5-2011

Tight Polyhedral Representations of Discrete Sets Using Projections, Simplices, and Base-2 Expansions

Stephen Henry
Clemson University, smhenry@clemson.edu

Tight Polyhedral Representations of Discrete Sets using Projections, Simplices, and Base-2 Expansions

---

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

---

by
Stephen M. Henry
May 2011

Accepted by:
Dr. Warren Adams, Committee Chair
Dr. Douglas Shier
Dr. Matthew Saltzman
Dr. Hervé Kerivin

Abstract

This research effort focuses on the acquisition of polyhedral outer-approximations to the convex hull of feasible solutions for *mixed-integer linear* and *mixed-integer nonlinear programs.* The goal is to produce desirable formulations that have superior size and/or relaxation strength. These two qualities often have great influence on the success of underlying solution strategies, and so it is with these qualities in mind that the work of this dissertation presents three distinct contributions.

The first studies a family of relatively unknown polytopes that enable the linearization of polynomial expressions involving two discrete variables. Projections of higher-dimensional convex hulls are employed to reduce the dimensionality of the requisite linearizing polyhedra. For certain lower dimensions, a complete characterization of the convex hull is obtained; for others, a family of facets is acquired. Furthermore, a novel linearization for the product of a bounded continuous variable and a general discrete variable is obtained.

The second contribution investigates the use of simplicial facets in the formation of novel convex hull representations for a class of mixed-discrete problems having a subset of their variables taking on discrete, affinely independent realizations. These simplicial facets provide new theoretical machinery necessary to extend the *reformulation-linearization technique* (RLT) for mixed-binary and mixed-discrete programs. In doing so, new insight is provided which allows for the subsumation of previous mixed-binary and mixed-discrete RLT results.

The third contribution presents a novel approach for representing functions of discrete variables and their products using logarithmic numbers of 0-1 variables in order to economize on the number of these binary variables. Here, base-2 expansions are used within linear restrictions to enforce the appropriate behavior of functions of discrete variables. Products amongst functions are handled by scaling these linear restrictions. This approach provides insight into, improves upon, and subsumes recent related linearization methods from the literature.

Dedication

Once again to Toni. Your tireless passion for integer programming was an inspiration.

Acknowledgments

The work presented in this dissertation was made possible by the support of many people. Particularly, I would like to thank Dr. Douglas Shier, Dr. Matthew Saltzman, and Dr. Hervé Kerivin for being such sharp, insightful, and helpful committee members. Most of all, I would like to thank my advisor Dr. Warren Adams. Your incredibly hard work, guidance, and knowledge made this research effort extremely rewarding and enjoyable. I can hardly believe that our long afternoon meetings are coming to a close. Thank you so much.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Mixed-integer programs (MIPs) are a class of optimization problems having two sets of decision variables: a discrete set restricted to realize integer values and a continuous set that can take on a continuum of values. MIPs can be linear in the objective function and constraints, or they can include nonlinear terms involving the decision variables. Such nonlinear problems are typically referred to as *mixed-integer nonlinear programs* (MINLPs). While special problem instances are readily solvable, general MIPs and MINLPs have proven notoriously difficult to solve due primarily to the combinatorial explosion of feasible integer realizations.

This difficulty is unfortunate in light of the diverse contexts in which MIPs and MINLPs naturally arise. Their applications include supply-chain optimization [4, 2, 9], chemical engineering [6, 7, 5], transportation [1, 10], and portfolio management [8, 3], to name a few. While there exists an equally diverse collection of solution techniques, the robustness of these algorithms tends to lag behind real-world requirements, leaving many important industrial problems unsolved.

A critical component in the derivation of effective solution strategies is the acquisition of tight polyhedral outer-approximations to the convex hull of feasible solutions. Convex hull representations are useful in that they allow difficult combinatorial optimization problems to be reduced to much simpler linear programs. Tight approximations of the convex hull can provide superior bounds within enumerative strategies over the integer variables and thereby allow entire subsets of candidate solutions to be implicitly disregarded. These approximations can appear in the original variable space, a higher-dimensional variable space via the introduction of auxiliary variables, or an entirely different variable space through suitable variable transformations. Obtaining such approximations is the goal of a large number of discrete optimization techniques including cutting planes, coefficient adjustment, reformulation-linearization, lift and project, constraint aggregation/disaggregation, and variable redefinition.

To illustrate the concept of polyhedral outer-approximations, consider Figure 1.1 which presents two such representations for an integer program having two variables. The feasible region of the integer program is the collection of ten points that satisfy the five linear constraints. The shaded region on the left is the typical continuous relaxation obtained by ignoring the integrality restrictions. While simple to identify, a drawback to this outer-approximation is that it may be weak and allow many non-integral extreme points. In contrast, the polyhedral set given by the shaded region on the right has all integral extreme points and is the convex hull of the ten feasible points. The convex hull is the strongest possible representation and, assuming a linear objective, reduces the integer program to a linear program. A potential drawback, however, is that in general it may be very difficult to explicitly characterize the convex hull. Moreover, the number of defining constraints may be exponential in terms of the number of variables.

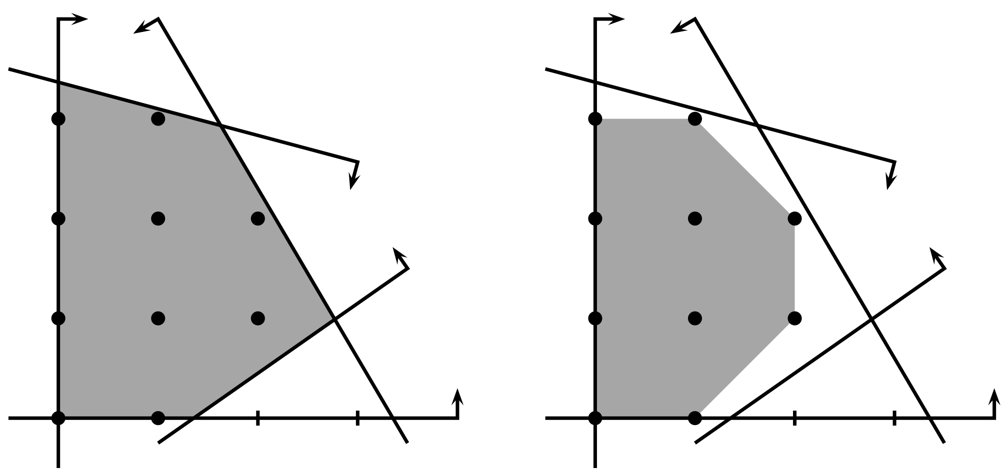

Figure 1.1: A comparison of the continuous relaxation (left) and the convex hull (right).

It is unfortunate that conciseness and strength, two very desirable properties of polyhedral outer-approximations, are frequently in conflict. Consequently, researchers often seek approximations that balance the strength and size of the formulation. This dissertation presents three distinct approaches for generating polyhedral outer-approximations with such a balance in mind.

A fundamental tool used throughout Chapters 2 and 3 is a methodology known as the *reformulation-linearization technique* (RLT) which operates by recasting a discrete problem into new, higher-dimensional regions so as to partially eradicate the discretizations, nonlinearities, and non-convexities that complicate the original formulation. Figure 1.2 illustrates this underlying idea. On the left is the feasible region of a two-dimensional mixed-discrete set given by

$$\{(x, y) : 1 \leq x \leq 4 - y,\ 3 - y \leq x \leq 4,\ y \in \{0, 2\}\}$$

which enforces that if $y = 0$ then $3 \leq x \leq 4$ and if $y = 2$ then $1 \leq x \leq 2$. The constraints defining the polyhedron on the right are automatically generated by the RLT by multiplying the original constraints by *functional factors* involving the discrete variable y, enforcing special *simplifying identities*, and introducing a new auxiliary variable w which represents the product xy. In this way, the convex hull of the feasible solutions of the original set is captured, albeit in a higher-dimensional space. For a more detailed description of general RLT methodology, see Sections 2.2 and 3.1.

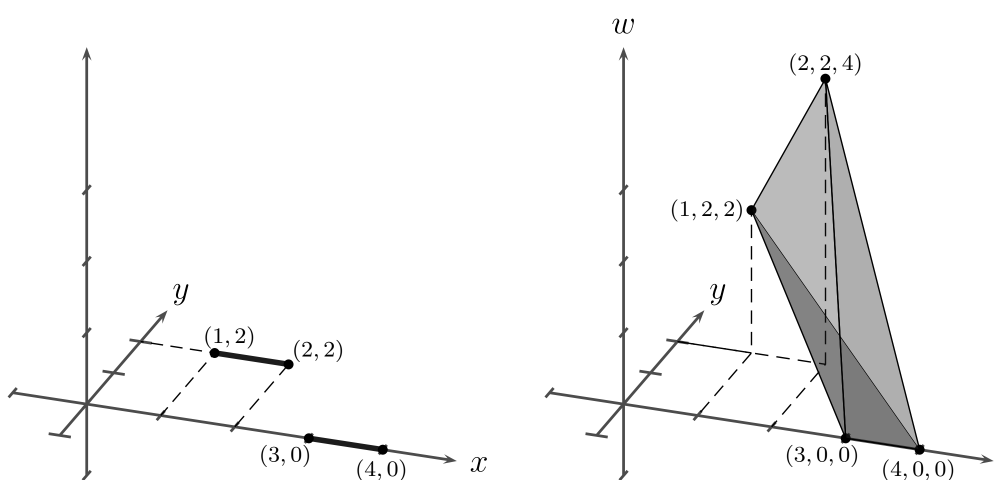

Figure 1.2: A mixed-integer feasible region in $\mathbb{R}^2$ (left) and its convex hull in $\mathbb{R}^3$ given by the RLT (right).

The three main contributions of this work, which deal with distinct strategies for computing polyhedral outer-approximations, appear in the next three chapters. To aid the reader, each chapter is fully self-contained; the chapters may be read in any order or independently. A brief outline of these chapter follows.

Chapter 2 focuses on optimization problems having quadratic expressions involving general discrete variables. Traditionally, quadratic expressions of *binary* variables are often linearized by defining an auxiliary variable for each quadratic term, and by then forming special polytopes which enforce that the new variables equal their intended products at all extreme points. Such polytopes similarly formed for quadratic expressions of continuous variables enable global optimization methods, and have sparked recent interest in characterizing associated convex hulls. A family of related polytopes, for which little is known, arises via the RLT using special Lagrange interpolating polynomials (LIPs) and can linearize products of general discrete variables. We study these polytopes, characterize desirable extreme point traits, establish fundamental properties relative to their dimensions and facial structures, and project these higher-dimensional LIP polytopes onto lower-dimensional subspaces in order to more efficiently express the linearization of these quadratic terms. The nature of these projections yields linearized expressions that represent polynomial terms in the two variables. In particular, for the special cases wherein one of the variables is binary and the other is discrete, we completely characterize all facets of the convex hulls of the feasible realizations in lower dimensions. For the more general case having the product of two discrete variables, these same projections provide families of facets that partially describe the lower-dimensional convex hulls. We also obtain new polytopes that allow for the linearization of polynomial expressions involving a bounded *continuous* variable and a general discrete variable.

Chapter 3 extends the underlying RLT constructs to develop a much richer convex hull theory. These extensions are developed using the facets of a special class of polytopes known as simplices, which are formed as the convex hull of $n+1$ affinely independents points in $\mathbb{R}^n$. As its name suggests, the RLT is composed of the two key steps of reformulation and linearization. Given an MIP or MINLP, the reformulation step consists of multiplying the problem constraints by product factors of the discrete variables, and employing a simplification that exploits the discrete structure. The linearization step then transforms the problem into a higher-dimensional variable space by substituting a continuous variable for each distinct nonlinear term. For the case in which the discrete variables are binary, the product factors consist of products of the 0-1 variables with their complements. For the general discrete case, special functions of these variables, known as Lagrange interpolating polynomials, are instead used. Chapter 3 shows that all these products factors are special cases of the more general simplices. Specifically, the simplicial product factors generalize those from the LIPs, which in turn generalize the products of binary variables and their complements. As such, the results of Chapter 3 can be envisioned as unifying and subsuming the convex hull theory of the RLT.

Chapter 4 presents an approach for representing functions of discrete variables, and their products, using logarithmic numbers of binary variables. In contrast to Chapters 2 and 3, this chapter does not rely on RLT constructs to generate polyhedral outer-approximations. Instead, it focuses concise binary transformations. Given a univariate function whose domain consists of n distinct values, a typical binary representation employs n 0-1 variables, one for each value in the domain, to model the function. In contrast, we employ a variable transformation that uses a base-2 expansion to express the function in terms of $\lceil \log_2 n \rceil$ binary and n continuous variables. This approach is novel in that it requires fewer linear restrictions than related approaches in the literature. The model relies on a simple observation relative to the unit hypercube which states that a binary vector can be represented as a convex combination of a subset of distinct extreme points of the unit hypercube if and only if the vector is itself one of these extreme points, with a single convex multiplier equaling 1, and the remaining equaling 0. Furthermore, by employing this observation we linearize products of m such functions by multiplying the linear restrictions associated with any one function by a scaled version of the product of $p-1$ remaining functions in an inductive fashion from $p=2$ to m. These representations are important for reformulating general discrete variables as binary, and also for linearizing mixed-integer generalized geometric and discrete nonlinear programs, where it is desired to economize on the number of binary variables. It provides insight into, improves upon, and subsumes related linearization methods for products of functions of discrete variables.

Bibliography

[1] Abara, J., “Applying Integer Linear Programming to the Fleet Assignment Problem,” *Interfaces* Vol. 19, No. 4, 20-28, 1989.

[2] Arntzen, B.C., Brown, G.G., Harrison, T.P., and Trafton, L.L., “Global Supply Chain Management at Digital Equipment Corporation,” *Interfaces*, Vol. 25, No. 1, 69-93, 1995.

[3] Bertsimas, D., Darnell, C., and Soucy, R., “Portfolio Construction Through Mixed-Integer Programming at Grantham, Mayo, Van Otterloo and Company,” *Interfaces*, Vol. 29, No. 1, 49-66, 1999.

[4] Cohen, M.A. and Lee, H.L., “Resource Deployment Analysis of Global Manufacturing and Distribution Networks,” *Journal of Manufacturing and Operations Management*, Vol. 2, 81-104, 1989.

[5] Gounaris, C.E., Misener, R., and Floudas, C.A., “Computational Comparison of Piecewise-Linear Relaxations for Pooling Problems,” *Industrial and Engineering Chemistry Research*, Vol. 48, Issue 12, 5742-5766, 2009.

[6] Grossmann, I.E., “Mixed-Integer Optimization Techniques for Algorithmic Process,” *Chemical Engineering*, Vol. 23, Process Synthesis, 171-246, 1996.

[7] Grossmann, I.E., Caballero, J.A., and Yeomans, H., “Advances in Mathematical Programming for Automated Design, Integration and Operation of Chemical Processes,” *Proceedings of the International Conference on Process Integration*, Copenhagen, Denmark (1999).

[8] Markowitz, H., *Portfolio Selection: Efficient Diversification and Investments*, John Wiley and Sons, New York, 1959.

[9] Melachrinoudis, E. and Min H., “The Dynamic Relocation and Phase-Out of a Hybrid, Two-Echelon Plant/Warehousing Facility: A Multiple Objective Approach,” *European Journal of Operational Research*, Vol. 123, Issue 1, 1-15, 2000.

[10] Rushmeier, R., Homan, K., Padberg, M., *Recent Advances in Exact Optimization of Airline Scheduling Problems*, Technical Report, Department of Operations Research and Operations Engineering, George Mason University, 1995.

Chapter 2

On Polytopes Associated with Products of Discrete Variables

2.1 Linearization Background

Polytopes associated with products of binary variables were introduced over fifty years ago in the context of 0-1 quadratic programming. Given two binary variables x_1 and x_2, the papers [11, 12] and later [13] substitute a continuous variable w_{12} for the product x_1x_2, and then use the following four linear inequalities to enforce that $w_{12} = x_1x_2$ for all binary x_1 and x_2

$$w_{12} \geq 0,\ w_{12} \geq x_1 + x_2 - 1,\ w_{12} \leq x_1,\ w_{12} \leq x_2. \tag{2.1}$$

It is straightforward to show that these four inequalities define the facets of the *polytope*

$$P(J) = \text{conv}\left\{(x_i, x_j, w_{ij}) \in \{0,1\}^3 : w_{ij} = x_ix_j\right\}, \tag{2.2}$$

where $J \equiv \{i, j\}$ with $i = 1$ and $j = 2$, and where conv $\{\bullet\}$ denotes the convex hull of the set $\bullet$.

Inequalities (2.1) allow for the linearization of *unconstrained quadratic programs* in 0-1 variables of the form

$$\text{minimize} \left\{ \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} C_{ij} x_i x_j : \boldsymbol{x} \in \{0,1\}^n \right\}. \tag{UQP}$$

Each product term $x_i x_j$ in the objective function is replaced with a distinct w_{ij}, and constraints of the form (2.1) are enforced for each (i, j) pair with $i < j$. This allows Problem UQP to be equivalently rewritten as the *linearized quadratic program*

$$\text{minimize} \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} C_{ij} w_{ij} : \boldsymbol{x} \in \{0,1\}^n, \\ w_{ij} \geq 0, \; w_{ij} \geq x_i + x_j - 1, \; w_{ij} \leq x_i, \; w_{ij} \leq x_j \; \forall \, (i,j), \; i < j \end{array} \right\}. \tag{LQP}$$

Linear and/or quadratic constraints can be included within UQP to form a *constrained quadratic program*, and LQP will remain an equivalent form provided these same constraints are preserved, with the substitution $w_{ij} = x_i x_j$ similarly applied. The continuous relaxation of LQP obtained by relaxing the restrictions $\boldsymbol{x} \in \{0,1\}^n$ to $\boldsymbol{x} \in [0,1]^n$ has been studied. For the special case in which every C_{ij} is nonpositive, the first two families of inequalities are redundant at optimality, so that [18] was able to solve this relaxation as a network and obtain an optimal binary solution to UQP. For general C_{ij}, UQP is NP-hard, but the relaxation of LQP (yielding potentially fractional extreme points) can be transformed to a network, with concise forms found in [1, 23].

Generalizations of (2.2) have proven important in both discrete and continuous, nonconvex optimization. The *boolean quadric polytope* in n binary variables $\boldsymbol{x}$ is defined as

$$BP_n = \text{conv} \left\{ (\boldsymbol{x}, \boldsymbol{w}) \in \{0,1\}^{n \times \frac{n(n-1)}{2}} : w_{ij} = x_i x_j \; \forall \, (i,j), \; i < j \right\},$$

which reduces to (2.2) when $n = 2$. This polytope was introduced in [17], and has since attracted considerable interest, including [10, 22]. It has been shown equivalent [9], via a nonsingular linear transformation, to the cut polytope [6].

The significance of BP_n is that the binary optimization problem UQP reduces to the *linear program*

$$\text{minimize} \left\{ \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} C_{ij} w_{ij} : (\boldsymbol{x}, \boldsymbol{w}) \in BP_n \right\}. \tag{LP}$$

Consequently, an explicit description of BP_n allows for the solving of UQP as a linear program.

It is instructive to note the difference between the feasible regions to Problem LP and the above-mentioned continuous relaxation to LQP. The feasible region to this relaxation is equivalent, by (2.2), to $(\boldsymbol{x}, \boldsymbol{w}) \in \bigcap_{\substack{J \subseteq N \\ |J|=2}} P(J)$, where $N \equiv \{1, 2, \ldots, n\}$. It is readily verified that $\bigcap_{\substack{J \subseteq N \\ |J|=2}} P(J) \subseteq BP_n$, with equality holding if and only if $n = 2$. For the case in which $n = 3$, the paper [17] shows that four additional "triangle inequalities" are needed to achieve equality.

A second generalization of (2.1) arises in global optimization relative to the approximation of non-convex functions. Given the product x_1x_2 in *continuous* variables x_1 and x_2 with $l_1 \leq x_1 \leq u_1$ and $l_2 \leq x_2 \leq u_2$, we can construct the inequalities

$$\begin{aligned} w_{12} &\geq l_2x_1 + l_1x_2 - l_1l_2, \qquad & w_{12} &\geq u_1x_2 + u_2x_1 - u_1u_2, \\ w_{12} &\leq u_2x_1 + l_1x_2 - l_1u_2, \qquad & w_{12} &\leq l_2x_1 + u_1x_2 - u_1l_2, \end{aligned} \tag{2.3}$$

which reduce to (2.1) when $l_1 = l_2 = 0$ and $u_1 = u_2 = 1$. (These inequalities are motivated in the next section from the perspective of a *reformulation-linearization technique* (RLT).) The paper [16] notes that the first two inequalities of (2.3) give the convex envelope of the function x_1x_2 when $l_1 \leq x_1 \leq u_1$ and $l_2 \leq x_2 \leq u_2$. The work [4] uses this result, and also that $x_1x_2 = \max\{l_2x_1 + l_1x_2 - l_1l_2, u_1x_2 + u_2x_1 - u_1u_2\}$ when either x_1 or x_2 is at its lower or upper bound, to develop solution strategies for biconvex programs.

More recent investigations into the product x_1x_2 for continuous variables x_1 and x_2 include the following two works. The paper [5] provides semi-definite inequalities for the polytopes

$$CP_n = \text{conv}\left\{(\boldsymbol{x}, \boldsymbol{w}) \in [0,1]^{n \times n^2} : w_{ij} = x_ix_j \ \forall\, (i,j),\ i \leq j\right\},$$

and shows that these inequalities completely describe CP_n if and only if $n = 1$ or $n = 2$. Furthermore, they describe CP_3 by forming a triangulation of the unit cube and applying simplicial results. The contribution of [7] extends this study by focusing on the structure of CP_n for general $n \geq 3$, and by examining connections with the boolean quadric polytope.

In this chapter, we explore a natural generalization of (2.2) arising in polynomial integer optimization where the variables x_1 and x_2 are general discrete, as opposed to binary. Specifically, suppose that x_1 and x_2 are restricted to realize one of the k_1 and k_2 values in the discrete sets

$S_1 \equiv \{\theta_{11}, \theta_{12}, \ldots, \theta_{1k_1}\}$ and $S_2 \equiv \{\theta_{21}, \theta_{22}, \ldots, \theta_{2k_2}\}$ respectively, where it is assumed without loss of generality that the elements in each set are distinct and arranged in increasing order so that $\theta_{11} < \theta_{12} < \cdots < \theta_{1k_1}$ and $\theta_{21} < \theta_{22} < \cdots < \theta_{2k_2}$. Consider a polynomial function in these variables of the form

$$p(x_1, x_2) \equiv \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j, \tag{2.4}$$

where each a_{ij} is a real number and where d_1 and d_2 denote the maximum degrees of x_1 and x_2 respectively in (2.4). Of interest are the cases having $d_1 + d_2 \geq 2$ so that $p(x_1, x_2)$ is nonlinear. For these cases, we focus on the *discrete polytope*

$$DP(d_1, d_2) = \text{conv} \left\{ \begin{array}{l} (x_1, x_2, \boldsymbol{w}) \in \mathbb{R}^{(d_1+1)(d_2+1)-1} : \quad x_1 \in S_1, x_2 \in S_2, \\ w_{ij} = x_1^i x_2^j \ \forall \ (i,j) \ni i + j \geq 2, i \in \{0, \ldots, d_1\}, j \in \{0, \ldots, d_2\} \end{array} \right\}. \tag{2.5}$$

Note here that the variable w_{ij} effectively records the product of x_1^i and x_2^j via the definition $w_{ij} = x_1^i x_2^j$, as opposed to the product of the two variables x_i and x_j defined earlier by $w_{ij} = x_i x_j$. This polytope (2.5) relates to (2.4) in that the nonlinear discrete program to optimize $p(x_1, x_2)$ over $x_1 \in S_1$ and $x_2 \in S_2$ reduces to the linear program that optimizes this same function over $DP(d_1, d_2)$ under the substitution $w_{ij} = x_1^i x_2^j$.

Observe how $DP(d_1, d_2)$ relates to (2.1) and (2.3) for the special case having $k_1 = k_2 = 2$ and $d_1 = d_2 = 1$ so that (2.4) simplifies to $p(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$. When $S_1 = S_2 = \{0, 1\}$, then $DP(1, 1)$ is defined by inequalities (2.1). When $S_1 = \{\ell_1, u_1\}$ and $S_2 = \{\ell_2, u_2\}$, then $DP(1, 1)$ is described by inequalities (2.3). Thus, $DP(d_1, d_2)$ can be envisioned as motivating a richer family of polytopes than either (2.1) or (2.3).

The concern of this study is to characterize the polytopes $DP(d_1, d_2)$, which by definition are expressed in terms of the *original* variables x_1 and x_2. This is in contrast to earlier works that rely on higher-dimensional spaces expressed in terms of suitable binary expansions of x_1 and x_2. The paper [8] focuses on the special cases of x_1 and x_2 having $S_1 = \{0, \ldots, k_1 - 1\}$ and $S_2 = \{0, \ldots, k_2 - 1\}$ where, for simplicity, $k_1 = 2^{n_1}$ and $k_2 = 2^{n_2}$ for positive integers n_1 and n_2. In this manner, x_1 and x_2 are integer variables satisfying $0 \leq x_1 \leq k_1 - 1$ and $0 \leq x_2 \leq k_2 - 1$. The product $x_1 x_2$ is first expressed as

$$x_1 x_2 = \left(\sum_{i=1}^{n_1} 2^{i-1} \lambda_{1i} \right) \left(\sum_{j=1}^{n_2} 2^{j-1} \lambda_{2j} \right)$$

where all such λ_{1i} and λ_{2j} are binary variables. Letting w_{ij} replace the quadratic term $\lambda_{1i}\lambda_{2j}$ within this expression for each (i, j), they then have that

$$x_1 x_2 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} 2^{i+j-2} w_{ij}$$

by enforcing $4n_1n_2$ inequalities of the form found in (2.1), one set of four for each (i, j) pair, as

$$w_{ij} \geq 0,\ w_{ij} \geq \lambda_{1i} + \lambda_{2j} - 1,\ w_{ij} \leq \lambda_{1i},\ w_{ij} \leq \lambda_{2j}\ \ \forall\, (i, j),\ i = 1, \ldots, n_1,\ j = 1, \ldots, n_2.$$

Alternate approaches that employ binary expansions to linearize products of discrete functions, and consequently discrete variables, are found in [2] and [15].

The chapter is organized as follows. The next section briefly reviews the RLT methodology for mixed-discrete problems. The RLT relies on special functions of x_1 and x_2, called Lagrange interpolating polynomials (LIPs), which provide an explicit characterization of $DP(k_1 - 1, k_2 - 1)$. These LIPs allow us to establish a relationship between the parameters k_1 and k_2, and the degrees of the variables x_1 and x_2 found within $p(x_1, x_2)$ of (2.4); specifically, we can assume without loss of generality that $d_1 \leq k_1 - 1$ and $d_2 \leq k_2 - 1$ in (2.4) and (2.5). Our study continues in Section 2.3 with the cases where $k_1 = 1$, so that $d_1 = 0$ and only the variable x_2 is present in (2.4) and (2.5). We obtain an explicit characterization of the polytopes $DP(0, d_2)$, for each $d_2 \in \{1, \ldots, k_2 - 2\}$, via a suitably-defined projection operation from the higher-dimensional space $DP(0, k_2 - 1)$. Included here is a characterization of all facets. Section 2.4 identifies the sets $DP(k_1 - 1, d_2)$ for any $d_2 \in \{1, \ldots, k_2 - 1\}$ and any k_1. This last result is particularly useful when x_1 is binary and x_2 is general discrete, so that $k_1 = 2$. In fact, convex hull representations are also obtained when the binary variable x_1 is relaxed to be continuous. For the most general case of $DP(d_1, d_2)$, the convex hull is not obtained, but families of facets are identified. Concluding remarks are found in Section 2.5.

2.2 Reformulation-Linearization Constructs

The paper [3] uses Lagrange interpolating polynomials (LIPs) to generalize a reformulation-linearization technique [19, 20, 21] for mixed 0-1 polynomial programs so as to handle problems containing general discrete variables. The LIP constructs play a critical role in our study of the polytopes $DP(d_1, d_2)$, and are therefore briefly summarized below.

As pointed out in [3], a crucial observation for extending (2.1) to handle discrete variables x_1 and x_2 is that, given a binary variable x_j, the expressions $1-x_j$ and x_j are Lagrange interpolating polynomials. To explain, consider the two discrete variables $x_j,\ j \in \{1,2\}$, that can realize values in the sets $S_j = \{\theta_{j1}, \theta_{j2}, \ldots, \theta_{jk_j}\}$ introduced in the previous section. Then there exist k_j LIPs associated with each x_j, see [14], with every polynomial of degree $k_j - 1$. The polynomials take the following forms, where $K_j \equiv \{1, \ldots, k_j\}$.

$$L_{jk}(x_j) = \frac{\displaystyle\prod_{i \in (K_j - \{k\})} (x_j - \theta_{ji})}{\displaystyle\prod_{i \in (K_j - \{k\})} (\theta_{jk} - \theta_{ji})} \qquad k \in K_j \tag{2.6}$$

These polynomials have the property that for each $x_j \in S_j$,

$$L_{jk}(x_j) = \begin{cases} 1 & \text{if } x_j = \theta_{jk} \\ 0 & \text{otherwise} \end{cases} \qquad k \in K_j, \tag{2.7}$$

so that

$$x_j = \sum_{k=1}^{k_j} \theta_{jk} L_{jk}(x_j), \tag{2.8}$$

and

$$L_{jk}(x_j) L_{j\ell}(x_j) = \begin{cases} 0 & \text{if } \ell \neq k \\ L_{jk}(x_j) & \text{if } \ell = k \end{cases} \quad \forall\ (k,\ell),\ k \in \{1,\ldots,k_j\},\ \ell \in \{1,\ldots,k_j\},\ \ell \neq k. \tag{2.9}$$

The property (2.7) implies the LIPs of (2.6) to be nonnegative for all $x_j \in S_j$. This nonnegativity can be expressed in matrix notation as $C_j \boldsymbol{x}^j \geq \mathbf{0}$, where C_j is a $k_j \times k_j$ matrix whose $(k,q)^{th}$ element is the coefficient in $L_{jk}(x_j)$ on x_j^{q-1}, $\boldsymbol{x}^j$ is a column vector in $\mathbb{R}^{k_j}$ whose q^{th} entry is x_j^{q-1}, and $\mathbf{0}$ is a column vector of zeros in $\mathbb{R}^{k_j}$. (Here, $x_j^0 \equiv 1$.)

