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Fractal Jackson Networks

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FRACTAL JACKSON NETWORKS

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Abstract

In this dissertation, Gaussian random measures that arise as limits of Jackson networks have been studied. The support of the random measure is a fractal having Hausdorff dimension δ. The variance measure is the Hausdorff measure also of dimension δ.

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Table of Contents

List of Figures

Chapter 1

Introduction

THE problem of finding equilibrium in an infinite stochastic network is a problem of great discussions. Having many properties transferred from finite case to infinite case can have systematic errors. On the other hand there are properties of the finite stochastic systems which could be transferred to the infinite case [10]. Some good references in studying finite queues and stochastic networks are [18], [11], [19] and [20]. In this writing we will look into limit of a Jackson network based on a fractal system which has self repeating patterns. I should note convergence in the stochastic process could be read from [23], [3], [1]. And resources such as [24], [25], [2] and [14] are good references for the limit theorems.

1.1 Fractals

The concept of self similarity appears in the different areas of mathematics. Julia sets and fractals are types of subsets complex plane and of $\mathbb{R}^n$ that are related to this concept. Fractals were popularized by Mandelbrot in the [15]. The Cantor set is one of the prototype forms of fractals which has well known properties such as self similarity.

A simple understanding of fractals can be seen in [15]. Although fractals have been studied through different aspect, our main focus here is on the self similarity part of them [13] and [22]. Self similarity could be seen in many mathematical objects including Cantor set. As we can see in the following sections many of the properties could be transferred to the smaller parts. For example, consider a highway which is serving cars are served everywhere and its nodes are when it reaches to

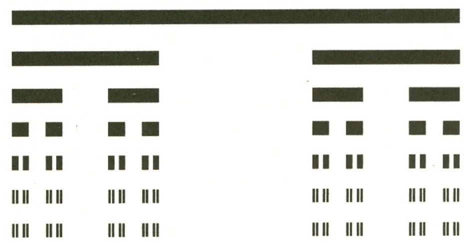

Figure 1.1: Cantor Set

local streets and some of the nodes are congested, in the meaning that there is a waiting for service at each node, or as in the figure 1.2.

Another example is a stochastic network which looks like a fractal and shows basic structure of the network we will study here is in figure 1.3.

Section 2.1 contains a review of fractals. Here we follow Hutchinson [7]. The main results relating to our paper are one the fractal K is a compact subset of finite dimensional Euclidean space having finite Hausdorff measure, H, which we can normalize so that $H(K) = 1$. Second, for some positive integer J there is, for each positive integer p, a natural way to partition the fractal into J^p sets, $K_{p,\mathbf{i}}$, $\mathbf{i} \in \{1, \dots, J\}^p$, of equal Hausdorff measure $1/J^p$. Each such set is the location of a node in the p-th Jackson network.

One of the other things we should consider is how to look into an infinite system. There are parts of each infinite stochastic system which can be studied through convergence on the finite cases. In [6] scaling of the network is discussed which is useful for studying infinite networks. Other approaches in heavy traffic system using central martingale approach are discussed in [17] and using functional central limit theorem in [2].

1.2 Jackson Networks

Jackson networks initiated with the work of Jackson [8], who constructed a queueing network whose limiting queue length distribution has product form, that is, the limiting queue lengths are independent.. An example of open Jackson Network could be described as a network with N nodes, independent Poisson input to the node i with rate $\lambda_i > 0$, exponential service time μ_i at the same node, and first in first out service. After a customer finishes the service at node i, he departs to node j with probability p_{ij} and with probability p_{i0} he leaves the network. Obviously $p_{i0} = 1 - \sum_{i=0}^{N} p_{ij}$.

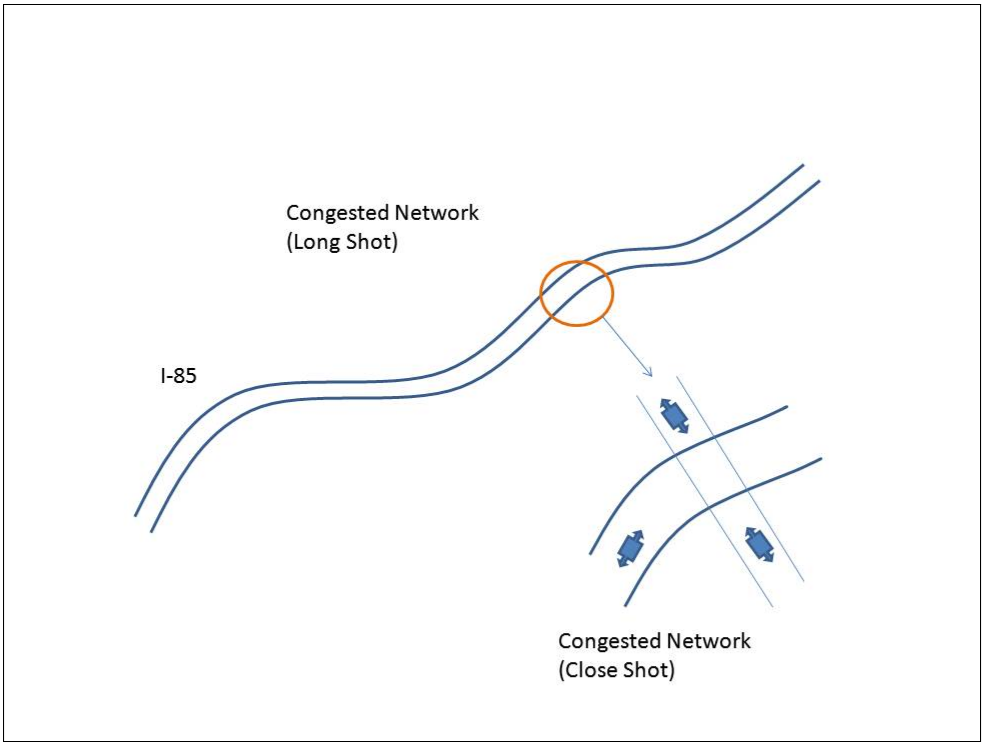

Figure 1.2: A Congested Network

In a closed Jackson network there is no exit or entering to or from outside. In the next chapter,the existence of a random field on a particular closed Jackson network and the convergence of measures in that network is shown. In addition to that the limit process is obtained. Also the Markov property of the limit process is shown.

Gordon and Newell [5] demonstrated a similar result holds for close queueing networks. The networks constructed in this paper are closed networks. There is now a voluminous literature devoted to queueing networks and in particular networks having product form limiting distributions [12]. Serfozo [21] provides an excellent overview of these results. Queueing networks have been extensively applied to model communication systems, population systems and traffic systems, as well as, many other systems. In these examples, one certainly can imagine the number of queues extremely large. This suggests the possibility of constructing a “limiting “ network which is infinite dimensional.

In this paper, we develop a Markov process which is the limit of a sequence of Jackson queueing networks. The p-th network has J^p nodes so that as $p \to \infty$ the number of queues becomes uncountable. The limit process process then has an infinite dimensional state space which

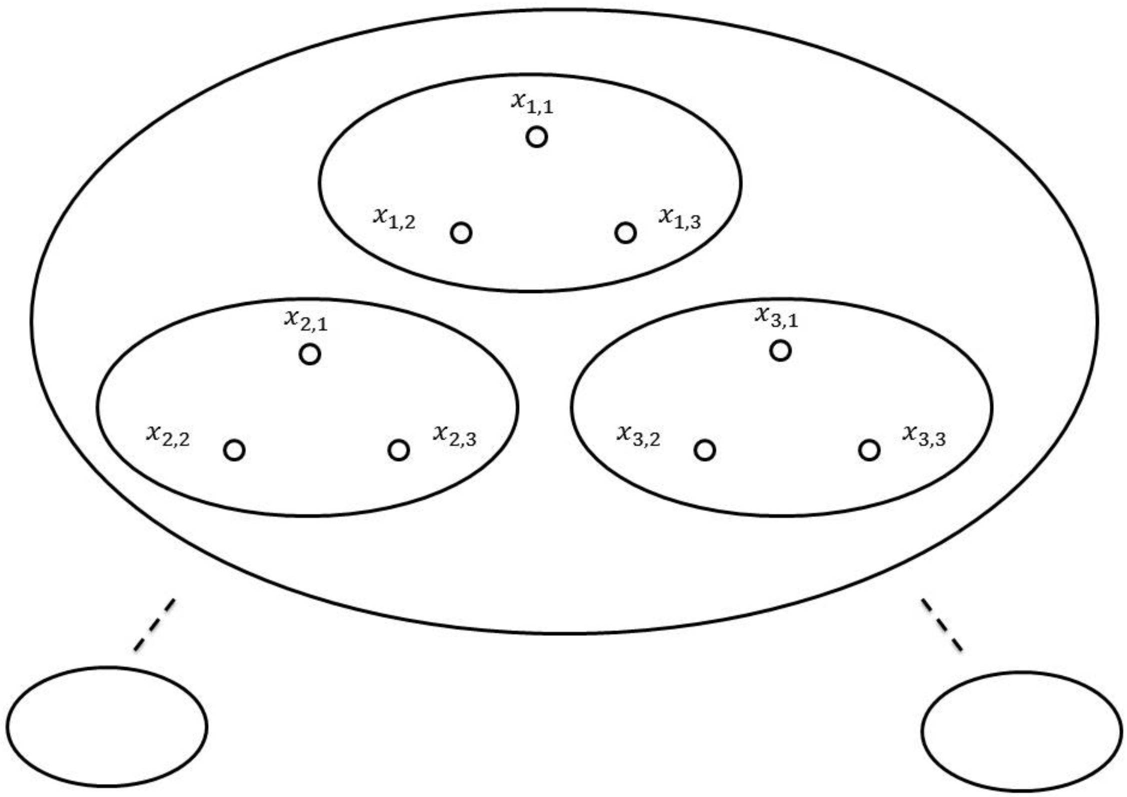

Figure 1.3: A Stochastic Fractal Network

is a complete separable metric space. We construct the networks so that the $p+q$-th network can be lumped by combining nodes into the pth network. In this way, the limiting process can be thought of as being self-similar. Additionally we give the network a geometry in that as $p \to \infty$ the locations of nodes converge to a fractal. Since fractals are self-similar sets, the limiting process has an additional self-similar property. When the initial distribution is the Gaussian limit of the normalized stationary queue length distributions of the Jackson networks, the limiting process is a stationary Gaussian Markov process.

In Section 2.2, we introduce the Jackson networks that we use in the construction of the limit process. We start with a closed network containing J infinite server queues having unit exponential service times, MJ customers and a routing matrix P. The joint queue length process for the network is known to be a J-dimensional finite state Markov process and its limiting distribution is easy to deduce. The pth network is much like the original network, only that there are J^p queues, MJ^p customers and the routing matrix is the Kronecker product of P with itself p times. Thus the joint queue length process will be a J^p-dimensional finite state Markov process. Moreover, and this key to developing the results in Sections 2.4 ,2.5 and 2.6, we can combine queues in a way so that the $p+q$th network behaves like the pth network only with MJ^{p+q} customers, see Lemma 2.2.4.

The state space E for the limit process will be a product space having the product topology and the σ-algebra, $\mathcal{E}$ will be the Borel subsets of E. From the sets $K_{p,\mathbf{i}}$, $p = 1, 2, \ldots$ and $\mathbf{i} \in \{1, \ldots, J\}^p$ we build an orthonormal basis for $L^2(K) = L^2(K, \mathcal{B}(K), H)$. The product space E is then constructed from this orthonormal set in a natural way. Let E^* denote the dual space of E. The space is constructed so that each $\Lambda \in E^*$ belongs to $L^2(K)$ and to Λ we associate a mean 0 variance $||\Lambda||_2^2$ random variable. Each member of the basis will belong to E^* and thus is a mean 0 variance 1 random variable on E. These random variable induce a product product measure γ on E. The transition function for the limiting Markov process, constructed in Section 6, will be a strongly continuous contraction semigroup on $L^2(E, \mathcal{E}, \gamma)$.

The measure γ also appears as a limit distribution. In Section 2.3, we show that γ is the limit distribution of the a normalized version of the stationary joint queue length distributions π^p as $p \to \infty$. So to positive integer p and each $\mathbf{i} \in \{1, \ldots, J\}^p$, is a queue and its normalized stationary queue length distribution is normal with mean 0 variance $1/J^p$. But $1_{K_{p,\mathbf{i}}} \in E^*$ and its distribution is also normal with mean 0 and variance $1/J^p$.

In Section 2.4, we construct for positive integers p and q, processes $\mathbf{X}^{(p,q)}$ that are normalized queue length processes of a network with J^p queues and MJ^p customers. As $q \to \infty$,we show that $\mathbf{X}^{(p,q)}$ converges in distribution to a linear diffusion process $\mathbf{X}^{(p)}$. The drift term of the generator generator $L^{(p)}$ for $\mathbf{X}^{(p)}$ is the limit of the expected infinitesimal change of $\mathbf{X}^{(p,q)}$ and the diffusion term is the variance of the infinitesimal change of $\mathbf{X}^{(p,q)}$. Both the expected value and variance of the infinitesimal change in $\mathbf{X}^{(p,q)}$ can be calculated from the generators of the queue length process.

