8-2010

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Electromechanical Modeling and Analysis of a Self-excited Micro-power Generator

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Masters of Science
Mechanical Engineering

by
Amin Bibo
July 2010

Accepted by:
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Amin Bibo

(ABSTRACT)

Micro-power generators (MPGs) are compact, scalable, and low-maintenance energy harvesting devices that capture and transform wasted ambient energy into electricity. Such devices, which are currently being researched as a possible replacement for batteries, can act as a power source to maintain and allow autonomous operations of remote low-power consumption sensors. This thesis introduces a novel MPG which transforms wind energy into electricity via wind-induced self-excited oscillations of piezoelectric cantilever beams. The operation concept of the device is simple: similar to music-playing harmonica that create tones via oscillations of reeds when subjected to air blow, the proposed device uses flow-induced self-excited oscillations of a piezoelectric beam embedded within a cavity to generate electric power. When the volumetric flow rate of air past the beam exceeds a certain threshold, the energy pumped into the structure via nonlinear pressure forces offsets the intrinsic damping in the system setting the beam into self-sustained limit-cycle oscillations as a result of a Hopf bifurcation. The vibratory energy is then converted into electricity through principles of piezoelectricity.

The objectives of this thesis are two folds: The first investigates the development of an analytical aero-electromechanical model to describe the response behavior of the device, and the second deals with understanding the influence of the design parameters on its cut-on wind speed and the generated power.

To achieve the first objective, we obtain a mathematical model describing the dynamic evolution of the four essential system’s parameters. These are the spatial and temporal dynamics of the beam deflection, the temporal dynamics of the voltage
developed across the electric load, the temporal evolution of the exciting pressure on the surface of the beam, and the flow rate through the aperture between the beam and the support. The modeling is carried out at three successive levels. First, we employ Hamilton’s principle in combination with the nonlinear Euler-Bernoulli’s beam theory and the linear constitutive equations of piezoelectricity to obtain the nonlinear partial differential equation relating the flexural dynamics of the beam to the output voltage and the exciting pressure. Second, we use basic electric circuits theories to obtain the nonlinear ordinary differential equation relating the output voltage of the harvester to the strain rate in the piezoelectric layer. Third, assuming that the flow rate through the aperture is irrotational, two dimensional, and steady; we utilize the steady Bernoulli’s equation in conjunction with the continuity equation to relate the exciting pressure on the surface of the beam to the in- and outflow rates of air.

Subsequently, we use a Galerkin expansion to discretize the partial differential equation into a set of nonlinearly-coupled ordinary differential equations. We carry a convergence analysis and determine that a single-mode reduced-order model can predict the static, linear, and nonlinear dynamic responses of the device. Additionally, we study the influence of neglecting the beam’s geometric and inertia nonlinearities on the response behavior showing that such nonlinearities can be safely ignored within the operation range of the device. We validate the resulting reduced-order model against experimental data demonstrating good agreement for two different configurations.

To achieve the second objective, we utilize the resulting analytical model to understand the influence of the design parameters (e.g., beam’s thickness, length, chamber’s volume, aperture’s width, and electric load) on the device’s response with the goal of minimizing the cut-on wind speed and maximizing the output power of the MPG. Results indicate that for a beam of a given thickness and length, there exists an optimal volume that minimizes the cut-on wind speed of the device. This
optimal volume is inversely proportional to the beam’s first modal frequency. Results also indicate that the cut-on wind speed can be decreased significantly as the aperture’s width is decreased. However, it is observed that minimizing the cut-on wind speed does not always correspond to an increase in the output power. As such, we use the resulting model to construct design charts that aid in designing a MPG with optimal design parameters for a given known average wind speed. Finally, in an attempt to increase the output power of the device, we explore the prospect of designing the harvester such that the Hopf bifurcation responsible for the onset of the beam’s oscillation is sub-critical. Towards that end, we utilize the method of multiple scales to obtain the bifurcation’s normal form, then use it to demonstrate that the resulting bifurcation will always be super-critical.
Dedication

To my parents, for all the love, guidance, and support you have given me, I am forever grateful.
Acknowledgments

First and foremost, I thank God for all the blessings he bestowed on me during my life, and for granting me this great opportunity along with the patience and enthusiasm necessary to complete it.

Secondly, I especially thank my Thesis advisor, Dr. Mohammed F. Daqaq, for his invaluable support and advice throughout my time in Clemson. I am very blessed to have him as a friend and a mentor. Words cannot describe how indebted I am for the priceless amount of knowledge I have gained from him. Without his thorough guidance, the completion of this work would not have been possible.

I would also like to thank and acknowledge my committee members, Drs. Ardalan Vahidi and Gang Li, for devoting the time to serve on my committee and for their invaluable guidance and patience throughout the process. I especially appreciate their tremendous dedication in teaching the courses I have taken with them. I also thank Dr. Li for the brilliant idea upon which the work of this thesis is based.

Without doubt, my deepest gratitude goes to my family, without whom, I would not have been able to reach this point: My father, Mr. Saleh Bibo, and my mother Mrs. Iotedal Bibo, for their endless love, kindness, support, encouragement, and continuous care. My brothers and sisters, Amani, Eman, Amneh, Ala, Arwa, Ayman, and Ashraf for their endless love and encouragement.

I would also like to express my appreciation to my research group with whom I have
studied and worked: Miss Tugba Demir, Mr. Thiago Osorio, Mr. Chris Stabler, Mr. Yousef Qaroush, Mr. Abdrouf Abuson, Mr. Clodoaldo DaSilva, Mr. Ravendra Masana, Mr. Shyam Penyam, and Mr. Keyur Shah. Especial thanks are also due to Mr. Venkata Sennakesavababu and Mr. Daniel St. Clair for generating the experimental data.

Special thanks are due to the following persons: Dr. Mohammed Al-Nimr for being more than a brother and for his continuous encouragement and belief in me; Mr. Yousef Qaroush for his great friendship and endless help; Dr. Jawad Meziane, Mr. Mahmoud Abdel-Hamid and Mr. Ali Al-Almar for their companionship; and Mr. Hassan Odeh for being an exciting and fun friend.
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Chapter 1

Introduction

1.1 Motivations

Today, many critical electronics, such as health-monitoring sensors [1, 2], pace makers [3], spinal stimulators [4], electric pain relievers [5], wireless sensors [6, 7, 8], micro-electromechanical systems [9, 10], etc., require minimal amounts of power to function. A wireless transponder for data transmission can operate efficiently with less than 1 mW of power [11, 12]. A sensor interface chip for health monitoring that consists of a sensor and a microcontroller has an average power consumption of 48 µW [13, 14]. Such devices have, for long time, relied on batteries that have not kept pace with the devices’ demands, especially in terms of energy density [15]. In addition, batteries have a finite life span, adverse environmental impacts, and require regular replacement or recharging, which, in many of the previously mentioned examples, is a very cumbersome and expensive process. One area that is currently suffering from battery technology’s shortcomings is active implantable medical devices [16]. The long-anticipated artificial pancreas to treat diabetes operates on batteries that must be replaced every nine months posing a significant risk of infection that can claim lives, thusly rendering this life-saving technology inefficient.
Other devices, like cochlear ear implants are too small to contain batteries [16].

In light of such challenges, scavenging otherwise wasted energy from the environment can provide a solution to lower our dependance on batteries and advance many life-saving technologies. While the process of harnessing energy, also known as energy harvesting, is not new and has been historically practised by humans in the form of windmills, sailing ships, and waterwheels; today, and due to many recent and critical advances in manufacturing electronics that made low-power consumption devices a reality; researchers are taking this same old approach into new domains where the goal is to design compact and scalable generators that can harvest minute amounts of energy to run and maintain low-power consumption electronics [17, 18, 19].

1.2 Current Approaches for Micro-power Generation

To power, maintain, and allow autonomous operations of low-power consumption devices, the concept of micro-power generators (MPGs) was introduced [20, 21, 22, 23, 24]. Micro-power generators are essentially compact and scalable energy harvesting devices that can transform wasted ambient energy, e.g., thermal, solar, wind, and vibrations into electricity. In the last couple of years, wind and vibrations have attracted specific attention due to their abundance. Today, large wind turbines that are highly efficient span different areas in the United States and many countries around the world. Unfortunately, traditional wind turbine designs that are based on rotary-type generation concepts suffer from two critical problems. First, they have scalability issues because their performance drops significantly as their size decreases. Mitcheson et al. [25] reported that the power coefficient can drop from 0.59 which corresponds to the Betz limit to less than 0.1 as the size of the turbine gets smaller. This is a result of i) relatively high viscous drag on the blades at low Reynolds
numbers [26], ii) bearing and thermal losses which increase significantly as size decreases, and iii) high electromagnetic interferences. In addition to performance issues, the design and fabrication of traditional small-scale rotary-type generators that require a rotor, a stator, magnets, wirings, and blades is a very complex and expensive process. This makes their actual implementation for compact applications such as those mentioned previously an astounding task.

Over the last decade, vibratory energy harvesting has also flourished as a major thrust area of micro power generation. Various devices have been developed to transform mechanical motions directly into electricity by exploiting the ability of active materials and some mechanisms to generate an electric potential in response to mechanical stimuli and external vibrations [17, 18, 19]. However, this energy harvesting concept has a critical shortcoming in its operation concept. Vibratory energy harvesters operate efficiently only within a narrow frequency bandwidth where the excitation frequency is very close to the fundamental frequency of the harvester (Resonance Condition). Small variations in the excitation frequency around the harvester’s fundamental frequency drop its small energy output even further making the energy harvesting process inefficient [27, 28, 29, 30, 31, 32]. This becomes an even more pressing issue when one realizes that most environmental excitations have a broad-band or time-dependent characteristics in which the energy is distributed over a wide spectrum of frequencies or the dominant frequencies drift with time. As such, many viable excitation sources such as structural and machine vibrations, ocean waves, acoustic excitations, running, walking, among other motions are considered impractical due to their inherent randomness or non-stationarities.
1.3 Proposed Approach

Motivated by the obvious need for a compact, scalable, cheap, and low-maintenance micro-power generator, this work introduces a new concept for an energy harvester which uses wind energy to maintain remote low-power consumption sensors. Inspired by music playing harmonica, the harvester shown in Fig. 1.1 consists of a piezoelectric cantilever uni-morph structure embedded within a cavity to mimic the vibrations of the reeds in a harmonica when subjected to air blow. The operation principle of the harvester is simple. Wind blows into the chamber and tries to escape through the small aperture between the cantilever (reed) and the supporting structure. The sudden change in area causes the flow to separate from the cantilever at the sharp edge which causes the velocity to increase rapidly. This, in turn, produces a pressure drop across the cantilever. The resulting pressure drop bends the cantilever which causes the aperture area to increase. Consequently, the flow velocity drops and the pressure drop decreases. The mechanical restoring force pulls the beam back decreasing the aperture area and the process is repeated. These periodic fluctuations in the pressure cause the beam to undergo self-sustained oscillations. The resulting periodic strain in the piezoelectric layer produces an electric field which can be channeled as a current to an electric device.

The significance of this novel concept for micro power generation stems from its ability to eliminate the shortcomings of traditional vibratory energy harvesters and rotary type generators while, at the same time, combining aerodynamics with vibrations (flow-induced vibrations) to generate power. On one hand, this concept is based on transforming vibrations to electricity but does not require an external vibration source eliminating the bandwidth issues associated with resonant vibratory energy harvesters [27, 28, 29, 30, 31, 32]. On the other hand, while this device depends on the presence of an aerodynamic energy field, it does not suffer from the scalability issues that hinder the efficiency of small size wind turbines [25]. This
Figure 1.1: Schematic for a simplified model representing the operation concept of the harvester.

The idea of utilizing flow-induced oscillations for energy extraction is new and has been recently explored by various researchers. In one demonstration, Allen and Smits [33], investigated the feasibility of placing piezoelectric membrane in the wake of a bluff body and using the induced oscillations due to the formed vortex street behind the body to provide a power source. In another demonstration, Tang et al. [34] developed a flutter-mill which consists of a two-dimensional cantilevered flexible plate mounted in axial flow. In their work, they investigated the flow-induced vibrations due to fluid-structural interactions and the key design parameters that influence the power-extraction capacity. The performance of a flow-energy harvester based on oscillating foils was also investigated by Zhu et al. [35].