Example 2.1

Given a variable x_j that can realize values in $S_j = \{0, 1\}$, the LIPs are as follows.

$$\begin{aligned} L_{j1}(x_j) = & \tfrac{(x_j-1)}{(0-1)} = & 1 - x_j \\ L_{j2}(x_j) = & \tfrac{(x_j-0)}{(1-0)} = & x_j \end{aligned}$$

Nonnegativity of these LIPs is expressed in matrix notation as

$$C_j \boldsymbol{x}^j = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example 2.2

Given a variable x_j that can realize values in $S_j = \{0, 1, 2\}$, the LIPs are as follows.

$$\begin{aligned} L_{j1}(x_j) = & \tfrac{(x_j-1)(x_j-2)}{(0-1)(0-2)} = & 1 - \frac{3}{2}x_j + \frac{1}{2}x_j^2 \\ L_{j2}(x_j) = & \tfrac{(x_j-0)(x_j-2)}{(1-0)(1-2)} = & 0 + 2x_j - x_j^2 \\ L_{j3}(x_j) = & \tfrac{(x_j-0)(x_j-1)}{(2-0)(2-1)} = & 0 - \frac{1}{2}x_j + \frac{1}{2}x_j^2 \end{aligned}$$

Nonnegativity of these LIPs is expressed in matrix notation as

$$C_j \boldsymbol{x}^j = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 2 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ x_j \\ x_j^2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Relative to the product of two discrete variables, insight can be gained by reconsidering the inequalities in (2.1), and by expressing these restrictions in terms of Kronecker products of LIPs. Recall that, given two matrices A_1 and A_2, where A_1 is $m_1 \times n_1$ and A_2 is $m_2 \times n_2$, the Kronecker product of A_1 and A_2, denoted by $A_1 \otimes A_2$, is the $m_1 m_2 \times n_1 n_2$ matrix defined as $A_1 \otimes A_2 = \begin{bmatrix} a_{11}A_2 & \dots & a_{1n_1}A_2 \\ \vdots & \ddots & \vdots \\ a_{m_1 1}A_2 & \dots & a_{m_1 n_1}A_2 \end{bmatrix}$, where a_{ij} represents the $(i, j)^{th}$ entry of A_1. Then (2.1) can be written using Kronecker products as follows

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.10}$$

This thought process for expressing (2.1) as (2.10) is extendable to general discrete variables. Given such x_1 and x_2 associated with sets S_1 and S_2 respectively, compute the corresponding LIPs, and express them in matrix notation as above to obtain $C_1\boldsymbol{x}^1 \geq \mathbf{0}$ and $C_2\boldsymbol{x}^2 \geq \mathbf{0}$. Here, C_1 and C_2 are of sizes $k_1 \times k_1$ and $k_2 \times k_2$ respectively, and the vectors $\boldsymbol{x}^j$ for $j \in \{1, 2\}$, are columns in $\mathbb{R}^{k_j}$ whose q^{th} entry is x_j^{q-1}. (Throughout the remainder of this chapter, we let $\mathbf{0}$ denote suitably-dimensioned column vectors of zeros.) Following [3], and in the spirit of (2.10), compute the products of the LIPs associated with $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, and set these products nonnegative to obtain

$$C_1\boldsymbol{x}^1 \otimes C_2\boldsymbol{x}^2 \geq \mathbf{0} \otimes \mathbf{0}. \tag{2.11}$$

A property of Kronecker products is that, for any matrices A, B, C, and D such that the multiplications AB and CD are defined, we have

$$AB \otimes CD = (A \otimes C)(B \otimes D). \tag{2.12}$$

Using this property, (2.11) can be rewritten as follows

$$(C_1 \otimes C_2)\left(\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right) \geq \mathbf{0} \otimes \mathbf{0}.$$

We linearize these inequalities by substituting a continuous variable for each distinct nonlinear term, adopting the notation of (2.5) that $w_{ij} = x_1^i x_2^j$ for all (i, j) such that $i + j \geq 2$. Consistent with [3], let $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right\}_L$ denote the linearized form of $\boldsymbol{x}^1 \otimes \boldsymbol{x}^2$ obtained by performing such a substitution. The following polyhedral set P results:

$$P = \left\{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right\}_L : (C_1 \otimes C_2)\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right\}_L \geq \mathbf{0} \otimes \mathbf{0}\right\}. \tag{2.13}$$

The set P is, in fact, the polytope $DP(k_1 - 1, k_2 - 1)$ of (2.5). This equivalence was established in a different setting in [3], but is stated formally below for completeness.

Theorem 2.1: *Given any $k_1 \geq 2$ and $k_2 \geq 2$, the set P of (2.13) is the polytope $DP(k_1 - 1, k_2 - 1)$ of* (2.5).

The argument in [3] for establishing Theorem 2.1 is the following. Identity (2.7) gives us that each of the matrices C_1 and C_2 is invertible with, for $j \in \{1, 2\}$, $C_j^{-1} = V_j^T$, where V_j represents the $k_j \times k_j$ Vandermonde matrix whose $(p, q)^{th}$ entry is θ_{jp}^{q-1}, for $j \in \{1, 2\}$. (We let $0^0 = 1$ for convenience.) Then $(C_1 \otimes C_2)^{-1} = V_1^T \otimes V_2^T$ by (2.12) so that (2.13) can be rewritten as

$$P = \left\{ \left\{ \boldsymbol{x}^1 \otimes \boldsymbol{x}^2 \right\}_L : \left\{ \boldsymbol{x}^1 \otimes \boldsymbol{x}^2 \right\}_L = \left(V_1^T \otimes V_2^T \right) \boldsymbol{z} \text{ for some } \boldsymbol{z} \geq \boldsymbol{0} \right\},$$

where $\boldsymbol{z}$ is a column vector in $\mathbb{R}^{k_1 k_2}$. As the first entry of $\left\{ \boldsymbol{x}^1 \otimes \boldsymbol{x}^2 \right\}_L$ is 1 and the first row of $V_1^T \otimes V_2^T$ has all ones, the set P can again be rewritten as

$$P = \left\{ \left(V_1^T \otimes V_2^T \right) \boldsymbol{z} : \boldsymbol{z} \geq \boldsymbol{0}, \boldsymbol{e}^T \boldsymbol{z} = 1 \right\}, \tag{2.14}$$

where $\boldsymbol{e}$ is a column vector of ones in $\mathbb{R}^{k_1 k_2}$. Theorem 2.1 follows since the extreme points to $\{\boldsymbol{z} \geq \boldsymbol{0}, \boldsymbol{e}^T \boldsymbol{z} = 1\}$ are the unit vectors in $\mathbb{R}^{k_1 k_2}$. This gives us that each extreme point of the polytope P of (2.14) has $x_j \in S_j$ for $j \in \{1, 2\}$, and $w_{ij} = x_1^i x_2^j$ for all $i \in \{0, \ldots, d_1\}$ and $j \in \{0, \ldots, d_2\}$ with $i + j \geq 2$. Moreover, each inequality defining (2.13) is a facet.

The example below, taken from [3], illustrates this argument and demonstrates how the columns of $V_1^T \otimes V_2^T$ correspond to the vectors $\boldsymbol{x}^1 \otimes \boldsymbol{x}^2$ for all possible realizations of $x_1 \in S_1$ and $x_2 \in S_2$.

Example 2.3

Consider discrete variables x_1 and x_2 that realize values in the sets $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2\}$ respectively. Then $k_1 = 2$ and $k_2 = 3$ with $C_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 2 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $V_1^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $V_2^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$. The polytope P of (2.13) is given by

$$P = \left\{ \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \end{array} \right\}_L : \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & -1 & 0 & -2 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \hline 0 & 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \end{array} \right\}_L \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

which can be rewritten in the form of (2.14) as

$$P = \left\{ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} : \sum_{j=1}^{6} z_j = 1, z_j \geq 0 \text{ for } j = 1, \ldots, 6 \right\}.$$

Given any $k_1 \geq 2$ and $k_2 \geq 2$, Theorem 2.1 gives P of (2.13) as an explicit description of $DP(k_1 - 1, k_2 - 1)$ from (2.5), with each defining inequality a facet. The special case for which $k_1 = k_2 = 2$ has $DP(1, 1)$ defined in Section 2.1 by (2.3) when $\theta_{11} = l_1$, $\theta_{1k_1} = u_1$, $\theta_{21} = l_2$, and $\theta_{2k_2} = u_2$. But other polytopes $DP(d_1, d_2)$ for general k_1 and k_2 are not known. This is unfortunate since the degrees of the variables x_1 and x_2 in the polynomial $p(x_1, x_2)$ of (2.4) can be far smaller than k_1 and k_2 respectively, permitting a convex hull representation with far fewer variables. Such smaller representations are the concern of this chapter.

Before proceeding to the next section, we make two observations. First, the result found in Section 2.1 stating that the initial two inequalities of (2.3) define the convex envelope of the function x_1x_2 over $l_1 \leq x_1 \leq u_1$ and $l_2 \leq x_2 \leq u_2$, as provided in [16], follows from the logic of the theorem and its proof. Suppose we temporarily generalize our definition of the matrices C_1 and C_2 for this case to be $C_1 = \begin{bmatrix} u_1 & -1 \\ -l_1 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} u_2 & -1 \\ -l_2 & 1 \end{bmatrix}$, and we let $\boldsymbol{x}^1 = \begin{bmatrix} 1 \\ x_1 \end{bmatrix}$ and $\boldsymbol{x}^2 = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}$. Then the set P of (2.13) takes the form

$$\begin{bmatrix} u_1 & -1 \\ -l_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \end{bmatrix} \otimes \begin{bmatrix} u_2 & -1 \\ -l_2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is (2.2) with $w_{12} = x_1x_2$. As $C_1^{-1} = \frac{1}{u_1-l_1}\begin{bmatrix} 1 & 1 \\ l_1 & u_1 \end{bmatrix}$ and $C_2^{-1} = \frac{1}{u_2-l_2}\begin{bmatrix} 1 & 1 \\ l_2 & u_2 \end{bmatrix}$, the proof of the theorem gives us that the set $\{(x_1, x_2, w_{12}) : (x_1, x_2, w_{12}) \text{ satisfies } (2.2)\}$ is the polytope whose extreme points are $(l_1, l_2, l_1l_2)^T$, $(l_1, u_2, l_1u_2)^T$, $(u_1, l_2, u_1l_2)^T$, and $(u_1, u_2, u_1u_2)^T$. Therefore, the only linear inequalities under-approximating the function x_1x_2 over $l_1 \leq x_1 \leq u_1$ and $l_2 \leq x_2 \leq u_2$ are the first two of (2.2), establishing the result.

Second, the properties of LIPs allow us to establish an upper bound on the parameters d_1 and d_2 defining $p(x_1, x_2)$ of (2.4) in terms of the numbers of permissible realizations k_1 and k_2 for x_1 and x_2 respectively. Consider the following lemma.

Lemma 2.1: *Any polynomial $p(x_1, x_2)$ of the form (2.4) with $x_1 \in S_1 \equiv \{\theta_{11}, \theta_{12}, \ldots, \theta_{1k_1}\}$ and $x_2 \in S_2 \equiv \{\theta_{21}, \theta_{22}, \ldots, \theta_{2k_2}\}$ can be expressed so that $d_1 \leq k_1 - 1$ and $d_2 \leq k_2 - 1$.*

Proof. Given $j \in \{1, 2\}$, the proof is to show that x_j^p for $p \geq k_j$ can be written in terms of x_j^i for $i \leq k_j - 1$. Toward this end, note that $x_j^p = \left[\sum_{k=1}^{k_j} \theta_{jk} L_{jk}(x_j)\right]^p = \sum_{k=1}^{k_j} \theta_{jk}^p L_{jk}(x_j)$, where the first equality is by (2.7) and the second equality follows from (2.8) and (2.9). As each L_{jk} is of degree $k_j - 1$, the result follows. □

By Lemma 2.1, we henceforth assume throughout the remainder of the chapter that (2.4) has $d_1 \leq k_1 - 1$ and $d_2 \leq k_2 - 1$.

2.3 Projected Convex Hull Forms for One Variable

Given any $k_1 \geq 2$ and $k_2 \geq 2$, an explicit description of $DP(k_1 - 1, k_2 - 1)$ from (2.5) is given by P of (2.13). This section considers, the cases having $k_1 = 1$ and general k_2, and focuses on the polytopes $DP(0, d_2)$ for which $d_2 \in \{2, \ldots, k_2 - 2\}$. (Observe that by symmetry this is equivalent to studying the polytopes $DP(d_1, 0)$ for $d_1 \in \{2, \ldots, k_1 - 2\}$.) For such cases, the variable x_1 is a constant, so that only x_2 is present. Less attention is given to the set $DP(0, 1)$ as it is trivially defined by $\theta_{21} \leq x_2 \leq \theta_{2k_2}$.

For simplicity of notation within this section, and since only the single variable x_2 is being considered, we suppress the subscript of 2 on x, S, k, d, and K, and the first subscript on θ. We also suppress the superscript on the k-dimensional column vector $\boldsymbol{x}$, the subscript on C, the first subscript on the auxiliary variables w within $DP(0, d)$, and let $K \equiv \{1, \ldots, k\}$.

Thus, consider a single discrete variable x realizing values in $S = \{\theta_1, \theta_2, \ldots, \theta_k\}$, where $\theta_1 < \theta_2 < \cdots < \theta_k$. Similar to the previous section, we compute the k LIPs associated with x, set these expressions to be nonnegative, and then linearize by substituting a continuous variable for each product term. The resulting polyhedral set $DP(0, k-1)$ is given by

$$DP(0, k-1) = \{\{\boldsymbol{x}\}_L : C\{\boldsymbol{x}\}_L \geq \mathbf{0}\}, \tag{2.15}$$

where $\{\boldsymbol{x}\}_L$ is the linearized version of $\boldsymbol{x}$, and where the matrix C represents the LIP coefficients as in (2.13) defined relative to a single x. The polytope $DP(0, k-1)$ was obtained by linearizing polynomials of degree $k-1$. As such, it inherits the property of Theorem 2.1 that there exists k extreme points, with each extreme point $\{\boldsymbol{x}\}_L$ equal to $\boldsymbol{x}$ evaluated at some θ_j.

The task of computing $DP(0, d)$ for $d \in \{2, \ldots, k-2\}$ is equivalent to defining projections onto lower-dimensional spaces of $DP(0, k-1)$. Specifically, to compute $DP(0, d)$, we desire to project $DP(0, k-1)$ onto the lower-dimensional space which corresponds to the first $d+1$ entries of $\{\boldsymbol{x}\}_L$, for $d \in \{2, \ldots, k-2\}$. Denote the first $d+1$ entries of $\{\boldsymbol{x}\}_L$ by $\{\boldsymbol{x}_d\}_L$ and the last $k-d-1$ entries by $\{\boldsymbol{x}'_d\}_L$. We choose the notation $\{\boldsymbol{x}_d\}_L$ instead of $\{\boldsymbol{x}_{d+1}\}_L$ for its simplicity and since the leading element of $\{\boldsymbol{x}\}_L$ is the constant 1. This choice of notation conveniently enforces that $\{\boldsymbol{x}_d\}_L$ is a vector holding the linearized forms of x raised to the nonnegative integer powers up to d. Formally, the task is to compute the projection of $C\{\boldsymbol{x}\}_L \geq \mathbf{0}$ onto the space of the variables $\{\boldsymbol{x}_d\}_L$.

Such a projection, denoted by $\text{proj}_{\{\boldsymbol{x}_d\}_L} \{DP(0, k-1)\}$, is defined to be the set of all $\{\boldsymbol{x}_d\}_L$ for which there exists a $\{\boldsymbol{x}'_d\}_L$ so that $\{\boldsymbol{x}\}_L \equiv \left\{ \begin{array}{c} \boldsymbol{x}_d \\ \boldsymbol{x}'_d \end{array} \right\}_L$ satisfies (2.15). Let us partition C defining $DP(0, k-1)$ in (2.15) into $\left[\begin{array}{c|c} C_d & C'_d \end{array} \right]$, where C_d represents the first $d+1$ columns of C and C'_d represents the remaining $k-d-1$ columns. Then $C\{\boldsymbol{x}\}_L \geq \mathbf{0}$ can be written as

$$\left[\begin{array}{c|c} C_d & C'_d \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{x}_d \\ \boldsymbol{x}'_d \end{array} \right\}_L \geq \mathbf{0}.$$

Now, consider the *projection cone*

$$\Pi = \left\{ \boldsymbol{\pi} \in \mathbb{R}^k : \boldsymbol{\pi}^T C'_d = \mathbf{0},\ \boldsymbol{\pi} \geq \mathbf{0} \right\}. \tag{2.16}$$

It is well known that a linear inequality in the variables $\{\boldsymbol{x}_d\}_L$ is valid for $\text{proj}_{\{\boldsymbol{x}_d\}_L} \{DP(0, k-1)\}$

if and only if it can be obtained as a linear combination of the inequalities $C\{\boldsymbol{x}\}_L \geq \mathbf{0}$ using some $\boldsymbol{\pi} \in \Pi$. It is also well known that it is not necessary to consider every $\boldsymbol{\pi} \in \Pi$ to fully define the projection; it is sufficient to examine only the extreme directions.

In general, it is not a simple task to compute the projection of a polyhedral set onto a lower-dimensional subspace. The extreme directions of the projection cone may not be readily available, and the number of such directions can be exponential. However, in this case, the set $DP(0, k-1)$ has a special structure that allows for an explicit description.

Given the desired $d \in \{1, \ldots, k-2\}$, we introduce polynomial inequalities of degree d. To do this, we form sets R having $R \subset K$ and $|R| = d$. Then, given a binary v, these inequalities take the form

$$(-1)^v \left(\prod_{j \in R} (\theta_j - x) \right) \geq 0. \tag{2.17}$$

We begin by characterizing those sets R and binary v for which (2.17) is satisfied for all $x \in S$. Define a matrix A_d in the following manner. Each row has k entries such that d entries are value 1 and the remaining $k - d$ entries are value 0. The matrix has a row for each binary vector that satisfies one of the following two properties.

- *Property 0.* For every entry of value 0, there is an *even* number of entries of value 1 to the left.
- *Property 1.* For every entry of value 0, there is an *odd* number of entries of value 1 to the left.

A row of the matrix A_d that results from a binary vector satisfying Property 0 is called a *type-0 row* while a row that results from a binary vector satisfying Property 1 is called a *type-1 row*. An inequality of the form (2.17) is generated for each row of A_d by letting R denote the index set of row entries having value 1, and by setting the parameter v to value 0 for type-0 rows, and to value 1 for type-1 rows. See Examples 2.4–2.6 for instances of A_d with $k = 5$ and d equal to 3, 2, and 1.

Now consider the following lemma.

Lemma 2.2: *A polynomial inequality of the type (2.17) is satisfied by all realizations of $x \in S$ if and only if it is generated by either a type-0 row of A_d with $v = 0$ or a type-1 row of A_d with $v = 1$.*

Proof. The expression $\prod_{j \in R}(\theta_j - x)$ will be nonnegative (non-positive) for all $x \in S$ if and only if, for every $u \in K - R$, the set $R' \equiv \{j \in R : \theta_j < \theta_u\}$ has even (odd) cardinality. But since

$\theta_1 < \theta_2 < \cdots < \theta_k$, the set R' has even (odd) cardinality if and only if the associated inequality (2.17) is generated from a type-0 (type-1) row of A_d. The parameter v thus ensures that (2.17) is satisfied. □

Let E denote the number of rows of A_d and assume that the rows have been arranged so the first $\tilde{E}$ are type-0 rows and the remaining $E - \tilde{E}$ are type-1 rows. We denote the index set of row entries having value 1 in row e by R_e for $e = 1, \ldots, E$. Then, by Lemma 2.1, we can describe the set of all polynomial inequalities of the type (2.17) that are valid for all realizations of $x \in S$ as follows:

$$\prod_{j \in R_e} (\theta_j - x) \geq 0, \quad e = 1, \ldots, \tilde{E}, \quad \text{and} \quad -\prod_{j \in R_e} (\theta_j - x) \geq 0, \quad e = \tilde{E}+1, \ldots, E. \tag{2.18}$$

Now, in the same manner as was used to obtain (2.15) from $C\boldsymbol{x} \geq \mathbf{0}$, let us linearize the E inequalities in (2.18) by substituting a continuous variable for each nonlinear expression. Then, as demonstrated by Theorem 2.2, the polyhedral set $DP(0, d)$ is given by:

$$DP(0,d) = \left\{ \begin{array}{l} \{\boldsymbol{x}_d\}_L \in \mathbb{R}^{d+1} : \\ \qquad \left\{ \prod_{j \in R_e} (\theta_j - x) \right\}_L \geq 0, \quad e = 1, \ldots, \tilde{E}, \\ \qquad -\left\{ \prod_{j \in R_e} (\theta_j - x) \right\}_L \geq 0, \quad e = \tilde{E}+1, \ldots, E \end{array} \right\}. \tag{2.19}$$

Theorem 2.2: *Given any $d \in \{1, \ldots, k-2\}$, we have $DP(0,d) = proj_{\{\boldsymbol{x}_d\}_L} \{DP(0, k-1)\}$, with each of the E inequalities of (2.19) defining a facet of $DP(0,d)$.*

Proof. The proof is established in three steps. First, we show that each extreme direction of the projection cone (2.16) generates an inequality of the type found in (2.19). Second, we show that every inequality of (2.19) can be generated from a direction of (2.16). Finally, we show that every inequality defining (2.19) is a facet of $DP(0,d)$.

To begin, consider any extreme direction $\hat{\boldsymbol{\pi}}$ of Π. Observe that the matrix C'_d has k rows and $k - d - 1$ columns so that $\hat{\boldsymbol{\pi}}$ must have at least d entries of value 0. Also, it follows from (2.7) for each $j \in K$, that the polynomial $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ must realize value $\hat{\pi}_j$ when $x = \theta_j$. So $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ equals 0 when $x = \theta_j$ for each of the (at least) d entries of $\hat{\pi}_j$ that equal 0. Furthermore, since $\hat{\boldsymbol{\pi}} \in \Pi$ of

(2.16), the expression $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ has degree at most d. This uniquely defines $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ to be of the form $(-1)^v \left(\prod_{i\in R}(\theta_i - x)\right)$. Thus, $\hat{\boldsymbol{\pi}}^T C\{\boldsymbol{x}\}_L \geq 0$ is of the form $(-1)^v \left\{\prod_{i\in R}(\theta_i - x)\right\}_L \geq 0$.

Now, consider any inequality defining (2.19): assume without loss of generality that it is one of the first $\tilde{E}$ inequalities of the form $\left\{\prod_{i\in R_e}(\theta_i - x)\right\}_L \geq 0$. Define $\hat{\pi}_j = \prod_{i\in R_e}(\theta_i - \theta_j)$ for each $j \in K$. Each such $\hat{\pi}_j$ is nonnegative by (2.18), with $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x} = \prod_{j=1}^{k} \hat{\pi}_j L_j(x) = \prod_{i\in R_e}(\theta_i - x)$, where the first equality is by definition of the matrix C and the second is due to (2.7). Thus, $\hat{\boldsymbol{\pi}} \in \Pi$ of (2.16), and $\hat{\boldsymbol{\pi}}^T C\{\boldsymbol{x}\}_L = \left\{\prod_{i\in R_e}(\theta_i - x)\right\}_L$. (The proof follows analogously for each of the last $E - \tilde{E}$ inequalities of (2.19) by defining $\hat{\pi}_j = -\prod_{i\in R_e}(\theta_i - \theta_j)$ for each $j \in K$.)

Finally, to show that every inequality defining (2.19) is a facet, we first show that $DP(0, d)$ has dimension d. It cannot have dimension $d + 1$ since the first component of every feasible point has value 1. Now, select any subset of K having cardinality $d + 1$, say the first $d + 1$ elements. By construction of $DP(0, d)$, the $d + 1$ points $\{\boldsymbol{x}_d\}_L = \boldsymbol{x}_d$ evaluated at $x = \theta_i$ for $i = 1, \ldots, d + 1$, are feasible to $DP(0, d)$. Form the $(d+1) \times (d+1)$ Vandermonde matrix V_{d+1} whose $(i, q)^{th}$ entry is θ_i^{q-1} so that the i^{th} row is the transpose of $\boldsymbol{x}_d$ at $x = \theta_i$. The matrix V_{d+1} is invertible with determinant $\prod_{1\leq j<i}^{d+1}(\theta_i - \theta_j)$ (see page 29 of [14]), so the $d + 1$ points generating V_{d+1} are affinely independent, giving that $DP(0, d)$ has dimension d. Now consider any $e \in \{1, \ldots, E\}$ and the linear expression $\left\{\prod_{i\in R_e}(\theta_i - x)\right\}_L$ found in constraint e of (2.19). For $e \in \left\{1, \ldots, \tilde{E}\right\}$, this expression is positive at all $\{\boldsymbol{x}_d\}_L = \boldsymbol{x}_d$ evaluated at $x = \theta_i$ for the $k - d$ points having $i \notin R_e$. For $e \in \left\{\tilde{E} + 1, \ldots, E\right\}$, this expression is negative when evaluated at the same points. In either case, it equals 0 for the d points having $i \in R_e$. Since these latter d points are affinely independent, we have that the associated constraint in (2.19) is a facet of $DP(0, d)$. □

Three comments relative to the above theorem and proof are in order. First, the initial assertion of the theorem stipulates that the projection of the set $DP(0, k - 1)$ onto the space of the variables $\{\boldsymbol{x}_d\}_L$ is defined by the inequalities of (2.19), while the latter shows that every inequality of (2.19) is needed. Thus, no more concise representation exists. Second, the proof uses the fact that every extreme direction $\hat{\boldsymbol{\pi}}$ of the projection cone Π in (2.16) has at least d entries of value 0, and that each such entry corresponds to a root of $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$. It also uses the fact that $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ is of degree no greater than d. Consequently, each extreme direction must have exactly d entries of 0 and $\hat{\boldsymbol{\pi}}^T C\boldsymbol{x}$ must be of degree d since the number of real roots of a polynomial is bounded above by its degree. Lastly, it follows from the theorem that, given any $\{\tilde{\boldsymbol{x}}\}_L \in DP(0, k - 1)$ of (2.15), the truncated vector $\{\tilde{\boldsymbol{x}}_d\}_L$ is feasible to $DP(0,d)$ of (2.19). Since $DP(0,k-1)$ of (2.15) is bounded, it also follows that every extreme point $\{\tilde{\boldsymbol{x}}_d\}_L$ of $DP(0,d)$ can be obtained by projecting the extreme point $\{\tilde{\boldsymbol{x}}\}_L$ of $DP(0,k-1)$ onto the space of the variables $\boldsymbol{x}_d$, where $\{\tilde{\boldsymbol{x}}\}_L = \tilde{\boldsymbol{x}}$ evaluated at $\tilde{x}$. However, we have not shown that the projection of every extreme point $\{\boldsymbol{x}\}_L$ of $DP(0,k-1)$ yields an extreme point of $DP(0,d)$; that is, that $\boldsymbol{x}_d$ evaluated at $\tilde{x}$ is an extreme point of $DP(0,d)$ for all $\tilde{x} \in S$. Theorem 2.3 proves this result for $d \in \{2,\ldots,k-2\}$. We preface this theorem with Lemma 2.3 which gives a further characterization of the set $DP(0,2)$ and is used in the proof of Theorem 2.3.

Lemma 2.3: *Given any* $(1,\tilde{x},\tilde{w}_2)^T \in DP(0,2)$*, it follows that* $\tilde{w}_2 \geq \tilde{x}^2$*, where* $\tilde{w}_2 = \{\tilde{x}^2\}_L$*.*

Proof. Given any $(1,\tilde{x},\tilde{w}_2)^T \in DP(0,2)$, select $j \in K$ so that $\tilde{x} \in [\theta_j,\theta_{j+1}]$. Then

$$\tilde{w}_2 \geq -\{(\theta_j-\tilde{x})(\theta_{j+1}-\tilde{x})\}_L + \tilde{w}_2 = -(\theta_j-\tilde{x})(\theta_{j+1}-\tilde{x}) + \tilde{x}^2 \geq \tilde{x}^2,$$

where $\{(\theta_j-\tilde{x})(\theta_{j+1}-\tilde{x})\}_L$ is defined to be $\{(\theta_j-x)(\theta_{j+1}-x)\}_L$ evaluated at $(1,\tilde{x},\tilde{w}_2)^T$. The first inequality follows from $\{(\theta_j-x)(\theta_{j+1}-x)\}_L \geq 0$ being a restriction of $DP(0,2)$. The equality is due to $\tilde{w}_2 = \{\tilde{x}^2\}_L$. The second inequality holds since $\tilde{x} \in [\theta_j,\theta_{j+1}]$. □

Theorem 2.3: *Given any* $d \in \{2,\ldots,k-2\}$*, the point* $\tilde{\boldsymbol{x}}_d$ *evaluated at* $\tilde{x}$ *for each* $\tilde{x} \in S$ *is an extreme point of* $DP(0,d)$*.*

Proof. It is sufficient to show the result for $d=2$ since, for each $d \in \{3,\ldots,k-2\}$, the set $DP(0,2)$ is the projection of the set $DP(0,d)$ onto the space of the variables $\{\boldsymbol{x}_2\}_L$. Thus, let $d=2$ and consider $\tilde{\boldsymbol{x}}_2 = (1,\tilde{x},\tilde{w}_2)^T = \left(1,\tilde{x},\tilde{x}^2\right)^T$ for any $\tilde{x} \in S$. The linear function $w_2 - 2\tilde{x}x$ has $(1,x,w_2)^T = \left(1,\tilde{x},\tilde{x}^2\right)^T$ as the unique minimum over $\{\tilde{\boldsymbol{x}}_2\}_L \in DP(0,2)$. This follows since, for $(1,x,w_2)^T \in DP(0,2)$, the inequality $w_2 \geq x^2$ from Lemma 2.2 gives $w_2-2\tilde{x}x+\tilde{x}^2 \geq x^2-2\tilde{x}x+\tilde{x}^2 = (x-\tilde{x})^2 \geq 0$. Thus, $w_2-2\tilde{x}x$ reaches its minimum of $-\tilde{x}^2$ at the unique point $(x,w_2)^T = \left(\tilde{x},\tilde{x}^2\right)^T$. □

Theorems 2.2 and 2.3 combine to show that the sets $DP(0,d)$ of (2.19), for $d \in \{2,\ldots,k-2\}$, are the desired projections of the set $DP(0,k-1)$ defined in (2.15). Recall from our earlier discussion that the paper [3] establishes a one-to-one correspondence between the extreme points of the set $DP(0,k-1)$ and the vectors $\boldsymbol{x}$ evaluated at θ_j for $j \in K$. We, on the other hand, show that for each $d \in \{2,\ldots,k-2\}$, the set $DP(0,d)$ is a projection of $DP(0,k-1)$ that preserves the same correspondence, while requiring fewer variables.

Observe that Theorem 2.3 does not consider $d \in \{0, 1, k-1\}$. For $d = k-1$, no projection emerges. For $d = 0$, the projection is trivial since the first entry of each $\{\boldsymbol{x}\}_L$ is 1. The case for $d = 1$ is more interesting, as Theorem 2.2 holds but Theorem 2.3 does not. Examples 2.4 and 2.5 below illustrate Theorems 2.2 and 2.3 by projecting a discrete variable allowed to realize $k = 5$ values onto the set $DP(0,3)$ and $DP(0,2)$, respectively. Following this, Example 2.6 shows that Theorem 2.3 does not hold for $DP(0,1)$.

Example 2.4

Consider a variable x which takes on values in $S = \{-2, -1, 0, 1, 2\}$. Here, $k = 5$ so the inequalities defining $DP(0, k-1) = DP(0,4)$ of (2.15) are given by:

$$\begin{bmatrix} 0 & \frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} & \frac{1}{24} \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ 1 & 0 & -\frac{5}{4} & 0 & \frac{1}{4} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{12} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{24} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{Bmatrix}_L \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let $d = 3$ in order to project this polytope onto the space $\left\{1, x, x^2, x^3\right\}_L^T$. The associated projection cone of (2.16) is:

$$\Pi = \left\{ \boldsymbol{\pi} \in \mathbb{R}^5 : [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] \begin{bmatrix} \frac{1}{24} \\ -\frac{1}{6} \\ \frac{1}{4} \\ -\frac{1}{6} \\ \frac{1}{24} \end{bmatrix} = 0, \boldsymbol{\pi} \geq \mathbf{0} \right\}.$$

This cone has six extreme points given by $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)^T \in \{(0,0,4,6,0), (24,0,0,6,0),$ $(24,6,0,0,0), (0,0,0,6,24), (0,6,0,0,24), (0,6,4,0,0)\}$. Surrogating the inequalities of $DP(0,4)$ with these directions yields:

$$\begin{aligned}
\{(-2-x)(-1-x)(2-x)\}_L &\geq 0, \\
\{(-1-x)(0-x)(2-x)\}_L &\geq 0, \\
\{(0-x)(1-x)(2-x)\}_L &\geq 0, \\
\{-(-2-x)(-1-x)(0-x)\}_L &\geq 0, \\
\{-(-2-x)(0-x)(1-x)\}_L &\geq 0, \text{ and} \\
\{-(-2-x)(1-x)(2-x)\}_L &\geq 0.
\end{aligned}$$

These inequalities are exactly the facets which define the polytope $DP(0,3)$ of (2.19) where the first three inequalities are of type-0 and the last three are of type-1. Observe that we can avoid characterizing the extreme directions of the projection cone Π and instead directly acquire these facets by forming all the type-0 and type-1 rows of the matrix A_d for $d = 3$ given by

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Using the procedure of Lemma 2.2, this matrix gives rise to the same six inequalities defining the facets of $DP(0,3)$. For example, the first row of A_3 is of type-0, since every 0 entry has an even number of 1's to the left. The index set of the elements of this row equaling 1 is given by $R_1 = \{1,2,5\}$. Hence, this row of A_3 gives rise to the constraint $\{(\theta_1 - x)(\theta_2 - x)(\theta_5 - x)\}_L = \{(-2-x)(-1-x)(2-x)\}_L \geq 0$ which is exactly the facet defined by projecting along the extreme direction $(0,0,4,6,0)$ of Π. The other five facets are formed in an analogous manner.

