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The Steiner Linear Ordering Problem: Application to resource-constrained scheduling problems

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Abstract

When examined through polyhedral study, the resource-constrained scheduling problems have always dealt with processes which have the same priority. With the Steiner Linear Ordering problem, we can address systems where the elements involved have different levels of priority, either high or low. This allows us greater flexibility in modeling different resource-constrained scheduling problems. In this paper, we address both the linear ordering problem and its application to scheduling problems, and provide a polyhedral study of the associated polytopes.

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Chapter 1

Introduction

This paper focuses on the Steiner Linear Order Problem, which arises in the context of distributed systems. A distributed system can be defined as a collection of independent *components* (e.g., processors, servers, stations, units) that appear to function as a single system. Examples of distributed systems are common in daily life, in every area from banking to videoconferencing, avionics to cellular phone systems. Because of the crucial role that these systems play, it is important that they maintain their operability even if there is some failure due to a defect in the system; this is called fault-tolerance. To maintain fault-tolerance, the most widely accepted approach is the primary/backup redundancy approach. In the primary/backup redundancy approach, a task is represented by two processes, the primary process and the backup process, which are present on two different components. The backup process is run only if there is a failure in the primary process. Generally, the entire approach is controlled by fault-tolerance software that handles the actions of these processes, such as the creation of the primary and the fully-synchronized backup processes, and termination or move of a process from one component to another. Fault-tolerance software attempts a strategy of minimum disruption while it works to restore the load distribution on the components as close as possible to the initial distribution. Due to this strategy, the repetition of faults and recoveries slowly deteriorates the structure of the load distribution and eventually arrives at a system whose performance may be far from optimal. At this point, the system must have some down-time, during which serious reconfiguration procedures can be applied to the distribution. For this paper, we study a reconfiguration procedure which will likely require a non-neglible number of both process moves and possible temporary outages of some tasks [23].

During the reconfiguration procedure, a process which is moved is either moved by migration or moved by interruption. If a process is migrated, it consumes resource on the component it is moving from (its source) and the component it is moving to (its target) for the duration of the move. However, the migration of a process is not trivial, and so it is desirable to keep the number of migrations to one per process. If a process is interrupted, it is removed from its source component and so consumes no resource. At a later point, that process will be restarted on a different component. This move by interruption causes a disruption in service, and so this is not the preferred method of moving a process. Since a process which is interrupted is no longer in the distribution (for the time being), and each migrated process is only moved once, we can state that each process is moved (migrated or interrupted) exactly once. Also, throughout the reconfiguration, it is required that no movement of a process creates an overflow on any component, and as such, the final state must always be feasible. This reconfiguration procedure yields the following load balancing problem:

> Given an arbitrary load distribution of the processes, find the least disruptive sequence of process moves which fulfills the foregoing constraints, and at the end of which the system ends up with another predefined load distribution.

Although a first inclination might be to solve the load balancing problem all at once, it is better, from a theoretical and computational aspect, to solve the problem in two stages: first determining the set of process moves, which is an assignment problem, and then scheduling those process moves. The second stage is called the *Process Move Programming* (PMP) problem, and is an application that we will examine in detail in Chapter 3.

Before continuing, we must define the notation and phrasing that will be used in the discussion of the PMP, the majority of which is borrowed from [15] and [26]. Consider a distributed system with a set U of components where each unit u offers a set amount $c_u \in \mathbb{N}$ of resource. Also in this system is a set P of processes which consume the resource available through each unit. For each process p in P, we denote the set amount of resource used by that process as $w_p \in \mathbb{N}$. A state is *admissible* if there is a mapping $f : P \to U \cup \{u_\infty\}$ so that $\sum_{p \in P(u,f)} w_p \leq c_u$ for every $u \in U \cup \{u_\infty\}$, where u_∞ denotes a "dummy" component that has infinite resource, and $P(u, f) = \{p \in P : f(p) = u\}$ for every unit u of U. If we are given two admissible states, f_s and f_t, define M to be the set of processes which have to be migrated to get from f_s to f_t. That is, $M = \{p \in P : \exists u \in U \text{ so that } p \in P(u, f_s) \setminus P(u, f_t)\}$. Note that $P(u_\infty, f_i) = \emptyset$ for all states i. If a process p in P is interrupted, that interruption will incur a cost, or penalty of $i_p \in \mathbb{R}_+$. Furthermore, if a process i is scheduled to be migrated before j, we will denote that as $i \prec j$.

When the PMP problem has previously been studied, all processes have been given the same priority: all processes were eligible for interruption [22, 24, 25]. A similar problem was also studied, in which all processes are not allowed to be interrupted [11]. In that case, solving the PMP problem is equivalent to scheduling the migrations of all processes in P. As each process is moved exactly once, this scheduling must be a linear ordering, and this problem is known as the *Linear Ordering* (LO) problem. Finding the optimal linear ordering of n elements is NP-complete [11], and this problem has been studied by Reinelt [20] and Grötschel et al. [11] using a polyhedral approach. If all processes are eligible for interruption, we must first find the set of interrupted processes, and then a partial linear ordering for the processes not interrupted. Such is the case of the *Partial Linear Ordering* (PLO) problem, which has been investigated by Sirdey et al. [22]. Kerivin [15], Sirdey and Kerivin [24, 25, 26] have also studied the problem through the use of the associated polytope. The PLO is NP-complete in the strong sense [22], although there are some polynomial-time solvable cases [22].

In this paper, we are approaching the PMP problem from a different vantage point. For the *Steiner Linear Ordering* (SLO) problem, we assume that the set of processes N can be partitioned into two sets, L and H, where L is the set of low-priority processes that are interruptible, and H is the set of high-priority processes that cannot be interrupted. By characterizing the processes as such, we can better model systems that have processes of different priority levels, making our models more accurate and our results more helpful. We will call the convex hull of the feasible solutions of the SLO the *Steiner linear ordering polytope* and denote it as $P_{SLO}(H, L)$. Notice that if the set L is empty, we are dealing with $SLO(H, \emptyset)$ where no process is allowed to be interrupted, and so this problem is equivalent to the *Linear Ordering* Problem ($LO(H, L)$) with $|H| = n$, in which we have to find a linear ordering for all processes. If the set H is empty, so we are working with $SLO(\emptyset, L)$ with $|L| = n$, all processes are candidates for interruption and this is equivalent to working with the *Partial Linear Ordering* problem ($PLO(H, L)$).

The *Steiner Linear Ordering* problem is so named for its similarities to the *Steiner Tree Problem*, (STP). The STP is defined on a simple, undirected graph $G = (V, E)$, where V is the set of vertices, E is the set of edges, and there exists an edge weight function $w : E \to \mathbb{Q}_+$. For the STP, you are also given a partition of V: $T \subseteq V$ called terminals and $S = V \setminus T$ called Steiner vertices. A *Steiner Tree* on $[G, T]$ is a minimal weight tree covering T [19]. Steiner vertices can be used to reduce the cost of the tree, or to provide connectivity when appropriate. This problem is NP-hard [17], although there are some instances of the STP that can be solved in polynomial time. Similar to the STP, an instance of a SLO provides a set H of “Terminal Processes”, and a set $L = P \setminus H$ of “Steiner Processes”. The goal of the SLO is to find a linear ordering for the terminal processes, a linear ordering which may include Steiner processes in the linear ordering or not, although there is a penalty for not ordering the low-priority processes.

In this paper, we will cover the Steiner Linear Ordering problem in detail, examining its formulation, the dimension of its convex hull, and several facet-defining inequalities. We will also discuss how the SLO is related to the LO and PLO problems, with respect to dimension and facet-defining inequalities. Chapter 3 will focus on applying the SLO to the *Process Move Programming* (PMP) problem, and several additional inequalities that apply. The third chapter will also examine the existence of feasible solutions, special structures that yield feasible solutions, and necessary conditions for the dimension of the convex hull of the PMP with respect to the SLO. In the conclusion, we will call attention to further areas of study, as well as additional applications that the SLO can be useful for.

Chapter 2

The Steiner Linear Ordering Problem

2.1 Formulation

Let N be the set of elements in the Steiner Linear Ordering problem SLO with $|N| = n$. The set N can be partitioned into two sets, L and H. The set H is the set of terminal elements which must be ordered, and L is the set of Steiner elements that may be in the ordering. We can define M to be the set of elements in N that are ordered, and define I to be the elements in $N \setminus M$, those elements not in the ordering. Let X be the set of feasible solutions. We say that a solution is in the set X if and only if it satisfies the total linear ordering relation on M. That is, for all $i, j \in M$, the solution must be

- Reflexive: $i \prec i$
- Antisymmetric: If $i \prec j$ and $j \prec i$, then $i = j$
- Transitive: If $i \prec j$ and $j \prec k$, then $i \prec k$
- Total: For every (i, j) pair, either $i \prec j$ or $j \prec i$

First, we must define our variables $x \in \{0, 1\}^p$ with $p = |M|^2 - |H|$ to be such that

$$x_{ll} = \begin{cases} 1 & \text{if element } l \text{ is not ordered} \\ 0 & \text{otherwise} \end{cases}$$

for all l in L, and

$$x_{ij} = \begin{cases} 1 & \text{if element } i \text{ is ordered before element } j \\ 0 & \text{otherwise} \end{cases}$$

for all i, j in $N \setminus I$.

With these variables, we can define the set $I(x)$ as $I(x) = \{l \in L | x_{ll} = 1\}$ and we can define a one-to-one function $\sigma(x) : M \to \{1, \ldots, |N \setminus I(x)|\}$ where $\sigma_i(x) = 1 + \sum_{j \in M} x_{ji}$ for $i \in M$. Thus we can define the set of feasible solutions mathematically as $X = \{x \in \{0,1\}^p \big| (I(x), \sigma(x)) \text{ is feasible}\}$, where feasiblity implies that the total linear ordering and priority restrictions are satisfied. Let $R = \{x \in \mathbb{R}^{\,p} \big| A^{=}x = b^{=},\ Ax \leq b,\ x \geq 0\}$, where $A^{=}x = b^{=}$ is given by

$$x_{ij} + x_{ji} = 1 \quad \forall \{i,j\} \text{ with } i,j \in H, i \neq j \tag{2.1}$$

$$x_{ij} + x_{ji} + x_{jj} = 1 \quad \forall \{i,j\} \text{ with } i \in H, j \in L \tag{2.2}$$

and $Ax \leq b$ is composed of

$$x_{ij} + x_{ji} + x_{jj} \leq 1 \quad \forall \{i,j\} \text{ with } i,j \in L, i \neq j \tag{2.3}$$

$$x_{ij} + x_{ji} + x_{jj} + x_{ii} \geq 1 \quad \forall \{i,j\} \text{ with } i,j \in L, i \neq j \tag{2.4}$$

$$x_{ij} + x_{jk} - x_{ik} \leq 1 \quad \forall \{i,j,k\} \text{ with } i,j,k \in M \tag{2.5}$$

(Note that we maintained the direction of (2.4) to avoid dealing with all non-positive variable coefficients.)

We now show that R is a formulation for X, by showing that if x is a solution in X, then $x \in R \cap \{0,1\}^p$, and also the reverse [30].

Claim 1 *R is a formulation for X.*

Proof: We begin by demonstrating that the inequalities (2.1) through (2.5) are valid for our formulation. That is, $x \in X$ implies that $x \in R \cap \{0,1\}$. Let x be a feasible solution of X. We know $I(x)$ and $\sigma(x)$, and we know $I(x) \cap H = \emptyset$. Since $\sigma(x)$ is a one-to-one mapping, we know that, for $h, h' \in H$, $h \neq h'$, either $h \prec h'$ or $h' \prec h$, and so (2.1) is satisfied. We turn our attention to $l \in L$. If $x_{ll} = 1$, then $l \in I(x)$ and l is not assigned an index in $\sigma(x)$, so $x_{lj} = x_{jl} = 0$ for all $j \in M$, as l is not part of the ordering. Thus (2.2) is valid. If $x_{ll} = 0$, l must be part of the ordering, and so (2.3) is valid. For $l, l' \in L$, if both l and l' are not in $I(x)$, they must be ordered, and $x_{ll'} + x_{l'l} = 1$, and $l, l' \notin I(x)$, so (2.4) is satisfied. If one or both of l and l' are in $I(x)$, $x_{ll'} = x_{l'l} = 0$, and so the inequality is also satisfied. Thus (2.4) is valid. As transitivity must hold for $x \in X$, we know that if $i, j, k \in M$, and if $i \prec j$ and $j \prec k$, it must be the case that $i \prec k$, then clearly (2.5) must hold. Thus $x \in X$ implies that $x \in R \cap \{0, 1\}$.