Inspired by fish ability to extract energy from unsteady flows and vortical structures, Liao et al. and Simpson et al. [36, 37] theoretically and experimentally investigated energy extraction from the sinusoidal heave and pitch motion of flapping foils. Rob-
bins [38] also investigated the feasibility of harvesting energy from a flapping flag-like membranes composed of flexible piezoelectric materials, while Barrero-Gil et al. [39] studied the use of transverse galloping of different structures for energy harvesting and the influence of geometric and material properties on the energy conversion factor.

1.4 Operation Concept

The operation concept of this device and all other flow-induced energy harvesting concepts is based on a nonlinear phenomenon knows as self-excited or self-sustained oscillations. This phenomenon can be best explained by studying the dynamics of the long-celebrated Van der Pol oscillator whose equation of motion can be written as [40]

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu \geq 0. \quad (1.1)$$

Equation (1.1) is a simple harmonic oscillator but with a linear negative damping term $-\mu \dot{x}$ and a nonlinear positive damping term $\mu x^2 \dot{x}$. Note that negative damping pumps energy into the system while positive damping pumps energy out of the system. As such, for small oscillations, $|x| < 1$, the nonlinear positive damping is very small and the effective damping of the system is negative causing small amplitude oscillations to grow. However, as $|x| > 1$, the nonlinear damping becomes large and the effective damping becomes positive causing large amplitudes to decay. At one point, the energy dissipated over one cycle balances the energy pumped and the system settles into self-sustained fixed-amplitude oscillations that are called limit cycles.

In the case of this MPG concept, self-excited oscillations occur when the volumetric flow rate past the cantilever is large enough such that the energy pumped into the structure via nonlinear pressure forces offsets the intrinsic linear damping in the
system which consists of the structural damping and electric damping due to electric energy generation. One can think of this process as a nonlinear feedback mechanism in which the motion of the cantilever produces a disturbance in the potential flow that feeds enough energy back to the structure to overcome the internal damping [41, 42, 43].

![Figure 1.2: Two scenarios for the voltage response of the MPG as the flow rate increases.](image)

The onset of the limit-cycle oscillations necessary for energy harvesting (cut-on wind speed) is governed by a threshold combination of the flow and design parameters known as the *Hopf bifurcation* (HB) point. Below that point, the energy pumping mechanism cannot overcome the damping mechanism and the structure settles at a static equilibrium and hence no power can be harvested as shown in Fig. 1.2(a). Beyond that threshold, the nonlinear pressure forces overcome the intrinsic damping in the system and the beam undergoes limit-cycle oscillations.

From a mathematical perspective, this bifurcation threshold represents a point at which two or more complex-conjugate eigenvalues associated with the Jacobian of the system dynamics transversally cross the imaginary axis from the left- to the right-half of the complex plane. With this understanding, it becomes evident that the ability of this device to generate energy depends on the onset of the bifurcation which has a *complex* and, as of today, *unknown* dependence on the design parameters.
and flow characteristics.

Not only does the combination of the design parameters determine the cut-on wind speed but they also determine the nature of the response beyond it (bifurcation nature). As shown in Fig. 1.2(a), when the bifurcation is supercritical and the flow rate exceeds the threshold value, small-amplitude limit-cycle oscillations about the former static position are born. On the other hand, when the bifurcation is subcritical as shown in Fig. 1.2(b), the output voltage jumps to a distant attractor which can be another fixed point, a large-amplitude limit cycle, or even a chaotic attractor. In most engineering applications, subcritical bifurcations are considered dangerous because they can cause structural failure. In our study however, a subcritical Hopf bifurcation means large amplitude oscillations at lower wind speeds which implies an enhanced performance of the MPG.

It is desired then, through this thesis, to understand how the flow characteristics and design parameters influence the onset of the bifurcation, its nature, as well as the amplitude of the resulting limit cycles and hence the output power of the device. Once this understanding is established, the design parameters can be altered to maximize the output power for a given wind speed.

1.5 Thesis Objectives and Organization

In two previous studies [44, 45], we introduced the basic physics of this new generator and proved its feasibility. However, the design parameters used in the previous experiments were all chosen arbitrarily and are far from being the optimal parameters necessary to maximize the performance and minimize the cut-on wind speed. The reason for this arbitrary choice of parameters is the lack of a mathematical model that describes the dependance of the output power on these parameters. For instance, the dimensions, shape, and material properties of the beam; the volume
and cross-sectional area of the chamber; the gap width and electric load, among many other design parameters, play an interconnected and unknown role to determine the cut-on wind speed and output power of this MPG. To resolve this unknown dependence, the objectives of this thesis are

- Combining basic theories in continuous-systems vibrations, piezoelectricity, and fluid dynamics, to obtain an analytical model of the harvest system. The model invokes several assumptions on the fluid-structural interactions to obtain a set of nonlinear and coupled equations that govern the qualitative behavior of the MPG. Towards that end, in Chapter 2, we adopt the nonlinear Euler-Bernoulli beam’s theory to express the beam’s strain-deflection relationship and the linear constitutive relations to construct the strain-stress equations for both of the structural and piezoelectric elements. Subsequently, we use Hamilton’s principle in conjunction with Kirchhoff’s laws and electric circuits theory, to derive the coupled partial differential equation that captures the dynamics of the beam and the ordinary differential equation governing the dynamics of the harvesting circuit. Using the steady Bernoulli equation and the continuity equation, we develop the relationship between the exciting pressure at the surface of the beam, the flow of air past the aperture, and the inflow rate. Employing a Galerkin’s expansion, we reduce the order of the model by discretizing the resulting partial differential equation into a set of nonlinear ordinary differential equations. We carry a convergence analysis to determine the minimum number of modes to be kept in the reduced-order model, and perform dimensional analysis to identify the important parameters that affect the system’s response. Finally, we provide a brief description of the experimental setup and validate the theoretical model for two parametric case studies.

- Implementing a systematic analysis to understand the role of the design param-
eters in the transduction of this MPG concept. Since, the basic phenomenon responsible for beam oscillations is nonlinear, we will carry a nonlinear analysis to describe how the design parameters affect the response characteristics and the cut-on wind speed of the generator. To that end, in Chapter 3, we use the Routh-Hurwitz criterion to describe the conditions under which the Hopf bifurcation occurs. We then study the effect of the chamber volume on the onset of limit-cycle oscillations for different beam lengths and thicknesses. We also investigate the effect of the gap width on the cut-on wind speed. Subsequently, we use the method of multiple scales to determine the normal form of the bifurcation (sub- or super-critical) and investigate the effect of beam’s length, thickness, chamber volume, and load resistance on the output power. Using the resulting understanding, we develop design charts to assist in choosing the harvester’s optimal design parameters for known average inflow wind speeds.

- Presenting critical conclusions with regards to the modeling and optimal design of the MPG as well as providing directions for future work. These conclusions and future recommendations are given in Chapter 4.
Chapter 2

Modeling of the Wind Energy Harvester

In this chapter, we present and validate a nonlinear aero-electro-mechanical model that describes the response of the MPG. We adopt the nonlinear Euler-Bernoulli beam’s theory to express the beam’s strain-deflection relationship and the linear constitutive relations to construct the strain-stress equations for both of the structural and piezoelectric elements. Subsequently, we use Hamilton’s principle in conjunction with Kirchhoff’s laws to derive the coupled partial differential equation that captures the dynamics of the beam and the ordinary differential equation governing the dynamics of the harvesting circuit. Furthermore, using the steady Euler-Bernoulli equation and the continuity equation, we develop the relationship between the exciting pressure at the surface of the beam, the flow of air past the aperture, and the inflow rate. Employing a Galerkin’s expansion, we reduce the order of the model by discretizing the resulting partial differential equation into a set of nonlinear-ordinary differential equations. We carry a convergence analysis to determine the minimum number of modes to be kept in the reduced-order model. Finally, we present two experimental studies to validate the theoretical model.
To obtain a nonlinear mathematical model that governs the response of the system shown in Fig. 2.1, we assume that air of a flow rate, $U_0$, blows into one side of a large reservoir. As the flow enters the reservoir, its speed drops causing a pressure $P_A(t)$ to build on the top side of the cantilever which forces the cantilever to deflect by $w(s,t)$ and elongate by $u(s,t)$. Air escapes with a flow rate $U(t)$ through the aperture between the cantilever and the support. Variations in the pressure produce a time-varying strain in the piezoelectric layer which produces a voltage $V(t)$ across an electric load, $R$.

Five equations are necessary to describe the evolution of the system dynamics which is governed by five parameter, namely, the exciting pressure, $P_A(t)$, the beam’s deflection and elongation, $w(s,t)$ and $u(s,t)$, the flow rate through the aperture, $U(t)$, and the voltage developed across the load, $V(t)$; with the last being the critical parameter necessary to calculate the output power of the harvester. To obtain these equations, we divide the problem into two parts. The first describes the electromechanical response while the second describes the flow characteristics past the cantilever.
2.1 The Electromechanical Model

2.1.1 Strain-displacement Relationship

For a slender beam unimorph similar to the one considered here, shear deformations and rotary inertia can be neglected allowing for the adoption of the nonlinear Euler-Bernoulli’s beam theory to model the beam’s response. According to Euler’s theory, the two dimensional dynamics of the beam can be described using a longitudinal displacement $u(s,t)$ and a transversal displacement $w(s,t)$, Fig. 2.2(b), where $s$ denotes the arclength and $t$ denotes time. To describe a beam element before and
after deformation, two cartesian coordinate systems are utilized: the \((x, y, z)\) is considered to be global, while the \((\bar{x}, \bar{y}, \bar{z})\) is a local system, and they are related through a transformation matrix corresponding to the rotation around the \(\bar{y}\)-axis. Using Fig. 2.2(b), it follows that the longitudinal elongation of the beam element can be written as

\[
e = \sqrt{(ds + du)^2 + dw^2} - ds. \tag{2.1}
\]

Dividing Equation (2.1) by the element length, \(ds\), the strain along the neutral axis of the differential element becomes

\[
\epsilon_0 = \sqrt{(1 + u')^2 + w'^2} - 1, \tag{2.2}
\]

where the over prime denotes a derivative with respect to the arclength, \(s\). Using a quadratic Taylor expansion of Equation (2.2), we obtain

\[
\epsilon_0 = u' + \frac{w'^2}{2}. \tag{2.3}
\]

Due to rotation of a differential beam element, the strain at a point having the coordinates \((\bar{x}, \bar{y}, \bar{z})\) relative to the neutral axis can be written as function of the beam’s curvature using

\[
\epsilon = -\bar{z} \frac{d\psi}{ds}. \tag{2.4}
\]

By referring to Fig. 2.2(b), the rotation angle, \(\psi(s, t)\), can be described as

\[
\psi(s, t) = \tan^{-1} \left[ \frac{w'(s, t)}{1 + w'(s, t)} \right]. \tag{2.5}
\]

Substituting Equation (2.5) back into Equation (2.4), then expanding the outcome in a Taylor expansion up to cubic terms, yields

\[
\epsilon = -\bar{z} \left[ w'' - w''u' - w'u'' - w''w'^2 \right]. \tag{2.6}
\]

Adding Equations (2.3) and (2.6), the total axial strain can be written as

\[
\epsilon_x = u' + \frac{w'^2}{2} - \bar{z} \left[ w'' - w''u' - w'u'' - w''w'^2 \right]. \tag{2.7}
\]
### 2.1.2 Stress-strain Relationship

The stress-strain relationships of the beam and the piezoelectric layer are assumed to follow the linear constitutive equations:

\[
\sigma^b_x = Y^b_x \epsilon^b_x, \quad (2.8)
\]
\[
\sigma^p_x = Y^p \left[ \epsilon^p_x - d_{31} E_3 \right], \quad (2.9)
\]

where \( \sigma_x \) and \( \epsilon_x \) are the stress and the strain in the axial direction, respectively; \( Y \) is Young’s modulus, \( d_{31} \) is the piezoelectric constant, and \( E_3 \) is the electric field developed in the piezoelectric layer. Here, the superscript \( p \) and \( b \) stand for the piezoelectric and substructure layers, respectively.

