefficient)/100 when n=1 and h/a=0.001

Figure 4.30 Contour plot for Mode-II (k coefficient)/100 when n=1 and h/a=0.001
Figure 4.31 Contour plot for Mode-I (T coefficient)/1000 when n=1 and h/a=0.001

Figure 4.32 Contour plot for Mode-II (T coefficient)/1000 when n=1 and h/a=0.001
Figure 4.33 Contour plot for Mode-I k coefficient when n=0 and h/a=0.0001

Figure 4.34 Close up view of the left side of Figure 4.33.
Figure 4.35 Close up view of the right side of Figure 4.33 which shows a region in the upper right hand corner where the mode I stress intensity factor is zero.
Figure 4.36 Contour plot for Mode-II k coefficient when n=0 and h/a=0.0001

Figure 4.37 Close up view of the left side of Figure 4.36.
Figure 4.38 Close up view of the right side of Figure 4.36.
Figure 4.39 Contour plot for Mode-I T coefficient when n=0 and h/a=0.0001

Figure 4.40 Close up view of the left side of Figure 4.39.
Figure 4.41 Close up view of the right side of Figure 4.39.
Figure 4.42 Contour plot for Mode-II T coefficient when n=0 and h/a=0.0001

Figure 4.43 Close up view of the left side of Figure 4.42.
Figure 4.44 Close up view of the right side of Figure 4.42.

The most interesting feature in these results is that for $h/a = 0.0001$ the value of the mode I stress intensity factor in Figure 4.35 becomes zero even though the loading is far-field tension as shown in Figure 1.1. This means that the crack tip is pinching closed. This feature is studied in the next section.

**4.3. Crack Opening Displacement and Comninou Contact Zones**

England (1965) was the first to point out the anomalous result of crack surface interpenetration very near the tip of an interface crack (case of $h = 0$ in Figure 2.1). The complex singularity that results suggests an oscillatory behavior that is not physically possible. Within the limitations of the linear theory of elasticity, Comninou (1977)
obtained the mathematically correct solution which is simply a closed crack tip with a very short contact zone, $\Delta_0$ as illustrated in Figure 4.45.

Figure 4.45. A Comninou contact zone at the tip of an interface crack.

Other important contributions include Comninou and Dundurs (1980), Gautesen (1992). In particular Gautesen provides an analytical asymptotic solution that shows, for example, the extent of the contact zone as a function of the Dundurs parameter, $\beta$, which is presented in Figure 4.46.
Figure 4.46. A Comninou contact zone of length $\Delta_0$, and its value as a function of the Dundurs parameter, $\beta$.

In the current study this same phenomena occurs for a crack parallel and very close to an interface, and is illustrated schematically in Figure 4.47. It is noted that the occurrence of such a closed crack tip is defined by the “cusp” of Figure 4.45, where the mode I stress intensity factor is zero as first seen in Figure 4.35 in the upper right-hand corner of the $\alpha-\beta$ plot. Within the context of the current study, the special value of $h/a$, i.e., $h_1^*/a$, as shown in Figure 4.47, is determined such that the stress intensity factor becomes zero. As seen in Figure 4.46, the larger the value of $\beta$, the larger the contact zone and therefore the easier it is to determine this special location numerically.
The crack opening displacement, $V$, defined by Equation (3-3) and given numerically by (3-30), is presented in Figure 4.48 for several values of $h/a$ for $\alpha = 0.98$, $\beta = 0.495$, which is a case where a relatively large value of $h_1^*$ is expected. Since the contact zone is so small, it is necessary to look very closely at the crack tip to see the behavior noted in Figure 4.47. This is presented in Figure 4.49 where it is shown that $h_1^* = 0.00012357$. Another version of this plot is presented in Figure 4.50 with the weight function removed from the displacement.
Figure 4.48. Normalized crack opening displacement for a range of $h/a$ for the material pair, $\alpha = 0.98$, $\beta = 0.495$

Figure 4.49. A close-up view of the normalized crack opening displacement showing that for the material pair, $\alpha = 0.98$, $\beta = 0.495$, the case of $h_1^*$ from Figure 4.47 has been obtained.
Figure 4.50. Another close-up view of the normalized crack opening displacement without the square root weight function that shows more clearly that the case of $h_1^*$ from Figure 4.47 has been obtained.