We will now show that $x \in R \cap \{0, 1\}^p$ implies that $x \in X$. Let $\bar{x}$ be a feasible solution of $R \cap \{0, 1\}$. Then it satisfies (2.1) through (2.5). For $h, h' \in H$ with $h \neq h'$, it is the case that either $h \prec h'$ or $h' \prec h$, as $\bar{x}$ must satisfy (2.1). Similarly, if $l \in L$ is such that $l \notin I(x)$, it must be the case that either $l \prec h$ or $h \prec l$ for every (l, h) pair. Finally, supposing that $l' \in L$ is such that $l' \notin I(x)$, with $l' \neq l$, we see from (2.3) and (2.4) that either $l \prec l'$ or $l' \prec l$, and thus a solution $\bar{x}$ must be antisymmetric with respect to all element pairs $(i, j) \in M$, and also totally ordered with respect to all elements of M, and thereby reflexive. For all cases of $i \neq j$, we see that if an element $l \in L$ is such that $l \in I(x)$, l is not assigned to a position in $\sigma(x)$, by (2.2) and (2.3). Inequality (2.5) clearly enforces transitivity of all elements of M, and so $x \in R \cap \{0, 1\}^p$ implies that $x \in X$, and thus our formulation is complete. ∎

2.2 Dimension and Technical Lemmas

In order to better understand our problem, we can examine

$$P_{SLO}(H, L) := \text{conv}\{x \in \{0, 1\}^p \big| A^= x = b^=,\ Ax \leq b,\ x \geq 0\}$$

which is the convex hull of our polytope. The goal with any polyhedral study is to obtain a complete description of the convex hull of a polytope, thereby easing our work in finding a solution [7]. Part of understanding a polytope is discerning its dimension, which is useful for identifying the facet-defining inequalities which make up the description of our convex hull. For $P_{SLO}(H, L)$, as long as the set H is not empty, the Steiner element move polytope is not full-dimensional, due to the system of equations $(A^=, b^=)$. Before we examine the facet-defining properties of $Ax \leq b$, and other related inequalities, we have some technical lemmas that will ease our representation of the proofs that follow.

We begin by introducing some notation that will be used in the technical lemmas. Let x be a point in $P_{SLO}(H, L)$. We denote $I(x)$ as the subset of elements of L which are not ordered for a point x. That is, $I(x) = \{l \in L \big| x_{ll} = 1\}$. Consider two distinct elements i and j in N. We denote $i \prec_x j$ whenever i is ordered before j for point x; that is, whenever $x_{ij} = 1$. We denote $i \prec\prec_x j$ whenever i is immediately ordered before j; that is, $x_{ij} = 1$ and $x_{ik} + x_{kj} = 0$ for all $k \in N \setminus \{i, j\}$.

Lemma 2 *Let $\mathcal{F} = \{x \in P_{SLO}(H,L) \big| A^0 x = b^0\}$ be a face of $P_{SLO}(H,L)$, and let i and j be two distinct elements in N. If there exist two points in $\mathcal{F}$, say x^1 and x^2, so that*

(a) $i \prec\prec_{x^1} j$,

(b) $j \prec\prec_{x^2} i$,

then $\alpha_{ij} = \alpha_{ji}$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0).

Proof:

$$\begin{aligned} \alpha_0 &= \alpha^T x^1 \\ &= \alpha^T x^2 - \alpha_{ji} + \alpha_{ij} \\ &= \alpha_0 - \alpha_{ji} + \alpha_{ij} \end{aligned}$$

and so we see that $\alpha_{ij} = \alpha_{ji}$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0), for $i, j \in N$, $i \neq j$. ∎

Lemma 3 *Let $\mathcal{F} = \{x \in P_{SLO}(H,L) \big| A^0 x = b^0\}$ be a face of $P_{SLO}(H,L)$, and let l be an element in L. If there exist two points in $\mathcal{F}$, say x^1 and x^2 so that*

(a) $I(x^1) = I(x^2) \cup \{l\}$ *and* $I(x^1) \neq I(x^2)$

(b) $i \prec_{x^2} j$ *if and only if* $i \prec_{x^1} j$ *for distinct i and j in* $N \setminus I(x^1)$

(c) $l \prec_{x^2} i$ *for* $i \in N \setminus I(x^1)$

then $\alpha_{ll} = \sum_{i \in N \setminus I(x^1)} \alpha_{li}$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0).

Proof:

$$\begin{aligned} \alpha_0 &= \alpha^T x^1 \\ &= \alpha^T x^2 - \sum_{i \in N \setminus I(x^1)} \alpha_{li} + \alpha_{ll} \\ &= \alpha_0 - \sum_{i \in N \setminus I(x^1)} \alpha_{li} + \alpha_{ll} \end{aligned}$$

and so $\alpha_{ll} = \sum_{i \in N \setminus I(x^1)} \alpha_{li}$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0), for all $l \in L$. ∎

Corollary 4 *Let $\mathcal{F} = \{x \in P_{SLO}(H,L) \big| A^0 x = b^0\}$ be a face of $P_{SLO}(H,L)$, and let l and l' be two distinct elements in L. If there exists three points in $\mathcal{F}$, say x^1, x^2, and x^3 so that*

(a) x^1 and x^3 satisfy conditions (a), (b), and (c) of Lemma 3 with respect to l'

(b) x^2 and x^3 satisfy conditions (a), (b), and (c) of Lemma 3 with respect to l'

(c) x^1 and x^2 satisfy condition (a) of Lemma 2

then $\alpha_{l'l} = 0$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0).

Proof: From Lemma 3 for x^1, x^3, with respect to some l', we see that

$$\alpha_{l'l'} = \sum_{i \in N \setminus I(x^1)} \alpha_{l'i}$$

From Lemma 3 for x^2, x^3, with respect to some l', we see that

$$\alpha_{l'l'} = \sum_{i \in N \setminus I(x^2)} \alpha_{l'i}$$

Since $I(x^1) = I(x^2) \cup \{l\}$, we also have that

$$\alpha_{l'l'} = \sum_{i \in N \setminus I(x^2)} \alpha_{l'i} + x_{l'l}$$

and therefore, then $\alpha_{l'l} = 0$ for every equation $\alpha^T x = \alpha_0$ of (A^0, b^0), for $l, l' \in L$, $l \neq l'$. ■

When discussing the partitions of $N = H \cup L$, we let $|H| = n_H$ and $|L| = n_L$, and define $[n_H] = \{1, \ldots, n_H\}$ and $[n_L] = \{1, \ldots, n_L\}$. With this notation in mind, we approach the next lemma.

Lemma 5 *The system $(A^=, b^=)$ is a minimal system for $P_{SLO}(H,L)$ with rank of $\frac{n_H(n_H-1)}{2} + n_L n_H$*

Proof: There are $\frac{n_H(n_H-1)}{2}$ equations in (2.1) and $n_L n_H$ equations in (2.2). Each equation of $(A^=, b^=)$ introduces a variable which is not present in the other equations of $(A^=, b^=)$. Therefore, the matrix $(A^=, b^=)$ is of full row rank. Thus $\text{rank}(A^=, b^=) = \frac{n_H(n_H-1)}{2} + n_L n_H$.

To prove that $(A^=, b^=)$ is minimal for $P_{SLO}(H,L)$, assume there exists another equation, say $\alpha^T x = \alpha_0$, not included in $(A^=, b^=)$, and so that $P_{SLO}(H,L) \subseteq \mathcal{F} = \{x \in P_{SLO}(H,L) \big| \alpha^T x = \alpha_0\}$ and α is a non-zero vector. Since $P_{SLO}(H,L)$ is a face for itself, Lemma 2 can be applied to any pair of distinct elements in N, that is,

$$\alpha_{ij} = \alpha_{ji} \text{ for distinct } i,\, j \in N \tag{1}$$

Moreover, we can also apply Lemma 3 to any element in L and get

$$\alpha_{ll} = \sum_{h \in H} \alpha_{lh} \tag{2}$$

Finally, using Corollary 4 for every pair of distinct elements in L, we then obtain

$$\alpha_{ll'} = 0 \text{ for distinct } l,\, l' \in L \tag{3}$$

From (1), (2), and (3), we can write

$$\alpha^T x = \sum_{i=1}^{n_H - 1} \sum_{j=i+1}^{n_H} \alpha_{h_i h_j}(x_{h_i h_j} + x_{h_j h_i}) + \sum_{l \in L} \sum_{h \in H} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll})$$

and $\alpha^T x = \alpha_0$ is a linear combination of the equations in $(A^=, b^=)$. Consequently, $(A^=, b^=)$ is a minimal system for $P_{SLO}(H, L)$. ■

We are now prepared to examine the dimension of our polytope $P_{SLO}(H, L)$.

Claim 6 *Let $|H| = n_H$ and $|L| = n_L$, so that $|N| = n_H + n_L$. Then* $\dim P_{SLO}(H, L) = (n_L + n_H)^2 - n_H - \left(\frac{n_H(n_H-1)}{2} + n_H n_L\right)$.

Proof: We know that the dimension of a polyhedron is at most equal to the difference between the number of variables and the rank of the matrix representing a minimal system of equalities satisfied by every point in the system [17]. The number of variables in our polyhedron is $(n_H + n_L)^2 - n_H$, and so $\dim P_{SLO}(H, L) \leq (n_H + n_L)^2 - n_H$. As we have shown in the previous lemma, the system $(A^=, b^=)$, generated by (2.1) and (2.2), is a minimal system with $\text{rank}(A^=, b^=) = \left(\dfrac{n_H(n_H-1)}{2} + n_H n_L\right)$. Thus the dimension of our polytope is written $\dim P_{SLO}(H, L) = (n_L + n_H)^2 - n_H - \left(\dfrac{n_H(n_H-1)}{2} + n_H n_L\right)$. ■

2.3 Facet-Defining Property Inequalities

We now delve more deeply into our investigation of the $P_{SLO}(H, L)$. We will begin that investigation by examining whether some inequalities do or do not define facets for the $P_{SLO}(H, L)$. Before we start, however, as we have shown that our polytope is not full-dimensional, it is important to note that there may be sets of inequalities that define equivalent facets.

Proposition 7 *Inequalities* $\pi^1 x \leq \pi_0^1$ *and* $\pi^2 x \leq \pi_0^2$*, denoted* (π^1, π_0^1) *and* (π^2, π_0^2)*, are equivalent with respect to our polytope if we can find some* $(\lambda, \mu) \in (\mathbb{R}_+ \times \mathbb{R}^m)$ *such that* $(\pi^2, \pi_0^2) = \lambda(\pi^1, \pi_0^1) + \mu(A^=, b^=)$ *where* $m = \text{rank}(A^=, b^=)$*. [17]*

We note some obvious instances of Proposition 7, although we will not list all equivalent inequalities. Also note that we will define $H = \{h_1, \dots, h_{n_H}\}$ to be an ordered set where $|H| = n_H$, and if $h_i \prec h_j$, the indices are ordered $i < j$, and similarly for L.

Claim 8 *Let* i *and* j *be two distinct elements in* N*. The inequality* $x_{ij} \geq 0$ *defines a facet for* $P_{SLO}(H, L)$*.*

Proof: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} = 0\}$, and assume that there exists a valid inequality $\alpha^T x \leq \alpha_0$ so that $\mathcal{F} \subseteq \mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| \alpha^T x = \alpha_0\}$. Lemma 2 can be applied to any pair of distinct elements in $N \setminus \{i, j\}$, and then

$$\alpha_{uv} = \alpha_{vu} \qquad \text{for distinct } \{u, v\} \in N,\ \{u, v\} \neq \{i, j\} \tag{1}$$

Using Lemma 3 and Corollary 4, we see that

$$\alpha_{ll} = \sum_{h \in H} \alpha_{lh} \qquad \text{for } l \in L,\ l \neq i \tag{2}$$

$$\alpha_{ii} = \sum_{h \in H} \alpha_{hi} \qquad \text{if } i \in L,\ l \neq i \tag{3}$$

We must examine three separate cases. First consider the case where i and j belong to H. As $i, j \in H$, we see that Corollary 4 can be applied to any pair of distinct elements in L, and then

$$\alpha_{ll'} = 0 \qquad \text{for distinct } \{l, l'\} \in L \tag{4}$$

Using the coefficients we have determined from $(1) - (4)$, we can write the equality $\alpha^T x = \alpha_0$ as

$$\alpha^T x = \sum_{h, h' \in H, \{h, h'\} \neq \{i, j\}, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L} \alpha_{ll} x_{ll} + \alpha_{ji}(x_{ij} + x_{ji})$$

As $x_{ij} = 0$, we can drop $\alpha_{ij}x_{ij}$. Also, we can make substitution and we have

$$\begin{aligned}\alpha^T x &= \sum_{h,h' \in H, \{h,h'\} \neq \{i,j\}, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L} \sum_{h \in H} \alpha_{hl} x_{ll} + \alpha_{ji} x_{ji} \\ &= \sum_{h,h' \in H, \{h,h'\} \neq \{i,j\}, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L} \alpha_{hl}(x_{hl} + x_{lh} + x_{ll}) + \alpha_{ji} x_{ji}\end{aligned}$$

Thus the equality in $\mathcal{F}^*$ can be written as a linear combination of equations from our minimal system $(A^=, b^=)$, and one additional equality, $x_{ji} = 1$, which is equivalent to $x_{ij} = 0$, for $i, j \in H$. Thus $\mathcal{F}$ defines a facet for $P_{SLO}(H, L)$.

We now consider the case where i and j are both in L. As $j \in L$, we can apply Corollary 4, and we see that

$$\alpha_{uv} = 0 \qquad \text{for all } \{u, v\} \neq \{i, j\}, \text{ with } \{u, v\} \in L, \tag{5}$$

(2.6)

and,

$$\alpha_{ji} = 0 \tag{6}$$

Using what we have learned about the variable coefficients from $(1) - (3)$ and $(5) - (6)$, we can write $\alpha^T x$ as

$$\begin{aligned}\alpha^T x &= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L \setminus \{i,j\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i,j\}} \alpha_{ll} x_{ll} + \alpha_{ii} x_{ii} + \alpha_{jj} x_{jj} \\ &= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L \setminus \{i,j\}} \alpha_{hl}(x_{hl} + x_{lh} + x_{ll}) + \alpha_{ii} x_{ii} + \alpha_{jj} x_{jj} + \alpha_{ji} x_{ji} \\ &= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L \setminus \{i,j\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i,j\}} \sum_{h \in H} \alpha_{hl} x_{ll} + \sum_{h \in H} \alpha_{hi} x_{ii} \\ &\quad + \sum_{h \in H} \alpha_{hi} x_{jj} + \alpha_{ji} x_{ji} \\ &= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L} \alpha_{hl}(x_{hl} + x_{lh} + x_{ll}) + \alpha_{ji} x_{ji}\end{aligned}$$

Thus the equality in $\mathcal{F}^*$ can be written as a linear combination of equations from our minimal system $(A^=, b^=)$, and one additional equality, $x_{ji} = 1$, which is equivalent to $x_{ij} = 0$, for $i, j \in H$. Thus $\mathcal{F}$ defines a facet for $P_{SLO}(H, L)$.

