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Asymptotics of Families of Polynomials and Sums of Hurwitz Class Numbers

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Asymptotics of Families of Polynomials and Sums of Hurwitz Class Numbers

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Abstract

In a note in the American Mathematical Monthly in 1960, Strodt mentions a way to prove both the Euler-Maclaurin summation formula and the Boole summation formula using operators. In a 2009 article in the Monthly, Borwein, Calkin, and Manna expand on this idea. Therein, they define Strodt operators and Strodt polynomials and show that the classical Bernoulli polynomials and Euler polynomials are examples of Strodt polynomials.

It is well known that both Bernoulli polynomials and Euler polynomials on a fixed interval are asymptotically sinusoidal. Borwein, Calkin, and Manna show that a similar result holds for the uniform Strodt polynomials. We extend this idea to other generalized families of Strodt polynomials. We state and prove several theorems which make explicit the asymptotic behavior of these families. We also explain our experiments and give examples of Strodt polynomials with unknown, non-sinusoidal asymptotics.

We also consider an identity for sums of Hurwitz class numbers and we state a theorem arising from taking subsums of this identity. The Hurwitz class numbers can be defined by counting classes of binary quadratic forms. They can also be related to isomorphism classes of elliptic curves over a finite field via Deuring's Theorem. We use a combinatorial proof to prove this theorem.

Dedication

To Heather and to my parents

Acknowledgments

First and foremost, I would like to thank my advisor, Neil Calkin. He has worked tirelessly with me over the past several years and has provided more help than I could possibly describe. He brought the Strodt polynomial topic to me before the journal article was even complete and he saw this end result from that day. At each step he has given me new things to think about, new experiments to investigate, and new perspectives to consider. I never walk away from meeting with him without an idea of what to try next. He also helped me correct and edit this dissertation.

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I also give many thanks to my other two committee members, Wayne Goddard and Gretchen Matthews. They were understanding when I changed topics after my original proposal and they each offered valuable editorial comments regarding content, presentation and tone.

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Brittany Brown and Amy Stout, participants in a 2003 Clemson REU, spent many hours looking through data I gave them and are responsible for conjecturing the results in Chapter 6. They were also my audience for the early versions of the proofs in that chapter. Explaining the steps to them helped me to better understand them myself. The final versions of those proofs owe a great deal to Ethan Smith who offered improvements to the organization and notation used therein.

Finally, I would like to thank Heather, my wife. Through her job at Cooper Library she has been invaluable in finding all of the books, journals, and articles I needed. She also helped keep my references organized and helped format a bibliography. Most importantly, she has been my biggest fan and a constant source of encouragement and love.

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Chapter 1

Introduction

In the late seventeenth century, the Swiss mathematician Jakob Bernoulli considered sums of the form

$$\sum_{n=1}^{k} n^r$$

for positive integer r. In his famous work *Ars conjectandi* (1713), which was published several years after his death, Bernoulli presented a solution to this problem. His formula involved a series of constants he used to compute the sums. Today we call these constants the Bernoulli numbers, B_n. They are related to the Bernoulli polynomials. We will begin with the definition.

The generating function for the *Bernoulli polynomials*, $B_n(x)$, is

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ . \tag{1.1}$$

The first few Bernoulli polynomials are

$$B_0(x) = 1$$

$$B_1(x) = x - 1/2$$

$$B_2(x) = x^2 - x + 1/6$$

$$B_3(x) = x^3 - 3/2\,x^2 + 1/2\,x$$

$$B_4(x) = x^4 - 2\,x^3 + x^2 - 1/30$$

$$B_5(x) = x^5 - 5/2\,x^4 + 5/3\,x^3 - 1/6\,x$$

$$B_6(x) = x^6 - 3\,x^5 + 5/2\,x^4 - 1/2\,x^2 + 1/42$$

$$B_7(x) = x^7 - 7/2\,x^6 + 7/2\,x^5 - 7/6\,x^3 + 1/6\,x$$

$$B_8(x) = x^8 - 4\,x^7 + 14/3\,x^6 - 7/3\,x^4 + 2/3\,x^2 - 1/30\ .$$

The *Bernoulli numbers* are the constant terms of these polynomials. There are numerous applications and generalizations of these numbers and polynomials (see, for example, [16], [27], or [18]).

The *Euler polynomials*, $E_n(x)$, have generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}\ . \tag{1.2}$$

The first few Euler polynomials are

$$\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - 1/2 \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - 3/2\,x^2 + 1/4 \\
E_4(x) &= x^4 - 2\,x^3 + x \\
E_5(x) &= x^5 - 5/2\,x^4 + 5/2\,x^2 - 1/2 \\
E_6(x) &= x^6 - 3\,x^5 + 5\,x^3 - 3\,x \\
E_7(x) &= x^7 - 7/2\,x^6 + \frac{35}{4}\,x^4 - 21/2\,x^2 + \frac{17}{8} \\
E_8(x) &= x^8 - 4\,x^7 + 14\,x^5 - 28\,x^3 + 17\,x\ .
\end{aligned}$$

The Euler numbers satisfy $E_n = 2^n E_n(1/2)$. Euler polynomials and Euler numbers do not appear as often in the literature.

In the next chapter we will define the Strodt polynomials and give several properties of them. We will also show that the Bernoulli polynomials and Euler polynomials can be viewed as members of this family of polynomials. In Chapters 3 and 4 we will consider the asymptotic behavior of Strodt polynomials. We will prove asymptotic results for several families and also state some examples of Strodt polynomials whose asymptotics are unknown.

In Chapter 5 we switch our focus to topics in elementary and modern number theory. In particular, we will define the Hurwitz class number, $H(N)$, which satisfies the identity

$$\sum_{r^2<4p} H(4p - r^2) = 2p\ .$$

In Chapter 6 we will apply Deuring's Theorem and facts about elliptic curves over a finite field to prove a theorem about taking subsums of this sum. We will also state some similar theorems and conjectures.

Chapter 2

Strodt Polynomials

In this chapter, we will follow the presentation in [4, §2] to outline the definition and properties of Strodt polynomials. In Section 4, we will show that both Bernoulli polynomials and Euler polynomials are examples of Strodt polynomials. In Section 5 we consider the summation formulas which were the focus of Strodt's note [25]. We begin, however, with a review of some elementary tools for the analysis of series.

2.1 Some Elementary Analysis of Power Series

The exponential and trigonometric functions can be defined as power series. In this context, one can prove many familiar properties and identities of the functions. This analytic approach can be found in many elementary texts (e.g. [23], [15]). We give a short summary of these ideas here.

We begin by defining the exponential function, $\exp(x)$, as

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ . \tag{2.1}$$

This series converges for every x (ratio test). Now, take the derivative of the power series

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp(x)\right] = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{2.2}$$

to obtain the result $\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp(x)\right] = \exp(x)$. Define the exponential number e by $e = \exp(1)$.

We have the following rule for multiplying absolutely convergent power series:

$$\sum_k a_k x^k \sum_k b_k x^k = \sum_k c_k x^k \text{ where } c_k = \sum_j a_j b_{k-j}. \tag{2.3}$$

We apply this fact and use the Binomial Theorem to verify a familiar property of the exponential function.

$$\begin{aligned}
\exp(x) \cdot \exp(y) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{k=0}^{\infty} \frac{y^k}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{x^j y^{k-j}}{j!(k-j)!} \\
&= \sum_{k=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{k} \binom{n}{j} x^j y^{k-j} \\
&= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} \\
&= \exp(x+y).
\end{aligned}$$

We may also conclude that $\exp(x) = e^x$ (see [23] for details).

We proceed now to the trigonometric function, sine. Analytically, $\sin(x)$ is defined as the power series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{2.4}$$

or more familiarly as

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Similarly, for cosine

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \ , \tag{2.5}$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Each of these series converge for all x [23].

We can extend the definitions of exponential and trigonometric functions to the complex numbers by replacing the variable x with a complex variable, z. We use i as the imaginary number and recall the elementary identities $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$.

The preceding can be used to verify the following classical result.

Fact 2.1.1 (Euler's formula)**.** *$e^{iz} = \cos(z) + i\sin(z)$.*

In the next two chapters we will use this fact frequently in our proofs. We will also need the following corollary.

Fact 2.1.2. *Let z be a complex variable and let a and b be non-zero real numbers. Then*

$$e^{i(a-b)z} \frac{\sin(az)}{\sin(bz)} = \frac{(e^{i2az} - 1)}{(e^{i2bz} - 1)} \ . \tag{2.6}$$

The following trigonometric identity is an immediate corollary of the angle sum and angle difference formulas.

Fact 2.1.3.

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Finally, we state a power series result.

Fact 2.1.4. *Let a be real and fixed and let z be a variable, $|z| < \sqrt{a}$. Then power series for even term polynomials and odd term polynomials are given by*

$$\frac{1}{a^2 - z^2} = \sum_{n \geq 0} a^{-(2n+2)} z^{2n},$$

and

$$\frac{z}{a^2 - z^2} = \sum_{n \geq 0} a^{-(2n+2)} z^{2n+1}.$$

2.2 Strodt Operators

In a note in the American Mathematical Monthly in 1960, Strodt [25] mentions a way to prove both the Euler-Maclaurin summation formula and the Boole summation formula using operators. In a 2009 article in the Monthly, Borwein, Calkin, and Manna [4] expand on this idea. We will not give details here about the specific operators Strodt introduced in [25]. Rather, we will focus on the more general operators we will use in subsequent chapters.

For $n \in \mathbb{N}$, define the *uniform interpolation Strodt operator*[4], $\mathcal{S}_n$, by

$$\mathcal{S}_n(f)(x) := \sum_{j=0}^{n-1} \frac{1}{n} \cdot f\left(x + \frac{j}{n-1}\right) \ . \tag{2.7}$$

We note that this operator will map a continuous function to a continuous function. We will state more properties of $\mathcal{S}_n$ shortly.

As an example, let $n = 2$ and consider the operator $\mathcal{S}_2$. Apply this operator to an Euler polynomial (defined in Chapter 1). In particular,

$$\begin{aligned}\mathcal{S}_2(E_4)(x) &= \sum_{j=0}^{1} \frac{1}{2} \cdot E_4\left(x + \frac{j}{1}\right) \\ &= \frac{1}{2}\big(x^4 - 2x^3 + x + (x+1)^4 - 2(x+1)^3 + x + 1\big) \\ &= x^4 \ .\end{aligned}$$

Similarly, for the 5th Euler polynomial,

$$\begin{aligned}\mathcal{S}_2(E_5)(x) &= \frac{1}{2}\left(E_5(x) + E_5(x+1)\right) \\ &= x^5 \ .\end{aligned}$$

These results are not coincidental, as we will see.

Now we define a more general operator. Consider a finite set of points

$$\{x_i\}_{i=1}^{N} \subset [0,\, 1] \ ,$$

and a probability weight function

$$w \ : \ \{x_i\}_{i=1}^{N} \mapsto (0,\, 1) \tag{2.8}$$

where w has the property

$$\sum_{i=1}^{N} w(x_i) = 1 \ .$$

We use this probability weight function to determine a corresponding *finite Strodt operator* [4] by

$$S_w(f)(x) := \sum_{i=1}^{N} f(x + x_i) w(x_i) \ .$$

This operator can also be written as an integral. Set

$$g(u) = \sum_{i=1}^{N} w(x_i)\delta_{x_i}(u)$$

and define

$$\mathcal{S}_g := \int f(x+u)g(u)\mathrm{d}u \; .$$

In [4], the $\mathcal{S}_g$ operator is even more general, but this definition will suffice for our purposes.

Define $P_k := \left\{\sum_{i=0}^{k} a_i x^i : a_i \in \mathbb{R}\right\} \cong \mathbb{R}^{k+1}$.

Proposition 2.2.1. *([4, Prop.2.2]) Let g be a probability density function as above. For all $h \in P_k$, there is a unique $f \in P_k$ so that $\mathcal{S}_g(f) = h$.*

Thus, every degree n polynomial has a unique preimage under the generalized operator. For a probability density function g, we define the *Strodt polynomials* [4] to be $\mathcal{S}_g^{-1}(x^n)$ for $n \in \mathbb{N}_0$. We usually denote this family of polynomials as $P_n(x)$. Note that each choice of g changes the family of polynomials. So, it is important to know in context which density we are using.

For us, the best example of a g is a probability weight function as in 2.8. We can think of building a family of Strodt polynomials as follows:

- Choose a finite set of points from the interval [0, 1].
- Assign a weight to each point
- For any n, the polynomial $P_n(x)$ is now uniquely determined.

In the next section we will take this a step further and describe how to get a generating function for these new polynomials.

2.3 Generating Functions for Strodt Polynomials

We begin with another result from [4].

Proposition 2.3.1. *([4, Theorem.2.3]) For $n \in \mathbb{N}_0$, let $P_n^g(x)$ be the Strodt polynomials associated with given density function g; that is, for all $x \in \mathbb{R}$, $P_n^g(x)$ is defined implicitly by*

$$\mathcal{S}_g\left(P_n^g(x)\right) = x^n$$

for $n \in \mathbb{N}_0$ where $\mathcal{S}_g$ is a Strodt operator. Then for all $n \in \mathbb{N}$,

$$\frac{d}{dx} P_n^g(x) = n P_{n-1}^g(x). \tag{2.9}$$

A sequence of polynomials with the property 2.9 is, by definition, an Appell sequence [22]. This knowledge is helpful because all Appell sequences $A_n(x)$ have an exponential generating function of the form

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{e^{xt}}{G(t)}$$

(we refer the reader to [22] for more details).

As a corollary, [4] shows that the exponential generating function of a Strodt polynomial is

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = \frac{e^{xt}}{Q_g(t)}$$

where Q_g depends on the density function g and

$$Q_g(t) := \int e^{ut} g(u) \mathrm{d}u$$

.

Let us return to the situation 2.8 where g is determined by picking points $\{x_i\}_{i=1}^{N} \subset [0,\, 1]$ and weights $w(x_i)$. Then we can see that

$$\begin{aligned} Q &= \int e^{ut} g(u) \mathrm{d}u \\ &= \int e^{ut} \left(\sum_{i=1}^{N} w(x_i) \delta_{x_i}(u) \right) \mathrm{d}u \\ &= \sum_{i=1}^{N} w(x_i) e^{x_i t}. \end{aligned}$$

Therefore, the choice of a finite number of points and their corresponding weights leads to a unique exponential generating function for their associated Strodt polynomials as follows:

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = \frac{e^{xt}}{\sum_{i=1}^{N} w(x_i) e^{x_i t}}. \tag{2.10}$$

This is the generating function we will use throughout the next chapter.

2.4 Euler Polynomials and Bernoulli Polynomials

Define the *Euler operator* as $\mathcal{S}_E(f)(x) := \mathcal{S}_2(f)(x)$. This corresponds to picking the two points 0 and 1 and assigning them each a weight of $\frac{1}{2}$. Using the result from the previous section, the generating function for the Strodt polynomials associated with this operator is

$$\frac{e^{xt}}{\frac{1}{2}(e^0) + \frac{1}{2}e^t} = \frac{2e^{xt}}{1 + e^t}$$

which is exactly the generating function for the Euler polynomials, $E_n(x)$. In fact, the unique polynomials which satisfy $\mathcal{S}_E\left(E_n(x)\right) = x^n$ is a valid definition for the Euler polynomials (see [4] for proof).

Similarly, we can define the *Bernoulli operator* as

$$\mathcal{S}_B(f)(x) := \int_0^1 f(x+u)\mathrm{d}u$$

and deduce the generating function for the Bernoulli polynomials. Also, finding the unique polynomials which satisfy $\mathcal{S}_B\left(B_n(x)\right) = x^n$ is a sufficient definition for the Bernoulli polynomials. We can also think of $\mathcal{S}_B$ as $\mathcal{S}_n$ as $n \to \infty$. From this perspective, $B_n(x)$ and $E_n(x)$ arise from the same family. We will revisit this idea in the next chapter.

2.5 Summation Formulas

The following theorem gives an asymptotic expression of the Gregory series approximation for $\frac{\pi}{2}$.

Theorem 2.5.1. *[3, Thm 1.a] The following asymptotic expansion holds:*

$$\begin{aligned}\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} &\sim \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}} \\ &= \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \frac{61}{N^7} + \dots\end{aligned}$$

Consider setting N to be a power of 10. Then this theorem not only gives which decimal places in the expansion will have an error, but also exactly what that error will be. Thus, the digits in the approximation differ from the digits of $\frac{\pi}{2}$ by exactly the values of the Euler numbers. This fun result can be verified by an application of the Boole summation formula (see [3] for details).

The *periodic Euler polynomials*, $\tilde{E}_n(x)$, are defined by setting $\tilde{E}_n(x) = E_n(x)$ for $0 \leq x < 1$ and $\tilde{E}_n(x+1) = -\tilde{E}_n(x)$ for all other x [19, 24.2.12]. It is clear that these polynomials are periodic with period 2. The Boole summation formula is

$$\sum_{j=a}^{N-1}(-1)^j f(j+h) = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(h)}{k!}\left((-1)^{N-1}f^{(k)}(N)+(-1)^a f^{(k)}(a)\right) + \frac{1}{2(m-1)!}\int_a^N f^{(m)}(x)\tilde{E}_{m-1}(h-x)\mathrm{d}x$$

for $0 < h < 1$ and where f has m absolutely integrable derivatives. This formula gives an expression for an alternating sum of values of a function.

The *periodic Bernoulli polynomials*, $\tilde{B}_n(x)$, are defined by considering $\{x\}$, the fraction part of x, and setting $\tilde{B}_n(x) = B_n(\{x\})$ [19, 24.2.11]. It is clear that they are periodic with period 1. The Euler-Maclaurin summation formula is

$$\sum_{j=a}^{N-1} f(j) = \int_a^N f(x)\,\mathrm{d}x + \sum_{k=1}^{m}\frac{B_k}{k!}\left(f^{(k-1)}(N) - f^{(k-1)}(a)\right) + \frac{(-1)^{m+1}}{m!}\int_a^N \tilde{B}_m(y) f^{(m)}(y)\mathrm{d}y$$

where f has m absolutely integrable derivatives.

We noted in Section 2 of this chapter that the inspiration for the definition of the Strodt polynomials in [4] was a comparison in [25] between the Boole summation and Euler-Maclaurin summation formulas. In [4, §5] there are proofs for these two formulas using Strodt operators. Each formula can also be derived by starting with an expression like

$$f(j) = \int_{j-1}^{j} f(x)\mathrm{d}x + \int_{j-1}^{j} f(j) - f(x)\mathrm{d}x\ .$$

Then integrate the latter integral by parts and iterate. We control the growth by using antiderivatives which integrate to give 0 over one period. This is where the periodic Bernoulli and Euler polynomials come into play. The final result can be obtained when we recall that $\tilde{B}_n(x)$ and $\tilde{E}_n(x)$ each have the property 2.9. Complete details of the Euler-Maclaurin case can be found in [2, Appx.D] or [21, §8.1].