![Schematic of the harvesting circuit.](image)

Assuming that the charge has a homogeneous distribution along the piezoelectric layer, the electric field can be related to the voltage developed across the load, \( V(t) \), and the layer thickness, \( t_p \), using \( E_3 = -V(t)/t_p \). Furthermore, using Ohm’s law, the voltage can be further related to the current using \( V(t) = R \dot{Q}_R(t) \), where \( R \) is the load resistance, and \( \dot{Q}_R \) is the current passing through the load, see Fig. 2.3. Here, the over-dot indicates a derivative with respect to time. Substituting the aforementioned relations back into Equation (2.9), yields

\[
\sigma^p_x = Y^p \left[ \epsilon^p_x + \frac{d_{31}}{t_p} R \dot{Q}_R(t) \right]. \quad (2.10)
\]
2.1.3 Equations of Motion and Boundary Conditions

To obtain the equations of motion, we use Hamilton's variational principle which states that

$$\int_{t_1}^{t_2} \delta L + \delta W_{\text{ext}} dt = 0,$$

(2.11)

where $t_1$ to $t_2$ is any arbitrary time interval, $\delta$ is the virtual operator, $L = T - U$ is the Lagrangian, and $W_{\text{ext}}$ is a non-conservative work term. The kinetic energy, $T$, of the system can be expressed as

$$T = \frac{1}{2} \int_0^L M(s) (\dot{u}^2 + \dot{w}^2) ds,$$

(2.12)

where $M(s)$ is the mass per unit length of the beam given by

$$M(s) = W_b \rho_b t_b + W_p \rho_p t_p [H(s) - H(s - L_p)].$$

(2.13)

Here, $\rho$ is the mass density of the layer, $t$ and $W$ are the associated thickness and width of the layer, and $H(s)$ is the Heaviside function.

The total potential energy of the system, $U$, consists of the strain energy of the composite beam and the electric potential stored in the capacitive piezoelectric layer and can be written as

$$U = \frac{1}{2} \int_{\mathcal{V}} \left( \sigma_x^b \epsilon_x^b + \sigma_x^p \epsilon_x^p \right) d\mathcal{V} - \frac{1}{2} \int_{\mathcal{V}} E_3 D_3 d\mathcal{V},$$

(2.14)

where $\mathcal{V}$ is the domain and $D_3$ is the electric displacement given by the following linear piezoelectric constitutive relation:

$$D_3 = d_{31} Y_p e_x^p - e_{33} E_3,$$

(2.15)

where $e_{33}$ is the permittivity at constant strain.

Replacing the electric field, $E_3$, in Equation (2.15) again by $-R \dot{Q}_R/t_p$, then substituting Equations (2.7), (2.8), (2.10), and (2.15) back into Equation (2.14), and
carrying the integration over the thickness of each layer, we obtain

\[ U = \frac{1}{2} \int_0^L \left[ YA(s) \left( u'^2 + u'^2 + \frac{1}{4} w'^4 \right) + YI(s) \left( w''^2 - 2w''^2 u' - 2w' w'' u'' \right) \right. \]

\[ + 2\theta(s) \left( w'' - w'' u' - w' u'' - w'' w'^2 \right) R\dot{Q}_R(t) ds - \frac{1}{2} C_p (R\dot{Q}_R(t))^2 \],

(2.16)

where \( YA(s), YI(s), \theta(s), \text{ and } C_p \) are the axial stiffness, the bending stiffness, the electromechanical coupling, and the piezoelectric capacitance, respectively, given by

\[ YA(s) = W_p t_p Y_p \left[ H(s) - H(s - L_p) \right] + W_b t_b Y_b, \]

\[ YI(s) = \frac{1}{3} \left( W_b Y_b \left( h_b^3 - h_a^3 \right) + W_p Y_p \left( h_c^3 - h_b^3 \right) \right) \left[ H(s) - H(s - L_p) \right] \]

\[ + \frac{W_b Y_b t_b^3}{12} \left[ H(s - L_p) - H(s - L) \right], \]

\[ \theta(s) = -\frac{W_p Y_p d_{31}}{2t_p} \left( h_c^2 - h_b^2 \right) \left[ H(s) - H(s - L_p) \right], \]

\[ C_p = e_{33} W_p L_p. \]

Here, \((h_a, h_b), (h_b, h_c)\) are the thickness boundaries measured from the neutral axis of the substructure and the piezoelectric layer, respectively, as depicted in Fig. 2.2(a).

The neutral axis location is determined relative to the bottom surface of the composite beam by recalling that stresses through the cross section must be in equilibrium. Summing forces in the x-direction gives,

\[ \sum F_x = \int_{A_b} \sigma_x^b dA + \int_{A_p} \sigma_x^p dA = 0. \]

(2.17)

Relating the bending stress to the radius of curvature, \( \rho \), as \( \sigma_x = -\frac{Y z}{\rho} \) and knowing that the curvature at all locations of a given cross section is the same, equation (2.17) becomes

\[ Y^b W_b \int_{h_a}^{t_b + h_a} z dz + Y^p W_p \int_{t_b + h_a}^{t_b + h_a + t_p} z dz = 0. \]

(2.18)
Carrying out the integrals in equation (2.18) and solving for \( h_a \) yields the location of the neutral axis measured from the bottom of the substructure layer as

\[
h_a = -\frac{1}{2} \frac{Y_b W_b^2 t_b^2 + 2Y^p W_p^2 t_p t_b + Y^p W_p^2 t_p^2}{Y^b W_b t_b + Y^p W_p t_p}.
\]  

(2.19)

Next, we assume that the beam is inextensible, i.e., the elongation \( \varepsilon ds \) along the neutral axis is assumed to be zero. This condition is valid when the beam has a zero geometric boundary condition at one end only; similar to the cantilever beam considered here. With that, Equation (2.2) yields

\[
(1 + u')^2 + w'^2 = 1.
\]  

(2.20)

Integrating Equation (2.20) twice with respect to the arclength \( s \), taking into account the boundary conditions at the clamped end, we obtain

\[
u = -\frac{1}{2} \int_0^s w'^2 ds.
\]  

(2.21)

Using Equations (2.12), (2.16), and (2.20), the Lagrangian of the system can be written as

\[
\mathcal{L} = \frac{1}{2} \int_0^L \left\{ M(s) \left( \dot{u}^2 + \dot{w}^2 \right) - YA(s) \left( u'^2 + u'w'^2 + \frac{1}{4}w'^4 \right) - YI(s) \left( w''^2 - 2w''w'^2 - 2w'^2u' - 2w'w''u'' \right) - 2\theta(s) \left( w'' - w''u' - w'u'' - w''w'^2 \right) R\dot{Q}_R(t) + \lambda(s, t) \left[ 1 - (1 + u')^2 - w'^2 \right] \right\} ds + \frac{1}{2} C_p(R\dot{Q}_R(t))^2,
\]  

(2.22)

where \( \lambda(s, t) \) is a Lagrange multiplier introduced to account for the inextensibility constraint. The influence of the non-conservative forces can be captured by introducing the general virtual-work term

\[
\delta W^{ext} = \int_0^L \left( F^*_{u} \delta u + F^*_{w} \delta w + F^*_{\lambda} \delta \lambda \right) ds + F^*_{Q_R} \delta Q_R,
\]  

(2.23)
where

\[ F_u^* = 0, \quad F_w^* = [WP_A(t) - c_a\dot{w}], \quad F_\lambda^* = 0, \quad F_{QR}^* = -R\dot{Q}_R(t). \]

The term associated with \( F_w^* \) represents the work done by the pressure forces on the beam surface and mechanical viscous damping. Here, the gauge pressure, \( P_A(t) \), is assumed to be uniform over the beam area and \( c_a \) denotes a viscous damping coefficient. The term associated with \( F_{QR}^* \) is used to account for the electric damping.

Substituting Equations (2.22) and (2.23) back into Equation (2.11) yields the equations of motion and boundary conditions in the following general form [46]:

\[
\frac{\partial \ell}{\partial q_i} - \frac{\partial}{\partial s} \left( \frac{\partial \ell}{\partial q_i'} \right) + \frac{\partial^2}{\partial s^2} \left( \frac{\partial \ell}{\partial q_i''} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \ell}{\partial \dot{q}_i} \right) = -F_i^*, \quad (2.24)
\]

\[
B1_i = \left\{ \left[ \frac{\partial \ell}{\partial q_i'} - \frac{\partial}{\partial s} \left( \frac{\partial \ell}{\partial q_i''} \right) \right] \delta q_i \right\}_{s=0}^{s=L} = 0,
\]

\[
B2_i = \left\{ \left[ \frac{\partial \ell}{\partial q_i''} \right] \delta q_i \right\}_{s=0}^{s=L} = 0, \quad (2.25)
\]

where \( \mathcal{L} = \int_0^L \ell(s,t)ds \) and \( q_i \equiv (u(s,t), w(s,t), Q_R(t)) \). When \( q_i \equiv u(s,t) \), Equations (2.24) and (2.25) yield

\[
- [YI(s)w''']' - \frac{1}{2}R\dot{Q}_R [2\theta(s)w'']' + \left[ \lambda(1 - \frac{1}{2}w'^2) \right]' + [YI(s)w''' + \theta(s)R\dot{Q}_R w]'''
+ M(s) \left[ \frac{1}{2} \int_0^s (\ddot{w})^2 ds \right] = 0,
\]

\[
(2.26)
\]

and the boundary conditions

\[
u = 0 \quad \text{at} \quad s = 0, \quad (2.27)
\]

\[
YI(s)w''' + \theta(s)w''R\dot{Q}_R - \lambda(1 - \frac{1}{2}w'^2) - [YI(s)w''' + \theta(s)R\dot{Q}_R w]' = 0
\]

at \( s = L \),

\[
(2.28)
\]
Using Equation (2.26) and (2.28), we can solve for $\lambda(s, t)$ and obtain to second order

$$
\lambda(s, t) = YI(s)w''^2 - \left[ YI(s)w'w'' + \theta(s)RQ_Rw' \right]' + \theta(s)w''RQ_R
$$

$$
- \int_{L}^{s} M(s) \left[ \frac{1}{2} \int_{0}^{s} (\dot{w})^2 ds \right] ds.
$$

(2.29)

When $q_i \equiv w(s, t)$, Equations (2.21), (2.24), (2.25), and (2.29) yield the following equation of motion governing the transversal vibrations of the beam

$$
M(s)\ddot{w} + \left[ YI(s)w'' + w'(YI(s)w') \right]' + \left[ \frac{1}{2} \int_{L}^{s} \frac{1}{L} \left( \dot{\theta}(s) \right) \cdot (\dot{w})^2 ds \right] ds
$$

$$
+ c_\omega \dot{w} + 3 \left[ \theta(s)w'w''V(t) \right]' + \left[ \theta'(s)V(t)(1 + \frac{1}{2}w'^2) \right]' = W_bP_A(t),
$$

(2.30)

and the associated boundary conditions

$$
w = w' = 0, \quad \text{at} \quad s = 0; \quad w'' = w''' = 0 \quad \text{at} \quad s = L.
$$

(2.31)

Now, setting $q_i \equiv Q_R(t)$ in Equation (2.24) and (2.25), we obtain the following equation for the electric circuit dynamics:

$$
\frac{\partial}{\partial t} \left[ \int_{0}^{L} \theta(s)(w'' + \frac{1}{2}w''w'^2) ds \right] - C_p\dot{V}(t) = \dot{Q}_R.
$$

(2.32)

Since the piezoelectric element and the resistive load are connected in parallel, Fig. 2.3, we can replace the current passing through the resistor, $\dot{Q}_R$, by $V(t)/R$. With that, Equation (2.32) becomes

$$
C_p\dot{V}(t) + \frac{1}{R}V(t) = \frac{\partial}{\partial t} \left[ \int_{0}^{L} \theta(s)(w'' + \frac{1}{2}w''w'^2) ds \right].
$$