We conclude our proof by considering the case where i is in L and j is in H. Consider the points x' with $x'_{ii} = 1$, and x'' with $x''_{ii} = 0$ and $x''_{ji} = 1$. As these points are feasible and therefore in $\mathcal{F}^*$, we see that

$$\alpha_{ii} = \alpha_{ji} + \sum_{h \in H \setminus \{j\}} \alpha_{ih} \tag{7}$$

Using the information gathered from (1) – (3) and (7), we can now write $\alpha^T x$ as

$$\begin{aligned}
\alpha^T x &= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}, l \in L \setminus \{i\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i\}} \alpha_{ll} x_{ll} + \alpha_{ii} x_{ii} + \alpha_{ji} x_{ji} \\
&= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}, l \in L \setminus \{i\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i\}} \sum_{h \in H} \alpha_{hl} x_{ll} \\
&\quad + \sum_{h \in H \setminus \{j\}} \alpha_{ih} ii + \alpha_{ji} x_{ji} \\
&= \sum_{h,h' \in H, h \neq h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}, l \in L} \alpha_{hl}(x_{hl} + x_{lh} + x_{ll}) + \alpha_{ji} x_{ji}
\end{aligned}$$

Thus the equality in $\mathcal{F}^*$ can be written as a linear combination of equations from our minimal system $(A^=, b^=)$, and one additional equality, $x_{ji} = 1$, which is equivalent to $x_{ij} = 0$, for $i, j \in H$. Thus $\mathcal{F}$ defines a facet for $P_{SLO}(H, L)$. ■

Corollary 9 *Let i and j be distinct elements in H. The inequality $x_{ji} \leq 1$ defines the same facet as the inequality $x_{ij} \geq 0$ for $P_{SLO}(H, L)$.*

Proof: We apply Proposition 7, and let $(\bar{A}^=, \bar{b}^=)$ be a subset of $(A^=, b^=)$ such that $\bar{A}^= x = x_{ij} + x_{ji}$ and $\bar{b}^= = 1$, and let $\pi^1 x = x_{ij}$ and $\pi^1_0 = 0$. Then, with $(\lambda, \mu) = (1, -1)$, we see that $x_{ji} \leq 1$ is an inequality equivalent to $x_{ij} \geq 0$. ■

Claim 10 *Let i be an element in L, and let j be an element in N. The inequality $x_{ij} \leq 1$ does not define a facet for $P_{SLO}(H, L)$.*

Proof: We must examine two cases. First, let $i \neq j$. Then we can define $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} = 1\}$, and for all $x \in \mathcal{F}$, $x_{ii} = 0$. Thus we can also define $\mathcal{F}' = \{x \in P_{SLO}(H, L) \big| x_{ii} = 0\}$, and we see $\mathcal{F} \subsetneq \mathcal{F}'$, as there exists a point x' such that $x'_{ji} = 0$, so $x' \in \mathcal{F}'$, but $x' \notin \mathcal{F}$. We can also examine a point x^* such that $x^*_{ii} = 1$. Clearly $x^* \notin \mathcal{F}'$, but $x^* \in P_{SLO}(H, L)$. Then $\mathcal{F} \subsetneq \mathcal{F}' \subsetneq P_{SLO}(H, L)$, and the inequality $x_{ij} \leq 1$ does not define a facet if $i \neq j$.

Additionally, we must examine the case where $i = j$. Then we can define $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ii} = 1\}$, and for all $x \in \mathcal{F}$, $x_{ij} = 0$. Thus we can also define $\mathcal{F}' = \{x \in P_{SLO}(H, L) \big| x_{ij} = 0\}$, and we see $\mathcal{F} \subsetneq \mathcal{F}'$, as there exists a point x' such that $x'_{ji} = 0$, so $x' \in \mathcal{F}'$, but $x' \notin \mathcal{F}$. We can also examine a point x^* such that $x^*_{ji} = 1$. Clearly $x^* \notin \mathcal{F}'$, but $x^* \in P_{SLO}(H, L)$. Then $\mathcal{F} \subsetneq \mathcal{F}' \subsetneq P_{SLO}(H, L)$, and the inequality $x_{ij} \leq 1$ does not define a facet if $i = j$. ■

Claim 11 *Let l be an element of L. The inequality $x_{ll} \geq 0$ defines a facet if and only if $L = \{l\}$.*

Proof:

If $n_L \geq 2$: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ii} = 1\}$. For every $\bar{x} \in \mathcal{F}$, $\bar{x}_{ii} = 0$, and so we can define, for $j \in L \backslash \{i\}$, $\mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{ji} + x_{ii} + x_{jj} = 1\}$, and notice that $\mathcal{F} \subseteq \mathcal{F}^*$. However, for $\bar{x}^*$ corresponding to interrupting every element in $l \in L \setminus \{j\}$, $\bar{x}^* \in \mathcal{F}^*$ but $\bar{x}^* \notin \mathcal{F}$, and thus $\mathcal{F} \subsetneq \mathcal{F}^*$. Additionally, we can consider a point x' corresponding to interrupting all elements $l \in L$. Then $x' \in P_{SLO}(H, L)$, but $x' \notin \mathcal{F}^*$. Thus $\mathcal{F} \subsetneq \mathcal{F}^* \subsetneq P_{SLO}(H, L)$, and the constraint $x_{ii} \geq 0$ does not define a facet for $P_{SLO}(H, L)$.

If $n_L = 1$: Lemma 2 applied for any pair of distinct elements in N yields $\alpha_{ij} = \alpha_{ji}$. Then

$$\begin{aligned}
\alpha^T x &= \sum_{i=1}^{n_h - 1} \sum_{j=i+1}^{n_H} \alpha_{h_i h_j}(x_{h_i h_j} + x_{h_j h_i}) + \sum_{h \in H} \alpha_{hl}(x_{hl} + x_{lh}) + \alpha_{ll} x_{ll} \\
&= \sum_{i=1}^{n_h - 1} \sum_{j=i+1}^{n_H} \alpha_{h_i h_j}(x_{h_i h_j} + x_{h_j h_i}) + \sum_{h \in H} \alpha_{hl}(x_{hl} + x_{lh}) + \alpha_{ll} x_{ll} + \sum_{h \in H} (\alpha_{hl} - \alpha_{lh}) x_{ll} \\
&= \sum_{i=1}^{n_h - 1} \sum_{j=i+1}^{n_H} \alpha_{h_i h_j}(x_{h_i h_j} + x_{h_j h_i}) + \sum_{h \in H} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{h \in H} \alpha_{hl}(x_{hl} + x_{lh} + x_{ll}) \\
&\quad + (\alpha_{ll} - \sum_{h \in H} \alpha_{hl}) x_{ll}
\end{aligned}$$

so α is a linear combination of the equations in $\mathcal{F}$. ■

Claim 12 *Let i and j be distinct elements of L. The inequality $x_{ij} + x_{ji} + x_{jj} \leq 1$ defines a facet for $P_{SLO}(H, L)$.*

Proof: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{ji} + x_{jj} = 1\}$, and assume that there exists an inequality $\alpha^T x \leq \alpha_0$ such that $\mathcal{F} \subseteq \mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| \alpha^T x = \alpha_0\}$. Lemma 2 can be applied for any pair of distinct elements in $N \setminus \{i, j\}$, yielding

$$\alpha_{uv} = \alpha_{vu} \qquad \text{for distinct } \{u, v\} \in N,\ \{u, v\} \neq \{i, j\} \tag{1}$$

From application of Lemma 3 and Corollary 4, we see that

$$\alpha_{ll} = \sum_{h \in H} \alpha_{lh} \qquad \text{for } l \in L \setminus \{i\} \tag{2}$$

$$\alpha_{ij} = \alpha_{ji} \tag{3}$$

$$\alpha_{uv} = 0 \qquad \text{for distinct } \{u, v\} \in L,\ \{u, v\} \neq \{i, j\} \tag{4}$$

Using the information gathered from (1) – (4), we can now complete our proof by showing that we can write $\alpha^T x$ as

$$\begin{aligned} \alpha^T x &= \sum_{h,h' \in T, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L \setminus \{j\}} \alpha_{ll} x_{ll} + \sum_{h \in H, l \in L} \alpha_{lh}(x_{lh} + x_{hl}) + \alpha_{ij} x_{ij} \\ &\quad + \alpha_{ji} x_{ji} + \alpha_{jj} x_{jj} \\ &= \sum_{h,h' \in T, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H, l \in L \setminus \{j\}} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll}) + \alpha_{ij}(x_{ij} + x_{ji} + x_{jj}) \end{aligned}$$

Thus $\mathcal{F}$ defines a facet for $P_{SLO}(H, L)$. ■

Claim 13 *Let i and j be distinct elements of L. The inequality $x_{ij} + x_{ji} + x_{ii} + x_{jj} \geq 1$ defines a facet for $P_{SLO}(H, L)$.*

Proof: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{ji} + x_{jj} = 1\}$, and assume that there exists the inequality $\alpha^T x \leq \alpha_0$ such that $\mathcal{F} \subseteq \mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| \alpha^T x = \alpha_0\}$. We begin by applying Lemma 2 to all distinct pairs of N, not equal to $\{i, j\}$, and we see that

$$\alpha_{uv} = \alpha_{vu} \qquad \text{for all distinct } \{u, v\} \in N,\ \{u, v\} \neq \{i, j\} \tag{1}$$

Additionally, through application of Lemma 3, we see that

$$\alpha_{ij} = \alpha_{ji} \tag{2}$$

When we consider Lemma 3 in conjunction with Corollary 4, we find that

$$\alpha_{ll'} = 0 \qquad \text{for all } \{l, l'\} \in L,\ \{l, l'\} \neq \{i, j\} \tag{3}$$

We use this information about the variable coefficients from (1) − (3), and now write out $\alpha^T x$ as

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L} \sum_{h \in H} \alpha_{lh}(x_{lh} + x_{hl}) + \sum_{l \in L \setminus \{i,j\}} \alpha_{ll} x_{ll} + \alpha_{ii} x_{ii} + \alpha_{jj} x_{jj} \\
&+ \alpha_{ij} x_{ij} + \alpha_{ji} x_{ji} \\
= &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L \setminus \{i,j\}} \sum_{h \in H} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll}) + \alpha_{ij} x_{ii} + \sum_{h \in H} \alpha_{ih} x_{ii} \\
&+ \alpha_{ij} x_{jj} + \sum_{h \in H} \alpha_{jh} x_{jj} + \alpha_{ij}(x_{ij} + x_{ji}) \\
= &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L} \sum_{h \in H} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll}) + \alpha_{ij}(x_{ij} + x_{ji} + x_{ii} + x_{jj})
\end{aligned}$$

We see that $\alpha^T x = \alpha_0$ can be written as a linear combination of equations from the minimal system and the one additional equality defined by our face $\mathcal{F}$, and thus $\mathcal{F}$ is facet-defining for $P_{SLO}(H, L)$. ■

Claim 14 *Let i, j, and k be distinct elements, with $j \in H$ and $i, k \in P$. The inequality $x_{ij} + x_{jk} - x_{ik} \leq 1$ defines a facet for $P_{SLO}(H, L)$.*

Proof: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{jk} - x_{ik} = 1\}$, and assume that there exists an inequality $\alpha^T x \leq \alpha_0$ such that $\mathcal{F} \subseteq \mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| \alpha^T x = \alpha_0\}$. We begin by applying Lemma 2 to all applicable $\{u, v\}$ pairs in N, and we see that

$$\alpha_{uv} = \alpha_{vu} \qquad \text{for all } \{u, v\} \in N \text{ with } \{u, v\} \notin \big\{\{i, j\}, \{j, k\}, \{i, k\}\big\} \tag{1}$$

By an argument similar to the proof for Lemma 3, we see that

$$\alpha_{ll} = \sum_{h \in H \cup \{i,k\}} \alpha_{lh} \qquad \text{for all } l \in L \setminus \{i, k\}, \tag{2}$$

and Corollary 4, we know

$$\alpha_{ll'} = 0 \qquad \text{for all } \{l, l'\} \in L \setminus \{i, k\} \tag{3}$$

The remainder of our proof is divided into three cases. We must now concern ourselves with the ordering of $\{i, j, k\}$. We will examine three points x^1, x^2, and x^3 where the orderings are $k \prec i \prec j$, $j \prec k \prec i$, and $i \prec j \prec k$ respectively. As all three of these points are valid for $\mathcal{F}$, we know that $\alpha_{kj} - \alpha_{jk} = \alpha_{ji} - \alpha_{ij} = \alpha_{ik} - \alpha_{ki}$. We need to make some modifications, and so we will add a particular form of zero to each pair of $\{i, j, k\}$, yielding the following representation

$$\begin{aligned}
\alpha_{ij}x_{ij} + \alpha_{ji}x_{ji} &= \alpha_{ij}x_{ij} + \alpha_{ji}x_{ji} + \alpha_{ji}x_{ij} - \alpha_{ji}x_{ij} \\
\alpha_{jk}x_{jk} + \alpha_{kj}x_{kj} &= \alpha_{jk}x_{jk} + \alpha_{kj}x_{kj} + \alpha_{kj}x_{jk} - \alpha_{kj}x_{jk} \\
\alpha_{ik}x_{ik} + \alpha_{ki}x_{ki} &= \alpha_{ik}x_{ik} + \alpha_{ki}x_{ki} + \alpha_{ki}x_{ik} - \alpha_{ki}x_{ik}.
\end{aligned}$$

The remainder of our work is divided into three cases.