The preceding plots show that program has the ability to study cracks very close to the interface and in particular to see if the stress field that causes the crack tip to close can be revealed.
4.4 Stresses along the line of the crack:

In this section the normalized stresses $\sigma_{xx}$, $\sigma_{yy}$ and $\tau_{xy}$ are plotted along the line of the crack and compared for $h/a=1.0, 0.1, 0.01, 0.001$ and $0.0001236$, the latter value selected since it corresponds to $h_1^*$ where the mode I stress intensity factor becomes zero. The material constants in all cases in this section are $\alpha=0.98$ and $\beta=0.495$. The $\sigma_{xx}$ stress can be plotted in three regions: the upper crack surface, the lower crack surface and in front of the crack. The other two stress components are zero on the crack surfaces so are only plotted in front of the crack tip. In each Figure the stresses are plotted over two ranges. The first range is for $0 < x/a < 1$, which is from the center of the crack to the right crack tip. The second plot presents a close-up of the crack tip and is scaled with $h$ by using the coordinate $(x-a)/h$. When this quantity is -1, the location is $h$ to the left of the crack tip, and when it is +1, the location is $h$ to the right of the crack tip. In Figure 4.51 and 4.52, $\sigma_{xx}$ is plotted on the upper and lower crack surfaces, respectively.

Figure 4.51 Normalized $\sigma_{xx}$ stress on upper surface of the crack for different $h/a$ and a material pair corresponding to $\alpha=0.98$, $\beta=0.495$. The figure on the right is a close-up view of the crack tip.
Figure 4.52 Same as Figure 4.51 for the lower crack surface.

It is observed that these stresses are very different from each other due to the delta function term from the expression (3-14). Furthermore, in Figure 4.52, the plot on the right shows how the stresses scale with $h$, since, for example, at $2h$ to the left of the crack tip ($(x-a)/h = -2$), the stress is approximately independent of $h/a$.

In Figures 4.53-4.55 the stresses in front of the crack tip are presented. Again the two ranges are provided to show the details near the crack tip. For the special case of $h/a = 0.0001236$ where the mode I stress intensity factor is zero, the normalized stresses $\sigma_{xx}$ in Figure 4.53 and $\sigma_{yy}$ in Figure 4.55, become zero, which is understood from Equations (3-82) and (3-83). The boundary layer nature of the solution is once again evident in the way the solution scales with $h$. These five figures compare the effect of $h/a$ on each of the five stresses of interest along the line of the crack.
Figure 4.53 Normalized $\sigma_{xx}$ stress in front of the crack for different $h/a$ and a material pair corresponding to $\alpha=0.98$, $\beta=0.495$. The figure on the right is a close-up view of the crack tip.

Figure 4.54 Same as 4.53 for the shear stress, $\tau_{xy}$.

Figure 4.55 Same as 4.53 for $\sigma_{yy}$. 
Now for each value of $h/a$, the full-field solution is compared to asymptotic solutions to show that the higher order terms obtained are indeed correct. The more terms taken, the better the comparison in the limit as $r/a$ approaches zero. In each of Figures 4.56-4.60 four plots are given, which correspond to $h/a = 1$ in the upper left, $h/a = 0.1$ in the upper right, $h/a = 0.01$ in the lower left and 0.001 in the lower right. The asymptotic solutions for stresses in front of the crack are obtained from Equations (3-82 – 3-84), while for the $\sigma_{xx}$ stress on the crack surfaces, Equation (3-86) is used. The Figures 4.56 – 4.60 correspond respectively to $\sigma_{xx}$ on the upper surface, $\sigma_{xx}$ on the lower surface, followed by $\sigma_{xx}$, $\sigma_{xy}$ and $\sigma_{yy}$ in front of the crack.
Figure 4.56 Full field solution for $\sigma_{xx}$ on the upper surface of the crack compared to the asymptotic solutions for $h/a = 1.0, 0.1, 0.01$ and 0.001.
Figure 4.57 Same as Figure 4.56 for the lower crack surface.
Figure 4.58. Full field solution for $\sigma_{xx}$ in front of the crack compared to the asymptotic solutions for $h/a = 1.0, 0.1, 0.01$ and 0.001.
Figure 4.59 Full field solution for $\tau_{xy}$ in front of the crack compared to the asymptotic solutions for $h/a = 1.0, 0.1, 0.01$ and 0.001.
Figure 4.60. Full field solution for $\sigma_{yy}$ in front of the crack compared to the asymptotic solutions for $h/a = 1.0, 0.1, 0.01$ and $0.001$.