(2.33)

2.2 Flow Characteristics

The only remaining unknown in the model is the pressure distribution on the cantilever surface. This distribution which we denoted as $P_A(t)$ depends on the inflow
rate of air, $U_0$, the beam deflection, $w(s,t)$, and the flow rate through the aperture, $U(t)$. To obtain an equation that governs the dynamics of the pressure exerted on the cantilever, we invoke several assumptions on the flow field near the cantilever. First, we assume that the flow near the beam and through the aperture is inviscid knowing the very low viscosity of air at normal temperatures. Also, following studies by Ricot et al. [47] on the dynamics of harmonica, we assume that the flow stream through the aperture is irrotational, two-dimensional and laminar. With these assumptions, we can utilize the steady Euler-Bernoulli equation to relate the air pressure on the surface, $P_A(t)$, to the volumetric air flow rate through the aperture, $U(t)$, as following:

$$P_A(t) = \frac{1}{2} \rho_a \frac{U^2(t)}{C_c^2 A(t)} ,$$

(2.34)

where $\rho_a$ is the density of air, $C_c$ is the flow contraction coefficient for flow through a sharp edged slit [48], and $A(t)$ is the total exit area of the aperture given by

$$A(t) = 2 \int_0^L \left[ \frac{w^2(s,t) + b^2}{2} \right] \frac{1}{2} ds + W_b \left[ \frac{w^2(L,t) + b^2}{2} \right]^{\frac{1}{2}}.$$  

(2.35)

Here, $b$ is the width of the clearance gap around the beam. Using the continuity equation, we can also relate the pressure to the steady inflow $U_0$, the outflow $U(t)$, and the change in reservoir volume caused by beam vibration using

$$\dot{P}_A(t) = \frac{\rho_a c^2}{V_r} \left[ U_0 - U(t) - W_b \frac{\partial}{\partial t} \int_0^L \frac{w(s,t) ds}{2} \right],$$

(2.36)

where $c$ is the speed of sound, and $V_r$ is the volume of the chamber. Solving Equation (2.34) for $U(t)$, then substituting into Equation (2.36), we obtain

$$\dot{P}_A(t) = \frac{\rho_a c^2}{V_r} \left[ U_0 - C_c A(t) \left( \frac{2}{\rho_a} P_A(t) \right)^{\frac{1}{2}} \right. - W_b \frac{\partial}{\partial t} \int_0^L w(s,t) ds \right].$$

(2.37)

For known $U_0$ and system design parameters, the response characteristics of the harvester can now be determined by solving Equations (2.30), (2.33), and (2.37).
2.3 Reduced-order Modeling

We utilize a Galerkin expansion to discretize the partial differential equation governing the motion of the system. We express the spatio-temporal function representing the transversal vibrations of the beam, $w(s,t)$, in the form of a convergent series of eigenfunctions multiplied by unknown temporal coordinates, i.e., we let

$$w(s,t) = \sum_{i=1}^{\infty} \phi_i(s)q_i(t), \quad (2.38)$$

where $q_i(t)$ is the unknown temporal coordinates and $\phi_i(s)$ are chosen as the orthonormal mass-normalized mode shapes of a cantilever beam. These can be written as

$$\phi_i(s) = C_i \left[ \cosh \frac{\lambda_i}{L}s - \cos \frac{\lambda_i}{L}s - \sigma_i \left( \sinh \frac{\lambda_i}{L}s - \sin \frac{\lambda_i}{L}s \right) \right], \quad (2.39)$$

where $\sigma_i$ is expressed as

$$\sigma_i = \frac{\sinh \lambda_i - \sin \lambda_i}{\cosh \lambda_i + \cos \lambda_i}, \quad (2.40)$$

and the $\lambda_i$ and $C_i$ are obtained via

$$1 + \cosh \lambda_i \cos \lambda_i = 0, \quad (2.41)$$

$$\int_0^L M(s)\phi_i^2(s) \, ds = 1. \quad (2.42)$$

Substituting Equation (2.38) into Equation (2.30), multiplying by $\phi_n(s)$, integrating over the length of the beam, and using the orthonormality properties of the chosen mode shapes yields the following set of nonlinear ordinary differential equations:

$$\ddot{q}_n + 2\zeta_n \omega_n q_n + \omega_n^2 q_n + \sum_{i,j,k} A_{nijk} q_i q_j q_k + \sum_{i,j,k} B_{nijk} (\ddot{q}_j q_k + 2\dot{q}_j \dot{q}_k + q_j \ddot{q}_k)$$

$$+ \left[ D_n + \sum_{i,j} C_{nij} q_i q_j \right] V(t) = \mathcal{E}_n P_A(t), \quad n = 1, 2, 3, ... \quad (2.43)$$
\[
\omega_n^2 = \int_0^L \phi_n [YI(s)\phi'''] ds,
\]
\[
A_{nijk} = \int_0^L \phi_n [\phi'_i(YI(s)\phi'_j\phi'_k)]' ds,
\]
\[
B_{nijk} = \frac{1}{2} \int_0^L \phi_n \left[ \phi'_i \int M(s) \left( \int_0^s \phi'_j\phi'_k ds \right) ds \right]' ds,
\]
\[
C_{nij} = 3 \int_0^L \phi_n [\theta(s)\phi'_j\phi'''] ds + \frac{1}{2} \int_0^L \phi_n [\theta'(s)\phi'_j\phi'_j]' ds,
\]
\[
D_n = \int_0^L \phi_n \theta''(s) ds,
\]
\[
E_n = W_b \int_0^L \phi_n ds.
\]

Also, substituting Equation (2.38) into Equations (2.33) and (2.37), we obtain
\[
C_I V(t) + \frac{1}{R} V(t) = \sum_n \mathcal{F}_n \dot{q}_n + \sum_{nij} \mathcal{G}_{nij} (q_n q_i q_j + q_n \dot{q}_i q_j + q_n \dot{q}_j q_i),
\]
(2.44)

and
\[
\hat{P}_A(t) = \frac{\rho_a c^2}{V_r} \left[ U_0 - C_c A(t) \left( \frac{2}{\rho_a} P_A(t) \right)^{\frac{1}{2}} - \sum_n E_n \dot{q}_n \right],
\]
(2.45)

where
\[
\mathcal{F}_n = \int_0^L \theta(s)\phi'''' ds,
\]
\[
\mathcal{G}_{nij} = \frac{1}{2} \int_0^L \theta(s)\phi'''\phi'_i\phi'_j ds,
\]
(2.46)

\[
A(t) = 2 \int_0^L \left[ \sum_{ni} \phi_n \phi_i q_n q_i + b^2 \right]^{\frac{1}{2}} ds + W_b \left[ \sum_{ni} \phi_n (L)\phi_i (L) q_n q_i + b^2 \right]^{\frac{1}{2}}.
\]
2.4 Convergence Analysis

When the air flows into the chamber with flow rate $U_0$, the beam deflects into a new static position determined by the flow rate. If the flow rate is less than a certain threshold which we will denote as the bifurcation or cut-on flow rate, the pressure forces cannot overcome the intrinsic damping in the system and the beam settles at the equilibrium position. On the other hand, when the flow rate exceeds the bifurcation threshold, the equilibrium solution loses stability via a Hopf bifurcation giving way to limit-cycle oscillations around the static position. Before we delve into the experimental validations of the derived model and its ability to predict this behavior, we seek to determine the minimum number of modes necessary for convergence. In other words, we want to determine the minimum number of modes to be kept in the series such that the addition of anymore modes does not affect i) the static response, ii) the cut-on flow rate (linear dynamics), and iii) the amplitude of the limit cycles around it (nonlinear dynamics).

Towards that end, we express Equations (2.43), (2.44) and (2.45) in the state-space form as following:

$$\dot{\bar{x}} = f(\bar{x}, U_0)$$  \hspace{1cm} (2.47)

where $\bar{x} = \begin{bmatrix} q_1, \dot{q}_1, q_2, \dot{q}_2, ..., q_i, \dot{q}_i, \dot{P}_A, \dot{V} \end{bmatrix}$, $U_0$ is the input flow rate (bifurcation parameter), and $f(\bar{x}, U_0)$ is the nonlinear vector field. To study the convergence of the static solution, the equilibrium points of the system are obtained by setting the right-hand side of Equation (2.47) to zero and solving the resulting nonlinear algebraic equations, $f(\bar{x}_0, U_0) = 0$, for $\bar{x}_0$.

Figure 3.1 depicts variation of the static tip deflection of the beam with the input flow rate using a single-mode and three-mode approximations. Results are obtained for the numerical parameters listed in Table 1 and associated with an Aluminum beam. It can be seen clearly that the algebraic system yields only one physical
solution and that the two curves are in excellent agreement. This implies that the static position is well-estimated using a single-mode approximation.

Figure 2.4: Static tip deflection of the beam using a single-mode approximation (solid) and three-mode reduced-order model (dashed).

To determine the minimum number of modes necessary to capture the linear dynamic response of the system, we obtain the eigenvalues of the response by finding the determinant of the Jacobian matrix evaluated at the fixed points. This yields a characteristic equation having $2n + 2$ roots. Two of the resulting eigenvalues are always real and are associated with the harvesting circuit and pressure dynamics, respectively. The remaining $2n$ eigenvalues represent $n$ pair of complex conjugate roots that describe the mechanical vibrations of the beam. The convergence of the reduced-order model is investigated by keeping a single mode in the series ($n = 1$) and calculating the first four eigenvalues. The number of modes is then gradually
Table 2.1: Geometric and material properties of the beam and piezoelectric layer.

<table>
<thead>
<tr>
<th>Properties/Beam material</th>
<th>Aluminum</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of elasticity, $Y^b[Pa]$</td>
<td>70</td>
<td>210</td>
</tr>
<tr>
<td>Density, $\rho^b[kg/m^3]$</td>
<td>2700</td>
<td>7900</td>
</tr>
<tr>
<td>Length, $L_b[mm]$</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Width, $W_b[mm]$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Thickness, $t_b[mm]$</td>
<td>0.25</td>
<td>0.16</td>
</tr>
<tr>
<td>Gap width, $b[mm]$</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Piezoelectric layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho^p[kg/m^3]$</td>
</tr>
<tr>
<td>Length, $L_p[mm]$</td>
</tr>
<tr>
<td>Width, $W_b[mm]$</td>
</tr>
<tr>
<td>Modulus of elasticity, $Y^p[Pa]$</td>
</tr>
<tr>
<td>Thickness, $t_p[mm]$</td>
</tr>
<tr>
<td>Electromechanical coupling, $d_{31}[pm/V]$</td>
</tr>
<tr>
<td>Permittivity, $e_{33}[nF/m]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air density, $\rho_a[kg/m^3]$</td>
</tr>
<tr>
<td>Speed of sound, $c[m/s]$</td>
</tr>
<tr>
<td>Contraction Coefficient, $C_c$</td>
</tr>
<tr>
<td>Chamber volume, $V_r[L]$</td>
</tr>
<tr>
<td>Electrical load, $R[\Omega]$</td>
</tr>
</tbody>
</table>

increased and variation of these eigenvalues is monitored. Using a flow rate of $U_0 = 0.25 \text{ L/s}$, we calculate the following eigenvalues by keeping one mode: $71.60 \pm 588.02i, -366.00,$ and $-997.00$. Next, we keep an additional mode and find that the
eigenvalues change to $72.99 \pm 587.80i$, $-360.69$, and $-977.37$. This yields a maximum error of less than 3% in all of the resulting eigenvalues. By adding a third mode, the maximum error drops to less than 0.5%. Furthermore, the bifurcation point representing the cut-on flow rate of the device was determined to be $U_0 = 0.2 \, L/s$ using a single-mode assumption and numerically at $U_0 = 0.1965 \, L/s$ using the three-mode approximation. Such results indicate that a single-mode approximation is sufficient to predict the local dynamics of the response around the static equilibria.