If i, j, and k are all in H, our examination is complete, and using the information found in (1) – (4), we can now write $\alpha^T x$ as

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H \setminus \{i,j,k\}, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L, h \in H\}} \alpha_{lh}(x_{lh} + x_{hl}) \\
&+ \sum_{l \in L} \alpha_{ll}x_{ll} + (\alpha_{ij} - \alpha_{ji})(x_{ij} + x_{jk} - x_{ik}) + \alpha_{ji}(x_{ij} + x_{ji}) + \alpha_{jk}(x_{jk} + x_{kj}) + \alpha_{ik}(x_{ik} + x_{ki}) \\
= &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L, h \in H\}} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll}) + (\alpha_{ij} - \alpha_{ji})(x_{ij} + x_{jk} - x_{ik})
\end{aligned}$$

Thus $\alpha^T x = \alpha_0$ can be written as a linear combination of equations from the minimal system and the one additional equality defined by our face $\mathcal{F}$, and thus $\mathcal{F}$ is facet-defining for $P_{SLO}(H, L)$.

We also examine the case where i is in L and k is in H. We must consider a fourth point x^4 where i is interrupted and $j \prec k$. As this point is valid for our face, we see that, when compared to x^3, we can write

$$\alpha_{ii} = \sum_{h \in H \cup \{k,j\}} \alpha_{ih} + \alpha_{ij} + \alpha_{ik} \tag{4}$$

Using information from (1) – (4), we can now write $\alpha^T x$ as

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H \setminus \{j,k\}, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L \setminus \{i\}, h \in H\}} \alpha_{lh}(x_{lh} + x_{hl}) + \sum_{l \in L} \alpha_{ll}x_{ll} \\
&+ (\alpha_{ij} - \alpha_{ji})(x_{ij} + x_{jk} - x_{ik}) + \alpha_{ji}(x_{ij} + x_{ji}) + \alpha_{jk}(x_{jk} + x_{kj}) + \alpha_{ik}(x_{ik} + x_{ki}) + \alpha_{ii}x_{ii} \\
= &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L, h \in H\}} \alpha_{lh}(x_{lh} + x_{hl} + x_{ll}) + (\alpha_{ij} - \alpha_{ji})(x_{ij} + x_{jk} - x_{ik})
\end{aligned}$$

Thus $\alpha^T x = \alpha_0$ can be written as a linear combination of equations from the minimal system and the one additional equality defined by our face $\mathcal{F}$, and thus $\mathcal{F}$ is facet-defining for $P_{SLO}(H, L)$.

If we examine the case where $k \in L$, $i \in H$, we see that this case is similar to the previous case. ■

Corollary 15 *Let i, k be distinct elements in H, and j in L. The face defined by $x_{ij} + x_{jk} + x_{ki} \leq 2$ is equivalent to the face defined by $x_{ij} + x_{jk} - x_{ik} \leq 1$.*

Proof: Let $(\bar{A}^{=}, \bar{b}^{=})$ be a subset of $(A^{=}, b^{=})$, such that $\bar{A}^{=}x = x_{ik} + x_{ki}$ and $\bar{b}^{=} = 1$, and let $\pi^1 x = x_{ij} + x_{jk} - x_{ik}$ and $\pi_0^1 = 1$. Then, with $(\lambda, \mu) = (1, 1)$, we see that $x_{ij} + x_{jk} + x_{ki} \leq 2$ is an inequality equivalent to $x_{ij} + x_{jk} - x_{ik} \leq 1$. ■

Remark: As the following inequality has not been previously studied, we first show that it is valid for $P_{SLO}(H, L)$, and then prove that it is facet-defining.

Claim 16 *Let i, j, and k be distinct elements with $j \in H$ and $i, k \in L$. The inequality $x_{ij} + x_{jk} + x_{ki} + x_{ii} + x_{kk} \leq 2$ is valid for and defines a facet of $P_{SLO}(H, L)$.*

Proof:

Validity: Let $\bar{x}$ be a point in $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{jk} + x_{ki} + x_{ii} + x_{kk} = 2\}$, with $\bar{x}_{ii} = \bar{x}_{kk} = 1$. Then by (2.2) and (2.3), $\bar{x}_{ij} = \bar{x}_{jk} = \bar{x}_{ki} = 0$, and the left-hand side has at most a value of two. Now let $\tilde{x}$ be a point in $\mathcal{F}$ with $\tilde{x}_{ii} = 1$ and $\tilde{x}_{kk} = 0$. Again, by (2.2) and (2.3), we see that $\tilde{x}_{ij} = \tilde{x}_{ki} = 0$, and the left-hand side has at most a value of two, with a similar argument for $\tilde{x}'$ with $\tilde{x}_{ii} = 0$ and $\tilde{x}_{kk} = 1$. Finally, we examine x' where $x'_{kk} = x'_{ii} = 0$. Due to the transitivity inequality (2.5), we know that the left-hand side has value at most two, and our inequality is valid for $P_{SLO}(H, L)$.

Facet-Defining: Let $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| x_{ij} + x_{ji} + x_{jj} = 1\}$, and assume that there exists an inequality $\alpha^T x \leq \alpha_0$ such that $\mathcal{F} \subseteq \mathcal{F}^* = \{x \in P_{SLO}(H, L) \big| \alpha^T x = \alpha_0\}$. We begin by applying Lemma 2 to all applicable $\{u, v\}$ pairs in N, and we see that

$$\alpha_{uv} = \alpha_{vu} \qquad \text{for all } \{u, v\} \in N \text{ with } \{u, v\} \notin \big\{\{i, j\}, \{j, k\}, \{i, k\}\big\} \tag{1}$$

By application of Lemma 3, we see that

$$\alpha_{ll} = \sum_{h \in H \cup \{i, k\}} \alpha_{lh} \qquad \text{for all } l \in L, \tag{2}$$

and from Corollary 4, we know

$$\alpha_{ll'} = 0 \qquad \text{for all } \{l, l'\} \in L. \tag{3}$$

If we assume i and k are not interrupted, we must concern ourselves with the ordering of $\{i, j, k\}$. Through examination of $\mathcal{F}$, we see that there are three feasible orderings: either $i \prec\prec j \prec\prec k$ or $k \prec\prec i \prec\prec j$ or $j \prec\prec k \prec\prec i$. By examining all three of these orderings, we see that it must be the case that

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ik} = \alpha_{ki} + \alpha_{ij} + \alpha_{kj} = \alpha_{jk} + \alpha_{ki} + \alpha_{ji}$$

which in turn yields $\alpha_{kj} - \alpha_{jk} = \alpha_{ki} - \alpha_{ik} = \alpha_{ji} - \alpha_{ij}$. With this information, along with (1) – (3), we now turn to writing $\alpha^T x$ as

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L \setminus \{i,k\}} \alpha_{ll} x_{ll} + \alpha_{ii} x_{ii} + \alpha_{kk} x_{kk} \\
&+ \sum_{h \in H \setminus \{j\}} \sum_{l \in L \setminus \{i,k\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i,k\}} \alpha_{jl}(x_{jl} + x_{lj}) + \sum_{h \in H \setminus \{j\}} \alpha_{ih}(x_{ih} + x_{hi}) \\
&+ \sum_{h \in H \setminus \{j\}} \alpha_{hk}(x_{hk} + x_{kh}) + \alpha_{ij} x_{ij} + \alpha_{ji} x_{ji} + \alpha_{ik} x_{ik} + \alpha_{ki} x_{ki} + \alpha_{ij} x_{ij} + \alpha_{jk} x_{jk} + \alpha_{kj} x_{kj}
\end{aligned}$$

By intelligently adding particular forms of zero as seen below, we have

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{l \in L \setminus \{i,k\}} \alpha_{ll} x_{ll} + \alpha_{ii} x_{ii} + \alpha_{kk} x_{kk} \\
&+ \sum_{h \in H \setminus \{j\}} \sum_{l \in L \setminus \{i,k\}} \alpha_{hl}(x_{hl} + x_{lh}) + \sum_{l \in L \setminus \{i,k\}} \alpha_{jl}(x_{jl} + x_{lj}) + \sum_{h \in H \setminus \{j\}} \alpha_{ih}(x_{ih} + x_{hi}) \\
&+ \sum_{h \in H \setminus \{j\}} \alpha_{hk}(x_{hk} + x_{kh}) + \alpha_{ij} x_{ij} + \alpha_{ji} x_{ji} + \alpha_{ik} x_{ik} + \alpha_{ki} x_{ki} + \alpha_{ij} x_{ij} \\
&+ \alpha_{jk} x_{jk} + \alpha_{kj} x_{kj} + \big(\alpha_{ji} x_{ij} - \alpha_{ji} x_{ij} + \alpha_{ik} x_{ki} - \alpha_{ik} x_{ki} + \alpha_{kj} x_{jk} - \alpha_{kj} x_{jk}\big)
\end{aligned}$$

After much algebra, this condenses to

$$\begin{aligned}
\alpha^T x = &\sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}} \sum_{l \in L \setminus \{i,k\}} \alpha_{hl}(x_{hl} + x_{lh} + \alpha_{ll}) \\
&+ \sum_{l \in L \setminus \{i,k\}} \alpha_{jl}(x_{jl} + x_{lj}) + \sum_{h \in H \setminus \{j\}} \alpha_{ih}(x_{ih} + x_{hi} + x_{ii}) + \sum_{h \in H \setminus \{j\}} \alpha_{hk}(x_{hk} + x_{kh} + x_{kk}) \\
&+ \alpha_{ii} x_{ii} + \alpha_{kk} x_{kk} + (\alpha_{ij} - \alpha_{ji}) x_{ij} + \alpha_{ji}(x_{ij} - x_{ji}) + (\alpha_{ki} - \alpha_{ik}) x_{ki} \\
&+ \alpha_{ik}(x_{ik} + x_{ki}) + (\alpha_{jk} + \alpha_{kj}) x_{jk} + \alpha_{kj}(x_{kj} + x_{jk})
\end{aligned}$$

Again, we add a convenient form of zero, and we have

$$\alpha^T x = \sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}} \sum_{l \in L \setminus \{i,k\}} \alpha_{hl}(x_{hl} + x_{lh} + \alpha_{ll})$$
$$+ \sum_{l \in L \setminus \{i,k\}} \alpha_{jl}(x_{jl} + x_{lj}) + \sum_{h \in H \setminus \{j\}} \alpha_{ih}(x_{ih} + x_{hi} + x_{ii}) + \sum_{h \in H \setminus \{j\}} \alpha_{hk}(x_{hk} + x_{kh} + x_{kk})$$
$$+ \alpha_{ii}x_{ii} + \alpha_{kk}x_{kk} + (\alpha_{ij} - \alpha_{ji})x_{ij} + \alpha_{ji}(x_{ij} - x_{ji}) + (\alpha_{ki} - \alpha_{ik})x_{ki}$$
$$+ \alpha_{ik}(x_{ik} + x_{ki}) + (\alpha_{jk} + \alpha_{kj})x_{jk} + \alpha_{kj}(x_{kj} + x_{jk}) + \big((\alpha_{ij} - \alpha_{ji})x_{ii} - (\alpha_{ij} - \alpha_{ji})x_{ii}$$
$$+ (\alpha_{ik} - \alpha_{ki})x_{kk} - (\alpha_{ik} - \alpha_{ki})x_{kk}\big)$$

This simplifies to

$$\alpha^T x = \sum_{h,h' \in H, h \prec h'} \alpha_{hh'}(x_{hh'} + x_{h'h}) + \sum_{h \in H \setminus \{j\}} \sum_{l \in L \setminus \{i,k\}} \alpha_{hl}(x_{hl} + x_{lh} + \alpha_{ll})$$
$$+ \sum_{l \in L \setminus \{i,k\}} \alpha_{jl}(x_{jl} + x_{lj}) + \sum_{h \in H \setminus \{j\}} \alpha_{ih}(x_{ih} + x_{hi} + x_{ii}) + \sum_{h \in H \setminus \{j\}} \alpha_{hk}(x_{hk} + x_{kh} + x_{kk})$$
$$+ (\alpha_{ij} - \alpha_{ji})(x_{ij} + x_{jk} + x_{ki} + x_{ii} + x_{kk}) + \alpha_{ji}(x_{ij} + x_{ji} + x_{ii}) + \alpha_{kj}(x_{jk} + x_{kj} + x_{kk})$$

Thus $\alpha^T x = \alpha_0$ can be written as a linear combination of equations from the minimal system and the one additional equality defined by our face $\mathcal{F}$, and thus $\mathcal{F}$ is facet-defining for $P_{SLO}(H, L)$. ■

Notice that using Claims 14 and 16, we have shown that transitivity is a facet-defining property for our problem.

2.4 Trivial Lifting

In Grötschel et al. [11], we are introduced to trivial lifting as a method for extending facets of a polytope of degree n to be facets for a polytope of degree $n+1$ for the P^n_{LO}. This subject also comes up in Sirdey and Kerivin's paper [24] with respect to the P^n_{PLO}. For our paper, we examine a modified form of trivial lifting that satisfies the requirements of our polytope $P_{SLO}(H, L)$.