In each of the Figures 4.56 – 4.60, it is evident that the asymptotic solutions fit the full-field solutions so this proves that they are correct. Another important observation is that their validity is over a fraction of $h$. This means of course that as $h/a$ approaches zero, these solutions do not represent the stress state over a physically realistic range. However, the method presented can be used to define the mathematical range over which the asymptotic solutions are valid.
4.5 Full-field polar plots for stresses around the crack tip

To study the stresses around the crack tip, several polar plots are made using $\alpha=0.98$, $\beta=0.495$ as the bi-material constants. Plots are made for the non-dimensional crack lengths, $h/a=0.1, 0.01, 0.001$ and the stresses are determined at distances from the crack tip corresponding to $r/a=0.2, 0.02, 0.002$ respectively, such that both the materials fall in the area of focus. The radius for a particular value of $h/a$ is chosen and the stresses for these parameters are plotted all around the crack tip, from 0 to 360 degrees. Figures 4.61-4.72 show the plots for the different stress components $\sigma_{xx}/\sigma_0$, $\tau_{xy}/\sigma_0$, $\sigma_{yy}/\sigma_0$ and $\sigma_e/\sigma_0$, where the equivalent stress $\sigma_e/\sigma_0$ is calculated using the formula:

$$\sigma_e = \frac{1}{\sqrt{2}}\left[\left(\sigma_r - \sigma_\theta\right) + \sigma_r^2 + \sigma_\theta^2 + 6\tau_{r\theta}^2\right]^{\frac{1}{2}}$$

(4-4)
Figure 4.61: Plot for normalized stress $\sigma_{rr}/\sigma_0$ for $h/a=0.1$ and $r/a=0.2$ and $\alpha=0.98$, $\beta=0.495$
Figure 4.62: Plot for normalized stress $\tau_{r\theta}/\sigma_0$ for $h/a=0.1$ and $r/a=0.2$ and $\alpha=0.98$, $\beta=0.495$
Figure 4.63: Plot for normalized stress $\sigma_{00}/\sigma_0$ for $h/a =0.1$ and $r/a=0.2$ and $\alpha=0.98$, $\beta=0.495$
Figure 4.64: Plot for normalized stress $\sigma_e/\sigma_0$ for $h/a = 0.1$ and $r/a = 0.2$ and $\alpha = 0.98$, $\beta = 0.495$
Figure 4.65: Plot for normalized stress $\sigma_{rr}/\sigma_0$ for $h/a = 0.01$ and $r/a = 0.02$ and $\alpha=0.98$, $\beta=0.495$
Figure 4.66: Plot for normalized stress $\tau_{\theta\theta}/\sigma_0$ for $h/a = 0.01$ and $r/a = 0.02$ and $\alpha = 0.98$, $\beta = 0.495$
Figure 4.6: Plot for normalized stress $\sigma_{\theta \theta}/\sigma_0$ for $h/a = 0.01$ and $r/a = 0.02$ and $\alpha = 0.98$, $\beta = 0.495$.
Figure 4.68: Plot for normalized stress $\sigma_c/\sigma_0$ for $h/a = 0.01$ and $r/a = 0.02$ and $\alpha = 0.98$, $\beta = 0.495$
Figure 4.69: Plot for normalized stress $\sigma_{rr}/\sigma_0$ for $h/a = 0.001$ and $r/a = 0.002$ and $\alpha = 0.98$, $\beta = 0.495$.
Figure 4.7: Plot for normalized stress $\tau_{r\theta}/\sigma_0$ for $h/a = 0.001$ and $r/a = 0.002$ and $\alpha = 0.98$, $\beta = 0.495$
Figure 4.7: Plot for normalized stress $\sigma_{00}/\sigma_0$ for $h/a = 0.001$ and $r/a = 0.002$ and $\alpha = 0.98$, $\beta = 0.495$.
In all the plots in this section, since the $\sigma_{rr}$ stress is discontinuous at the interface, all four of the stress components are discontinuous where $r/a$ intersects the interface, which for all cases considered is at the angles, $210^0$ and $330^0$. In addition, since the $\sigma_{rr}$ stress is discontinuous from the upper to the lower crack surfaces, there is a discontinuity in both this stress component and the equivalent stress. Since the geometry was selected such
that for each crack length, $r/h = 2$, it can be observed that for each of the four stress components, the shapes of the plots are similar, although the magnitude increases as $h/a$ decreases. This increase in magnitude can be explained by assuming a fixed $a$, as $h$ becomes smaller, $r$ must also become smaller. Stresses increase as the crack tip is approached.
4.6 Comparison of full-field and asymptotic solution around the crack tip:

The following polar plots in this section compare the full field solution (equivalent stress $\sigma_e/\sigma_0$) to the asymptotic solution for different number of terms around the crack tip. These plots are made for $h/a=0.1, 0.01 \text{ and } 0.001 \text{ and } r/h =0.3$ respectively. As in the previous section, the material constants are defined by $\alpha=0.98, \beta=0.495$. The asymptotic solution is obtained from Equation 1.1 for up to 4 terms, where a “term” includes both a k and a T coefficient.

Figure 4.73 Plot comparing the stress $\sigma_e/\sigma_0$ with the asymptotic solution for $h/a =0.1$ and $r/h=0.3$ and $\alpha=0.98, \beta=0.495$
Figure 4.74 Plot comparing the stress $\sigma/e_0$ with the asymptotic solution for $h/a = 0.01$ and $r/h = 0.3$ and $\alpha = 0.98, \beta = 0.495$
Figures 4.73-4.75 show that the asymptotic solution improves with the number of terms. As the number of terms increases the lines in the plots come closer to the full field solution, which is expected, but validates the two solutions.
CHAPTER - 5
SUMMARY AND CONCLUSION

An accurate method to calculate the higher order terms in the asymptotic expansion of stresses around the crack tip is presented. Singular integral equation approach is chosen for a mixed mode problem and the stress intensity factor and T-stress coefficient are determined. The above method is applied to a crack parallel to interface problem.

The coefficients are difficult to obtain as n value increases because of convergence and double precision problems. K’s are obtained for accurately for n=6 and T’s are obtained till n=11. Then the analysis for the coefficients as a function of material properties $\alpha$ and $\beta$ is performed. Various plots for the first two terms of the coefficients are plotted to understand their behavior with respect to $\alpha$ and $\beta$. For $h/a=0.0001$ the stress intensity factor becomes zero, which is a very interesting feature.

The crack opening displacement is studied for different crack distances from the interface. The closing of crack tip is studied. In this problem, closing of the crack tip occurs for a crack parallel and very close to the interface. A value for $h_1^*/a$ is determined where the stress intensity factor becomes zero and the closing of crack tip occurs.

The stresses along the line of the crack are also studied and they are compared with the stresses obtained by asymptotic solutions. Stresses in front of the crack and upper and lower surfaces of the crack are compared for different $h/a$. The comparison with asymptotic solution makes sure that the higher order terms obtained are correct. As
the number of terms increases the solution becomes better when we are very close to the crack tip.

Polar plots are made to study the stresses around the crack tip. All the three stresses and equivalent stress are potted to see how the stresses vary as we go from one material to another around the crack tip. At the interface, the stresses are discontinuous as there is a change in material at the interface and it is evident from the plots.