![Figure 2.5: Voltage response of the beam using a single-mode approximation (solid) and a three-mode reduced-order model (dashed).](image)

To see how these trends are reflected in the nonlinear system response, Equations (2.43), (2.44), and (2.45) were solved numerically using a single- and three-mode expansion. The resulting bifurcation diagram shown in Fig. 2.5 clearly demon-
strates negligible differences between the single- and three-mode response. This again demonstrates the accuracy of the single-mode approximation. As such, further analysis presented in this manuscript will be based on a reduced-order model consisting of a single mode.

2.5 Dimensional Analysis

To identify the important system parameters and understand their effect on the response of the system, we non-dimensionalize the coupled ordinary differential equations governing the response. Towards that end, we introduce the following dimensionless quantities

\[
\bar{q} = \frac{q}{L}, \quad \bar{t} = t\omega_n, \quad \bar{V} = \frac{C_p}{\theta} V, \quad \bar{P}_A = \frac{P_A}{P_\infty}, \quad \bar{U} = \frac{U_0}{U_{cr}}, \tag{2.48}
\]

where \( P_\infty \) is the ambient pressure and \( U_{cr} \) is the critical inflow rate. Introducing Equations (2.48) into Equations (2.43), (2.44) and (2.45) yields

\[
\ddot{\bar{q}} + 2\zeta \dot{\bar{q}} + \bar{q} + \Gamma \bar{q}^3 + \Lambda \left[ \bar{q}^2 \ddot{\bar{q}} + \bar{q}\dot{\bar{q}}^2 \right] + \theta_1 \bar{V} + \theta_2 \bar{q}^2 \bar{V} = \Delta \bar{P}_A, \tag{2.49}
\]

\[
\dot{\bar{V}} = -\alpha \bar{V} + \theta_3 \dot{\bar{q}} + \theta_4 \bar{q}^2 \dot{\bar{q}}, \tag{2.50}
\]

\[
\dot{\bar{P}}_A = \beta_1 \left[ \bar{U} - C_c \bar{A}(\bar{t}) \left( \frac{2P_\infty}{\rho} \bar{P}_A \right) \frac{1}{2} - \Pi \dot{\bar{q}} \right], \tag{2.51}
\]

where

\[
\begin{align*}
\Gamma &= \frac{AL^2}{\omega_n^2}, & \Lambda &= BL^2, & \theta_1 &= \frac{D\theta}{L\omega_n^2 C_p}, \\
\theta_2 &= \frac{CL\theta}{C_p \omega_n^2}, & \Delta &= \frac{\varepsilon P_\infty}{L\omega_n^2}, & \alpha &= \frac{1}{RC_p \omega_n}; \\
\theta_3 &= \frac{FC_p L}{\theta}, & \theta_4 &= \frac{GL^3 C_p}{\theta}, & \beta_1 &= \frac{\rho S^2}{V_{cr} \omega_n P_\infty}, & \Pi &= \varepsilon L\omega_n, \\
\bar{A}(\bar{q}) &= 2L\int_0^L \left[ \phi^2(s)L^2 \bar{q}^2 + b^2 \right] \frac{1}{2} ds + W_b \left[ \phi^2(L)L^2 \bar{q}^2 + b^2 \right] \frac{1}{2} \tag{2.52}
\end{align*}
\]
2.6 Area Approximation

From the knowledge of the form of $\phi(s)$, we can approximate the integral in the first term in Equation (2.53), so that the total exit area of the aperture would be a function of the tip deflection as

$$\bar{A}(\bar{q}) = 2L \left[ 0.16\phi^2(L)\bar{q}^2L^2 + b^2 \right]^{\frac{1}{2}} + W_b \left[ \phi^2(L)\bar{q}^2L^2 + b^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (2.54)

Figure 2.6 shows a comparison between the exact and the approximated values of the aperture area. Two sets of curves are generated by evaluating Equations (2.53) and (2.54) for a given range of tip deflections. The first set is obtained for a small gap width, $b = 0.1\text{mm}$, while the second set is calculated for a relatively large gap width, $b = 0.5\text{mm}$. It can be seen that in both cases, the approximated aperture
area is well-estimated using Equation (2.54), and that the error in the estimation increases by increasing the gap width.

### 2.7 Influence of the Nonlinearity

Next, we assess the influence of the nonlinear terms in Equations (2.49) and (2.50). Towards that end, we integrate the equations numerically and obtain the time history for the non-dimensional voltage, $\bar{V}$, and the displacement $\bar{q}$ with and without including the nonlinear terms. Fig. 2.7 demonstrates that the error resulting from neglecting the nonlinear terms remains less than 0.01% for both $\bar{q}$ and $\bar{V}$. This implies that these cubic nonlinearity terms have a negligible influence on the system response and thereby can be neglected.

![Figure 2.7](image)

Figure 2.7: Steady-state error resulting from neglecting the nonlinearity as function of $U_o/U_{cr}$: $\bar{q}$ (solid), $\bar{V}$ (dashed).
2.8 Experimental Validations

Figure 2.8 depicts the experimental configuration employed to investigate the validity of the proposed model. A 2.4 liter air chamber is constructed using a PVC pipe with inside diameter of 76.2 mm, closed with two PVC caps. On one end cap, the cantilever beam is mounted over the aperture/slot as shown in the figure. To accommodate the beam, the aperture is made slightly larger than the beam. From the other end of the chamber, air is supplied using an air pump through a small hole at the center of the cap.

To validate the model and compare the performance of different configurations, beams are cut from a 0.3 mm thick aluminum sheet and a 0.15 mm thick steel sheet. The geometric and material properties of the beams and the piezoelectric layer are listed in Table 1.

![Figure 2.8: Wind-driven autonomous beam vibrations.](image_url)

In the experiments, the air pressure is increased incrementally from 0 to 100 Pa. At each step, the mean pressure at the beam surface is measured using a pressure gauge and the corresponding output voltage at different pressures is recorded. The
air pressure is then slowly decreased from 100 Pa until the beam vibration ceases. Once again, the output voltage at different pressures is recorded. It was observed that there are not much differences between the voltage values measured at both directions of the pressure sweep. As such, only the average value is reported. The natural frequency of the limit cycle was observed to have slight variation with the inflow rate and was recorded at about 85 Hz for the Aluminum beam and at about 53 Hz for the Steel beam. Figures 2.9 and 2.10 depict variation of the voltage and output power of the device with the wind speed over an electric load of 50kOhm. The experimental results are also compared to numerical simulations obtained via long-time integration of Equations (2.43), (2.44), and (2.45).

As shown in the figures, there is a fairly good agreement between the theory and experimental findings especially in the Aluminum beam case. The Hopf bifurcation point is well estimated in both cases. The Steel and Aluminum beams are activated at moderate cut-on wind speeds of approximately 6.45-6.95 m/s, respectively. The agreement of the model with the experimental data is more pronounced in the Aluminum case. We believe that the deviations in the Steel case is due to imperfect clamping at the fixed end which tends to soften the beam. This conclusion is based on the observation that the first-modal frequency as obtained experimentally is 10Hz less than the one obtained theoretically in the Steel beam case. On the other hand, there is almost a perfect match between the theoretical and experimental values obtained for the first modal frequency of the Aluminum beam. A theoretical overestimation of the frequencies of a vibrating beam is usually attributed to imperfect clamping. This conclusion is further confirmed by the fact that the theoretical output voltage overestimates the experimental values. Imperfect clamping at the fixed end reduces the strain in the piezoelectric layer which drops the voltage from its theoretical values.

The experiments further reveal that the transition from the static position to the limit-cycle oscillations is continuous without sudden jumps. This indicates that the
bifurcation is supercritical for the chosen design parameters which agrees with the predictions of the mathematical model. The 0.1 – 0.8 milliwatts of output power attained using the Aluminum beam at wind speeds ranging between 7.5 and 12.5 m/sec clearly demonstrate the potential for using such concept to power and operate many microcontroller chips, health monitoring sensors, and wireless transponders [12, 13]. However, such results by no means represent the optimal performance of this device. The model and the experiments clearly demonstrate the interconnected dependence of the output power on the design parameters of the harvester. For instance, decreasing the stiffness to reduce the cut-on wind speed decreases the output power of the device; whereas increasing the stiffness to increase the power output has an adverse influence on the cut-on wind speed. As such, it is believed that the output power as well as the cut-on wind speed of the device can be significantly enhanced using a comprehensive optimization analysis of the theoretical model. This will be the scope of the next chapter.
Figure 2.9: Variation of the voltage and output power with the wind speed for the Aluminum beam. Asterisks represent experimental data.

Figure 2.10: Variation of the voltage and output power with the wind speed for the Steel beam. Asterisks represent experimental data.
Chapter 3

Influence of the Design Parameters on the Cut-on Wind Speed and Output Power

In this chapter, we carry a comprehensive nonlinear analysis to describe how the design parameters affect the response characteristics and the cut-on wind speed of the generator. To that end, we use the Routh-Hurwitz criterion to describe the conditions under which the Hopf bifurcation occurs. We then study the effect of the chamber volume on the onset of limit-cycle oscillations for different beam lengths and thicknesses. We investigate the effect of the gap width on the cut-on wind speed. Subsequently, we use the method of multiple scales to determine the normal form of the bifurcation (sub- or super-critical) and investigate the effect of beam’s length, thickness, chamber volume, and load resistance on the output power. Using the resulting understanding, we develop design charts to assist in choosing the harvester’s optimal design parameters for known average inflow wind speeds.
3.1 Cut-on Wind Speed

3.1.1 Hopf Bifurcation

In traditional structural vibration problems maximizing the wind speed corresponding to the onset of the bifurcation (flutter) is usually desired to avoid harmful large-amplitude oscillations. In this application, however, we are interested in finding the design parameters that will minimize the bifurcation parameter (cut-on wind speed). This will aid in exciting the harvester at lower inflow rates which in turn can maximize the output power of the device.

To investigate the influence of the harvester’s design parameters on the cut-on wind speed of the device, it is necessary to study the bifurcations that the equilibria of Equations (2.49), (2.50), and (2.51) undergo. To that end, we set the time derivatives in these equations equal to zero and obtain the following nonlinear algebraic equations:

\[
\begin{align*}
q_s + \Gamma q_s^3 &= \Delta P_s, \\
P_s &= \left[ \frac{U_0}{\beta_2 A(q_s)} \right]^2, \\
V_s &= 0,
\end{align*}
\]  

(3.1)

where \( \beta_2 = C_c \sqrt{2 P_s / \rho} \) and \( q_s, P_s, \) and \( V_s \) represent, respectively, the static deflection, pressure, and output voltage for a given input flow rate, \( U_0 \). To study the stability of the resulting fixed points, we enlarge the phase space to include \( \dot{q} \) and construct the Jacobian of the system as \(^1\)

---

\(^1\)It is worth noting that the influence of nonlinearities resulting from the nonlinear strain-deflection relation is assumed to be negligible in the Jacobian calculations. Please, refer to Chapter 2 for further details.
\[
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
-1 & -2\zeta & \Delta & \theta_1 \\
J_{31} & C_{10} & J_{33} & 0 \\
0 & \theta_3 & 0 & \alpha
\end{bmatrix},
\]

where
\[
J_{31} = C_2 + C_3 P_s + 2C_5 q_s + 2C_6 P_s q_s + C_7 P_s^2 + 3C_9 q_s^2,
\]
\[
J_{33} = C_1 + C_3 q_s + 2C_4 P_s + C_6 q_s^2 + 2C_7 P_s q_s + 3C_8 P_s^2.
\]

and
\[
C_1 = \frac{\beta_1 \beta_2 \alpha_0}{2P_s}, \quad C_2 = \beta_1 \beta_2 \alpha_1, \quad C_3 = \frac{\beta_1 \beta_2 \alpha_1}{2P_s}, \quad C_4 = -\frac{\beta_1 \beta_2 \alpha_0}{8P_s^2},
\]
\[
C_5 = \beta_1 \beta_2 \alpha_2, \quad C_6 = \beta_1 \beta_2 \alpha_3, \quad C_7 = -\frac{\beta_1 \beta_2 \alpha_1}{8P_s^2}, \quad C_8 = \frac{\beta_1 \beta_2 \alpha_2}{2P_s},
\]
\[
C_9 = \frac{\beta_1 \beta_2 \alpha_0}{16P_s^3}, \quad C_{10} = \beta_1 \Pi, \quad \alpha_n = \sqrt{P_s} \left| \frac{d^n A(q)}{dq^n} \right| \bigg|_{q_s}, \quad n = 0, 1, 2, 3
\]