Proposition 17 *Let $\alpha^T x \leq \alpha_0$ be a facet-defining inequality for $P_{SLO}(H, K)$. If we introduce a new element*

$u \in H$, and set

$$\bar{\alpha}_{ij} = \begin{cases} \alpha_{ij} & \text{for all } i, j \in \{1, \ldots, n\}, i \neq j, \\ \alpha_{ii} & \text{for all } i \in L, \\ 0 & \text{if } i = u \text{ or } j = u \end{cases}$$

then the inequality $\bar{\alpha}^T x \leq \alpha_0$ defines a facet of $P_{SLO}(H \cup \{u\}, L)$.

Proof: When we add an element u into H, we increase the dimension of our polytope by $|N| = |H \cup L| = n$, as evidenced by Proposition 6. Let $\dim P_{SLO}(H, L) = k$. We first show that we can extend the affinely independent points in $\mathcal{F} = \{x \in P_{SLO}(H, L) \big| \alpha^T x \leq \alpha_0\}$ to be points in $P_{SLO}(H \cup \{u\}, L)$, and still retain the affine independence of these points. Suppose we extend these points by letting $x_{up} = 1$ for all $p \in N$, for all $x \in \mathcal{F}$. We know that the points in $\mathcal{F}$ are affinely independent, and so if $\sum_{i=1}^k \alpha^i x^i = 0$ and $\sum_{i=1}^k \alpha^i = 0$, then $\alpha^i = \vec{0}$ for all $i = 1, \ldots, k$. If, by the proposed extension, we lose the affine independence of these points, that would mean that even if $\sum_{i=1}^k \bar{\alpha}^i x^i = 0$ and $\sum_{i=1}^k \bar{\alpha}^i = 0$, it could be the case that $\bar{\alpha}^i \neq \vec{0}$ for all $i = 1, \ldots, k$. However, by definition, we know that $\bar{\alpha}^i_{up} = 0$ for all α^i, for all $p \in N$, and this implies that we could have $\sum_{i=1}^k \alpha^i x^i = 0$ and $\sum_{i=1}^k \alpha^i = 0$, but $\alpha^i \neq \vec{0}$. This is clearly a contradiction to starting with affinely independent points, and so our method of extension must maintain the independence of the points. We now want to show that we can create n more affinely independent points. Begin by partitioning N into $N_1, N_2, \ldots, N_k$ where

$$N_i = \left\{ p_0 \in N \Big| \sum_{p \in N \setminus \{p_0\}} x_{p_0 p} = n - i, \text{ and } p_0 \notin \cup_{j=1}^{i-1} N_j \right\}.$$

For each $p_0 \in N_i$, let $p_0 \prec u$ and $u \prec p$ for all $p \in N \setminus \{p_0\}$. This point is the first point for which $x_{p_0 u} = 1$, and thus it must be affinely independent of the previous points. Also, we have created n new points, as $N_1, N_2, \ldots, N_k$ creates a complete partitioning of N. If it did not, there would be a $p \in N$ with $p \notin \cup_{i=1}^k N_i$. Thus there would not be a point where $x_{pu} = 1$, and since from Claim 10, we know that the inequality $x_{ii} \leq 1$ does not define a face, the facet created by the inequality $\alpha^T x \leq \alpha_0$, would be included within the face where $x_{pu} = 0$. This contradicts $\alpha^T x \leq \alpha_0$ defining a facet for $N \cup \{u\}$, and thus we must create n new points. ∎

2.4.1 Additional Facet-Defining Inequalities

We now examine two families of inequalities, the *k-clique* inequality and the *k-unicycle* inequality, presented in [24] with respect to the P^n_{PLO}.

Let $\mathcal{I} \subset \{1, \ldots, p\}$ with $|\mathcal{I}| = k$. The *k-clique* inequality is

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} x_{ij} \geq |\mathcal{I}| - 1 \tag{2.7}$$

Let $\mathcal{I} \subset \{1, \ldots, p\}$ with $|\mathcal{I}| = k$ and $i_0 \in \{1, \ldots, n\} \setminus \{\mathcal{I}\}$. The *k-unicycle* inequality is

$$x_{i_0 i_0} + \sum_{i \in \mathcal{I}} (x_{i i_0} + x_{i_o i}) - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I} \setminus \{i\}} x_{ij} \leq 1 \tag{2.8}$$

Claim 18 *If $\mathcal{I} \subseteq L$ the k-clique inequality is valid for the $P_{SLO}(H, L)$.*

Proof: Observe that the P^n_{PLO} is equivalent to an instance of the $P_{SLO}(H, L)$ where $H = \emptyset$, and so it is clearly feasible if $H = \emptyset$ and $\mathcal{I} \subseteq L$. If H is not empty, it does not effect the inequality at all, and thus it remains valid. ■

Claim 19 *If $\mathcal{I} \subseteq L$ the k-unicycle inequality is valid for the $P_{SLO}(H, L)$.*

Proof: Observe that the P^n_{PLO} is equivalent to an instance of the $P_{SLO}(H, L)$ where $H = \emptyset$, and so it is clearly feasible if $H = \emptyset$ and $\mathcal{I} \subseteq L$. If H is not empty, if $i_0 \notin H$, then H does not influence the inequality at all, and the inequality remains valid. If $i_0 \in H$, we cannot have $x_{i_0 i_0}$ as the variable does not exist, so we confine ourselves to examining only the interactions between i_0 and elements of $\mathcal{I}$. Let $I(x)$ denote the elements of x that are not ordered. Then by (2.2), we know that the sum

$$\sum_{i \in \mathcal{I}} (x_{i i_0} + x_{i_0 i}) = |\mathcal{I}| - |I(x)| \tag{2.9}$$

When we examine the next sum, we see that, by (2.3),

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I} \setminus \{i\}} x_{ij} \leq |\mathcal{I}| - |I(x)| - 1$$

and so the inequality is valid. ■

We will focus on applying the trivial lifting technique to the *k-clique* and *k-unicycle* inequalitieslNotice that we have already been introduced to the basic form of the *k-clique* and *k-unicycle* inequalities in (2.4) and (2.3), respectively. We desire to show that we can apply Proposition 17 to make the *k-clique* and *k-unicycle* inequalities facet-defining for $P_{SLO}(H, L)$. However, following the statement of the proposition, we must first know that the inequalities are facet-defining for the lower-dimensional polytope. To that effect, we will reproduce the necessary proofs from [24] for the two inequalities.

Proposition 20 *The k-clique inequality is a facet of* P^n_{PLO}.

Proof: Let D_I denote the complete digraph having N as a node set, with $|N| = n$. Let us consider the following points of P^n_{PLO}:

1. The points which only order node k, for each $k \in N$ (i.e., $x_{kk} = 0$ and $x_{ll} = 1$ for all $l \in N \setminus \{k\}$). There are $|N|$ such points, and so it is clear that inequality (2.7) is tight for them.

2. The points which only order node k_1 before k_2 for each k_1, $k_2 \in N$, $k_1 \neq k_2$. There are $|N|(|N| - 1)$ such points and again, it is clear that inequality (2.7) is tight for them.

Linear, and hence affine, independence of the above $|N|^2$ points is straightforward. Hence, as they all belong to P^n_{PLO}, inequality (2.7) is facet-defining for P^n_{PLO}. ■

Proposition 21 *The k-unicycle inequality is a facet of* P^n_{PLO}.

Proof: Let us consider the following points of P^n_{PLO} where $i_0 \in N$ and $|N| = n$:

1. The point which orders no nodes (i.e., $x_{ii} = 1$ for all $i \in N$), and for which inequality (2.8) is obviously tight.

2. The points which only order node k, for each $k \in N$. There are n such points and inequality (2.8) is obviously tight for them (only the 1-valued loop of node i_0 is selected). Linear independence follows from the fact that a point in this set is the only point so far such that $x_{kk} = 0$.

3. The points which order either only node i_0 before node k or only node k before i_0, for each $k \in N$. There are $2n$ such points and inequality (2.8) is obviously tight for them (only one 1-valued arc is selected, in-between i_0 and k). Linear independence follows from the fact that a point in this set is the only point so far such that either $x_{ki_0} = 1$ or $x_{i_0k} = 1$.

4. The points which order only node i_0 before node k_1, node i_0 before node k_2 and either node k_1 before node k_2 or node k_2 before node k_1, for each $\{k_1, k_2\} \subseteq N$ (i.e., $x_{i_0k_1} = x_{i_0k_2} = 1$, $x_{k_1k_2} = 1$, or $x_{k_2k_1} = 1$, and $x_{ii} = 1$ for all $i \in N \setminus \{k_1, k_2\}$). There are $n(n-1)$ such points and inequality (2.8) is tight for them (two 1-valued arcs and one (-1)-valued arc are selected). Linear independence follows from the fact that a point in this set is the only point so far such that either $x_{k_1k_2} = 1$ or $x_{k_2k_1} = 1$.

Hence, we have we have exhibited n^2 linearly, and therefore affinely, independent points of P^n_{PLO} for which inequality (2.8) is tight. It follows that inequality (2.8) is facet-defining for P^n_{PLO}. ■

Claim 22 *Let $\mathcal{I} \subset \{1, \ldots, p\}$ with $|\mathcal{I}| = k$. If $I \subseteq L$, the k-clique inequality,$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}} x_{ij} \geq |\mathcal{I}| - 1$ is facet-defining for $P_{SLO}(H, L)$.*

Proof: Let it be noted that the *k-clique* inequality is facet-defining for P^n_{PLO}. We can write this instance of the PLO as an instance of the SLO, and so the *k-clique* inequality is facet-defining for $P_{SLO}(\emptyset, L)$ where $L = N$. By application of Proposition 17, we can add elements into H until we reach the final (H, L) set, and the inequality remains facet-defining. ■

Claim 23 *Let $\mathcal{I} \subset \{1, \ldots, p\}$ with $|\mathcal{I}| = k$ and $i_0 \in \{1, \ldots, n\} \setminus \mathcal{I}$. If $I \subseteq L$, the k-unicycle inequality $x_{i_0i_0} + \sum_{i\in\mathcal{I}}(x_{ii_0} + x_{i_oi}) - \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}\setminus\{i\}} x_{ij} \leq 1$ is facet-defining for $P_{SLO}(H, L)$.*

Proof: Let it be noted that the *k-unicycle* inequality is facet-defining for P^n_{PLO}. We can write this instance of the PLO as an instance of the SLO, and so the *k-unicycle* inequality is facet-defining for $P_{SLO}(\emptyset, L)$ where $L = N$. By application of Proposition 17, we can add elements into H until we reach the final (H, L) set, and the inequality remains facet-defining. ■

Chapter 3

Process Move Programming Problem

3.1 Introducing the PMP with respect to H and L

The *Process Move Programming* (PMP) problem is the study of a resource-constrained scheduling problem which arises when studying the fault-tolerance and operability of certain distributed systems [22]. Sirdey et al. described it as, starting with an arbitrary distribution of processes on processors in a distributed system, we try to find a least-disruptive sequence of moves (interruptions or migrations) which ends with the system in another predefined arbitrary state. Recall that a migration is performed by removing a process from one component, and restarting it on a different component, where for the time of the migration, the process consumes capacity on both components. An interruption is performed by removing a process from one component, and not restarting until after the reconfiguration. When phrased in the context of our two sets H and L, we say that H must be migrated, and L can be migrated or interrupted. The goal of the PMP is to reach the predefined final state with a minimum of interruptions, as those moves have a cost associated with them. One of the most important requirements of this problem is that we do not exceed the capacity of any processor at any point in the reconfiguration, and we assume that the final state is always feasible (i.e., no processor's capacity is exceeded). We will now discuss how the PMP can be formulated with respect to the SLO, present some inequalities which are specific to the $PMP(H, L)$, examine the existence of feasible solutions and dimension of the convex hull of those feasible solutions, and end our discussion with a specific class of feasible solutions, the Unitary case.

We begin the formulation for the $PMP(H, L)$ by defining F to be the set of feasible solutions to the Process Move Programming problem. A solution is in F if

1. Every process in N is moved exactly once

2. All interruptions are performed before the first migration

3. Two migrations cannot be performed simultaneously

4. the capacity on all components is not exceeded at any point in the reconfiguration.

Notice that a feasible solution to (1), (2), and (3) is a feasible solution for the $SLO(H, L)$. Let N be the complete set of processes, partitioned into two sets H and L, defined as for the SLO. We can let $M \subseteq N$ be the set of processes which are migrated, $I \subseteq L$ be the set of processes which are interrupted. Let $p = |N|^2 - |H|$, and define the set X to be $X = \{x \in \{0,1\}^p \big| (1), (2), (3), \text{ and } (4) \text{ are satisfied}\}$. We will retain the definition of the variables from the SLO, where

$$x_{mm} = \begin{cases} 1 & \text{if process } m \text{ is interrupted} \\ 0 & \text{otherwise} \end{cases}$$

for all m in L, and

$$x_{mm'} = \begin{cases} 1 & \text{if process } m \text{ is migrated before process } m' \\ 0 & \text{otherwise} \end{cases}$$

for all m, m' in $M = N \setminus I$, as well as the retaining w_p to denote the amount of resource consumed by process p. We also define K_u to denote the amount of resource available on component u. Define $I = \{m \in N \big| x_{mm} = 1\}$. For every processor $p \in P$, with $p \notin I$, there exists $\mathcal{S}(p)$, the source component, and $\mathcal{T}(p)$, the target component of p. Let R denote the set of vectors in $\mathbb{R}^{\,p}$, such that for all processes $m_0 \in H$,

$$k_{t_{m_0}} + \sum_{h \in \mathcal{S}(t_{m_0}) \cap H} w_h x_{hm_0} + \sum_{l \in \mathcal{S}(t_{m_0}) \cap L} w_l (x_{ll} + x_{lm_0}) - \sum_{p \in \mathcal{T}(t_{m_0})} w_p x_{pm_0} \geq w_{m_0} \tag{3.1}$$

and for all processes $m_0 \in L$,

$$K_{t_{m_0}} + \sum_{h \in \mathcal{S}(t_{m_0}) \cap H} w_h x_{hm_0} + \sum_{l \in \mathcal{S}(t_{m_0}) \cap L} w_l (x_{ll} + x_{lm_0}) - \sum_{p \in \mathcal{T}(t_{m_0})} w_p x_{pm_0} \geq w_{m_0}(1 - x_{m_0 m_0}) \tag{3.2}$$

As any feasible solution to $PMP(H, L)$ must also be a feasible solution to $SLO(H, L)$, all that remains to be shown is that inequalities (3.1), (3.2) enforce that no unit's capacity is exceeded. We will denote the convex hull of the feasible points of the $PMP(H, L)$ to be $P_{PMP}(H, L)$ where $N = H \cup L$, and $|N| = n$.