Earlier in this chapter we defined the Strodt polynomials. Also, recall from the previous section that Euler polynomials and Bernoulli polynomials are both Strodt. This leads us to pose the following question.

Open Question 2.5.2. *Is it possible to deduce a new summation formula (similar to Euler-Maclaurin summation and Boole summation) which uses a different Strodt polynomial?*

Chapter 3

Strodt Asymptotics

We turn our attention to the asymptotics of Strodt polynomials. We begin with the long-known asymptotics for $B_n(x)$ and $E_n(x)$, as well as a recent result from [4]. In section 2 we discuss Darboux's Method and then use it in Section 3 to prove asymptotic behavior of a particular Strodt polynomial family. We will use this proof as a guide to the proofs in the following chapter.

3.1 Asymptotics of Uniform Strodt Polynomials

Another fact about the Bernoulli polynomials which has been known for a long time is their tendency toward a periodic wave as n goes to infinity. In fact, the $B_n(x)$ are asymptotic to a sine (or cosine) curve with period 1. One way to state this, taken from [19, 24.11.5], gives the following asymptotic formula. As $n \to \infty$,

$$(-1)^{\lfloor n/2 \rfloor - 1} \frac{(2\pi)^n}{2(n!)} B_n(x) \to \begin{cases} \cos 2\pi x, & n \text{ even}, \\ \sin 2\pi x, & n \text{ odd}, \end{cases} \tag{3.1}$$

where the convergence is uniform in x on compact subsets of $\mathbb{C}$. For example, Figure

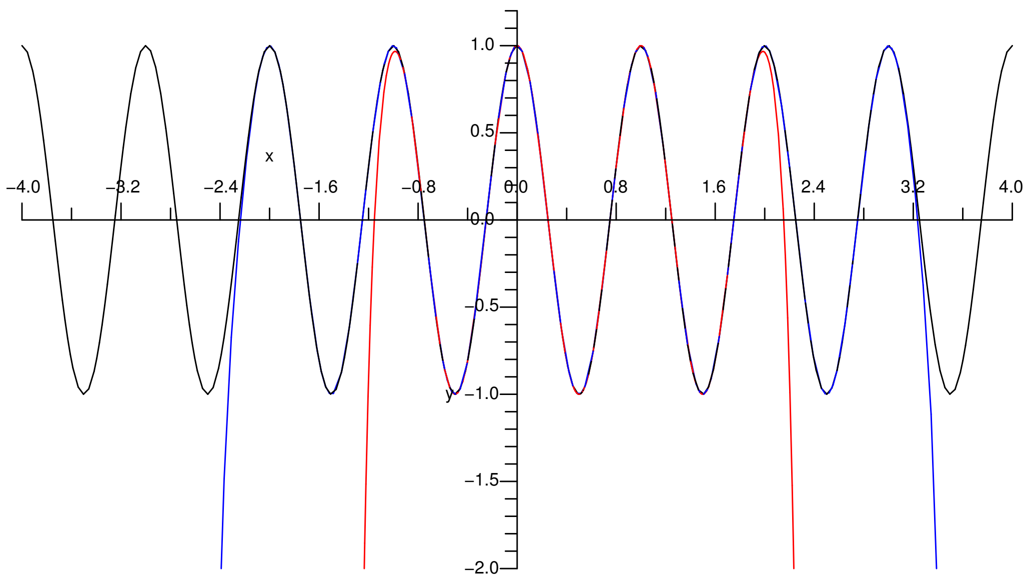

Figure 3.1: $\cos 2\pi x$, $B_{20}(x)$, and $B_{40}(x)$ scaled as in (3.1)

3.1 shows $B_{20}(x)$ and $B_{40}(x)$ (scaled appropriately) with the cosine curve.

Similarly, [19, 24.11.6] gives an asymptotic formula for $E_n(x)$. As $n \to \infty$, we have

$$(-1)^{\lfloor (n+1)/2 \rfloor} \frac{\pi^{n+1}}{4(n!)} E_n(x) \to \begin{cases} \sin \pi x, & n \text{ even}, \\ \cos \pi x, & n \text{ odd}, \end{cases} \tag{3.2}$$

where the convergence is uniform in x on compact subsets of $\mathbb{C}$. We see that the Euler polynomials (Figure 3.2) are asymptotic to a curve of period 2.

In [4], there is another way to get the above two results. First, we define the *uniform Strodt polynomials* on m points [4] for $m \in \mathbb{N}$ and $m \geq 2$. We choose m distinct points evenly distributed between 0 and 1 and including the endpoints (e.g. for $m = 4$ the points would be 0, 1/3, 2/3, and 1). The value of the weight function for each point is $\frac{1}{m}$. Then, from 2.10 we know that the exponential generating function

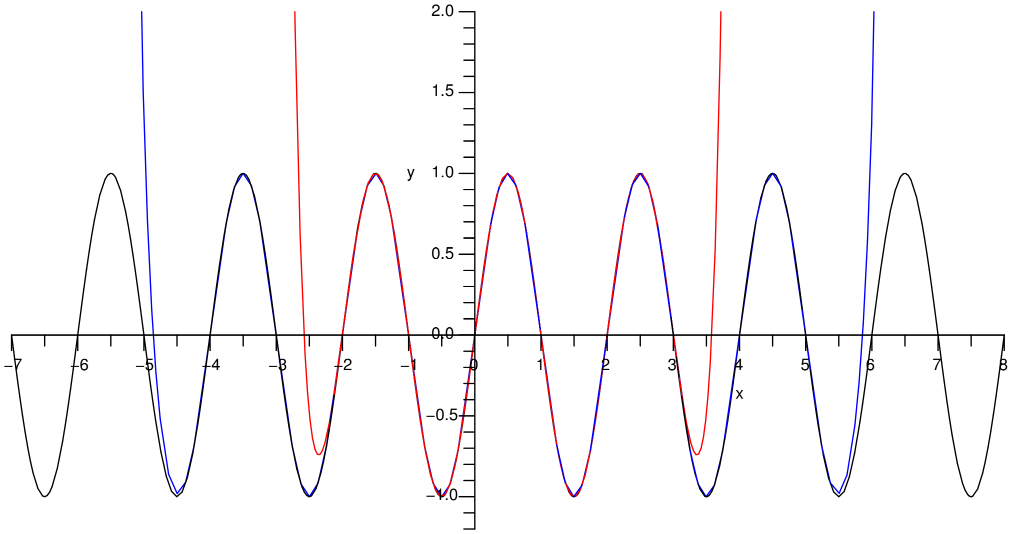

Figure 3.2: $\sin \pi x$, $E_{20}(x)$, $E_{40}(x)$ scaled as in (3.2)

for the uniform Strodt polynomials on these m points is

$$\sum_{n \geq 0} P_n^m(x) \frac{t^n}{n!} = \frac{m e^{xt}}{\sum_{j=0}^{m-1} e^{\frac{jt}{m-1}}} \ .$$

[4] gives a proof for the asymptotic behavior of this family. We state their result here.

Theorem 3.1.1. *[4, Thm 7.1] Let $x \in \mathbb{R}$. For all $m \geq 2$, as $n \to \infty$,*

$$\frac{\csc\left(\frac{\pi}{m}\right)}{2m} \frac{\pi^{n+1}}{n!} \left(\frac{2m-2}{m}\right)^n P_n^m(x) \sim \cos\left(\frac{2m-2}{m}\pi x + \frac{n\pi}{2}\right) \ . \tag{3.3}$$

Note that when $m = 2$, we get the generating function for the Euler polynomials and that the asymptotics match 3.2 above. Also, as $m \to \infty$, we get the generating function for Bernoulli polynomials and the asymptotics match 3.1 above.

At this point it is natural to wonder if all Strodt polynomials will behave in this fashion. That is the topic for the remainder of this chapter and the next.

3.2 Darboux's Method

Studying asymptotic behavior of functions can be very difficult at times. There are many places where the details of describing behavior near infinity can be troublesome. It is helpful to have powerful tools and theorems at your disposal to navigate these potential pitfalls.

In the following sections we will be employing Darboux's Method. Loosely speaking, suppose $F = \sum f_n x^n$ is a meromorphic function. If you can find a "comparison function" with certain properties, then Darboux's Method allows you to deduce the asymptotic behavior of f_n by studying the asymptotic behavior of the terms of the comparison function.

The exact method is in the following theorem. Note that we follow the presentation of Olver [21, §8.9] and only give one version of the method. Olver gives more details, less restrictive versions of the method, several examples, and a proof. Other examples of the method can be found in [20, §11], [26, §VIII.4], and [28, §5.3].

Theorem 3.2.1 (Darboux's Method [8], see also [21])**.** *Let $F(t)$ be a given function with*

$$F(t) = \sum_{n=-\infty}^{\infty} f_n t^n$$

and suppose F is analytic in $0 < |t| < r$. Let r (or $-r$) be real and the location of F's singularity nearest the origin. Suppose that we can find a comparison function $G(t)$,

$$G(t) = \sum_{n=-\infty}^{\infty} g_n t^n \ ,$$

which has the following properties:

i $G(t)$ is holomorphic in $0 < |t| < r$.

ii $F(t) - G(t)$ *is continuous in* $0 < |t| \leq r$.

iii *The coefficients* g_n *have known asymptotic behavior.*

Further, suppose $F - G$ *is* b *times continuously differentiable. Then*

$$f_n = g_n + o(r^{-n}n^{-b})$$

and $f_n \sim g_n$ *as* $n \to \infty$.

The "little o" notation used in this theorem is interpreted to mean that $\frac{f_n - g_n}{r^{-n}n^{-b}} \to 0$ as $n \to \infty$ (see [21, §1.2]).

We will apply Darboux's method to the context of the Strodt polynomials. We will be able to deduce the asymptotic behavior of a family of polynomials by studying the behavior of its exponential generating function *if* we can find a suitable comparison function. In the next few sections, we will demonstrate ways to do this.

3.3 Strodt Polynomial Asymptotics - An Example

In the next chapter we will state and prove theorems about the asymptotic behavior of several general families of Strodt polynomials. First, however, we will deduce the asymptotics of one particular family. The proofs in the subsequent sections use the same techniques we will demonstrate here. The goal is to elaborate on the steps in this section to enable to reader to follow and understand the methods being used.

The overall structure of our proofs is modeled after the proof in [4, §7]. However, their proof was not originally constructive. They experimented to conjecture a result and built a proof toward reaching this desired goal. We have taken their ideas and adapted them to use in our proofs.

To begin, recall that any probability weight function as in 2.8 leads to a unique Strodt polynomial family (see Proposition 2.2.1). In this section, we will take a specific weight function as an example and then deduce the asymptotic behavior of the resulting polynomials.

Consider the probability weight function defined by

$$\left\{\frac{1}{4}, \frac{3}{4}\right\} \text{ where } w\left(\frac{1}{4}\right) = w\left(\frac{3}{4}\right) = \frac{1}{2},$$

and let $P_n(x)$ be the Strodt polynomials determined by these points and weights. According to 2.10, the exponential generating function for $P_n(x)$ is

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^{\frac{1}{4}t} + e^{\frac{3}{4}t}} . \tag{3.4}$$

To save space, let q be the right hand side of 3.4.

Our goal here is to show that as $n \to \infty$, the polynomials $P_n(x)$ are sinusoidal. In order to get q from an expression with exponentials to something involving sine, we will apply Fact 2.1.2. First, we use algebraic manipulation and notice that

$$\begin{aligned} q &= \frac{2e^{xt}}{e^{\frac{1}{4}t} + e^{\frac{3}{4}t}} \\ &= \frac{2e^{xt}}{e^{\frac{1}{4}t}\left(1 + e^{\frac{1}{2}t}\right)} \\ &= \frac{2e^{xt}}{e^{\frac{1}{4}t}\left(\frac{e^t - 1}{e^{\frac{1}{2}t} - 1}\right)}. \end{aligned}$$

Now we make a change of variables, replacing t by πiz, and apply Fact 2.1.2 to obtain

$$\begin{aligned} q &= \frac{2e^{x\pi iz}\, e^{-\frac{1}{4}\pi iz}}{\left(\dfrac{e^{\pi iz}-1}{e^{\frac{1}{2}\pi iz}-1}\right)} \\ &= \frac{2e^{x\pi iz}\, e^{-\frac{1}{4}\pi iz}}{e^{\frac{1}{4}\pi iz}\,\dfrac{\sin\left(\frac{\pi}{2}z\right)}{\sin\left(\frac{\pi}{4}z\right)}} \\ &= \frac{2e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\,\sin\left(\frac{\pi}{4}z\right)}{\sin\left(\frac{\pi}{2}z\right)}. \end{aligned}$$

We also need to do the same change of variables on the left hand side of 3.4, resulting in

$$\sum_{n=0}^{\infty} P_n(x)\frac{(\pi i)^n}{n!}\, z^n = q\ .$$

Because of the i^n in the sum, we know that for odd n we have imaginary terms and for even n we have real terms. Also, we can apply Fact 2.1.1 to split q into real and imaginary parts. Thus, the real part gives the generating function for the even polynomials

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}(x)z^{2n} = \frac{2\cos\left(\pi xz-\frac{\pi}{2}z\right)\sin\left(\frac{\pi}{4}z\right)}{\sin\left(\frac{\pi}{2}z\right)}, \tag{3.5}$$

and the imaginary part gives the generating function for the odd polynomials

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}P_{2n+1}(x)z^{2n+1} = \frac{2\sin\left(\pi xz-\frac{\pi}{2}z\right)\sin\left(\frac{\pi}{4}z\right)}{\sin\left(\frac{\pi}{2}z\right)}. \tag{3.6}$$

Take the even case first. Define $F(z)$ to be the right hand side of 3.5. Thus,

$$F(z) := \frac{2\cos\left(\pi xz-\frac{\pi}{2}z\right)\sin\left(\frac{\pi}{4}z\right)}{\sin\left(\frac{\pi}{2}z\right)}.$$

Note that F has singularities at $z = \pm 2$. To apply Darboux's Method we will need a comparison function, call it $G(z)$, with the same singularities as $F(z)$. But we also need to know the asymptotic behavior of this comparison function. It will be logical to use something of the form

$$\frac{1}{(2-z)(2+z)} = \frac{1}{4-z^2} = \sum_{n\geq 0} \frac{1}{2^{2n+2}} z^{2n} \tag{3.7}$$

(see 2.1.4). Now we need to determine what we should use as the numerator of $G(z)$.

We compute the limit

$$\begin{aligned}
&\lim_{z\to 2^-} (2-z)(2+z)F(z) \\
&\quad = \left(\lim_{z\to 2^-} \frac{(2-z)}{\sin\left(\frac{\pi}{2}z\right)}\right) \cdot \left(\lim_{z\to 2^-} (2+z)2\cos\left(\pi x z - \frac{\pi}{2}z\right)\sin\left(\frac{\pi}{4}z\right)\right) \\
&\quad = \left(\frac{2}{\pi}\right)(4)(2)\left(sin\frac{\pi}{2}\right) \cdot \cos\left(2\pi x - \pi\right) \\
&\quad = \left(\frac{16}{\pi}\right)\cos\left(2\pi x - \pi\right)
\end{aligned}$$

by using L'Hôpital's Rule on the first limit and substitution on the second. Also, notice that the limit as $z \to -2^+$ is equal to the above result.

Now we construct our comparison function. Set

$$G(z) = \frac{\left(\frac{16}{\pi}\right)\cos\left(2\pi x - \pi\right)}{4-z^2}$$

and notice that

$$g_{2n} = \left(\frac{1}{2^{2n+2}}\right)\left(\frac{16}{\pi}\right)\cos\left(2\pi x - \pi\right).$$

This G has the desired properties. Namely, with $r = 2$, we have G with no singularities between 0 and 2 and we have $F - G$ with removable singularities at $z = \pm 2$. Also,

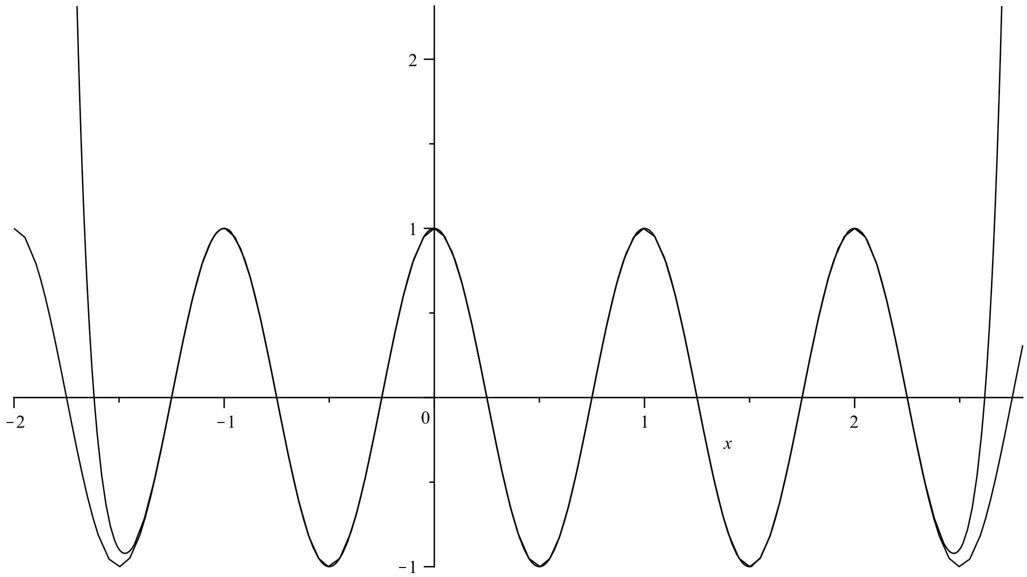

Figure 3.3: Illustration of Theorem 3.3.1 for $n = 30$

$F - G$ is b times differentiable for any b (because it consists of polynomials and trigonometric functions).

Thus, we apply Darboux's Method to get that

$$\frac{(-1)^n \pi^{2n}}{(2n)!} P_{2n}(x) \sim \left(\frac{1}{2^{2n+2}}\right) \left(\frac{16}{\pi}\right) \cos(2\pi x - \pi) + o(r^{-n} n^{-b})$$

holds for any $b \in \mathbb{N}$.