We plot the polytope $DP(0,3)$ in Figure 2.1 where the continuous variables w_2 and w_3 represent $\{x^2\}_L$ and $\{x^3\}_L$, respectively. Note that the five extreme points of $DP(0,3)$ each have $(x, w_2, w_3) = (x, x^2, x^3)$ evaluated at $x \in \{-2,-1,0,1,2\}$.

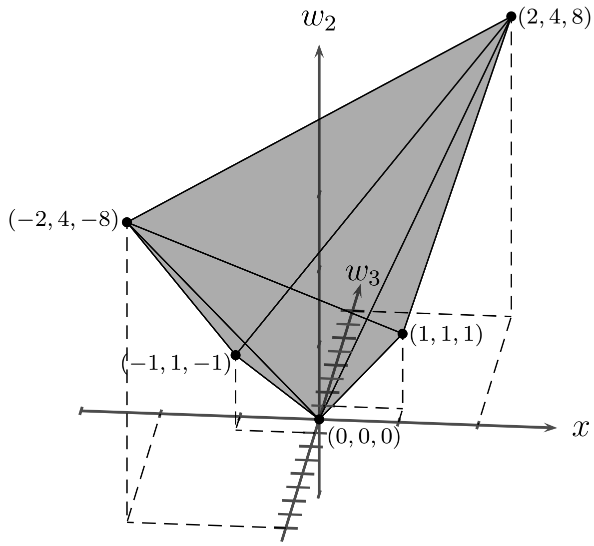

Figure 2.1: Polytope $DP(0,3)$ in (x, w_2, w_3) space for $x \in S = \{-2, -1, 0, 1, 2\}$ with $w_2 = \left\{x^2\right\}_L$ and $w_3 = \left\{x^3\right\}_L$.

Example 2.5

Again, let x realize values in $S = \{-2, -1, 0, 1, 2\}$. Let $d = 2$ so as to project the polytope $DP(0,4)$ onto the space $\left\{1, x, x^2\right\}_L^T$. Theorem 2.2 ensures that the facets of the desired projection, given by the set $DP(0,2)$, arise from the type-0 and type-1 rows of A_2 given by:

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the first four rows are type-0 and the fifth row is type-1. This matrix gives rise to the facets of $DP(0,2)$ given by:

$$\begin{aligned} \{(-2-x)(-1-x)\}_L &\geq 0, \\ \{(-1-x)(0-x)\}_L &\geq 0, \\ \{(0-x)(1-x)\}_L &\geq 0, \\ \{(1-x)(2-x)\}_L &\geq 0, \quad \text{and} \\ \{-(-2-x)(2-x)\}_L &\geq 0. \end{aligned}$$

Figure 2.2 plots $DP(0,2)$ where the continuous variables w_2 represents $\{x^2\}_L$. Note that, as proved by Theorem 2.3, $DP(0,2)$ has five extreme points each satisfying $(x, w_2) = (x, x^2)$ evaluated for some $x \in \{-2,-1,0,1,2\}$. Also observe that all $(x, w_2) \in DP(0,2)$ have $w_2 \geq x^2$ as proved by Lemma 2.3.

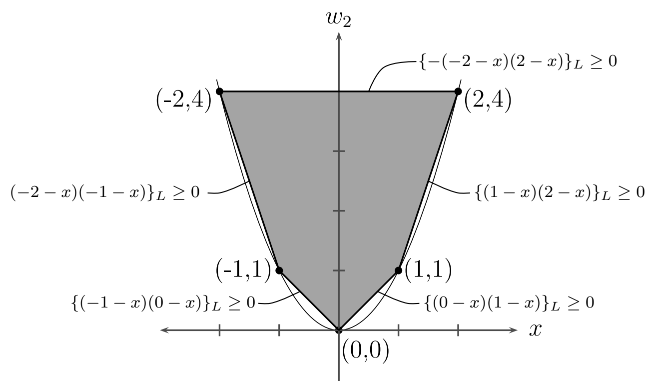

Figure 2.2: Polytope $DP(0,2)$ in (x, w_2) space for $x \in S = \{-2,-1,0,1,2\}$ with $w_2 = \{x^2\}_L$.

Example 2.6

Once again, let x realize values in $S = \{-2,-1,0,1,2\}$. Let $d = 1$ so that we project the polytope $DP(0,4)$ onto the $\{1, x\}_L^T$ space. Theorem 2.2 gives that the facets of $DP(0,1)$ arise from

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and are given by:

$$\{(2-x)\}_L \geq 0, \quad \text{and} \quad \{-(-2-x)\}_L \geq 0.$$

These inequalities, simply stated as $-2 \leq x \leq 2$, demonstrate that the polytope $DP(0,1)$ has the two extreme points corresponding to $\{x\}_L = x$ evaluated at $x \in \{-2,2\}$. Thus, the one-to-one correspondence between the extreme points of the projected polytope and the permissible values in S has been lost, exhibiting that Theorem 2.3 does not hold for $d = 1$.

As a final observation for this section, we provide a count on the number of facets defining $DP(0,d)$ of (2.19) for any $d \in \{2,\ldots,k-2\}$. Due to the definition of the type-0 and type-1 constraints, these counts are dependent on the parity of d as shown in the Table 2.1.

Table 2.1: Counts for the number of type-0 and type-1 facets of $DP(0,d)$.

	d even	d odd
# type-0 facets	$\binom{k-\frac{d}{2}}{\frac{d}{2}}$	$\binom{k-1-\frac{d-1}{2}}{\frac{d-1}{2}}$
# type-1 facets	$\binom{k-2-\frac{d-2}{2}}{\frac{d-2}{2}}$	$\binom{k-1-\frac{d-1}{2}}{\frac{d-1}{2}}$

To lend intuition to these counts, recall that Example 2.4 has $d=3$ and $x \in \{-2,-1,0,1,2\}$ so that $k=5$. Since d is odd, the number of type-0 or type-1 facets is $\binom{5-1-\frac{3-1}{2}}{\frac{3-2}{2}} = \binom{3}{1} = 3$ so that the total count is 6. Similarly, Example 2.5 has $d=2$ and $k=5$ so that the number of type-0 facets is $\binom{5-\frac{2}{2}}{\frac{2}{2}} = \binom{4}{1} = 4$ and the the number of type-1 facets is given by $\binom{5-2-\frac{2-2}{2}}{\frac{2-2}{2}}\binom{3}{0} = 1$ so that the total count is 5.

Interestingly, the combinatorial nature of these facet counts implies that if *only a few* or *almost all* higher-dimensional product variables are projected out, then the number of facets defining $DP(0,d)$ is relatively small. The greatest number of facets are present when projecting to levels roughly halfway between the quadratic level ($d=2$) and the full level ($d=k-1$). Figure 2.3 plots the number of facets defining each $DP(0,d)$ for the case with a discrete variable taking on 21 realizations ($k=21$) and $d \in \{2,\ldots,19\}$. Notice that $DP(0,12)$ is characterized by 7007 facets while $DP(0,2)$ has only 21 facets.

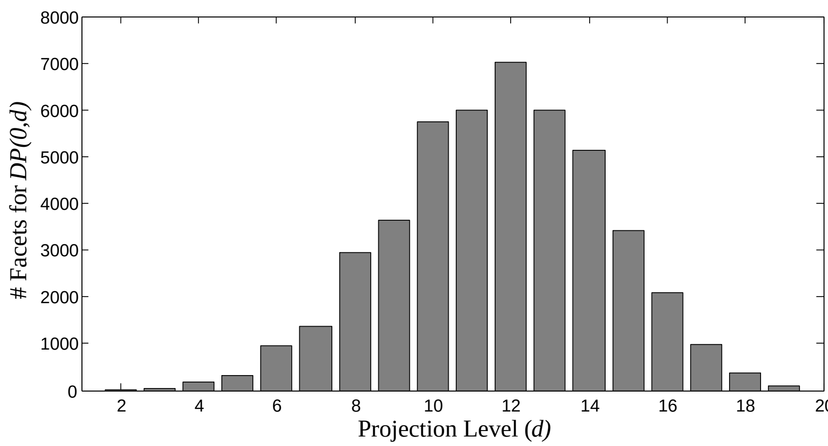

Figure 2.3: Number of facets of $DP(0,d)$ for $k=21$ and $d \in \{2,\ldots,19\}$.

2.4 Projected Convex Hull Forms for Two Variables

As in Section 2.2, let us now consider two discrete variables x_1 and x_2 that realize values in the sets $S_1 = \{\theta_{11}, \theta_{12}, \ldots, \theta_{1k_1}\}$ and $S_2 = \{\theta_{21}, \theta_{22}, \ldots, \theta_{2k_2}\}$, respectively, and ordered so that $\theta_{11} < \theta_{12} < \cdots < \theta_{1k_1}$ and $\theta_{21} < \theta_{22} < \cdots < \theta_{2k_2}$. Recall that Theorem 2.1 characterizes the set P of (2.13) as having k_1k_2 extreme points, with each extreme point defined by a column of the matrix $V_1^T V_2^T$. Hence, P is equivalent to $DP(k_1-1, k_2-1)$ and has the two properties that at each extreme point the linearized vector $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2$ with $x_1 \in S_1$ and $x_2 \in S_2$ and that there exists a one-to-one correspondence between the extreme points of P and the possible realizations of x_1 and x_2. In particular, the first property gives us that element $k_2 + 2$ of $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2\right\}_L$, say w_{11}, has $w_{11} = x_1x_2$ at all k_1k_2 realizations of x_1 and x_2. The challenge is to project P onto lower-dimensional subspaces without losing these properties. In the previous section, we successfully defined a similar projection for the linearized LIPs of a single variable via the polytope $DP(0, d)$ in (2.19). This result can be combined in a novel manner with results from [3] to obtain a desired projection of P.

With this in mind, rewrite $DP(0, d)$ in matrix form by defining a matrix D_d so that

$$DP(0,d) = \left\{\{\boldsymbol{x}_d\}_L : D_d \{\boldsymbol{x}_d\}_L \geq \boldsymbol{0}\right\}.$$

As in Section 2.3, let $C_p \{\boldsymbol{x}^p\}_L \geq \boldsymbol{0}$ represent the linearized LIPs associated with x_p for $p \in \{1, 2\}$. Then $D^p_{d_p} \left\{\boldsymbol{x}^p_{d_p}\right\}_L \geq \boldsymbol{0}$ represents our projection of $C_p \{\boldsymbol{x}^p\}_L \geq \boldsymbol{0}$ onto the first $d_p + 1$ entries of $\{\boldsymbol{x}^p\}_L$ for $p \in \{1, 2\}$. We can now define a polytope $DP(k_1 - 1, d_2)$ as:

$$DP(k_1-1, d_2) = \left\{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L : \left(C_1 \otimes D^2_{d_2}\right)\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L \geq \boldsymbol{0} \otimes \boldsymbol{0}\right\}. \tag{2.20}$$

Observe that by symmetry the study of $DP(k_1-1, d_2)$ and $DP(d_1, k_2-1)$ are equivalent. Theorem 2.4 demonstrates that this new polyhedral set preserves the two desired extreme point properties and that $DP(k_1 - 1, d_2)$ is a projection of P onto a lower-dimensional space.

Theorem 2.4: *$DP(k_1 - 1, d_2) = proj_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L} \{P\}$ for $d_2 \in \{2, \ldots, k_2 - 2\}$ and has k_1k_2 extreme points given by $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}$ evaluated at $x_1 \in S_1$ and $x_2 \in S_2$.*

Proof. That $DP(k_1 - 1, d_2)$ has k_1k_2 extreme points given by $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}$ evaluated at $x_1 \in S_1$ and $x_2 \in S_2$ follows directly from Theorems 2.2 and 2.3 and Theorem 3 of [3], since this set is formed as the Kronecker product of the full linearized LIPs for x_1 with a polytope that preserves the desired extreme point properties for x_2. To show equivalence between the two sets, we demonstrate that each is a subset of the other. Begin by choosing any extreme point $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L$ of $DP(k_1-1, d_2)$. Then the completion of this point, given by $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2$ evaluated for some $x_1 \in S_1$ and $x_2 \in S_2$ is certainly in P. Thus, $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L \in \text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\}$. Since all extreme points of $DP(k_1-1, d_2)$ are in $\text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\}$, then $DP(k_1-1, d_2) \subseteq \text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\}$. Next, choose any extreme point $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L$ of $\text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\}$. Given the structure of the extreme points of P, the point $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L$ must satisfy $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}$ for some $x_1 \in S_1$ and $x_2 \in S_2$. Thus $\left\{\tilde{\boldsymbol{x}}^1 \otimes \tilde{\boldsymbol{x}}^2_{d_2}\right\}_L$ is also an extreme point of $DP(k_1-1, d_2)$. Since all extreme points of $\text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\}$ are in $DP(k_1-1, d_2)$, then $\text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L}\{P\} \subseteq DP(k_1-1, d_2)$. □

Example 2.7

Let x_1 and x_2 take on values in $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2, 3\}$ respectively. Then $k_1 = 2$ and $k_2 = 4$ with $C_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 1 & -\frac{11}{6} & 1 & -\frac{1}{6} \\ 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$. To form the polytope $DP(1, 2)$, project $C_2\left\{\boldsymbol{x}^2\right\}_L$ onto the $\left\{1, x, x^2\right\}^T_L$ space so that $D^2_2 = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 6 & -5 & 1 \\ 0 & 3 & -1 \end{bmatrix}$. Then, $DP(1, 2)$ is given by:

$$DP(1,2) = \left\{ \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \end{array} \right\}_L : \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 0 & 1 & -1 \\ 2 & -3 & 1 & -2 & 3 & -1 \\ 6 & -5 & 1 & -6 & 5 & -1 \\ 0 & 3 & -1 & 0 & -3 & 1 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 6 & -5 & 1 \\ 0 & 0 & 0 & 0 & 3 & -1 \end{array} \right] \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \end{array} \right\}_L \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This polytope is bounded and has 8 extreme points given by $\left\{1, x_2, x_2^2, x_1, x_1x_2, x_1x_2^2\right\}^T_L \in \{(1,0,0,0,0,0), (1,1,1,0,0,0), (1,2,4,0,0,0), (1,3,9,0,0,0), (1,0,0,1,0,0), (1,1,1,1,1,1), (1,2,4,1,2,4), (1,3,9,1,3,9)\}$. Thus, all extreme points correspond to $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_2\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2_2$ evaluated at all $x_1 \in S_1$ and $x_2 \in S_2$.

A note regarding Theorem 2.4 is in order. P is a $(k_1k_2 - 1)$-dimensional polytope, but

$DP(k_1-1,d_2)$ preserves the characteristics of P in only $(k_1d_2+k_1-1)$ dimensions. It is prudent to define x_2 as the discrete variable which realizes the most values and then choose $d_2 \in \{2,\ldots,k_2-2\}$ as small as possible. For example, if $k_1=10$ and $k_2=100$, then $P=DP(9,99)$ has 999 dimensions. Yet if we let $d_2=2$, then $DP(9,2)$ has only 29 dimensions. In this manner, the polytope $DP(k_1-1,d_2)$ can facilitate a substantial reduction in the number of requisite auxiliary variables.

As a further extension in the spirit of [3], it seems logical to individually project the LIPs for *both* variables and then take the Kronecker product of these projections. This can be accomplished via a new polyhedral set P_{d_1,d_2} defined as:

$$P_{d_1,d_2}=\left\{\left\{\boldsymbol{x}_{d_1}^1\otimes\boldsymbol{x}_{d_2}^2\right\}_L:\left(D_{d_1}^1\otimes D_{d_2}^2\right)\left\{\boldsymbol{x}_{d_1}^1\otimes\boldsymbol{x}_{d_2}^2\right\}_L\geq\mathbf{0}\otimes\mathbf{0}\right\}.$$

Whereas Theorem 2.2 characterized new polytopes $DP(0,d)=\text{proj}_{\{\boldsymbol{x}_d\}_L}\{DP(0,k-1)\}$ for a single variable with $d\in\{2,\ldots,k-2\}$, we similarly desire that $P_{d_1,d_2}=DP(d_1,d_2)=\text{proj}_{\left\{\boldsymbol{x}_{d_1}^1\otimes\boldsymbol{x}_{d_2}^2\right\}_L}\{P\}$ for two variables with $d_1\in\{2,\ldots,k_1-2\}$ and $d_2\in\{2,\ldots,k_2-2\}$. However, this is not the case, and $P_{d_1,d_2}\neq DP(d_1,d_2)$ as Example 2.8 demonstrates.

Example 2.8

Let the discrete variables x_1 and x_2 both take on values in $S_1=S_2=\{0,1,2,3\}$ and form the polytope $P_{2,2}$. To accomplish this, we project the linearized LIPs for each individual variable onto the $\left\{1,x_j,x_j^2\right\}_L^T$ space so that $d_1=d_2=2$ and $D_2^1=D_2^2=\begin{bmatrix}0&-1&1\\2&-3&1\\6&-5&1\\0&3&-1\end{bmatrix}$. Then, the polytope

$P_{2,2}$ is given by:

$$P_{2,2} = \left\{ \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2 \\ x_1^2x_2 \\ x_1^2x_2^2 \end{array} \right\}_L : \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & 3 & -1 & 2 & -3 & 1 \\ 0 & 0 & 0 & -6 & 5 & -1 & 6 & -5 & 1 \\ 0 & 0 & 0 & 0 & -3 & 1 & 0 & 3 & -1 \\ \hline 0 & -2 & 2 & 0 & 3 & -3 & 0 & -1 & 1 \\ 4 & -6 & 2 & -6 & 9 & -3 & 2 & -3 & 1 \\ 12 & -10 & 2 & -18 & 15 & -3 & 6 & -5 & 1 \\ 0 & 6 & -2 & 0 & -9 & 3 & 0 & 3 & -1 \\ \hline 0 & -6 & 6 & 0 & 5 & -5 & 0 & -1 & 1 \\ 12 & -18 & 6 & -10 & 15 & -5 & 2 & -3 & 1 \\ 36 & -30 & 6 & -30 & 25 & -5 & 6 & -5 & 1 \\ 0 & 18 & -6 & 0 & -15 & 5 & 0 & 3 & -1 \\ \hline 0 & 0 & 0 & 0 & -3 & 3 & 0 & 1 & -1 \\ 0 & 0 & 0 & 6 & -9 & 3 & -2 & 3 & -1 \\ 0 & 0 & 0 & 18 & -15 & 3 & -6 & 5 & -1 \\ 0 & 0 & 0 & 0 & 9 & -3 & 0 & -3 & 1 \end{array} \right] \left\{ \begin{array}{c} 1 \\ x_2 \\ x_2^2 \\ x_1 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2 \\ x_1^2x_2 \\ x_1^2x_2^2 \end{array} \right\}_L \geq \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \right\}.$$

This polytope is bounded and has 24 extreme points. The first 16 extreme points correspond to $\left\{\boldsymbol{x}_2^1 \otimes \boldsymbol{x}_2^2\right\}_L = \boldsymbol{x}_2^1 \otimes \boldsymbol{x}_2^2$ evaluated at all $x_1 \in S_1$ and $x_2 \in S_2$. Unfortunately, the remaining 8 extreme points are fractional, indicating that $P_{2,2} \neq DP(2,2) = \text{proj}_{\left\{\boldsymbol{x}_2^1 \otimes \boldsymbol{x}_2^2\right\}_L} \{P\}$.

The previous example illustrates that while all inequalities of P_{d_1,d_2} are valid, they do not completely characterize $DP(d_1,d_2) = \text{proj}_{\left\{\boldsymbol{x}_{d_1}^1 \otimes \boldsymbol{x}_{d_2}^2\right\}_L} \{P\}$. An ongoing challenge is to find a closed form for this characterization.

Interestingly, the above arguments can be generalized to accommodate polytopes which linearize the product of a continuous variable x_1 with a discrete variable x_2. To see this, consider the case where $x_1 \in \{l, u\}$ within P of (2.13) so that $k_1 = 2$, $\theta_{11} = l$, and $\theta_{12} = u$. Then for any $d_2 \in \{2, \ldots, k_2 - 2\}$, Theorem 2.4 shows that the polytope $DP(1, d_2)$ of (2.20) characterizes $\text{proj}_{\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}_{d_2}^2\right\}_L} \{P\}$ and that $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}_{d_2}^2\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}_{d_2}^2$ at each extreme point of $DP(1, d_2)$. Notably for this case, these polytopes $DP(1, d_2)$ possess an even stronger result; provided *any* $x_1 \in [l, u]$ and that $\left\{\boldsymbol{x}_{d_2}^2\right\}_L = \boldsymbol{x}_{d_2}^2$ evaluated at $x_2 \in S_2$, it follows that $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}_{d_2}^2\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}_{d_2}^2$. This allows for the variable x_1 to realize any value in the interval $[l, u]$ and yet still have a valid linearization of polynomial terms amongst the two variables. To prove this generalization, we present the lemma and theorem below. In keeping with previous notation, the the vector $\left\{\boldsymbol{x}_{n-1}^2\right\}_L = \left\{1, x_2, x_2^2, \ldots, x_2^{n-1}\right\}_L^T$.

Lemma 2.4: *Consider a linear expression of the form* $\left\{x_1 \prod_{i\in N}(\theta_i - x_2)\right\}_L$ *having distinct realizations* θ_i *for* $i \in \{1,\ldots,n\} \equiv N$. *If* $\left\{x_1 \otimes \boldsymbol{x}^2_{n-1}\right\}_L = x_1 \otimes \boldsymbol{x}^2_{n-1}$ *evaluated at* $x_2 = \theta_j$ *for some* $j \in N$, *then it follows that* $\left\{x_1 \prod_{i\in N}(\theta_i - x_2)\right\}_L = x_1\theta_j^n - \{x_1x_2^n\}_L$.

Proof. Begin by re-expressing $\left\{x_1 \prod_{i\in N}(\theta_i - x_2)\right\}_L$ in vector form as $\boldsymbol{f}\left\{x_1 \otimes \boldsymbol{x}^2\right\}_L$ where $\boldsymbol{f} = [f_0, f_1, \ldots, f_n]$ and $\left\{x_1 \otimes \boldsymbol{x}^2\right\}_L = \left\{x_1, x_1x_2, x_1x_2^2, \ldots, x_1x_2^n\right\}_L^T$ and note that the inherent structure of $\left\{x_1 \prod_{i\in N}(\theta_i - x_2)\right\}_L$ gives that $f_n = (-1)^n$. Observe that the *nonlinear* expression $x_1 \prod_{i\in N}(\theta_i - x_2)$ equals zero if $x_2 = \theta_j$ for some $j \in N$. This gives that $f_0x_1 + \sum_{i=1}^{n-1} f_ix_1\theta_j^i = -f_nx_1\theta_j^n = (-1)^{n+1}x_1\theta_j^n$. Thus, if $\left\{x_1 \otimes \boldsymbol{x}^2_{n-1}\right\}_L = x_1 \otimes \boldsymbol{x}^2_{n-1}$ evaluated at $x_2 = \theta_j$ for some $j \in N$, then $\boldsymbol{f}\left\{x_1 \otimes \boldsymbol{x}^2\right\}_L = \left\{f_0x_1 + \sum_{i=1}^{n-1} f_ix_1x_2^i + f_nx_1x_2^n\right\}_L = \left\{(-1)^{n+1}x_1\theta_j^n + (-1)^n x_1x_2^n\right\}_L = x_1\theta_j^n - \{x_1x_2^n\}_L$. □

Theorem 2.5: *For* $S_1 = \{l, u\}$, $S_2 = \{\theta_{21}, \ldots, \theta_{2k_2}\}$, *and* $d_2 \in \{2, \ldots, k_2 - 2\}$, *given any feasible solution to* $DP(1, d_2)$ *having* $x_1 \in [l, u]$ *and* $\left\{\boldsymbol{x}^2_{d_2}\right\}_L = \boldsymbol{x}^2_{d_2}$ *evaluated at some* $x_2 \in S_2$, *then it follows that* $\left\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}\right\}_L = \boldsymbol{x}^1 \otimes \boldsymbol{x}^2_{d_2}$.

Proof. Assume $d_2 \in \{2, \ldots, k_2 - 2\}$ and $\left\{\boldsymbol{x}^2_{d_2}\right\}_L = \boldsymbol{x}^2_{d_2}$ evaluated at some $x_2 = \theta_{2j} \in S_2$. Observe that Theorem 2.4 implies that $DP(1, d_2)$ is a projection of the set $DP(1, d_2 + 1)$, since both of these sets are projections of the higher-dimensional set P of (2.13). This guarantees that any valid implications made by the set $DP(1, d_2)$ automatically hold in the higher-dimensional space $DP(1, d_2+1)$. Hence, this proof proceeds by induction on the value d_2, beginning with $d_2 = 2$. Then, in higher dimensions with $d_2 \in \{3, \ldots, k_2 - 2\}$, it is assumed that the lower-dimensional product terms $\{x_1x_2\}_L$ through $\left\{x_1x_2^{d_2-1}\right\}_L$ equal their intended values and it is shown that $\left\{x_1x_2^{d_2}\right\}_L$ equals its intended value.

Base Case: Let $d_2 = 2$. Since this case utilizes the projection of the LIPs for x_2 onto the quadratic space $\left\{\boldsymbol{x}^2_2\right\}_L = \left\{1, x_2, x_2^2\right\}_L$, any $\theta_{2j} \in S_2$ satisfies exactly one of two conditions regarding the set $DP(0, 2)$ of (2.19). The first condition is that the point $\left\{\boldsymbol{x}^2_2\right\}_L = \left(1, \theta_{2j}, \theta_{2j}^2\right)^T$ lies on two type-0 facets. The second condition is that $\left\{\boldsymbol{x}^2_2\right\}_L = \left(1, \theta_{2j}, \theta_{2j}^2\right)^T$ lies on one type-0 and one type-1 facet. Proven below is the result that $x_1\theta_{2j} = \{x_1x_2\}_L$ and $x_1\theta_{2j}^2 = \left\{x_1x_2^2\right\}_L$ under both condition 1 and condition 2.

Condition 1: Assume $\left\{\boldsymbol{x}_2^2\right\}_L = \left(1, \theta_{2j}, \theta_{2j}^2\right)^T$ lies on two type-0 facets. Judiciously choose four of the constraints of $DP(1,2)$ given by

$$\{(-l + x_1)(\theta_{2j-1} - x_2)(\theta_{2j} - x_2)\}_L \geq 0,$$
$$\{(u - x_1)(\theta_{2j-1} - x_2)(\theta_{2j} - x_2)\}_L \geq 0,$$
$$\{(-l + x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - x_2)\}_L \geq 0, \qquad \text{and}$$
$$\{(u - x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - x_2)\}_L \geq 0$$

which, since $\left\{\boldsymbol{x}_2^2\right\}_L = \boldsymbol{x}_2^2$ evaluated at $x_2 = \theta_{2j}$, reduce to

$$F_1 \equiv \{(x_1)(\theta_{2j-1} - x_2)(\theta_{2j} - x_2)\}_L \geq 0,$$
$$F_2 \equiv \{(-x_1)(\theta_{2j-1} - x_2)(\theta_{2j} - x_2)\}_L \geq 0,$$
$$F_3 \equiv \{(x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - x_2)\}_L \geq 0, \qquad \text{and}$$
$$F_4 \equiv \{(-x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - x_2)\}_L \geq 0.$$

Surrogating F_2 and F_3 with multiples of 1 results in the inequality

$$\{(x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - \theta_{2j-1})\}_L \geq 0. \tag{2.21}$$

Similarly, surrogating F_1 and F_4 with multiples of 1 results in the inequality

$$\{(x_1)(\theta_{2j} - x_2)(\theta_{2j+1} - \theta_{2j-1})\}_L \leq 0. \tag{2.22}$$

Thus (2.21) and (2.22) enforce that $0 \leq \{(x_1)(\theta_{2j} - x_2)\}_L \leq 0$ so that $x_1\theta_{2j} = \{x_1x_2\}_L$ as desired. Now, since $x_1\theta_{2j} = \{x_1x_2\}_L$, Lemma 2.4 gives that the constraint F_1 reduces to $x_1\theta_{2j}^2 - \left\{x_1x_2^2\right\}_L \geq 0$ and F_2 reduces to $x_1\theta_{2j}^2 - \left\{x_1x_2^2\right\}_L \leq 0$ so that $x_1\theta_{2j}^2 = \left\{x_1x_2^2\right\}_L$.

Condition 2: Assume $\left\{\boldsymbol{x}_2^2\right\}_L = \left(1, \theta_{2j}, \theta_{2j}^2\right)^T$ lies on one type-0 facet and one type-1 facet. This implies that θ_{2j} either equals θ_{21} or θ_{2k_2}. Without loss of generality, assume that $\theta_{2j} = \theta_{21}$

and judiciously choose four constraints of $DP(1,2)$ given by

$$\{(-l+x_1)(\theta_{21}-x_2)(\theta_{22}-x_2)\}_L \geq 0,$$
$$\{(u-x_1)(\theta_{21}-x_2)(\theta_{22}-x_2)\}_L \geq 0,$$
$$\{-(-l+x_1)(\theta_{21}-x_2)(\theta_{2k_2}-x_2)\}_L \geq 0, \qquad \text{and}$$
$$\{-(u-x_1)(\theta_{21}-x_2)(\theta_{2k_2}-x_2)\}_L \geq 0.$$

Since $\left\{\boldsymbol{x}_2^2\right\}_L = \boldsymbol{x}_2^2$ evaluated at $x_2 = \theta_{21}$, these constraints reduce to

$$F_1 \equiv \{(x_1)(\theta_{21}-x_2)(\theta_{22}-x_2)\}_L \geq 0,$$
$$F_2 \equiv \{(-x_1)(\theta_{21}-x_2)(\theta_{22}-x_2)\}_L \geq 0,$$
$$F_3 \equiv \{(-x_1)(\theta_{21}-x_2)(\theta_{2k_2}-x_2)\}_L \geq 0, \qquad \text{and}$$
$$F_4 \equiv \{(x_1)(\theta_{21}-x_2)(\theta_{2k_2}-x_2)\}_L \geq 0.$$

Surrogating F_2 and F_4 with multiples of 1 results in the inequality

$$\{(x_1)(\theta_{2j}-x_2)(\theta_{2k_2}-\theta_{22})\}_L \geq 0. \tag{2.23}$$

Similarly, surrogating F_1 and F_3 with multiples of 1 results in the inequality

$$\{(x_1)(\theta_{2j}-x_2)(\theta_{2k_2}-\theta_{22})\}_L \leq 0. \tag{2.24}$$

Thus (2.23) and (2.24) enforce that $0 \leq \{(x_1)(\theta_{21}-x_2)\}_L \leq 0$ so that $x_1\theta_{21} = \{x_1x_2\}_L$ as desired. Now, since $x_1\theta_{21} = \{x_1x_2\}_L$, Lemma 2.4 gives that the constraint F_1 reduces to $x_1\theta_{21}^2 - \left\{x_1x_2^2\right\}_L \geq 0$ and F_2 reduces to $x_1\theta_{21}^2 - \left\{x_1x_2^2\right\}_L \leq 0$ so that $x_1\theta_{21}^2 = \left\{x_1x_2^2\right\}_L$ as desired.