Claim 24 *Inequalities* (3.1) *and* (3.2) *are sufficient to ensure that the resource consumed on each unit is never more than the available capacity.*

Proof: Let x be a solution to $P_{PMP}(H, L)$. If $m_0 \in L$ and $m_0 \in I$, then m_0 cannot create an overflow problem. We know that our final state is always feasible, and we see that (3.2) enforces that the capacity freed by interrupting the processes on the target of m_0, t_{m_0}, is always greater than or equal to zero. If $m_0 \in L \setminus I$, or $m_0 \in H$, we need to check that the combination of the processes that migrate from t_{m_0} before m_0, and the processes that migrate to t_{m_0} before m_0 leave enough resource to accommodate m_0. Without loss of generality, we see that (3.1) requires that the residual capacity on t_{m_0} plus the sum of the capacity that is freed by processes in H migrating before m_0 and the capacity that is freed by processes in L being interrupted or migrating before m_0, minus the capacity used by processes in $H \cup L$ that migrate to t_{m_0} before m_0 must be less than the resource required for m_0. Thus we see that the inequalities (3.1) and (3.2) do ensure that there is never any overflow. ■

3.2 Cover and Cover-based Inequalities

We now examine the general topic of covers, and also more problem-specific applications of covers. First, however, we must define what a cover is. Covers are generally discussed in terms of the 0-1 knapsack problem, where the 0-1 knapsack polytope is defined as

$$\mathcal{P} := \text{conv}\{x \in \{0, 1\}^n \big| a^t x \leq b\},$$

a set $C \subseteq N = \{1, \ldots, n\}$ defines a cover if it satisfies $\sum_{i \in C} a_i > b$. That is, in terms of the knapsack problem, all elements in C cannot be chosen at the same time, as their combined weights exceed the maximum value allowed by the constraint. We now examine the definition of a cover in our problem with respect to a component u. Without loss of generality, let there be two distinct processes i and j, where the source processor of i is also the target processor of j, so $\mathcal{S}(i) = \mathcal{T}(j) = u$. Suppose that the amount of resource consumed by j is greater than the sum of the residual resource on u plus the amount of resource consumed by process i, and so both i and j cannot exist on u at the same time. Then the two processes, i and j, define a *cover*, with respect to the unit u, and we write $(i, j) \in C$. Thus the process j cannot migrate to u before i migrates from u for any feasible solution of the PMP. We can expand the generalities of this case, but the same idea holds if we assume i to be the set of processes in the cover for which $s_i = u$, and similarly for j. In our notation, we would write: $j \not\prec i$ and so $x_{ji} = 0$ for all $x \in P_{SLO}(H, L)$. Derived from the idea of cover, either directly or indirectly, we now have four sets of inequalities from [15] that we consider in detail.

3.2.1 Source and Target Cover Inequality

Let m_0 be a process move of M and u be a processor of $\{s_{m_0}, t_{m_0}\}$. Consider two subsets $S \subseteq S(u)$ and $T \subseteq T(u)$ so that (m_0, S, T) induces a cover; that is, $(S \cup \{m_0\}, T)$ or $(S, T \cup \{m_0\})$ is a cover with respect to $u = s_{m_0}$ or $u = t_{m_0}$ respectively. If $m_0 \in H$, the *cover inequality* is given as

$$\sum_{m \in S} x_{m_0 m} + \sum_{m \in T} x_{m m_0} \leq |S| + |T| - 1, \tag{3.3}$$

whereas, if $m_0 \in L$, the *cover inequality* is given as

$$\sum_{m \in S} x_{m_0 m} + \sum_{m \in T} x_{m m_0} + (|S| + |T| - 1)x_{m_0 m_0} \leq |S| + |T| - 1, \tag{3.4}$$

We will show that these cover inequalities are valid for $P_{PMP}(H, L)$. These inequalities express that if all processes that share a target with m_0 migrate to u before m_0, at least one process whose source is u must migrate before m_0 can migrate, thus enforcing the principle of cover.

Proposition 25 *Inequalities* (3.3) *and* (3.4)*, known as the cover inequalities, are valid for the* $P_{PMP}(H, L)$*.*

Proof: Let x be a solution of $P_{PMP}(H, L)$. If $m_0 \in L$, and $x_{m_0 m_0} = 1$, validity is obvious, as $x_{m m_0} = x_{m_0 m} = 0$ for all $m \in N \setminus \{m_0\}$. If $m_0 \in L$, but $x_{m_0 m_0} = 0$, or if $m_0 \in H$, the maximum value of the right-hand side is $|S| + |T|$. However, if it takes that value, it must be the case that all process moves of S have been migrated before any process moves of T, which violates (m_0, S, T) as cover-inducing. Thus the maximum value of the left-hand side is strictly less than $|S| + |T|$, and the inequality is valid. ∎

3.2.2 Overload Inequalities

Let u be a processor of U, and consider two non-empty subsets $S \subseteq S(u)$ and $T \subseteq T(u)$ so that (S, T) induces a cover with respect to u. Then for some processes $m_s \in S$ and $m_t \in T$, the *overload inequality* is given as

$$\sum_{m \in S \setminus \{m_s\}} x_{m_s m} + \sum_{m \in T \setminus \{m_t\}} x_{m m_t} + x_{m_t m_s} \leq |S| + |T| - 2 \tag{3.5}$$

if m_s, $m_t \in H$,

$$\sum_{m \in S \setminus \{m_s\}} x_{m_s m} + \sum_{m \in T \setminus \{m_t\}} x_{m m_t} + x_{m_t m_s} + (|T| - 1) x_{m_t m_t} \leq |S| + |T| - 2 \tag{3.6}$$

if $m_s \in H$, $m_t \in L$,

$$\sum_{m \in S \setminus \{m_s\}} x_{m_s m} + \sum_{m \in T \setminus \{m_t\}} x_{m m_t} + x_{m_t m_s} + (|S| - 1) x_{m_s m_s} \leq |S| + |T| - 2 \tag{3.7}$$

if $m_s \in L$, $m_t \in H$, and

$$\sum_{m \in S \setminus \{m_s\}} x_{m_s m} + \sum_{m \in T \setminus \{m_t\}} x_{m m_t} + x_{m_t m_s} + (|S| - 1) x_{m_s m_s} + (|T| - 1) x_{m_t m_t} \leq |S| + |T| - 2 \tag{3.8}$$

if m_s, $m_t \in L$. We will present the proof for the validity of inequalities (3.5), (3.6), (3.7), and (3.8) in the same four cases as above.

Claim 26 *If $m_s, m_t \in H$, then inequality* (3.5) *is valid for $P_{PMP}(H, L)$.*

Proof: Let x be a solution of $P_{PMP}(H, L)$. If $x_{m_s m_t} = 1$, then validity follows from the maximum possible sums. That is,

$$\sum_{m \in S \setminus \{m_s\}} x_{m_s m} \leq |S| - 1, \text{ and } \sum_{m \in T \setminus \{m_t\}} x_{m m_t} \leq |T| - 1,$$

and so the maximum value of the left-hand side is $|S| + |T| - 2$. If instead, $x_{m_t m_s} = 1$, then at least one of the previous inequalities is not tight, else all process moves of S would be migrated before any process moves of T, contradicting (S, T) as cover-inducing. Therefore the inequality must be valid. ■

Claim 27 *If $m_s \in H$, $m_t \in L$, then inequality* (3.6) *is valid for $P_{PMP}(H, L)$.*

Proof: Let x be a solution of $P_{PMP}(H, L)$. If $x_{m_t m_t} = 1$, then $x_{m m_t} = 0$ for all $m \in T \setminus \{m_t\}$, and $x_{m_t m_s} = 0$. Thus the maximum value of the left-hand side is the maximum value of $\sum_{m \in S \setminus \{m_s\}} x_{m_s m}$ plus $|T| - 1$. As seen in the previous case, the maximum value of the sum is $|S| - 1$, and so the inequality is valid. If $x_{m_t m_t} = 0$, we are in the same situation as Claim 26, and thus the inequality must be valid. ■

Claim 28 *If $m_s \in L$, $m_t \in H$, then inequality* (3.7) *is valid for $P_{PMP}(H, L)$.*

Proof: The proof is similar to claim 27. ■

Claim 29 *If $m_s, m_t \in L$, then inequality* (3.8) *is valid for $P_{PMP}(H, L)$.*

Proof: Let x be a solution of $P_{PMP}(H,L)$. If $x_{m_s m_s} = x_{m_t m_t} = 1$, then the validity is obvious, as $x_{m_s m} = x_{m m_t} = 0$ for all $m \in N \setminus \{m_s, m_t\}$, and so the two sides are equal. All other cases have been covered above, and thus the inequality must be valid. ■

In all possible cases, the *overload inequality* is valid for the $P_{PMP}(H,L)$.

3.2.3 Source Excess Inequality

Let u be a processor of U. Consider two non-empty subsets $S \subseteq S(u)$ and $T \subseteq T(u)$ so that $(S \setminus \{m\}, T)$ induces a cover with respect to u for every $m \in S$ and $(S, T \setminus \{m\})$ induces a cover with respect to u for every $m \in T$. For two distinct processes $m_0, m_s \in S$, if $m_0 \in H$, we have the following inequality

$$\sum_{m \in S \setminus \{m_0} x_{m_0 m}\} + \sum_{m \in T} x_{m m_0} + x_{m_s m_0} - \sum_{m \in T} x_{m_s m} \leq |S| + |T| - 3, \tag{3.9}$$

whereas if $m_0 \in L$, we have

$$\sum_{m \in S \setminus \{m_0} x_{m_0 m}\} + \sum_{m \in T} x_{m m_0} + x_{m_s m_0} - \sum_{m \in T} x_{m_s m} + (|S| + |T| - 3) x_{m_0 m_0} \leq |S| + |T| - 3. \tag{3.10}$$

Proposition 30 *Inequalities* (3.9) *and* (3.10)*, called the source excess inequalities, are valid for* P^n_{PMP}.

Proof: Let x be a solution of P^M_{PMP}. If $m_0 \in L$ and $x_{m_0 m_0} = 1$, then as $x_{m_0 m} = x_{m m_0} = 0$ for all $m \in N \setminus \{m_0\}$, the maximum value of the left-hand side for (3.9) and (3.10) is equal to the right-hand side value, and so both inequalities are valid. Let $m_0 \in L$ and $x_{m_0 m_0} = 0$, or $m_0 \in H$. Then if $x_{m_s m_s} = 1$, as $(S \setminus \{m_s\}, T)$ defines a cover with respect to u, the cover inequalities (3.3) and (3.4), with a cover induced by $(m_o, S \setminus \{m_o, m_s\}, T)$, implies validity.

Assume $x_{m_s m_s} = 0$. If at least two processes from $(S \setminus \{m_0, m_s\}) \cup T$ are interrupted, then the left-hand side of the inequality is bounded above by

$$\sum_{m \in S, m \neq m_0} x_{m_0 m} + \sum_{m \in T} x_{m m_0} + x_{m_s m_0} \leq |S| + |T| - 3$$

as the sum of the first three terms cannot exceed $|S| + |T| - 3$. If exactly one process among $(S \setminus \{m_0, m_s\}) \cup T$ is interrupted, say $m' \in S$, then as $(S \setminus \{m'\}, T)$ defines a cover, the above inequality continues to hold.

If $\sum_{m\in T} x_{m_s m} > 1$, then validity obviously holds, as it decreases the left-hand side value, and at least negates the possible addition of one by $x_{m_s m_0} = 1$. Suppose that $\sum_{m\in S, m\neq m_0} x_{m_0 m} + \sum_{m\in T} x_{m m_0} + x_{m_s m_0} = |S| + |T| - 2$, and $\sum_{m\in T} x_{m_s m} = 0$. This would mean that we have all process moves of T migrated to u before any process moves in $S \setminus \{m'\}$ are migrated, which contradicts $(S \setminus \{m'\}, T)$ being a cover.

Now assume that no process in $S \cup T$ is interrupted. If all of the process moves in $S \setminus \{m_0\}$ are migrated after m_0, we cannot have more than $|T| - 2$ process moves in T migrated to u before m_0, since $(S, T \setminus \{m\})$ is a cover. Thus the validity of the source excess inequality holds. If all process moves in $S \setminus \{m_0\}$ but $m' \neq m_s$ are migrated after m_0, we have $\sum_{m\in S, m\neq m_0} x_{m_0 m} = |S| - 2$ and $x_{m_s m_0} = 0$, and as $(S \setminus \{m'\}, T)$ induces a cover, we also have $\sum_{m\in T} x_{m m_0} \leq |T| - 1$.