In Section 2.5, we construct the limiting Markov process $\mathbf{Y}$. We would like this process to be the limit of the $\mathbf{X}^{(p)}$ constructed in Section 2.4. As mentioned earlier this process will have state space E. The components of E are represented in terms its orthonormal basis and not $1_{K_{p,\mathbf{i}}}$. The processes $\mathbf{X}^{(p)}$ are presented in terms of the $1_{K_{p,\mathbf{i}}}$. By constructing change of bases of the we show that $\mathbf{Y}$ is the limit in probability of transformed versions of the $\mathbf{X}^{(p)}$. The convergence to the limit process and the space of the limit process are discussed in 2.6 and 2.7.

Chapter 2

Results

THE concept of self similar sets is involved with self similar functions. Hutchinson [7] discusses the self similarity subject. We follow the same here.

2.1 Fractals and Self Similarity

Let K be a compact set in $\mathbb{R}^d$. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a *similitude* if for some fixed $r > 0$, $|S(x) - S(y)| = r|x - y|$ for all x, y in $\mathbb{R}^d$. From [7], it is known that $S : \mathbb{R}^d \to \mathbb{R}^d$ is a similitude if and only if $S = \mu_r \circ \tau_b \circ \mathbf{O}$ where

(i) $\mu_r : \mathbb{R}^d \to \mathbb{R}^d$ is the homothety $\mu_r(x) = rx$, $r > 0$.

(ii) $\tau_b : \mathbb{R}^d \to \mathbb{R}^d$ is the translation $\tau_b(x) = x - b$.

(iii) $\mathbf{O} : \mathbb{R}^d \to \mathbb{R}^d$ is an orthonormal transformation.

Let J be a positive integer and $S_1, \ldots, S_J$ be similitudes where for $j = 1, \ldots, J$, $|S_j(x) - S_j(y)| = r_j|x - y|$ with $0 < r_j < 1$. For $j_1, \ldots, j_p$ in $\{1, \ldots, J\}$, define $S_{j_1,\cdots,j_p} = S_{j_1} \circ \cdots \circ S_{j_p}$. For $A \subset \mathbb{R}^d$, set $\mathcal{S}(A) = \cup_{j=1}^J S_j(A)$, where $S_j(A) = \{S_j(x); x \in A\}$. Let $\mathcal{S}^0(A) = A$, $\mathcal{S}^1(A) = \mathcal{S}(A)$ and, for $p = 2, 3, \ldots$, $\mathcal{S}^p(A) = \mathcal{S}(\mathcal{S}^{p-1}(A))$. A compact set $K \subset \mathbb{R}^d$ is *self similar* with respect to $\mathcal{S}$ if

(i) K is invariant with respect to $\mathcal{S}$, that is, $\mathcal{S}(K) = K$.

(ii) $\mathcal{H}^k(K) > 0$, $\mathcal{H}^k(K_i \cap K_j) = 0$ if $i \neq j$, where $k = \dim K$, $K_i = \mathcal{S}_i(K)$ and $\mathcal{H}^k$ is the Hausdorff measure of dimension k.

We will assume additionally that $K_i \cap K_j = \emptyset$ when $i \neq j$. So if $i_1 \cdots i_p \neq j_1 \cdots j_p$, then $K_{i_1 \cdots i_p} \cap K_{j_1 \cdots j_p} = \emptyset$. Since the sets are compact and disjoint, $\inf\{|x-y|; x \in K_{i_1 \cdots i_p}, y \in K_{j_1, \cdots, j_p}\} > 0$. For any positive integer p, $K = \cup K_{j_1 \cdots j_p}$, where the union is over all $(j_1 \cdots, j_p) \in \{1, \ldots, J\}^p$. It follows that if $f : K \to \mathbb{R}$ is piecewise constant on each $K_{j_1 \cdots j_p}$, then it is a continuous function.

Let $C(K)$ be the Banach space of continuous functions on K where for $x \in C(K)$,$|x| = \sup_{t \in K} |x(t)|$. For $x \in C(K)$, the modulus of continuity of x is defined for $\delta > 0$ by

$$w_x(\delta) = w(x, \delta) = \sup_{|s-t|<\delta} |x(s) - x(t).$$

For $x \in C(K)$ and positive integers p, let

$$w_x(p) = \max \sup\{|x(s) - x(t)|; s, t \in K_{j_1 \cdots j_p}\},$$

where the maximum is taken over all $(j_1, \ldots, j_p) \in \{1, \ldots, J\}^p$.

For $A \subset K$, the diameter of A is given by

$$\mathrm{diam}(A) = \sup\{|s-t|; s, t \in A\}.$$

For subsets A and B of K, we define

$$\mathrm{dist}(A, B) = \inf\{|s-t|; s \in A, t \in B\}.$$

Set $r = \max\{r_1, \ldots, r_J\}$. Then $r < 1$ and for all $(j_1, \ldots, j_p) \in \{1, \ldots, J\}^p$, $\mathrm{diam}(K_{j_1, \ldots, j_p}) < r^p$.

Lemma 2.1.1 *Let $x \in C(K)$. For every $\delta > 0$, there exists a positive integer p_0 such that for $p \geq p_0$, $w_x(p) \leq w_x(\delta)$. Conversely for every positive integer p there is a $\delta_0 > 0$ such that for all $\delta \leq \delta_0$, $w_x(\delta) \leq w_x(p)$.*

Proof. Let $\delta > 0$. Choose p_0 so that $r^{p_0} < \delta$. Then for all $p \geq p_0$ and $(j_1, \ldots, j_p) \in \{1, \ldots, J\}^p$, $\mathrm{diam}(K_{j_1, \ldots, j_p}) < \delta$. Since $s, t \in K_{j_1, \ldots, j_p}$ imply $|s-t| < r^p < \delta$,

$$\sup\{|x(s) - x(t)|; s, t \in K_{j_1, \ldots, j_p}\} < w_x(\delta).$$

It now follows that for $p \geq p_0$, $w_x(p) < w_x(\delta)$.

Conversely, let p be a positive integer. For distinct $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_p)$ in $\{1, \ldots, J\}^p$, $\text{dist}(K_{i_1,\ldots,i_p}, K_{j_1,\ldots,j_p}) > 0$. Let

$$2\delta_0 = \min\{\text{dist}(K_{i_1,\ldots,i_p}, K_{j_1,\ldots,j_p}); (i_1, \ldots, i_p) \neq (j_1, \ldots, j_p) \in \{1, \ldots, J\}^p\}.$$

Then for $\delta \leq \delta_0$ and for $s, t \in K$ such that $|s - t| < \delta$, s and t must belong to the same $K_{j_1,\ldots,j_p}$. Hence

$$|x(s) - x(t)| < w_x(p).$$

Thus, $w_x(\delta) < w_x(p)$ and the proof is complete.

A subset $\mathcal{A}$ of $C(K)$ is equicontinuous if

$$\lim_{\delta \to 0} \sup_{x \in \mathcal{A}} w_x(\delta) = 0.$$

From Lemma 2.1, $\mathcal{A}$ is equicontinuous if and only if

$$\lim_{p \to \infty} \sup_{x \in \mathcal{A}} w_x(p) = 0.$$

A subset $\mathcal{A}$ of $C(K)$ is said to be pointwise bounded if for each $t \in K$, $\{x(t); x \in \mathcal{A}\}$ is a bounded subset of $\mathbb{R}$.

Lemma 2.1.2 *Let $\mathcal{A} \subset C(K)$. Suppose there exists a positive integer p so that*

$$\sup_{x \in \mathcal{A}} w_x(p) < 1.$$

Also suppose that for each $(j_1, \ldots, j_p) \in \{1, \ldots, J\}^p$, there is a $t_{j_1,\ldots,j_p} \in K_{j_1,\ldots,j_p}$ for which $\{x(t_{j_1,\ldots,j_p}); x \in \mathcal{A}\}$ is bounded. Then $\mathcal{A}$ is pointwise bounded.

Proof. Assume the conditions stated in the lemma and let $t \in K$. Then there is a unique $(j_1, \ldots, j_p)$ for which $t \in K_{j_1,\ldots,j_p}$. For each $x \in \mathcal{A}$, $|x(t)| < |x(t_{j_1,\ldots,j_p}| + 1$. It follows that

$$\sup\{|x(t)|; x \in \mathcal{A}\} < \sup\{|x(t_{j_1,\ldots,j_p})|; x \in \mathcal{A}\} + 1.$$

Since $\sup |x(t_{j_1,\ldots,j_p})|; x \in \mathcal{A}\}$ is bounded so is $\{|x(t)|; x \in \mathcal{A}\}$. That is, $\mathcal{A}$ is pointwise bounded and the proof is complete.

A subset $\mathcal{A}$ of $C(K)$ is *relatively compact* if it pointwise bounded and equicontinuous. Lemmas 2.1 and 2.2 demonstrate that we can formulate conditions for $\mathcal{A}$ to be relatively compact in terms of the $w_x(p)$, $x \in \mathcal{A}$ and p a positive integer.

Lemma 2.1.3 *A subset $\mathcal{A}$ of $C(K)$ is relatively compact*

(i) if

$$\lim_{p\to\infty} \sup_{x\in\mathcal{A}} w_x(p) = 0$$

and

(ii) if for any positive integer p for which $\sup_{x\in\mathcal{A}} w_x(p) < 1$ and for each $(j_1, \ldots, j_p) \in \{1, \ldots, J\}^p$, there is a $t_{j_1,\ldots,j_p} \in K_{j_1,\ldots,j_p}$ such that $\{x(t_{j_1,\ldots,j_p}); x \in \mathcal{A}\}$ is bounded.

Proof. By Lemma 2.1, (i) implies that $\mathcal{A}$ is equicontinuous. Condition (i) also implies that there is a positive integer p for which $\sup_{x\in\mathcal{A}} w_x(p) < 1$. Hence (ii) together with Lemma 2.2 imply that $\mathcal{A}$ is pointwise bounded. Thus $\mathcal{A}$ is relatively compact and the proof is complete.

2.2 Jackson Networks

Consider a closed Jackson network of J infinite server nodes with MJ customers. A customer departing node i goes next to node j with probability p_{ij}. The $J \times J$ matrix P whose i, j-th entry is p_{ij} is called the routing matrix. P is a stochastic matrix, which we assume is irreducible, aperiodic and doubly stochastic. We assume the service times are exponentially distributed with mean 1.

For $j = 1, \ldots, J$ and $t \geq 0$, let $Q_j(t)$ be the number of customers present at node j at time t. Let $\mathbf{Q}(t) = (Q_1(t), \ldots, Q_J(t))$ and $\mathbf{Q} = \{\mathbf{Q}(t); t \geq 0\}$. Then $\mathbf{Q}$ is a Markov process. The state space is $E_Q = \{(n_1, \ldots, n_J) \in \mathbb{N}_0^J; n_1 + \cdots n_J = MJ\}$ For i and j in $\{1, \ldots, J\}$ define operators $T_{i,j} : E_Q/\{\mathbf{n} \in E_Q; n_i > 0\} \to E_Q$ by $T_{ij}(\mathbf{n}) = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ where $\mathbf{e}_i$ is a vector whose ith element is one and all other elements are equal to 0. The generator A for $\mathbf{Q}$ is given by

$$A_{\mathbf{nm}} = \begin{cases} -MJ & \text{if } \mathbf{m} = \mathbf{n} \\ n_i P_{ij} & \text{if } n_i > 0 \text{and } \mathbf{m} = T_{ij}(\mathbf{n}) \end{cases} .$$

Let $\tilde{P}$ be the matrix defined by

$$\tilde{P}_{\mathbf{nm}} = \begin{cases} 0 & \text{if } \mathbf{m} = \mathbf{n} \\ A_{\mathbf{nm}}/(MJ) & \text{otherwise} \end{cases} .$$

The transition $P(t)$ function for $\mathbf{Q}$ can be written now as

$$P_{\mathbf{nm}}(t) = \sum_{k=0}^{\infty} \frac{(MJ)^k}{k!} \tilde{P}^k_{\mathbf{nm}} e^{(MJ)}.$$

The limiting probability that a customer is at node j is $1/J$. The limiting probability that there are $n_1, ...n_J$ customers at nodes 1,...,J is

$$\pi(n_1, \ldots, n_J) = \frac{(MJ)!}{n_1! \cdots n_J!} J^{-(MJ)},$$

where $n_1 + \cdots + n_J = MJ$.

We now turn to extending the network to one having J^p infinite server nodes with MJ^p customers. The routing matrix is $P^{\otimes p}$ the Kronecker product of P with itself p times. The service times are independent and exponentially distributed with mean 1. Before continuing we state some properties of the Kronecker product.