Now that we have finished the even terms, we go back and look at the odd terms. Setting F equal to the right hand side of 3.6 we get

$$F(z) = \frac{2 \sin\left(\pi x z - \frac{\pi}{2} z\right) \sin\left(\frac{\pi}{4} z\right)}{\sin\left(\frac{\pi}{2} z\right)}.$$

Notice that only the numerator has changed. So we use $r = 2$ again. Also we can check that the limit does not change. So, the only changes we need to make for our new comparison function are changing the cosine function to a sine function and

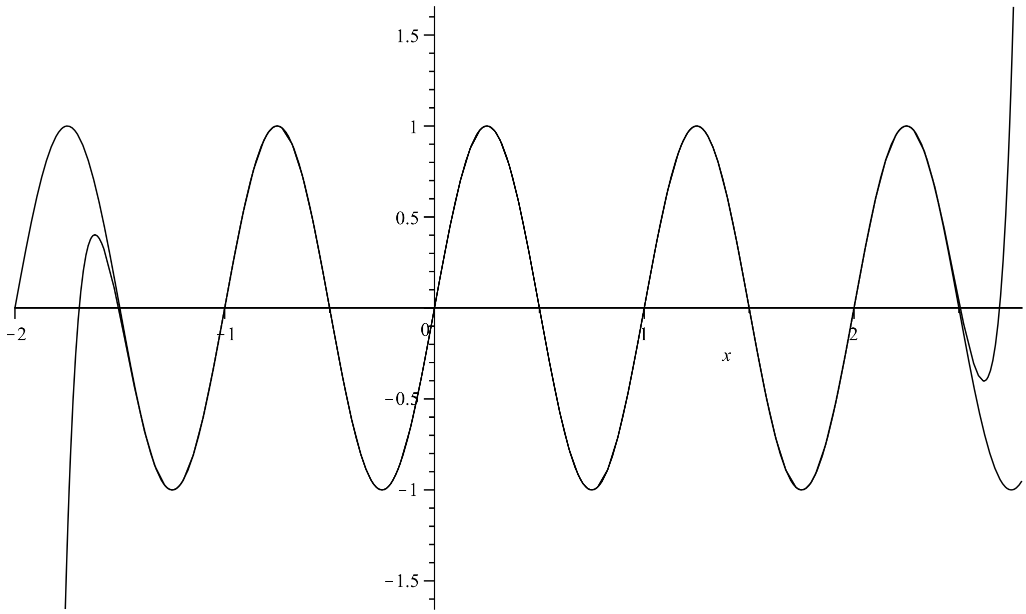

Figure 3.4: Illustration of Theorem 3.3.1 for $n = 31$

adjusting the series to have odd terms instead. For the latter change we note from Fact 2.1.4 that

$$\frac{z}{(2-z)(2+z)} = \frac{z}{4-z^2} = \sum_{n\geq 0} \frac{1}{2^{2n+2}} z^{2n+1}$$

Thus, we will set

$$G(z) = \frac{z\left(\frac{16}{\pi}\right)\sin\left(2\pi x - \pi\right)}{4-z^2}.$$

For this choice of G, we observe that $F - G$ has the desired properties to apply Darboux's Method again. The result is

$$\frac{(-1)^n \pi^{2n+1}}{(2n+1)!} P_{2n+1}(x) \sim \left(\frac{1}{2^{2n+1+2}}\right)\left(\frac{16}{\pi}\right)\sin\left(2\pi x - \pi\right) + o(r^{-n}n^{-b}),$$

which holds for any $b \in \mathbb{N}$.

It remains to combine the results from the even and odd cases into a final asymptotic expression. We can change the sine in the odd case to cosine by shifting an odd multiple of π at every odd term. The natural shift, then, is πn. In addition, we can change the alternating signs into a shift by adding $\frac{1}{2}\pi n$. Thus, we will insert a shifting term of $\frac{3}{2}\pi n$.

Combining all of this together, therefore, we have proved the following:

Theorem 3.3.1. *We take the points* $\left\{\frac{1}{4}, \frac{3}{4}\right\}$ *each weighted* $1/2$ *and let* $P_n(x)$ *be the resulting family of Strodt polynomials. Then*

$$\left(\frac{\pi}{16}\right)\left(2^{n+2}\right)\frac{\pi^n}{(n)!}P_n(x) \sim \cos\left(2\pi x - \pi + \frac{3}{2}\pi n\right)$$

as $n \to \infty$.

Chapter 4

Asymptotics of Strodt Families

In the next two sections we will state and prove several theorems about the asymptotic behavior of some general Strodt families. We end the chapter with some experimental evidence that not all Strodt polynomials have the same asymptotics.

4.1 Fixed Point Families

Theorem 3.3.1 gives the asymptotics of the Strodt polynomials which arise from the points 1/4 and 3/4 when each are assigned a weight of 1/2. The following theorem broadens this result and establishes the asymptotics of a more general 2-point family.

Theorem 4.1.1. *For some real number $k > 2$, we choose the points $\{\frac{1}{k}, \frac{k-1}{k}\}$ with each weighted 1/2 and let $P_n^k(x)$ be the resulting family of Strodt polynomials. Then*

$$\left(\frac{\pi(k-2)^2}{4k^2}\right)\left(\frac{k}{k-2}\right)^{n+2}\frac{\pi^n}{(n)!}P_n^k(x) \sim \cos\left(\frac{k\pi}{k-2}x - \frac{k\pi}{2(k-2)} + \frac{3}{2}\pi n\right) \tag{4.1}$$

as $n \to \infty$.

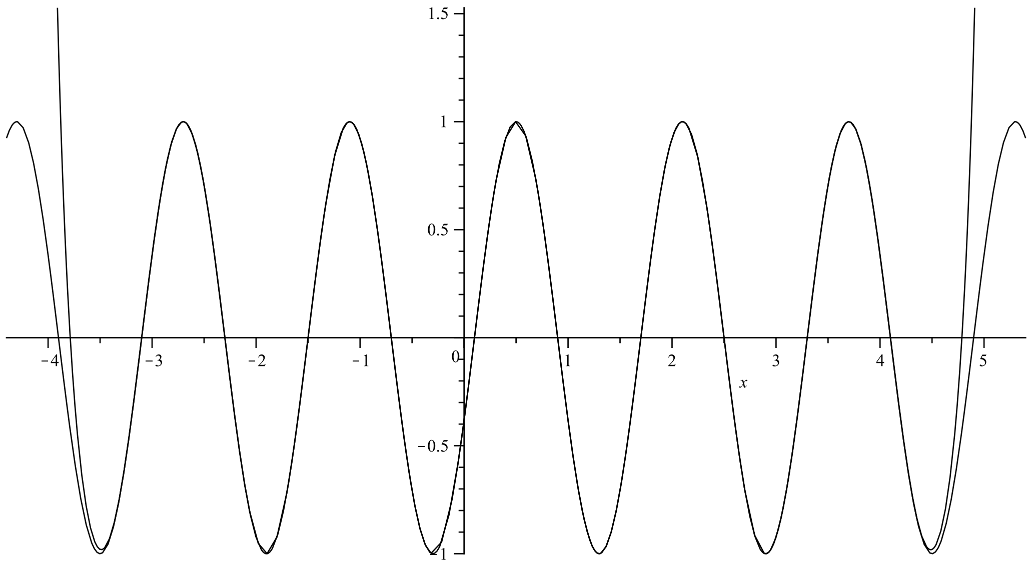

Figure 4.1: Illustration of Theorem 4.1.1 for $k = 10$ and $n = 40$

Proof. Let q be the exponential generating function for $P_n^k(x)$. Then

$$\sum_{n\geq 0} P_n^k(x)\frac{t^n}{n!} = q$$

Applying 2.10, we can write

$$\begin{aligned} q &= \frac{2e^{xt}}{e^{\frac{1}{k}t} + e^{\frac{k-1}{k}t}} \\ &= \frac{2e^{xt}}{e^{\frac{1}{k}t}\left(1 + e^{\frac{k-2}{k}t}\right)} \\ &= \frac{2e^{xt}}{e^{\frac{1}{k}t}\left(\frac{e^{\frac{2(k-2)}{k}t} - 1}{e^{\frac{k-2}{k}t} - 1}\right)}. \end{aligned}$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain

the following:

$$
\begin{aligned}
q &= \frac{2e^{x\pi iz}\,e^{-\frac{1}{k}\pi iz}}{\left(\dfrac{e^{\frac{2(k-2)}{k}\pi iz}-1}{e^{\frac{k-2}{k}\pi iz}-1}\right)} \\
&= \frac{2e^{x\pi iz}\,e^{-\frac{1}{k}\pi iz}}{e^{\frac{k-2}{2k}\pi iz}\,\dfrac{\sin\left(\frac{(k-2)\pi}{k}z\right)}{\sin\left(\frac{(k-2)\pi}{2k}z\right)}} \\
&= \frac{2e^{x\pi iz}\,e^{-\frac{1}{2}\pi iz}\,\sin\left(\frac{(k-2)\pi}{2k}z\right)}{\sin\left(\frac{(k-2)\pi}{k}z\right)}.
\end{aligned}
$$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$
\sum_{n\geq 0}\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}(x)z^{2n} = \frac{2\sin\left(\frac{(k-2)\pi}{2k}z\right)\cos\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{(k-2)\pi}{k}z\right)},
$$

and

$$
\sum_{n\geq 0}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}P_{2n+1}(x)z^{2n+1} = \frac{2\sin\left(\frac{(k-2)\pi}{2k}z\right)\sin\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{(k-2)\pi}{k}z\right)}.
$$

Set the right hand side of the even sum equal to $F(z)$ and notice that F has singularities at $z=\pm\frac{k}{k-2}$. For convenience, we will set $r=\frac{k}{k-2}$. Now we compute the limits

$$
\begin{aligned}
&\lim_{z\to r^-}\left(\frac{k}{k-2}-z\right)\left(\frac{k}{k-2}+z\right)F(z) \\
&\quad=\left(\frac{k}{\pi(k-2)}\right)\left(\frac{2k}{k-2}\right)(2)\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{k\pi}{k-2}x-\frac{k\pi}{2(k-2)}\right) \\
&\quad=\lim_{z\to -r^+}\left(\frac{k}{k-2}-z\right)\left(\frac{k}{k-2}+z\right)F(z)
\end{aligned}
$$

Let a comparison function be

$$G(z) = \frac{\left(\frac{4k^2}{\pi(k-2)^2}\right)\cos\left(\frac{k\pi}{k-2}x - \frac{k\pi}{2(k-2)}\right)}{\left(\frac{k}{k-2} - z\right)\left(\frac{k}{k-2} + z\right)}.$$

We note that $F - G$ has removable singularities, is continuous on $0 < |z| \leq r$, and is b times differentiable for any $b \in \mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n \pi^{2n}}{(2n)!} P_{2n}^k(x) \sim \left(\frac{k-2}{k}\right)^{n+2} \left(\frac{4k^2}{\pi(k-2)^2}\right) \cos\left(\frac{k\pi}{k-2}x - \frac{k\pi}{2(k-2)}\right) + o(r^{-n}n^{-b})$$

for $b \in \mathbb{N}$. For the odd case, we use the comparison function

$$G(z) = \frac{z\left(\frac{4k^2}{\pi(k-2)^2}\right)\sin\left(\frac{k\pi}{k-2}x - \frac{k\pi}{2(k-2)}\right)}{\left(\frac{k}{k-2} - z\right)\left(\frac{k}{k-2} + z\right)}.$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows. □

Next, we state a result for a generalized 3-point family.

Theorem 4.1.2. *For some real $k > 2$, choose the points $\{\frac{1}{k}, \frac{1}{2}, \frac{k-1}{k}\}$ with each weighted $1/3$ and let $P_n^k(x)$ be the resulting family of Strodt polynomials. Then*

$$\left(\frac{3\pi(k-2)^2}{16k^2\sqrt{3}}\right)\left(\frac{4k}{3(k-2)}\right)^{n+2} \frac{\pi^n}{(n)!} P_n^k(x) \sim \cos\left(\frac{4k\pi}{3(k-2)}x - \frac{2k\pi}{3(k-2)} + \frac{3}{2}\pi n\right)$$

as $n \to \infty$.

Proof. Let q be the exponential generating function for $P_n^k(x)$. Then

$$\sum_{n \geq 0} P_n^k(x) \frac{t^n}{n!} = q$$

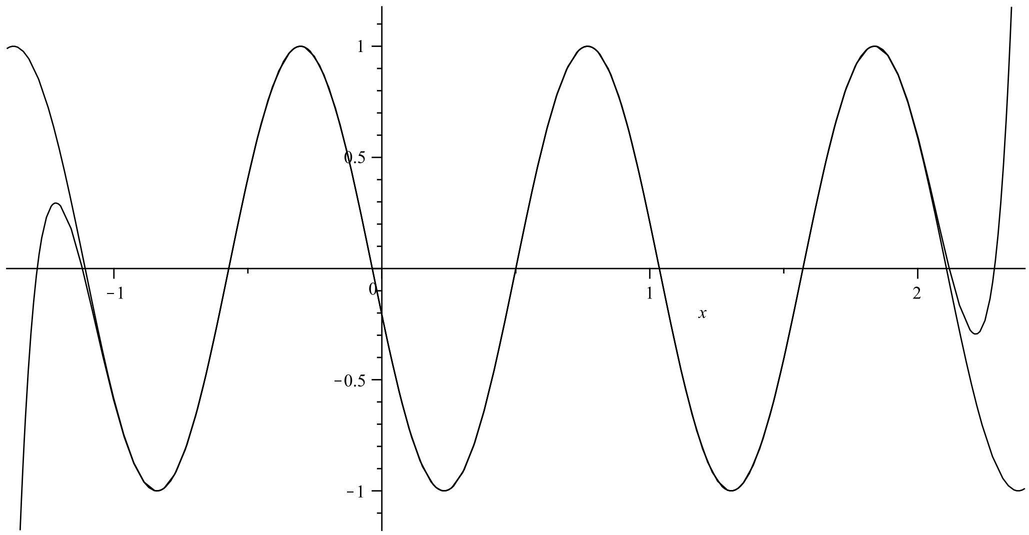

Figure 4.2: Illustration of Theorem 4.1.2 for $k = 7$ and $n = 21$

Applying 2.10, we can write

$$\begin{aligned} q &= \frac{3e^{xt}}{e^{\frac{1}{k}t} + e^{\frac{1}{2}t} + e^{\frac{k-1}{k}t}} \\ &= \frac{3e^{xt}}{e^{\frac{1}{k}t}\left(1 + e^{\frac{k-2}{2k}t} + e^{\frac{k-2}{k}t}\right)} \\ &= \frac{3e^{xt}}{e^{\frac{1}{k}t}\left(\dfrac{e^{\frac{3(k-2)}{2k}t} - 1}{e^{\frac{k-2}{2k}t} - 1}\right)}. \end{aligned}$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain the following:

$$\begin{aligned} q &= \frac{3e^{x\pi iz}}{e^{\frac{1}{k}\pi iz}\, e^{\frac{k-2}{2k}\pi iz}\, \dfrac{\sin\left(\frac{3(k-2)\pi}{4k}z\right)}{\sin\left(\frac{(k-2)\pi}{4k}z\right)}} \\ &= \frac{3e^{x\pi iz}\, e^{-\frac{1}{k}\pi iz}\, e^{-\frac{k-2}{2k}\pi iz} \sin\left(\frac{(k-2)\pi}{4k}z\right)}{\sin\left(\frac{3(k-2)\pi}{4k}z\right)}. \end{aligned}$$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}(x)z^{2n}=\frac{3\sin\left(\frac{(k-2)\pi}{4k}z\right)\cos\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{3(k-2)\pi}{4k}z\right)},$$

and

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}P_{2n+1}(x)z^{2n+1}=\frac{3\sin\left(\frac{(k-2)\pi}{4k}z\right)\sin\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{3(k-2)\pi}{4k}z\right)}.$$

Set the right hand side of the even sum equal to $F(z)$ and notice that F has singularities at $z=\pm\frac{4k}{3(k-2)}$. For convenience, we will set $r=\frac{4k}{3(k-2)}$. Now we compute the limits

$$\begin{aligned}
&\lim_{z\to r^-}\left(\frac{4k}{3(k-2)}-z\right)\left(\frac{4k}{3(k-2)}+z\right)F(z)\\
&=\left(\frac{4k}{3\pi(k-2)}\right)\left(\frac{8k}{3(k-2)}\right)(3)\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{4k\pi}{3(k-2)}x-\frac{2k\pi}{3(k-2)}\right)\\
&=\lim_{z\to -r^+}\left(\frac{4k}{3(k-2)}-z\right)\left(\frac{4k}{3(k-2)}+z\right)F(z).
\end{aligned}$$

Set

$$C_k=\left(\frac{16k^2\sqrt{3}}{3\pi(k-2)^2}\right),$$

and let a comparison function be

$$G(z)=\frac{C_k\cos\left(\frac{4k\pi}{3(k-2)}x-\frac{2k\pi}{3(k-2)}\right)}{\left(\frac{4k}{3(k-2)}-z\right)\left(\frac{4k}{3(k-2)}+z\right)}.$$

We note that $F-G$ has removable singularities, is continuous on $0<|z|\leq r$, and is b times differentiable for any $b\in\mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n\pi^{2n}}{(2n)!}P^k_{2n}(x) \sim \left(\frac{3(k-2)}{4k}\right)^{n+2} C_k \cos\left(\frac{4k\pi}{3(k-2)}x - \frac{2k\pi}{3(k-2)}\right) + o(r^{-n}n^{-b})$$

for $b \in \mathbb{N}$. For the odd case, we use the comparison function

$$G(z) = \frac{zC_k \sin\left(\frac{4k\pi}{3(k-2)}x - \frac{2k\pi}{3(k-2)}\right)}{\left(\frac{4k}{3(k-2)} - z\right)\left(\frac{4k}{3(k-2)} + z\right)}.$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows. □

The next theorem describes the asymptotics of a generalized 5-point family of Strodt polynomials.

Theorem 4.1.3. *For some rational $k > 4$, we choose the points $\{\frac{k-4}{2k}, \frac{k-2}{2k}, \frac{1}{2}, \frac{k+2}{2k}, \frac{k+4}{2k}\}$ with each weighted $1/5$ and let $P^k_n(x)$ be the resulting family of Strodt polynomials. Then*

$$\left(\frac{5\pi}{8k^2\sin\left(\frac{\pi}{5}\right)}\right)\left(\frac{2k}{5}\right)^{n+2}\frac{\pi^n}{(n)!}P^k_n(x) \sim \cos\left(\frac{2k\pi}{5}x - \frac{k\pi}{5} + \frac{3}{2}\pi n\right) \qquad (4.2)$$

as $n \to \infty$.