If all the processes in $S \setminus \{m_0\}$ but m_s are migrated after m_0, we then have $\sum_{m\in S, m\neq m_0} x_{m_0 m} = |S| - 2$ and $x_{m_s m_0} = 1$ and as $(S \setminus \{m_s\}, T)$ induces a cover, we also have $\sum_{m\in T} x_{m m_0} \leq |T| - 1$ and $\sum_{m\in T} x_{m_s m} \geq 1$. If two processes in $S \setminus \{m_0\}$ are migrated before m_0, one of them being m_s, we then $\sum_{m\in S, m\neq m_0} x_{m_0 m} = |S| - 3$ and $x_{m_s m_0} = 1$ and again, as $(S \setminus \{m_s\}, T)$ defines a cover, $\sum_{m\in T} x_{m m_0} \leq |T|$ and $\sum_{m\in T} x_{m_s m} \geq 1$, and the validity then follows.

Let us examine a solution where one process move in T, say $\bar{m}$ is interrupted, and no process move of S is interrupted. Since $(S, T \setminus \{\bar{m}\})$ defines a cover, we have

$$\sum_{m\in S, m\neq m_0} x_{m_0 m} + \sum_{m\in T} x_{m_0} + x_{m_s m_0} \leq |S| + |T| - 2$$

Let the above inequality be tight, and $\sum_{m\in T} x_{m_s m} = 0$. If $x_{m_s m_0} = 1$, then $\sum_{m\in S, m\neq m_0} x_{m_0 m} = |S| - 2$, and we have $|T| - 1$ processes in T migrated to u before any process moves of S leave u, which contradicts $(S, T \setminus \{\bar{m}\})$ inducing a cover. Thus the source excess inequalities are valid. ■

3.2.4 Target Excess Inequality

Let u be a processor of U. Consider two non-empty subsets $S \subset S(u)$ and $T \subset T(u)$ so that $(S \setminus \{m\}, T)$ defines a cover for every $m \in S$ and $(S, T \setminus \{m\})$ defines a cover for every $m \in T$. For two distinct processes $m_0, m_t \in T$, if $m_0 \in H$, we have the following inequality

$$\sum_{m\in S} x_{m_0 m} + \sum_{m\in T\setminus\{m_0\}} x_{m m_0} + x_{m_0 m_t} - \sum_{m\in S} x_{m m_t} \leq |S| + |T| - 3, \tag{3.11}$$

whereas, if $m_0 \in L$, we have

$$\sum_{m \in S} x_{m_0 m} + \sum_{m \in T \setminus \{m_0\}} x_{m m_0} + x_{m_0 m_t} - \sum_{m \in S} x_{m m_t} + (|S| + |T| - 3) x_{m_0 m_0} \leq |S| + |T| - 3 \tag{3.12}$$

These inequalities guarantee that when both m_0 and m_t are migrated to their target u, enough resource has been freed on u to allow the migration.

Proposition 31 *Inequalities* (3.11) *and* (3.12)*, called the target excess inequalities, are valid for* $P_{PMP}(H, L)$*.*

Proof: Let x be a solution of $P_{PMP}(H, L)$. If $m_0 \in L$ and $x_{m_0 m_0} = 1$, validity is obvious as $x_{m m_0} = x_{m_0 m} = 0$ for all $m \in N \setminus \{m_0\}$, and thus the maximum value of the left-hand side is equal to the right-hand side value. Let $m_0 \in L$ and $x_{m_0 m_0} = 0$, or $m_0 \in H$. If $x_{m_t m_t} = 1$, then, as $(S, T \setminus \{m_t\})$ defines a cover with respect to u, then the cover inequality from above induced by $(m_0, S, T \setminus \{m_0, m_t\})$ implies validity.

Assume $x_{m_t m_t} = 0$. Then from the cover inequality induced by either $(m_0, S, T \setminus \{m_0\})$ if $x_{m_t m_0} = 1$ or by $(m_0, S, T \setminus \{m_0, m_t\})$ if $x_{m_0 m_t} = 1$, we have

$$\sum_{m \in S} x_{m_0 m} + \sum_{m \in T \setminus \{m_0\}} x_{m m_0} + x_{m_0 m_t} \leq |S| + |T| - 2. \qquad (*)$$

If (*) is tight, then there exists a process move m' of $S \cup T$ such that either $x_{m' m_0} = 1$ if $m' \in T \setminus \{m_0, m_t\}$, or $x_{m_0 m'} = 0$ if $m' \in S$. Since validity is obvious if $\sum_{m \in S} x_{m m_t} > 0$, we complete our proof by considering the case where $\sum_{m \in S} x_{m m_t} = 0$. From (*) being tight, the definition of m', and setting $\sum_{m \in S} x_{m m_t} = 0$, we deduce that all the process moves of $(S \cup T) \setminus \{m'\}$ are present at the same time on the processor u. If $m' \in S$ (respectively $m' \in T \setminus \{m_0, m_t\}$), the vector x is not feasible, since $(S \setminus \{m'\}, T)$ (respectively $(S, T \setminus \{m'\})$) defines a cover with respect to u. Thus validity holds. ∎

There is a situation to be considered: Let $(i, j) \in C(N, 2)$, so $j \not\prec i$. Suppose, however, that at the time j needs to migrate, i is unable to. If $i \in L$, this is not a problem, as we can interrupt i, thus creating enough capacity for j to move, and the problem still has a feasible solution. However, if $i \in H$, the process cannot be interrupted, and there is no solution. This leads us to wonder exactly what situations guarantee a feasible solution.

3.3 Existence of Feasible Solutions

It is important to be able to discern when a feasible solution to the $PMP(H, L)$ exists. One necessary condition is that, if we interrupt all processes in L, we need to be able to feasibly migrate all processes in H. This question is the same question asked by the *Zero-Impact Process Move Programming* (ZIPMP) problem. Similar to what Sirdey et al. stated in [22], given a set of moves, H we want to know if there exists a bijection $\sigma : H \rightarrow \{1, \dots, n_H\}$ such that for all $m \in H$,

$$w_m \leq K_{t_m} + \sum_{m' \in S(t_m), \sigma(m') \leq \sigma(m)} w_{m'} - \sum_{m' \in T(t_m), \sigma(m') < \sigma(m)} w_{m'} \tag{3.13}$$

This problem was shown to be NP-complete in the strong sense, as the classic 3-partition problem (which is known to be NP-complete in the strong sense [9]) can be solved by an algorithm able to solve the ZIPMP problem, and the ZIPMP can be restricted to the 3-partition problem [22].

There are two polynomially solvable special cases of the PMP problem which have been studied by Sirdey et al., although not in the context of the $PMP(H, L)$. Below we introduce those cases, adapting the wording and some of the proofs to fit the $PMP(H, L)$ instead of the general PMP. However, we must first introduce notation, also borrowed from Sirdey et al. [22], that allows us to associate an instance of the $PMP(H, L)$ with a directed multigraph. Let $D = (H, A_H)$ be the directed multigraph, called the *transfer multigraph*, whose vertices represent the all processes in H, and all arcs are ordered (s_h, t_h) for all processes $h \in H$. We are only interested in determining when the process moves in H have a feasible solution, since it is unavoidable for these processes to consume resource, and the consumption of resource is the cause of infeasible solutions.

3.3.1 The Acyclic Transfer Digraph

The first case we desire to examine is the case where the transfer multigraph is acyclic. Recall that there exists a topological ordering of the vertices of any acyclic graph $G = (V, A)$, defined as a bijection $\phi : V \rightarrow \{1, \dots, |V|\}$ such that $\phi(v) < \phi(w)$ for all arcs $(v, w) \in A$. We will refer to D as the transfer graph, where D is defined as $D = (N, A_N)$ where A_N is the set of all (s_p, t_p) pairs for all $p \in N$. We will also denote $|N| = n$. Any similar notation is likewise defined.

Claim 32 *If $D = (H, A_H)$ is acyclic, a zero-impact process move program exists and can be found in linear time.*

Proof: Let the graph $D = (H, A_H)$ be acyclic. Then we know that D has a topological ordering of its vertices. By definition of a topological ordering, the final vertex in the ordering, call it u_{n_h}, has only incoming arcs, and so $S(\phi^{-1}(n_h)) = \emptyset$. As our final state is always feasible, there is sufficient capacity on u_{n_h} to accommodate all processes m for which $t_m = u_{n_h}$. Perform that migration. Then u_{h_n} has no more incoming arcs, and $S(\phi^{-1}(n_{h-1})) = \emptyset$. We repeat our argument, migrating moves until we reach $u_2 = \phi(2)$. Once we migrate all incoming arcs to u_2, we are left with u_1. By definition of a topological ordering, the root vertex, in this case u_1 does not have any incoming arcs. Thus we have reached a point where $S(\phi^{-1}(1)) = T(\phi^{-1}(1)) = \emptyset$, and we have performed all migrations. This reconfiguration is equivalent to obtaining a topological ordering, which is known to be derivable in linear time [9]. ■

Before we look at our next case, we conclude this topic of acyclic digraphs by recalling that a directed multigraph $G = (V, A)$ is strongly connected if either

a. $|V| = 2$, or

b. G contains a path from v to w and from w to v for all $(v, w) \in V, v \neq w$.

We can extend the above idea, knowing that even if $D = (H, A_H)$ contains some cycles, we may still be able to obtain a topological ordering on $H \subset N$ by examining the strongly connected components of D. If we let $C_1, \ldots, C_m$ denote the strongly connected components of D and assume that they are topologically ordered, Sirdey et al. [22] proved the following:

Proposition 33 *Assuming that given $1 < i \leq n$, the moves having both their source and target in $\bigcup_{j=i+1}^{m} C_j$ have been performed and that the corresponding arcs have been removed from $D = (H, A_H)$. Then a process move program over H which first schedules the moves having their source in C_i and target not in C_i, then moves internal to C_i followed by the remaining moves, dominates any other program not satisfying this property.*

3.3.2 The Homogeneous PMP

The next case that we concern ourselves with is the *Homogeneous* case, in which all processes $m \in N$ consume the same amount of resource. Without loss of generality, we can scale that set amount so that $w_m = 1$ for all $m \in N$. To discuss the special aspects of this case, we must first recall that a directed multigraph $G = (V, A)$ is Eulerian if it is connected (without respect to direction) and the in-degree of a vertex is equal to its out-degree, for all vertices $v \in V$.

Algorithm 34 *An algorithm for the homogeneous case when $D = (N, A_N)$ is strongly connected and non-Eulerian.*

While $N \neq \emptyset$

Let C denote the set of vertices in the last of the topologically ordered strongly connected components of D.

(a) If C contains only one vertex, say v, then perform all the moves targeting v in an arbitrary order, remove them from M, remove the corresponding arcs from A_N, and remove v from N.

(b) Else choose a vertex, say v_0, in C whose remaining capacity is non-zero and a maximal Eulerian sub-digraph rooted at v_0. Perform the moves in the sub-digraph in reverse order of an Eulerian tour, removing them from M and removing the corresponding arcs from N.

End.

The above algorithm is very useful, and allows us to solve the homogeneous case, if D is non-Eulerian and strongly connected; we can not only solve it, but can do so in polynomial time, as shown by the following.

Corollary 35 *Assume that $D = (H, A_H)$ is connected. Then, if D is non-Eulerian, a zero-impact admissible process move program exists and can be found in polynomial time.*

Proof: If $D = (H, A_H)$ is Eulerian, then we proceed to Proposition 36. Let us assume that D is connected and not Eulerian, and let $C_1, \ldots, C_n$ denote the strongly connected components of D, topologically ordered. Algorithm 34 considers these components of D as implied by Proposition 33. Assume that $|C_n| > 1$. If the transfer multigraph associated with the moves internal to C_n, call it D'_n, is not Eulerian, we can use Algorithm 34 to find a zero-impact process move program of D'_n. Otherwise, if D'_n is Eulerian, then the in-degree is equal to the out-degree of each vertex in D'_n. However, since D is connected, at least one vertex in C_n, say v_0, is the head of an arc whose tail is not in C_n, and it follows that the in-degree of v_0 is greater than its out-degree, and thus $K_{v_0} > 0$. This provides a vertex from which an Eulerian tour can be started.

When the moves internal to C_i $(i < n, |C_i| > 1)$ are considered, as D is connected, at least one move with source in C_i and target not in C_i has been performed, ensuring that one unit of resource is available on at least one of the vertices of C_i. Let D'_i denote the transfer digraph associated with the moves internal to C_i.

It follows that a zero-impact process move program is given by either an Eulerian tour (if D'_i is Eulerian) or by Algorithm 34 otherwise, which is clearly polynomial. ■

If D is Eulerian, we have a feasible solution only if the following proposition holds.

Proposition 36 *If $D = (H, A_H)$ is Eulerian, then the homogeneous case can be solved in linear time if and only if there exists a processor $u \in U$ such that $K_u \geq 1$.*

Proof: If there exists a processor $u \in U$ such that $K_u \geq 1$, then a zero-impact process move program over H is obtained by performing the moves in the reverse order of an Eulerian tour on $D = (H, A_H)$, starting with any of the moves targeting u. Otherwise, as no process of H can be interrupted, the problem is infeasible. ■

Although we have been dealing with the Homogeneous case, we can similarly consider a non-homogeneous case under special circumstances.