The eigenvalues can then be obtained by setting the determinant of the Jacobian matrix equals to zero. This yields the following characteristic equation:
\[
\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0,
\]

where
\[
a = -\alpha + 2\zeta - J_{33},
\]
\[
b = -2\zeta\alpha - \theta_3 \theta_1 - C_{10} \Delta - 2\zeta J_{33} + 1 + J_{33} \alpha,
\]
\[
c = -\alpha + \theta_3 \theta_1 J_{33} + 2\zeta J_{33} \alpha - J_{33} - J_{31} \Delta + C_{10} \Delta \alpha,
\]
\[
d = J_{33} \alpha + J_{31} \Delta \alpha.
\]

To assess the sign of the resulting eigenvalues, we construct the Routh-Hurwitz array.
as
\[
\begin{pmatrix}
\lambda^4 & 1 & b & d \\
\lambda^3 & a & c & 0 \\
\lambda^2 & b - \frac{c}{a} & d & 0 \\
\lambda^1 & c - \frac{ad}{b-a} & 0 & 0 \\
\lambda^0 & d & 0 & 0
\end{pmatrix}.
\]

It follows from the first column in the array that the fixed points experience a bifurcation when \(c - \frac{ad}{b-a} = 0\), where a row of zeros appears. These fixed points correspond to a critical inflow rate, \(U_{cr}\). At this point, the Jacobian has a pair of purely imaginary eigenvalues of the form \(\lambda_{1,2} = \pm iw\) while the other two eigenvalues are both real and negative. Through further analysis not shown here, one can also prove that near \(U_0 = U_{cr}\), this pair of eigenvalues has a traversal (nonzero velocity) crossing of the imaginary axis. With these conditions satisfied, one can correctly surmise that a periodic solution of period \(\frac{2\pi}{\omega}\) is born as a result of a Hopf bifurcation at \((P_s, q_s, V_s, U_{cr})\).

Now that we have shown that the system’s equilibria can undergo a Hopf bifurcation at a critical inflow rate, \(U_0\), and having obtained the conditions under which this bifurcation occurs, we can use them to minimize the cut-on wind speed of the device. Specifically, in the next sections we study the influence of the chamber’s volume and the gap width on the cut-on wind speed for different beam lengths and thicknesses.

### 3.1.2 Optimal Chamber’s Volume

We make use of the Hopf bifurcation condition defined earlier as
\[
c^2 - abc + a^2d = 0 \tag{3.4}
\]
and try to understand the effect of the chamber volume \(V_r\) on the critical inflow rate. Figure 3.1 depicts variation of the critical flow rate with the chamber volume,
\( V_r \), and the beam length, \( L_b \). It is evident that, for each \( L_b \), there exists an optimal volume at which \( U_{cr} \) can be minimized. Designing the harvester with a chamber volume that is relatively far from the optimal one can dramatically increase the cut-on wind speed. Thus, it is recommended to use the optimal \( V_r \) as a first design step. This fact was also observed experimentally when the length of the chamber, and, hence its volume is varied for the setup employing the steel beam. When the volume is increased from 0.92\( L \) to 2.33\( L \) and 4.4\( L \), the value of the threshold pressure which is proportional to the cut-on wind speed for the chamber was recorded at \( > 1 \) \text{ inch} – \( H_2O \), \( 0.1 \) \text{ inch} – \( H_2O \), and \( 0.32 \) \text{ inch} – \( H_2O \), respectively as shown in Fig. 3.2.

Figure 3.1: Critical inflow rate, \( U_{cr} \), as function of chamber volume, \( V_r \), and beam length, \( L_b \).

Figures 3.3 and 3.4 depict, respectively, variation of the optimal chamber volume and the associated critical inflow rate as the beam’s thickness and length are varied within the specified range shown in the figures. To minimize the calculation time, the subroutine searching for the optimal volume was limited to a maximum volume
Figure 3.2: Threshold pressure as function of chamber volume, $V_r$: Theoretical (solid line), and asterisks represent experimental data.

of $25L$. This explains the constant volume behavior shown in Fig. 3.3 for larger values of $L_b$. The figures clearly illustrate that the optimal volume decreases as the beam thickness is increased and increases with its length. Such results lead to the conclusion that the optimal volume is inversely proportional to the first modal frequency of the beam as shown in Fig. 3.5. This also leads us to a critical conclusion with regards to the scalability of the optimal design. Specifically, due to the fact that the optimal chamber volume decreases as the first modal frequency increases, and that the natural frequency increases as the size of the beam decreases, one can conclude that the optimal volume of the chamber will be smaller for smaller beams. For instance, the optimal chamber’s volume and the critical inflow rate for a small-sized MPG ($\omega = 265 \ Hz$: $L_b = 1.5 \ cm$, $t_b = 0.05 \ mm$, $W_b = 4 \ mm$, and $b = 0.05 \ mm$) were found to be $0.035 \ L$ and $9.3 \times 10^{-3} \ L/s$, respectively. This constitutes a very advantageous characteristic of the device as maintaining an
optimal volume does not impede the scalability of the device.

The critical inflow rate, on the other hand, increases as the beam thickness is increased and decreases with its length at the optimal volume. Note that the presence of a well in Fig. 3.4 is not due to the presence of a minimum flow rate for certain optimal volumes, but is rather due to the increase in the inflow rate for the values where the chamber’s volume is forced to be constant at $25L$.

![Figure 3.3: Optimal chamber volume, $V_r$, as function of beam’s length, $L_b$, and thickness, $t_b$.](image)

3.1.3 Gap Width

In addition to the chamber’s volume, the gap width seems to play a critical role in determining the cut-on wind speed of the MPG. To elucidate this role, the gap width is varied between 0.15 mm and 0.3 mm for the Aluminum beam, and different chamber volumes. The results clearly demonstrate that the cut-on wind speed
Figure 3.4: Critical inflow rate, $U_{cr}$, as function of beam’s length, $L_b$, and thickness, $t_b$, for an optimal chamber volume.

Figure 3.5: Optimal $V_r$, solid, and the associated $U_{cr}$, dashed, as function of beam’s first modal frequency.
decreases with the gap width for any chamber volume indicating that, at the har-
vester’s optimal volume, the gap width should be minimized. Achieving smaller gaps
however, can be limited by manufacturing processes especially at the macroscale. At
the microscale, very small gaps with a sub-micron resolution can be easily realized
using a simple etching process.

Figure 3.6: Cut-on wind speed curves as function of the chamber’s volume for dif-
ferent gap widths.

Using the previous results, we simulate the voltage response of the Aluminium beam
using the optimal volume $V_r = 0.65 \, L$ and a smaller gap width $b = 0.15 \, mm$ while
keeping the other parameters associated with the beam constant. The results shown
in Fig. 3.7 are then compared to those obtained and validated previously for the
Aluminum beam with $V_r = 2.4 \, L$ and $b = 0.2 \, mm$. It is evident that the cut-on
wind speed of the device can be reduced by around 50% from $6m/sec$ to $3m/sec$
when a smaller gap width and the optimal volume are utilized.
Figure 3.7: Variation of the output voltage with the wind speed for the Aluminum beam using $V = 2.4 \, L, b = 0.2 \, mm$ (solid line), and $V = 0.65 \, L, b = 0.15 \, mm$ (dashed lines).

### 3.2 Maximizing the Output Power

The process of minimizing the cut-on wind speed of the device does not necessarily guarantee an increase in the output power beyond it. For instance consider Fig. 3.8, where we plot the output voltage of the harvester at the associated optimal volume that minimizes the cut-on wind speed for different beam lengths. It is evident that, if the average inflow rate at the location where the harvester is designed to operate is larger than $U_0 = 0.08 \, L/s$, then the shorter beam which has the highest cut-on wind speed, will provide maximum output voltage. As such, it is important to maximize the output power of the device in addition to minimizing the cut-on wind speed. Towards that end, we explore various ways to optimize the output power of
the device.

![Bifurcation diagram comparison](image)

Figure 3.8: Bifurcation diagram comparison: solid ($L_b = 60 \text{ mm}, t_b = 0.13 \text{ mm}$), dashed ($L_b = 66 \text{ mm}, t_b = 0.13 \text{ mm}$), and dash-dot ($L_b = 72 \text{ mm}, t_b = 0.13 \text{ mm}$).

### 3.2.1 Exploring the Sub-critical Bifurcation

Not only does the combination of the design parameters determine the cut-on wind speed but they also determine the nature of the response beyond it (*bifurcation nature*). As shown in Fig. 3.9(a), when the bifurcation is *supercritical* and the flow rate exceeds the threshold value, small-amplitude limit-cycle oscillations about the former static position are born. On the other hand, when the bifurcation is *subcritical* as shown in Fig. 3.9(b), the output voltage jumps to a distant attractor which can be another fixed point, a large-amplitude limit cycle, or even a chaotic attractor. In most engineering applications, sub-critical bifurcations are considered dangerous because they can cause structural failure. In our study however, a sub-critical Hopf bifurcation means large amplitude oscillations at lower wind speeds.
which could imply an enhanced performance of the harvester.

Figure 3.9: Two scenarios for the voltage response of the harvester as the flow rate increases.

In this section, we utilize the method of multiple scales to obtain the normal form of the Hopf bifurcation responsible for the onset of limit-cycle oscillations of the harvester. We try to identify any combination of parameters that can make the bifurcation sub-critical to further assist in enhancing the output power of the device. To that end, we express the system’s response as the sum of static and dynamic components as follows:

\[ \bar{q} = q_s + q_d, \]
\[ P_A = P_s + p_d, \]
\[ \bar{V} = V_s + V_d, \]

where the subscript \( s \) denotes the static part and \( d \) denotes the dynamic component. Substituting Equation (3.5) back into Equations (2.49), (2.50) and (2.51), expanding the aperture area, \( \bar{A}(\bar{q}) \), in a Taylor series up to cubic terms, and separating the dynamic and static terms, yield the following set of equations:

\[
\begin{align*}
\ddot{q}_d + 2\zeta \dot{q}_d + q_t V_d &= \Delta p_d, \\
\dot{V}_d &= -\alpha V_d + \theta_3 \dot{q}_d, \\
\dot{p}_d &= -\left( C_1 p_d + C_2 q_d + C_3 p_d q_d + C_4 p_d^2 + C_5 q_d^2 + C_6 p_d^3 + C_7 p_d^2 q_d \\
&\quad + C_8 p_d q_d^2 + C_9 p_d^3 + C_{10} \dot{q}_d \right),
\end{align*}
\]
where the $C_i$'s are given by Equation (3.2).