Proof. Let q be the exponential generating function for $P^k_n(x)$. Then

$$\sum_{n\geq 0} P^k_n(x)\frac{t^n}{n!} = q$$

Applying 2.10, we can write

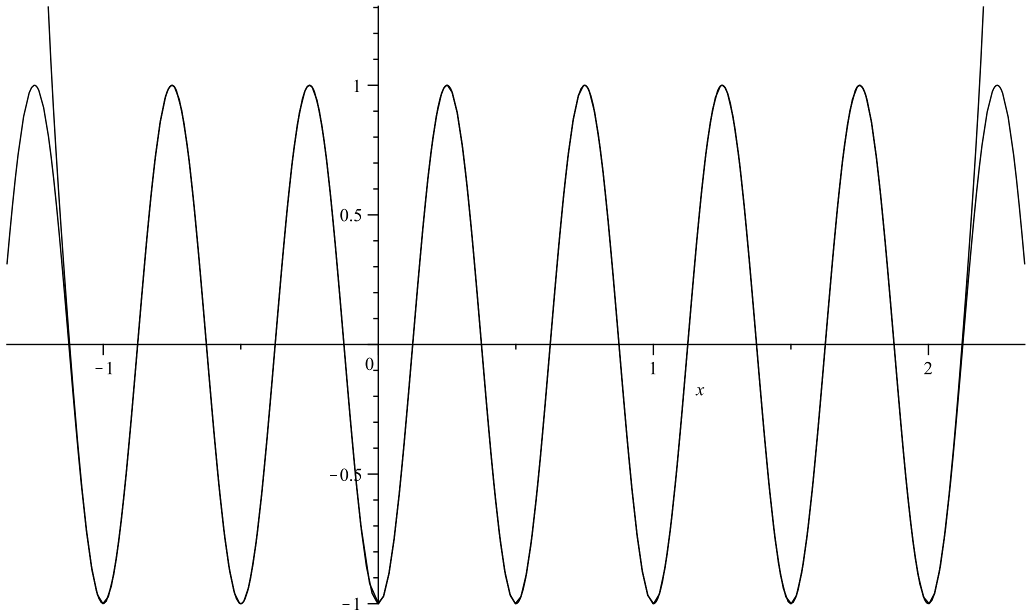

Figure 4.3: Illustration of Theorem 4.1.3 for $k = 10$ and $n = 50$

$$\begin{aligned} q &= \frac{5e^{xt}}{e^{\frac{k-4}{2k}t} + e^{\frac{k-2}{2k}t} + e^{\frac{1}{2}t} + e^{\frac{k+2}{2k}t} + e^{\frac{k+4}{2k}t}} \\ &= \frac{5e^{xt}}{e^{\frac{k-4}{2k}t}\left(1 + e^{\frac{1}{k}t}e^{\frac{2}{k}t} + e^{\frac{3}{k}t} + e^{\frac{4}{k}t}\right)} \\ &= \frac{5e^{xt}}{e^{\frac{k-4}{2k}t}\left(\dfrac{e^{\frac{5}{k}t}-1}{e^{\frac{1}{k}t}-1}\right)}. \end{aligned}$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain

the following:

$$\begin{aligned} q &= \frac{5e^{x\pi iz}\, e^{-\frac{k-4}{2k}\pi iz}}{\left(\dfrac{e^{\frac{5}{k}\pi iz}-1}{e^{\frac{1}{k}\pi iz}-1}\right)} \\ &= \frac{5e^{x\pi iz}\, e^{-\frac{k-4}{2k}\pi iz}}{e^{\frac{2}{k}\pi iz}\,\dfrac{\sin\left(\frac{5\pi}{2k}z\right)}{\sin\left(\frac{\pi}{2k}z\right)}} \\ &= \frac{5e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\,\sin\left(\frac{\pi}{2k}z\right)}{\sin\left(\frac{5\pi}{2k}z\right)}. \end{aligned}$$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}(x)z^{2n} = \frac{5\sin\left(\frac{\pi}{2k}z\right)\cos\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{5\pi}{2k}z\right)},$$

and

$$\sum_{n\geq 0}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}P_{2n+1}(x)z^{2n+1} = \frac{5\sin\left(\frac{\pi}{2k}z\right)\sin\left(\pi xz-\frac{1}{2}\pi z\right)}{\sin\left(\frac{5\pi}{2k}z\right)}$$

Set the right hand side of the even sum equal to $F(z)$ and notice that F has singularities at $z=\pm\frac{2k}{5}$. For convenience, we will set $r=\frac{2k}{5}$. Now we compute the limits

$$\begin{aligned} &\lim_{z\to r^-}\left(\frac{2k}{5}-z\right)\left(\frac{2k}{5}+z\right)F(z) \\ &\quad = \left(\frac{2k}{5\pi}\right)\left(\frac{4k}{5}\right)(5)\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{2k\pi}{5}x-\frac{k\pi}{5}\right) \\ &\quad = \lim_{z\to -r^+}\left(\frac{2k}{5}-z\right)\left(\frac{2k}{5}+z\right)F(z). \end{aligned}$$

Let a comparison function be

$$G(z) = \frac{\left(\frac{8k^2 \sin\left(\frac{\pi}{5}\right)}{5\pi}\right) \cos\left(\frac{2k\pi}{5}x - \frac{k\pi}{5}\right)}{\left(\frac{2k}{5} - z\right)\left(\frac{2k}{5} + z\right)}$$

We note that $F - G$ has removable singularities, is continuous on $0 < |z| \leq r$, and is b times differentiable for any $b \in \mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n \pi^{2n}}{(2n)!} P_{2n}^k(x) \sim \left(\frac{5}{2k}\right)^{n+2} \left(\frac{8k^2 \sin\left(\frac{\pi}{5}\right)}{5\pi}\right) \cos\left(\frac{2k\pi}{5}x - \frac{k\pi}{5}\right) + o(r^{-n}n^{-b})$$

for $b \in \mathbb{N}$. For the odd case, we use the comparison function

$$G(z) = \frac{z\left(\frac{8k^2 \sin\left(\frac{\pi}{5}\right)}{5\pi}\right) \sin\left(\frac{2k\pi}{5}x - \frac{k\pi}{5}\right)}{\left(\frac{2k}{5} - z\right)\left(\frac{2k}{5} + z\right)}$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows. □

There are additional plots in Appendix A which give examples of the theorems in this section.

4.2 Families with m Points

The next theorem is a result similar to Theorem 3.1.1. The difference here is we do not include the endpoints (0 and 1) as two of the m points. We know from Section 2.2 that each different probability density function leads to a unique family of Strodt polynomials. The asymptotics of this particular family are as follows.

Theorem 4.2.1. *Suppose we have m points with $m \geq 2$. Let the points be*

$$\left\{\frac{1}{2m}, \frac{3}{2m}, \ldots, \frac{2m-1}{2m}\right\}$$

each assigned a weight of $1/m$. Let $P_n^m(x)$ be the resulting family of Strodt polynomials. Then

$$\left(\frac{\pi}{8m\sin\left(\frac{\pi}{m}\right)}\right)(2)^{n+2}\frac{\pi^n}{(n)!}P_n^m(x) \sim \cos\left(2\pi x - \pi + \frac{3}{2}\pi n\right)$$

as $n \to \infty$.

Proof. Let q be the exponential generating function for $P_n^m(x)$. Then

$$\sum_{n \geq 0} P_n^m(x)\frac{t^n}{n!} = q$$

Applying 2.10, we can write

$$\begin{aligned} q &= \frac{me^{xt}}{\sum_{j=0}^{m-1} e^{\frac{2j+1}{2m}t}} \\ &= \frac{me^{xt}}{e^{\frac{1}{2m}t}\left(1 + e^{\frac{1}{m}t} + e^{\frac{2}{m}t} + \ldots + e^{\frac{m-1}{m}t}\right)} \\ &= \frac{me^{xt}}{e^{\frac{1}{2m}t}\left(\frac{e^t - 1}{e^{\frac{1}{m}t} - 1}\right)} \end{aligned}$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain

the following:

$$
\begin{aligned}
q &= \frac{me^{x\pi iz}\, e^{-\frac{1}{2m}\pi iz}}{\left(\dfrac{e^{\pi iz}}{e^{\frac{1}{m}\pi iz}-1}\right)} \\
&= \frac{me^{x\pi iz}\, e^{-\frac{1}{2m}\pi iz}}{e^{\frac{m-1}{2m}\pi iz}\, \dfrac{\sin\left(\frac{\pi}{2}z\right)}{\sin\left(\frac{\pi}{2m}z\right)}} \\
&= \frac{me^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\, \sin\left(\frac{\pi}{2m}z\right)}{\sin\left(\frac{\pi}{2}z\right)}
\end{aligned}
$$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$
\sum_{n\geq 0} P_{2n}^{m}(x)\frac{(-1)^n\pi^{2n}}{(2n)!}z^{2n} = \frac{m\cos\left(\pi xz - \frac{\pi}{2}z\right)\sin\left(\frac{\pi}{2m}z\right)}{\sin\left(\frac{\pi}{2}z\right)},
$$

and

$$
\sum_{n\geq 0} \frac{(-1)^n\pi^{2n+1}}{(2n+1)!}P_{2n+1}^{m}(x)z^{2n+1} = \frac{m\sin\left(\pi xz - \frac{\pi}{2}z\right)\sin\left(\frac{\pi}{2m}z\right)}{\sin\left(\frac{\pi}{2}z\right)}.
$$

Set the right hand side of the even sum equal to $F(z)$ and notice that F has singularities at $z = \pm 2$. For convenience, we will set $r = 2$. Now we compute the limits

$$
\begin{aligned}
&\lim_{z\to r^-}(2-z)(2+z)F(z) \\
&\quad = \left(\frac{2}{\pi}\right)(4)(m)\sin\left(\frac{\pi}{m}\right)\cos\left(2\pi x - \pi\right) \\
&\quad = \lim_{z\to -r^+}(2-z)(2+z)F(z).
\end{aligned}
$$

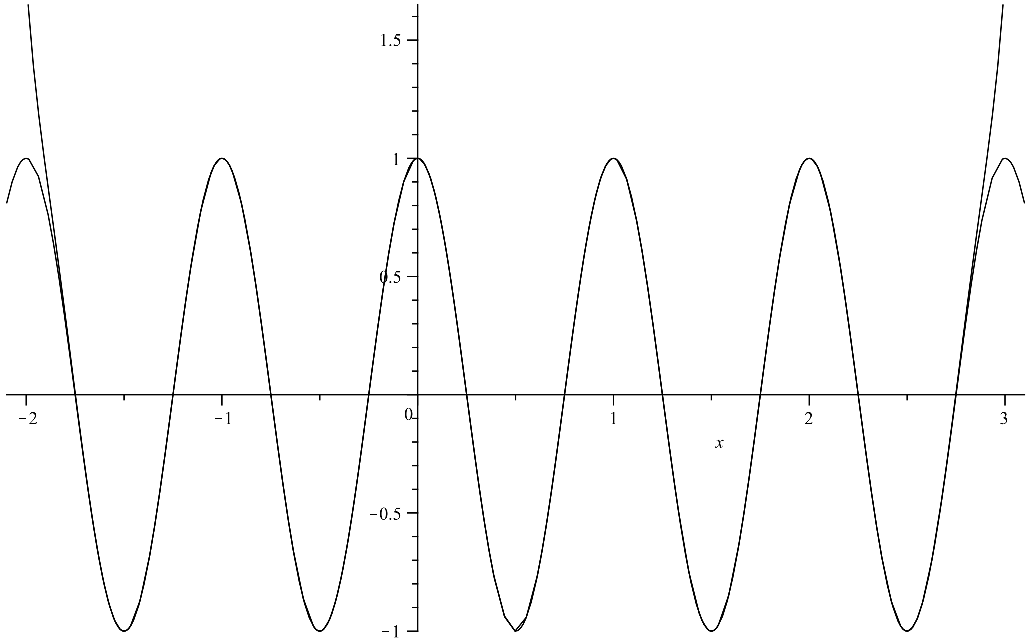

Figure 4.4: Illustration of Theorem 4.2.1 for $m = 10$ and $n = 34$

Let a comparison function be

$$G(z) = \frac{\left(\frac{8m}{\pi}\right)\sin\left(\frac{\pi}{m}\right)\cos\left(2\pi x - \pi\right)}{(2-z)(2+z)}.$$

We note that $F - G$ has removable singularities, is continuous on $0 < |z| \leq r$, and is b times differentiable for any $b \in \mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n \pi^{2n}}{(2n)!} P_{2n}^m(x) \sim \left(\frac{8m}{\pi}\right)\sin\left(\frac{\pi}{m}\right) 2^{n+2}\cos\left(2\pi x - \pi\right) + o(r^{-n}n^{-b})$$

for $b \in \mathbb{N}$. For the odd case, we use the comparison function

$$G(z) = \frac{z\left(\frac{8m}{\pi}\right)\sin\left(\frac{\pi}{m}\right)\sin\left(2\pi x - \pi\right)}{(2-z)(2+z)},$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows. □

In each of the preceding theorems we have considered the situation where all of the points are weighted equally. We now state a theorem using the points from Theorem 4.2.1, but we assign extra weight to two of the points. The result is similar although the method differs slightly as shown below.

Theorem 4.2.2. *Let m be even and suppose we have the m points given by*

$$\left\{\frac{1}{2m}, \frac{3}{2m}, \ldots, \frac{2m-1}{2m}\right\}$$

where the points $\frac{m-1}{2m}$ and $\frac{m+1}{2m}$ are weighted $\frac{2}{m+2}$ and all other points are weighted $\frac{1}{m+2}$. Let $P_n^m(x)$ be the resulting family of Strodt polynomials. Set

$$C_m = \frac{\pi(m-2)^2 \sin\left(\frac{\pi(m+2)}{2(m-2)}\right)}{8m^2(m+2)\sin\left(\frac{\pi}{m-2}\right)}.$$

Then when $m \geq 8$ (and even) we have

$$C_m \left(\frac{2m}{m-2}\right)^{n+2} \frac{\pi^n}{(n)!} P_n^m(x) \sim \cos\left(\frac{2m\pi}{m-2}x - \frac{m\pi}{m-2} + \frac{3}{2}\pi n\right)$$

as $n \to \infty$. In the special case where $m = 4$, we have

$$\left(\frac{3\pi}{128\sqrt{3}}\right)\left(\frac{8}{3}\right)^{n+2} \frac{\pi^n}{(n)!} P_n^m(x) \sim \cos\left(\frac{8\pi}{3}x - \frac{4\pi}{3} + \frac{3}{2}\pi n\right)$$

as $n \to \infty$.

Proof. Let q be the exponential generating function for $P_n^m(x)$. Then

$$\sum_{n\geq 0} P_n^m(x)\frac{t^n}{n!} = q$$

Applying 2.10, we can write

$$\begin{aligned}
q &= \frac{(m+2)e^{xt}}{e^{\frac{m-1}{2m}t} + e^{\frac{(m+1)}{2m}t} + \sum_{j=0}^{m-1} e^{\frac{2j+1}{2m}t}} \\
&= \frac{(m+2)e^{xt}}{e^{\frac{m-1}{2m}t}\left(1 + e^{\frac{1}{m}t}\right) + e^{\frac{1}{2m}t}\left(1 + e^{\frac{1}{m}t} + \ldots + e^{\frac{m-1}{m}t}\right)} \\
&= \frac{(m+2)e^{xt}}{e^{\frac{m-1}{2m}t}\left(1 + e^{\frac{1}{m}t}\right) + e^{\frac{1}{2m}t}\left(\frac{e^t-1}{e^{\frac{1}{m}t}-1}\right)} \\
&= \frac{(m+2)e^{xt}\left(e^{\frac{1}{m}t} - 1\right)}{e^{\frac{m-1}{2m}t}\left(e^{\frac{2}{m}t} - 1\right) + e^{\frac{1}{2m}t}\left(e^t - 1\right)}
\end{aligned}$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain the following:

$$\begin{aligned}
q &= \frac{(m+2)e^{x\pi iz}\, e^{-\frac{1}{2m}\pi iz}\left(\dfrac{e^{\frac{1}{m}\pi iz} - 1}{e^{\pi iz} - 1}\right)}{e^{\frac{m-2}{2m}\pi iz}\left(\dfrac{e^{\frac{2}{m}\pi iz} - 1}{e^{\pi iz} - 1}\right) + 1} \\
&= \frac{(m+2)e^{x\pi iz}\, e^{-\frac{1}{2m}\pi iz}\, e^{-\frac{m-1}{2m}\pi iz}\dfrac{\sin\left(\frac{\pi}{2m}z\right)}{\sin\left(\frac{\pi}{2}z\right)}}{\dfrac{\sin\left(\frac{\pi}{m}z\right)}{\sin\left(\frac{\pi}{2}z\right)} + 1} \\
&= \frac{(m+2)e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\sin\left(\frac{\pi}{2m}z\right)}{\sin\left(\frac{\pi}{m}z\right) + \sin\left(\frac{\pi}{2}z\right)}
\end{aligned}$$

Apply Fact 2.1.3 to get

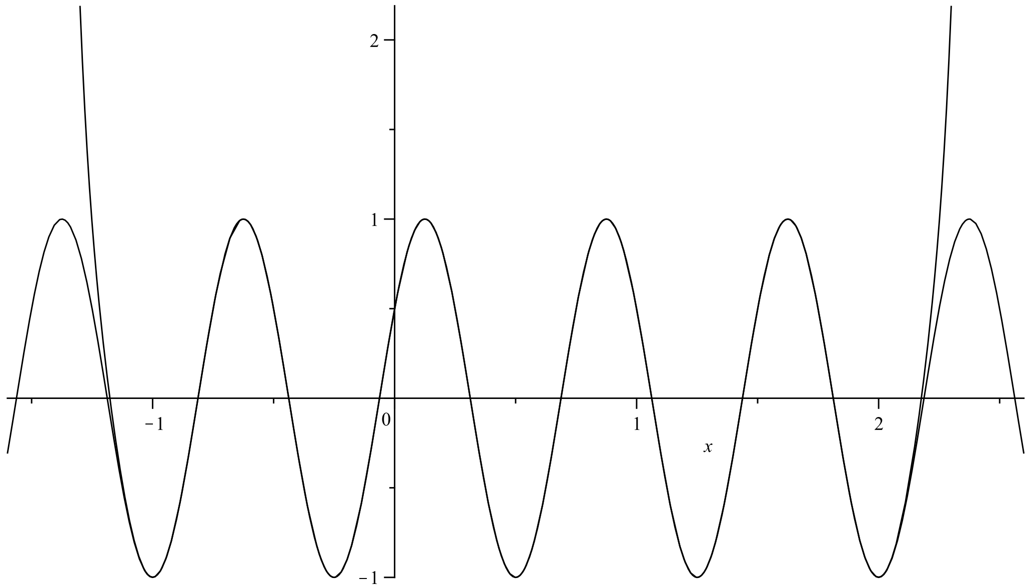

Figure 4.5: Illustration of Theorem 4.2.2 for $m = 4$ and $n = 30$

$$q = \frac{\frac{(m+2)}{2} e^{x\pi i z}\, e^{-\frac{1}{2}\pi i z} \sin\left(\frac{\pi}{2m} z\right)}{\sin\left(\frac{(m+2)\pi}{4m} z\right) \cdot \cos\left(\frac{(m-2)\pi}{4m} z\right)} \tag{4.3}$$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n}}{(2n)!} P^m_{2n}(x) z^{2n} = \frac{\frac{m+2}{2} \sin\left(\frac{\pi}{2m} z\right) \cos\left(\pi x z - \frac{1}{2}\pi z\right)}{\sin\left(\frac{\pi(m+2)}{4m} z\right) \cos\left(\frac{\pi(m-2)}{4m} z\right)},$$

and

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} P^m_{2n+1}(x) z^{2n+1} = \frac{\frac{m+2}{2} \sin\left(\frac{\pi}{2m} z\right) \sin\left(\pi x z - \frac{1}{2}\pi z\right)}{\sin\left(\frac{\pi(m+2)}{4m} z\right) \cos\left(\frac{\pi(m-2)}{4m} z\right)}.$$