Lemma 2.2.1 *For positive integers p, q and n,*

$$P^{\otimes(p+q)} = P^{\otimes p} \otimes P^{\otimes q},$$

$$(P^{\otimes p} \otimes P^{\otimes q})^n = (P^{\otimes p})^n (P^{\otimes q})^n.$$

Lemma 2.2.2 *For $p = 1, 2, \ldots$, $P^{\otimes p}$ is irreducible, aperiodic and doubly stochastic.*

Proof. The proof that $P^{\otimes(p+q)}$ is doubly stochastic is a straightforward calculation. For fixed $\mathbf{i} = (i_1, \ldots, i_p)$

$$\sum_{\mathbf{j}} P^{\otimes}_{\mathbf{ij}} = \sum_{j_1,\ldots,j_p} P_{i_1,j_1} \cdots P_{i_p j_p} = 1$$

and for fixed $\mathbf{j} = (j_1, \ldots, j_p)$

$$\sum_{\mathbf{i}} P^{\otimes p}_{\mathbf{ij}} = \sum_{i_1,\ldots,i_p} P_{i_1,j_1} \cdots P_{i_p j_p} = 1.$$

To show that $P^{\otimes(p+q)}$ is irreducible and aperiodic it suffices to show that there is an n so that for all states $\mathbf{i}$ and $\mathbf{j}$, $(P^{\otimes p})^{(n)}_{\mathbf{ij}} > 0$. By assumption a positive integer n exists so that for all i and j in $\{1, \ldots, J\}$, $P^{(n)}_{ij} > 0$. The result now follows from Lemma 3.1 since for all $\mathbf{i} = (i_1, \ldots, i_p$ and $\mathbf{j} = (j_1, \ldots, j_p)$

$$(P^{\otimes p})^{(n)}_{\mathbf{ij}} = P^{(n)}_{i_1 j_1} \cdots P^{(n)}_{i_p j_p} > 0.$$

Here we recall a notion from Markov chains known as lumpability. Let P^1 and P^2 be stochastic matrices with state spaces E_1 and E_2 respectively. Let $f : E_1 \to E_2$ be onto. We will say that P^1 is lumpable to P^2 under the mapping f if for each i_2, $i_1 \in f^{-1}(\{i_2\})$ and j_2 in E_2

$$\sum_{j_1 \in f^{-1}(\{j_2\})} P^1_{i_1 j_1} = P^2_{i_2 j_2}.$$

Let p and q be positive integers. For $\mathbf{i} \in \{1, \ldots J\}^{p+q}$, write $\mathbf{i} = (\mathbf{i}_p \mathbf{i}_q)$ where $\mathbf{i}_p = (i_1, \ldots, i_p)$ and $\mathbf{i}_q = (i_{p+1}, \ldots, \imath_{p+q})$. Define $f : \{1, \ldots, J\}^{p+q} \to \{1, \ldots, J\}^p$ by $f(\mathbf{i}) = \mathbf{i}_p$.

Lemma 2.2.3 *Let p and q be positive integers. The matrix $P^{\otimes(p+q)}$ is lumpable to the matrix $P^{\otimes p}$ under f..*

Proof. For $\mathbf{i} = (i_1, \ldots, i_{p+q}) \in \{1, \ldots, J\}^{(p+q)}$, we write $\mathbf{i} = (\mathbf{i}_p, \mathbf{i}_q)$ were $\mathbf{i}_p = (i_1, \ldots, i_p) \in \{1, \ldots, J\}^p$ and $\mathbf{i}_q = (i_{p+1}, \ldots, i_{p+q} \in \{1, \ldots, J\}^q$. Then for fixed $\mathbf{i} \in \{1, \ldots, J\}^{p+q}$

$$\sum_{\mathbf{j}_q} P^{\otimes(p+q)}_{\mathbf{ij}} = P^{\otimes p}_{\mathbf{i}_p \mathbf{j}_p} \sum_{\mathbf{j}_q} P^{\otimes q}_{\mathbf{i}_q \mathbf{j}_q} = P^{\otimes p}_{\mathbf{i}_p \mathbf{j}_p}.$$

This completes the proof of the lemma.

Let p be a positive integer. For $\mathbf{j} \in \{1, \ldots, J\}^p$ and $t \geq 0$ let $Q^{(p)}_{\mathbf{j}}(t)$ be the number of customers at node $\mathbf{j}$ at time t. Let $\mathbf{Q}^{(p)}(t) = (Q^{(p)}_{\mathbf{j}}(t); \mathbf{j} \in \{1, \ldots, J\}^P)$ and $\mathbf{Q}^{(p)} = \{\mathbf{Q}^{(p)}(t); t \geq 0\}$. As before $\mathbf{Q}^{(p)}$ is a Markov process. The state space is the set $E_{Q^p} = \{\mathbf{n} \in \mathbb{N}_0^{J^p} : \sum_{\mathbf{j} \in \{1,\ldots,J\}^p} n_{\mathbf{j}} =$

MJ^p }. For $\mathbf{i}$ and $\mathbf{j}$ in $\{1,\ldots,J\}^p$, define $T^p_{\mathbf{ij}} : E_{Q^p}/\{\mathbf{n} \in E_{Q^p}; n_{\mathbf{i}} > 0\} \to E_{Q^p}$ by

$$T^p_{\mathbf{ij}}(\mathbf{n}) = \mathbf{n} - \mathbf{e_i} + \mathbf{e_j},$$

where as before $\mathbf{e_i}$ is a vectors of zeros expect for a one in the $\mathbf{i}$th place. The generator A^p for $\mathbf{Q}^{(p)}$ is given by

$$A^p_{\mathbf{nm}} = \begin{cases} -MJ^p & \text{if } \mathbf{m} = \mathbf{n} \\ n_{\mathbf{i}} P^{\otimes p}_{\mathbf{ij}} & \text{if } \mathbf{m} = T^p_{\mathbf{ij}}(\mathbf{n}) \end{cases}.$$

Let $\tilde{P}^{(p)}$ be the matrix defined by

$$\tilde{P}^{(p)}_{\mathbf{nm}} = \begin{cases} 0 & \text{if } \mathbf{m} = \mathbf{n} \\ A_{\mathbf{nm}}/(MJ^p) & \text{otherwise} \end{cases}.$$

The transition $P^{(p)}(t)$ function for $\mathbf{Q}^{(p)}$ can be written now as

$$P^{(p)}_{\mathbf{nm}}(t) = \sum_{k=0}^{\infty} \frac{(MJ^p)^k}{k!} (\tilde{P}^{(p)}_{\mathbf{nm}})^k e^{-MJ^p}.$$

Let p and q be positive integers. For $t \geq 0$ and $\mathbf{j}_p \in \{1,\ldots,J\}^p$, set

$$Q^{(p,q)}_{\mathbf{j}_p}(t) = \sum_{\mathbf{j}_q \in \{1,\ldots,J\}^q} Q^{(p+q)}_{(\mathbf{j}_p\mathbf{j}_q)}(t),$$

$\mathbf{Q}^{(p,q)}_{\mathbf{j}}(t) = (Q^{(p,q)}_{\mathbf{j}_p}(t); \mathbf{j}_p \in \{1,\ldots,J^p\})$ and $\mathbf{Q}^{(p,q)} = \{\mathbf{Q}^{(p,q)}(t); t \geq 0\}$.

Lemma 2.2.4 *For every positive integer p, $\mathbf{Q}^{(p,q)}$ is a Markov process having state space $E_{Q^{p+q}} = \{\mathbf{n} \in \mathbb{N}_0^{J^p} : \sum_{\mathbf{j} \in \{1,\ldots,J\}^p} n_{\mathbf{j}} = MJ^{p+q}\}$. with generator $A^{(p,q)}$ given by*

$$A^{(p,q)}_{\mathbf{nm}} = \begin{cases} -MJ^{p+q} & \textit{if } \mathbf{m} = \mathbf{n} \\ \sum_{\mathbf{i}_q \in \{1,\ldots,J\}^q} n_{(\mathbf{i}_p\mathbf{i}_q)} P^{\otimes p}_{\mathbf{i}_p\mathbf{j}_p} & \textit{if } \mathbf{m} = T^p_{\mathbf{ij}}(\mathbf{n}) \textit{for some } \mathbf{i} \textit{and } \mathbf{j} \textit{in } \{1,\ldots,J\}^p \end{cases}.$$

Proof. The system described in the lemma is a closed Jackson network in which there are J^p infinite server queues, MJ^{p+q} customers, routing matrix $P^{\otimes p}$ and the service times are independent exponentially distributed with mean one. We show that $\mathbf{Q}^{(p,q)}$ is the queue length process corresponding to this system. Suppose the current state of $\mathbf{Q}^{p+q}$ is $\mathbf{n} = (\mathbf{n}_{(\mathbf{i}_p\mathbf{i}_q)}; (\mathbf{i}_p\mathbf{i}_q) \in \{1,\ldots,J\}^{p+q})$. Then the current state of $\mathbf{Q}^{(p,q)}$ is $\hat{\mathbf{n}}$ where $\hat{n}_{\mathbf{i}_p} = \sum_{\mathbf{i}_q} n_{(\mathbf{i}_p\mathbf{i}_q)}$. The probability that a randomly selected customer is from node $\mathbf{i}_p$ in the hat system is $\hat{n}_{\mathbf{i}_p}/(MJ^{p+q})$. The probability that customer now chooses node $\mathbf{j}_p$ to go next is

$$\sum_{\mathbf{i}_q \in \{1,\ldots,J\}^q} \frac{n_{(\mathbf{i}_p\mathbf{i}_q)}}{\hat{n}_{\mathbf{i}_p}} \sum_{\mathbf{j}^q \in \{1,\ldots,J\}^q} P^{\otimes(p+q)}_{(\mathbf{i}_p\mathbf{i}_q)(\mathbf{j}_p\mathbf{j}^q)}.$$

By the definition of $\hat{n}_{\mathbf{i}_p}$ and Lemma 3.1 we get that sum is equal to $P^{\otimes p}_{\mathbf{i}_p\mathbf{j}_p}$. Thus the probability the hat system makes a jump from $\hat{\mathbf{n}}$ to $T^p_{\mathbf{i}_p\mathbf{j}_p}(\hat{\mathbf{n}})$ is just

$$\frac{\hat{n}_{\mathbf{i}_p}}{MJ^{(p+q)}} P^{\otimes p}_{\mathbf{i}_p\mathbf{j}_p}.$$

Since rate a which any jump in made is MJ^{p+q}, multiplying the previous expression by MJ^{p+q} gives the generator and completes the proof of the lemma.

2.3 Convergence

Let p be a positive integer and lexicographically order the set $\mathcal{J}_p$. Let $\{X_{p,\mathbf{i}}; \mathbf{i} \in \mathcal{J}_p\}$ be independent and identically distributed random variables having expected value 0 and variance $1/J^p$. For $\mathbf{i} \in \mathcal{J}_p$, set

$$S_{p,\mathbf{i}} = \sum_{\mathbf{j} \leq \mathbf{i}} X_{p,\mathbf{i}}.$$

Each $x \in K$ belongs to a unique $K_{p,\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{J}_p$. For $\mathbf{i} \in \mathcal{J}_p$ and $x \in K_{\mathbf{i}}$, set

$$Y_p(x) = S_{p,\mathbf{i}}.$$

By assumption the sets $K_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{J}_p$ are disjoint compact sets of K and hence of $\mathbb{R}^n$. Thus, there are disjoint open sets $O_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{J}_p$, in $\mathbb{R}^n$, such that $K_{\mathbf{i}} \subset O_{\mathbf{i}}$. Hence there is a $\delta > 0$, such that for x and y in K with $d(x, y) < \delta$ then x and y belong to the same $K_{\mathbf{i}}$ and hence $Y_p(x) = Y_p(y)$. So for any $\epsilon > 0$ there is a $\delta > 0$ so that if $d(x, y) < \delta$, then $|Y_p(x) - Y_p(y)| < \epsilon$. Hence, the random field $\{Y_p(x) : x \in K\}$ is continuous.

Theorem 2.3.1 *There exists a continuous random field $\{Y(x); x \in K\}$ such that $Y_p \Rightarrow Y$ and for each positive integer k, $x_1 < \cdots < x_k$ (ordered lexicographically) $Y(x_1), Y(x_2) - Y(x_1), \ldots, Y(x_k) -$*

$Y(x_{k-1})$ *are independent random variables with* $Y(x_j) - Y(x_{j-1}) \stackrel{d}{=} N(0, H^d(\mathcal{A}))$, *where* $\mathcal{A} = \{x \in K; x_{j-1} < x \leq x_j\}$.

The proof of the theorem follows from the three lemmas below. The first shows the convergence of the finite dimensional distributions of $\{Y_p(x); x \in K\}$. The second gives conditions for the sequence $\{Y_p(x); x \in K\}$ to be tight and the third shows that the sequence satisfies these conditions. Before proceeding to the lemmas we calculate the variances of the $S_{p,\mathbf{i}}$ for p a positive integer and $\mathbf{i} \in \mathcal{J}_p$.