The time dependence is also expanded into multiple time scales as

$$T_n = \epsilon^n t, \quad n = 0, 1, 2,$$

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + O(\epsilon^3), \quad (3.7)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + 2\epsilon^2 D_0 D_2 + O(\epsilon^3),$$

where $\epsilon$ is a bookkeeping parameter that will be set to unity at the end of the analysis and $D_n = \frac{\partial}{\partial T_n}$. We seek a solution in the form

$$q_d(t; \epsilon) = q_0(T_0, T_1, T_2) + \epsilon q_1(T_0, T_1, T_2) + \epsilon^2 q_2(T_0, T_1, T_2) + ...$$

$$V_d(t; \epsilon) = V_0(T_0, T_1, T_2) + \epsilon V_1(T_0, T_1, T_2) + \epsilon^2 V_2(T_0, T_1, T_2) + ... \quad (3.8)$$

$$p_d(t; \epsilon) = p_0(T_0, T_1, T_2) + \epsilon p_1(T_0, T_1, T_2) + \epsilon^2 p_2(T_0, T_1, T_2) + ...$$

We scale the terms with quadratic and cubic nonlinearities in Equation (3.6c), as well as the damping, voltage, and pressure in Equation (3.6a) to appear at the second-order of the perturbation problem. Towards that end, we let $C_i = \epsilon C_i$, where $i = 3..9$, $\zeta = \epsilon \zeta$, $\Delta = \epsilon \Delta$, and $\theta_1 = \epsilon \theta_1$. Now, substituting Equations (3.7) and (3.8) into Equations (3.6), and equating coefficients of like powers of $\epsilon$ yield

$$\epsilon^0:$$

$$D_0^2 q_0 + q_0 = 0, \quad (3.9a)$$

$$D_0 V_0 - \alpha V_0 = \theta_3 D_0 q_0, \quad (3.9b)$$

$$D_0 p_0 + C_1 p_0 = - (C_2 q_0 + C_{10} D_0 q_0), \quad (3.9c)$$

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The solution of the first-order problem, Equation (3.9a), can be expressed as

\[ q_0 = A(T_1, T_2)e^{iT_0} + cc, \] (3.12)

where \( A \) is a complex-valued function that will be determined at a later stage in
the analysis and \( cc \) is the complex conjugate of the preceding term. Substituting
Equation (3.12) into Equations (3.9b) and (3.9c), and solving for the corresponding
variable, one obtains

\[ V_0 = X Ae^{iT_0} + cc, \] (3.13)
\[ p_0 = \frac{C_2 + iC_{10}}{C_1 + i} Ae^{iT_0} + cc, \] (3.14)

where

\[ X = \frac{i\theta_3}{i - \alpha}. \]
Substituting Equations (3.12), (3.13) and (3.14) into Equation (3.10a) yields
\[ D_0^2 q_1 + q_1 = - \left( 2iD_1 A + 2i\zeta A - \Delta \frac{C_2 + IC_{10}}{C_1 + i} A + \theta_1 X A \right) e^{iT_0} + cc. \] (3.15)

Next, we eliminate any secular terms (terms having the factor \( e^{\pm iT_0} \)), and obtain
\[ D_1 A = \frac{1}{2} i \left[ 2i\zeta A - \Delta \frac{C_2 + IC_{10}}{C_1 + i} A + \theta_1 X A \right]. \] (3.16)

Considering Equation (3.16), the solution of the second-order problem, Equation (3.10a), is \( q_1 = 0 \). Substituting \( q_0, V_0, p_0, \) and \( q_1 \) into Equations (3.10b) and (3.10c), then solving for \( V_1 \) and \( p_1 \) yields, respectively
\[ V_1 = - \frac{\theta_3 \alpha}{(i - \alpha)^2} D_1 Ae^{iT_0} + cc, \] (3.17)
\[ P_1 = - \frac{1}{C_1} \left[ C_3 X + C_4 X \bar{X} + C_5 \right] A \bar{A} - \left[ \frac{(X + IC_{10})D_1 A + Y_1 A^2 \bar{A}}{C_1 + i} \right] e^{iT_0} \]
\[ - \frac{Y_2 A^2}{C_1 + 2i} e^{2iT_0} - \frac{Y_3 A^3}{C_1 + 3i} e^{3iT_0}, \] (3.18)

where
\[ Y_1 = 2C_6 X + C_6 \bar{X} + 2C_7 X \bar{X} + C_7 X^2 + 3C_8 X^2 \bar{X} + 3C_9, \]
\[ Y_2 = C_3 X + C_4 X^2 + C_5, \]
\[ Y_3 = C_6 X + C_7 X^2 + C_8 X^3 + C_9, \]
and \( \bar{A} \) and \( \bar{X} \) are the complex conjugate of \( A \) and \( X \). Substituting \( q_0, q_1, V_1, \) and \( p_1 \) into Equation (3.11a) and eliminating the terms that lead to secular terms yields
\[ D_2 A = F_1 \bar{A} A^2 + F_2 A, \] (3.19)

where \( F_1 \) and \( F_2 \) are functions of \( \zeta, \alpha, \Delta, \theta_1, \theta_3, U_0 \) and \( C_n \). Using the method of reconstitution [49], one can write
\[ \frac{dA}{dt}(t; \epsilon) = \epsilon D_1 A + \epsilon^2 D_2 A + \ldots \] (3.20)

Substituting Equations (3.16) and (3.19) into Equation (3.20), expressing \( A \) in the polar form \( A = \frac{1}{2} ae^{i\beta} \), where \( a \) and \( \beta \) are real-valued amplitude and phase, separating real and imaginary parts, and setting \( \epsilon = 1 \), we obtain
\[ \dot{a} = \mu_1 a^3 + \mu_2 a, \] (3.21)
where $\mu_1$, $\mu_2$, $\gamma_1$, $\gamma_2$ are functions of the design parameters and the inflow rate. Equations (3.21) and (3.22) represent the normal form for a Hopf bifurcation with $\mu_2$ equal to zero at the critical inflow rate, $U_{cr}$. To solve for the steady-state amplitude of the response, we set $\dot{a} = 0$ in Equation (3.21) and obtain

$$a_0 = 0, \pm \sqrt{-\frac{\mu_2}{\mu_1}} \text{, when } \frac{\mu_2}{\mu_1} < 0$$

and

$$a_0 = 0, \text{ when } \frac{\mu_2}{\mu_1} > 0$$

Substituting the non-zero fixed point back into Equation (3.12), one can obtain the approximation for the limit-cycle solution of (3.6) as

$$q_d = a_0 \cos(\omega_m t + \beta_0) + O(a_0^3),$$

$$V_d = a_0 \mathcal{F} \cos(\omega_m t + \beta_0) + O(a_0^3),$$

where $\beta_0$ and $\mathcal{F}$ are constants, and

$$\omega_m = (1 + \dot{\beta}) + O(a_0^3) = 1 + \gamma_1 a^2 + \gamma_2 + O(a_0^3).$$

It is important to bear in mind that the solutions acquired via the method of multiple scales is accurate for small range of the inflow rate beyond its critical value. Hence, the accuracy is expected to deteriorate as $U_0$ becomes much larger than the critical value. In Fig. 3.10, we compare the analytical approximation (dashed lines) with the numerical solutions (solid lines) of Equations (3.6) for different values of $U_0$. The curves are constructed for the steel beam and the numerical solutions are obtained by using a long-time Runge-Kutta integration routine. The figure demonstrates good agreement between the analytical and numerical solutions for moderate values of $U_0/U_{cr}$. As the ratio between $U_0$ and $U_{cr}$ increases beyond 1.5 the analytical solution starts to deviate from the numerical integration. This fact is further confirmed in
Figure 3.10: Bifurcation diagrams constructed by the method of multiple scales (dashed) and numerically (solid): left for $q_d$ and right for $V_d$.

Fig. 3.11 which depicts a comparison between the limit cycles generated numerically and analytically at different inflow rates.

Equation (3.26) is used to investigate variations of the response frequency, $\omega_m$, with the inflow rate ratio for different dimensions of the steel beam as shown in Fig. 3.12. It can be clearly seen that the beam oscillates at a frequency slightly higher than its first modal frequency (short circuit) at the bifurcation point. This is attributed to the electromechanical coupling which tends to increases the oscillation frequency from its short circuit value. As the inflow rate increases, the oscillation frequency, $\omega_m$, increases slightly and almost linearly with the inflow rate.

To determine the nature of the Hopf bifurcation, the stability of the steady-state solutions is determined by evaluating the Jacobian of the modulation equations,
Equation (3.21) and (3.22) at the fixed points. This yields

\[ J(a_0) \equiv \left. \frac{d\dot{a}}{da} \right|_{a_0} = \begin{cases} \mu_2, & a_0 = 0 \\ -2\mu_2, & a_0 = \pm \sqrt{-\frac{\mu_2}{\mu_1}} \end{cases} \quad (3.27) \]

When \( \mu_1 \mu_2 > 0 \), only the trivial solution exists and it is stable for \( \mu_2 < 0 \) and unstable for \( \mu_2 > 0 \). On the other hand, when \( \mu_1 \mu_2 < 0 \), three fixed points exist; for \( \mu_2 > 0 \) the trivial fixed point is unstable while the nontrivial fixed points are stable resulting in a super-critical Hopf bifurcation as illustrated in Fig. 3.13(a). On the other hand, when \( \mu_2 < 0 \), the zero fixed point is stable while the nonzero fixed points are unstable yielding a sub-critical Hopf bifurcation as illustrated in Fig. 3.13(b).

By virtue of the previous discussion, it becomes evident that the signs of \( \mu_1 \) determines the nature of the bifurcation at the linear stability boundary. Figure 3.14
Figure 3.12: Response frequency, $\omega_m$, as function of inflow rate ratio for different beam lengths.

Figure 3.13: Sketches of fixed points and their stability: (a) $\mu_1 < 0$ and (b) $\mu_1 > 0$.

(a) and (c) depict, respectively, variations of $\mu_1$ and $\mu_2$ for a range of the chamber’s volume, $V_r$, and beam’s length, $L_b$. The other design parameters and material properties of the steel beam and the piezoelectric layer are kept constant as listed in Table 2.1. Results indicate that $\mu_1$ is always negative while $\mu_2$ remains positive throughout the range considered in the figure. This implies that a supercritical Hopf bifurcation always occurs for the values of $L_b$ and $V_r$ considered.
Figure 3.14: Variation of $\mu_1$ and $\mu_2$ with the chamber’s volume and beam length:
(a) $\mu_1$, (b) Projection of $\mu_1$, (c) $\mu_2$ and (d) Projection of $\mu_2$.

It is worth mentioning that the values of $\mu_1$ and $\mu_2$ exist only when Equation (3.4) yields a solution for $U_{cr}$. Otherwise, no Hopf bifurcation occurs in the first place. Figure 3.14 (b) and (d) depict two-dimensional projection of Figs. 3.14 (b) and (d),
respectively. The colored regions show the combinations of \((L_b, V_r)\) which yield a real solution of Equation (3.4) for \(U_{cr}\), whereas the unshaded regions represents combinations of \((L_b, V_r)\) for which no bifurcation occurs. In those regions, the beam does not oscillate regardless of how large the inflow rate is. Long-time numerical integration of the original equations of motion for different initial conditions and inflow rates yielded similar conclusions.

### 3.2.2 Optimizing the Electric Load

Optimizing the electric load of an energy harvester represents an important step in maximizing the flow of energy from the environment to the load [27, 50]. Based on impedance matching conditions for linear systems, while assuming negligible mechanical damping, the optimal electric load can be approximated as

\[
R_{opt} = \frac{1}{C_p \omega_n}
\]

where \(C_p\) is the capacitance of the piezoelectric element and \(\omega_n\) is the natural frequency of the oscillating structure. For the nonlinear system at hand, where the frequency, \(\omega_m\), of the limit-cycle oscillations resulting from the bifurcation depends on the inflow rate, the optimal electric load is expected to change with \(\omega_m\). Figure 3.15 shows the optimal resistance as a function of the inflow rate ratio for the steel beam. It can be clearly seen that, \(R_{opt}\) is not very sensitive to variations in the flow rate because, as shown in Fig. 3.12, \(\omega_m\) itself is not very sensitive to variations in the inflow rate. Indeed, one can clearly observe that the optimal load obtained by optimizing the output power, solid line, closely matches the quantity \(1/(C_p \omega_m)\), dashed line, as obtained from the method of multiple scales using Equation (3.26).

This result implies that, for a MPG of a given design, there is no need to change the electric load as the wind speed is changing because near optimal power can be harnessed using the bifurcation’s optimal load.

On the other hand, as the design parameters are varied, the optimal electric load varies significantly. Figure 3.16 depicts variations of the optimal load calculated at
Figure 3.15: Optimal load as a function of the inflow rate. Solid line is obtained by numerically optimizing the output power. Dashed lines are obtained using the approximation $R_{opt} = 1/(C_p\omega_m)$.

1.2$U_{cr}$ for different beam lengths, thicknesses, and the associated optimal volume. The figure clearly indicates that the optimal load is inversely proportional to the beam’s first modal frequency.