Induction Step: Assume $d_2 \in \{3,\ldots,k_2-2\}$ and that $\left\{x_1 \otimes \boldsymbol{x}_{d_2-1}^2\right\}_L = x_1 \otimes \boldsymbol{x}_{d_2-1}^2$ evaluated at $x_1 \in [l,u]$ and $x_2 = \theta_{2j} \in S_2$. Choose any two constraints of $DP(1,d_2)$ having the form

$$\{(-l+x_1)(\theta_{2j}-x_2)(\theta_{2j_1}-x_2)\cdots(\theta_{2j_n}-x_2)\}_L \geq 0, \qquad \text{and}$$
$$\{(u-x_1)(\theta_{2j}-x_2)(\theta_{2j_1}-x_2)\cdots(\theta_{2j_n}-x_2)\}_L \geq 0$$

where $n = d_2 - 1$ and the the elements $j_1, j_2, \ldots, j_n \in \{1, \ldots, k_2\}$ appropriately define the chosen constraints. Since $\left\{\boldsymbol{x}_{d_2}^2\right\}_L = \boldsymbol{x}_{d_2}^2$ for $x_2 = \theta_{2j} \in S_2$, these constraints reduce to

$$\{(x_1)(\theta_{2j} - x_2)(\theta_{2j_1} - x_2) \cdots (\theta_{2j_n} - x_2)\}_L \geq 0, \qquad \text{and}$$
$$\{(-x_1)(\theta_{2j} - x_2)(\theta_{2j_1} - x_2) \cdots (\theta_{2j_n} - x_2)\}_L \geq 0.$$

Finally, since we assume $\left\{x_1 \otimes \boldsymbol{x}_{d_2-1}^2\right\}_L = x_1 \otimes \boldsymbol{x}_{d_2-1}^2$, Lemma 2.4 gives that these two inequalities reduce to

$$x_1\theta_{2j}^{d_2} - \left\{x_1 x_2^{d_2}\right\}_L \geq 0, \qquad \text{and}$$
$$x_1\theta_{2j}^{d_2} - \left\{x_1 x_2^{d_2}\right\}_L \leq 0.$$

Thus, $x_1\theta_{2j}^{d_2} = \left\{x_1 x_2^{d_2}\right\}_L$ as desired. □

2.5 Conclusions

Given two discrete variables x_1 and x_2 that can realize k_1 and k_2 distinct values respectively, this chapter focused on the construction of convex hull representations that can be used to model polynomial functions of these variables. It was shown that, without loss of generality, such functions can be assumed to have maximum degrees of $k_1 - 1$ and $k_2 - 1$ on x_1 and x_2 respectively. In order to accurately model the polynomial expressions in such a function, the problem was recast in a higher-dimensional variable space by defining a new continuous variable for each distinct nonlinear term. Such polytopes naturally generalize known results relative to the product of two binary variables, the product of a binary and continuous variable, and outer-approximations for the product of two continuous variables.

The polytopes under study have two special properties. First, there is a one-to-one correspondence between the set of extreme points and the k_1k_2 possible pairwise-realizations of x_1 and x_2. Second, at every such extreme point, each of x_1 and x_2 realizes one of its permissible values, and every auxiliary variable equals to its intended product. These properties are desirable since explicit descriptions of such polytopes allow the motivating polynomial functions, through similar variable substitutions, to be optimized as a linear program.

The theoretical foundation of our constructions is a special family of functions, known as Lagrange interpolating polynomials, that have previously been used to derive convex hull forms within a reformulation-linearization technique (RLT). The novelty of this chapter is the projection of these higher-dimensional RLT spaces onto lower-dimensional counterparts. We completely characterized such projections for the case of a single variable, and in turn used this characterization to motivate special projections for the two-variable case. Included here is the complete and explicit description of all lower-dimensional spaces for the product of a discrete and binary variable (general k_1 and $k_2 = 2$). The resulting polyhedral structure also allows for the representation of the product of a discrete and bounded-continuous variable. Interestingly, for both of these latter cases, the number of facets is only $2(k_1 + 1)$ when the projected space has the variable x_1 of degree at most 2. For the general case in which $k_1 \geq 3$ and $k_2 \geq 3$, facets are provided but the convex hull representation is not known.

Future research includes both an extension of the theory and the computer implementation within solution algorithms. Relative to theory, the key challenge is the acquisition of the convex hull representation for the general cases having $k_1 \geq 3$ and $k_2 \geq 3$. For these cases, the projection operation is not fully defined. The insights gained from the special instances may shed light in this regard. From a computational point of view, the known convex hull representations should give rise to more efficient solution strategies. Such forms have been extensively used for the case of binary x_1 and x_2, and it is expected that similar successes may be realized for the discrete case. Of particular interest here is the product of a discrete x_1 and binary x_2 where x_1 has degree at most 2 since, as mentioned above, the convex hull forms are available and concise in size.

Bibliography

[1] Adams, W.P. and Hadavas, P.T., "A Network Approach for Specially-Structured Linear Programs Arising in 0-1 Quadratic Optimization," *Discrete Applied Mathematics*, Vol. 156, Issue 11, 2142-2165, 2008.

[2] Adams, W.P. and Henry, S.M., "Base-2 Expansions for Linearizing Products of Functions of Discrete Variables," *Operations Research*, submitted.

[3] Adams, W.P. and Sherali, H.D., "A Hierarchy of Relaxations Leading to the Convex Hull Representation for General Discrete Optimization Problems," *Annals of Operations Research*, Vol. 140, No. 1, 21-47, 2005.

[4] Al-Khayyal, F.A. and Falk, J.E., "Jointly Constrained Biconvex Programming," *Mathematics of Operations Research*, Vol. 8, No. 2, 273-286, 1983.

[5] Anstreicher, K.M. and Burer, S., "Computable Representations for Convex Hulls of Low-Dimensional Quadratic Forms," *Working Paper*, Department of Management Sciences, University of Iowa, 2007.

[6] Barahona, F., "The Max-Cut Problem on Graphs Not Contractible to K_5," *Operations Research Letters*, Vol. 2, Issue 3, 107-111, 1983.

[7] Burer, S. and Letchford, A.N., "On Non-Convex Quadratic Programming with Box Constraints," *SIAM Journal on Optimization*, Vol 20, Issue 2, 1073-1089, 2009.

[8] Coppersmith, D., Lee, J., and Leung, J., "A Polytope for a Product of Real Linear Functions in 0/1 Variables," *IBM Research Report RC21568*, 1999.

[9] DeSimone, C., "The Cut Polytope and the Boolean Quadric Polytope," *Discrete Mathematics*, Vol. 79, Issue 1, 71-75, 1990.

[10] DeSimone, C., "A Note on the Boolean Quadric Polytope," *Operations Research Letters*, Vol. 19, Issue 3, 115-116, 1996.

[11] Fortet, R. "L'algèbre de Boole et ses Applications en Recherche Opérationelle," *Cahiers du Centre d'Études de Recherche Opérationelle*, Vol. 1, No. 4, 5-36, 1959.

[12] Fortet, R., "Applications de l'algèbre de Boole en Recherche Opérationelle," *Revue Française d'Informatique et de Recherche Opérationelle*, Vol. 4, No. 14, 17-26, 1960.

[13] Glover, F., "Improved Linear Integer Programming Formulations of Nonlinear Integer Programs," *Management Science*, Vol. 22, No. 4, 455-460, 1975.

[14] Horn, R. A. and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, New York, NY 10011, 1985.

[15] Li, H.L. and Lu, H.C., "Global Optimization for Generalized Geometric Programs with Mixed Free-Sign Variables," *Operations Research*, Vol. 57, No. 3, 701-713, 2009.

[16] McCormick, G.P., "Computability of Global Solutions to Factorable Nonconvex Programs: Part I - Convex Underestimating Problems," *Mathematical Programming*, Vol. 10, 146-175, 1976.

[17] Padberg, M., "The Boolean Quadric Polytope: Some Characteristics, Facets, and Relatives," *Mathematical Programming*, Vol. 45, 139-172, 1989.

[18] Rhys, J.M.W., "A Selection Problem of Shared Fixed Costs and Network Flows," *Management Science*, Vol. 17, No. 3 200-207, 1970.

[19] Sherali, H.D. and Adams, W.P., "A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-One Programming Problems," *SIAM Journal on Discrete Mathematics*, Vol. 3, No. 3, 411-430, 1990.

[20] Sherali, H.D. and Adams, W.P., "A Hierarchy of Relaxations and Convex Hull Characterizations for Mixed-Integer Zero-One Programming Problems," *Discrete Applied Mathematics* Vol. 52, No. 1, 83-106, 1994.

[21] Sherali, H.D. and Adams, W.P., "A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems," Kluwer Academic Publishers, Norwell, MA, 1999.

[22] Sherali, H.D., Lee, Y., and Adams, W.P., "A Simultaneous Lifting Strategy for Identifying New Classes of Facets for the Boolean Quadric Polytope," *Operations Research Letters*, Vol. 17, Issue 1, 19-26, 1995.

[23] Sun, X., "Combinatorial Algorithms for Boolean and Pseudo-Boolean Functions," PhD Dissertation, Rutgers University, 1992.

Chapter 3

Exploiting Simplices in Computing Convex Hulls

A fundamental challenge in optimizing *mixed-integer programs* (MIPs) is the procurement of strong polyhedral outer-approximations to the convex hull of feasible solutions. These outer-approximations can provide strong bounds in general enumeration strategies and greatly improve computational performance. In this chapter, we focus on a method known as the *reformulation-linearization technique* (RLT) which operates by recasting original problem descriptions into new, higher-dimensional regions. In doing so, the RLT can partially or completely eliminate discretizations, nonlinearities, and non-convexities that complicate the original formulations.

Generally speaking, the RLT operates by performing the following two steps. First, the *reformulation step* multiplies the constraints of the original problem by special "functional factors" which vary depending on the nature of the discrete variables. Then, using the inherent structure of these functional factors, certain "simplifying identities" can be applied where, again, the nature of these simplifications depends on the nature of the functional factors. Second, the *linearization step* replaces all nonlinear terms with new continuous variables. Depending on the manner in which the constraints and functional factors are multiplied, the strength of the resulting higher-dimensional polyhedron can range anywhere between the continuous relaxation and the convex hull.

The purpose of this chapter is to introduce new functional factors and simplifying identities that subsume and generalize previous RLT results for mixed-binary and mixed-discrete programs and give rise to richer convex hull theory. These generalizations are made by employing judiciously scaled facets of a class of special polytopes known as simplices. The remainder of this chapter is arranged as follows: Section 3.1 reviews RLT results for mixed-binary and mixed-discrete sets, Section 3.2 introduces the simplicial structures that form the foundation for our new RLT results, Section 3.3 develops novel convex hull proofs that generalize previous results, Section 3.4 relates our new insights to classical RLT ideas, and Section 3.5 summarizes these results.

3.1 Reformulation-Linearization Technique Background

This section provides a brief survey of the RLT as it relates to mixed-binary and mixed-discrete problems and introduces the underlying theoretic machinery. The RLT for binary and mixed-binary programs was introduced over 20 years ago in [2, 3] and was only recently generalized to mixed-discrete sets in [1]. These papers demonstrate that the RLT methodology provides the ability to generate a *hierarchy* of successively tighter representations where the lowest level of this hierarchy yields a continuous relaxation for the original problem and the highest level gives an explicit convex hull representation of the feasible solutions. In this chapter, we are concerned with the development of new convex hull representations which generalize the convex hulls obtained by the RLT for mixed-binary and mixed-discrete sets. Hence, this section focuses on the *highest* level of the RLT hierarchy and provides an overview of how this representation is formed.

3.1.1 Kronecker Products

We begin by examining a matrix operation known as the *Kronecker product*. This product allows for an elegant description of the RLT machinery; its properties form the backbone of several proofs that appear later in the chapter. The use of Kronecker products was first introduced in [1] for generating RLT representations of mixed-discrete sets, but it can also be used to form the mixed-binary RLT representations of [2, 3]. This product, denoted by $\otimes$, is defined as follows. Consider an $m \times n$ matrix A and a second matrix B of any dimension. Then $A \otimes B \equiv \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$ where a_{ij} represents the $(i,j)^{th}$ element of A.

A useful interpretation of the Kronecker product as it relates to the RLT can be intuitively observed when both A and B are column vectors. In this case, the resulting product $A \otimes B$ is a column vector wherein every element of A is multiplied by every element of B. This interpretation has application to the reformulation step of the RLT where we desire to multiply every constraint of a fomulation by every functional product associated with the discrete variables.

Three important properties arise from the definition of the Kronecker product. Take a scalar k and matrices A, B, C, and D with appropriate dimensions so that the standard products AC and BD are defined and where A and B are invertible. Then we have:

Property 1: $(A \otimes B)(C \otimes D) = AC \otimes BD$,

Property 2: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,

Property 3: $k(A \otimes B) = kA \otimes B = A \otimes kB$.

Note that the invertibility of A and B plays no part in validity of Properties 1 and 3. Also, observe that Property 1 can be applied recursively if the matrices A, B, C, and D are themselves formed as the Kronecker product of matrices with appropriate sizes. The same is true for Property 2 if A and B are composed of Kronecker products of *invertible* matrices.

3.1.2 RLT for Mixed-Binary Programs

We now demonstrate how the RLT generates convex hull representations for mixed-binary programs. Begin by considering the set

$$X_B \equiv \{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, x \in \{0, 1\}\} \tag{3.1}$$

where only a single variable x is restricted to take on binary values and the remaining m variables of $\boldsymbol{y}$ are continuous between 0 and 1. Assuming $Ax+B\boldsymbol{y} \geq \boldsymbol{d}$ describes a system of p linear constraints, observe that $A \in \mathbb{R}^{p\times 1}$ since there is a single variable x but $B \in \mathbb{R}^{p\times m}$ since $\boldsymbol{y}$ contains m variables. We focus on this relatively simple case to clarify the RLT description and notation and to draw more explicit parallels between this binary case and the new, more general convex hulls results that appear in Section 3.3. In Section 3.4 we demonstrate how the RLT operates for mixed-binary problems having more than one binary variable and how the work of this chapter generalizes those results.

Since the variable x is restricted to be binary, we have that the inequalities $1 - x \geq 0$ and $x \geq 0$ are implicitly satisfied by any solution to X_B. In fact, the expressions $1 - x$ and x are exactly the *functional products* used by the RLT for mixed-binary problems. They can be expressed in vector form as the system

$$P\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

For notational convenience throughout this chapter, given any column vector $\boldsymbol{x}$ we let $\boldsymbol{x}'$ denote the related vector $\begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix}$ having a 1 appended to the beginning of the original vector. For the specific case where x is scalar, then the notation $\boldsymbol{x}'$, as in $P\boldsymbol{x}'$ above, denotes the 2-dimensional vector $\begin{pmatrix} 1 \\ x \end{pmatrix}$.

Now that we have defined the functional products of the binary variable x in (3.2), consider the associated simplifying identity which arises from the nature of x. Observe that since x is binary, it naturally exhibits the idempotent property

$$x^2 = x. \tag{3.3}$$

This equality is exactly the *simplifying identity* used by the RLT for mixed-binary programs.

Using the functional products of (3.2) and the simplifying identity of (3.3), we now apply the two steps of reformulation and linearization to generate a higher-dimensional polytope that produces the convex hull for the original set X_B. Begin by noting that since any solution to X_B implicitly satisfies $P\boldsymbol{x}' \geq \mathbf{0}$, we can equivalently rewrite X_B as

$$X_B \equiv \left\{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \mathbf{0} \leq \boldsymbol{y} \leq \mathbf{1}, P\boldsymbol{x}' \geq \mathbf{0}, x \in \{0,1\}\right\}. \tag{3.4}$$

Then, using (3.4), the two RLT steps operate as follows.

Step 1. Reformulation

Compute the Kronecker product of $P\boldsymbol{x}'$ with the constraints $Ax + B\boldsymbol{y} \geq \boldsymbol{d}$ and $\mathbf{0} \leq \boldsymbol{y} \leq \mathbf{1}$ of (3.4). Then, enforce the simplifying identity $x^2 = x$. Note that it is unnecessary to compute the Kronecker product of functional products with themselves since $P\boldsymbol{x}' \otimes P\boldsymbol{x}'$ reduces to $P\boldsymbol{x}'$ upon enforcement of $x^2 = x$. Hence, it is sufficient to enforce $P\boldsymbol{x}' \geq \mathbf{0}$.

Step 2. Linearization

For each of the m distinct product terms in the vector $\boldsymbol{x}' \otimes \boldsymbol{y}$ substitute a continuous variable and denote the resulting linearized vector as $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L$. Notice that the single product term in $\boldsymbol{x}' \otimes x$

can be substituted out of the problem via the identity $x^2 = x$. Thus, enforcing $\{\boldsymbol{x}' \otimes x\}_L = (x, x)^T$ removes the need to explicitly enforce $x^2 = x$. By denoting the new linearized variables in $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L$ by $\boldsymbol{w}$ and using Property 1 of Kronecker products, we have the resulting polyhedral set given by

$$\Omega_B = \left\{ \begin{array}{l} (x, \boldsymbol{y}, \boldsymbol{w}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m : \\ \qquad (P \otimes A)(x, x)^T + (P \otimes B)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L \geq (P \otimes \boldsymbol{d})\boldsymbol{x}', \\ \qquad \boldsymbol{0} \leq (P \otimes I_m)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L \leq (P \otimes \boldsymbol{1}_m)\boldsymbol{x}', \\ \qquad P\boldsymbol{x}' \geq \boldsymbol{0} \end{array} \right\}$$

where I_m is a $m \times m$ identity matrix and $\boldsymbol{1}_m$ is a $m \times 1$ vector of ones.

The paper [3] demonstrates that this set Ω_B is a convex hull representation for X_B. Specifically, the paper proves:

1) for any $(x, \boldsymbol{y}, \boldsymbol{w}) \in \Omega_B$ with $x \in \{0, 1\}$, then $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L = \boldsymbol{x}' \otimes \boldsymbol{y}$, and

2) $\text{conv}\{\Omega_B \cap x \in \{0, 1\}\} = \Omega_B$

where $\text{conv}\{\bullet\}$ denotes the convex hull of the set $\bullet$. These results allow the original binary space X_B to be equivalently modeled by the continuous linear space Ω_B.

3.1.3 RLT for Mixed-Discrete Programs

Similarly to the previous subsection, here we demonstrate how the RLT generates convex hull representations for mixed-discrete sets. Begin by considering the set

$$X_D \equiv \{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, x \in S\} \tag{3.5}$$

where x is restricted to realize discrete values in the set $S = \{\theta_1, \theta_2, \ldots, \theta_k\}$. As before, we concentrate on this relatively simple case having a single discrete variable in order to clarify notation/explanations and to enable easy comparison of these results with our more general results in Section 3.3. Section 3.4 discusses convex hull representations of mixed-discrete problems having more than one discrete variable.

Now, to form the set of functional products associated with the allowable values of the discrete x, we turn to a special set of expressions known as Lagrange interpolating polynomials (LIPs). Letting $K \equiv \{1, 2, \ldots, k\}$, there are k distinct $(k-1)$-degree LIPs, denoted $L_i(x)$ for each

$i \in K$, which are given by

$$L_i(x) = \frac{\displaystyle\prod_{j \in K-\{i\}} (x - \theta_j)}{\displaystyle\prod_{j \in K-\{i\}} (\theta_i - \theta_j)} \quad \forall i \in K. \tag{3.6}$$

These k LIPs are exactly the *functional products* used by the RLT for mixed-discrete programs. For convenience, we can write this system of LIPs using matrix notation as

$$C\dot{\boldsymbol{x}} \geq \mathbf{0} \tag{3.7}$$

where the $k \times k$ matrix C has the $(i,j)^{th}$ entry given by the coefficient of x^{j-1} in $L_i(x)$ and the k-dimensional vector $\dot{\boldsymbol{x}}$ has each element equal to the variable x raised to successively higher integer powers, starting with power 0. Thus $\dot{\boldsymbol{x}}$ appears as $\dot{\boldsymbol{x}} = (1, x, x^2, x^3, \ldots, x^{k-1})^T$.

Note that for any $x \in S$, the system (3.7) is naturally nonnegative since the LIPs are constructed in such a way as to ensure that if x takes on the i^{th} realization in S, then the i^{th} LIP will take on value 1 and the rest will take on value 0. Hence, for any $x \in S$ we have the property that

$$L_i(x) = [C]_i\dot{\boldsymbol{x}} = \begin{cases} 1 & \text{if } x = \theta_i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in K, \tag{3.8}$$

where $[\bullet]_i$ denotes the i^{th} row of the matrix $\bullet$. As shown in [1], the LIP system $C\dot{\boldsymbol{x}}$ generalizes and subsumes the functional products x and $1 - x$ for a binary variable x. Observe that if $S = \{0, 1\}$, then $\dot{\boldsymbol{x}} = (1, x)^T$ so that $[C]_1\dot{\boldsymbol{x}} = \frac{x-1}{0-1} = 1 - x$ and $[C]_2\dot{\boldsymbol{x}} = \frac{x-0}{1-0} = x$. Thus the system (3.7) reduces to the system (3.2) when x is binary.

The LIP property (3.8) ensures that if $x \in S$, then we have that

$$([C]_i\dot{\boldsymbol{x}})x = ([C]_i\dot{\boldsymbol{x}})\theta_i \quad \forall i \in K. \tag{3.9}$$

These are exactly the *simplifying identities* used by the RLT for mixed-discrete programs. Stated in words, the identities of (3.9) guarantee that whenever $x \in S$, then the i^{th} LIP times x is equal to the i^{th} LIP times that single realization θ_i which causes the LIP to equal 1. Note that the identity $x^2 = x$

of (3.3) for binary x is a special case of the identities (3.9), as can be observed when $S = \{0, 1\}$.

Using the functional products (3.7) and simplifying identities (3.9) above, we now apply the two steps of reformulation and linearization to generate a higher-dimensional polytope which yields the convex hull for X_D. Since any solution to X_D implicitly satisfies $C\dot{\boldsymbol{x}} \geq \mathbf{0}$, the set can be equivalently rewritten as

$$X_D \equiv \{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \mathbf{0} \leq \boldsymbol{y} \leq \mathbf{1}, C\dot{\boldsymbol{x}} \geq \mathbf{0}, x \in S\}. \tag{3.10}$$

Using (3.10), the two RLT steps operate as follows.

Step 1. Reformulation

Compute the Kronecker product of $C\dot{\boldsymbol{x}}$ with the constraints $Ax + B\boldsymbol{y} \geq \boldsymbol{d}$ and $\mathbf{0} \leq \boldsymbol{y} \leq \mathbf{1}$ of (3.10). Then enforce the simplifying identities of (3.9). It is possible to simultaneously enforce all k of these simplifying identities by defining

$$C\dot{\boldsymbol{x}} \otimes x = \boldsymbol{\theta}^* \tag{3.11}$$

where $\boldsymbol{\theta}^*$ is a k-dimensional vector of the form $\boldsymbol{\theta}^* = \begin{bmatrix} ([C]_1\dot{\boldsymbol{x}})\theta_1 \\ \vdots \\ ([C]_k\dot{\boldsymbol{x}})\theta_k \end{bmatrix}$. Finally, observe that due to the property (3.8) and identities (3.9), it is not necessary to compute the Kronecker product $C\dot{\boldsymbol{x}} \otimes C\dot{\boldsymbol{x}}$ since this equals $C\dot{\boldsymbol{x}}$. Instead, it is sufficient to enforce $C\dot{\boldsymbol{x}} \geq \mathbf{0}$.

Step 2. Linearization

For each of the $(k-1)m$ distinct product terms in the vector $\dot{\boldsymbol{x}} \otimes \boldsymbol{y}$ substitute a continuous variable and denote the resulting linearized vector as $\{\dot{\boldsymbol{x}} \otimes \boldsymbol{y}\}_L$. Note that $k-1$ product terms in $\dot{\boldsymbol{x}} \otimes x$ can be substituted out of the problem in the reformulation step via (3.11). Thus it is only necessary to linearize the $k-2$ nonlinear terms in $\dot{\boldsymbol{x}}$. Denote this linearized vector as $\{\dot{\boldsymbol{x}}\}_L$. Then, using Property 1 of Kronecker products and by denoting the new linearized variables in $\{\dot{\boldsymbol{x}} \otimes \boldsymbol{y}\}_L$ and $\{\dot{\boldsymbol{x}}\}_L$ by $\boldsymbol{w}$

and $\boldsymbol{z}$ respectively, we have the resulting polyhedral set given by

$$\Omega_D = \left\{ \begin{array}{l} (x, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(k-1)m} \times \mathbb{R}^{k-2} : \\ \qquad (I_k \otimes A)\boldsymbol{\theta}^* + (C \otimes B)\{\dot{\boldsymbol{x}} \otimes \boldsymbol{y}\}_L \geq (C \otimes \boldsymbol{d})\{\dot{\boldsymbol{x}}\}_L, \\ \qquad \boldsymbol{0} \leq (C \otimes I_m)\{\dot{\boldsymbol{x}} \otimes \boldsymbol{y}\}_L \leq (C \otimes \boldsymbol{1}_m)\{\dot{\boldsymbol{x}}\}_L, \\ \qquad C\{\dot{\boldsymbol{x}}\}_L \geq \boldsymbol{0} \end{array} \right\}$$

where I_k and I_m are $k \times k$ and $m \times m$ identity matrices, respectively, and $\boldsymbol{1}_m$ is a $m \times 1$ vector of ones.

The paper [1] demonstrates that the set Ω_D is a convex hull representation for X_D. Specifically, it proves:

1) for any $(x, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}) \in \Omega_D$ with $x \in S$, then $\{\dot{\boldsymbol{x}} \otimes \boldsymbol{y}\}_L = \dot{\boldsymbol{x}} \otimes \boldsymbol{y}$, $\{\dot{\boldsymbol{x}}\}_L = \dot{\boldsymbol{x}}$, and

2) $\text{conv}\,\{\Omega_D \cap x \in S\} = \Omega_D$.

These results allow the original discrete space X_D to be equivalently modeled by the continuous linear space Ω_D.

3.2 Simplicial Structure

In this section, we examine several properties of a class of special polytopes known as simplices. These simplices and their associated properties form the basis for the RLT generalizations presented in the next section. Geometrically, a simplex in $\mathbb{R}^n$ is the convex hull of $n+1$ affinely independent points. We define the collection of these $n+1$ points as

$$\Theta \equiv \left\{ \boldsymbol{\theta}^i \in \mathbb{R}^n, i \in \{1, \ldots, n+1\} : \boldsymbol{\theta}^i \text{ are affinely independent} \right\}. \tag{3.12}$$

Then, we use elements of the set Θ to form the associated n-dimensional simplex as follows:

$$SP \equiv \left\{ \boldsymbol{x}' \in \mathbb{R}^{n+1} : F\boldsymbol{x}' \geq \boldsymbol{0} \right\} \text{ where } F^{-1} = \left[(\boldsymbol{\theta}^1)' \cdots (\boldsymbol{\theta}^{n+1})' \right] \tag{3.13}$$

where, as before, $\boldsymbol{x}'$ denotes a column vector $\boldsymbol{x}$ with a 1 appended in the first position so that $\boldsymbol{x}' = \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix}$. Observe that the $(n+1) \times (n+1)$ matrix F^{-1} of (3.13) is invertible since the vectors $\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^{n+1} \in \Theta$ are affinely independent. Thus, forming the simplex SP in this manner appropriately scales the defining inequalities $F\boldsymbol{x}' \geq \mathbf{0}$ to ensure that if $\boldsymbol{x}$ takes on any value in Θ, then one of the constraints will have a slack value equal 1 and the rest will have slacks of 0. That is, if we let $[\bullet]_i$ be the i^{th} row of matrix $\bullet$, then for $\boldsymbol{x} = \boldsymbol{\theta}^i \in \Theta$ we have that $[F]_i\boldsymbol{x}' = 1$ and $[F]_j\boldsymbol{x}' = 0$ for all $j \neq i$. The extreme points of the polyhedral set SP are exactly the points $\boldsymbol{\theta}^1$ through $\boldsymbol{\theta}^{n+1}$ given in Θ and the constraints $F\boldsymbol{x}' \geq \mathbf{0}$ are exactly the facets defining the convex hull of these points. The example below demonstrates these properties for a two dimensional simplex.

Example 3.1

Consider the simplex in $\mathbb{R}^2$ with variables $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ defined by the three affinely independent points $\boldsymbol{\theta}^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\boldsymbol{\theta}^2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $\boldsymbol{\theta}^3 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ so that $\Theta = \{\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta}^3\}$. Then the matrix $F^{-1} = \left[(\boldsymbol{\theta}^1)' \;\; (\boldsymbol{\theta}^2)' \;\; (\boldsymbol{\theta}^3)'\right] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ gives the simplex $SP \equiv \{\boldsymbol{x}' \in \mathbb{R}^3 : F\boldsymbol{x}' \geq \mathbf{0}\}$ of (3.13) as

$$SP \equiv \left\{ \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} : F\boldsymbol{x}' = \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Figure 3.1 plots this simplex and labels the defining facets. Note that by the definition of F, these facets have been automatically scaled so that each slack variable takes on value 1 when $\boldsymbol{x}$ realizes the extreme point not lying on that facet and value 0 when $\boldsymbol{x}$ realizes one of the other two extreme points. For example, if $\boldsymbol{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then we have that

$$F\boldsymbol{x}' = \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

since $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ lies on the second and third facets but not the first.

Using the formulation (3.13), we prove the following two lemmas regarding simplicial struc-

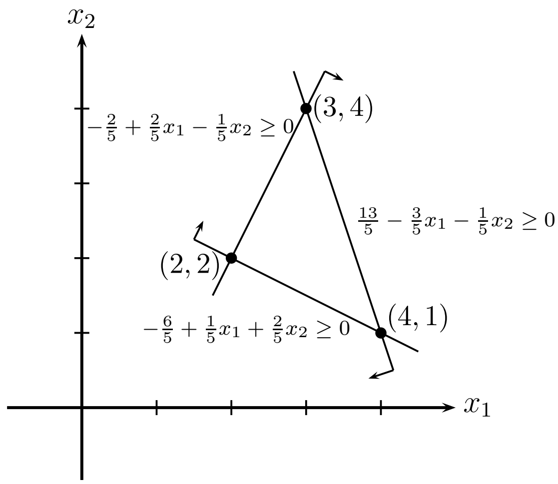

Figure 3.1: The simplex in $\mathbb{R}^2$ associated with the points $(2,2)$, $(3,4)$ and $(4,1)$.

tures. These lemmas will be used in the next section for establishing our convex hull results.

Lemma 3.1: *Given any system* $F\boldsymbol{x}' \geq \mathbf{0}$ *defined in* (3.13)*, we have that* $\mathbf{1}^T F\boldsymbol{x}' = 1$ *where* $\mathbf{1} \in \mathbb{R}^{n+1}$ *is a vector of ones.*

Proof. This result follows directly from the definition of F^{-1} in (3.13), as the first entry of each vector $(\boldsymbol{\theta}^i)'$ is 1, for $i \in \{1, \ldots, n+1\}$. □

Lemma 3.2: *For* $F\boldsymbol{x}' \geq \mathbf{0}$ *and* $\boldsymbol{\theta}^i \in \Theta$ *for* $i \in \{1, \ldots, n+1\}$ *as defined above, it follows that* $\sum_{i=1}^{n+1} \left([F]_i \boldsymbol{x}'\right) \boldsymbol{\theta}^i = \boldsymbol{x}$.

Proof. By the definition of F^{-1} in (3.13) we have that $\left[(\boldsymbol{\theta}^1)' \cdots (\boldsymbol{\theta}^{n+1})'\right] F\boldsymbol{x}' = \boldsymbol{x}'$. Eliminating the first equations gives the desired result. □

We now come to the main result of this section. Consider two full-dimensional simplices in *disjoint variables* given by $SP_1 \in \mathbb{R}^{n_1}$ and $SP_2 \in \mathbb{R}^{n_2}$ whose extreme points are given by the sets Θ_1 and Θ_2 from (3.12) and with the facets defining these simplices given by $F_1 \left(\boldsymbol{x}^1\right)' \geq \mathbf{0}$ and $F_2 \left(\boldsymbol{x}^2\right)' \geq \mathbf{0}$ as in (3.13). Apply the RLT steps of *reformulation* and *linearization* as follows. In the reformulation step, compute all pairwise products of the $n_1 + 1$ facets defining the first simplex with the $n_2 + 1$ facets defining the second. In the linearization step, substitute a continuous variable for each distinct product term. Then the new polyhedral set in $\mathbb{R}^{(n_1+1)(n_2+1)-1}$ is defined in terms of $(n_1+1)(n_2+1)$ inequalities.

The main result of this section, embodied by Theorem 3.1, is that this higher-dimensional region is itself a simplex, and has the desirable property that there exists a one-to-one correspondence between its extreme points and the extreme points of the original sets S_1 and S_2, with the linearized product variables equal to their intended products at each such point. That is, any extreme point in this higher-dimensional region has the form $\left\{(\boldsymbol{x}^1)' \otimes (\boldsymbol{x}^2)'\right\}_L = \left(\boldsymbol{\theta}^{1,i}\right)' \otimes \left(\boldsymbol{\theta}^{2,j}\right)'$ where $\boldsymbol{\theta}^{1,i} \in \Theta_1$ for $i \in \{1, \ldots, n_1+1\}$ and $\boldsymbol{\theta}^{2,j} \in \Theta_2$ for $j \in \{1, \ldots, n_2+1\}$. In fact, as shown in Theorem 3.1, this same extreme point property will hold for any finite number m of such simplices. As will be explained in the following section, this simplicial structure gives rise to important linearization consequences for mixed 0-1 and mixed-discrete polynomial programs.