Equivalent stress around the crack tip is also compared with asymptotic solution. As the number of terms increases the solution gets better.
APPENDICES
APPENDIX A

ASYMPTOTIC STRESSES AND DISPLACEMENTS FOR MODES I AND II.

Following the eigenfunction expansion approach of Williams (1952), the in plane stress and displacement components near the tip of a stress free crack can be expressed as follows:

\[ \sigma_y(r, \theta) = \sum_{n=0}^{\infty} (2r)^{-\frac{1}{2}} \left[ k_n^I f_{\theta}^I (n, \theta) + k_n^H f_{\theta}^H (n, \theta) \right] + T^I_0 f_{\theta}^{II} (0, \theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T^I_n f_{\theta}^{II} (n, \theta) + T^H_n f_{\theta}^{II} (n, \theta) \right], \quad i = r, \theta; \quad j = r, \theta, \]  

(A.1)

\[ 2\mu u_r(r, \theta) = \sum_{n=0}^{\infty} (2r)^{-\frac{1}{2}} \left[ k_n^I g_r^I (n, \theta) + k_n^H g_r^H (n, \theta) \right] + T^I_0 (2r) g_r^{II} (0, \theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T^I_n g_r^{II} (n, \theta) + T^H_n g_r^{II} (n, \theta) \right] + E \cos(\theta) + F \sin(\theta), \]  

(A.2)

\[ 2\mu u_\theta(r, \theta) = \sum_{n=0}^{\infty} (2r)^{-\frac{1}{2}} \left[ k_n^I g_\theta^I (n, \theta) + k_n^H g_\theta^H (n, \theta) \right] + T^I_0 (2r) g_\theta^{II} (0, \theta) + \sum_{n=1}^{\infty} (2r)^n \left[ T^I_n g_\theta^{II} (n, \theta) + T^H_n g_\theta^{II} (n, \theta) \right] + G2r \frac{\nu+1}{8} - E \sin(\theta) + F \cos(\theta). \]

The angular functions with a superscript of “I” are symmetric functions which correspond to mode I, while the superscript of “II” is for the antisymmetric case of mode II. Rigid body displacement and rotation in (A.2) are accounted for by the \( E, F \) and \( G \) constants. From the point of view of displacement, the rigid body rotation constant, \( G \), appears to play the role of \( T^H_n \) for \( n = 0 \). The mode I coefficients, \( k_n^I \) and \( T^I_n \), are defined by the normalizations,

\[ f_{\theta \theta}^I (n, \theta = 0) = 1 \text{ and } f_{\theta}^{II} (n, \theta = 0) = 1, \quad n = 0, ..., \infty, \quad (A.3) \]

which give
\[ f_{rr}^{\text{Ik}} = \frac{1}{4} \left\{ - (2n - 5) \cos \left[ \frac{(2n-1)\theta}{2} \right] + (2n-1) \cos \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ f_{\theta\theta}^{\text{Ik}} = \frac{1}{4} \left\{ - (n-2) \cos(n\theta) + (n+2) \cos[(n+2)\theta] \right\} \]
\[ f_{\theta\theta}^{\text{IIk}} = \frac{1}{4} \left\{ (2n + 3) \cos \left[ \frac{(2n-1)\theta}{2} \right] - (2n-1) \cos \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ f_{\theta\theta}^{\text{IIr}} = \frac{n+2}{4} \left\{ \cos(n\theta) - \cos[(n+2)\theta] \right\} \]
\[ f_{rr}^{\text{IIk}} = \frac{2n-1}{4} \left\{ \sin \left[ \frac{(2n-1)\theta}{2} \right] - \sin \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ f_{rr}^{\text{IIr}} = \frac{1}{4} \left\{ n \sin(n\theta) - (n+2) \sin[(n+2)\theta] \right\} \]
\[ g_{rr}^{\text{Ik}} = \frac{1}{4(2n+1)} \left\{ (2\kappa - 2n-1) \cos \left[ \frac{(2n-1)\theta}{2} \right] + (2n-1) \cos \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ g_{rr}^{\text{IIk}} = \frac{1}{8(n+1)} \left\{ (\kappa - n-1) \cos(n\theta) + (n+2) \cos[(n+2)\theta] \right\} \]
\[ g_{\theta\theta}^{\text{IIk}} = \frac{1}{4(2n+1)} \left\{ (2\kappa + 2n+1) \sin \left[ \frac{(2n-1)\theta}{2} \right] - (2n-1) \sin \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ g_{\theta\theta}^{\text{IIr}} = \frac{1}{8(n+1)} \left\{ (\kappa + n+1) \sin(n\theta) - (n+2) \sin[(n+2)\theta] \right\} \].