Set the right hand side of the even sum equal to $F(z)$. In the case where $m \geq 8$, the relevant singularity is involving the cosine factor of the denominator. So, F has singularities at $z = \pm\frac{2m}{m-2}$. For convenience, we will set $r = \frac{2m}{m-2}$. Now we compute the limits

$$\begin{aligned}
\lim_{z\to r^-}(r-z)(r+z)F(z) &= \left(\frac{4m}{\pi(m-2)}\right)\left(\frac{m+2}{2}\right)\left(\frac{4m}{m-2}\right) \\
&\quad \sin\left(\frac{\pi}{m-2}\right)\csc\left(\frac{\pi(m+2)}{2(m-2)}\right)\cos\left(\frac{2m\pi}{m-2}x-\frac{m\pi}{m-2}\right) \\
&= C_m^{-1}\cos\left(\frac{2m\pi}{m-2}x-\frac{m\pi}{m-2}\right) \\
&= \lim_{z\to -r^+}(r-z)(r+z)F(z).
\end{aligned}$$

Let a comparison function be

$$G(z)=\frac{C_m^{-1}\cos\left(\frac{2m\pi}{m-2}x-\frac{m\pi}{m-2}\right)}{\left(\frac{2m}{m-2}-z\right)\left(\frac{2m}{m-2}+z\right)}.$$

We note that $F-G$ has removable singularities, is continuous on $0<|z|\leq r$, and is b times differentiable for any $b\in\mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}^m(x)\sim\left(\frac{2m}{m-2}\right)^{n+2}C_m^{-1}\cos\left(\frac{2m\pi}{m-2}x-\frac{m\pi}{m-2}\right)+o(r^{-n}n^{-b})$$

for $b\in\mathbb{N}$. For the odd case, we use the comparison function

$$G(z)=\frac{zC_m^{-1}\sin\left(\frac{2m\pi}{m-2}x-\frac{m\pi}{m-2}\right)}{\left(\frac{2m}{m-2}-z\right)\left(\frac{2m}{m-2}+z\right)}.$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows for the case where $m\geq 8$. To handle the case where $m=4$, we note that the singularities occur in the sine factor at the step where we split into even and odd sums. In that case, we must set $r=8/3$ and proceed through the steps from there. □

The next theorem is similar to the previous one. However, we now include the endpoints as two of the points. We still add extra weight to the two points closest to 1/2.

Theorem 4.2.3. *Let m be even and suppose we have the m points given by*

$$\left\{0, \frac{1}{m-1}, \frac{2}{m-1}, \ldots, 1\right\}$$

where the points $\frac{m-2}{2(m-1)}$ and $\frac{m}{2(m-1)}$ are weighted $\frac{2}{m+2}$ and all other points are weighted $\frac{1}{m+2}$. Let $P_n^m(x)$ be the resulting family of Strodt polynomials. Set

$$C_m = \frac{\pi(m-2)^2 \sin\left(\frac{\pi(m+2)}{2(m-2)}\right)}{8m(m-1)(m+2)\sin\left(\frac{\pi}{m-2}\right)}$$

Then when $m \geq 8$ (and even) we have

$$C_m \left(\frac{2(m-1)}{m-2}\right)^{n+2} \frac{\pi^n}{(n)!} P_n^m(x) \sim \cos\left(\frac{2\pi(m-1)}{m-2}x - \frac{(m-1)\pi}{m-2} + \frac{3}{2}\pi n\right)$$

as $n \to \infty$. In the special case where $m = 4$, we have

$$\left(\frac{\pi}{24\sqrt{3}}\right)(2)^{n+2}\frac{\pi^n}{(n)!}P_n^m(x) \sim \cos\left(2\pi x - \pi + \frac{3}{2}\pi n\right)$$

as $n \to \infty$.

Proof. Let q be the exponential generating function for $P_n^m(x)$. Then

$$\sum_{n\geq 0} P_n^m(x)\frac{t^n}{n!} = q$$

Applying 2.10, we can write

$$
\begin{aligned}
q &= \frac{(m+2)e^{xt}}{e^{\frac{m-2}{2(m-1)}t} + e^{\frac{m}{2(m-1)}t} + \sum_{j=0}^{m-1} e^{\frac{j}{m-1}t}} \\
&= \frac{(m+2)e^{xt}}{e^{\frac{m-2}{2(m-1)}t}\left(1 + e^{\frac{1}{m-1}t}\right) + \left(1 + e^{\frac{1}{m-1}t} + \ldots + e^{t}\right)} \\
&= \frac{(m+2)e^{xt}}{e^{\frac{m-2}{2(m-1)}t}\left(1 + e^{\frac{1}{m-1}t}\right) + \left(\dfrac{e^{\frac{m}{m-1}t} - 1}{e^{\frac{1}{m-1}t} - 1}\right)} \\
&= \frac{(m+2)e^{xt}\left(e^{\frac{1}{m-1}t} - 1\right)}{e^{\frac{m-2}{2(m-1)}t}\left(e^{\frac{2}{m-1}t} - 1\right) + \left(e^{\frac{m}{m-1}t} - 1\right)}
\end{aligned}
$$

Next, we change variables and replace t by πiz. Then we apply Fact 2.1.2 to obtain the following:

$$
\begin{aligned}
q &= \frac{(m+2)e^{x\pi iz}\left(\dfrac{e^{\frac{1}{m-1}\pi iz} - 1}{e^{\frac{m}{m-1}\pi iz} - 1}\right)}{e^{\frac{m-2}{2(m-1)}\pi iz}\left(\dfrac{e^{\frac{2}{m-1}\pi iz} - 1}{e^{\frac{m}{m-1}\pi iz} - 1}\right) + 1} \\
&= \frac{(m+2)e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\,\dfrac{\sin\left(\frac{\pi}{2(m-1)}z\right)}{\sin\left(\frac{m\pi}{2(m-1)}z\right)}}{\dfrac{\sin\left(\frac{\pi}{m-1}z\right)}{\sin\left(\frac{m\pi}{2(m-1)}z\right)} + 1} \\
&= \frac{(m+2)e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\sin\left(\frac{\pi}{2(m-1)}z\right)}{\sin\left(\frac{\pi}{m-1}z\right) + \sin\left(\frac{m\pi}{2(m-1)}z\right)}
\end{aligned}
$$

Now apply Fact 2.1.3 to get

$$
q = \frac{\frac{(m+2)}{2}e^{x\pi iz}\, e^{-\frac{1}{2}\pi iz}\sin\left(\frac{\pi}{2(m-1)}z\right)}{\sin\left(\frac{(m+2)\pi}{4(m-1)}z\right)\cdot\cos\left(\frac{(m-2)\pi}{4(m-1)}z\right)} \tag{4.4}
$$

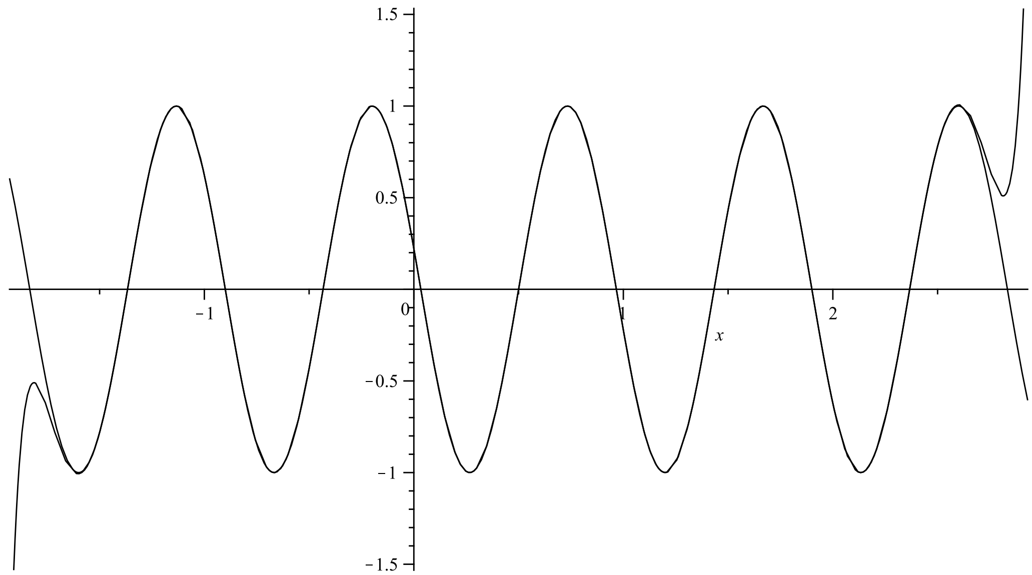

Figure 4.6: Illustration of Theorem 4.2.3 for $m = 16$ and $n = 33$

Now separate sum and generating function into real and imaginary parts (using Fact 2.1.1) to get the even and odd cases, respectively. Namely,

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n}}{(2n)!} P_{2n}^m(x) z^{2n} = \frac{\frac{m+2}{2} \sin\left(\frac{\pi}{2(m-1)} z\right) \cos\left(\pi x z - \frac{1}{2}\pi z\right)}{\sin\left(\frac{\pi(m+2)}{4(m-1)} z\right) \cos\left(\frac{\pi(m-2)}{4(m-1)} z\right)},$$

and

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} P_{2n+1}^m(x) z^{2n+1} = \frac{\frac{m+2}{2} \sin\left(\frac{\pi}{2(m-1)} z\right) \sin\left(\pi x z - \frac{1}{2}\pi z\right)}{\sin\left(\frac{\pi(m+2)}{4(m-1)} z\right) \cos\left(\frac{\pi(m-2)}{4(m-1)} z\right)}.$$

Set the right hand side of the even sum equal to $F(z)$. In the case where $m \geq 8$, the relevant singularity is involving the cosine factor of the denominator. So, F has singularities at $z = \pm\frac{2(m-1)}{m-2}$. For convenience, we will set $r = \frac{2(m-1)}{m-2}$. Now we compute the limits

$$\begin{aligned}
\lim_{z\to r^-}(r-z)(r+z)F(z) &= \left(\frac{4(m-1)}{\pi(m-2)}\right)\left(\frac{m+2}{2}\right)\left(\frac{4m}{m-2}\right)\sin\left(\frac{\pi}{m-2}\right) \\
&\quad \csc\left(\frac{\pi(m+2)}{2(m-2)}\right)\cos\left(\frac{2(m-1)\pi}{m-2}x-\frac{(m-1)\pi}{m-2}\right) \\
&= C_m^{-1}\cos\left(\frac{2(m-1)\pi}{m-2}x-\frac{(m-1)\pi}{m-2}\right) \\
&= \lim_{z\to -r^+}(r-z)(r+z)F(z).
\end{aligned}$$

Let a comparison function be

$$G(z)=\frac{C_m^{-1}\cos\left(\frac{2(m-1)\pi}{m-2}x-\frac{(m-1)\pi}{m-2}\right)}{\left(\frac{2(m-1)}{m-2}-z\right)\left(\frac{2(m-1)}{m-2}+z\right)}.$$

We note that $F-G$ has removable singularities, is continuous on $0<|z|\le r$, and is b times differentiable for any $b\in\mathbb{N}$. Therefore, we apply Darboux's Method to get the asymptotic expression

$$\frac{(-1)^n\pi^{2n}}{(2n)!}P_{2n}^m(x)\sim\left(\frac{2(m-1)}{m-2}\right)^{n+2}C_m^{-1}\cos\left(\frac{2(m-1)\pi}{m-2}x-\frac{(m-1)\pi}{m-2}\right)+o(r^{-n}n^{-b})$$

for $b\in\mathbb{N}$. For the odd case, we use the comparison function

$$G(z)=\frac{zC_m^{-1}\sin\left(\frac{2(m-1)\pi}{m-2}x-\frac{(m-1)\pi}{m-2}\right)}{\left(\frac{2(m-1)}{m-2}-z\right)\left(\frac{2(m-1)}{m-2}+z\right)}.$$

and the asymptotics are similar. Using the properties of sine and cosine to shift appropriately, the result follows for the case where $m\ge 8$. To handle the case where $m=4$, we note that the singularities occur in the sine factor at the step where we split into even and odd sums. In that case, we must set $r=2$ and proceed through the steps from there. □

There are figures in Appendix A which give additional examples of the theorems in this section.

4.3 Experiments and a Conjecture

One thing we notice from the proofs in the previous section is that two of them had a "missing" case. Theorems 4.2.2 and 4.2.3 each established a result for even $m \geq 8$ and for $m = 4$. Neither mentioned the case when $m = 6$. In fact, we have experimental evidence which suggests that these polynomials are not asymptotic to a cosine curve as in the other cases.

Consider choosing 6 points in the manner of Theorem 4.2.1 and Theorem 4.2.2. The 6 points would be

$$\{\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{7}{12}, \frac{9}{12}, \frac{11}{12}\}. \tag{4.5}$$

If all the points are weighted equally then Theorem 4.2.1 gives the asymptotic behavior of the resulting polynomials. Figures 8 through 14 in Appendix B show these polynomials (and their corresponding cosine curve) for several values of n. We notice that as n increases, the width of the interval on which the two graphs coincide increases as well. In addition, we observe that within these intervals the two functions share a period and amplitude.

Now, consider taking the same six points in 4.5 where $\frac{5}{12}$ and $\frac{7}{12}$ are assigned a weight of $\frac{1}{4}$ and all other points are weighted $\frac{1}{8}$. Let $P_n^*(x)$ be the resulting family of Strodt polynomials. Figure 4.7 shows a (scaled) plot of the polynomial for $n = 31$. Contrast this and Figures 15 through 22 in Appendix B with the plots in the example above. The lines drawn at $y = \pm 1$ are for comparison. We observe that for each value

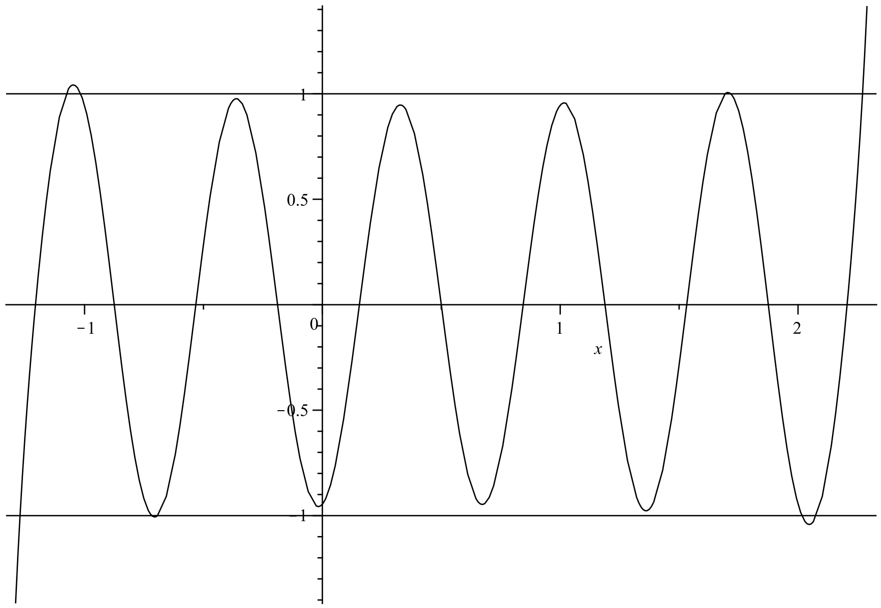

Figure 4.7: $P^*_{31}(x)$ and lines at $y = \pm 1$ for comparison

of n, the amplitude (and possibly the period) is not constant. The center peak appears to be the shortest and each peak and valley appears to increase in amplitude as x increases and decreases. Each of these observations lead us to conclude that these polynomials are not asymptotic to a sine wave at all. Our best guess is that the asymptotic behavior is actually the product of a cosine curve and another function.

Theorem 4.2.3 also has a "missing" case when $m = 6$. Consider the probability density function defined by the six points 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1. We assign a weight of $\frac{1}{4}$ to the points $\frac{2}{5}$ and $\frac{3}{5}$ and a weight of $\frac{1}{8}$ to the other four points. Call the resulting family of polynomials $P^{**}_n(x)$. We do not have an asymptotic expression for these polynomials either. Figure 23 in Appendix C shows one example.

This result is surprising. We had previously conjectured that when the points and weights were symmetric to each other about 1/2 that the resulting polynomials would always behave like a sine curve. In the proofs themselves, the problem with letting $m = 6$ was that it caused 2 singularities in the generating function denominator. We do not yet know how we could predict this from the original points and weights.

In addition to all the cases mentioned above, we did numerous experiments with graphs of Strodt polynomials arising from points and weights which were *not* symmetric about 1/2. Many of these curves were not sinusoidal nor did they look like the plots of $P_n^*(x)$ and $P_n^{**}(x)$. Often these graphs looked like the examples in Figures 24 and 25 in Appendix C.

Open Question 4.3.1. *Is there a general result for the asymptotic behavior of Strodt polynomials? Can we determine which probability weight functions will give polynomials asymptotic to sine curves and which will not?*

Chapter 5

Class Numbers and Elliptic Curves

In this chapter we will change gears and turn our attention toward topics in modern number theory. We begin with an introduction to class numbers of binary quadratic forms. In section 2 we define a refinement called the Hurwitz class number and state a theorem for their sum. Section 3 will be a brief introduction to elliptic curves over a finite field and we close the chapter with an overview of the Legendre symbol in Section 4.

5.1 Quadratic Forms and Class Numbers

A quadratic form in n variables is a homogeneous polynomial of degree 2. Here, we are interested in quadratic forms with two variables called *binary* quadratic forms. We can write a general binary quadratic form as

$$f(x, y) = ax^2 + bxy + cy^2 \tag{5.1}$$

where a, b, and c are all integers.

Now, suppose for some form f, $\gcd(a,\, b,\, c) = m \neq 1$. Then we can write

$$f(x,\, y) = ax^2 + bxy + cy^2 = ma'x^2 + mb'xy + mc'y^2 = m\, g(x,\, y) \tag{5.2}$$

where g is a binary quadratic form. A form which has all of its coefficients relatively prime is called *primitive*. So, every form is a multiple of a primitive form.