Let $\mathbf{i} = (i_1, \ldots, i_p) \in \mathcal{J}_p$. Then

$$|\{\mathbf{j} \in \mathcal{J}_p | \mathbf{j} \leq \mathbf{i}\}| = (i_1 - 1)J^{p-1} + \cdots + (i_{p-1} - 1)J + i_p.$$

Then $S_{p,\mathbf{i}}$ has expectation 0 and variance

$$\sigma^2_{p\mathbf{i}} = ((i_1 - 1)J^{p-1} + \cdots + (i_{p-1} - 1)J + i_p)/J^p.$$

Now let $\mathbf{i}_p = (i_1, \ldots, i_p) \in \mathcal{J}_p$ and $\mathbf{q} = (j_1, \ldots, j_q) \in \mathcal{J}_q$ and $\mathbf{i}_{p+q} = (\mathbf{i}_p, \mathbf{j}_q) = (i_1, \ldots, i_p, j_1, \ldots, j_q) \in \mathcal{J}_{p+q}$. Then

$$|\{\mathbf{j}_{p+q} \in \mathcal{J}_{p+q} | \mathbf{j}_{p+q} \leq \mathbf{i}_{p+q}\}| = J^q(|\{\mathbf{h} \in \mathcal{J}_p | \mathbf{h} \leq \mathbf{i}_p\}| - 1) + |\{\mathbf{h} \in \mathcal{J}_q | \mathbf{h} \leq \mathbf{j}_q\}|.$$

Thus,

$$\begin{aligned}\sigma^2_{p+q,\mathbf{i}_{p+q}} &= \sigma^2_{p,\mathbf{i}_p} - J^{-p} + J^{-p-q}((j_1 - 1)J^{q-1} + \cdots + (j_{q-1} - 1)J + j_q) \\ &\leq \sigma^2_{p,\mathbf{i}_p} \qquad (2.1)\end{aligned}$$

Lemma 2.3.2 *For each integer* n, *let* $x_1 < \ldots x_n$ *be in* K. *Then* $(Y_p(x_1), \ldots, Y_p(x_n) \Rightarrow (Y(x_1), \ldots, Y(x_n))$ *as* $p \to \infty$ *where* $Y(x_1), Y)x_2) - Y(x_1), \ldots, Y(x_n) - Y(x_{n-1})$ *are independent Gaussian distributed having expectation 0 and the variance of* $Y(x_j) - Y(x_{j-1}) = \sigma^2(x_l) - \sigma^2(x_{j-1})$.

Proof. Let $\alpha \in C(J)$ and let $x = k_\alpha$. If $\alpha = (1, 1, \ldots)$, then for every positive integer p, $Y_p(x) = X_{p,(1,1,\ldots)}$ which has expectation 0 and variance J^{-p}. Since $J^{-p} \to 0$ as $p \to \infty$, $Y_p(x) \Rightarrow 0$. If $\alpha \neq (1, 1, \ldots)$ and $x = k_\alpha$, $|\{y \in K | y \leq x\}| = \infty$. For p a positive integer, let $\alpha_p = (\alpha_1, \ldots, \alpha_p)$.

Then $|\{\mathbf{j} \in \mathcal{J} | \mathbf{j} \leq \alpha_p\}| \to \infty$ as $p \to \infty$. Then $Y_p(x) = S_{p,\alpha_p}$ has expectation 0 and variance σ^2_{p,α_p}. An earlier calculation shows that the sequence $\{\sigma^2_{p,\alpha_p}\}$ is Cauchy and hence converges. Let $\sigma(x)$ denote the limit. The Lindeberg-Feller Theorem shows $Y_p(x) \Rightarrow \mathcal{N}(0, \sigma^2(x))$.

More generally, for a positive integer n, let $\alpha^1 < \cdots < \alpha^n$ be in $C(J)$ and for $j = 1, \ldots, n$, set $x_j = k_{\alpha^j}$. For each positive integer p and each $j = 1, \ldots, n$, set $\alpha^j_p = (\alpha^j_1, \ldots, \alpha^j_p)$. Then $Y_p(x_j) = S_{p,\alpha^j_p}$. Moreover the random variables $Y_p(x_1), Y_p(x_2) - Y_p(x_1), \ldots, Y_p(x_n) - Y_p(x_{n-1})$ are independent with expectation variance and the variance of $Y_p(x_j) - Y_p(x_{j-1}) = \sigma^2_{\alpha^j_p} - \sigma^2_{\alpha^{j-1}_p}$ which converges to $\sigma^2(x_j) - \sigma^2(x_{j-1})$. Applying again the Lindeberg-Feller Theorem and continuous mapping theorem $Y_p(x_1), \ldots, Y_p(x_n)$ converges in distribution to random variables $Y(x_1), \ldots, Y(x_n)$ where $Y(x_1), Y)x_2) - Y(x_1), \ldots, Y(x_n) - Y(x_{n-1})$ are independent Gaussian distributed having expectation 0 and the variance of $Y(x_j) - Y(x_{j-1}) = \sigma^2(x_l) - \sigma^2(x_{j-1})$.

The next step is to show that the family of random fields $\{Y_p(x); x \in K\}_{p=1}^{\infty}$ is tight. Recall that each $\{Y_p(x); x \in K\}$ is tight in $C(K)$, $\{Y_p(x); x \in K\}_{p=1}^{\infty}$ is tight if for each $\epsilon > 0$ there is a compact subset F of $C(K)$ such that

$$\limsup_{p\to\infty} P\{Y_p \in F\} > 1 - \epsilon.$$

Lemma 2.3.3 *For each $\epsilon > 0$, suppose that we can find for each positive integer k a positive integer q_k with $q_1 < q_2 < \cdots$ and an $a > 0$ so that*

$$(i) \qquad \limsup_{p\to\infty} P\{|Y_p(x_{\mathbf{i}})| > a \textit{for some } \mathbf{i} \in \mathcal{J}_{q_1}\} < \epsilon/2.$$

and

$$(ii) \qquad \limsup_{p\to\infty} P\{\max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y\in K_{\mathbf{i}}} |Y_p(x) - Y_p(y)| \geq k^{-1}\} < \epsilon/2^{k+1}.$$

Then $\{Y_p(x); x \in K\}_{p=1}^{\infty}$ is tight.

Proof. Let $\epsilon > 0$ and for each positive integer k choose a positive integer q_k so that $q_1 < q_2 < \ldots$ and an $a > 0$ so that both (1) and (2) hold. Choose a positive integer p_0 so that for all $p \geq p_0$,

$$P\{|Y_p(x_{\mathbf{i}})| > a\text{for some } \mathbf{i} \in \mathcal{J}_{q_1}\} \quad < \quad \epsilon/2. \tag{2.2}$$

and

$$P\{\max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y\in K_{\mathbf{i}}} |Y_p(x) - Y_p(y)| \geq k^{-1}\} \quad < \quad \epsilon/2^{k+1}. \tag{2.3}$$

Let

$$\begin{aligned} F &= \{Y \in C(K) : |Y(x_{\mathbf{i}})| \leq a \text{for each } \mathbf{i} \in \mathcal{J}_{q_1}\} \\ &\cap \cap_{k=1}^{\infty} \{Y \in C(K) : \max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y\in K_{\mathbf{i}}} |Y(x) - Y(y)| < k^{-1}\}. \end{aligned}$$

Let $Y \in F$. For every $x \in K$ there is an $\mathbf{i} \in \mathcal{J}_{q_1}$ such that $x \in K_{\mathbf{i}}$. It follows that

$$|Y(x)| \leq |Y(x) - Y(x_{\mathbf{i}})| + |Y(x_{\mathbf{i}})| < 1 + a.$$

Since Y was arbitrarily chosen

$$\sup_{Y\in F} |Y(x)| < 1 + a$$

and F is pointwise bounded. Again let $Y \in F$ and let $\epsilon > 0$. Let k be a positive integer such that $k^{-1} < \epsilon$. Then whenever x and y are in $K_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{J}_{q_k}$,

$$|Y(x) - Y(y)| < k^{-1} < \epsilon.$$

Since $Y \in F$ was arbitrarily chosen the inequality above holds for every $Y \in F$ and F is equicontinuous. Since F is both pointwise bounded and equicontinuous, by the Arzel-Ascoli Theorem, F is a compact subset of $C(K)$.

Let $p \geq p_0$. Suppose $Y_p \in F^c$. Then either $|Y_p(x_i)| > a$ for some $\mathbf{i} \in \mathcal{J}_{q_1}$ or for some positive integer k,

$$\max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y\in K_{\mathbf{i}}} |Y_p(x) - Y_p(y)| \geq k^{-1}.$$

Thus by (3) and (4)

$$\begin{aligned}
&P\{Y_p \in F^c\} \\
&= P(\{|Y_p(x_{\mathbf{i}})| > a \text{for some } \mathbf{i} \in \mathcal{J}_{q_1}\} \cup \cup_{k=1}^{\infty} \{\max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y \in K_{\mathbf{i}}} |Y_p(x) - Y_p(y)| \geq k^{-1}\}) \\
&\leq P\{|Y_p(x_{\mathbf{i}})| > a \text{for some } \mathbf{i} \in \mathcal{J}_{q_1}\} \\
&+ \sum_{k=1}^{\infty} P\{\max_{\mathbf{i}} \in \mathcal{J}_{q_k} \max_{x,y \in K_{\mathbf{i}}} |Y_p(x) - Y_p(y)| \geq k^{-1}\} \\
&< \epsilon.
\end{aligned}$$

Since $p \geq p_0$ is arbitrarily chosen,

$$\limsup_{p \to \infty} P\{Y_p \in F^c\} < \epsilon$$

and hence $\{Y_p(x); x \in K\}_{p=1}^{\infty}$ is tight.

Lemma 2.3.4 *The sequence $\{Y_p(x); x \in K\}_{p=1}^{\infty}$ is tight.*

Proof. Let $\epsilon > 0$ and let k be a positive integer. We find a positive integer q_k so that (ii) holds. For now, fix q_k and assume $p > q_k$. Suppose $x \in K_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{J}_{q_k}$. Then there is a $\mathbf{j}_x \in \mathcal{J}_{p-q_k}$ such that $x \in K_{\mathbf{i}\mathbf{j}_x}$. Thus,

$$Y_p(x) = \sum_{\mathbf{l} < \mathbf{i}\mathbf{1}_{p-q_k}} X_{p,\mathbf{l}} + \sum_{\mathbf{i}\mathbf{1}_{p-q_k} \leq \mathbf{l} \leq \mathbf{i}\mathbf{j}_x} X_{p,\mathbf{l}}.$$

Hence for x and y in $K_{\mathbf{i}}$,

$$\begin{aligned}
|Y_p(x) - Y_p(y)| &\leq |\sum_{\mathbf{i}\mathbf{1}_{p-q_k} \leq \mathbf{l} \leq \mathbf{i}\mathbf{j}_x} X_{p,\mathbf{l}} - \sum_{\mathbf{i}\mathbf{1}_{p-q_k} \leq \mathbf{l} \leq \mathbf{i}\mathbf{j}_y} X_{p,\mathbf{l}}| \\
&\leq 2 \max_{\mathbf{j} \in \mathcal{J}_{p-q_k}} |\sum_{\mathbf{i}\mathbf{1}_{p-q_k} \leq \mathbf{i}\mathbf{j}} X_{p,\mathbf{l}}| \\
&\stackrel{d}{=} 2 \max_{\mathbf{j} \in \mathcal{J}_{p-q_k}} |\sum_{\mathbf{j} \in \mathcal{J}_{p-q_k}} X_{p,\mathbf{1}_{q_k}\mathbf{j}}| \\
&= 2 \max_{\mathbf{j} \in \mathcal{J}_{p-q_k}} |S_{p,\mathbf{1}_{q_k}\mathbf{j}}|
\end{aligned}$$

From the inequality above, it follows that the probability in (ii) is no greater than

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{J}_{q_k}} P\{\max_{x,y\in K_{\mathbf{i}}}|Y_p(x)-Y_p(y)|>k^{-1}\} &\le \sum_{\mathbf{i}\in\mathcal{J}_{\amalg_{\|}}} P\{\max_{\mathbf{j}\in\mathcal{J}_{p-q_k}}|S_{p,\mathbf{1}_{q_k}\mathbf{j}}|>k^{-1}/2\}\\ &= J^{q_k}P\{\max_{\mathbf{j}\in\mathcal{J}_{p-q_k}}|S_{p,\mathbf{1}_{q_k}\mathbf{j}}|>k^{-1}/2\}\\ &\le J^{q_k}3\max_{\mathbf{j}\in\mathcal{J}_{p-q_k}}P\{|S_{p,\mathbf{1}_{q_k}\mathbf{j}}|>k^{-1}/6\}\end{aligned}$$

where the second inequality is Etemadi's inequality.