(A.4)

The normalizations used for the mode II coefficients are
\[ f_{\theta\theta}^{\text{IIr}}(n, \theta = 0) = 1, \ n = 0, \ldots, \infty \quad \text{and} \quad g_{\theta\theta}^{\text{IIr}}(n, \theta = \pm \pi) = \frac{\kappa + 1}{8(n+1)}(-1)^n, \ n = 1, \ldots, \infty, \]

(A.6)

which result in the mode II eigenfunctions:
\[ f_{rr}^{\text{II}} = \frac{1}{4} \left\{ - (2n - 5) \sin \left[ \frac{(2n-1)\theta}{2} \right] + (2n+3) \sin \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ f_{\theta\theta}^{\text{II}} = \frac{1}{4} \left\{ (n-2) \sin(n\theta) - n \sin[(n+2)\theta] \right\} \]
\[ f_{\theta\theta}^{\text{IIk}} = \frac{2n+3}{4} \left\{ \sin \left[ \frac{(2n-1)\theta}{2} \right] - \sin \left[ \frac{(2n+3)\theta}{2} \right] \right\} \]
\[ f_{\theta\theta}^{\text{IIr}} = \frac{1}{4} \left\{ - (n+2) \sin(n\theta) + n \sin[(n+2)\theta] \right\} \]
\[ f_{r0}^{\text{nk}} = \frac{1}{4} \left\{ -(2n - 1)\cos \left[ \frac{(2n - 1)\theta}{2} \right] + (2n + 3)\cos \left[ \frac{(2n + 3)\theta}{2} \right] \right\} \]
\[ f_{r0}^{\text{irr}} = \frac{n}{4} \left\{ \cos(n\theta) - \cos((n + 2)\theta) \right\}. \quad (A.7) \]
\[ g_{r}^{\text{nk}} = \frac{1}{4(2n + 1)} \left\{ (2\kappa - 2n - 1)\sin \left[ \frac{(2n - 1)\theta}{2} \right] + (2n + 3)\sin \left[ \frac{(2n + 3)\theta}{2} \right] \right\} \]
\[ g_{r}^{\text{irr}} = \frac{1}{8(n + 1)} \left\{ -(\kappa - n - 1)\sin(n\theta) - n \sin[(n + 2)\theta] \right\} \]
\[ g_{r}^{\text{nk}} = \frac{1}{4(2n + 1)} \left\{ -(2\kappa + 2n + 1)\cos \left[ \frac{(2n - 1)\theta}{2} \right] + (2n + 3)\cos \left[ \frac{(2n + 3)\theta}{2} \right] \right\} \]
\[ g_{r}^{\text{irr}} = \frac{1}{8(n + 1)} \left\{ (\kappa + n + 1)\cos(n\theta) - n \cos[(n + 2)\theta] \right\}. \quad (A.8) \]