### 3.2.3 Optimal Design Charts

Using the analytical solution obtained by the method of multiple scales, we investigate the effect of different design parameters on the steady-state output voltage. A steel beam is considered and six design specifications are studied: $(L_b = 60 \text{ mm}, t_b = 0.13 \text{ mm})$, $(L_b = 60 \text{ mm}, t_b = 0.15 \text{ mm})$, $(L_b = 60 \text{ mm}, t_b = 0.17 \text{ mm})$, $(L_b = 66 \text{ mm}, t_b = 0.13 \text{ mm})$, $(L_b = 72 \text{ mm}, t_b = 0.13 \text{ mm})$, and $(L_b = 66 \text{ mm}, t_b = 0.15 \text{ mm})$. The first beam is considered as the reference beam. The second
Figure 3.16: Optimal load as a function of the beam’s length, $L_b$, and thickness, $t_b$.

and the third beams represent incremental changes in $t_b$. The fourth and the fifth beams represent incremental change in $L_b$. The sixth beam represents an incremental change in both $t_b$ and $L_b$. The steady-state voltage response is then calculated at the optimal volume for each beam which is found to be 8.5 $L$, 6.5 $L$, 5 $L$, 14 $L$, 22 $L$ and 10.5 $L$ respectively. Figure 3.17 (a) shows the voltage diagram for the reference beam, first case, compared to the thicker beams, second and third cases. It is obvious that, while maintaining an optimal volume, increasing the beam thickness shifts the whole bifurcation diagram towards higher critical flow rates. This implies that a thicker beam always requires a higher wind speed to produce the same output of voltage at the optimal volume.

A comparison between the steady-state voltage curves of the reference beam and a longer beam, third and fourth cases, is shown in Fig. 3.17 (b). We can see that increasing the length of the beam decreases the slope of the voltage diagram and causes a shift to the left along the flow rate axis. This indicates that a smaller inflow
rate is necessary to activate the harvester for a longer beam. However, beyond a certain value of the inflow rate, the voltage curves intersect and the steady-state output voltage for the shorter beam becomes larger. This clearly illustrate that prior knowledge of the average wind speed at the location where the harvester is designed to operate is necessary in order to select the design parameters that maximizes the output power of the harvester. Figure 3.17 (c) demonstrates how changing both the beam’s length and thickness affect the bifurcation diagram. It can be simply explained as being a superposition of the effects of the thickness and length illustrated in Figs. 3.17 (a) and (b).

As seen throughout the preceding analysis, there are many parameters that can be optimized to enhance the harvester’s performance. As an effort to provide a quick methodology for the selection of these design parameters, we generate a family of design charts that can be used to maximize the output voltage of the harvester subjected to the user’s constraints, see Figs. 3.18 and 3.19. Prior to using these charts, a designer should have an idea about the average wind speed at the location where the harvester is designed to operate and the space limitations which can constraint the maximum size of the harvester represented by its chamber’s volume $V_r$. For the sake of illustration, the output voltage of the harvester is calculated at four different locations having an average wind speeds of 4 m/s, 5 m/s, 6 m/s, and 7 m/s, respectively, as shown in Figs. 3.18 (a) and (b), and Figs. 3.19 (a) and (b). The background color, filled contours, shows the steady-state output voltage, $V_d$, that is calculated at the optimal $V_r$ and $R_{opt}$ associated with each $(L_b, t_b)$. Note that $V_r$ is kept constant at 25 $L$ for beams that require larger optimal volumes. The solid-lined contours represent the bifurcation inflow rate, $U_{cr}$, required to activate the harvester. The dash-lined contours display the associated optimal volume, $V_r$.

The charts indicate the presence of regions in the $(t_p, L_p, V_r)$ space that maximize the output voltage for a given average wind speed. These large-amplitude voltage regions shift to the left along the beam’s thickness axis for the higher average wind
Figure 3.17: Bifurcation diagram comparison: (a) solid ($L_b = 60 \text{ mm}$, $t_b = 0.13 \text{ mm}$), dashed ($L_b = 60 \text{ mm}$, $t_b = 0.15 \text{ mm}$), and dash-dot ($L_b = 60 \text{ mm}$, $t_b = 0.17 \text{ mm}$). (b) solid ($L_b = 60 \text{ mm}$, $t_b = 0.13 \text{ mm}$), dashed ($L_b = 66 \text{ mm}$, $t_b = 0.13 \text{ mm}$), and dash-dot ($L_b = 72 \text{ mm}$, $t_b = 0.13 \text{ mm}$). (c) solid ($L_b = 60 \text{ mm}$, $t_b = 0.13 \text{ mm}$) and dashed ($L_b = 66 \text{ mm}$, $t_b = 0.15 \text{ mm}$).

speed. As such, it can be seen clearly by comparing chart Figs. 3.18 (a) and 3.19 (d) that a shorter beam provides a higher voltage at the higher wind speeds.
Figure 3.18: Design charts obtained at different average wind speeds: (a) 4 m/s, and (b) 5 m/s.
Figure 3.19: Design charts obtained at different average wind speeds: (a) 6 m/s, and (b) 7 m/s.
Chapter 4

Conclusions and Future Work

4.1 Aero-Electro-Mechanical Modeling of the MPG

In this thesis, we developed a nonlinear reduced-order aero-electro-mechanical model to capture the response behavior of a self-excited wind generator that consists of a piezoelectric uni-morph beam embedded within a cavity to mimic vibrations of harmonica’s reeds when subjected to air flow. The analytical model describes the dynamic evolution of the four essential system’s parameters. These are the spatial and temporal dynamics of the beam deflection, the temporal dynamics of the voltage developed across the electric load, the temporal evolution of the exciting pressure on the surface of the beam, and the flow rate through the aperture between the beam and the support. The model is obtained at three successive levels. First, we employed Hamilton’s principle in combination with the nonlinear Euler-Bernoulli’s beam theory and the linear constitutive equations of piezoelectricity to obtain the nonlinear partial differential equation (PDE) relating the flexural dynamics of the beam to the output voltage and the exciting pressure. In addition to the traditional linear inertia, stiffness, and damping terms, the resulting PDE includes the effects of the beam’s geometric and inertia nonlinearities, the linear and nonlinear
backward electromechanical coupling terms, and the nonlinear exciting pressure. Second, we modeled the piezoelectric layer as a capacitor connected in parallel to a resistive load and used Kirchoff’s laws to obtain the nonlinear ordinary differential equation (ODE) relating the output voltage of the harvester to the strain rate in the piezoelectric layer. The resulting equation relates the current passing through the piezoelectric capacitance and the load current to the current generated due to the time-varying strain in the piezoelectric layer. Third, assuming that the flow rate through the aperture between the beam and the support is irrotational, two dimensional, and steady; we utilized the steady Bernoulli’s equation in conjunction with the continuity equation to relate the exciting pressure on the surface of the beam to the in- and outflow rates of air. After the full model was established, we used a Galerkin expansion to discretize the PDE into a set of nonlinearly-coupled ODEs. The descretization was performed using a cantilevered beam mechanical mode shapes. We carried a convergence analysis to determine the minimum number of mechanical modes to be kept in the reduced-order model. We found that a single-mode reduced-order model is accurate enough to predict the static, linear, and nonlinear dynamic responses of the MPG for a large range of input flow rates. Additionally, we investigated the influence of neglecting the beam’s geometric and inertia nonlinearities on the response behavior showing that neglecting such nonlinearities yields less than 0.01% error in the output voltage calculations. We validated the resulting reduced-order model against experimental data using both a steel and aluminum beams. Results demonstrated excellent match between the model and experiments in the aluminum beam’s case and qualitative agreement in the steel beam’s case. Further analysis of the experimental results in the steel case revealed that there was a significant mismatch between the experimental and theoretical first modal frequencies. Specifically, it was observed that the first-modal frequency as obtained experimentally is 10Hz less than the one obtained theoretically. This deviation is usually attributed to the imperfect clamping at the fixed end, which tends
4.2 Influence of the Design Parameters on the Generator’s Performance

Enhancing the performance of the MPG requires minimizing the cut-on wind speed and maximizing the generated power. To achieve these two objectives, we used the experimentally-validated model to study the influence of the design parameters on the cut-on wind speed and output power. Using the Routh-Hurwitz stability criterion, we established a condition for the onset of beam’s limit-cycle oscillations (Hopf Bifurcation point). We used the resulting condition to study the influence of the chamber’s volume, beam’s length and thickness, and gap’s width on the cut-on wind speed of the MPG. We observed that, for a given beam’s length and thickness, there exists an optimal chamber volume at which the cut-in wind speed of the device can be minimized. This optimal volume decreases as the beam’s first-modal frequency increases, i.e., as the beam size decreases. This implies that designing an optimal device does not impede the scalability of the proposed concept. We also observed that the cut-on wind speed decreases significantly with the gap’s width. As such, designing a MPG which can be activated at lower wind speeds requires minimizing the gap between the beam and the supporting structure. At the macroscale, the gap size is usually limited by the accuracy and tolerance of the manufacturing processes used in the making the gap. Such issues can be alleviated at the microscale using a simple itching process which can be used to realize gap widths with sub-micrometer resolution. Voltage bifurcation diagrams showing variation of the output voltage of the device with the inflow rate were constructed and compared for different beams at the associated optimal volume. For typical values of the physical parameters involved, we observed that i) increasing the thickness of the beam shifts the bifur-
cation point towards higher critical inflow rates, and ii) increasing the length of the beam decreases the slope of the bifurcation diagram and reduces the cut-on wind speed. This behavior causes the voltage curves to intersect at a certain inflow rate, making the shorter beams, which have a higher cut-on wind speed, more efficient in harvesting energy at flow rates that are higher than the intersection flow rate and vice versa. This clearly implies that lower cut-on wind speeds do not always yield higher output power. Based on the preceding critical conclusion, we used the resulting model to construct design charts that aid in designing a MPG with optimal design parameters for a given known average wind speed. These charts depict the optimal output voltage as a function of beam length and thickness at the optimal volume. By knowing the average wind speed at the location where the harvester is designed to operate and the space limitations which can constrain the maximum size of the harvester represented by its chamber volume, the designer can choose the beam dimensions that maximize the output voltage. To maximize the flow of energy from the vibrating structure to the electric load, we also obtained the optimal load resistance which maximizes the output power at different inflow rates. We observed that the optimal resistance is not very sensitive to variations in the inflow rate and that it is well-approximated using impedance matching conditions but with the frequency of the limit cycle replacing the first-modal frequency of the beam. Finally, in an attempt to increase the output power of the device, we explored the prospect of designing the harvester such that the Hopf bifurcation responsible for the onset of the beam’s oscillation is sub-critical. Towards that end, we utilized the method of multiple scales to obtain the bifurcation’s normal form. Using the resulting normal form, the stability of the resulting steady-state trivial and non-trivial solutions was analyzed near the bifurcation point. We observed that the trivial solutions always lose stability via a super-critical Hopf bifurcation.
4.3 Recommendations for Future Work

Having developed and validated an analytical model of the MPG, then studied the effect of the design parameters on the cut-on wind speed and output power of the device, our future work will be directed towards:

- Developing and fabricating a prototype of a new MPG with the optimal design parameters for a known average wind speed at a pre-determined location. The performance of this optimal device will be evaluated in a wind tunnel. Subsequently, the device will be installed at the desired location where its performance in powering and maintaining a wireless sensor will be assessed.

- Since the most important property of this device is its scalability, we will develop and fabricate a prototype at the MEMS scale. The MEMS scale design will be optimized based on design charts obtained via the validated model. The performance of this device will also assessed using the Microsystem Analyzer (MSA-400) available in the Nonlinear Vibrations and Energy Harvesting Laboratory.

- Studying the influence of the beam’s geometry, especially tapered beams, on the cut-on wind speed and output power of the device. This requires a new analytical model that accounts for variations in the beam’s width.

- Finding exact or approximate analytical solutions to study amplitude variation’s and bifurcations of the resulting limit cycles with the inflow rate. Such analysis, which can be carried out using the Floquet theory in conjunction with the method of harmonic balance, is critical to assess whether the limit cycles undergo additional bifurcations that can lead to complex dynamic responses, e.g. period doubling, quasi-periodicity, or even chaotic motions.
Bibliography