Let $\mathbf{j}=(j_1,\ldots,j_{p-q_k})\in\mathcal{J}_{p-q_k}$ be such that there is a positive integer $h<p-q_k$ for which $j_h\ge 2$. Then $S_{p,\mathbf{j}}$ is the sum of at least J^{p-q_k-h} terms. For $\mathbf{j}\le\mathbf{1}_{q_k+h}\mathbf{J}_{p-q_k-h}$, $S_{p,\mathbf{j}}$ is the sum of at most J^{p-q_k-h} of the $X_{p,\mathbf{i}}$'s. For such $\mathbf{j}$, Chebyshev's inequality implies that

$$3J^{q_k}P\{|S_{p,\mathbf{j}}|>k^{-1}/6\}\le 3J^{q_k}\frac{6k}{J^p}J^{p-q_k-h}=\frac{18k}{J^h}.$$

Choose h so that $18kJ^{-h}<\epsilon/2^{k+1}$

We now choose q_k so that $3J^{q_k}\exp(-J^{-q_k}k^{-1}/6)<\epsilon/2^{k+2}$. For each $\mathbf{j}=(j_1,\ldots,j_{p-q_k})\in\mathcal{J}_{p-q_k}$ such that $j_h\ge 2$, let

$$\tilde{S}_{p,\mathbf{1}_{q_k}\mathbf{j}}=\sqrt{\frac{J^p}{|\{\mathbf{i}\in\mathcal{J}_{p-q_k};\mathbf{i}\le\mathbf{j}\}|}}S_{p,\mathbf{1}_{q_k}\mathbf{j}}$$

which has expectation 0 and variance 1. Using lexicographical ordering for each positive integer p, we can order the set

$$\mathcal{A}=\{(p,\mathbf{1}_{q_k}\mathbf{j});p>q_k+h,\mathbf{1}_{q_k}\mathbf{j}\in\mathcal{J}_{p-q_k};j_h\ge 2\}.$$

Thus we can construct a triangular array of random variables where each $(p,\mathbf{1}_{q_k}\mathbf{j})\in\mathcal{A}$ is a row and the random variables $\sqrt{\frac{J^p}{|\{\mathbf{i}\in\mathcal{J}_{p-q_k};\mathbf{i}\le\mathbf{j}\}|}}X_{p,\mathbf{i}}$, $\mathbf{i}\le\mathbf{1}_{q_k}\mathbf{j}$ are elements of that row. The Lindeberg-Feller Theorem implies that

$$\tilde{S}_{p,\mathbf{1}_{q_k}\mathbf{j}}\Rightarrow\chi,$$

where χ has a standard normal distribution. Thus there is a p_0 such that for all $p\ge p_0$ for which

$$3J^{q_k}|P\{|\tilde{S}_{P,\mathbf{1}_{q_k}\mathbf{j}}|>\sqrt{J^{q_k}}k^{-1}/6\}-P\{|\chi|>\sqrt{J^{q_k}}k^{-1}/6\}|<\epsilon/2^{k+2}.$$

Since we also have that

$$3J^{q_k}P\{|\chi| > \sqrt{J^{q_k}}\} < 3J^{q_k}\exp(-J^{-q_k}k^{-1}/6) < \epsilon/2^{k+2},$$

$$\begin{aligned}
3J^{q_k}P\{|S_{p,\mathbf{1}_{q_k}\mathbf{j}}| > k^{-1}/6\} &= 3J^{q_k}P\{|\tilde{S}_{p,\mathbf{1}_{q_k}\mathbf{j}}| > \sqrt{\frac{J^p}{|\{\mathbf{i}\in\mathcal{J}_{p-q_k};\mathbf{i}\le\mathbf{j}\}|}}k^{-1}/6\} \\
&\le 3J^{q_k}P\{|\tilde{S}_{p,\mathbf{1}_{q_k}\mathbf{j}}| > k^{-1}/6\} \\
&\le 3J^{q_k}|P\{|\tilde{S}_{P,\mathbf{1}_{q_k}\mathbf{j}}| > \sqrt{J^{q_k}}k^{-1}/6\} - P\{|\chi| > \sqrt{J^{q_k}}k^{-1}/6\}| \\
&+ 3J^{q_k}P\{|\chi| > \sqrt{J^{q_k}}\} \\
&< \epsilon/2^{k+1}
\end{aligned}$$

It now follows that for all $p \ge p_0$ that

$$J^{q_k}3\max_{\mathbf{j}\in\mathcal{J}_{p-q_k}} P\{|S_{p,\mathbf{1}_{q_k}\mathbf{j}}| > k^{-1}/6\} < \epsilon/2^{k+1}.$$

Thus condition (ii) is satisfied.

To show condition (i), let $\mathbf{j}\in\mathcal{J}_{q_1}$ and let p be a integer greater than q_1. Then

$$|\{\mathbf{i}\in\mathcal{J}_p;\mathbf{i}\le\mathbf{j}\mathbf{J}_{p-q_1}\}| = (i_1-1)J^{p-1}\cdots(i_{q_1-1}-1)J^{p-q_1+1}i_{q_1}J^{p-q_1}.$$

Hence the variance of $S_{p,\mathbf{j}\mathbf{J}_{p-q_k}}$ is

$$(i_1-1)J^{-1}\cdots(i_{q_1-1}-1)J^{-q_1+1}i_{q_1}J^{-q_1},$$

which is independent of p. For each $\mathbf{j}\in\mathcal{J}_{q_1}$ set $x_{\mathbf{j}} = (\mathbf{j}JJ\ldots)$. Then for each integer $p > q_1$, $Y_p(x_{\mathbf{j}}) = S_{p,\mathbf{j}\mathbf{J}_{p-q_1}}$. Hence the variance of $Y_p(x_{\mathbf{j}})$ is independent of p and only depends on $\mathbf{j}$. Given $\epsilon > 0$, using Chebychev's inequality we can find for each $\mathbf{j}\in\mathcal{J}_{q_1}$ an $a_{\mathbf{j}}$ so that

$$P\{|Y_p(x_{\mathbf{j}})| > a_{\mathbf{j}}\} < \frac{\epsilon}{2J^{q_1}}.$$

Condition (i) now follows from

$$
\begin{aligned}
P\{|Y_p(x_{\mathbf{j}})| > a_{\mathbf{j}} \text{for some } j \in \mathcal{J}_{q_1}\} &\leq \sum_{\mathbf{j}\in\mathcal{J}_{q_1}} P\{|Y_p(x_{\mathbf{j}})| > a_{\mathbf{j}}\} \\
&< \epsilon/2
\end{aligned}
$$

and the proof is complete.

2.4 A Diffusion Limit

For positive integers p and q, we recall the process $\mathbf{Q}^{(p,q)}$ from Section 3. Define for $\mathbf{i}_p \in \{1,..,J\}^p$,

$$
Q^{(p,q)}_{\mathbf{i}_p}(t) = \sum_{\mathbf{i}_q\in\{1,..,J\}^q} Q_{(\mathbf{i}_p,\mathbf{i}'_q)}(t)
$$

where $(\mathbf{i}_p, \mathbf{i}'_q) = (i_1, ..., i_p, i'_1, .., i'_q)$. Let $Q^{(p,q)}(t) = \{Q^{(p,q)}_{\mathbf{i}_p}(t); \mathbf{i}_p \in \{1, ..., J\}^P$. By Lemma 3.4 the process process $\mathbf{Q}^{(p,q)}$ is the queue length process for a closed Jackson network of J^p infinite server queues having exponential service times with unit expectation, MJ^{p+q} customers and routing matrix $P^{\otimes p}$.

Define

$$
X^{(p,q)}_{\mathbf{i}_p}(t) = \frac{Q^{(p,q)}_{\mathbf{i}_p}(t) - (MJ^{p+q})\alpha^{\otimes p}_{\mathbf{i}_p}}{\sqrt{MJ^{p+q}}}.
$$

Let $\mathbf{X}^{(p,q)} = (X^{(p,q)}_{\mathbf{i}_p}(t); \mathbf{i}_p \in \{1, \ldots, J\}^p)$ and $\mathbf{X}^{(p,q)} = \{\mathbf{X}^{(p,q)}(t); t \geq 0\}$. Set $E_{X^{(p,q)}} = \{\mathbf{x}; \sqrt{MJ^{p+q}}\mathbf{x} + MJ^{p+q}\alpha^{\otimes p} \in E_{Q^{(p,q)}}\}$. Define a mapping $\mathbf{n} : E_{X^{(p,q)}} \to E_{Q^{(p,q)}}$ by $\mathbf{n}(\mathbf{x}) = \sqrt{MJ^{p+q}}\mathbf{x} + MJ^{p+q}\alpha^{\otimes p}$ Then $\mathbf{X}^{(p,q)}$ is a Markov process on the finite state space $E_{X^{(p,q)}}$ with generator

$$
\hat{A}^{(p,q)}_{\mathbf{xy}} = A^{(p,q)}_{\mathbf{n(x)n(y)}}.
$$

Set

$$
C^{(p)} = (I - P^{\otimes p})^t \qquad B^{(p)} = I - \frac{P^{\otimes p_0} + (P^{\otimes p_0})^t}{2}.
$$

Let $v_{p,\mathbf{i}}; \mathbf{i} \in \{1, \ldots, J\}^p$ be the standard basis in $\mathbb{R}^{J^p}$. For $F \in C^2(\mathbb{R}^{J^p})$ and $x \in \mathbb{R}^{J^p}$ with

$$x = \sum_{\mathbf{i}} x_{\mathbf{i}} v_{\mathbf{i}},$$

set

$$\begin{aligned}(L^{(p)}F)(x) &= -\sum_{\mathbf{j}} < C^{(p)}x, v_{\mathbf{j}} > \frac{\partial}{\partial x_{\mathbf{j}}} F(x) \\ &+ 1/2 \sum_{\mathbf{i},\mathbf{j}} < B^{(p)}v_{\mathbf{i}}, v_{\mathbf{j}} > \frac{\partial^2}{\partial x_{\mathbf{i}} \partial x_{\mathbf{j}}} F(x).\end{aligned}$$

We note that $L^{(p)}$ is the generator for a diffusion process $\mathbf{X}^p$ on $\mathbb{R}^{J^p}$.

Theorem 2.4.1 *For $p = 1, 2, \ldots,$*

$$\mathbf{X}^{(p,q)} \Rightarrow \mathbf{X}^{(p)} \qquad as\ q \to \infty$$

Proof. To prove the theorem we apply Theorem 7.1 from Chapter 8 of Durrett [3] (pages 297-298.) Our proof is verify the conditions (i),(ii),(iii) and B all hold. To this end first fix p and for the moment q. Note that for $\mathbf{i} \in \{1, \ldots, J\}^p$ and $\mathbf{x} \in E_{X^{(p,q)}}$,

$$\begin{aligned}\sum_{\mathbf{j} \in \{1,\ldots,J\}^p} A^{(p,q)}_{\mathbf{n}(\mathbf{x}) T^p_{\mathbf{ji}}(\mathbf{n}(\mathbf{x}))} &= \sum_{\mathbf{j} \neq \mathbf{i}} (MJ^{p+q} \alpha^{\otimes p}_{\mathbf{j}} + \sqrt{MJ^{p+q}} \mathbf{x}_j) P^{\otimes p}_{\mathbf{j},\mathbf{i}} \\ &= MJ^{p+q} \alpha^{\otimes p}_{\mathbf{i}} (1 - P^{\otimes p}_{\mathbf{ii}}) + \sqrt{MJ^{p+q}} \sum_{\mathbf{j} \neq \mathbf{i}} \mathbf{x}_{\mathbf{j}} P^{\otimes p}_{\mathbf{ji}}\end{aligned}$$

and

$$\sum_{\mathbf{j} \in \{1,\ldots,J\}^p} A^{(p,q)}_{\mathbf{n}(\mathbf{x}) T^p_{\mathbf{ij}}(\mathbf{n}(\mathbf{x}))} = MJ^{p+q} \alpha^{\otimes p}_{\mathbf{i}} (1 - P^{\otimes p}_{\mathbf{ii}}) + \sqrt{MJ^{p+q}} \mathbf{x}_{\mathbf{i}} (1 - P^{\otimes p}_{\mathbf{ii}}).$$

Then

$$\begin{aligned}C^{(p,q)}_{\mathbf{i}}(x) &= \frac{\mathbf{v}_{\mathbf{i}}}{\sqrt{MJ^{p+q}}} \left(\sum_{\mathbf{j} \in \{1,\ldots,J\}^p} A^{(p,q)}_{\mathbf{n}(\mathbf{x}) T^p_{\mathbf{ji}}(\mathbf{n}(\mathbf{x}))} - \sum_{\mathbf{j} \in \{1,\ldots,J\}^p} A^{(p,q)}_{\mathbf{n}(\mathbf{x}) T^p_{\mathbf{ij}}(\mathbf{n}(\mathbf{x}))} \right) \\ &= - \sum_{\mathbf{j} \in \{1,\ldots\}^p} \mathbf{x}_{\mathbf{j}} (\delta_{\mathbf{ji}} - P^{\otimes p}_{\mathbf{ji}})\end{aligned}$$