Theorem 3.1: *Given any integer $m \geq 1$, for each $j \in M \equiv \{1, \ldots, m\}$, let SP_j denote an n_j-dimensional simplex in variables $\boldsymbol{x}^j \in \mathbb{R}^{n_j}$ defined as the convex hull of n_j+1 affinely independent points $\boldsymbol{\theta}^{j,1}, \ldots, \boldsymbol{\theta}^{j,n_j+1} \in \Theta_j$ of* (3.12) *and whose facets are given by $F_j(\boldsymbol{x}^j)' \geq \mathbf{0}$ as in* (3.13). *Then, for each $J \subseteq M$,*

$$SP_J \equiv \left\{\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L \in \mathbb{R}^{\prod_{j\in J}(n_j+1)} : \left\{\otimes_{j\in J}F_j(\boldsymbol{x}^j)'\right\}_L \geq \mathbf{0}\right\} \tag{3.14}$$

defines the $\left(\prod_{j\in J}(n_j+1)-1\right)$-dimensional simplex whose extreme points are given by the columns of the $\prod_{j\in J}(n_j+1) \times \prod_{j\in J}(n_j+1)$ matrix $\otimes_{j\in J}F_j^{-1}$, less the first row.

Proof. To begin, for any chosen $J \subseteq M$, the convex hull of the columns of the matrix $\otimes_{j\in J}F_j^{-1}$, less the first row, forms a simplex because $\otimes_{j\in J}F_j^{-1}$ is invertible by property 2 of the Kronecker products of matrices, and the first row consists entirely of ones by the definition of Kronecker products. To show that the columns of this truncated matrix constitute the extreme points of the set SP_J of

(3.14), observe that

$$\begin{aligned} SP_J &= \left\{\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L : \left\{\otimes_{j\in J}F_j(\boldsymbol{x}^j)'\right\}_L \geq \mathbf{0}\right\} \\ &= \left\{\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L : \left\{\otimes_{j\in J}F_j(\boldsymbol{x}^j)'\right\}_L = \boldsymbol{\lambda} \text{ for some } \boldsymbol{\lambda} \geq \mathbf{0}\right\} \\ &= \left\{\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L : \otimes_{j\in J}F_j \left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L = \boldsymbol{\lambda} \text{ for some } \boldsymbol{\lambda} \geq \mathbf{0}\right\} \\ &= \left\{\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L : \left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L = \otimes_{j\in J}F_j^{-1}\boldsymbol{\lambda} \text{ for some } \boldsymbol{\lambda} \geq \mathbf{0}\right\}, \end{aligned}$$

where the first equation is by definition of SP_J, the second equation follows trivially, and the third and fourth equations follow from the stated properties 1 and 2, respectively, of the Kronecker products of matrices. As the first entry of $\left\{\otimes_{j\in J}(\boldsymbol{x}^j)'\right\}_L$ is 1 and the first row of $\otimes_{j\in J}F_j^{-1}$ consists entirely of ones, the result follows. □

A useful consequence of Theorem 3.1 is that, for each $J \subseteq M$, the set SP_J defined in (3.14) characterizes the convex hull of the region $\otimes_{j\in J}\left(\boldsymbol{x}^j\right)'$ when $\left(\boldsymbol{x}^j\right)'$ is restricted to have $\left(\boldsymbol{x}^j\right)' \in SP_j$ for all $j \in J$. A formal statement is given below.

Corollary 3.1: *Given any integer $m \geq 1$ and any $J \subseteq M$, let SP_J be as defined in (3.14) and let*

$$T_J \equiv \left\{\otimes_{j\in J}(\boldsymbol{x}^j)' \in \mathbb{R}^{\prod_{j\in J}(n_j+1)} : (\boldsymbol{x}^j)' \in SP_j \text{ for all } j \in J\right\}. \tag{3.15}$$

Then $\text{conv}\{T_J\} = SP_J$.

Proof. We have $\text{conv}\{T_J\} \subseteq SP_J$ since every point feasible to T_J is by construction also feasible to SP_J, and since SP_J is a convex set. Also, $\text{conv}\{T_J\} \supseteq SP_J$ since Theorem 3.1 shows SP_J to be a polytope whose extreme points are given by the columns of the $\prod_{j\in J}(n_j+1) \times \prod_{j\in J}(n_j+1)$ matrix $\otimes_{j\in J}F_j^{-1}$, with each column of this matrix a feasible point to T_J since, for each $j \in J$, every column of the matrix F_j^{-1} is by definition in the set SP_j. □

Observe that since $\text{conv}\{T_J\} = SP_J$, and these polytopes are bounded, then they must have the same set of extreme points; namely, the columns of the matrix $\otimes_{j\in J}F_j^{-1}$.

We illustrate Theorem 3.1 and Corollary 3.1 with an example.

Example 3.2

For $m = 2$, let SP_1 denote the 1-simplex in the variable $\boldsymbol{x}^1 = \left(x_1^1\right) \in \mathbb{R}^1$ defined as the convex hull of the two affinely independent points $\boldsymbol{\theta}^{1,1} = (2)$ and $\boldsymbol{\theta}^{1,2} = (3)$, and let SP_2 denote the two-simplex in the variables $\boldsymbol{x}^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \in \mathbb{R}^2$ defined as the convex hull of the three affinely independent points $\boldsymbol{\theta}^{2,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\boldsymbol{\theta}^{2,2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\boldsymbol{\theta}^{2,3} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then $F_1^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ with $F_1 = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, and $F_2^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ with $F_2 = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Now, letting $J = \{1\}$ and then $J = \{2\}$, we obtain that $SP_1 \equiv \left\{ \begin{pmatrix} 1 \\ x_1^1 \end{pmatrix} : \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and that $SP_2 \equiv \left\{ \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} : \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$, defining the respective simplices as desired. Next, letting $J = M = \{1, 2\}$, we obtain that

$$
\begin{aligned}
SP_J &= \left\{ \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L : \left\{ \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^1 \end{pmatrix} \otimes \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \right\}_L \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L : \left\{ \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^1 \end{pmatrix} \otimes \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \right\}_L = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}, \boldsymbol{\lambda} \geq \mathbf{0} \right\} \\
&= \left\{ \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L : \left[\begin{array}{ccc|ccc} 3 & -1 & -1 & -1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 2 & -1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & -1 & 2 & 0 & \frac{1}{3} & -\frac{2}{3} \\ \hline -2 & \frac{2}{3} & \frac{2}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{4}{3} & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{4}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \end{array} \right] \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}, \boldsymbol{\lambda} \geq \mathbf{0} \right\} \\
&= \left\{ \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L : \left\{ \begin{matrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{matrix} \right\}_L = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 4 & 2 & 0 & 6 & 3 \\ 0 & 2 & 4 & 0 & 3 & 6 \end{array} \right] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}, \boldsymbol{\lambda} \geq \mathbf{0} \right\},
\end{aligned}
$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^T$. Observe in the last expression that the 6×6 matrix $F_1^{-1} \otimes F_2^{-1}$ has the first row consisting entirely of ones, thus providing the 5-dimensional simplex whose 6 extreme points are given by the columns of this matrix, less the first row. Each such extreme point has $\{x_1^1 x_1^2\}_L = x_1^1 x_1^2$ and $\{x_1^1 x_2^2\}_L = x_1^1 x_2^2$, as desired. Moreover, and relative to Corollary 3.1, for $J = \{1\}$ and $J = \{2\}$, it follows directly from (3.15) that $T_J = SP_J$, giving $\text{conv}\{T_J\} = SP_J$. For $J = M = \{1, 2\}$, (3.15) gives us that

$$T_J = \left\{ \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \end{pmatrix} : \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

with $\text{conv}\{T_J\} = SP_J$ by Corollary 3.1.

The simplicial structure of the sets SP_j for $j = \{1, \ldots, m\}$ used in Theorem 3.1 ensures that each extreme point of the higher-dimensional polytopes SP_J have all the linearized variables equal to their intended products. It is important to note that this extreme point property is not necessarily true when the sets SP_j are general polyhedra rather than simplices, as demonstrated in Example 3.3.

Example 3.3

For $m = 2$ and $j \in M$, temporarily define $SP_j \equiv \left\{ \begin{pmatrix} x_1^j \\ x_2^j \end{pmatrix} : 0 \leq x_1^j \leq 1, 0 \leq x_2^j \leq 1 \right\}$ or, equivalently, $SP_j \equiv \left\{ \begin{pmatrix} 1 \\ x_1^j \\ x_2^j \end{pmatrix} : \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^j \\ x_2^j \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. Note that the sets SP_j for $j \in \{1, 2\}$ define unit squares in $\mathbb{R}^2$, which are not simplices. The set $J = M = \{1, 2\}$ gives

$$SP_J = \left\{ \left\{ \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \\ x_1^1 \\ x_1^1 x_1^2 \\ x_1^1 x_2^2 \\ x_2^1 \\ x_2^1 x_1^2 \\ x_2^1 x_2^2 \end{pmatrix} \right\}_L : \left\{ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^1 \\ x_2^1 \end{pmatrix} \otimes \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \right\}_L \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This polytope can be verified to have 24 extreme points, with one such point given by $\left\{1, x_1^2, x_2^2, x_1^1, x_1^1x_1^2, x_1^1x_2^2, x_2^1, x_2^1x_1^2, x_2^1x_2^2\right\}_L = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Observe that this point does not have the linearized product terms equal to their intended products as, for example, $\left\{x_1^1x_1^2\right\}_L = \frac{1}{2} \neq \frac{1}{4} = x_1^1x_1^2$.

Theorem 3.1, Corollary 3.1, and Example 3.2 addressed properties associated with products of disjoint simplices, while Example 3.3 showed that these properties are not shared by general polyhedral sets.

3.3 Convex Hull Representations

In this section, we generalize the convex hull representations for mixed binary and mixed-discrete programs obtained by the highest level of the RLT hierarchy. In doing so, we generalize the identities $x^2 = x$ of (3.3) from the 0-1 case where x is binary and $([C]_i\dot{\boldsymbol{x}})x = ([C]_i\dot{\boldsymbol{x}})\theta_i$ of (3.9) from the discrete case where $x \in \{\theta_1, \dots \theta_n\}$. To accomplish this, we employ the defining constraints $F\boldsymbol{x}' \geq \mathbf{0}$ of (3.13) which form facets of the n-dimensional simplex associated with the affinely independent points in $\Theta = \left\{\boldsymbol{\theta}^1, \dots, \boldsymbol{\theta}^{n+1}\right\}$ of (3.12). As discussed in the previous section, these constraints are defined in such a way that for each $\boldsymbol{x} \in \Theta$ we have

$$[F]_i\boldsymbol{x}' = \begin{cases} 1 & \text{if } \boldsymbol{x}' = \left(\boldsymbol{\theta}^i\right)' \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

where, again, $[\bullet]_i$ is the i^{th} row of the matrix $\bullet$. Hence, for each $\boldsymbol{x} \in \Theta$ we have the important identity

$$([F]_i\boldsymbol{x}')\,\boldsymbol{x} = ([F]_i\boldsymbol{x}')\,\boldsymbol{\theta}^i \qquad \forall i \in \{1, \dots, n+1\}. \tag{3.17}$$

An insightful interpretation of (3.17) is that, given SP of (3.13), the product of any facet $[F]_i\boldsymbol{x}'$ of SP with any vector $\boldsymbol{x}$ that realizes an extreme point of SP is equal to the product of $[F]_i\boldsymbol{x}'$ with that single extreme point $\boldsymbol{\theta}^i$ that does not satisfy $[F]_i\boldsymbol{x}' = 0$. The example below demonstrates these simplifying identities.

Example 3.4

Consider the two-dimensional simplex from Example 3.1 having variables $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with extreme points $\boldsymbol{\theta}^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\boldsymbol{\theta}^2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $\boldsymbol{\theta}^3 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. The matrix $F^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ yields the appropriately scaled facets of this simplex which are given by the system

$$F\boldsymbol{x}' = \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using these facets, the simplifying identities of (3.17) are given by

$$\left(\frac{13}{5} - \frac{3}{5}x_1 - \frac{1}{5}x_2\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{13}{5} - \frac{3}{5}x_1 - \frac{1}{5}x_2\right)\begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

$$\left(-\frac{6}{5} + \frac{1}{5}x_1 + \frac{2}{5}x_2\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(-\frac{6}{5} + \frac{1}{5}x_1 + \frac{2}{5}x_2\right)\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \text{ and}$$

$$\left(-\frac{2}{5} + \frac{2}{5}x_1 - \frac{1}{5}x_2\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(-\frac{2}{5} + \frac{2}{5}x_1 - \frac{1}{5}x_2\right)\begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

As demonstrated later, the identities of (3.17) generalize the simplifying identities of (3.3) and (3.9) for binary and discrete problems, respectively. In fact, the functional products x and $1 - x$ for the binary case and LIPs for the discrete case can be viewed as facets of the form $[F]_i\boldsymbol{x}'$ for specially structured simplices. For now, though, we turn or attention to the development of generalized convex hull arguments using the new identity (3.17).

To begin, consider a mixed-discrete region with variables $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ and where $\boldsymbol{x}$ is restricted to realize values in a set $\Theta \equiv \left\{\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^{n+1}\right\}$ with $\boldsymbol{\theta}^1$ through $\boldsymbol{\theta}^{n+1}$ affinely independent points in $\mathbb{R}^n$ as described in (3.12). This region is given by

$$X \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, \boldsymbol{x} \in \Theta\}. \tag{3.18}$$

Because X enforces that $\boldsymbol{x}$ is discrete and lies in the set Θ and since the inequalities $F\boldsymbol{x}' \geq \boldsymbol{0}$ define the facets of the convex hull of these points, these facets are redundant to X. Thus, X can be equivalently written as

$$X \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, F\boldsymbol{x}' \geq \boldsymbol{0}, \boldsymbol{x} \in \Theta\} \tag{3.19}$$

where $F\boldsymbol{x}' \geq \boldsymbol{0}$ is explicitly yet redundantly enforced. Using (3.19), we present a new, generalized RLT methodology which appears as follows.

Step 1. Reformulation

Compute the Kronecker product of $F\boldsymbol{x}'$ with the constraints $A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}$ and $\boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}$ from (3.19). Since, by (3.16) when $\boldsymbol{x} \in \Theta$, the expressions $[F]_i\boldsymbol{x}'$ are binary for $i \in \{1, \ldots, n+1\}$ so that $([F]_i\boldsymbol{x}')([F]_i\boldsymbol{x}') = ([F]_i\boldsymbol{x}')$ and $([F]_i\boldsymbol{x}')([F]_j\boldsymbol{x}') = 0$ whenever $i \neq j$, it is unnecessary to compute the Kronecker product of $F\boldsymbol{x}'$ with itself and is instead sufficient to enforce $F\boldsymbol{x}' \geq \boldsymbol{0}$. The identities (3.17) for all $i \in \{1, \ldots, n+1\}$ can be enforced using

$$F\boldsymbol{x}' \otimes \boldsymbol{x} = \boldsymbol{\theta}^* \tag{3.20}$$

where $\boldsymbol{\theta}^*$ is a $\left(n^2 + n\right)$-dimensional vector of the form $\boldsymbol{\theta}^* = \begin{bmatrix} ([F]_1\boldsymbol{x}')\boldsymbol{\theta}^1 \\ \vdots \\ ([F]_{n+1}\boldsymbol{x}')\boldsymbol{\theta}^{n+1} \end{bmatrix}$.

Step 2. Linearization

For each of the $n \times m$ product terms in the vector $\boldsymbol{x}' \otimes \boldsymbol{y}$ substitute a distinct continuous variable and denote the resulting linearized vector as $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L$. Note that all product terms in the vector $\boldsymbol{x}' \otimes \boldsymbol{x}$ are substituted out in the reformulation step by enforcing (3.20) since $\boldsymbol{\theta}^*$ contains no products amongst the variables of $\boldsymbol{x}$. Hence, letting $\boldsymbol{w}$ refer only to the linearized product terms in $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L$, we have the resulting polyhedral set given by

$$\Omega = \left\{ \begin{array}{l} (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} : \\ \qquad (I_{n+1} \otimes A)\boldsymbol{\theta}^* + (F \otimes B)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L \geq (F \otimes \boldsymbol{d})\boldsymbol{x}', \\ \qquad \boldsymbol{0} \leq (F \otimes I_m)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L \leq (F \otimes \boldsymbol{1}_m)\boldsymbol{x}', \\ \qquad F\boldsymbol{x}' \geq \boldsymbol{0} \end{array} \right\} \tag{3.21}$$

where I_{n+1} and I_m are identity matrices in $\mathbb{R}^{n+1,n+1}$ and $\mathbb{R}^{m,m}$, respectively, and $\boldsymbol{1}_m$ is a vector of ones in $\mathbb{R}^m$.

We now show that the polyhedral set Ω of (3.21) produced by our new RLT procedure is exactly the convex hull of the discrete set X of (3.18) in a higher-dimensional space. We demonstrate this in the following two theorems. Theorem 3.2 proves that the new auxiliary variables $\boldsymbol{w}$ in Ω take on their intended product values whenever the discrete variables $\boldsymbol{x}$ take on one of the realizations in Θ. Then Theorem 3.3 shows that Ω is equivalent to the convex hull of the set of points where $(\boldsymbol{x}, \boldsymbol{y}) \in X$ together with $\boldsymbol{w} = \boldsymbol{x}' \otimes \boldsymbol{y}$. This characterization allows the mixed-discrete set X to be equivalently modeled by the continuous linear region Ω.

Theorem 3.2: *Every solution to Ω with $\boldsymbol{x} \in \Theta$ must have $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L = \boldsymbol{x}' \otimes \boldsymbol{y}$.*

Proof. Consider a single variable y_j of the vector $\boldsymbol{y}$ where $j \in \{1, \ldots, m\}$. The constraints $\mathbf{0} \leq (F \otimes I_m)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L \leq (F \otimes \mathbf{1}_m)\boldsymbol{x}'$ of (3.21) involving y_j can be written as

$$\mathbf{0} \leq \{F\boldsymbol{x}' \otimes y_j\}_L \leq F\boldsymbol{x}'. \tag{3.22}$$

For a single facet, say $[F]_i\boldsymbol{x}' \geq 0$ for some $i \in \{1, \ldots, n+1\}$ the constraints of (3.22) include the specific constraint

$$0 \leq \{[F]_i\boldsymbol{x}' \otimes y_j\}_L \leq [F]_i\boldsymbol{x}'. \tag{3.23}$$

By Lemma 3.1 which gives $\mathbf{1}^T F\boldsymbol{x}' = 1$, we can surrogate the remaining constraints of (3.22) not including (3.23) to obtained the redundant restriction

$$0 \leq \{(1 - [F]_i\boldsymbol{x}') \otimes y_j\}_L \leq 1 - [F]_i\boldsymbol{x}'$$

which can equivalently be written as

$$0 \leq y_j - \{[F]_i\boldsymbol{x}' \otimes y_j\}_L \leq 1 - [F]_i\boldsymbol{x}'. \tag{3.24}$$

Now, if a solution to Ω has $\boldsymbol{x} = \boldsymbol{\theta}^i \in \Theta$, then by the property (3.16) we have $[F]_i\boldsymbol{x}' = 1$ so that (3.24) gives $\{[F]_i\boldsymbol{x}' \otimes y_j\}_L = y_j$. On the other hand, if $\boldsymbol{x} = \boldsymbol{\theta}^k \in \Theta$ where $k \neq i$, then we have $[F]_i\boldsymbol{x}' = 0$ so that (3.23) gives $\{[F]_i\boldsymbol{x}' \otimes y_j\}_L = 0$. In either case, it follows that $\{[F]_i\boldsymbol{x}' \otimes y_j\}_L = [F]_i\boldsymbol{x}' \otimes y_j$

for all $i \in \{1, \ldots, n+1\}$ and $j \in \{1, \ldots, m\}$ which gives

$$(F \otimes I_m)\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L = (F \otimes I_m)(\boldsymbol{x}' \otimes \boldsymbol{y}).$$

Since F is invertible by construction and by property 2 of Kronecker products, left-multiplying this system by $F^{-1} \otimes I_m$ results in $\{\boldsymbol{x}' \otimes \boldsymbol{y}\}_L = \boldsymbol{x}' \otimes \boldsymbol{y}$ which completes the proof. $\square$

Theorem 3.3: *The set* Ω *of* (3.21) *satisfies conv* $\big\{\Omega \cap \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) : \boldsymbol{x} = \boldsymbol{\theta}^i \ \forall \ \boldsymbol{\theta}^i \in \Theta\}\big\} = \Omega$.

Proof. If $\Omega = \emptyset$, then the proof is trivial. Assume $\Omega \neq \emptyset$ and arbitrarily select $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}}) \in \Omega$. Let $\boldsymbol{\alpha} = F\tilde{\boldsymbol{x}}'$ and note that $\boldsymbol{\alpha} \geq \boldsymbol{0}$ and $\sum_{i=1}^{n+1} \alpha_i = 1$ since $\tilde{\boldsymbol{x}}'$ satisfies $F\boldsymbol{x}' \geq \boldsymbol{0}$. Let $E = \{i = 1, \ldots, n+1 : \alpha_i > 0\}$. Then, for each $i \in E$, define $\boldsymbol{x}^i = \boldsymbol{\theta}^i$, $\boldsymbol{y}^i = \frac{\{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L}{\alpha_i}$ evaluated at $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$, and $\boldsymbol{w}^i = \boldsymbol{x}^i \otimes \boldsymbol{y}^i$. Then it remains to show that $(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i) \in \Omega$ such that $\sum_{i\in E} \alpha_i(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$.

We first show that for each $i \in E$, $(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i)$ is feasible to Ω. Since $\boldsymbol{w}^i = \boldsymbol{x}^i \otimes \boldsymbol{y}^i$, proving feasibility reduces to showing that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in X$ of (3.19). Towards this end, remember that $[F]_i \tilde{\boldsymbol{x}}' = \alpha_i$ so that $B\{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L \geq \alpha_i(\boldsymbol{d} - A\boldsymbol{\theta}^i)$ and $\boldsymbol{0} \leq \{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L \leq \alpha_i \boldsymbol{1}$ for $\{\boldsymbol{y}, \boldsymbol{w}\}_L = (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$. Dividing both inequalities by α_i gives $B\frac{\{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L}{\alpha_i} \geq \boldsymbol{d} - A\boldsymbol{\theta}^i$ and $\boldsymbol{0} \leq \frac{\{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L}{\alpha_i} \leq \boldsymbol{1}$ for $\{\boldsymbol{y}, \boldsymbol{w}\}_L = (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$. Hence, we have $A\boldsymbol{x}^i + B\boldsymbol{y}^i \geq \boldsymbol{d}$ and $\boldsymbol{0} \leq \boldsymbol{y}^i \leq \boldsymbol{1}$ so that $(\boldsymbol{x}^i, \boldsymbol{y}^i)$ is feasible to X. Thus $(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i) \in \Omega$.

Now it remains to show that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$ is a convex combination of the points $(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i)$. First, by construction of the $\boldsymbol{\alpha}$ and E, we have $\sum_{i\in E} \alpha_i = 1$ and $\sum_{i\in E} \alpha_i \boldsymbol{x}^i = \tilde{\boldsymbol{x}}$. For $\{\boldsymbol{y}\}_L = \tilde{\boldsymbol{y}}$ we have

$$\begin{aligned}
\sum_{i\in E} \alpha_i \boldsymbol{y}^i &= \sum_{i\in E} \{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L \\
&= \sum_{i=1}^{n+1} \{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L \\
&= \left\{ \left(\sum_{i=1}^{n+1} [F]_i \boldsymbol{x}' \right) \boldsymbol{y} \right\}_L \\
&= \{(\boldsymbol{1}^T F \boldsymbol{x}')\boldsymbol{y}\}_L \\
&= \tilde{\boldsymbol{y}}.
\end{aligned}$$

Here, the first equality follows from the construction of $\boldsymbol{y}^i$ and the second since $\{([F]_i \boldsymbol{x}')\boldsymbol{y}\}_L = 0$ for all $i \neq E$. The third is due to the nature of linearization operations where, given polynomial expressions Ψ_1 through Ψ_n, we have $\sum_{i=1}^{n} \{\Psi_i\}_L = \{\sum_{i=1}^{n} \Psi_i\}_L$. The fourth equality holds by definition and the fifth by Lemma 3.1 and since $\{\boldsymbol{y}\}_L = \tilde{\boldsymbol{y}}$. Finally, for $\{\boldsymbol{w}\}_L = \tilde{\boldsymbol{w}}$ we have

$$\begin{aligned}
\sum_{i\in E} \alpha_i \boldsymbol{w}^i &= \sum_{i\in E} \alpha_i \left(\boldsymbol{x}^i \otimes \frac{\{([F]_i\boldsymbol{x}')\boldsymbol{y}\}_L}{\alpha_i} \right) \\
&= \sum_{i\in E} \boldsymbol{x}^i \otimes \{([F]_i\boldsymbol{x}')\boldsymbol{y}\}_L \\
&= \sum_{i=1}^{n+1} \boldsymbol{x}^i \otimes \{([F]_i\boldsymbol{x}')\boldsymbol{y}\}_L \\
&= \sum_{i=1}^{n+1} \{(([F]_i\boldsymbol{x}')\boldsymbol{x}^i) \otimes \boldsymbol{y}\}_L \\
&= \left\{ \sum_{i=1}^{n+1} (([F]_i\boldsymbol{x}')\boldsymbol{x}^i) \otimes \boldsymbol{y} \right\}_L \\
&= \{\boldsymbol{x} \otimes \boldsymbol{y}\}_L \\
&= \tilde{\boldsymbol{w}}.
\end{aligned}$$

The first equality follows from the definition of $\boldsymbol{w}^i$ and $\boldsymbol{y}^i$. The second is due to Property 3 of Kronecker products. The third holds since $\{([F]_i\boldsymbol{x}')\boldsymbol{y}\}_L = 0$ for all $i \neq E$. The fourth again follows from Property 3 of Kronecker products since $[F]_i\boldsymbol{x}'$ is scalar. The fifth is given since $\sum_{i=1}^n \{\Psi_i\}_L = \{\sum_{i=1}^n \Psi_i\}_L$ as seen above for polynomial expressions Ψ_i. The sixth equality is due to Lemma 3.2 since $\boldsymbol{x}^i = \boldsymbol{\theta}^i$. Lastly, the seventh equality follows since $\{\boldsymbol{w}\}_L = \tilde{\boldsymbol{w}}$. Thus, we have $\sum_{i\in E} \alpha_i(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{w}^i) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{w}})$, which completes the proof. □

Example 3.5

Consider the discrete three-dimensional set where variables $(x_1, x_2, y_1)^T$ are all bounded between values 0 and 5 and where $(x_1, x_2)^T$ are further restricted to realize one of the affinely independent points in the set $\Theta = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$. In the manner of (3.18), this set can be expressed as

$$X = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix} : A\boldsymbol{x} + B\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{bmatrix} 0 \\ -5 \\ 0 \\ -5 \end{bmatrix} = \boldsymbol{d}, 0 \leq y_1 \leq 5, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Theta \right\}.$$

From Example 3.1, we have that the facets defining the convex hull of the points in Θ are given by $\boldsymbol{F x'} = \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$. Thus, letting w_{11} and w_{21} represent the linearized products $x_1 y_1$ and $x_2 y_1$, respectively, then the linear system Ω given in (3.21) appears as

$$\Omega = \left\{ \begin{array}{l} (x_1, x_2, y_1, w_{11}, w_{21}) \in \mathbb{R}^5 : \\ \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \begin{pmatrix} (\frac{13}{5} - \frac{3}{5}x_1 - \frac{1}{5}x_2)\, 2 \\ (\frac{13}{5} - \frac{3}{5}x_1 - \frac{1}{5}x_2)\, 2 \\ (-\frac{6}{5} + \frac{1}{5}x_1 + \frac{2}{5}x_2)\, 3 \\ (-\frac{6}{5} + \frac{1}{5}x_1 + \frac{2}{5}x_2)\, 4 \\ (-\frac{2}{5} + \frac{2}{5}x_1 - \frac{1}{5}x_2)\, 4 \\ (-\frac{2}{5} + \frac{2}{5}x_1 - \frac{1}{5}x_2)\, 1 \end{pmatrix} \geq \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ -13 & 3 & 1 \\ 0 & 0 & 0 \\ -13 & 3 & 1 \\ \hline 0 & 0 & 0 \\ 6 & -1 & -2 \\ 0 & 0 & 0 \\ 6 & -1 & -2 \\ \hline 0 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{array}\right] \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} y_1 \\ w_{11} \\ w_{21} \end{pmatrix} \leq \begin{bmatrix} 13 & -3 & -1 \\ -6 & 1 & 2 \\ -2 & 2 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}, \\ \begin{bmatrix} \frac{13}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \right\}.$$

This system has 6 extreme points given by $(x_1, x_2, y_1, w_{11}, w_{21}) \in \{(2,2,0,0,0), (3,4,0,0,0), (4,1,0,0,0), (2,2,5,10,10), (3,4,5,15,20), (4,1,5,20,5)\}$. Notice that each extreme point satisfies $w_{11} = x_1 y_1$ and $w_{21} = x_2 y_1$. If we project Ω onto the set of the three original variables, then the resulting polyhedron appears as the shaded region in Figure 3.2. As shown, this region is exactly the convex hull of $(x_1, x_2) \in \Theta$ together with y between 0 and 5.

As a final observation on the results of this section, notice that the RLT constructs and Theorems 3.2 and 3.3 employ only a *single* simplex. While this greatly simplifies the notation and proofs, it is *not* restrictive. As Theorem 3.1 from the previous section demonstrates, the Kronecker product of any family of simplices in disjoint variables is itself a simplex. Hence, by first computing the Kronecker product of a family of simplices and then treating the resulting product as a single simplex, the results of this section can be employed to produce the desired polyhedral region.

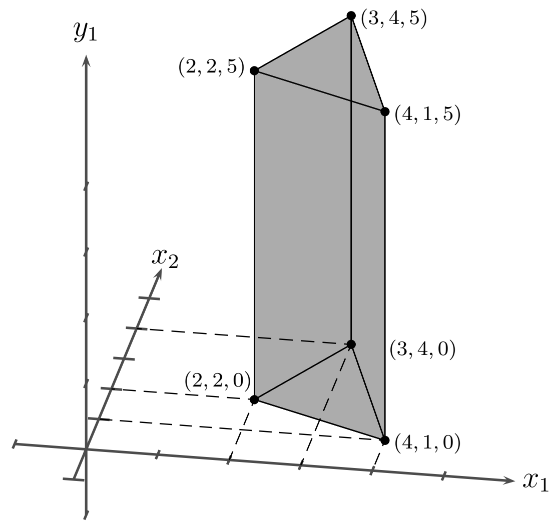

Figure 3.2: The shaded region represents the projection of Ω onto the space (x_1, x_2, y_1).

3.4 Insights for Classic RLT Results

In this section, we directly relate the mixed-binary and mixed-discrete RLT results from Section 3.1 to the new convex hull results from Sections 3.2 and 3.3. We demonstrate how the functional products x and $1 - x$ for binary and the LIPs for discrete cases can be equivalently interpreted as special simplicial facets. We show that the simplifying identities $x^2 = x$ of (3.3) and $([C]_i\dot{\boldsymbol{x}})x = ([C]_i\dot{\boldsymbol{x}})\theta_i$ of (3.9) are special cases of the more general identities $([F]_i\boldsymbol{x}')\boldsymbol{x} = ([F]_i\boldsymbol{x}')\boldsymbol{\theta}^i$ of (3.17). Using Theorem 3.1 from Section 3.2, which gives that the Kronecker product of disjoint simplices is itself a simplex, we illustrate how convex hull results for sets having multiple binary or discrete variables are also special cases of the convex hull proofs established in Section 3.3.

3.4.1 Insights for Mixed-Binary RLT

Reconsider from Section 3.1.2 the mixed-binary set (3.1) containing a single binary variable x. For ease of reading, this set is restated here:

$$X_B \equiv \{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, x \in \{0, 1\}\}.$$

Using the machinery developed in the previous section, we show how the functional products $P\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq \boldsymbol{0}$ of (3.2) and the simplifying identity $x^2 = x$ of (3.3) naturally arise via the facets of a particular 1-dimensional simplex. Adopting the notation of (3.12) and (3.13), this simplex, not surprisingly, is formed using the affinely independent, 1-dimensional points in the set $\Theta = \{\theta_1, \theta_2\} = \{0, 1\}$ which gives the associated matrix $F^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Hence, the resulting facets of this simplex are given by $F\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq \boldsymbol{0}$ which, since $F = P$, exactly produces the functional products of (3.2).