**Stresses and displacements along the line of a stress-free crack**

Stress and displacement components in front of the crack tip along \( \theta = 0 \) for arbitrary loading are:

\[ \sigma_{n}(r,0) = \sum_{n=0}^{\infty} k_{n}^{I}(2r)^{\frac{n-1}{2}} + \sum_{n=0}^{\infty} T_{n}^{I}(2r)^{n} \quad (A.9) \]
\[ \sigma_{r\theta}(r,0) = \sum_{n=0}^{\infty} k_{n}^{I}(2r)^{-\frac{n}{2}} \quad (A.10) \]
\[ \sigma_{n\theta}(r,0) = \sum_{n=0}^{\infty} k_{n}^{II}(2r)^{-\frac{n}{2}} \quad (A.11) \]
\[ 2\mu \alpha_{r}(r,0) = \frac{\kappa - 1}{2} \sum_{n=0}^{\infty} \frac{k_{n}^{I}}{2n + 1}(2r)^{\frac{n+1}{2}} + \frac{\kappa + 1}{8} \sum_{n=0}^{\infty} \frac{T_{n}^{I}}{n+1}(2r)^{n+1} + E \quad (A.12) \]
\[ 2\mu \alpha_{r\theta}(r,0) = -\frac{\kappa - 1}{2} \sum_{n=0}^{\infty} \frac{k_{n}^{II}}{2n + 1}(2r)^{\frac{n+1}{2}} + \frac{\kappa + 1}{8} \left\{ G2r + \sum_{n=1}^{\infty} \frac{T_{n}^{II}}{n+1}(2r)^{n+1} \right\} + F \quad (A.13) \]

Similarly, the non-zero stresses and displacements along the crack flanks where \( \theta = \pm \pi \) are:

\[ \sigma_{n}(r,\pm \pi) = \sum_{n=0}^{\infty} (-1)^{n} T_{n}^{I}(2r)^{n} + 2 \sum_{n=0}^{\infty} (-1)^{n} k_{n}^{II}(2r)^{-\frac{n}{2}} \quad (A.14) \]
\[ 2\mu u_r(r, \pm \pi) = \frac{\kappa + 1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n T_n^I}{(2n+1)} + \frac{\kappa}{2} \sum_{n=0}^{\infty} \frac{(-1)^n k_n^I}{(2n+1)} + \left( -E \right), \quad (A.15) \]

\[ 2\mu u_\theta(r, \pm \pi) = \frac{\kappa + 1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n k_n^I}{(2n+1)} + \frac{\kappa + 1}{8} \left( G2r + \sum_{n=1}^{\infty} \frac{(-1)^n T_n^II}{n+1} \right) - F. \quad (A.16) \]

In all of the quantities in (A.9-16), even though the loading is mixed, there is only one arbitrary constant associated with a given power of \( r \). Therefore any of these quantities can be used to determine a set of constants which appear in the expression. For example, \( u_r \) and \( u_\theta \) along the crack flanks can be used to determine all of the coefficients, i.e., \( T_n^I \) and \( k_n^I \) from (A.15) and \( k_n^I \) and \( T_n^II \) from (A.16).

Other important quantities along the line of the crack are the crack opening displacement for mode I and the crack shift displacement for mode II. These are given by:

\[ \text{COD} = u_y^+ - u_y^- = -u_\theta^+ + u_\theta^- = \frac{\kappa + 1}{2\mu} \sum_{n=0}^{\infty} \frac{(-1)^n k_n^I}{2n+1} \frac{}{} \]

\[ \text{CSD} = u_x^+ - u_x^- = -u_r^+ + u_r^- = \frac{\kappa + 1}{2\mu} \sum_{n=0}^{\infty} \frac{(-1)^n k_n^I}{2n+1} \frac{}{} \]

(A.17)

(A.18)
List of constants used in the expressions for stresses around the crack tip in chapter 2:

\[ a_1 = (\mu_2 + \mu_1 \kappa_2)(\mu_2 \kappa_1 + \mu_1)(\kappa_1 + 1) \]  
(B.1)

\[ a_2 = \kappa_1 + 1 \]  
(B.2)

\[ a_3 = 3\mu_2 \kappa_2 \mu_1 + 3\mu_2^2 - 3\mu_1 \mu_2 - 2\mu_1^2 \kappa_2 - \mu_1 \mu_2 \kappa_1 + \mu_1 \mu_2 \kappa_1 \kappa_2 - \mu_2^2 \kappa_1^2 \]  
(B.3)