It also follows that

$$\begin{aligned}
B^{(p,q)}_{\mathbf{ii}}(x) &= \frac{1}{MJ^{p+q}} \sum_{j\in\{1,\ldots,J\}^p} \left(A^{(p,q)}_{\mathbf{n}(\mathbf{x})T^p_{\mathbf{ji}}(\mathbf{n}(\mathbf{x}))} + A^{(p,q)}_{\mathbf{n}(\mathbf{x})T^p_{\mathbf{ij}}(\mathbf{n}(\mathbf{x}))} \right) \\
&= \frac{1}{MJ^{p+q}} [MJ^{p+q}\alpha^{\otimes p}_{\mathbf{i}}(1 - P^{\otimes p}_{\mathbf{ii}}) + \sqrt{MJ^{p+q}} \sum_{\mathbf{j}\neq\mathbf{i}} \mathbf{x}_{\mathbf{j}} P^{\otimes p}_{\mathbf{ji}} \\
&+ MJ^{p+q}\alpha^{\otimes p}_{\mathbf{i}}(1 - P^{\otimes p}_{\mathbf{ii}}) + \sqrt{MJ^{p+q}}\mathbf{x}_{\mathbf{i}}(1 - P^{\otimes p}_{\mathbf{ii}})] \\
&= [2\alpha^{\otimes p}_{\mathbf{i}}(1 - P^{\otimes p}_{\mathbf{ii}}) + \frac{1}{\sqrt{MJ^{p+q}}}(\sum_{\mathbf{j}\neq\mathbf{i}} \mathbf{x}_{\mathbf{j}} P^{\otimes p}_{\mathbf{ji}} + \mathbf{x}_{\mathbf{i}}(1 - P^{\otimes p}_{\mathbf{ii}}))].
\end{aligned}$$

We also know

$$\begin{aligned}
A^{(p,q)}_{\mathbf{n}(\mathbf{x})T^p_{\mathbf{ji}}(\mathbf{n}(\mathbf{x}))} &= (MJ^{p+q}\alpha^{\otimes p}_{\mathbf{j}} + \sqrt{MJ^{p+q}}\mathbf{x}_{\mathbf{j}}) P^{\otimes p}_{\mathbf{ji}} \\
&= (\alpha^{\otimes p}_{\mathbf{i}} - \alpha^{\otimes p}_{\mathbf{i}} P^{\otimes p}_{\mathbf{i,i}}).
\end{aligned}$$

So that

$$\begin{aligned}
B^{(p,q)}_{\mathbf{ij}}(x) &= \frac{1}{MJ^{p+q}} \left(A^{(p,q)}_{\mathbf{n}(\mathbf{x})T^p_{\mathbf{ji}}(\mathbf{n}(\mathbf{x}))} + A^{(p,q)}_{\mathbf{n}(\mathbf{x})T^p_{\mathbf{ij}}(\mathbf{n}(\mathbf{x}))} \right) \\
&= -\frac{1}{MJ^{p+q}} [(MJ^{p+q}\alpha^{\otimes p}_{\mathbf{j}} + \sqrt{MJ^{p+q}}\mathbf{x}_{\mathbf{j}}) P^{\otimes p}_{\mathbf{ji}} \\
&+ (MJ^{p+q}\alpha^{\otimes p}_{\mathbf{i}} + \sqrt{MJ^{p+q}}\mathbf{y}_{\mathbf{i}}) P^{\otimes p)}_{\mathbf{iJ}})] \\
&= - [\alpha^{\otimes p}_{\mathbf{j}} P^{\otimes p}_{\mathbf{ji}} + \alpha^{\otimes p}_{\mathbf{i}} P^{\otimes p}_{\mathbf{ij}} + \frac{1}{\sqrt{MJ^{p+q}}}(\mathbf{x}_{\mathbf{j}} P^{\otimes p}_{\mathbf{ji}} + \mathbf{x}_{\mathbf{i}} P^{\otimes p}_{\mathbf{ij}})]
\end{aligned}$$

Note for every q,

$$C^{(p,q)}(\mathbf{x}) = C^{(p)}\mathbf{x}.$$

This shows that condition (ii) of Theorem 7.1 holds. Fix $R > 0$. Then for all $\mathbf{i}$ and $\mathbf{j}$ in $\{1, \ldots, J\}^p$ and $|\mathbf{x}| \leq R$ that

$$\begin{aligned}
B^{(p,q)}_{\mathbf{ii}} - 1/2 B^{(p)}_{\mathbf{ii}} &\leq \frac{(R + 1)J^p}{\sqrt{MJ^{p+q}}} \\
&\to 0
\end{aligned}$$

as $q \to \infty$. This shows that condition (i) of Theorem 7.1 is satisfied. Condition (iii) follows from the observation for any $\epsilon > 0$ that once $1/\sqrt{MJ^{p+q}} < \epsilon$ any jump of $\mathbf{X}^{(p,q)}$ must have magnitude less than epsilon. Finally the condition (B) follows from the fact that for every q, the state space of $\mathbf{X}^{(p,q)}$ is finite. Thus we have that $\mathbf{X}^{(p,q)} \Rightarrow \mathbf{X}^{(p)}$ as $q \to \infty$. Since p is arbitrary the proof is complete.

2.5 The Limit Process

2.5.1 The state space E

In this section we construct the state space E for the Markov process $\{Y(t); t \geq 0\}$. The space will be homeomorphic to the space $\mathbb{R}^{\mathbb{N}}$ and thus will be a complete separable metric space. For $p = 1, 2, \ldots$, let $\mathcal{F}_p$ be the σ-algebra on K generated by the sets $K_{p,\mathbf{i}}$, $\mathbf{i} \in \{1, \ldots, J\}^p$ and $L^2_p(K) = L^2(K, \mathcal{F}_p, \mu)$. Next set $\mathcal{H}_1 = L^2_1$ and for $p = 2, 3, \ldots$ set $\mathcal{H}_p = (L^2_p \cap L^2_{p-1})^{\perp}$. Then $\mathcal{H}_p$ form a collection of finite dimensional orthogonal subspaces of $L^2(K)$ and $L^2(K)$ can be written as the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$. Let $e_{11}, \ldots e_{1J}$ be an orthonormal basis for $\mathcal{H}_1$ and for $p = 2, 3, \ldots$, let $e_{p1}, \ldots, e_{J^{p-1}(J-1)}$ be an orthonormal basis for $\mathcal{H}_p$. (Set $k_1 = J$ and for $p = 2, 3, \ldots$, set $k_p = J^{p-1}(J-1)$.)

Let E be all linear combinations of the form

$$\sum_{p=1}^{\infty} \sum_{k=1}^{k_p} \alpha_{pk} e_{pk},$$

where $\alpha_{pk} \in \mathbb{R}$ for all α and k. Thus E is the product space $\prod_{P=1}^{\infty} \prod_{k=1}^{k_p} \{c e_{pk}; c \in \mathbb{R}\}$. We give E the product topology. With this topology E is separable and metrizable. A metric d for which E with the product topology is a complete separable metric space is given by

$$d\Big(\sum_{p=1}^{\infty} \sum_{k=1}^{k_p} c_{pk} e_{pk}, \sum_{p=1}^{\infty} \sum_{k=1}^{k_p} d_{pk} e_{pk}\Big) = \sum_{p=1}^{\infty} \sum_{k=1}^{k_p} 2^{-(p+k)} (|c_{pk} - d_{pk}| \wedge 1).$$

The Borel σ-algebra $\mathcal{E}$ for E is generated by the e_{pk}'s. Define a probability measure γ as follows. For each p let γ_p be the distribution of the random vector on $\mathcal{H}_p \to \mathbb{R}^{k_p}$ given by $\sum_{k=1}^{k_p} \alpha_{pk} e_{pk} \to (\alpha_{p1}, \ldots, \alpha_{pk_p})$ which is normally distributed with expectation 0 and covariance function equal to the identity matrix. Then $\gamma = \Pi_{p=1}^{\infty} \gamma_p$.

For $p_0 = 1, 2, \ldots$, let $\mathcal{A}_{p_0}$ be the collection of all multi-indices $\alpha = \{\alpha_{pk}; p = 1, 2, \ldots, k = 1, \ldots k_p\}$ in $\prod_{p=1}^{\infty} \prod_{k=1}^{k_p} \mathbb{N}_0$ such that $\alpha_{pk} = 0$ for all $p > p_0$ and $\mathcal{A} = \cup_{p_0=1}^{\infty} \mathcal{A}_{p_0}$. Let $x \in L^2(K)$ and write

$$x = \sum_{p=1}^{\infty} \sum_{k=1}^{k_p} x_{kp} e_{kp}$$

and let $\alpha \in \mathcal{A}_{p_0}$. Set

$$H_\alpha(x) = \prod_{p=1}^{p_0} \prod_{k=1}^{k_p} H_{\alpha_{pk}}(x_{pk}),$$

where $H_0, H_1, \ldots$ are the Hermite polynomials on $\mathbb{R}$. We note that the polynomials $\{H_\alpha : \alpha \in \mathcal{A}\}$ are dense in the space $L^2(E, \mathcal{E}, \gamma)$.

2.5.2 The generator

We construct a generator for a strongly continuous contraction semigroup on $L^2(E, \mathcal{E}, \gamma)$. We actually construct a linear operator L on $\{H_\alpha : \alpha \in \mathcal{A}\}$ such that L is dissipative and that for some $\lambda > 0$ the range of $(\lambda - L)$ is also dense in $L^2(E, \mathcal{E}, \gamma)$. According to Theorem 1.2.12 in Ethier and Kurtz [4] closure of L is the generator of a strongly continuous contraction semigroup on $L^2(E, \mathcal{E}, \gamma)$. We use the the generators of the diffusion processes $\{X_p(t); t \geq 0\}$ whose state space is $\mathbb{R}^p$, $p = 1, 2, \ldots$ constructed earlier to approximate L. There are two steps involved; one is to put the diffusion on the state space $L_p^2(K)$ and the second is a change of basis.

Let $v_{p,\mathbf{i}}; \mathbf{i} \in \{1, \ldots, J\}^p$ be the standard basis in $\mathbb{R}^{J^p}$ and note the collection $\{1_{K_{p,\mathbf{i}}}; \mathbf{i} \in \{1, \ldots, J\}^p\}$ forms an orthogonal basis for L_p^2 which has dimension J^p. We identify $v_{p,\mathbf{i}}$ and $1_{K_{p,\mathbf{i}}}$ and thus identify $\mathbb{R}^{J^p}$ and $L_p^2(K)$.

Fix a positive integer p. Let the set $\{1, \ldots, J\}^p$ be ordered lexicographically. We map $\{1, \ldots, J\}^p$ to $\{1, \ldots, J^p\}$ by

$$\mathbf{i} = (i_1, \ldots, i_p) \to (i_1 - 1)J^{p-1} + (i_2 - 1)J^{p-2} + \cdots + (i_{p-1} - 1)J + i_p.$$

For $\mathbf{i} \in \{1, \ldots, J\}^p$, set $f_\mathbf{i} = 1_{K_{p,\mathbf{i}}}$. Then $\mathcal{B}_1 = \{f_\mathbf{i}; \mathbf{i} \in \{1, \ldots, J\}^p\}$ is an orthogonal basis for $L_p^2(K)$. (Note each $f_\mathbf{i}$ has norm $J^{p/2}$.) Let $\mathcal{B}_3 = \{v_i; i = 1, \ldots, J^p\}$ be the standard basis for $\mathbb{R}^{J^p}$. Let $\phi : L_p^2 \to \mathbb{R}^{J^p}$ be the linear mapping defined by

$$\phi(f_{(i_1 1, \ldots, i_p)}) = v_{(i_1-1)J^{p-1}+(i_2-1)J^{p-2}+\cdots+(i_{p-1}-1)J+i_p}.$$

The mapping ϕ shows that $L_p^2(K)$ a C^∞ manifold. Suppose L is a linear mapping from $\mathbb{R}^{J^p}$ to $\mathbb{R}^{J^p}$, then $\phi^{-1} \circ L \circ \phi$ is a linear mapping from $L^2_p(K)$ to $L^2_p(K)$ which we will also denote by L. At any $x \in L^2_p(K)$ we take the basis for the tangent space at x to be the set $\{\frac{\partial}{\partial x_{f_{\mathbf{i}}}}; \mathbf{i} \in \{1, \ldots, J\}^p\}$.

Set

$$A^{(p)} = (I - P^{\otimes p})^t \qquad B^{(p)} = I - \frac{P^{\otimes p_0} + (P^{\otimes p_0})^t}{2}.$$

We consider $A^{(p)}$ and $B^{(p)}$ as linear operators on $L^2_p(K)$. Let $C^2(L^2_p(K))$ be the collection of real-valued functions on $L^2_p(K)$ that are twice continuously differentiable. Define an operator L from $C^2(L^2_p(K))$ to $C(L^2_p(K))$, the continuous functions on $L^2_p(K)$ as follows. For $F \in C^2(L^2_p(K))$ and $x \in L^2_p(K)$ with

$$x = \sum_{\mathbf{i}} x_{\mathbf{i}} f_{\mathbf{i}},$$

set

$$\begin{aligned}(LF)(x) &= -\sum_{\mathbf{j}} < Ax, f_{\mathbf{j}} > \frac{\partial}{\partial x_{f_{\mathbf{j}}}} F(x) \\ &+ 1/2 \sum_{(q,k),(q',k')} < Bf_{\mathbf{i}}, f_{\mathbf{j}} > \frac{\partial^2}{\partial x_{f_{\mathbf{i}}} \partial x_{f_{\mathbf{j}}}} F(x).\end{aligned}$$

The thing that L extends to a generator for a strongly continuous contraction semi-group and that the generator corresponds to a diffusion process $\{X(t); t \geq 0\}$ where $X(t) = \{X_{\mathbf{i}}(t); \mathbf{i} \in \{1, \ldots, J\}^p\}$ on $L^2_p(K)$ is well known. In our setting, each $f_{\mathbf{i}}$ corresponds to a node in a queueing network. Then $X_{\mathbf{i}}(t)$ is the normalized workload at queue $f_{\mathbf{i}}$.