Let f and g be binary quadratic forms. Now, suppose there exist integers p, q, r, and s such that

$$ps - qr = \pm 1$$

and

$$f(x,\, y) = g(px + qy,\, rx + sy)\ .$$

Then we say f and g are *equivalent*. Note that equivalence of forms is an equivalence relation. To see this, notice that the condition $ps - qr = \pm 1$ is the same as requiring $\det \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \pm 1$. In other words, the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ must be invertible. The set of all 2×2 invertible integer matrices is a group, commonly referred to as $\mathrm{GL}(2,\, \mathbb{Z})$. Since it is a group, it is not too much work to show the equivalence, symmetric, and transitive properties hold as required for an equivalence relation. If $ps - qr = 1$, then the equivalence is called a *proper equivalence*. This is also an equivalence relation. [7]

For any binary quadratic form $ax^2+bxy+cy^2$, we can compute the *discriminant* of the form as $D = b^2 - 4ac$. A *positive definite* form is one where $D < 0$ and $a > 0$.

Fact 5.1.1. *Equivalent forms have the same discriminant.*

Proof. Let $f = a_1x^2 + b_1xy + c_1y^2$ and $g = a_2x^2 + b_2xy + c_2y^2$ be equivalent binary quadratic forms with discriminants D_f and D_g, respectively. So, there exist integers

p, q, r, and s with $f(x, y) = g(px + qy, rx + sy)$ and $ps - qr = \pm 1$. Thus,

$$\begin{aligned} a_1x^2 + b_1xy + c_1y^2 &= f(x, y) \\ &= g(px + qy, rx + sy) \\ &= a_2(px + qy)^2 + b_2(px + qy)(rx + sy) + c_2(rx + sy) \\ &= \left(a_2p^2 + b_2pr + c_2r^2\right) x^2 + (2a_2pq + b_2ps + b_2qr + 2c_2rs)\, xy \\ &\quad + \left(a_2q^2 + b_2qs + c_2s^2\right) y^2 \end{aligned}$$

Now, take the discriminant of both sides to conclude

$$\begin{aligned} D_f &= (ps - qr)^2(b_2^2 - 4a_2c_2) \\ &= (ps - qr)^2 D_g \\ &= D_g \end{aligned}$$

because $ps - qr = \pm 1$. □

Let $f(x, y) = ax^2 + bxy + cy^2$ be a primitive positive definite form. This form is *reduced* if both of the following are true:

- $|b| \leq a \leq c$,
- If either $|b| = a$ or $a = c$, then $b \geq 0$.

Now we can state a theorem due to Legendre [6]

Theorem 5.1.2. *Every primitive positive definite form is properly equivalent to a unique reduced form.*

If two forms are properly equivalent then they are in the same *class*. The number $h(D)$ is defined as the number of classes of primitive positive definite binary quadratic forms of discriminant D.

Theorem 5.1.3. *For a fixed $D < 0$, $h(D)$ is finite. Also, $h(D)$ is equal to the number of reduced forms of discriminant D. [6]*

For more information on quadratic forms, we refer the reader to [7] or [6].

5.2 Hurwitz Class Numbers

For an integer $N \geq 0$, the *Hurwitz class number* [7] $H(N)$ is defined as:

- $H(0) = -1/12$.
- If $N \equiv 1$ or $2 \pmod 4$, then $H(N) = 0$.
- Otherwise, $H(N)$ is equal to the number of classes of not necessarily primitive positive definite quadratic forms with discriminant $-N$, with the following exceptions:
  - Forms which are equivalent to a multiple of $x^2 + y^2$ should be counted with multiplicity $1/2$
  - Forms which are equivalent to a multiple of $x^2 + xy + y^2$ should be counted with multiplicity $1/3$.

The Hurwitz class numbers satisfy the identity

$$\sum_{r^2 < 4p} H(4p - r^2) = 2p, \tag{5.3}$$

for p a prime and where r varies through all permissible integers [7]. We will explore this sum further in the next chapter.

5.3 Elliptic Curves over a Finite Field

The study of elliptic curves is much too broad to give a complete overview here. We will review facts which will be relevant in later sections. An in depth introduction is available in [13] or in [24].

We will consider elliptic curves over a finite field $\mathbb{F}_p$. In this case all of the coefficients and all of the points on the curves are elements of this field. The set of points on E over $\mathbb{F}_p$ is a group. We refer the reader to [13, §III.3] and [24, §III.2] for a detailed description of the group law. In this particular case, the group E over $\mathbb{F}_p$ is a finite Abelian group. In fact, the number of points on E over $\mathbb{F}_p$ is given by $p+1-r$ where $-2\sqrt{p} < r < 2\sqrt{p}$.

Because we have an Abelian group, the group of points of order dividing m form a subgroup called the m-torsion subgroup. This torsion subgroup is non-trivial if and only if m divides the order of the whole group. Thus, if m is square-free, an elliptic curve over $\mathbb{F}_p$ has m-torsion if and only if $m \mid p+1-r$.

We also know something about the structure of the torsion subgroup. For example, if the 3-torsion subgroup of E over $\mathbb{F}_p$ is non-trivial, then the group is either $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. We will call the former case cyclic 3-torsion and we will call the latter case full 3-torsion.

5.4 The Legendre Symbol

Here we give a brief introduction to the Legendre symbol and list some specific facts which we will need in the next chapter. The following definitions (and more details) can be found in elementary number theory texts ([1] or [11] , for example).

Let n be an integer and let p be a prime with $\gcd(n,p) = 1$. Claiming $\sqrt{n} \in \mathbb{F}_p$

is equivalent to claiming there exists an integer b such that

$$b^2 \equiv n \pmod{p} .$$

If such a b can be found, then we say that n is a *quadratic residue* modulo p. The *Legendre symbol* of n above p is then defined to be

$$\left(\frac{n}{p}\right) := \begin{cases} 0, & \text{if } p \mid n \\ 1, & \text{if } n \text{ is a quadratic residue modulo } p \\ -1, & \text{if } n \text{ is a quadratic non-residue modulo } p \text{ .} \end{cases}$$

The specific values of Legendre symbols we will need later are in the following fact.

Fact 5.4.1.

$$\left(\frac{3}{p}\right) = \begin{cases} 1, & p \equiv \pm 1 \pmod{12} \\ -1, & p \equiv \pm 5 \pmod{12} \text{ ;} \end{cases}$$

$$\left(\frac{-1}{p}\right) = \begin{cases} 1, & p \equiv 1 \pmod{4} \\ -1, & p \equiv 3 \pmod{4} \text{ ;} \end{cases}$$

$$\left(\frac{-3}{p}\right) = \begin{cases} 1, & p \equiv 1 \pmod{3} \\ -1, & p \equiv 2 \pmod{3} \text{ .} \end{cases}$$

Rather than verify these identities here, we refer the reader to [1] or [11].

Chapter 6

Sums of Hurwitz Class Numbers

6.1 Experiments and Conjectures

In the summer of 2003, Brittany Brown and Amy Stout were participants in the Research Experience for Undergraduates at Clemson University under the direction of Neil Calkin and Kevin James. I worked with Brittany and Amy that summer as their graduate student “mentor.” Their project focused on the Hurwitz class number sum mentioned in the previous chapter. Recall from 5.3 that

$$\sum_{r^2<4p} H(4p-r^2) = 2p, \tag{6.1}$$

for any prime p. Their goal was to experiment with subsums of this series and see if there were any recognizable patterns. For example, what is the result of summing every other term, every 3rd term, etc.?

To facilitate their experiments, I wrote a program in C using the PARI/GP library to generate data. They formulated several conjectures and I wrote another program to test these conjectures. In the time since that summer, many of the conjectures have been proved. We will list most of the theorems and remaining conjectures in the last section of this chapter.

For now, we state the theorem about the situation where every 3rd term of the sum is considered. In turns out that the result depends on the congruence class of p modulo 3.

Theorem 6.1.1. *If $p > 3$ is prime and $p \equiv 1 \pmod 3$ then,*

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod 3}} H(4p-r^2) = \begin{cases} \frac{p+1}{2}, & \text{if } c=0 \\ \frac{3p-1}{4}, & \text{if } c=1 \\ \frac{3p-1}{4}, & \text{if } c=2. \end{cases}$$

If $p > 3$ is prime and $p \equiv 2 \pmod 3$ then,

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod 3}} H(4p-r^2) = \begin{cases} p-1, & \text{if } c=0 \\ \frac{p+1}{2}, & \text{if } c=1 \\ \frac{p+1}{2}, & \text{if } c=2. \end{cases}$$

In general, there is a symmetry to these subsums of Hurwitz class numbers which can be expressed as

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod m}} H(4p-r^2) = \sum_{\substack{r^2<4p,\\ r\equiv -c \pmod m}} H(4p-r^2). \tag{6.2}$$

This explains the symmetry in Theorem 6.1.1. In fact, combining this symmetry with the knowledge that the total sum must equal $2p$ tells us that we need only prove one of the subsums for each case and the other two will follow. What we need now is a way to prove one of the sums. This is discussed in the next section.

6.2 A Plan of Attack - Deuring's Theorem

The key idea we will use to prove Theorem 6.1.1 is based on a theorem of Deuring.

Theorem 6.2.1 (Deuring [9], [17])**.** *If r is an integer such that $r^2 < 4p$, then the number of isomorphism classes of elliptic curves over $\mathbb{F}_p$ with exactly $p+1-r$ points is equal to the number of equivalence classes of binary quadratic forms with discriminant $r^2 - 4p$.*

Recall the definition of Hurwitz class numbers and their relationship to equivalence classes of binary quadratic forms. This allows us to state the following:

Corollary 6.2.2. *For $r \in \mathbb{Z}$ and $r^2 < 4p$, the number of isomorphism classes of elliptic curves over $\mathbb{F}_p$ with exactly $p+1-r$ points is equal to $H(4p-r^2)+c_{r,p}$ where*

$$c_{r,p} = \begin{cases} \frac{2}{3}, & \text{if } \exists\, \alpha \in \mathbb{Z} \setminus \{0\} \text{ such that } r^2 - 4p = -3\alpha, \\ \frac{1}{2}, & \text{if } \exists\, \beta \in \mathbb{Z} \setminus \{0\} \text{ such that } r^2 - 4p = -4\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The Hurwitz class number counts forms equivalent to a multiple of x^2+xy+y^2 with multiplicity $1/3$. Forms equivalent to $\alpha x^2 + \alpha xy + \alpha y^2$ have discriminant $-3\alpha^2$. Suppose r and p are such that $r^2 - 4p = -3\alpha$ for some non-zero integer α. Then we must have $c_{r,p} = 2/3$ so that $H(4p - r^2) + c_{r,p}$ is an integer and correctly counts the number of isomorphism classes of elliptic curves with $p + 1 - r$ points.

The Hurwitz class number counts forms equivalent to a multiple of x^2+y^2 with multiplicity $1/2$. Forms equivalent to $\beta x^2 + \beta y^2$ have discriminant $-4\beta^2$. Suppose r and p are such that $r^2 - 4p = -4\beta^2$ for some non-zero integer β. Then we must have

$c_{r,p} = 1/2$ so that $H(4p - r^2) + c_{r,p}$ is an integer and correctly counts the number of isomorphism classes of elliptic curves with $p + 1 - r$ points.

The result now follows from Deuring's Theorem. □

In the last chapter we saw that the number of points on E over $\mathbb{F}_p$ is always of the form $p + 1 - r$ where $|r| < 2\sqrt{p}$. We also saw that a curve has 3-torsion if and only if 3 divides the number of points. So, E has 3-torsion if and only if $3 \mid p + 1 - r$. We will use $N_{3,p}$ to denote the number of isomorphism classes of elliptic curves over $\mathbb{F}_p$ which possess 3-torsion. Thus,

$$N_{3,p} = \sum_{\substack{r^2 < 4p, \\ r \equiv p+1 \pmod 3}} \left(H(4p - r^2) + c_{r,p} \right).$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{\substack{r^2 < 4p, \\ r \equiv 0 \pmod 3}} H(4p - r^2) = N_{3,p} - \sum_{\substack{r^2 < 4p, \\ r \equiv 0 \pmod 3}} c_{r,p}, \tag{6.3}$$

and if $p \equiv 1 \pmod 3$, then

$$\sum_{\substack{r^2 < 4p, \\ r \equiv 2 \pmod 3}} H(4p - r^2) = N_{3,p} - \sum_{\substack{r^2 < 4p, \\ r \equiv 2 \pmod 3}} c_{r,p}. \tag{6.4}$$

We will prove Theorem 6.1.1 using a combinatorial proof. First, we will need to count the number of isomorphism classes (which we do in the next section). Later we will need to compute the "correction term" (the sum of the $c_{r,p}$'s).

6.3 Counting Isomorphism Classes with 3-torsion

We want to consider only the elliptic curves with 3-torsion. We can follow the steps given in [13, §V.5] to construct the general form of such a curve. A curve will have 3-torsion if and only if it is isomorphic to a curve of the form

$$E_{a_1,a_3} : y^2 + a_1xy + a_3y = x^3 .$$

We will be considering this curve over the field $\mathbb{F}_p$ which means that the coefficients a_1 and a_3 are elements of $\mathbb{F}_p$. From [24, §III.1], we can compute the discriminant of E_{a_1,a_3}

$$\Delta_{a_1,a_3} = a_3^3(a_1^3 - 27a_3)$$

For $u \in \mathbb{F}_p^*$, we do a change of variables by setting $X = u^2x$ and $Y = u^3y$. The curve becomes

$$E_{a_1u,a_3u^3} : Y^2 + a_1uXY + a_3u^3Y = X^3$$

and we conclude that

$$\text{for any } u \in \mathbb{F}_p^* \text{ we have } E_{a_1,a_3} \cong E_{a_1u,a_3u^3} \tag{6.5}$$

Since $p > 3$, we can use the steps in [24, §III.1] to make another change of variables

$$y^2 = x^3 - 27c_4x - 54c_6 \tag{6.6}$$

with

$$c_4 = a_1^4 - 24a_1a_3, \tag{6.7}$$

$$c_6 = -a_1^6 + 36a_1^3a_3 - 216a_3^2 \tag{6.8}$$

Now, the curves in this form which are isomorphic [24, §III.1] to 6.6 can be written as

$$y^2 = x^3 - 27u^4c_4x - 54u^6c_6$$

for u nonzero.

Recall the change of variables and the resulting isomorphisms given in 6.5. From this we see that for any curve with $a_1 \neq 0$ then we can take $u = a_1^{-1}$ to get an isomorphic curve with $a_1 = 1$. Thus, each equivalence class must contain a curve of the form $E_{0,a}$ or $E_{1,a}$ for some $a \in \mathbb{F}_p$. However, we do not consider the singular curves $E_{0,0}$, $E_{1,0}$, or $E_{1,3^{-3}}$.

We want to know how many of these curves are isomorphic to each other. In other words, for a given curve we want to know how many pairs $(a_1, a_3) \in \mathbb{F}_p^2$ give isomorphic curves E_{a_1,a_3}. To count the number of these isomorphism classes it will often be helpful to know the size of the classes. Following the reasoning in [12, §2], we see that the size of the isomorphism class depends on whether the group E of points on the curve possesses full 3-torsion or cyclic 3-torsion. In particular, the number of isomorphism classes is given by

$$\begin{cases} \frac{4(p-1)}{3}, & \text{if } c_4 = 0,\ p \equiv 1 \pmod 3 \text{ and 3-torsion is full} \\ 2(p-1), & \text{if } c_6 = 0,\ p \equiv 1 \pmod 4 \text{ and 3-torsion is full} \\ 4(p-1), & \text{otherwise with full 3-torsion} \\ \frac{(p-1)}{3}, & \text{if } c_4 = 0,\ p \equiv 1 \pmod 3 \text{ and 3-torsion is cyclic} \\ \frac{p-1}{2}, & \text{if } c_6 = 0,\ p \equiv 1 \pmod 4 \text{ and 3-torsion is cyclic} \\ p-1, & \text{otherwise with cyclic 3-torsion} \end{cases} \tag{6.9}$$

(see [12, p.285]).

We can see in 6.9 that we will need to pay special attention to when c_4 or c_6 is zero. From 6.7 we can deduce that curves of the form $E_{0,a}$ must always have $c_4 = 0$ and c_6 non-zero. For curves of the form $E_{1,a}$, c_4 is zero when

$$1 - 24a = 0 \tag{6.10}$$

and c_6 is zero when

$$-1 + 36a - 216a^2 = 0\ . \tag{6.11}$$

The solution to 6.10 is $a = 24^{-1}$. Since $p > 3$, $p \not| 24$ and thus we know 24^{-1} exists. Then c_4 is zero for the curve $E_{1,24^{-1}}$. The solution to 6.11 is

$$a = \frac{3 \pm \sqrt{3}}{36}\ .$$

However, $\sqrt{3}$ is not always an element of $\mathbb{F}_p$. In particular, $\sqrt{3} \in \mathbb{F}_p$ if and only if 3 is a quadratic residue modulo p. From Fact 5.4.1 we conclude that $\mathbb{F}_p$ has this element and c_6 will be zero for the curve $E_{1,a}$ when $p \equiv \pm 1 \pmod{12}$. We summarize this in the following fact:

Fact 6.3.1. *The coefficient* $c_4 = 0$ *for* $E_{0,a}$ *and* $E_{1,24^{-1}}$. *The coefficient* $c_6 = 0$ *for the two curves* $E_{1,\frac{3\pm\sqrt{3}}{36}}$, *but only when* $p \equiv \pm 1 \pmod{12}$. *These are the only such curves with* $c_4 = 0$ *or* $c_6 = 0$.

We can see in 6.9 that we will also need to pay special attention to when E has cyclic or full 3-torsion. This is determined by the following proposition.

Proposition 6.3.2. *If* E *is an elliptic curve possessing full* m*-torsion over* $\mathbb{F}_p$, *then* $p \equiv 1 \pmod{m}$.

Proof. We follow the proof of Proposition 3 in [5]. Let $E[m]$ be the m-torsion subgroup of E. Let G be the Galois group of $\mathbb{F}_p(E[m])/\mathbb{F}_p$ and let ϕ be the Frobenius automorphism with

$$\phi : \mathbb{F}_p(E[m]) \to \mathbb{F}_p(E[m])$$

. We have the representation (see [24, pp.89-91])

$$\rho_m : G \hookrightarrow \mathrm{Aut}(E[m]) \cong \mathrm{Aut}(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \cong \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}) \ .$$

If E has full m-torsion, then $\mathbb{F}_p(E[m])/\mathbb{F}_p$ is a trivial extension and G is trivial. Then

$$\rho_m(\phi) = I \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}) \ .$$

Now, apply [24, Prop.V.2.3] to conclude

$$p \equiv \det(\rho_m(\phi)) \equiv 1 \pmod{m} \ .$$

□

Therefore, full 3-torsion over $\mathbb{F}_p$ is only possible when $p \equiv 1 \pmod 3$.