\[ a_4 = 2(\mu_1 - \mu_2)(\mu_2 + \mu_1 \kappa_2) \]  
(B.4)

\[ a_5 = 3\mu_2 \kappa_2 \mu_1 + 3\mu_2^2 - 3\mu_1 \mu_2 - 4\mu_1^2 \kappa_2 + \mu_1 \mu_2 \kappa_1 - \mu_1 \mu_2 \kappa_1 \kappa_2 + \mu_2^2 \kappa_1^2 \]  
(B.5)

\[ a_6 = -\mu_2 \kappa_2 \mu_1 - \mu_2^2 + \mu_2 \mu_1 + 2\mu_1^2 \kappa_2 - \mu_1 \mu_2 \kappa_1 + \mu_1 \mu_2 \kappa_1 \kappa_2 - \mu_2^2 \kappa_1^2 \]  
(B.6)

\[ a_7 = -\mu_2 \kappa_2 \mu_1 - \mu_2^2 + \mu_1 \mu_2 + \mu_1 \mu_2 \kappa_1 - \mu_1 \mu_2 \kappa_1 \kappa_2 + \mu_2^2 \kappa_1^2 \]  
(B.7)

\[ a_8 = -3\mu_1 \kappa_2 \mu_1 - 3\mu_1^2 \mu_2 + 3\mu_1 \mu_2 + 2\mu_1^2 \kappa_2 + \mu_1 \mu_2 \kappa_1 - \mu_1 \mu_2 \kappa_1 \kappa_2 + \mu_2^2 \kappa_1^2 \]  
(B.8)

\[ a_9 = -2\mu_2 \kappa_1 \mu_1 \kappa_2 - 2\mu_2^2 \kappa_1^2 - 2\mu_1 \mu_2 - 2\mu_1^2 \kappa_2 \]  
(B.9)

\[ a_{10} = 2(\mu_2 + \mu_1 \kappa_2)(\mu_1 + \mu_2 \kappa_1) \]  
(B.10)

\[ a_{11} = -3\mu_2 \kappa_2 \mu_1 - 3\mu_2^2 + 3\mu_1 \mu_2 + 4\mu_1^2 \kappa_2 - \mu_1 \mu_2 \kappa_1 + \mu_1 \mu_2 \kappa_1 \kappa_2 - \mu_2^2 \kappa_1^2 \]  
(B.11)

\[ a_{12} = -\mu_2 \kappa_2 \mu_1 - \mu_2^2 + \mu_1 \mu_2 + 2\mu_1^2 \kappa_2 - \mu_1 \mu_2 \kappa_1 + \mu_1 \mu_2 \kappa_1 \kappa_2 - \mu_2^2 \kappa_1^2 \]  
(B.12)

\[ a_{13} = -\mu_1 (\kappa_1 + 1)(-\mu_2 \kappa_1 - \mu_1 + \mu_1 \kappa_2) \]  
(B.13)

\[ a_{14} = (\mu_2 + \mu_1 \kappa_2)(\mu_1 + \mu_2 \kappa_1) \]  
(B.14)

\[ a_{15} = 3\mu_1 + 3\mu_2 \kappa_1 - \mu_2 - \mu_1 \kappa_2 \]  
(B.15)
\[ a_{16} = 2\mu_1 + 2\mu_2\kappa_1 - 2\mu_2 - 2\mu_1\kappa_2 \]  \hspace{1cm} (B.16)

\[ a_{17} = 2\mu_1 + 2\mu_2\kappa_1 \]  \hspace{1cm} (B.17)

\[ a_{18} = 3\mu_1 + 3\mu_2\kappa_1 + \mu_2 + \mu_1\kappa_2 \]  \hspace{1cm} (B.18)

\[ a_{19} = \mu_1 + \mu_2\kappa_1 + \mu_2 + \mu_1\kappa_2 \]  \hspace{1cm} (B.19)

\[ a_{20} = \mu_1 + \mu_2\kappa_1 - \mu_2 - \mu_1\kappa_2 \]  \hspace{1cm} (B.20)
REFERENCES