Our goal is to extend the Markov process to one that has state space $L^2(K)$. To this end we need to develop a strongly continuous semi-group on $L^2(L^2(K), \mathcal{B}(L^2(K)), \gamma)$ where γ is Wiener measure on $L^2(K)$. It turns out the basis $\mathcal{B}_1$ is not the most convenient one to make the extension from the finite dimensional spaces $L^2_p(K)$ to the infinite dimensional space $L^2(K)$. The basis $\mathcal{B}_2 = \{e_{q,k}; q = 1, \ldots, p, k = 1, \ldots, k_p\}\}$ is a better choice. This is a consequence of the fact that $\{e_{p,k}; p = 1, 2, \ldots, k = 1, \ldots, k_p\}$ is an orthonormal basis for $L^2(K)$. We now turn to writing the operator L in terms of the basis $\mathcal{B}_2$.

We have two bases for $L^2_p(K)$, $\mathcal{B}_1 = \{f_{\mathbf{i}}; \mathbf{i} \in \{1, \ldots, J\}^p\}$ and $\mathcal{B}_2 = \{e_{q,k}; q = 1, \ldots, p, k = 1, \ldots, k_p\}$. For each $e_{q,k} \in \mathcal{B}_2$, set

$$e_{q,k} = \sum_{\mathbf{i}} t_{(q,k),\mathbf{i}} f_{\mathbf{i}}.$$

Then

$$< e_{q,k}, f_{\mathbf{j}} >= t_{(q,k),\mathbf{j}} < f_{\mathbf{j}}, f_{\mathbf{j}} >,$$

so that

$$t_{(q,k),\mathbf{j}} = J^p < e_{q,k}, f_{\mathbf{j}} > .$$

Note that

$$\sum_{\mathbf{i}} t_{(q,k),\mathbf{i}} t_{(q',k'),\mathbf{i}} = J^{2p} < e_{q,k}, e_{q',k'} > .$$

For each $f_{\mathbf{i}} \in \mathcal{B}_2$, set

$$f_{\mathbf{i}} = \sum_{(q,k)} s_{\mathbf{i},(q,k)} e_{q,k}.$$

Then

$$< f_{\mathbf{i}}, e_{q,k} >= s_{\mathbf{i},(q,k)}$$

and

$$\sum_{(q,k)} s_{\mathbf{i},(q,k)} s_{\mathbf{j},(q,k)} =< f_{\mathbf{i}}, f_{\mathbf{j}} > .$$

Moreover,

$$\sum_{\mathbf{i}} t_{(q,k),\mathbf{i}} s_{\mathbf{i},(q',k')} =< e_{q,k}, e_{q',k'} >$$

and

$$\sum_{(q,k)} s_{\mathbf{i},(q,k)} t_{(q,k),\mathbf{j}} = J^p < f_{\mathbf{i}}, f_{\mathbf{j}} > .$$

Next let F be a differentiable function on $L^2_p(K)$. Let $x \in L^2_p(K)$, then for each $(q,k) \in \mathcal{B}_2$,

$$\begin{aligned} \frac{\partial}{\partial x_{e_{qk}}} F(x) &= \lim_{h\to 0} \frac{F(x + he_{q,k}) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{F(x + h\sum_{\mathbf{i}} t_{(q,k),\mathbf{i}} f_{\mathbf{i}}) - F(x)}{h} \\ &= \sum_{\mathbf{i}} t_{(q,k),\mathbf{i}} \frac{\partial}{\partial x_{f_{\mathbf{i}}}} F(x). \end{aligned}$$

For each $f_{\mathbf{i}} \in \mathcal{B}_1$,

$$\frac{\partial}{\partial x_{f_{\mathbf{i}}}} F(x) = \sum_{(q,k)} s_{\mathbf{i},(q,k)} \frac{\partial}{\partial x_{e_{q,k}}} F(x).$$

As for second derivatives we have,

$$\frac{\partial^2}{\partial x_{e_{q,k}} \partial x_{e_{q',k'}}} F(x) = \sum_{\mathbf{i},\mathbf{j}} t_{(q,k),\mathbf{i}} t_{(q',k'),\mathbf{j}} \frac{\partial^2}{\partial x_{f_\mathbf{i}} \partial x_{f_\mathbf{j}}} F(x),$$

and

$$\frac{\partial^2}{\partial x_{f_\mathbf{i}} \partial x_{f_\mathbf{j}}} F(x) = \sum_{(q,k),(q',k')} s_{\mathbf{i},(q,k)} s_{\mathbf{j},(q',k')} \frac{\partial^2}{\partial x_{e_{q,k}} \partial x_{e_{q',k'}}} F(x).$$

The operator L can be written terms of either $\mathcal{B}_1$ or $\mathcal{B}_2$. For $x \in L_p^2(K)$ with

$$x = \sum_{\mathbf{i}} x_\mathbf{i} f_\mathbf{i},$$

$$\begin{aligned} (LF)(x) &= -\sum_{\mathbf{i},\mathbf{j}} x_\mathbf{i} < Af_\mathbf{i}, f_\mathbf{j} > \frac{\partial}{\partial x_{f_\mathbf{j}}} F(x) \\ &+ 1/2 \sum_{(q,k),(q',k')} < Bf_\mathbf{i}, f_\mathbf{j} > \frac{\partial^2}{\partial x_{f_\mathbf{i}} \partial x_{f_\mathbf{j}}} F(x). \end{aligned}$$

For $y \in L_p^2(K)$ with

$$y = \sum_{(q,k)} y_{q,k} e_{q,k},$$

$$\begin{aligned} (\hat{L}F)(y) &= -\sum_{(q,k),(q',k')} y_{q,k} < Ae_{q,k}, e_{q',k'} > \frac{\partial}{\partial y_{e_{q',k'}}} F(x) \\ &+ 1/2 \sum_{(q,k),(q',k')} < Be_{q,k}, e_{q',k'} > \frac{\partial^2}{\partial y_{e_{q,k}} \partial y_{e_{q',k'}}} F(y). \end{aligned}$$

That

$$
\begin{aligned}
&\sum_{\mathbf{j}} < Ax, f_{\mathbf{j}} > \frac{\partial}{\partial x_{f_{\mathbf{j}}}} F(x) \\
&= \sum_{\mathbf{j}} < Ax, \sum_{(q,k)} s_{\mathbf{j},(q,k)} e_{q,k} > \sum_{(q',k')} s_{\mathbf{j},(q',k')} \frac{\partial}{\partial x_{e_{q',k'}}} F(x) \\
&= \sum_{(q,k)} < Ax, e_{q,k} > \sum_{(q',k')} J^{-2p} \sum_{\mathbf{j}} t_{(q,k),\mathbf{j}} t_{(q',k'),\mathbf{j}} \frac{\partial}{\partial x_{e_{q',k'}}} F(x) \\
&= \sum_{(q,k)} < Ax, e_{q,k} > \sum_{(q',k')} < e_{q,k}, e_{q',k'} > \frac{\partial}{\partial x_{e_{q',k'}}} F(x) \\
&= \sum_{(q,k)} < Ax, e_{q,k} > \frac{\partial}{\partial x_{e_{q,k}}} F(x)
\end{aligned}
$$

In a similar fashion we can show that

$$
\sum_{\mathbf{i},\mathbf{j}} < Bf_{\mathbf{i}}, f_{\mathbf{j}} > \frac{\partial^2}{\partial x_{f_{\mathbf{i}}} \partial x_{f_{\mathbf{j}}}} F(x) = \sum_{(q,k),(q',k')} < Be_{q,k}, e_{q',k'} > \frac{\partial^2}{\partial x_{e_{q,k}} \partial x_{e_{q',k'}}}.
$$

It now follows that

$$
(LF)(x) = (\hat{L}F)(x).
$$

We note that $H_\alpha = H_\alpha \circ \pi_{p_0}$ where $\pi_{p_0} : L^2(K) \to L^2_{p_0}$ is the projection mapping. Set $\mathcal{D}_1(L)$ to be the set H_α, $\alpha \in \mathcal{A}$.

$$
\begin{aligned}
L(H_\alpha)(x) \;\; = \;\; & \sum_{p=1}^{p_0} \sum_{k=1}^{k_p} - < ((I - P^{\otimes p_0})^t \pi_{p_0}(x)), e_{p,k} > \frac{\partial}{\partial x_{p,k}} H(\pi_{p_0}(x)) \\
+ \;\; & \sum_{p=1}^{p_0} \sum_{k=1}^{k_p} \sum_{p'=1}^{p_0} \sum_{k'=1}^{k_p} \\
& < \frac{1}{2J^{p_0}} (I - \frac{P^{\otimes p_0} + (P^{\otimes p_0})^t}{2} c) e_{p,k}, e_{p',k'} > \frac{\partial}{\partial x_{p,k} \partial x_{p',k'}} H(\pi_{p_0}(x)).
\end{aligned}
$$

where the vector $(\pi_{p_0}(x)(I - P^{\otimes p_0}))$ is written in terms of the basis $\{e_{p,k}; p = 1, \ldots, p_0, k = 1, \ldots, k_p\}$.

Suppose $p_0' > p_0$. Let $\alpha \in \mathcal{A}_{p_0}$ then $\alpha \in \mathcal{A}_{p_0'}$. Then we can write

$$\begin{aligned} L(H_\alpha)(x) &= \sum_{p=1}^{p_0'}\sum_{k=1}^{k_p} -(\pi_{p_0'}(x)(I - P^{\otimes p_0'}))_{pk} \frac{\partial}{\partial x_{p,k}} H(\pi_{p_0'}(x)) \\ &+ \sum_{p=1}^{p_0'}\sum_{k=1}^{k_p}\sum_{p'=1}^{p_0'}\sum_{k'=1}^{k_p} \\ &< \frac{1}{2J^{p_0'}}(I - \frac{P^{\otimes p_0'} + (P^{\otimes p_0'})^t}{2})e_{p,k}, e_{p',k'} > \frac{\partial}{\partial x_{p,k}\partial x_{p',k'}} H(\pi_{p_0'}(x)). \end{aligned}$$

So it appears that we have two representations of $L(H_\alpha)$. Since for any $x \in L^2(K)$ and $p > p_0$, $H_{\alpha_{pk}}(x_{p,k}) = 1$, it follows that

$$\frac{\partial}{\partial x_{p,k}} H_\alpha(x) = 0.$$

Let $x \in L^2_{p_0} \subset L^2_{p_0'}$. Since P and hence, for $p = 1, 2, \ldots$, $P^{\otimes p}$ is doubly stochastic we have $P^{\otimes p_0} x = P^{\otimes p_0'} x$ and $x P^{\otimes p_0} = x P^{\otimes p_0'}$. It then follows that the two representations of $L(H_\alpha)$ are the same. From this it follows that L is both well defined and linear on $\mathcal{D}_1(L)$.

2.5.3 The Markov Process

For $p = 1, 2, \ldots$, let $\{\mathbf{X}^p(t); t \geq 0\}$ be the Markov process on $L^2_p(K)$ having generator L_p and let T^p be the matrix whose $(q, k), \mathbf{i}$-th element, $q = 1, \ldots, p$, $k = 1, \ldots, k_p$ and $\mathbf{i} \in \{1, \ldots, J\}^p$ is $t^{(p)}_{(q,k),\mathbf{i}}$. For $p = 1, 2, \ldots$ and $t \geq 0$, set $\mathbf{Y}^p(t) = T^p \mathbf{X}^p(t)$. Note that $\mathbf{Y}^p(t) = (Y^p_{qk}(t); q = 1, \ldots, p, k = 1, \ldots, k_q)$. The analysis of the previous section shows that it $p' > p$ then

$$(Y^p_{qk}(t); q = 1, \ldots, p, k = 1, \ldots, k_q) = (Y^{p'}_{qk}(t); q = 1, \ldots, p, k = 1, \ldots, k_q).$$

Thus for each $t \geq 0$ we can unambiguously define a sequence of random variables $\mathbf{Y}(t) = (Y_{pk}(t); p = 1, 2, \ldots, k = 1, \ldots, k_p)$ where we can take $Y_{pk}(t) = Y^p_{pk}(t)$. Let $\mathbf{Y} = \{\mathbf{Y}(t); t \geq 0\}$.