We will now begin to count $N_{3,p}$ - the number of isomorphism classes of elliptic curves over $\mathbb{F}_p$ which possess 3-torsion. The results are in the next two propositions.

Proposition 6.3.3. *If* $p \equiv 2 \pmod 3$*, then* $N_{3,p} = p - 1$.

Proof. We want to count the number of isomorphism classes. With $p \equiv 2 \pmod 3$, 6.3.2 tells us that none of the curves over $\mathbb{F}_p$ possess full 3-torsion. Combining this fact with 6.9, we know that each of the isomorphism classes has either $p-1$ curves or $\frac{p-1}{2}$ curves.

From 6.5, curves of the form $E_{1,a}$ are isomorphic to the curves E_{u,u^3a} for any of the $p-1$ distinct $u \in \mathbb{F}_p^*$. This means we can not have any isomorphism classes here of size $\frac{p-1}{2}$. So, for any a where $E_{1,a}$ is non-singular, we get an isomorphism class of size $p-1$. Recall that there are $p-2$ such a's (because $E_{1,0}$ and $E_{1,3^{-3}}$ are singular). Thus, we have $p-2$ different isomorphism classes.

Now, we consider the curves $E_{0,a}$. Because $p \equiv 2 \pmod 3$ we know that $3 \nmid |\mathbb{F}_p^*| = p - 1$. Thus, cubing is a bijection mapping $\mathbb{F}_p^*$ onto itself. In particular, if we let u vary through all elements of $\mathbb{F}_p^*$, then u^3 will vary through all elements also. From 6.5, $E_{0,a} \cong E_{0,u^3a}$. Therefore, $E_{0,a} \cong E_{0,1}$ for every $a \in \mathbb{F}_p^*$. We can conclude that there is 1 isomorphism class from the curves $E_{0,a}$.

Combining all of above we see $N_{3,p} = p - 2 + 1 = p - 1$ as desired. □

Now we move to the case with $p \equiv 1 \pmod 3$. This will require a little more work than the previous proposition and actually involves sub-cases modulo 12.

Proposition 6.3.4. *When* $p \equiv 1 \pmod{12}$, $N_{3,p} = \frac{3p+9}{4}$. *When* $p \equiv 7 \pmod{12}$, $N_{3,p} = \frac{3p+7}{4}$.

Proof. We begin by looking at the case where the c_4 coefficient is 0. Recall from 6.3.1 that curves of the form $E_{0,a}$ have $c_4 = 0$. Then Prop. 6.9 tells us that each $E_{0,a}$ curve is either in an isomorphism class of size $\frac{p-1}{3}$ or one of size $\frac{4(p-1)}{3}$. Now if we apply the isomorphism from 6.5 to $E_{0,a}$ we get curves in the form E_{0,au^3}.

Since we are in the case $p \equiv 1 \pmod 3$, as u varies through $\mathbb{F}_p^*$, the $E_{0,a}$ are grouped into 3 groups (because cubing is no longer a bijection as in the previous proof). Each of these groups is of size $\frac{p-1}{3}$. From 6.3.1, we also know that $c_4 = 0$ for $E_{1,24^{-1}}$. So, this curve and all of those isomorphic to it must be isomorphic to the curves in 1 of the 3 groups. So, the size of that isomorphism class must be $\frac{4(p-1)}{3}$. The remaining two groups must be distinct isomorphism classes of size $\frac{p-1}{3}$ (recall that there were only 2 possible sizes in this case). Thus, we have a total of 3 isomorphism classes from the case where $c_4 = 0$.

Now we move to the case where $c_6 = 0$. From 6.3.1 we know there are no such curves of form $E_{0,a}$ and the only curves of the from $E_{1,a}$ are

$$E_{1,\frac{3+\sqrt{3}}{36}} \text{ and } E_{1,\frac{3-\sqrt{3}}{36}} .$$

Also recall that $\sqrt{3} \in \mathbb{F}_p$ when $p \equiv 1 \pmod{12}$ but not when $p \equiv 7 \pmod{12}$.

For notation purposes, we will define the set K to be

$$K = \{0, 3^{-3}, 24^{-1}, \frac{3 \pm \sqrt{3}}{36}\} .$$

To this point we have either eliminated or considered the curves of the form $E_{1,a}$ where $a \in K$. It remains to count the isomorphism classes of curves of the form $E_{1,a}$ where $a \in \mathbb{F}_p \setminus K$.

For such an $a \in \mathbb{F}_p \setminus K$, we know from 6.9 that the size of the isomorphism class of $E_{1,a}$ is either $p-1$ or $4(p-1)$. In addition, the only way for the size of the class to be $4(p-1)$ is if there exits some $b \neq a$ where $b \in \mathbb{F}_p \setminus K$ and $E_{1,a} \cong E_{1,b}$. We need to decide if this can ever happen. Put the curves in the form 6.6 and use 6.5 to find a possible isomorphism between them. This tells us that such an isomorphism will occur if there is some $u \in \mathbb{F}_p^*$ such that

$$\begin{aligned} 1 - 24a =& u^4 (1 - 24b) \\ -1 + 36a - 216a^2 =& u^6 \left(-1 + 36b - 216b^2\right) \ . \end{aligned}$$

We use Maple to solve these equations for a and b in terms of u and get two solutions:

$$\begin{aligned} a =& \frac{3 + \sqrt{-3}u - 3u^2 - \sqrt{-3}u^3}{72} \\ b =& \frac{(3u - \sqrt{-3})(u^2 - 1)}{72u^3} \ , \end{aligned}$$

and

$$\begin{aligned} a =& \frac{3 - \sqrt{-3}u - 3u^2 + \sqrt{-3}u^3}{72} \\ b =& \frac{(3u + \sqrt{-3})(u^2 - 1)}{72u^3} \ . \end{aligned}$$

Since we are in the case where $p \equiv 1 \pmod 3$, we know from Fact 5.4.1 that $\sqrt{-3} \in \mathbb{F}_p$. Further, we note that the second pair of solutions can be obtained from the first pair by replacing u with $-u$. Thus, we can consider only the first pair of solutions and we will find all the possible isomorphisms. For notational purposes, we will write the first pair of solutions as functions of u,

$$\begin{aligned} a(u) =& \frac{3 + \sqrt{-3}u - 3u^2 - \sqrt{-3}u^3}{72} \\ b(u) =& \frac{(3u - \sqrt{-3})(u^2 - 1)}{72u^3} \ . \end{aligned}$$

Before we solve these we need to take steps to ensure we do not find isomorphisms to any curves that have already been dealt with previously. In particular, we wish to continue to avoid the singular curves ($E_{1,0}$ and $E_{1,3^{-3}}$), so we do not want to use u values such that $a(u) = 0$ or $a(u) = 3^{-3}$. The set of solutions to these equations, call it S_0, is

$$S_0 = \left\{1, -1, \sqrt{-3}, \frac{\sqrt{-3}}{3}\right\} .$$

As mentioned above, we know -3 is a quadratic residue so $S_0 \subset \mathbb{F}_p^*$.

We dealt with the isomorphism class of $E_{1,24^{-1}}$ already so we will exclude the values of u which are solutions to $a(u) = 24^{-1}$. We will call this subset of u-vales S_1 and solving gives that $S_1 = \left\{\frac{\sqrt{-3}\pm1}{2}\right\}$.

When $p \equiv 1 \pmod{12}$, we know $\sqrt{3} \in \mathbb{F}_p$ and we have already dealt with the curves $E_{1,\frac{3\pm\sqrt{3}}{36}}$. This time we solve $a(u) = \frac{3\pm\sqrt{3}}{36}$ and call the set of solutions S_2. Then we have

$$S_2 = \left\{\pm\sqrt{-1}, \frac{\pm1 + \sqrt{-1} + \sqrt{-3} \pm \sqrt{3}}{2}, \frac{\mp1 + \sqrt{-1} + \sqrt{-3} \pm \sqrt{3}}{2}\right\}$$

Since $p \equiv 1 \pmod{12}$, Fact 5.4.1 tells us $\sqrt{-1} \in \mathbb{F}_p$ and then we see $S_2 \subset \mathbb{F}_p^*$.

Another type of u we need to omit is any which gives a trivial isomorphism. Here we solve $a(u) = b(u)$ and get $u = \frac{\sqrt{-3}\pm1}{2}$. Since these are in S_1 they have already been omitted.

Now we let S be the full set of the u values to remove from consideration. From above we see that when $p \equiv 1 \pmod{12}$, $S = S_0 \cup S_1 \cup S_2$ with $|S| = 12$. When

$p \equiv 7 \pmod{12}$, $S = S_0 \cup S_1$ with $|S| = 6$. In addition, we have that

$$|\mathbb{F}_p^* \setminus S| = \begin{cases} p - 13, & p \equiv 1 \pmod{12} \\ p - 7, & p \equiv 7 \pmod{12}. \end{cases} \tag{6.12}$$

Since we know what u cannot be, we will move on to decide what happens with the remaining u values.

We now let u vary through the remaining permissible values in $\mathbb{F}_p^* \setminus S$. However, we need to check whether or not we get any repeated value for a. In other words, are there any $v \in \mathbb{F}_p^* \setminus S$ where $a(u) = a(v)$? Solving for v gives the following 3 solutions:

$$v = u \quad \text{or} \quad \frac{\pm 1 + \sqrt{-3} - u(1 \mp \sqrt{-3})}{2}.$$

We can check that the only time these 3 solutions are not distinct is when $u = \sqrt{-3}$. Since $\sqrt{-3} \in S_0$, we know we get 3 distinct v values for every $u \in \mathbb{F}_p^* \setminus S$. The only other potential hang up here is if any of the isomorphisms repeat. However, it is straightforward to check (in Maple, say) that the only u values for which that can happen are ones we have already excluded in set S. Thus, as u varies through $\mathbb{F}_p^*$, each $E_{a,1}$ is isomorphic to 3 other curves of the same form.

Recall that we are counting the number of isomorphism classes of size $4(p-1)$ with a curve of the form $E_{1,a}$. The above steps lead us to conclude that there are a total of $|\mathbb{F}_p^* \setminus S|/3$ different curves of this form, each of which is isomorphic to 3 other curves of that form. Thus, there must be $|\mathbb{F}_p^* \setminus S|/12$ isomorphism classes of size $4(p-1)$ and this number depends on what p is modulo 12 (see 6.12).

The only possible isomorphism classes yet to be counted are those of size $p-1$ with a representative of the form $E_{1,a}$ which is not isomorphic to any other curves of that form. In lieu of counting these directly, we notice that there are p possible values a can take and we count how many of these are left. We subtract away the number of a's that give singular curves, make $c_4 = 0$, make $c_6 = 0$, or are in isomorphism classes of size $4(p-1)$. When $p \equiv 1 \pmod{12}$ this leaves

$$p - 2 - 1 - 2 - \frac{p-13}{3} = \frac{2p-2}{3}$$

isomorphism classes of size $p-1$; when $p \equiv 7 \pmod{12}$ this leaves

$$p - 2 - 1 - 0 - \frac{p-7}{3} = \frac{2p-2}{3}$$

isomorphism classes of size $p-1$.

Finally, to get $N_{3,p}$ we add up the total number of isomorphism classes of each possible size: $\frac{p-1}{3}$, $\frac{4(p-1)}{3}$, $2(p-1)$, $4(p-1)$, and $p-1$. Therefore, for $p \equiv 1 \pmod{12}$

$$N_{3,p} = 2 + 1 + 1 + \frac{p-13}{12} + \frac{2p-2}{3} = \frac{3p+9}{4} ,$$

and for $p \equiv 7 \pmod{12}$

$$N_{3,p} = 2 + 1 + 0 + \frac{p-7}{12} + \frac{2p-2}{3} = \frac{3p+7}{4}$$

as desired. □

6.4 Computing the Correction Term

Before we can complete the proof of Theorem 6.1.1, we need to compute the correction term. We do so in the following proposition.

Proposition 6.4.1. *For prime $p \equiv 2 \pmod 3$ and $r \equiv p+1 \equiv 0 \pmod 3$, the correction term is 0. For $p \equiv 1 \pmod 3$ and $r \equiv p+1 \equiv 2 \pmod 3$, the correction term is*

$$\sum_{\substack{r^2<4p \\ r\equiv 2 \pmod 3}} c_{r,p} = \begin{cases} 5/2, & p \equiv 1 \pmod{12} \\ 2, & p \equiv 7 \pmod{12} \end{cases}$$

Proof. Recall the definition of the Hurwitz class number (Section 5.2) and the proof of Corollary 6.2.2. We need only consider those forms which are proportional to x^2+xy+y^2 or proportional to x^2+y^2.

In the first case, forms proportional to x^2+xy+y^2 arise when r and p are such that $r \equiv p+1 \pmod 3$ and $r^2-4p = -3\alpha^2$ for some non-zero integer α. Rewriting, we see that we need p to be such that

$$p = \left(\frac{r+\alpha\sqrt{-3}}{2}\right)\left(\frac{r-\alpha\sqrt{-3}}{2}\right) .$$

A prime p can be written this way if and only if $p \equiv 1 \pmod 3$ (see Fact 5.4.1). So, we must have $r \equiv 2 \pmod 3$. For each such p, there are three r values that satisfy the above. Thus, for $p \equiv 1 \pmod 3$, we must add $3 \cdot \frac{2}{3} = 2$ into the correction term and for $p \equiv 2 \pmod 3$, we add 0 into the correction term.

In the latter case, forms proportional to x^2+y^2 arise when for some $r \equiv p+1 \pmod 3$ we have $r^2-4p = -4\beta^2$ for some non-zero integer β. Notice that this requires r to be even, so we let $r = 2r_0$. Rewriting, we see that we need p to be such that

$$p = (r_0+\sqrt{-1}\beta)(r_0-\sqrt{-1}\beta) .$$

A prime p can be written this way if and only if $p \equiv 1 \pmod 4$ (see Fact 5.4.1). We also need $2r_0 \equiv p+1 \pmod 3$ where $p = r_0^2+\beta^2$.

Now, given x and y such that $p = x^2 + y^2$ consider two subcases. If $p \equiv 1 \pmod 3$, then 3 divides exactly one of x and y. Without loss of generality, suppose $3 \mid x$. In this subcase, we need $r_0 \equiv 1 \pmod 3$, so take $r_0 = y$ if $y \equiv 1 \pmod 3$ (otherwise, take $r_0 = -y$). So, we have 1 form and need to add $\frac{1}{2}$ into the correction term. If instead we have $p \equiv 2 \pmod 3$, then we need $\gcd(3, x) = 1$ and $\gcd(3, y) = 1$, but also need $r_0 \equiv 0 \pmod 3$. These conditions contradict each other, so there are no forms proportional to $x^2 + y^2$ in this subcase.

Combining all the above, the result follows. □

6.5 Proof of Theorem 6.1.1

We now have all the tools we need to prove Theorem 6.1.1.

Proof. (Theorem 6.1.1) First, suppose $p \equiv 2 \pmod 3$. Apply Proposition 6.3.3 and Proposition 6.4.1 to the sum in 6.3 to obtain

$$\sum_{\substack{r^2 < 4p, \\ r \equiv 0 \pmod 3}} H(4p - r^2) = (p - 1) - (0) = p - 1.$$

We know the total sum over all r must be $2p$ and we know the sums when $r \equiv 1 \pmod 3$ and $r \equiv -1 \pmod 3$ are equal by the symmetry property (6.2). Thus,

$$\sum_{\substack{r^2 < 4p, \\ r \equiv 1 \pmod 3}} H(4p - r^2) = \sum_{\substack{r^2 < 4p, \\ r \equiv 2 \pmod 3}} H(4p - r^2) = \frac{1}{2}\left(2p - (p - 1)\right) = \frac{p + 1}{2}$$

Next, suppose $p \equiv 1 \pmod{12}$. Apply Prop. 6.3.4 and Prop.6.4.1 to the sum in 6.4

to obtain

$$\sum_{\substack{r^2<4p,\\ r\equiv 2 \pmod 3}} H(4p-r^2) = \left(\frac{3p+9}{4}\right) - \left(\frac{5}{2}\right) = \frac{3p-1}{4}.$$

Repeating for $p \equiv 7 \pmod{12}$ yields

$$\sum_{\substack{r^2<4p,\\ r\equiv 2 \pmod 3}} H(4p-r^2) = \left(\frac{3p+7}{4}\right) - (2) = \frac{3p-1}{4}.$$

Thus, whenever $p \equiv 1 \pmod 3$ we have

$$\sum_{\substack{r^2<4p,\\ r\equiv 2 \pmod 3}} H(4p-r^2) = \frac{3p-1}{4},$$

and by the symmetry property (6.2) we have

$$\sum_{\substack{r^2<4p,\\ r\equiv 1 \pmod 3}} H(4p-r^2) = \frac{3p-1}{4}$$

also. Finally we subtract from the total sum $2p$ and conclude that for $p \equiv 1 \pmod 3$

$$\sum_{\substack{r^2<4p,\\ r\equiv 0 \pmod 3}} H(4p-r^2) = 2p - 2\left(\frac{3p-1}{4}\right) = \frac{p+1}{2}$$

$\square$

Table 6.1:

$m = 5$	$c = 0$	$c = \pm 1$	$c = \pm 2$
$p \equiv 1 \pmod 5$	$\frac{(p+1)}{2}$	$\frac{(p+1)}{3}$	$\frac{(5p-7)}{12}$
$p \equiv 2 \pmod 5$	$\frac{(p+1)}{3}$	$\frac{(p+1)}{3}$	$\star\frac{\mathbf{(p-1)}}{\mathbf{2}}\star$
$p \equiv 3 \pmod 5$	$\frac{(p+1)}{3}$	$\star\frac{\mathbf{(p-1)}}{\mathbf{2}}\star$	$\frac{(p+1)}{3}$
$p \equiv 4 \pmod 5$	$\star\frac{\mathbf{(p-3)}}{\mathbf{2}}\star$	$\frac{(5p+5)}{12}$	$\frac{(p+1)}{3}$

6.6 Additional Theorems and Proof Techniques

In [5], the method we used to prove Theorem 6.1.1 is used to prove the following two identities for $p > 3$ a prime:

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod 2}} H(4p-r^2) = \begin{cases} \frac{4p-2}{3}, & \text{if } c = 0 \\ \frac{2p+2}{3}, & \text{if } c = 1. \end{cases}$$

and

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod 4}} H(4p-r^2) = \begin{cases} \frac{p+1}{3}, & \text{if } c \equiv \pm 1 \pmod 4 \\ \frac{5p-7}{6}, & \text{if } c \equiv p+1 \pmod 4 \\ \frac{p+1}{2}, & \text{if } c \equiv p-1 \pmod 4). \end{cases}$$

Also in [5], the authors prove Theorem 6.1.1 using an entirely different method. This approach has to do with coefficients of modular forms. They also suggest a generalization for p not necessarily prime. Finally, they use a third method which uses the Eichler-Selberg trace formula. The values in Table 6.6 are for the sum

$$\sum_{\substack{r^2<4p,\\ r\equiv c \pmod m}} H(4p-r^2)$$

when $m = 5$. The expression with a star are proved in [5]. Those without a star are still conjectures.