Proposition 37 *Consider the $PMP(H \cup \{m_0\}, L)$, and let the amount of resource consumed by all processes $m \in H \cup L$ be such that $w_m = 2k$ for some $k \in \mathbb{Z}_+$, and let $w_{m_0} = 1$. If there exists at least two components $u_i, u_j \in U$ with c_i, c_j odd, then m_0 is free. Additionally, solving a zero-impact process move problem for $PMP(H \cup \{m_0\}, L)$ equivalent to finding a feasible solution for $PMP(H, L)$ in the homogeneous case, and thus is polynomial solvable.*

Proof: Let $s_{m_0} = u_i$ with odd initial capacity. As m_0 is consuming resource on s_{m_0}, its residual capacity is even. Suppose we desire to migrate any process $m' \in H$ to u_i before migrating m_0. If there is sufficient residual capacity, there is no conflict. If there is not sufficient residual capacity on u_i, then the residual capacity on u_i must be $K_i \leq w_{m'} - 2$. If we let $m_0 \prec m'$, the updated residual capacity would be $K_i \leq w_{m'} - 1$, which we see is still not sufficient for the migration of m'. Let $t_{m_0} = u_j$ with odd initial capacity, and let $S_j = \{m \in H \setminus \{m_0\} \big| s_m = u_j\}$ and $T_j = \{m \in H \setminus \{m_0\} \big| t_m = u_j\}$. The residual capacity on u_j before the migration of m_0 can be written as $K_j = c_j - \sum_{m \in S_j} w_m x_{m_0 m} - \sum_{m \in T_j} w_m x_{m m_0}$. As $w_m = 2k$ for some $k \in \mathbb{Z}_+$, for any feasible solution, $K_j \geq 1$ after any set of migrations of $m \in H \setminus \{m_0\}$, and so there is always sufficient capacity for m_0 to migrate. Thus the process m_0 can never create any overflow issues, and the problem is equivalent to finding a feasible solution for $PMP(H, L)$, which is the homogeneous case and thereby polynomial, under the restrictions in the propositions above. ■

3.4 Dimension

Assume that we have a feasible solution with respect to H; that is, there is a ZIPMP for H. If we do not make this assumption, then $P_{SLO}(H, L) = \emptyset$, and we have nothing to work with. We begin by stating that $\dim P_{PMP}(H, L) \leq \dim P_{SLO}(H, L)$, as the PMP is a restriction of the SLO. The tightness of that bound needs to be considered, and so we will introduce some necessary conditions for the dimension of the $P_{PMP}(H, L)$. First, though, we need to introduce some notation, similar to that presented in Sirdey and Kerivin [25]. The process move polytope over (H, L) is written

$$P_{PMP}(H, L) = \text{conv}\{x^p \in \mathbb{R}^{\,p} \big| x \text{ satisfies items (1) through (4) in our formulation}\}$$

where $|H| = n_H$ and $|L| = n_L$, and $n_H + n_L = n = |N|$. We are given two subsets of moves $\{m_1, \ldots, m_r\} \subseteq M$ and $X \subseteq M$ with $1 \leq r \leq n$, $H \cap (X \cup \{m_1, \ldots, m_r\}) = H$, and $X \cap \{m_1, \ldots, m_r\} = \emptyset$. We denote an *incomplete process move program*, by $[m_1, \ldots, m_r; X]$, where the only specified ordering is on $\{m_1, \ldots, m_r\}$ with $m_i \prec m_j$ if $i < j$, and $X \subseteq M = N \setminus I$. If $X = \emptyset$, each incomplete program defines a unique point, and if $X \neq \emptyset$, it defines a family of points, not necessarily unique, satisfying the specified ordering on $\{m_1, \ldots, m_r\}$. An incomplete process move program $[m_1, \ldots, m_r; X]$ is *admissible* if and only if there exists a point $x \in P_{PMP}(H, K)$ such that

$$x_{mm} = \begin{cases} 0 & \text{if } m \in \{m_1, \ldots, m_r\} \cup X \\ 1 & \text{otherwise} \end{cases}$$

$$x_{m_i m_j} = \begin{cases} 1 & \text{if } m_i, m_j \in \{m_1, \ldots, m_r\} \text{ with } i < j \\ 0 & \text{if } m_i, m_j \in \{m_1, \ldots, m_r\} \text{ with } i \geq j \end{cases}$$

3.4.1 Necessary Conditions for Dimension

We now define several sets that partition the possible set of incomplete processes, and give some necessary conditions for them.

$$D_H = \{(h, h') \in H \text{ distinct ordered pairs} : [h, h'; H \setminus \{h, h'\}] \text{ is not admissible}\}$$

$$|D_H| = d_H$$

$$D_L = \{l \in L : [\emptyset; H \cup \{l\}] \text{ is not admissible}\}$$

$$|D_L| = d_L$$

$$D_{HL} = \{(h,l) \in H \times L : [h,l; H \setminus \{h\}] \text{ is not admissible, but } [\emptyset; H \cup \{l\}] \text{ is admissible}\}$$

$$|D_{HL}| = d_{HL}$$

$$D_{LH} = \{(l,h) \in L \times H : [l,h; H \setminus \{h\}] \text{ is not admissible, but } [\emptyset; H \cup \{l\}] \text{ is admissible}\}$$

$$|D_{LH}| = d_{LH}$$

$$D_{LL} = \{(l,l') \in L \text{ distinct ordered pairs} : [l,l'; H] \text{ is not admissible, but } l,\ l' \notin D_L\}$$

$$|D_{LL}| = d_{LL}$$

Proposition 38 *If there are two distinct $h,\ h' \in H$ such that $(h,h') \in D_H$, then*

1. $P_{PMP}(H,L) \subseteq \{x \in \mathbb{R}^{\,p} \big| x_{hh'} = 0\}$, *and*
2. $\dim P_{PMP}(H,L) \leq \dim P_{SLO}(H,L) - 1$

Proof: If the specified ordering is not admissible, then it never occurs in any feasible solution $\bar{x} \in P_{PMP}(H,L)$. Thus for all $\bar{x} \in P_{PMP}(H,L)$, $x_{hh'} = 0$, and so the polytope is included in the face $\{x \in \mathbb{R}^{\,p} \big| x_{hh'} = 0\}$.

If $x_{hh'} = 0$ for all feasible points of $P_{PMP}(H,L)$, then we add that equation to our minimal system $(A^=, b^=)$, thereby increasing its dimension by 1, as this equation is the first to mention only $x_{hh'}$. Thus our dimension decreases by 1, as shown in Claim 6. ∎

Proposition 39 *If there exists $l \in L$ such that $l \in D_L$, then*

1. $P_{PMP}(H,L) \subseteq \{x \in \mathbb{R}^{\,p} \big| x_{ll} = 1\}$, *and*
2. $\dim P_{PMP}(H,L) \leq \dim P_{SLO}(H,L) - (n_H + 1)$

Proof: If migrating process l is not admissible, then it never occurs in any feasible solution $\bar{x} \in P_{PMP}(H,L)$. Thus for all $\bar{x} \in P_{PMP}(H,L)$, $x_{ll} = 1$, and so the polytope is included in the face $\{x \in \mathbb{R}^{\,p} \big| x_{ll} = 1\}$.

If $x_{ll} = 1$ for all feasible points of $P_{PMP}(H,L)$, we know that this also specifies, without loss of generality, $x_{hl} = 0$ for all $h \in H$. We must add all of those equations and also $x_{ll} = 0$ to our minimal system $(A^=, b^=)$, thereby increasing its dimension by $n_H + 1$, as these equations are the first to mention only x_{hl} or x_{ll}. We need not specify $x_{lh} = 0$, as the equation already in our minimal system (2.2) takes care of that. Thus our dimension decreases by $(n_H + 1)$, as shown in Claim 6. ∎

Proposition 40 *If there are two distinct* l, $l' \in L$ *such that* $(l, l') \in D_{LL}$, *then*

1. $P_{PMP}(H, L) \subseteq \{x \in \mathbb{R}^{p} \big| x_{ll'} = 0\}$, *and*

2. $\dim P_{PMP}(H, L) \leq \dim P_{SLO}(H, L) - 1$

Proof: If the specified ordering is not admissible, then it never occurs in any feasible solution $\bar{x} \in P_{PMP}(H, L)$. Thus for all $\bar{x} \in P_{PMP}(H, L)$, $x_{ll'} = 0$, and so the polytope is included in the face $\{x \in \mathbb{R}^{p} \big| x_{ll'} = 0\}$.

If $x_{ll'} = 0$ for all feasible points of $P_{PMP}(H, L)$, then we add that equation to our minimal system $(A^{=}, b^{=})$, thereby increasing its dimension by 1, as this equation is the first to mention only $x_{ll'}$. Thus our dimension decreases by 1, as shown in Claim 6. ∎

Proposition 41 *If there exist* $h \in H$ *and* $l \in L$ *such that* $(h, l) \in D_{HL}$, *then*

1. $P_{PMP}(H, L) \subseteq \{x \in \mathbb{R}^{p} \big| x_{hl} = 0\}$, *and*

2. $\dim P_{PMP}(H, L) \leq \dim P_{SLO}(H, L) - 1$

Proof: If the specified ordering is not admissible, then it never occurs in any feasible solution $\bar{x} \in P_{PMP}(H, L)$. Thus for all $\bar{x} \in P_{PMP}(H, L)$, $x_{hl} = 0$, and so the polytope is included in the face $\{x \in \mathbb{R}^{p} \big| x_{hl} = 0\}$.

If $x_{hl} = 0$ for all feasible points of $P_{PMP}(H, L)$, then we add that equation to our minimal system $(A^{=}, b^{=})$, thereby increasing its dimension by 1, as this equation is the first to mention only x_{hl}. Thus our dimension decreases by 1, as shown in Claim 6. ∎

Proposition 42 *If there exist* $h \in H$ *and* $l \in L$ *such that* $(l, h) \in D_{LH}$, *then*

1. $P_{PMP}(H, L) \subseteq \{x \in \mathbb{R}^{p} \big| x_{lh} = 0\}$, *and*

2. $\dim P_{PMP}(H, L) \leq \dim P_{SLO}(H, L) - 1$

Proof: Proof is similar to that of Proposition 41. ∎

When we assemble all of these different parts of the set of incomplete process move programs, we arrive at the following bound on our dimension:

$$\dim P_{PMP}(H, L) \leq \dim P_{SLO}(H, L) - (d_H + d_L(n_H + 1) + d_{HL} + dLL) \tag{3.14}$$

This bound is helpful, but gives no insight as to how tight a bound it is. To gain some information on that, we examine an instance of the SLO problem where $H = \emptyset$. This problem is equivalent to an instance of the PLO where $|L| = n$, whose convex hull is known to have

$$\dim P^n_{PLO} = n^2 - |C(M, 2)|$$

from [15]. The notation $|C(M, 2)|$ denotes the covers of size two over the set M of processes which are migrated. As discussed previously, a cover of size two occurs between l and l' with respect to a component u when, without loss of generality, $l \prec l'$ for all $x \in P^n_{PLO}$ due to capacity restrictions. We draw the reader's attention to the fact that definition of covers of size two is equivalent to the definition for a set of processes $(l, l') \in L$ to be in D_{LL}. If we look to those sets to try and draw any other similarities, we see that, as $H = \emptyset$, $D_H = D_{HL} = \emptyset$ as well. Also, as our final state is always feasible, D_L is also empty. Thus we could rewrite our dimension in (3.14) as

$$\dim P_{PMP}(\emptyset, L) \leq \dim P_{SLO}(H, L) - (d_{LL}), \tag{3.14'}$$

which is exactly the dimension of the $P^{|L|}_{PLO}$, and so we see that our bound is quite good.

Chapter 4

Conclusion

This paper has introduced the *Steiner Linear Ordering* problem, and shown how it can be applied to resource-constrained scheduling problems, specifically the PMP. In the introduction, we first informally introduced the idea of handling two sets of processes, H and L, which have different priorities. That is, for scheduling issues and reconfigurations, the processes in H cannot be interrupted, while the processes in L can be interrupted. Chapter 2 focused on formally writing the SLO problem, as well as presenting facet-defining inequalities for the convex hull of the SLO. We introduced a modified form of trivial lifting, and applied it to some well-known inequalities for the *Linear Ordering* problem, in turn making them facet-defining for the SLO. In Chapter 3, we introduced the PMP with respect to the SLO's structure, namely, in terms of H and L. From the additional capacity restrictions imposed, we were able to present some cover-based inequalities that are known to be valid for the PMP with a partial linear ordering, and show that they are valid for the $P_{SLO}(H, L)$. Also in Chapter 3, we outlined necessary conditions for the existence of a feasible solution, shared some polynomial solvable instances of the problem, and developed a bound on our dimension, which we proved is quite good.

However, there are still many areas of the SLO which could be investigated. In Chapter 2, while we introduced a modified form of trivial lifting, we allowed an element to be added to H. While this let us show that some facet-defining inequalities for the P^n_{PLO} are facet-defining for the $P_{SLO}(\emptyset, N)$, there are other well-known facet-defining inequalities for the P^n_{PLO} for which this lifting can be applied. Also, we did not complete the trivial lifting that would allow us to trivially lift elements into L. Showing that such a trivial lifting is valid would enable us to lift inequalities that are facet-defining for the P^n_{LO} to be facet-defining for the $P_{SLO}(H, L)$, such as the *Möbius Ladder* inequalities and the *k-fence* inequalities, presented in [11]. We have already begun to work on finding facet-defining variations of the inequalities listed in the trivial lifting section, which may be included in a following paper.

Future research stemming from Chapter 3 includes examining the separation problem for the inequalities listed, as well as any ramifications that might arise due to the two priority levels. The cover and cover-based inequalities still need to be examined to determine what conditions make them facet-defining, for both the $P_{SLO}(H, L)$ and the P^n_{PLO}. While we did mention the homogeneous case with respect to feasible solutions, there is a wealth of discovery yet to be made with respect to that instance. Proposition 37 may be able to be generalized, and this opens up a whole area of future research for reconfiguration problems in which one or more processes are free to migrate at will, and others must be constrained.

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