We claim that $\mathbf{Y}$ is a Markov process having generator L. For $p = 1, 2, \ldots$ and probability measure μ on $L^2_p(K)$, $\mathbf{X}^p$ is a solution of the martingale problem for (μ, L^p). Since T^p is $1-1$, $\mathbf{Y}^p$ is a solution to the martingale problem for $(\mu \circ (T^p)^{-1}, L^p)$. For each $\alpha \in \mathcal{A}$ there exists a p_0 so that for all $p \geq p_0$,

$$H_\alpha(Y(t)) = \prod_{q=1}^{p}\prod_{k=1}^{k_q} H_{\alpha_{qk}}(Y_{qk}(t)).$$

Let $s \le t$ and Suppose $A \in \sigma(\mathbf{Y}^p(u); 0 \le u \le s)$ for some p. Then $A \in \sigma(\mathbf{Y}^p(u); 0 \le u \le s)$ for some $p \ge p_0$. It follows that

$$E[1_A(H_\alpha(\mathbf{Y}(t)) - \int_0^t LH_\alpha(\mathbf{Y}(u))du)] = E[1_A(H_\alpha(\mathbf{Y}(s)) - \int_0^s LH_\alpha(\mathbf{Y}(u))du)].$$

A monotone class theorem shows that the above argument works for all $A \in \sigma(\mathbf{Y}(u); 0 \le u \le s)$. Thus the process $\{H_\alpha(\mathbf{Y}(t)) - \int_0^t LH_\alpha(\mathbf{Y}(u))du; t \ge 0\}$ is a martingale. It follows that for any probability measure μ on E that $\mathbf{Y}$ solves the martingale problem for (μ, L). Theorem 4.4.1 of Ethier and Kurtz [4] now applies to show that $\mathbf{Y}$ is a Markov process with generator L.

2.6 A Limit Theorem

For $p = 1, 2, \ldots$, $t \ge 0$ and $\mathbf{i} \in \{1, \ldots, J\}^p$, set

$$\hat{X}_{\mathbf{i}}^{(p)}(t) = \frac{Q_{\mathbf{i}}^{(p)}(t) - MJ^p\alpha^{\otimes p}}{\sqrt{MJ^p}},$$

$$\hat{\mathbf{X}}^{(p)}(t) = (\hat{\mathbf{X}}_{\mathbf{i}}^{(p)}(t); \mathbf{i} \in \{1, \ldots, J\}^p),$$

and

$$\hat{\mathbf{X}}^{(p)} = \{\hat{\mathbf{X}}^{(p)}(t); t \ge 0\}.$$

For $p = 1, 2, \ldots$, define $\mathcal{T}^p : C(L_p^2(K)) \to C(L_p^2(K))$ by

$$\mathcal{T}^p(x) = \{T^p x(t); t \ge 0\}.$$

Since T^p is linear, $\mathcal{T}^p$ belongs to $C(L_p^2(K))$. For $q = 1, 2, \ldots$, define, $\pi_q : C(E) \to C(E)$ by

$$\pi_q((x_{(rk)}; r = 1, 2, \ldots, k = 1 \ldots, k_r)) = ((y_{(rk)}; r = 1, 2, \ldots, k = 1 \ldots, k_r))$$

where

$$y_{(rk)} = \begin{cases} x_{(rk)} & \text{if } r \le q \\ y_{(rk)} = 0 & \text{if } q < r \end{cases},$$

and for $p = 1, 2, \ldots$, define $\pi^p : C(L^2_p(K)) \to C(E)$ by

$$\pi^p((x_{(rk)}; r = 1, \ldots, p, k = 1 \ldots, k_r)) = ((y_{(rk)}; r = 1, 2, \ldots, k = 1 \ldots, k_r))$$

where

$$y_{(rk)} = \begin{cases} x_{(rk)} & \text{if } r \leq p \\ y_{(rk)} = 0 & \text{if } q < r \end{cases},$$

and for $p = 1, 2, \ldots$ and $q = 1, \ldots, p$, define $\pi^p_q : C(L^2_p(K)) \to C(E)$ by

$$\pi^p_q(x) = \pi_q \circ \pi^p(x).$$

Finally define for $p = 1, 2, \ldots$,

$$\hat{\mathbf{Y}}^p = \mathcal{T}^p(\hat{\mathbf{X}}^p).$$

Theorem 2.6.1 *The processes $\pi^p(\hat{\mathbf{Y}}^p)$ converge in distribution to the process $\mathbf{Y}$.*

Proof. For positive integers p and q,

$$\pi^{p+q}_q(\hat{\mathbf{Y}}^{(p+q)}) = \pi^q(\mathcal{T}^q(\mathbf{X}^{(q,p)})).$$

Since for each $q = 1, 2, \ldots$ $\mathbf{X}^{(q,p)} \Rightarrow (\mathbf{q})$ as $p \to \infty$ and $\mathcal{T}^q$ is continuous

$$\mathcal{T}^q(X^{(q,p)}) \Rightarrow \mathcal{T}^q(\mathbf{X}^{(p)}) \qquad \text{as } p \to \infty.$$

Since for all positive integers π^p, π_q and π^p_q are continuous

$$\pi^p_q(\hat{\mathbf{Y}}^{(q+p)}) \Rightarrow \pi_q(\mathbf{Y}) \qquad \text{as } p \to \infty.$$

So in particular for $q > p$

$$(\pi^{p-q}_q(\hat{\mathbf{Y}}^{(p)}))_{(qk)} \Rightarrow Y_{(qk)} \qquad \text{as } p \to \infty.$$

Then by Proposition 3.2.4 of Ethier and Kurtz [4] the result follows.

2.7 The space $L^2(K)$

For $p = 1, 2, \ldots$, let $\mathcal{F}_p = \sigma(K_{p,\mathbf{i}_p}; \mathbf{i}_p \in \{1, \ldots, J\}^p)$ and set $L^2_p(K) = L^2(K, \mathcal{F}_p, \mu)$. For each $p = 1, 2, \ldots$, $L^2_p(K)$ is a J^p dimensional subspace of $L^2(K)$ and as such is closed. Any $f \in L^2_p(K)$ can be written in the form

$$f = \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} 1_{K_{p,\mathbf{i}_p}}$$

with $c_{\mathbf{i}_p} \in \mathbb{R}$ and

$$||f|| = \left[\sum_{\mathbf{i}_p} c^2_{\mathbf{i}_p} \right]^{1/2}.$$

Let $f \in L^2(K)$. For $p = 1, 2, \ldots$, set

$$f_p = E[f|\mathcal{F}_p],$$

the projection of f onto $L^2_p(K)$. By the martingale convergence theorem f_p converges to f almost surely and in $L^2(K)$.

Since $L^1_1(K) \subset L^2_2(K) \subset \cdots$ we can construct a sequence of orthogonal subspaces $\mathcal{H}_p$, $p = 1, 2, \ldots$ such that $\mathcal{H}_1 = L^2_1(K)$ and for $p = 2, 3, \ldots$,

$$L^2_P(K) = L^2_{p-1}(K) \oplus \mathcal{H}_p.$$

It follows that we can write

$$L^2(K) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots.$$

Assuming that P is double stochastic we can write

$$A^{(p)} = \frac{2}{J^p} \left(I + \frac{P^{\otimes p} + (P^{\otimes p})^T}{2} \right).$$

Since $\frac{P^{\otimes p} + (P^{\otimes p})^T}{2}$ is a symmetric stochastic matrix, it's eigenvalues are a subset of $[-1, 1]$. Thus the eigenvalues of $A^{(p)}$ are contained in the interval $[0, 4/J^p]$. It follows that $A^{(p)}$ is nonnegative definite. It particular there is a set $\{\mathbf{v}_{\mathbf{i}_p}; \mathbf{i}_p \in \{1, \ldots, J\}^p\}$ of orthogonal eigenvectors of $A^{(p)}$ that span R^{J^p}. Any vector $\mathbf{x} \in \mathbb{R}^{J^p}$ can be identified with a unique element in $L^2_p(K)$. Writing $x = \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} \mathbf{e}_{\mathbf{i}_p}$ where the $e_{\mathbf{i}_p}$'s form the standard basis in $\mathbb{R}^{J^p}$, we write

$$x = \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} 1_{p,\mathbf{i}_p}$$

in $L^2_p(K)$. We view $\{\mathbf{v}_{\mathbf{i}_p}; \mathbf{i}_p \in \{1, \ldots, J\}^p\}$ as an orthonormal basis for $L^2_p(K)$.

We define an operator A on $L^2(K)$ as

$$Af = \sum_{p=1}^{\infty} A^{(p)} f_p$$

where $f_p \in \mathcal{H}_p$ and $f = \sum_{p=1}^{\infty} f_p$. For $p = 1, 2, \ldots$

$$f_p = \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} v_{p,\mathbf{i}_p}.$$

Let $\lambda_{p,\mathbf{i}_p}$ be the eigenvector corresponding to $v_{p,\mathbf{i}_p}$. Since

$$\begin{aligned} Af_p &= \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} A v_{p,\mathbf{i}_p} \\ &= \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} \lambda_{p,\mathbf{i}_p} v_{p,\mathbf{i}_p}, \end{aligned}$$

it follows from $\lambda_{p,\mathbf{i}_p} \leq 4/J^p$ that

$$||Af_p|| \leq \frac{4}{J^p} ||f_p||.$$

Theorem 2.7.1 *The operator A is a compact linear mapping on $L^2(K)$.*

Proof. Let $\{f_n\}$ be a bounded sequence in $L^2(K)$. Letting M be a bound for the sequence and letting f_n^p be the projection of f_n onto to $\mathcal{H}_p$ we have, for each n and p, that $||f_n^p|| \leq M$. Let $\{f_{1,k}\}$ be a subsequence of $\{f_n\}$ such that $Af_{1,k}^1$ converge, where $f_{1,k}^1$ is the projection of $f_{1,k}$ onto $\mathcal{H}_1$. Then inductively for $p = 2, 3, \ldots$, choose a subsequence $\{f_{p,k}\}$ of $\{f_{p-1,k}\}$ so that $Af_{p,k}^p$ converges where $f_{p,k}^p$ is the projection of $f_{p,k}$ onto $\mathcal{H}_p$. For $k = 1, 2, \ldots$, let f_{n_k} be f_{kk} and for $p = 1, 2, \ldots$, let $f_{n_k}^p$ be the projection of f_{n_k} onto $\mathcal{H}_p$. Then for each p the sequence $Af_{n_k}^p$ converges. Let $\epsilon > 0$, choose m so that

$$\sum_{p=m+1}^{\infty} \frac{8M}{J^p} < \epsilon/2.$$

For $p = 1, \ldots, m$ choose a positive integer k_p so that for all j and k greater than k_p,

$$||Af^p_{n_j} - Af^p_{n_k}|| < \epsilon/2m.$$

Let $k_0 = \max\{k_1, \ldots, k_p\}$. Then for all j and k greater than k_0,

$$\begin{aligned} ||Af_{n_k} - Af_{n_j}|| &\leq \sum_{p=1}^{\infty} ||Af^p_{n_k} - Af^p_{n_j}|| \\ &= \sum_{p=1}^{m} ||Af^p_{n_k} - Af^p_{n_j}|| + \sum_{p=m+1}^{\infty} ||Af^p_{n_k} - Af^p_{n_j}|| \\ &\leq \sum_{p=1}^{m} \epsilon/2m + \sum_{p=m+1}^{\infty} \frac{8m}{J^p} \\ &< \epsilon. \end{aligned}$$

Thus, Af_{n_k} is a Cauchy sequence and converges. Since every bounded sequence $\{f_n\}$ in $L^2(K)$ contains a subsequence $\{f_{n_k}\}$ for which $\{Af_{n_k}\}$ converges, A is compact and the proof is complete.

We now turn to constructing an operator B. Suppose λ is an eigenvalue of the matrix P with eigenvector $\mathbf{x}$. Then for $p = 1, 2, \ldots$, λ is an eigenvalue of the matrix $P^{\otimes p}$ with corresponding eigenvector $\mathbf{x}^{\otimes p}$. Since the eigenvalues of P lie in the disc in the complex plane of radius 1 centered at the origin, the eigenvalues of $I - P^{\otimes p}$ lie in the disc of radius 1 centered at the point (1,0). For $p = 1, 2, \ldots$, let $f_p \in \mathcal{H}_p$. Since $f_p \in L^2_p(K)$ we can write $f_p = \sum_{\mathbf{i}_p} c_{\mathbf{i}_p} 1_{K_{p,\rangle_p}}$. Let $\mathbf{c} = (c_{\mathbf{i}_p}; \mathbf{i}_p \in \{1, \ldots, J\}^p)$ and set $Bf_p = -\mathbf{c}^T(I - P^{\otimes p})$. Extend B to $\bigcup_{p=1}^{\infty} L^2_p(K)$ by linearity. It is not clear that B can be extended to all of $L^2(K)$. However since $\bigcup_{p=1}^{\infty} L^2_p(K)$ is dense in $L^2(K)$, each $L^2_p(K)$ is finite dimensional and $B : L^2_p(K) \to L^2_p(K)$, Proposition 3.5 in Chapter 1 of Ethier and Kurtz [4] shows that B is closable and its closure, which we shall also denote by B, generates a strongly continuous contraction semigroup $\{S(t); t \geq 0\}$ on $L^2(K)$.

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