Since the system $P\boldsymbol{x}' \geq \boldsymbol{0}$ defines the appropriately-scaled facets of a simplex, it follows that the equalities (3.17) are enforced so that if $x \in \{0, 1\}$ then

$$([P]_i\boldsymbol{x}')x = ([P]_i\boldsymbol{x}')\theta_i \quad \forall i \in \{1, 2\}.$$

Substituting the values of $[P]_i\boldsymbol{x}'$ and θ_i for $i \in \{1, 2\}$, these equalities appears as $(1-x)x = (1-x)0$ and $(x)x = (x)1$, respectively, and reduce to the single equality $x^2 = x$ which is the simplifying identity (3.3). Thus, the convex hull representation Ω_B from Section 3.1.2 is a special case of our more general convex hull representation Ω.

Now, suppose we redefine the set X_B so that instead of having a single binary variable x, it instead contains n binary variables in the vector $\boldsymbol{x}$. Then X_B appears as

$$X_B \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, \boldsymbol{x} \in \{0, 1\}^n\}.$$

In this case, letting $N = \{1, \ldots, n\}$, the RLT for mixed-binary programs yields a convex hull representation by forming functional products as $\otimes_{i \in N} P_i x_i'$ (the Kronecker product of the functional products of the individual variables) and by enforcing the simplifying identities $x_i^2 = x_i$ for all $i \in N$.

As shown below, this case can also be viewed as a special case of our simplicial facet results from the previous section. Since, for each x_i, the associated system of functional products $P_i\boldsymbol{x}_i' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x_i \end{pmatrix} \geq \boldsymbol{0}$ forms a simplex, it follows by Theorem 3.1 that the system $\{\otimes_{i \in N} P_i \boldsymbol{x}_i'\}_L \geq \boldsymbol{0}$ defines a (2^n-1)-dimensional simplex whose 2^n extreme points are given by the columns of $\otimes_{i \in N} P_i^{-1}$ less the first row. The structure of $\otimes_{i \in N} P_i^{-1}$ ensures that every extreme points has each x_i binary with each linearized product term equaling the product of the individual variables. Thus, if we define the matrix $F = \otimes_{i \in N} P_i$ and the vector $\boldsymbol{x}' = \otimes_{i \in N} \{\boldsymbol{x}_i'\}_L$, then the functional products $\otimes_{i \in N} P_i x_i'$

for the mixed-binary RLT are exactly equivalent to the facets $F\boldsymbol{x}'$ for all binary realizations of $\boldsymbol{x}$.

Finally, it remains to show that the general identities (3.17) result in the mixed-binary RLT identities $x_i^2 = x_i$ for all $i \in N$. To accomplish this, for any $i \in N$ consider the vector of facets $\{x_i \otimes_{j \in N-\{i\}} P_j \boldsymbol{x}'_j\}_L$. By Theorem 3.1, the system $\{\otimes_{j \in N-\{i\}} P_j \boldsymbol{x}'_j\}_L \geq \mathbf{0}$ gives the facets of a simplex. Lemma 3.1 states that the surrogation of these facets results in the scalar value 1. Hence, it follows that

$$\mathbf{1}^T \{x_i \otimes_{j \in N-\{i\}} P_j \boldsymbol{x}'_j\}_L = \{x_i\}_L = x_i. \tag{3.25}$$

Judiciously choosing the appropriate identities of (3.17) for this problem gives

$$\{x_i \otimes_{j \in N-\{i\}} P_j \boldsymbol{x}'_j\}_L \cdot (x_i) = \{x_i \otimes_{j \in N-\{i\}} P_j \boldsymbol{x}'_j\}_L \cdot (1). \tag{3.26}$$

Applying (3.25) to (3.26) results in the identity $x_i(x_i) = x_i(1)$ exactly as desired for each $i \in N$. Therefore, the convex hull results for the mixed-binary RLT are a special case of our more general results for simplicial facets.

As a final remark for this subsection, notice that for $\boldsymbol{x} \in \mathbb{R}^n$ the realizations $\boldsymbol{x} \in \{0,1\}^n$ *do not* define a set of affinely independent points. Instead, they define the extreme points of a n-dimensional hypercube, which is clearly not a simplex. However, the set $\{\otimes_{i \in N} P_i \boldsymbol{x}'_i\}_L \geq \mathbf{0}$ *is* a simplex in higher dimensions and yet preserves that $\boldsymbol{x} \in \{0,1\}^n$ at all extreme points. Thus, we can view the highest level of the mixed-binary RLT as an elegant method of lifting the original realizations $\boldsymbol{x} \in \{0,1\}^n$ into a higher- dimensional space so as to achieve affine independence and hence acquire a convex hull via the resulting simplicial facets.

3.4.2 Insights for Mixed-Discrete RLT

Recall from Section 3.1.3 the mixed-discrete set (3.5) having a single discrete x. For convenience, this set is restated below:

$$X_D \equiv \{(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^m : Ax + B\boldsymbol{y} \geq \boldsymbol{d}, \mathbf{0} \leq \boldsymbol{y} \leq \mathbf{1}, x \in S\}$$

where $S = \{\theta_1, \theta_2, \ldots, \theta_k\}$ defines the realizations of x. As in the previous subsection, we show how the machinery of Section 3.3 can be used to generate an equivalent, linearized version of the LIP functional products $C\dot{\boldsymbol{x}} \geq \mathbf{0}$ of (3.7) and the associated simplifying identities $([C]_i\dot{\boldsymbol{x}})x = ([C]_i\dot{\boldsymbol{x}})\theta_i$ of (3.9) via the facets of a particular k-dimensional simplex.

Begin by noting that if $k = 2$ then the points in S are affinely independent so that the associated LIPs naturally form a 1-dimensional simplex. However, if $k \geq 3$, meaning that x realizes 3 or more values, then the 1-dimensional points in S are certainly *not* affinely independent. Thus, we are tasked with "lifting" these points into a higher-dimensional space so that they acquire affine independence. One natural way to accomplish this is by defining a related set

$$\Theta = \{\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \ldots, \boldsymbol{\theta}^k\} = \left\{ \begin{pmatrix} \theta_1 \\ \theta_1^2 \\ \vdots \\ \theta_1^{k-1} \end{pmatrix}, \begin{pmatrix} \theta_2 \\ \theta_2^2 \\ \vdots \\ \theta_2^{k-1} \end{pmatrix}, \ldots, \begin{pmatrix} \theta_k \\ \theta_k^2 \\ \vdots \\ \theta_k^{k-1} \end{pmatrix} \right\} \tag{3.27}$$

in $k-1$ dimensions where the "extra" dimensions represent the original realizations raised to higher powers. In the manner of (3.12) and (3.13), the set Θ can be used to form the $k \times k$ inverse matrix $F^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \cdots & \theta_k \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{k-1} & \theta_2^{k-1} & \cdots & \theta_k^{k-1} \end{bmatrix}$, which has the structure of a transposed Vandermonde matrix. As shown in [1], the inverse of this Vandermonde matrix has the form of C of (3.7).

Now, recall that $\dot{\boldsymbol{x}} = (1, x, x^2, \ldots, x^{k-1})^T$ is a vector containing nonlinear terms. Thus, the LIP functional products $C\dot{\boldsymbol{x}} \geq \mathbf{0}$ of (3.7) are also nonlinear and hence are not facets. However, by defining the set of affinely independent points Θ as in (3.27), we have formed an associated $(k-1)$-dimensional simplicial system $C\boldsymbol{x}' \geq \mathbf{0}$ such that at every extreme point we have $\boldsymbol{x}' = \dot{\boldsymbol{x}}$ for $\dot{\boldsymbol{x}}$ evaluated at some $x \in S$. Thus, these facets are the linearized LIP functional products of (3.7).

It remains to show that the LIP simplifying identities $([C]_i\dot{\boldsymbol{x}})x = ([C]_i\dot{\boldsymbol{x}})\theta_i$ of (3.9) are equivalent to the identities $([C]_i\boldsymbol{x}')\boldsymbol{x} = ([C]_i\boldsymbol{x}')\boldsymbol{\theta}^i$ of (3.17) for general simplices. To see this, remember that at each of the k extreme points of the simplex we have $\boldsymbol{x}' = \dot{\boldsymbol{x}}$ for $\dot{\boldsymbol{x}}$ evaluated at some $x \in S$. We also have that the first element of each $\boldsymbol{\theta}^i$ is equal to the original $\theta_i \in S$. Hence, the simplifying identities (3.9) naturally result from (3.17) when the simplex is defined using (3.27), implying that the LIP functional products and simplifying identities are special cases of our simplicial facets results.

Next, suppose we redefine the set X_D so that instead of having a single discrete variable x, it instead contains n discrete variables in the vector $\boldsymbol{x}$. Then, given $N = \{1, 2, \ldots, n\}$ and

$S_j = \{\theta_{j1}, \ldots, \theta_{jk_j}\}$ for all $j \in N$, X_D appears as

$$X_D \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, x_j \in S_j \quad \forall j \in N\}.$$

In this case, the RLT for mixed-discrete programs yields a convex hull representation by forming functional products as $\otimes_{j \in N} C_j \dot{\boldsymbol{x}}_j$ (the Kronecker product of the LIP functional products of the individual variables) and by enforcing the simplifying identities $([C_j]_i \dot{\boldsymbol{x}}_j) x_j = ([C_j]_i \dot{\boldsymbol{x}}_j) \theta_{ji}$ for all $i \in K_j = \{1, \ldots, k_j\}$ and $j \in N$.

This case can also be viewed as a special case of our simplicial facet results from the previous section. Since, for each x_j, the associated system of LIP functional products $C_j \dot{\boldsymbol{x}}_j \geq \boldsymbol{0}$ is equivalent to the simplex $C_j \boldsymbol{x}'_j \geq \boldsymbol{0}$, it follows that the functional products $\otimes_{j \in N} C_j \dot{\boldsymbol{x}}_j \geq \boldsymbol{0}$ are equivalent to the system $\{\otimes_{j \in N} C_j \boldsymbol{x}'_j\}_L \geq \boldsymbol{0}$ which, by Theorem 3.1, defines a $(\prod_{j \in N} k_j - 1)$-dimensional simplex whose $\prod_{j \in N} k_j$ extreme points are given by the columns of $\otimes_{j \in N} C_j^{-1}$ less the first row. The structure of $\otimes_{j \in N} C_j^{-1}$ ensures that every extreme points has each $x_j \in S_j$ with each linearized product term equaling the product of the individual variables. Thus, if we define the matrix $F = \otimes_{j \in N} C_j$ and the vector $\boldsymbol{x}' = \otimes_{j \in N} \{\boldsymbol{x}'_j\}_L$, then the simplicial facets $F\boldsymbol{x}'$ are equivalent to the linearized form of the functional products $\otimes_{j \in N} C_j \dot{\boldsymbol{x}}_j$ for the mixed-discrete RLT.

Finally, it remains to show that the general identity (3.17) inherently gives rise to the mixed-binary RLT identities $([C_j]_i \dot{\boldsymbol{x}}_j) x_j = ([C_j]_i \dot{\boldsymbol{x}}_j) \theta_{ji}$ for all $i \in K_j$ and $j \in N$. To accomplish this, pick any $j \in N$ and $i \in K_j$. Then consider the vector of facets $\{([C_j]_i \boldsymbol{x}'_j) \otimes_{\ell \in N - \{j\}} C_\ell \boldsymbol{x}'_\ell\}_L$. By Theorem 3.1, the system $\{\otimes_{\ell \in N - \{j\}} C_\ell \boldsymbol{x}'_\ell\}_L \geq \boldsymbol{0}$ is itself a simplex and Lemma 3.1 states that the surrogation of the facets of this simplex results in the scalar value 1. Hence, it follows that

$$\boldsymbol{1}^T \{([C_j]_i \boldsymbol{x}'_j) \otimes_{\ell \in N - \{j\}} C_\ell \boldsymbol{x}'_\ell\}_L = \{[C_j]_i \boldsymbol{x}'_j\}_L = [C_j]_i \boldsymbol{x}'_j. \tag{3.28}$$

Judiciously choosing the appropriate identities of (3.17) for this problem gives

$$\{([C_j]_i \boldsymbol{x}'_j) \otimes_{\ell \in N - \{j\}} C_\ell \boldsymbol{x}'_\ell\}_L \cdot (x_j) = \{([C_j]_i \boldsymbol{x}'_j) \otimes_{\ell \in N - \{j\}} C_\ell \boldsymbol{x}'_\ell\}_L \cdot (\theta_{ji}). \tag{3.29}$$

Applying (3.28) to (3.29) results in the identity $([C_j]_i \boldsymbol{x}'_j)(x_j) = ([C_j]_i \boldsymbol{x}'_j)(\theta_{ji})$ exactly as desired for each $i \in K_j$ and $j \in N$. Via the equivalence of $C_j \dot{\boldsymbol{x}}_j \geq \boldsymbol{0}$ and $C_j \boldsymbol{x}'_j \geq \boldsymbol{0}$, we have established that the convex hull for the mixed-binary RLT is a special case of Ω of (3.21).

3.4.3 Insights for Special Structure RLT

As a final insight, we briefly turn our attention to the work of [4]. Here, the goal was to develop a new, unifying hierarchy for mixed-binary problems by utilizing explicit and/or implicit valid inequalities in the binary variables. In particular, the paper focused on a mixed-binary set

$$X_{SS} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : A\boldsymbol{x} + B\boldsymbol{y} \geq \boldsymbol{d}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{1}, \boldsymbol{1}^T\boldsymbol{x} \leq 1, \boldsymbol{x} \in \{0,1\}^n\}.$$

The paper demonstrated that by taking functional products of the form $1 - \boldsymbol{1}^T\boldsymbol{x} \geq 0$ and $x_i \geq 0$ for all $i \in N = \{1, \ldots, n\}$, and enforcing the simplifying inequalities $x_i^2 = x_i$ for all $i \in N$, then the convex hull of the set X_{SS} could be obtained. Note that this yields a much more efficient convex hull description than the original binary RLT where, as described in Section 3.4.1, the product factors are formed by taking the Kronecker product of the products factors of every individual variable.

This, too, can be viewed as a special case of our simplicial facets results in the following manner. Observe that, given binary $\boldsymbol{x}$, the constraint $\boldsymbol{1}^T\boldsymbol{x} \leq 1$ allows at most only a single x_i to realize value 1 while the rest must be 0. In the spirit of (3.12), these $n+1$ feasible points are given by the set

$$\Theta = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \tag{3.30}$$

and are trivially affinely independent. Using the set (3.30), the associated simplex description of (3.13) is given by $SP = \{\boldsymbol{x}' \in \mathbb{R}^{n+1} : F\boldsymbol{x}' \geq \boldsymbol{0}\}$ where $F = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$. Hence, the simplicial facets are of exactly the form $1 - \boldsymbol{1}^T\boldsymbol{x} \geq 0$ and $x_i \geq 0$ for all $i \in N = \{1, \ldots, n\}$. In the same manner as the previous two subsections, it can be shown that the simplifying identities $x_i^2 = x_i$ for all $i \in N$ result from the more general simplifying identities of the form (3.17). Thus, we can view these special structure RLT results from [4] as an implicit exploitation of the facets of a standard orthogonal simplex.

3.5 Conclusions

This chapter presents a new mechanism for generating convex hull representations of mixed-discrete sets where certain variables are restricted to realize a set of affinely independent vectors. This methodology hinges on the novel use of appropriately-scaled facets of a class of polytopes known as simplices. Given a set of n affinely independent realizations of the discrete variables, a simplex is the $(n-1)$-dimensional convex hull of these realizations. A convenient way to acquire the scaled simplicial facets is to form a $n \times n$ matrix, F^{-1}, whose columns consist of the n affinely independent realizations together with the value 1 appended in the first element. Then, the coefficients of the desired facets can be obtained from the rows of the inverse matrix F.

The procedure for generating convex hulls mirrors the approach of the mixed-binary and mixed-discrete RLT. Here, the simplicial facets described by F are used as functional products in a manner analogous to x and $1-x$ for the binary RLT and Lagrange interpolating polynomials for the discrete RLT. These facets enable strengthened representations via simplifying identities that, in essence, state that a facet times a discrete variable is equal to the facet times the value of the variable in that single discrete realization not lying on the facet. This property arises due to the scaling and structure of the simplicial facets which ensures that at every discrete realization a single facet has a slack value of 1 while all others have 0 slack. This property also holds for x and $1-x$ and the LIPs, implying that these structures can be viewed as special cases of the simplicial facets.

While this chapter subsumes and extends classic RLT results, it also provides new insights into the machinery promoting the older convex hull proofs. For example, the highest level of the mixed-binary RLT hierarchy provides a convex hull for $\boldsymbol{x} \in \{0,1\}^n$ despite the realizations of this set not being affinely independent for $n > 1$. In essence, the binary RLT operates by using product terms amongst the binary variables to lift the original realizations into the exact higher-dimensional space needed to achieve affine independence so that simplicial facets can be employed to obtain the convex hull. In a similar manner, the LIPs for a discrete variable operate by lifting a set $\{\theta_1, \ldots, \theta_k\}$ into higher dimensions by taking successively higher integer powers of the original realizations so that affine independence is achieved. Thus, it is this affine independence of the realizations, and the accompanying invertibility of the matrix F^{-1}, that gives rise to the convex hull representations afforded by the RLT.

Bibliography

[1] Adams, W.P. and Sherali, H.D., "A Hierarchy of Relaxations Leading to the Convex Hull Representation for General Discrete Optimization Problems," *Annals of Operations Research*, Vol. 140, No. 1, 21-47, 2005.

[2] Sherali, H.D. and Adams, W.P., "A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-One Programming Problems," *SIAM Journal on Discrete Mathematics*, Vol. 3, Issue 3, 411-430, 1990.

[3] Sherali, H.D. and Adams, W.P., "A Hierarchy of Relaxations and Convex Hull Characterizations for Mixed-Integer Zero-One Programming Problems," *Discrete Applied Mathematics* Vol. 52, No. 1, 83-106, 1994.

[4] Sherali, H.D., Adams, W.P., and Driscoll, P.J., "Exploiting Special Structures in Constructing a Hierarchy of Relaxations of 0-1 Mixed Integer Problems," *Operations Research*, Vol. 46, Issue 3, 396-405, 1998.

Chapter 4

Base-2 Expansions for Linearizing Products of Functions of Discrete Variables

Consider a discrete variable x that can realize values in the finite set $S = \{\theta_1, \theta_2, \ldots, \theta_n\}$. It is well known that x can be expressed in terms of n binary variables $\boldsymbol{\lambda}^T = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as

$$x = \sum_{j=1}^{n} \theta_j \lambda_j, \ \boldsymbol{\lambda} \in \Lambda, \tag{4.1}$$

where

$$\Lambda \equiv \left\{ \boldsymbol{\lambda} \in \mathbb{R}^n : \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \text{ binary for } j = 1, \ldots, n \right\}. \tag{4.2}$$

Moreover, given that x is an integer with $\theta_j = \theta_{j-1} + 1$ for $j = 2, \ldots, n$, then x can be alternately defined as in [6] by

$$x = \theta_1 + \sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k, \ x \leq \theta_n, \ u_k \text{ binary for } k = 1, \ldots, \lceil \log_2 n \rceil. \tag{4.3}$$

Of course, if $\lceil \log_2 n \rceil = \log_2 n$, then the inequality $x \leq \theta_n$ of (4.3) is not needed. (Throughout this chapter, we find it convenient to denote sums from 1 to n using the index j and sums from 1 to

$\lceil \log_2 n \rceil$ using the index k.)

An obvious difference between (4.1) and (4.3) is that the former requires n binary variables whereas the latter uses only $\lceil \log_2 n \rceil$. In this study, we represent *functions* of discrete variables in terms of logarithmic numbers of binary variables, and use these representations to linearize products of such functions. A recent work [4] has contributed two such linearizations by defining auxiliary continuous variables and linear constraints. The methods vary in their construction. This raises the following two-part question. Given a discrete variable x that can realize a finite number of values in some arbitrary set S, how can x be most economically represented, and how can such a representation be used to linearize products of discrete functions?

We use a simple observation relative to the unit hypercube to address this question so as to efficiently represent x and any associated function $f(x)$, and ultimately to represent products of such functions. As a consequence, we are able to improve upon the contributions of [4] relative to the linearization of monomial terms of discrete variables, as well as to mixed-integer generalized geometric programs. This chapter is in the spirit of [5], which presents an interesting study on the use of logarithmic numbers of binary variables to model disjunctive constraints, focusing on SOS1 and SOS2 type restrictions.

4.1 Base-2 Representations of Discrete Variables and Functions

In this section, we represent a discrete variable $x \in S = \{\theta_1, \theta_2, \ldots, \theta_n\}$ in terms of $\lceil \log_2 n \rceil$ binary variables, n nonnegative continuous variables, and $\lceil \log_2 n \rceil + 1$ linear equality restrictions. The representation is then shown to extend to functions of this variable, as well as to the product of such functions with a nonnegative variable. The study relies on the following elementary observation, stated without proof due to its simplicity.

Observation

Given any positive integer p, a binary vector $\boldsymbol{u} \in \mathbb{R}^p$ can be represented as a convex combination of a select subset of $n \leq 2^p$ distinct extreme points of the unit hypercube in $\mathbb{R}^p$ if and only if the vector $\boldsymbol{u}$ is itself one of the selected extreme points, with a single convex multiplier equaling 1, and the remaining $n-1$ multipliers equaling 0.

For our purposes, a useful implementation of this observation is the following. Consider the n extreme points $\boldsymbol{v}_j$, $j \in \{1, \ldots, n\}$, of the unit hypercube in $\mathbb{R}^{\lceil \log_2 n \rceil}$, defined as follows. Each vector $\boldsymbol{v}_j \in \mathbb{R}^{\lceil \log_2 n \rceil}$ is the base-2 expansion of the number $j-1$ where the entry i corresponds to the value 2^{i-1}. Let $\boldsymbol{\lambda} \in \mathbb{R}^n$ serve as convex multipliers of these points $\boldsymbol{v}_j$. Then the observation gives us, with $p = \lceil \log_2 n \rceil$, that $\boldsymbol{\lambda} \in \Lambda$ of (4.2) if and only if there exists a vector $\boldsymbol{u} \in \mathbb{R}^{\lceil \log_2 n \rceil}$ so that $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \Lambda'$, where

$$\Lambda' \equiv \left\{ (\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n : \sum_{j=1}^{n} \lambda_j = 1, \ \sum_{j=1}^{n} \boldsymbol{v}_j \lambda_j = \boldsymbol{u}, \ \boldsymbol{u} \text{ binary}, \ \boldsymbol{\lambda} \geq \boldsymbol{0} \right\}. \tag{4.4}$$

Consequently, (4.4) provides a mechanism for replacing the restrictions $\boldsymbol{\lambda} \in \Lambda$ of (4.2) in n binary variables with $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \Lambda'$ in $\lceil \log_2 n \rceil$ binary variables. This gives us that x described in (4.1) and (4.2) can be expressed with $\lceil \log_2 n \rceil$ binary variables $\boldsymbol{u}$, n nonnegative continuous variables $\boldsymbol{\lambda}$, and $\lceil \log_2 n \rceil + 1$ equality constraints from (4.4) in $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ as

$$x = \sum_{j=1}^{n} \theta_j \lambda_j, \ (\boldsymbol{u}, \boldsymbol{\lambda}) \in \Lambda'. \tag{4.5}$$

It is instructive to note cases of S for which (4.5) can be simplified so as to not include the $\boldsymbol{\lambda}$ variables. Define the $(\lceil \log_2 n \rceil + 1) \times n$ matrix V whose j^{th} column is given by $\begin{bmatrix} 1 \\ \boldsymbol{v}_j \end{bmatrix}$ so that the equations of Λ' can be written as

$$V \boldsymbol{\lambda} = \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}. \tag{4.6}$$

Now suppose that the vector $\boldsymbol{\theta}^T = (\theta_1, \theta_2, \ldots, \theta_n)$ can be written as a linear combination of the rows of V using multipliers $\boldsymbol{\alpha}^T \equiv (\alpha_0, \alpha_1, \ldots, \alpha_{\lceil \log_2 n \rceil})$ so that $\boldsymbol{\alpha}^T V = \boldsymbol{\theta}^T$. Then (4.5) simplifies to

$$x = \alpha_0 + \sum_{k=1}^{\lceil \log_2 n \rceil} \alpha_k u_k, \ \boldsymbol{u} \text{ binary}, \ (\boldsymbol{u}, \boldsymbol{\lambda}) \in \Lambda'. \tag{4.7}$$

As x in (4.7) is described entirely in terms of $\boldsymbol{u}$, the variables $\boldsymbol{\lambda}$ simply ensure that the $\boldsymbol{u}$ vector is a column $\boldsymbol{v}_j$ corresponding to the binary expansion of some integer between 0 and $n-1$. Then (4.7) can be rewritten as

$$x = \alpha_0 + \sum_{k=1}^{\lceil \log_2 n \rceil} \alpha_k u_k, \sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n - 1, \ \boldsymbol{u} \text{ binary.} \tag{4.8}$$

Similar to (4.3), if $\lceil \log_2 n \rceil = \log_2 n$, then the last inequality is unnecessary. For the special case where x is integer with $\theta_j = \theta_{j-1} + 1$ for $j = 2, \ldots, n$, we have $\alpha_0 = \theta_1$ and $\alpha_k = 2^{k-1}$ for $k = 1, \ldots, \lceil \log_2 n \rceil$, reducing (4.8) to (4.3).

Example 4.1

Let $x \in S \equiv \{2, 3, 5, 7, 8\}$ so that $n = 5$, $\lceil \log_2 n \rceil = 3$, and $\boldsymbol{\theta} = (2, 3, 5, 7, 8)^T$. Arranging the vectors $(1, \boldsymbol{v}_j)^T$ as the columns of V, we obtain that (4.5) can be written as

$$x = 2\lambda_1 + 3\lambda_2 + 5\lambda_3 + 7\lambda_4 + 8\lambda_5, \quad (\boldsymbol{u}, \boldsymbol{\lambda}) \in \Lambda'$$

where

$$\Lambda' = \left\{ (\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^3 \times \mathbb{R}^5 : V\boldsymbol{\lambda} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \ \boldsymbol{u} \text{ binary}, \ \boldsymbol{\lambda} \geq \boldsymbol{0} \right\}.$$

There exists no $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}^T V = \boldsymbol{\theta}^T$ and hence the $\boldsymbol{\lambda}$ variables cannot be removed. If, however, $S = \{2, 3, 5, 6, 8\}$, then $\boldsymbol{\alpha}^T V = \boldsymbol{\theta}^T$ for $\boldsymbol{\alpha}^T = (2, 1, 3, 6)$ and we can obtain (4.8) with

$$x = 2 + u_1 + 3u_2 + 6u_3, \ u_1 + 2u_2 + 4u_3 \leq 4, \ \boldsymbol{u} \text{ binary.}$$

Now, observe that (4.5) can be extended to express any function $f(x)$ of the discrete variable x, as well as the product of x and/or any such $f(x)$ with a nonnegative variable κ, in terms of the same $\lceil \log_2 n \rceil$ binary variables $\boldsymbol{u}$. Relative to the function $f(x)$, define a variable, say y, and include the linear equation

$$y = \sum_{j=1}^{n} f(\theta_j) \lambda_j \tag{4.9}$$

in (4.5). This equation forces y to equal $f(x)$ for binary $\boldsymbol{u}$. The products $x\kappa$ and $f(x)\kappa$ for nonnegative κ rely on a modification of (4.4). Suppose that each restriction in Λ' (exclusive of $\boldsymbol{u}$ binary) is multiplied by the nonnegative κ to obtain the system $\Gamma(\kappa)$ below, where we use variables $\boldsymbol{\gamma}$ to denote the scaled $\boldsymbol{\lambda}$.

$$\Gamma(\kappa) \equiv \left\{ (\boldsymbol{u}, \boldsymbol{\gamma}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n : \sum_{j=1}^{n} \gamma_j = \kappa,\ \sum_{j=1}^{n} \boldsymbol{v}_j \gamma_j = \boldsymbol{u}\kappa,\ \ \boldsymbol{u} \text{ binary},\ \boldsymbol{\gamma} \geq \mathbf{0} \right\}. \tag{4.10}$$

Then, since (4.10) is a scaling of the equations in (4.4), we have for any nonnegative realization of κ that the expressions $\sum_{j=1}^{n} \theta_j \gamma_j$ and $\sum_{j=1}^{n} f(\theta_j)\gamma_j$, which are scaled versions of that found in (4.5) and (4.9) respectively, will equal the products $x\kappa$ and $y\kappa$.

A drawback of (4.10) is that $\lceil \log_2 n \rceil$ of the equations contain quadratic terms, as found in the vector $\boldsymbol{u}\kappa$. These terms can be linearized via a procedure of Glover [2] that replaces $\boldsymbol{u}\kappa$ with a vector of continuous variables $\boldsymbol{w}$, and enforces $\boldsymbol{w} = \boldsymbol{u}\kappa$ using the $4\lceil \log_2 n \rceil$ inequalities below. Here κ^- and κ^+ are lower and upper bounds on the permissible values of κ, and $\mathbf{1}$ represents a vector of ones in $\mathbb{R}^{\lceil \log_2 n \rceil}$.

$$\kappa^- \boldsymbol{u} \leq \boldsymbol{w} \leq \kappa^+ \boldsymbol{u} \text{ and } \kappa \mathbf{1} - \kappa^+ (\mathbf{1} - \boldsymbol{u}) \leq \boldsymbol{w} \leq \kappa \mathbf{1} - (\mathbf{1} - \boldsymbol{u})\kappa^- \tag{4.11}$$

For each $k \in \{1, \ldots, \lceil \log_2 n \rceil\}$, if $u_k = 0$, the left-hand inequalities enforce $w_k = 0$ and the right-hand inequalities are redundant, while if $u_k = 1$, the right-hand inequalities enforce $w_k = \kappa$ and the left-hand inequalities are redundant.

We denote the linearized version of $\Gamma(\kappa)$ where $\boldsymbol{w}$ is substituted in (4.10) for $\boldsymbol{u}\kappa$ using (4.11) by $\Gamma'(\kappa)$, as given below.

$$\Gamma'(\kappa) \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, \boldsymbol{\gamma}, \boldsymbol{w}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n \times \mathbb{R}^{\lceil \log_2 n \rceil} : \\ \qquad \sum_{j=1}^{n} \gamma_j = \kappa,\ \sum_{j=1}^{n} \boldsymbol{v}_j \gamma_j = \boldsymbol{w},\ \boldsymbol{u} \text{ binary}, \boldsymbol{\gamma} \geq \mathbf{0}, \\ \qquad \kappa^- \boldsymbol{u} \leq \boldsymbol{w} \leq \kappa^+ \boldsymbol{u} \text{ and } \kappa \mathbf{1} - \kappa^+ (\mathbf{1} - \boldsymbol{u}) \leq \boldsymbol{w} \leq \kappa \mathbf{1} - (\mathbf{1} - \boldsymbol{u})\kappa^- \end{array} \right\} \tag{4.12}$$

Concise representations of the form given by (4.7) that do not require any variables $\boldsymbol{\lambda}$ can also be obtained for special cases of $f(x)$, and concise representations that do not require any variables $\boldsymbol{\gamma}$ can be similarly obtained for special cases of the functions $x\kappa$ and $f(x)\kappa$. Observe that $x\kappa$ can be expressed in such a concise form if and only if x can be so represented; that is, if and only if $\boldsymbol{\theta}^T$ can be expressed as a linear combination of the rows of V. In an analogous manner,

$f(x)$ and $f(x)\kappa$ can be expressed without variables $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ respectively if and only if the vector $\boldsymbol{f}^T = (f(\theta_1), f(\theta_2), \ldots, f(\theta_n))$ can be expressed as a linear combination of the rows of V. Of course, if it is desired to express either *both* x and $f(x)$ without variables $\boldsymbol{\lambda}$ and/or *both* $x\kappa$ and $f(x)\kappa$ without variables $\boldsymbol{\gamma}$, then *both* vectors $\boldsymbol{\theta}^T$ and $\boldsymbol{f}^T$ must be able to be expressed as linear combinations of the rows of V.

4.2 Base-2 Representations of Products of Discrete Functions

The strategy of (4.4) and (4.10) to transform the n binary $\boldsymbol{\lambda}$ and the n binary $\boldsymbol{\gamma}$ to non-negative continuous variables through the defining of $\lceil \log_2 n \rceil$ new binary $\boldsymbol{u}$, combined with the linearization of the expressions $\boldsymbol{u}\kappa$ of (4.10) via (4.11) to obtain (4.12), can be used to construct concise mixed 0-1 linear representations of products of functions of discrete variables. This construction yields representations that dominate the two methods of [4] in terms of numbers of constraints, while affording improved relaxation strength relative to the first approach and equivalent strength relative to the second.

Consider m functions $f_\ell(x_\ell)$, $\ell \in \{1, \ldots, m\}$, where $x_\ell \in S_\ell \equiv \{\theta_{\ell 1}, \theta_{\ell 2}, \ldots, \theta_{\ell n_\ell}\}$ and where n_ℓ denotes the number of realizations of x_ℓ. Here, we subscript the function $f(x)$, the variable x, the set S, and the multiplier κ of the previous sections with the index ℓ to denote the m different functions. Also, we let $\theta_{\ell j}$ denote the j^{th} realization of the variable x_ℓ. We further construct sets Λ'_ℓ and $\Gamma'_\ell(\kappa_\ell)$ of the form (4.4) and (4.12) respectively, one corresponding to each function $f_\ell(x_\ell)$, and accordingly apply the subscript ℓ to the variables $\boldsymbol{u}$, $\boldsymbol{\lambda}$, $\boldsymbol{\gamma}$, and $\boldsymbol{w}$, as well as to the vectors $\boldsymbol{v}_j$, to obtain the sets, for each $\ell \in \{1, \ldots, m\}$, given as

$$\Lambda'_\ell \equiv \left\{ (\boldsymbol{u}_\ell, \boldsymbol{\lambda}_\ell) \in \mathbb{R}^{\lceil \log_2(n_\ell) \rceil} \times \mathbb{R}^{n_\ell} : \sum_{j=1}^{n_\ell} \lambda_{\ell j} = 1,\ \sum_{j=1}^{n_\ell} \boldsymbol{v}_{\ell j} \lambda_{\ell j} = \boldsymbol{u}_\ell,\ \boldsymbol{u}_\ell \text{ binary},\ \boldsymbol{\lambda}_\ell \geq \mathbf{0} \right\},$$

and

$$\Gamma'_\ell(\kappa_\ell) \equiv \left\{ \begin{array}{l} (\boldsymbol{u}_\ell, \boldsymbol{\gamma}_\ell, \boldsymbol{w}_\ell) \in \mathbb{R}^{\lceil \log_2(n_\ell) \rceil} \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{\lceil \log_2(n_\ell) \rceil} : \\ \qquad \sum_{j=1}^{n_\ell} \gamma_{\ell j} = \kappa_\ell,\ \sum_{j=1}^{n_\ell} \boldsymbol{v}_{\ell j} \gamma_{\ell j} = \boldsymbol{w}_\ell,\ \boldsymbol{u}_\ell \text{ binary},\ \boldsymbol{\gamma}_\ell \geq \boldsymbol{0}, \\ \qquad \kappa_\ell^- \boldsymbol{u}_\ell \leq \boldsymbol{w}_\ell \leq \kappa_\ell^+ \boldsymbol{u}_\ell \text{ and } \kappa_\ell \boldsymbol{1} - \kappa_\ell^+ (\boldsymbol{1} - \boldsymbol{u}_\ell) \leq \boldsymbol{w}_\ell \leq \kappa_\ell \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{u}_\ell) \kappa_\ell^- \end{array} \right\} \quad (4.13)$$

where κ_ℓ^- and κ_ℓ^+ denote lower and upper bounds on the values of κ_ℓ.

By the logic of the previous sections, for each $\ell \in \{1, \ldots, m\}$, the variable x_ℓ and function $f_\ell(x_\ell)$ can be expressed as in (4.5) and (4.9) by

$$x_\ell = \sum_{j=1}^{n_\ell} \theta_{\ell j} \lambda_{\ell j} \text{ and } y_\ell = \sum_{j=1}^{n_\ell} f_\ell(\theta_{\ell j}) \lambda_{\ell j},\ (\boldsymbol{u}_\ell, \boldsymbol{\lambda}_\ell) \in \Lambda'_\ell, \quad (4.14)$$

where $y_\ell = f_\ell(x_\ell)$, and the products $x_\ell \kappa_\ell$ and $f_\ell(x_\ell)\kappa_\ell$ can be expressed by

$$x_\ell \kappa_\ell = \sum_{j=1}^{n_\ell} \theta_{\ell j} \gamma_{\ell j} \text{ and } f_\ell(x_\ell)\kappa_\ell = \sum_{j=1}^{n_\ell} f_\ell(\theta_{\ell j}) \gamma_{\ell j},\ (\boldsymbol{u}_\ell, \boldsymbol{\gamma}_\ell, \boldsymbol{w}_\ell) \in \Gamma'_\ell(\kappa_\ell). \quad (4.15)$$

If desired, the products $x_\ell \kappa_\ell$ and $f(x_\ell)\kappa_\ell$ can each be replaced in (4.15) by continuous variables.

We now focus on a representation of the product $\prod_{j=1}^m f_j(x_j)$ using the sets Λ'_ℓ and $\Gamma'_\ell(\kappa_\ell)$ from above. To begin, for each $\ell \in \{2, \ldots, m\}$, we represent the product $f_1(x_1)f_2(x_2)$ by a continuous variable y_{12}, the product $f_1(x_1)f_2(x_2)f_3(x_3)$ by a variable y_{123}, and so on up to the product $f_1(x_1)f_2(x_2)\cdots f_m(x_m)$ by a variable $y_{12\cdots m}$. For ease of notation, for each $\ell \in \{1, \ldots, m\}$, let $J_\ell = 1 \cdots \ell$ denote consecutive subscript indices so that $\prod_{j=1}^{\ell} f_j(x_j)$ is represented by the variable y_{J_ℓ} (with $y_1 = y_{J_1}$). As additional notation, for each $\ell \in \{1, \ldots, m-1\}$, denote computed lower and upper bounds on the product $\prod_{j=1}^{\ell} f_j(x_j)$ by $f_{J_\ell}^-$ and $f_{J_\ell}^+$ respectively. Continue by constructing Λ'_ℓ and expressing the variables x_ℓ and y_ℓ as in (4.14) for each $\ell \in \{1, \ldots, m\}$. Then compute $\Gamma'_\ell(\kappa_\ell)$ of (4.13) for each $\ell \in \{2, \ldots, m\}$ with the nonnegative scalar κ_ℓ given by $\kappa_\ell = \prod_{j=1}^{\ell-1} f_j(x_j) - f_{J_{\ell-1}}^-$. Such κ_ℓ have lower and upper bounds of $\kappa_\ell^- = 0$ and $\kappa_\ell^+ = f_{J_{\ell-1}}^+ - f_{J_{\ell-1}}^-$ respectively. The resulting system follows where, for each $\ell \in \{2, \ldots, m\}$, we have included explicit restrictions that $\kappa_\ell = y_{J_{\ell-1}} - f_{J_{\ell-1}}^-$, with $y_{J_{\ell-1}}$ substituted for the linearized version of $\prod_{j=1}^{\ell-1} f_j(x_j)$.

$$x_\ell = \sum_{j=1}^{n_\ell} \theta_{\ell j}\lambda_{\ell j},\ y_\ell = \sum_{j=1}^{n_\ell} f_\ell(\theta_{\ell j})\lambda_{\ell j},\ (\boldsymbol{u}_\ell, \boldsymbol{\lambda}_\ell) \in \Lambda'_\ell \ \ \forall\, \ell = 1, \ldots, m \tag{4.16}$$

$$\kappa_\ell = y_{J_{\ell-1}} - f^-_{J_{\ell-1}} \ \ \forall\, \ell = 2, \ldots, m \tag{4.17}$$

$$y_{J_\ell} = \sum_{j=1}^{n_\ell} f_\ell(\theta_{\ell j})\gamma_{\ell j} + y_\ell f^-_{J_{\ell-1}}, (\boldsymbol{u}_\ell, \boldsymbol{\gamma}_\ell, \boldsymbol{w}_\ell) \in \Gamma'_\ell(\kappa_\ell) \ \ \forall\, \ell = 2, \ldots, m \tag{4.18}$$

Note that the $\boldsymbol{u}_\ell$ binary restrictions for $\ell \in \{2, \ldots, m\}$ are found in both (4.16) and (4.18) but need only be stated once.

Upon substituting $\kappa_\ell = y_{J_{\ell-1}} - f^-_{J_{\ell-1}}$ for each $\ell \in \{2, \ldots, m\}$ from (4.17) into (4.18) and then removing (4.17), the counts on the types and numbers of variables in (4.16) and (4.18) are summarized in Table 4.1. Summing relevant entries, Table 4.1 gives that (4.16) and (4.18) have a total of $3m - 1 + n_1 + 2\sum_{\ell=2}^m n_\ell + \sum_{\ell=2}^m \lceil \log_2(n_\ell) \rceil$ continuous variables and $\sum_{\ell=1}^m \lceil \log_2(n_\ell) \rceil$ binary variables.

Table 4.1: Variable types and counts in (4.16) and (4.18).

Variable name	Variable type	Number of such variables
x_ℓ	continuous	m
y_ℓ	continuous	m
$y_{J_\ell}, \ell \neq 1$	continuous	$m - 1$
$\boldsymbol{\lambda}_\ell$	continuous	n_ℓ for each $\ell \in \{1, \ldots, m\}$
$\boldsymbol{\gamma}_\ell$	continuous	n_ℓ for each $\ell \in \{2, \ldots, m\}$
$\boldsymbol{w}_\ell$	continuous	$\lceil \log_2(n_\ell) \rceil$ for each $\ell \in \{2, \ldots, m\}$
$\boldsymbol{u}_\ell$	binary	$\lceil \log_2(n_\ell) \rceil$ for each $\ell \in \{1, \ldots, m\}$

Relative to the number of constraints in (4.16) and (4.18), a count is as follows. Each set Λ'_ℓ of (4.16) has $\lceil \log_2(n_\ell) \rceil + 1$ restrictions, while each set $\Gamma'_\ell(\kappa_\ell)$ of (4.18) with κ_ℓ as defined in (4.17) has $5\lceil \log_2(n_\ell) \rceil + 1$ restrictions. Including the additional $2m$ equalities defining x_ℓ and y_ℓ of (4.16) and the $m - 1$ equalities defining y_{J_ℓ} for $\ell \neq 1$ of (4.18), the total number of constraints is $5m - 2 + \lceil \log_2(n_1) \rceil + 6\sum_{\ell=2}^m \lceil \log_2(n_\ell) \rceil$.

The numbers of variables and constraints can be reduced, depending on the structure of the problem and the desired form of the resulting linearization. Four reduction strategies are listed below.

1. Since $\kappa^-_\ell = 0$ for each $\ell \in \{2, \ldots, m\}$, the inequalities $\kappa^-_\ell \boldsymbol{u}_\ell \leq \boldsymbol{w}_\ell$ of (4.13) become nonneg-

ativity on $\boldsymbol{w}_\ell$, reducing the number of constraints by $\sum_{\ell=2}^{m} \lceil \log_2(n_\ell) \rceil$. If some κ_ℓ is defined which allows for a strengthening of κ_ℓ^- from 0 to a positive value, then a transformation of variables $\boldsymbol{w}'_\ell = \boldsymbol{w}_\ell - \kappa_\ell^- \boldsymbol{u}_\ell$ (see [1, 3]) can be used.

2. If desired, the variables x_ℓ, y_ℓ, and y_{J_ℓ} can all be substituted from the linearization (as well as any encompassing optimization problem) by using the definition of variables in terms of $\lambda_{\ell j}$ and $\gamma_{\ell j}$ found in (4.16) and (4.18). This substitution reduces the number of variables and constraints by $3m - 1$ each.

3. Each of the sets Λ'_ℓ and $\Gamma'_\ell(\kappa_\ell)$ can be reduced in size by $\lceil \log_2(n_\ell) \rceil + 1$ variables via a transformation that changes the equality restrictions to inequality. To see this, consider Λ'_1. As the defining linear system of equations is of full rank (choose the columns corresponding to λ_{11} and $\lambda_{1(2^{p-1}+1)}$ for each $p \in \{1, \ldots, \lceil \log_2(n_1) \rceil\}$), a basis for $\mathbb{R}^{\lceil \log_2(n_1) \rceil + 1}$ can be obtained in terms of a subset of the columns of the defining system. Then the $\lceil \log_2(n_1) \rceil + 1$ basic variables can be expressed in terms of the nonbasic variables and subsequently eliminated from the formulation. Performing such a reduction on each Λ'_ℓ and $\Gamma'_\ell(\kappa_\ell)$ reduces the formulation by $2m - 1 + \lceil \log_2(n_1) \rceil + 2 \sum_{\ell=2}^{m} \lceil \log_2(n_\ell) \rceil$ continuous variables.

4. The order in which the functions are numbered and subsequently linearized affects the variable and constraint counts. The set $\Gamma'_1(\kappa_1)$ of (4.13) does not appear in (4.18), nor do the associated variables $\boldsymbol{\gamma}_1$ and $\boldsymbol{w}_1$. Therefore, selecting $f_1(x_1)$ so that $n_1 = \max\{n_\ell : \ell = 1, \ldots, m\}$ can yield a smaller formulation.

The lower and upper bounds $f_{J_\ell}^-$ and $f_{J_\ell}^+$ on the products $\prod_{j=1}^{\ell} f_j(x_j)$ for $\ell \in \{1, \ldots, m-1\}$ used in the construction of (4.16)–(4.18) can be computed in different ways. For each $\ell \in \{1, \ldots, m\}$, lower and upper bounds f_ℓ^- and f_ℓ^+ on the function $f_\ell(x_\ell)$ are readily obtained as $f_\ell^- = \min\{f_\ell(\theta_{\ell j}) : j = 1, \ldots, n_\ell\}$ and $f_\ell^+ = \max\{f_\ell(\theta_{\ell j}) : j = 1, \ldots, n_\ell\}$. Next consider the values $f_{J_\ell}^-$ and $f_{J_\ell}^+$ for $\ell \in \{2, \ldots, m-1\}$. If $f_j^- \geq 0$ for all $j \in \{1, \ldots, \ell\}$, then we can use $f_{J_\ell}^- = \prod_{j=1}^{\ell} f_j^-$ and $f_{J_\ell}^+ = \prod_{j=1}^{\ell} f_j^+$. If, however, $f_j^- < 0$ for some such j, then various options exist, including using $f_{J_\ell}^+ = \prod_{j=1}^{\ell} \max\{|f_j^-|, |f_j^+|\}$ and $f_{J_\ell}^- = -f_{J_\ell}^+$.

Three additional remarks relative to (4.16)–(4.18) are warranted. First, products of discrete variables (as opposed to products of functions of discrete variables) can be readily handled by having $f_\ell(x_\ell)$ serve as identity functions so that $f_\ell(\theta_{\ell j}) = \theta_{\ell j}$ for all $\ell \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_\ell\}$.

Then the first equation in (4.16) defining x_ℓ can be removed for each $\ell \in \{1,\ldots,m\}$, as $x_\ell = y_\ell$. Second, the linearization process that produces (4.16)–(4.18) does not depend on x_1 being discrete. This allows us to accommodate the expression $\prod_{j=1}^{m} f_j(x_j)$ when the function $f_1(x_1)$ is continuous. In this case, restrictions (4.16) with $\ell = 1$ are not used. Third, the approach of (4.16)–(4.18) does not make use of the product $f_1(x_1)\kappa_1$, so the value κ_1 and set $\Gamma_1'(\kappa_1)$ of (4.13) is not found in (4.18). Similarly, the lower and upper bounds $f_{J_m}^-$ and $f_{J_m}^+$ on $\prod_{j=1}^{m} f_j(x_j)$ are not needed.

We conclude this section with an example demonstrating the use of (4.16) and (4.18) in linearizing the monomial $x_1^3 x_2^{1.5}$.

Example 4.2

Consider the $m = 2$ functions $f_1(x_1) = x_1^3$ and $f_2(x_2) = x_2^{1.5}$, where $x_1 \in S_1 \equiv \{-1, 2, 5, 7\}$ and $x_2 \in S_2 \equiv \{2, 4, 8\}$, so that $n_1 = 4$ and $n_2 = 3$. The restrictions (4.16) and (4.18) have the continuous variables y_1, y_2, and y_{12} replacing $f_1(x_1)$, $f_2(x_2)$, and the product $f_1(x_1)f_2(x_2) = x_1^3 x_2^{1.5}$ respectively. Using matrices to simplify notation where possible, (4.16) is given by

$$x_1 = -1\lambda_{11} + 2\lambda_{12} + 5\lambda_{13} + 7\lambda_{14},\;\; y_1 = (-1)^3\lambda_{11} + 2^3\lambda_{12} + 5^3\lambda_{13} + 7^3\lambda_{14},\;\; (\boldsymbol{u}_1, \boldsymbol{\lambda}_1) \in \Lambda_1',$$

where

$$\Lambda_1' = \left\{ (\boldsymbol{u}_1, \boldsymbol{\lambda}_1) \in \mathbb{R}^2 \times \mathbb{R}^4 : \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix} = \begin{bmatrix} 1 \\ u_{11} \\ u_{12} \end{bmatrix}, \boldsymbol{u}_1 \text{ binary}, \boldsymbol{\lambda}_1 \geq \mathbf{0} \right\},$$

and

$$x_2 = 2\lambda_{21} + 4\lambda_{22} + 8\lambda_{23},\;\; y_2 = 2^{1.5}\lambda_{21} + 4^{1.5}\lambda_{22} + 8^{1.5}\lambda_{23},\;\; (\boldsymbol{u}_2, \boldsymbol{\lambda}_2) \in \Lambda_2',$$

where

$$\Lambda_2' = \left\{ (\boldsymbol{u}_2, \boldsymbol{\lambda}_2) \in \mathbb{R}^2 \times \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \\ \lambda_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ u_{21} \\ u_{22} \end{bmatrix}, \boldsymbol{u}_2 \text{ binary}, \boldsymbol{\lambda}_2 \geq \mathbf{0} \right\}.$$

Since $f_{J_1}^- = f_1^- = (-1)^3$, we have $\kappa_2 = x_1^3 - (-1)^3 = y_1 + 1$, with $\kappa_2^- = 0$ and

$\kappa_2^+ = f_1^+ - f_1^- = 7^3 - (-1)^3 = 344$. Then (4.18) becomes

$$y_{12} = 2^{1.5}\gamma_{21} + 4^{1.5}\gamma_{22} + 8^{1.5}\gamma_{23} - y_2, \ \ (\boldsymbol{u}_2, \boldsymbol{\gamma}_2, \boldsymbol{w}_2) \in \Gamma_2'(y_1+1),$$

where $\Gamma'(y_1+1)$ of (4.13) is expressed in matrix form as

$$\Gamma_2'(y_1+1) = \left\{ \begin{array}{l} (\boldsymbol{u}_2, \boldsymbol{\gamma}_2, \boldsymbol{w}_2) \in \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^2, \boldsymbol{\gamma}_2 \geq \mathbf{0} : \\ \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} y_1+1 \\ w_{21} \\ w_{22} \end{bmatrix}, \\ \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} \leq 344 \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}, \\ \qquad \begin{bmatrix} y_1+1 \\ y_1+1 \end{bmatrix} - 344 \begin{bmatrix} 1-u_{21} \\ 1-u_{22} \end{bmatrix} \leq \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} \leq \begin{bmatrix} y_1+1 \\ y_1+1 \end{bmatrix} \end{array} \right\},$$

with $\boldsymbol{u}_2$ binary not explicitly listed as it is found in Λ_2' above. Now, suppose that we change the problem so that the variable x_1 is redefined to be continuous in the interval $[-1, 7]$, and it is desired to have y_{12} represent the product of the continuous function x_1^3 having $-1 \leq x_1 \leq 7$ with the discrete-valued function $x_2^{1.5}$ having $x_2 \in S_2$; that is, $y_{12} = x_1^3 x_2^{1.5}$. Explicitly define y_1 to be x_1^3 via $y_1 = x_1^3$, and treat y_1 as a continuous function with $y_1 \in [-1, 343]$. In this case, none of the restrictions associated with (4.16) having $\ell = 1$ are needed (including Λ_1') and the values $f_1^- = -1$, $f_1^+ = 343$, $\kappa_2^- = 0$, and $\kappa_2^+ = 344$ are unchanged so that the set $\Gamma_2'(\kappa_2)$ remains the same.

4.3 Comparison with Other Methods

The size and relaxation strength of the system (4.16)–(4.18) compares favorably with alternate approaches. While there is considerable literature dealing with the linearization of nonlinear 0-1 programs and the representation of discrete variables in terms of binary variables, little attention has been given to modeling functions of discrete variables, and their products, in terms of logarithmic numbers of binary variables. We focus attention here on the two methods from Li and Lu [4], one per subsection below. These methods were reportedly designed for solving mixed-discrete generalized geometric programs.

4.3.1 Li & Lu Approach 1

Given a discrete variable x that can realize values in the set $S = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and a function $f(x)$ defined in terms of x, the first approach of [4] linearizes $f(x)$ using $\lceil \log_2 n \rceil$ binary variables and $2n+1$ linear inequalities, plus a single continuous variable y to represent $f(x)$. We temporarily adopt the notation of Section 4.1 that suppresses the subscript ℓ on the variable x, the function $f(x)$, the set S, the parameter n, the values θ_j for $j \in \{1, \ldots, n\}$, and the vectors $\boldsymbol{u}$, $\boldsymbol{\lambda}$, and $\boldsymbol{v}_j$ since a single function of a discrete variable is initially considered.

This approach of [4] can be explained in terms of ours as follows. It uses the same binary variables $\boldsymbol{u} \in \mathbb{R}^{\lceil \log_2 n \rceil}$ as (4.4) with (4.9), but in an altogether different manner. While not defining vectors $\boldsymbol{v}_j$ or variables $\boldsymbol{\lambda}$, it can be envisioned as also enforcing that $y = f(\theta_j)$ when $\boldsymbol{u} = \boldsymbol{v}_j$. (For now, we focus attention on the function $f(x)$ and later explain how the discrete variable x can be similarly handled. This method is unique in that it requires separate families of restrictions to handle each of x and $f(x)$.) For every $j \in \{1, \ldots, n\}$, it defines a linear function $A_j(\boldsymbol{u})$ of the binary variables $\boldsymbol{u}$ so that $A_j(\boldsymbol{u}) = 0$ if $\boldsymbol{u} = \boldsymbol{v}_j$ and $A_j(\boldsymbol{u}) \geq 1$ if $\boldsymbol{u} \neq \boldsymbol{v}_j$. For each such j, this is accomplished by adding to the sum $\sum_{k=1}^{\lceil \log_2 n \rceil} u_k$, the expression $1 - 2u_i$ for all i having the i^{th} component of $\boldsymbol{v}_j$ as 1. These functions can be computed using matrix multiplication as follows. Define the $(\lceil \log_2 n \rceil + 1) \times (\lceil \log_2 n \rceil + 1)$ invertible, symmetric matrix B whose $(i,j)^{th}$ element, denoted B_{ij} for all $i, j \in \{1, \ldots, \lceil \log_2 n \rceil + 1\}$, is given by

$$B_{ij} = \begin{cases} 1 & \text{if } (i = 1 \text{ and } j \neq 1) \text{ or } (i \neq 1 \text{ and } j = 1) \\ -2 & \text{if } i = j \neq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

so that

$$\begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}^T B \begin{bmatrix} 1 \\ \boldsymbol{v}_j \end{bmatrix} = A_j(\boldsymbol{u}) = \begin{bmatrix} 1 \\ \boldsymbol{v}_j \end{bmatrix}^T B \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix} \quad \forall\, j \in \{1, \ldots, n\}. \tag{4.20}$$

The left-hand equality becomes clear upon observing that the row vector $\begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}^T B \in \mathbb{R}^{\lceil \log_2 n \rceil + 1}$ has its first entry as $\sum_{k=1}^{\lceil \log_2 n \rceil} u_k$, and its i^{th} entry as $1 - 2u_{i-1}$ for each $i \in \{2, \ldots, \lceil \log_2 n \rceil + 1\}$. The right-hand equality follows from $\begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}^T B \begin{bmatrix} 1 \\ \boldsymbol{v}_j \end{bmatrix}$ being a 1×1 matrix, with B symmetric. Letting $M = f^+ - f^-$ with $f^- \equiv \min\{f(\theta_1), \ldots, f(\theta_n)\}$ and $f^+ \equiv \max\{f(\theta_1), \ldots, f(\theta_n)\}$, this formulation of [4] is as follows.

$$P \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, y) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R} : \\ \qquad f(\theta_j) - MA_j(\boldsymbol{u}) \leq y \leq f(\theta_j) + MA_j(\boldsymbol{u}) \;\; \forall\, j \in \{1, \ldots, n\}, \\ \qquad \displaystyle\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n - 1, \\ \qquad \boldsymbol{u} \text{ binary} \end{array} \right\}$$

The restrictions of P operate so that, given any binary $\boldsymbol{u}$ satisfying $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n-1$, the single $A_j(\boldsymbol{u})$ equaling 0, say $A_p(\boldsymbol{u})$, will have the two inequalities $f(\theta_p) - MA_p(\boldsymbol{u}) \leq f(x) \leq f(\theta_p) + MA_p(\boldsymbol{u})$ enforcing $y = f(\theta_p)$, and the remaining $2(n-1)$ inequalities with $A_j(\boldsymbol{u}) \geq 1$ being redundant.

Observe that P contains no variables $\boldsymbol{\lambda}$; it has a single continuous y and $\lceil \log_2 n \rceil$ binary $\boldsymbol{u}$. However, it requires $2n+1$ inequalities. In contrast, Λ' of (4.4) has n continuous $\boldsymbol{\lambda}$ and $\lceil \log_2 n \rceil$ binary $\boldsymbol{u}$, but only $\lceil \log_2 n \rceil + 1$ constraints. Recall, though, that reduction strategy 3 of Section 4.2 allows us to reduce the number of variables $\boldsymbol{\lambda}$ in Λ' by $\lceil \log_2 n \rceil + 1$. Thus, in summary, Λ' and P require the same number of binary variables, but the former uses $2n - \lceil \log_2 n \rceil$ fewer constraints at the expense of $n - \lceil \log_2 n \rceil - 2$ more continuous variables.

An important consideration when expressing any function of a discrete variable in terms of new binary variables in a mixed 0-1 linear form is the strength of the continuous relaxation. Let $\bar{\Lambda}'$ and $\bar{P}$ denote, respectively, the continuous relaxations of Λ' and P obtained by relaxing the $\boldsymbol{u}$ binary restrictions to $\mathbf{0} \leq \boldsymbol{u} \leq \mathbf{1}$. (Note that these $2\lceil \log_2 n \rceil$ inequalities are not needed in the set $\bar{\Lambda}'$, as they are implied by the other restrictions.) The theorem below shows that the set $\bar{\Lambda}'$ with (4.9) provides at least as tight a polyhedral representation, in terms of permissible values of y, as does $\bar{P}$.

Theorem 4.1: *Given any* $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in \bar{\Lambda}'$ *of (4.4), we have* $(\hat{\boldsymbol{u}}, \hat{y}) \in \bar{P}$*, where* $\hat{y} = \sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j$.

Proof. Let $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in \bar{\Lambda}'$ with $\hat{y} = \sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j$. Since for each $j \in \{1, \ldots, n\}$, $\boldsymbol{u} = \boldsymbol{v}_j$ satisfies $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n-1$, and since $\bar{\Lambda}'$ expresses $\hat{\boldsymbol{u}}$ as a convex combination $\hat{\boldsymbol{\lambda}}$ of the vectors $\boldsymbol{v}_j$, it follows that $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} \hat{u}_k \leq n-1$. Thus, the proof reduces to showing that

$$f(\theta_j) - MA_j(\hat{\boldsymbol{u}}) \leq \hat{y} \leq f(\theta_j) + MA_j(\hat{\boldsymbol{u}}) \; \forall\, j \in \{1, \ldots, n\}. \tag{4.21}$$

Toward this end, arbitrarily select any $p \in \{1, \ldots, n\}$ and consider (4.21) for $j = p$. Surrogate the equations of $\bar{\Lambda}'$, represented in matrix form as in (4.6), using the multipliers $\begin{bmatrix} 1 \\ \boldsymbol{v}_p \end{bmatrix}^T B$, and set $(\boldsymbol{u}, \boldsymbol{\lambda}) = (\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}})$, to obtain

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \hat{\lambda}_j \leq \sum_{j=1}^{n} A_j(\boldsymbol{v}_p)\hat{\lambda}_j = \begin{bmatrix} 1 \\ \boldsymbol{v}_p \end{bmatrix}^T BV\hat{\boldsymbol{\lambda}} = \begin{bmatrix} 1 \\ \boldsymbol{v}_p \end{bmatrix}^T B \begin{bmatrix} 1 \\ \hat{\boldsymbol{u}} \end{bmatrix} = A_p(\hat{\boldsymbol{u}}). \tag{4.22}$$

The inequality follows from the nonnegativity of $\hat{\boldsymbol{\lambda}}$ and because the function $A_j(\boldsymbol{v}_p)$ is defined to have $A_p(\boldsymbol{v}_p) = 0$ and $A_j(\boldsymbol{v}_p) \geq 1$ for $j \neq p$. The first equality is due to the left-hand equation of (4.20) with $\boldsymbol{u} = \boldsymbol{v}_p$, applied once for each $j \in \{1, \ldots, n\}$. The middle equality is the surrogation of the restrictions in $\bar{\Lambda}'$, and the last equality follows from the right-hand equation of (4.20) with $j = p$. Now, add the nonnegative multiple $(f^+ - f(\theta_p))$ of the inequality $\sum_{j=1, j\neq p}^{n} \hat{\lambda}_j \leq A_p(\hat{\boldsymbol{u}})$ of (4.22) to the multiple $f(\theta_p)$ of the equation $\sum_{j=1}^{n} \hat{\lambda}_j = 1$ from (4.4) to obtain

$$\sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j + \sum_{\substack{j=1 \\ j \neq p}}^{n} (f^+ - f(\theta_j))\hat{\lambda}_j \leq f(\theta_p) + (f^+ - f(\theta_p))A_p(\hat{\boldsymbol{u}})$$

which, by the nonnegativity of $(f^+ - f(\theta_j))\hat{\lambda}_j$ for all $j \neq p$ and the defining of $\hat{y} = \sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j$, establishes the right-hand inequality of (4.21) for $j = p$ because $f^+ - f(\theta_p) \leq f^+ - f^- = M$. Similarly, add the nonpositive multiple $(f^- - f(\theta_p))$ of the inequality $\sum_{j=1, j\neq p}^{n} \hat{\lambda}_j \leq A_p(\hat{\boldsymbol{u}})$ of (4.22) to the multiple $f(\theta_p)$ of the equation $\sum_{j=1}^{n} \hat{\lambda}_j = 1$ from (4.4) to obtain

$$\sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j + \sum_{\substack{j=1 \\ j \neq p}}^{n} (f^- - f(\theta_j))\hat{\lambda}_j \geq f(\theta_p) + (f^- - f(\theta_p))A_p(\hat{\boldsymbol{u}})$$

which, by the nonpositivity of $(f^- - f(\theta_j))\hat{\lambda}_j$ for all $j \neq p$ and the defining of $\hat{y} = \sum_{j=1}^{n} f(\theta_j)\hat{\lambda}_j$, establishes the left-hand inequality of (4.21) for $j = p$ since $f^- - f(\theta_p) \geq f^- - f^+ = -M$. This completes the proof. □

Note that the proof of Theorem 4.1 suggests a strengthening of the bound M used within P and $\bar{P}$. For each $j \in \{1, \ldots, n\}$, we can use $\underline{M}_j = f(\theta_j) - f^-$ and $\overline{M}_j = f^+ - f(\theta_j)$ to redefine the set P as

$$P \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, y) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R} : \\ \qquad f(\theta_j) - \underline{M}_j A_j(\boldsymbol{u}) \le y \le f(\theta_j) + \overline{M}_j A_j(\boldsymbol{u}) \;\; \forall\, j \in \{1, \ldots, n\}, \\ \qquad \displaystyle\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \le n - 1, \\ \qquad \boldsymbol{u} \text{ binary} \end{array} \right\}. \tag{4.23}$$

The set P remains unchanged with this adjustment but $\bar{P}$ is potentially tightened.

The representation of a discrete variable x, as opposed to a function $f(x)$, proceeds in an identical manner to the above. This is readily seen by defining $f(x)$ so that $f(x) = x$. The set P of (4.23) will then replace each $f(\theta_j)$ with θ_j, and each occurrence of y with x. If it is desired to represent *both* $f(x)$ and x, then $4n + 1$ associated inequalities are needed in the $\lceil \log_2 n \rceil$ binary variables $\boldsymbol{u}$, as the equation $\sum_{k=1}^{\lceil \log_2 n \rceil} u_k \le n - 1$ need not be repeated.

It is important to note that the converse of Theorem 4.1 is not true, even when the set $\bar{P}$ uses the improved values $\underline{M}_j$ and $\overline{M}_j$ as in (4.23). That is to say, there can exist a point $(\hat{\boldsymbol{u}}, \hat{y}) \in \bar{P}$ for which there exists no $\hat{\boldsymbol{\lambda}}$ having $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in \bar{\Lambda}'$ and $\hat{y} = \sum_{j=1}^{n} f(\theta_j) \hat{\lambda}_j$. An example illustrating Theorem 4.1 and the failure of its converse is below. For simplicity of presentation, we have $y = f(x) = x$ so that only one family of restrictions is required.

Example 4.3

Consider $f(x) = x$ with $x \in S \equiv \{1, 3, 5\}$ so that $n = 3$, $\lceil \log_2 n \rceil = 2$, $f^- = 1$, and $f^+ = 5$. Then (4.9) with the relaxed set $\bar{\Lambda}'$ is given by

$$y = \lambda_1 + 3\lambda_2 + 5\lambda_3, \quad (\boldsymbol{u}, \boldsymbol{\lambda}) \in \bar{\Lambda}',$$

where

$$\bar{\Lambda}' = \left\{ (\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^2 \times \mathbb{R}^3 : V\boldsymbol{\lambda} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}, \boldsymbol{\lambda} \ge \boldsymbol{0} \right\}.$$

The set $\bar{P}$, adjusted for the strengthened $\underline{M}_j$ and $\overline{M}_j$ as in (4.23), is

$$\bar{P} = \left\{ \begin{array}{rcl} (\boldsymbol{u}, y) \in \mathbb{R}^2 \times \mathbb{R} : & & \\ 1 & \leq y \leq & 1 + 4(u_1 + u_2) \\ 3 - 2(1 - u_1 + u_2) & \leq y \leq & 3 + 2(1 - u_1 + u_2) \\ 5 - 4(1 + u_1 - u_2) & \leq y \leq & 5 \\ u_1 + 2u_2 & \leq 2 & \\ 0 & \leq u_1 \leq & 1 \\ 0 & \leq u_2 \leq & 1 \end{array} \right\}.$$

For $\hat{\boldsymbol{u}}^T = (\hat{u}_1, \hat{u}_2) = (1, \frac{1}{2})$, every $\hat{y}$ satisfying $\hat{y} \in [2, 4]$ will have $(\hat{\boldsymbol{u}}, \hat{y}) \in \bar{P}$. However, there exists no $\boldsymbol{\lambda}$ with $(\hat{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \bar{\Lambda}'$ since the restrictions of $\bar{\Lambda}'$ enforce that the nonnegative $\boldsymbol{\lambda}$ must have $\lambda_2 = \hat{u}_1 = 1$, $\lambda_3 = \hat{u}_2 = \frac{1}{2}$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

The paper [4] extends this approach to products of univariate functions. Again consider the m functions $f_\ell(x_\ell)$, $\ell \in \{1, \ldots, m\}$, where $x_\ell \in S_\ell \equiv \{\theta_{\ell 1}, \theta_{\ell 2}, \ldots, \theta_{\ell n_\ell}\}$ and n_ℓ denotes the number of realizations of x_ℓ. Then the linearization of $\prod_{\ell=1}^m f_\ell(x_\ell)$ using our strengthened bounds of (4.23) is accomplished in two steps. First, for each $\ell \in \{1, \ldots, m\}$, form the set P_ℓ in the same manner as (4.23) to represent $f_\ell(x_\ell)$ as the variable y_ℓ using the binary variables $\boldsymbol{u}_\ell \in \mathbb{R}^{\lceil \log_2(n_\ell) \rceil}$. Here, for each such ℓ and for every $j \in \{1, \ldots, n_\ell\}$, the linear functions $A_{\ell j}(\boldsymbol{u}_\ell)$ are defined in the same manner as $A_j(\boldsymbol{u})$, and the bounds $\underline{M}_{\ell j}$ and $\overline{M}_{\ell j}$ replace $\underline{M}_j$ and $\overline{M}_j$ respectively so that $\underline{M}_{\ell j} = f_\ell(\theta_{\ell j}) - f_\ell^-$ and $\overline{M}_{\ell j} = f_\ell^+ - f_\ell(\theta_{\ell j})$, with $f_\ell^- \equiv \min\{f_\ell(\theta_{\ell 1}), \ldots, f_\ell(\theta_{\ell n_\ell})\}$ and $f_\ell^+ \equiv \max\{f_\ell(\theta_{\ell 1}), \ldots, f_\ell(\theta_{\ell n_\ell})\}$. In addition, each P_ℓ has the restriction $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_{\ell k} \leq n_\ell - 1$.

The second step is based on the following observation: for any given ℓ, by multiplying the functional values $f_\ell(\theta_{\ell j})$ found within P_ℓ by a variable, say ζ, the $2n_\ell$ inequalities involving $f_\ell(\theta_{\ell j})$ will enforce $y_\ell = \zeta f_\ell(x_\ell)$ provided that for each $j \in \{1, \ldots, n_\ell\}$, the values $\underline{M}_{\ell j}$ and $\overline{M}_{\ell j}$ are adjusted so that the associated inequalities are redundant when $A_{\ell j}(\boldsymbol{u}_\ell) \geq 1$; it is sufficient to have $\zeta f_\ell(\theta_{\ell j}) - \zeta f_\ell(x_\ell) \leq \underline{M}_{\ell j}$ and $\zeta f_\ell(x_\ell) - \zeta f_\ell(\theta_{\ell j}) \leq \overline{M}_{\ell j}$ for all possible realizations of ζ and $f_\ell(x_\ell)$. Now, using this observation and the notation from Section 4.2 that $J_\ell = 1 \cdots \ell$, we can inductively have $y_{J_\ell} = \prod_{j=1}^{\ell} f_j(x_j)$ for $\ell \geq 2$, beginning with $y_{12} = f_1(x_1) f_2(x_2) = y_1 f_2(x_2)$ and sequentially progressing to $y_{J_m} = f_1(x_1) f_2(x_2) \cdots f_m(x_m) = y_{J_{m-1}} f_m(x_m)$. The variable y_{12} is computed by forming a new set P_{12} using $\zeta = y_1$ within P_2 to obtain $y_{12} = y_1 y_2$. Then the variable y_{123} is computed by forming P_{123} using $\zeta = y_{12}$ within P_3 to obtain $y_{123} = y_1 y_2 y_3$. Continuing up to J_m, the variable y_{J_m} is computed by forming P_{J_m} using $\zeta = y_{J_{m-1}}$ within P_m to obtain $y_{J_m} = \prod_{j=1}^{m} y_j$. Here, each set P_{J_ℓ} has the same number $(2n_\ell + 1)$ of constraints and the same variables $\boldsymbol{u}_\ell$ as P_ℓ, but includes y_{J_ℓ} and $y_{J_{\ell-1}}$ instead of y_ℓ.

In the spirit of the above discussion, for each P_{J_ℓ} with $\ell \geq 2$, it is sufficient to have the adjusted $\underline{M}_{\ell j}$ and $\overline{M}_{\ell j}$, denoted $\underline{M}_{J_\ell j}$ and $\overline{M}_{J_\ell j}$ respectively, satisfy $\zeta f_\ell(\theta_{\ell j}) - \zeta f_\ell(x_\ell) \leq \underline{M}_{J_\ell j}$ and $\zeta f_\ell(x_\ell) - \zeta f_\ell(\theta_{\ell j}) \leq \overline{M}_{J_\ell j}$ for all possible realizations of $\zeta = y_{J_{\ell-1}} = \prod_{j=1}^{\ell-1} f_j(x_j)$ and $f_\ell(x_\ell)$. These values can be computed in various ways. One method is to have $\underline{M}_{J_\ell j} = \overline{M}_{J_\ell j} = f_{J_\ell}^+ - f_{J_\ell}^-$ where, as in Section 4.2, the terms $f_{J_\ell}^+$ and $f_{J_\ell}^-$ are upper and lower bounds on the product $\prod_{j=1}^{\ell} f_j(x_j)$. Different possibilities for these bounds exist. Again as in Section 4.2, if $f_j^- \geq 0$ for all $j \in \{1, \ldots, \ell\}$, then we can use $f_{J_\ell}^- = \prod_{j=1}^{\ell} f_j^-$ and $f_{J_\ell}^+ = \prod_{j=1}^{\ell} f_j^+$. If $f_j^- < 0$ for some j, then we can instead use $f_{J_\ell}^+ = \prod_{j=1}^{\ell} \max\{|f_j^-|, |f_j^+|\}$ and $f_{J_\ell}^- = -f_{J_\ell}^+$. Strengthened values for $\underline{M}_{J_\ell j}$ and $\overline{M}_{J_\ell j}$ can be computed based on problem structure and expended effort.

The size of the formulation is as follows. A count on each variable type is given in Table 4.2. Including the m original variables x_ℓ, there are $3m - 1$ continuous and $\sum_{\ell=1}^{m} \lceil \log_2(n_\ell) \rceil$ binary variables. Relative to constraints, each set P_ℓ for $\ell \in \{1, \ldots, m\}$ has $2n_\ell + 1$ restrictions and each set P_{J_ℓ} for $\ell \in \{2, \ldots, m\}$ has $2n_\ell$ additional restrictions. Also, $2n_\ell$ more inequalities are needed to handle the variables x_ℓ. The total number of constraints is then $m + 4n_1 + 6\sum_{\ell=2}^{m} n_\ell$.

Table 4.2: Variable types and counts in Approach 1 of [4].

Variable name	Variable type	Number of such variables
x_ℓ	continuous	m
y_ℓ	continuous	m
$y_{J_\ell}, \ell \neq 1$	continuous	$m - 1$
$\boldsymbol{u}_\ell$	binary	$\lceil \log_2(n_\ell) \rceil$ for each $\ell \in \{1, \ldots, m\}$

4.3.2 Li & Lu Approach 2

The second approach of [4] also represents functions of discrete variables, and their products, using logarithmic numbers of binary variables. For simplicity in presentation, we again begin by examining a single discrete variable $x \in S \equiv \{\theta_1, \theta_2, \ldots, \theta_n\}$ and function $f(x)$ so that we can temporarily drop the subscript ℓ.

While completely different in form and structure, this approach can be viewed as a blending of the first method of [4] that makes use of the linear functions $A_j(\boldsymbol{u})$ of (4.20) for binary $\boldsymbol{u} \in \mathbb{R}^{\lceil \log_2 n \rceil}$

with our method that employs a vector of nonnegative, continuous variables $\boldsymbol{\lambda} \in \mathbb{R}^n$ summing to unity. It operates by creating a nonlinear equation in $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ to enforce that $\boldsymbol{\lambda}$ is binary for $\boldsymbol{u}$ binary, and then sets $x = \theta_j$ and $y = f(\theta_j)$ for that single $\lambda_j = 1$. The nonlinear equation is subsequently linearized using [2]. Notably, our study will show that the resulting formulation allows for a substantial simplification that is achieved by identifying inequalities that can be set to equality, removing extraneous variables, and deleting redundant constraints. These simplifications render both the functions $A_j(\boldsymbol{u})$ and the linearization of [2] wholly unnecessary. In fact, the restrictions of the simplified form are directly obtainable by multiplying the equations $V\boldsymbol{\lambda} = \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}$ of (4.6) found in $\bar{\Lambda}'$ by the invertible matrix B of (4.19), thus establishing an equivalence between the resulting sets.

To begin, recall from the first approach of [4] in the previous section that the linear functions $A_j(\boldsymbol{u})$ of (4.20) were defined so that, for each $j \in \{1, \ldots, n\}$, $A_j(\boldsymbol{u}) = 0$ if $\boldsymbol{u} = \boldsymbol{v}_j$ and $A_j(\boldsymbol{u}) \geq 1$ if $\boldsymbol{u} \neq \boldsymbol{v}_j$. Also recall for each such j that the vector $\boldsymbol{v}_j$ denotes the base-2 expansion of $j - 1$, where entry i corresponds to the value 2^{i-1}. In this manner, $A_j(\boldsymbol{u})$ is defined for every binary $\boldsymbol{u}$ satisfying $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n - 1$. The second approach of [4] defines a vector of nonnegative, continuous variables $\boldsymbol{\lambda} \in \mathbb{R}^n$ that is restricted to have $\sum_{j=1}^{n} \lambda_j = 1$, and uses the nonlinear equation $\sum_{j=1}^{n} A_j(\boldsymbol{u})\lambda_j = 0$ to ensure that the single $j \in \{1, \ldots, n\}$, say p, having $A_p(\boldsymbol{u}) = 0$ must also have $\lambda_p = 1$. Then the equations

$$x = \sum_{j=1}^{n} \theta_j \lambda_j \text{ and } y = \sum_{j=1}^{n} f(\theta_j)\lambda_j, \tag{4.24}$$

which are identical to those found in (4.5) and (4.9), enforce $x = \theta_p$ and $y = f(\theta_p)$. The system is below.

$$Q \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n : \\ \qquad \sum_{j=1}^{n} \lambda_j = 1, \\ \qquad \sum_{j=1}^{n} A_j(\boldsymbol{u})\lambda_j = 0, \\ \qquad \sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1} u_k \leq n - 1, \\ \qquad \boldsymbol{u} \text{ binary}, \ \boldsymbol{\lambda} \geq \boldsymbol{0} \end{array} \right\}$$

The paper [4] linearizes the quadratic equation with the same method of [2] that was used to rewrite the nonlinear restrictions of (4.10) as (4.11). The first step is to factor the variables u_k

from $\boldsymbol{\lambda}$. Expressing this factorization in terms of earlier notation, by (4.20) we obtain

$$\sum_{j=1}^{n} A_j(\boldsymbol{u})\lambda_j = \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}^T BV\boldsymbol{\lambda},$$

where the matrix B is as defined in (4.19). For each $k \in \{1, \ldots, \lceil \log_2 n \rceil + 1\}$, denoting the k^{th} row of the vector $BV\boldsymbol{\lambda}$ by $g_{k-1}(\boldsymbol{\lambda})$ so that

$$\begin{bmatrix} g_0(\boldsymbol{\lambda}) \\ \vdots \\ g_{\lceil \log_2 n \rceil}(\boldsymbol{\lambda}) \end{bmatrix} = BV\boldsymbol{\lambda}, \tag{4.25}$$

the equation $\sum_{j=1}^{n} A_j(\boldsymbol{u})\lambda_j = 0$ in Q becomes

$$g_0(\boldsymbol{\lambda}) + \sum_{k=1}^{\lceil \log_2 n \rceil} g_k(\boldsymbol{\lambda})u_k = 0.$$

For each $k \in \{1, \ldots, \lceil \log_2 n \rceil\}$, the method of [2] substitutes a continuous variable δ_k for the product $g_k(\boldsymbol{\lambda})u_k$, and uses four inequalities to enforce $\delta_k = g_k(\boldsymbol{\lambda})u_k$ at binary $\boldsymbol{u}$. Using the fact that each such $g_k(\boldsymbol{\lambda})$ is lower and upper bounded by -1 and 1 respectively (since the coefficient on every λ_j in each function is $-1, 0$, or 1 and the sum of the λ_j equals 1), the formulation is as given below. The paper [4] does not include the restriction $\sum_{k=1}^{\lceil \log_2 n \rceil} 2^{k-1}u_k \leq n-1$ of Q; it can be shown redundant in the presence of the remaining constraints.

$$Q' \equiv \left\{ \begin{array}{ll} (\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\delta}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n \times \mathbb{R}^{\lceil \log_2 n \rceil} : & \\ \quad \sum_{j=1}^{n} \lambda_j = 1 & (4.26a) \\ \quad g_0(\boldsymbol{\lambda}) + \sum_{k=1}^{\lceil \log_2 n \rceil} \delta_k = 0 & (4.26b) \\ \quad g_k(\boldsymbol{\lambda}) - (1 - u_k) \leq \delta_k \leq g_k(\boldsymbol{\lambda}) + (1 - u_k) \ \forall \ k = 1, \ldots, \lceil \log_2 n \rceil & (4.26c) \\ \quad -u_k \leq \delta_k \leq u_k \ \forall \ k = 1, \ldots, \lceil \log_2 n \rceil & (4.26d) \\ \quad \boldsymbol{u} \text{ binary}, \ \boldsymbol{\lambda} \geq \mathbf{0} & \end{array} \right\}$$

While not noted in [4], the structure of Q' allows for a simplification that significantly reduces the numbers of variables and constraints. Consider the theorem below.

Theorem 4.2: *Every point* $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\delta}})$ *with* $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$ *and* $\mathbf{0} \leq \hat{\boldsymbol{u}} \leq \mathbf{1}$ *that satisfies (4.26a)–(4.26d) has* $-\hat{u}_k = \hat{\delta}_k = g_k(\hat{\boldsymbol{\lambda}}) - (1 - \hat{u}_k)$ *for all* $k \in \{1, \ldots, \lceil log_2 n \rceil\}$.

Proof. It is readily verified that the matrix B defined in (4.19) has the first row of B^{-1}, say $\boldsymbol{\rho}^T \in \mathbb{R}^{\lceil \log_2 n \rceil + 1}$, with $\frac{2}{\lceil \log_2 n \rceil}$ in the first entry and $\frac{1}{\lceil \log_2 n \rceil}$ elsewhere. Consequently,

$$\sum_{j=1}^{n} \lambda_j = \boldsymbol{\rho}^T B V \boldsymbol{\lambda} = \frac{2}{\lceil \log_2 n \rceil} g_0(\boldsymbol{\lambda}) + \frac{1}{\lceil \log_2 n \rceil} \sum_{k=1}^{\lceil \log_2 n \rceil} g_k(\boldsymbol{\lambda}), \tag{4.27}$$

where the first equality recognizes the first row of $V\boldsymbol{\lambda}$ from (4.6) as $\sum_{j=1}^{n} \lambda_j$, and the second equality follows from (4.25). Now, sum $\frac{2}{\lceil \log_2 n \rceil}$ times the equation in (4.26b) with $\frac{1}{\lceil \log_2 n \rceil}$ times the sum of the left-hand inequalities in (4.26c) and (4.26d) and invoke (4.27) to obtain

$$\sum_{j=1}^{n} \lambda_j \leq 1. \tag{4.28}$$

But (4.26a) enforces this restriction with equality for all $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\delta}) \in Q'$. Then the left-hand inequalities of both (4.26c) and (4.26d) must also hold with equality for all $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\delta}) \in Q'$. This completes the proof. □

The above theorem allows us to equivalently rewrite Q' with the left-hand inequalities of (4.26c) and (4.26d) satisfied with equality so that $\delta_k = g_k(\boldsymbol{\lambda}) - (1 - u_k)$ and $\delta_k = -u_k$ for each $k \in \{1, \ldots, \lceil \log_2 n \rceil\}$. This makes the right-hand inequalities redundant due to $\mathbf{0} \leq \boldsymbol{u} \leq \mathbf{1}$. Then we can substitute $\delta_k = -u_k$ throughout the problem so that the variables $\boldsymbol{\delta}$ and restrictions (4.26d) are no longer needed. The resulting reduced version of Q' is RQ' below.

$$RQ' \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^{\lceil \log_2 n \rceil} \times \mathbb{R}^n : \\ \qquad \sum_{j=1}^{n} \lambda_j = 1 \\ \qquad g_0(\boldsymbol{\lambda}) = \sum_{k=1}^{\lceil \log_2 n \rceil} u_k \\ \qquad g_k(\boldsymbol{\lambda}) = 1 - 2u_k, \ \forall \ k = 1, \ldots, \lceil \log_2 n \rceil \\ \qquad \boldsymbol{u} \text{ binary}, \ \boldsymbol{\lambda} \geq \mathbf{0} \end{array} \right\}$$

Denoting the continuous relaxations of Q' and RQ' where the binary restrictions on $\boldsymbol{u}$ are replaced with $\mathbf{0} \leq \boldsymbol{u} \leq \mathbf{1}$ by $\bar{Q}'$ and $\bar{RQ}'$ respectively, it directly follows that a point $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\delta}}) \in \bar{Q}'$ if and only if $\hat{\boldsymbol{\delta}} = -\hat{\boldsymbol{u}}$ and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in \bar{RQ}'$. Thus, $\bar{RQ}'$ can be viewed as an economical representation of $\bar{Q}'$ that is obtained by setting a subset of the inequalities to equality, and by removing redundant constraints and unnecessary variables.

The proof of Theorem 4.2 shows that RQ' can be further reduced in size by removing any one of the $\lceil \log_2 n \rceil + 2$ equality restrictions. This follows from (4.27), as each such restriction can be expressed as a linear combination of the others, with no multipliers of value 0.

Interestingly, the set $R\bar{Q}'$ provides exactly the same polyhedral region as $\bar{\Lambda}'$. This equivalence is addressed in the theorem below.

Theorem 4.3: *A point* $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in R\bar{Q}'$ *if and only if* $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\lambda}}) \in \bar{\Lambda}'$.

Proof. Multiply the restrictions $V\boldsymbol{\lambda} = \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}$ of $\bar{\Lambda}'$ by the invertible matrix B of (4.19). Then (4.25) and the structure of B gives that the equation $BV\boldsymbol{\lambda} = B \begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}$ yields the last $1 + \lceil \log_2 n \rceil$ equations found within $R\bar{Q}'$. As noted above, the restriction $\sum_{j=1}^{n} \lambda_j = 1$ is implied by the remaining equations of $R\bar{Q}'$, completing the proof. $\square$

Example 4.4

As in the previous Example 4.3, consider $f(x) = x$ with $x \in S = \{1, 3, 5\}$, so that again $n = 3$ with $\lceil \log_2 n \rceil = 2$. The set $\bar{\Lambda}'$ in three nonnegative continuous variables $\boldsymbol{\lambda}$, two binary variables $\boldsymbol{u}$, and three equality constraints is given in Example 4.3 where $V\boldsymbol{\lambda} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$. By (4.25), $\begin{bmatrix} g_0(\boldsymbol{\lambda}) \\ g_1(\boldsymbol{\lambda}) \\ g_2(\boldsymbol{\lambda}) \end{bmatrix} = \begin{bmatrix} \lambda_2 + \lambda_3 \\ \lambda_1 - \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 - \lambda_3 \end{bmatrix} = BV\boldsymbol{\lambda}$ with $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ so that the representation of [4] using $\bar{Q}'$ is

$$\bar{Q}' = \left\{ \begin{array}{rcl} \multicolumn{3}{l}{(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\delta}) \in \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^2, \boldsymbol{\lambda} \geq \mathbf{0} :} \\ \lambda_1 + \lambda_2 + \lambda_3 & = 1 & \\ \lambda_2 + \lambda_3 + \delta_1 + \delta_2 & = 0 & \\ \lambda_1 - \lambda_2 + \lambda_3 - 1 + u_1 & \leq \delta_1 \leq & \lambda_1 - \lambda_2 + \lambda_3 + 1 - u_1 \\ \lambda_1 + \lambda_2 - \lambda_3 - 1 + u_2 & \leq \delta_2 \leq & \lambda_1 + \lambda_2 - \lambda_3 + 1 - u_2 \\ -u_1 & \leq \delta_1 \leq & u_1 \\ -u_2 & \leq \delta_2 \leq & u_2 \\ 0 & \leq u_1 \leq & 1 \\ 0 & \leq u_2 \leq & 1 \end{array} \right\}.$$

Theorems 4.2 and 4.3 ensure that every point $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\delta}) \in \bar{Q}'$ must have $\boldsymbol{\delta} = -\boldsymbol{u}$, and that a point

$(\boldsymbol{u}, \boldsymbol{\lambda}) \in \bar{\Lambda}'$ if and only if $(\boldsymbol{u}, \boldsymbol{\lambda}, -\boldsymbol{u}) \in \bar{Q}'$. However, the form of $\bar{Q}'$ is larger than $\bar{\Lambda}'$. It uses the extra variables δ_1 and δ_2 and, not counting the lower bounds of 0 on u_1 and u_2, requires two equality and ten inequality constraints. To illustrate Theorem 4.2 that the four left-hand inequalities restricting δ_1 and δ_2 must hold with equality, sum the second constraint with $\frac{1}{2}$ times each of these four inequalities to obtain $\lambda_1 + \lambda_2 + \lambda_3 \leq 1$, as (4.28) was computed from (4.27). The first equation of $\bar{Q}'$ then establishes the result. The representation of $f(x)$ (equivalently x for this example) is achieved using (4.24).

The paper [4] notes that this approach can be combined with their first method to handle products of univariate functions. Given m functions $f_\ell(x_\ell)$ where $x_\ell \in S_\ell \equiv \{\theta_{\ell 1}, \theta_{\ell 2}, \ldots, \theta_{\ell n_\ell}\}$ for $\ell \in \{1, \ldots, m\}$, the product $\prod_{\ell=1}^{m} f_\ell(x_\ell)$ is linearized in an identical fashion to the previous section with the following exception. For each $\ell \in \{1, \ldots, m\}$, a set Q'_ℓ in the variables $\boldsymbol{u}_\ell$, $\boldsymbol{\lambda}_\ell$, and $\boldsymbol{\delta}_\ell$ is formed as in (4.26a)–(4.26d) so that x_ℓ and $f_\ell(x_\ell)$ can be expressed as in (4.24). Then the representations Q'_ℓ replace the sets P_ℓ. For each $\ell \in \{2, \ldots, m\}$, the set P_{J_ℓ} remains unchanged, having the variable y_{J_ℓ} represent the product $\prod_{j=1}^{\ell} f_j(x_j)$.

Relative to the number of constraints, for each $\ell \in \{1, \ldots, m\}$ the set Q'_ℓ and the corresponding expressions in (4.24) contain $4\lceil \log_2(n_\ell) \rceil + 4$ restrictions (noting that $\mathbf{0} \leq \boldsymbol{u} \leq \mathbf{1}$ is implied). For $\ell \in \{2, \ldots, m\}$ the set P_{J_ℓ} has $2n_\ell$ additional restrictions. In total, $4m + 4\sum_{\ell=1}^{m} \lceil \log_2(n_\ell) \rceil + 2\sum_{\ell=2}^{m} n_\ell$ constraints are required. (This is a savings beyond the first method in [4] of $4n_\ell - 4\lceil \log_2(n_\ell) \rceil - 3$ constraints for each $\ell \in \{1, \ldots, m\}$.) As for variables, Table 4.3 gives the names, types, and numbers required. Summing, there are $3m - 1 + \sum_{\ell=1}^{m}(n_\ell + \lceil \log_2(n_\ell) \rceil)$ continuous and $\sum_{\ell=1}^{m} \lceil \log_2(n_\ell) \rceil$ binary variables.

Table 4.3: Variable types and counts in Approach 2 of [4].

Variable name	Variable type	Number of such variables
x_ℓ	continuous	m
y_ℓ	continuous	m
$y_{J_\ell}, \ell \neq 1$	continuous	$m - 1$
$\boldsymbol{\lambda}_\ell$	continuous	n_ℓ for each $\ell \in \{1, \ldots, m\}$
$\boldsymbol{\delta}_\ell$	continuous	$\lceil \log_2(n_\ell) \rceil$ for each $\ell \in \{1, \ldots, m\}$
$\boldsymbol{u}_\ell$	binary	$\lceil \log_2(n_\ell) \rceil$ for each $\ell \in \{1, \ldots, m\}$

4.4 Conclusions

This chapter presents a strategy for expressing functions of discrete variables, and their products, in terms of logarithmic numbers of binary variables. The fundamental idea is an observation for writing a binary vector as a convex combination of extreme points of the unit hypercube. This observation allows us to treat n binary variables as continuous by defining a smaller number of $\lceil \log_2 n \rceil$ binary variables. Such collections of binary variables naturally arise in modeling general discrete variables, and functions thereof.

Our strategy provides a unifying perspective for two published approaches that are designed to use logarithmic numbers of binary variables. It compares favorably, in terms of the strengths of the continuous relaxations and formulation sizes, to both methods. We show for the case of a function $f(x)$ having x a discrete variable, that our continuous relaxation dominates one such method, and is theoretically equivalent to the other. For both competing approaches, our forms use markedly fewer constraints. Our proofs provide insight into relationships of the alternate approaches with each other, and improve upon the second by identifying (previously unnoticed) families of unnecessary constraints and extraneous variables.

Given a collection of m functions $f_\ell(x_\ell)$ for $\ell \in \{1, \ldots, m\}$, where each discrete variable x_ℓ can realize n_ℓ distinct values, Table 4.4 summarizes the numbers of continuous variables and constraints required to linearize the product $\prod_{\ell=1}^{m} f_\ell(x_\ell)$ for each of the three approaches. The first row of the table is the proposed method of Section 4.2, while rows two and three are the approaches of Sections 4.3.1 and 4.3.2. For readability, we let $N = \sum_{\ell=1}^{m} n_\ell$ and $L = \sum_{\ell=1}^{m} \lceil \log_2(n_\ell) \rceil$. Since *all three* approaches employ the same L binary variables, this count is not included in the table.

We also posed four reduction strategies based on variable substitutions and transformations. In order to perform more transparent comparisons, these strategies are not reflected in Table 4.4. However, it is interesting to note that, in addition to the proposed method, they can be selectively applied to the other two approaches. The substitution of variables $\boldsymbol{w}'_\ell = \boldsymbol{w}_\ell - \kappa_\ell^- \boldsymbol{u}_\ell$ in the first strategy for positive κ_ℓ^- is applicable to the second approach of [4], although it becomes unnecessary in light of Theorem 4.2. The second reduction strategy to eliminate the variables x_ℓ and y_ℓ is applicable to the second approach of [4]. But all variables in the first approach of [4], and the y_{J_ℓ} in the second approach, must be kept. The third reduction strategy that converts equality restrictions to inequalities can be applied to the second approach of [4], but will only save two variables, due to only two equality restrictions. Finally, the fourth reduction strategy dealing with the order of the functions considered can potentially reduce all formulations, though to different extents.

Table 4.4: Summary of variable and constraint counts

	Continuous Variables	Constraints
Proposed Method	$3m - 1 - n_1 + 2N + L - \lceil \log_2(n_1) \rceil$	$5m - 2 - 5\lceil \log_2(n_1) \rceil + 6L$
Li & Lu 1 [4]	$3m - 1$	$m + 6N - 2n_1$
Li & Lu 2 [4]	$3m - 1 + N + L$	$4m + 4L + 2N - 2n_1$

This study is theoretical in nature, focusing on representation size and relaxation strength, as well as establishing equivalences between, and improvements to, known techniques. Future research includes computational studies to determine the practical benefits made possible by reduced numbers of binary variables in concise model representations.

Bibliography

[1] Adams, W.P. and Forrester, R.J., "A Simple Recipe for Concise Mixed 0-1 Linearizations," *Operations Research Letters*, Vol. 33, Issue 1, 55-61, 2005.

[2] Glover, F., "Improved Linear Integer Programming Formulations of Nonlinear Integer Programs," *Management Science*, Vol. 22, No. 4, 455-460, 1975.

[3] Glover, F., "An Improved MIP Formulation for Products of Discrete and Continuous Variables," *Journal of Information and Optimization Sciences*, Vol. 5, Issue 1, 469-471, 1984.

[4] Li, H.L. and Lu, H.C., "Global Optimization for Generalized Geometric Programs with Mixed Free-Sign Variables," *Operations Research*, Vol. 57, No. 3, 701-713, 2009.

[5] Vielma, J.P. and Nemhauser, G.L., "Modeling Disjunctive Constraints with a Logarithmic Number of Binary Variables and Constraints," *Mathematical Programming, Series A*, DOI 10.1007/s10107-009-0295-4.

[6] Watters, L.J. "Reduction of Polynomial Programming Problems to Zero-One Linear Programming Problems," *Operations Research*, Vol. 15, No. 6, 1171-1174, 1967.