Chapter 7

Conclusion

In a way, each of the proofs in Sections 4.1 and 4.2 are constructive. In each case, we begin with a choice of points and weights for a probability weight function and use this to build a generating function for the associated Strodt polynomials. We then manipulate the generating function until it contains a sine or cosine curve in the numerator and take note of the resulting singularities. We then choose a comparison function, but even this "choice" is directly determined by a prescribed manner of choosing a polynomial and taking a limit. Finally, we apply Darboux's Method to obtain asymptotics.

There are pro's and con's to this method. On the one hand, it is unnecessary to know details about the asymptotic behavior of the polynomials ahead of time. On the other hand, this approach requires a great amount of structure in the generating function denominator in order to work. Furthermore, the glaring weakness of this method is that it does not work in every case and certainly cannot be used to explain the behavior of the other polynomials described in Section 4.3. If an appropriate answer to Question 4.3.1 is to be found and proved, we will need a more broad, less constructive, approach.

Appendices

Appendix A   Examples of Strodt Polynomials

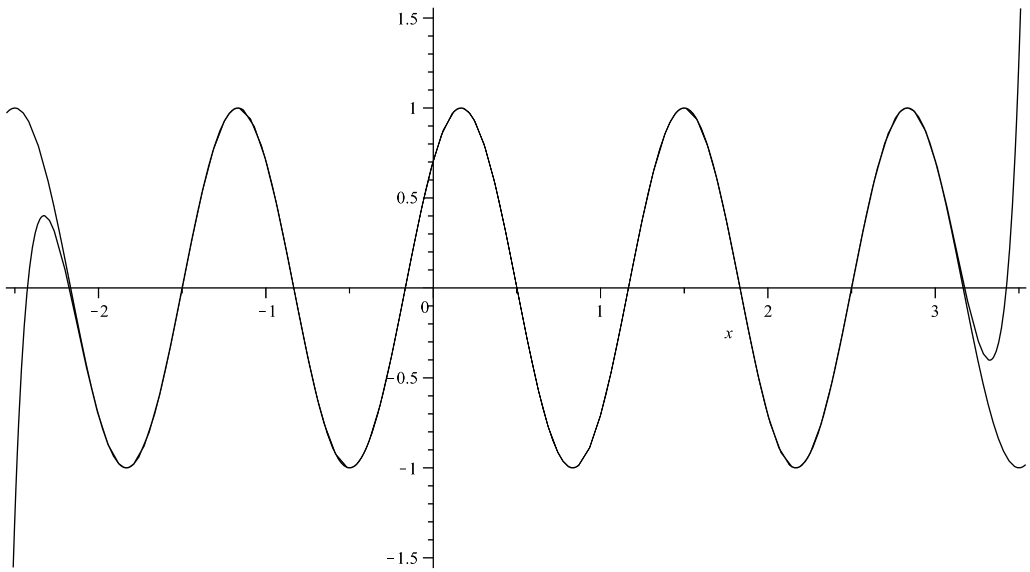

Figure 1: Illustration of Theorem 4.1.1 for $k = 6$ and $n = 31$

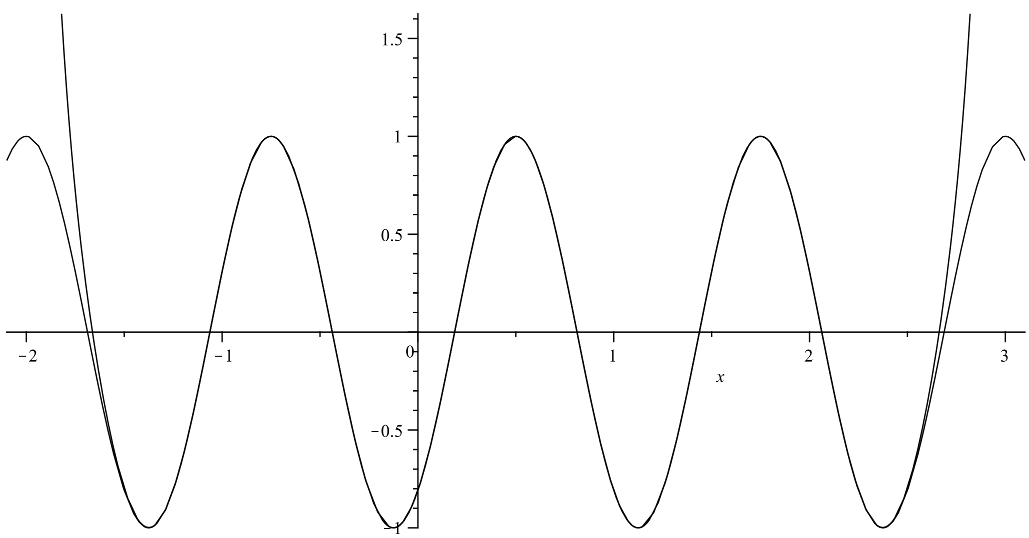

Figure 2: Illustration of Theorem 4.1.2 for $k = 12$ and $n = 24$

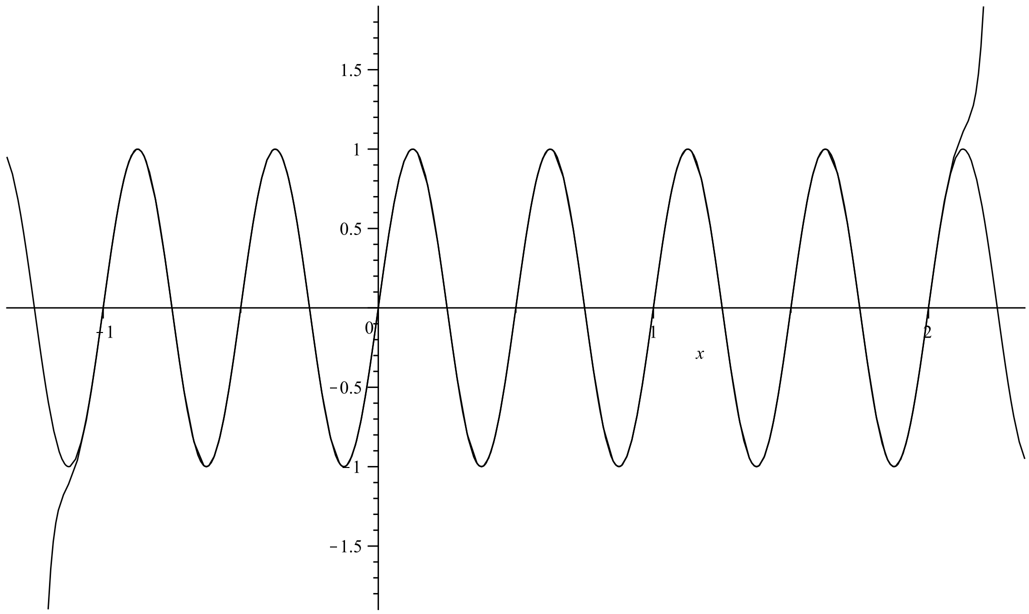

Figure 3: Illustration of Theorem 4.1.3 for $k = 10$ and $n = 49$

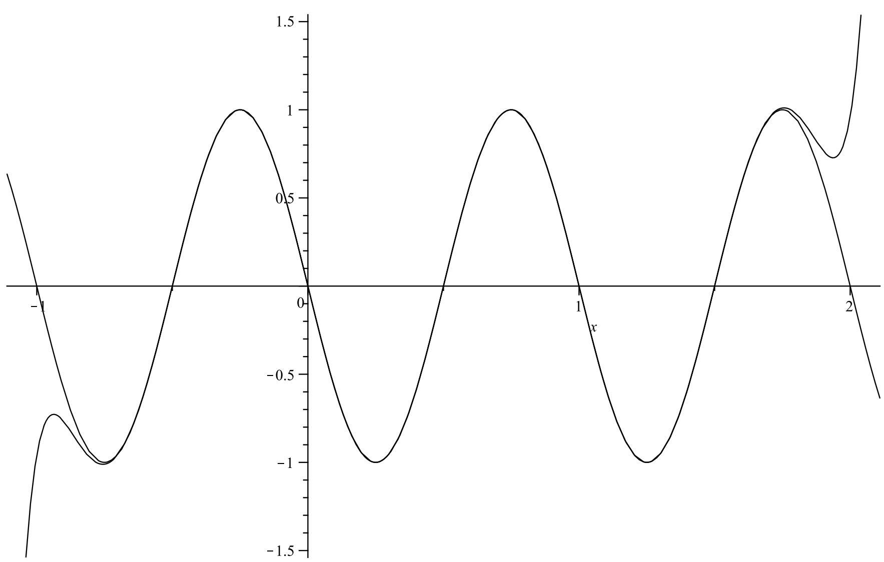

Figure 4: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 17$

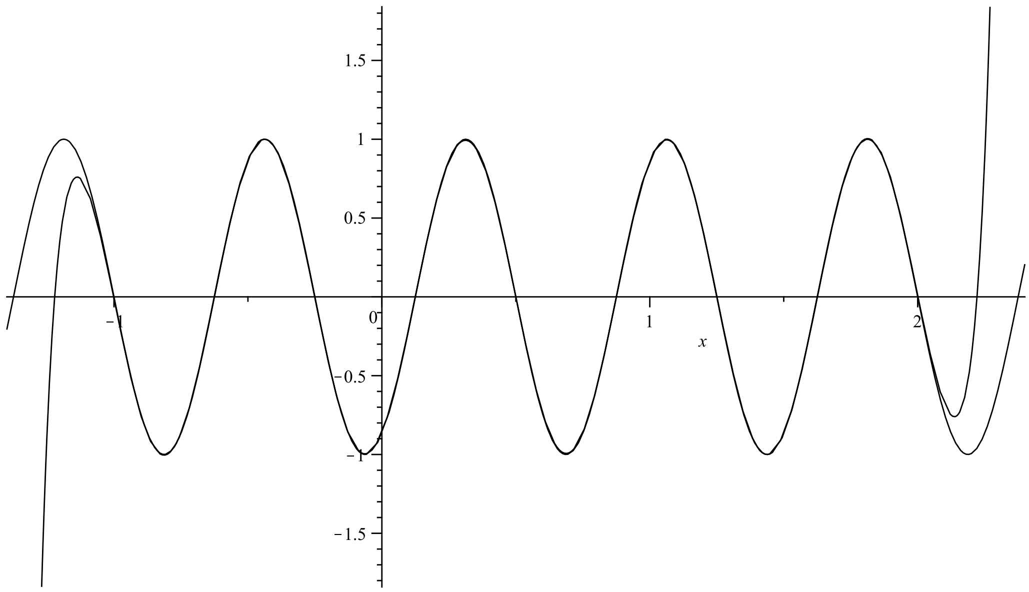

Figure 5: Illustration of Theorem 4.2.2 for $m = 8$ and $n = 27$

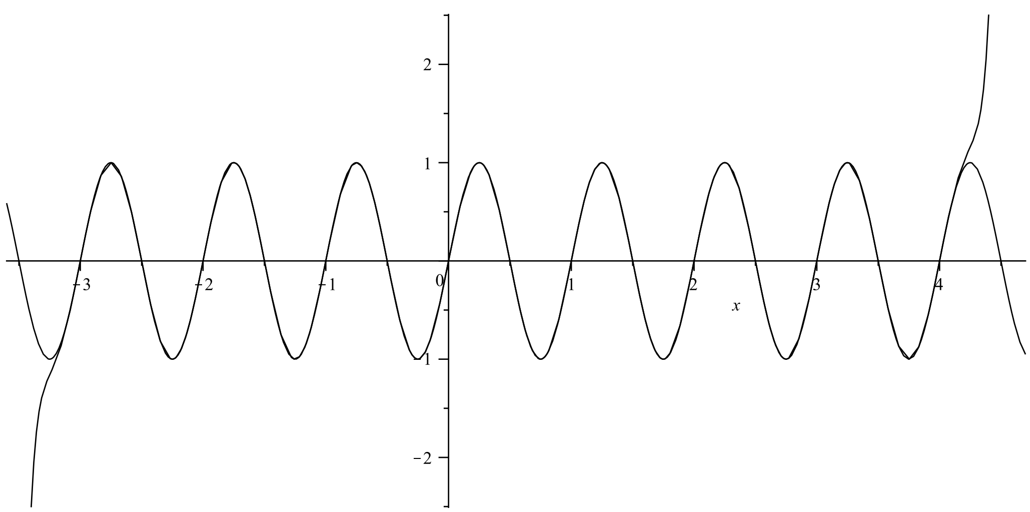

Figure 6: Illustration of Theorem 4.2.3 for $m = 4$ and $n = 55$

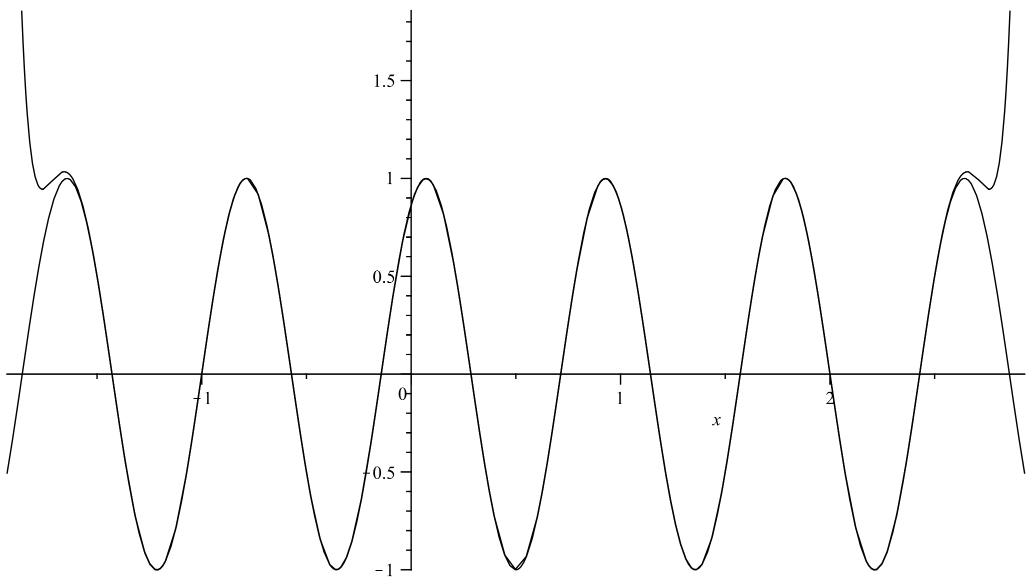

Figure 7: Illustration of Theorem 4.2.3 for $m = 8$ and $n = 34$

Appendix B Comparing Two Families as n Increases

The figures on this page and the subsequent pages accompany Section 4.3. Figures 8 through 14 are of the polynomials $P_n^6(x)$ from Theorem 4.2.1 and their corresponding cosine curve for various values of n. The graphs in Figures 15 through 22 are of the polynomials $P_n^*(x)$ for several values of n. These figures include lines at $y = \pm 1$ for comparison.

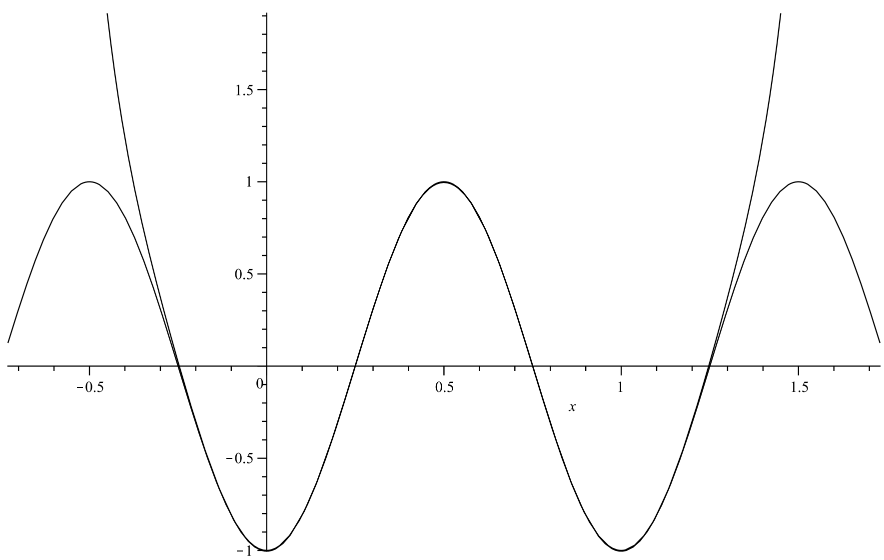

Figure 8: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 8$

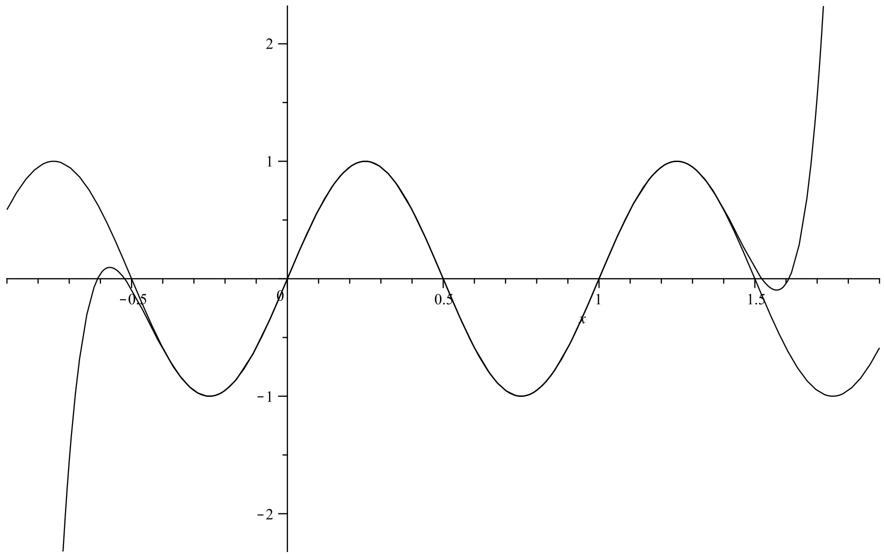

Figure 9: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 11$

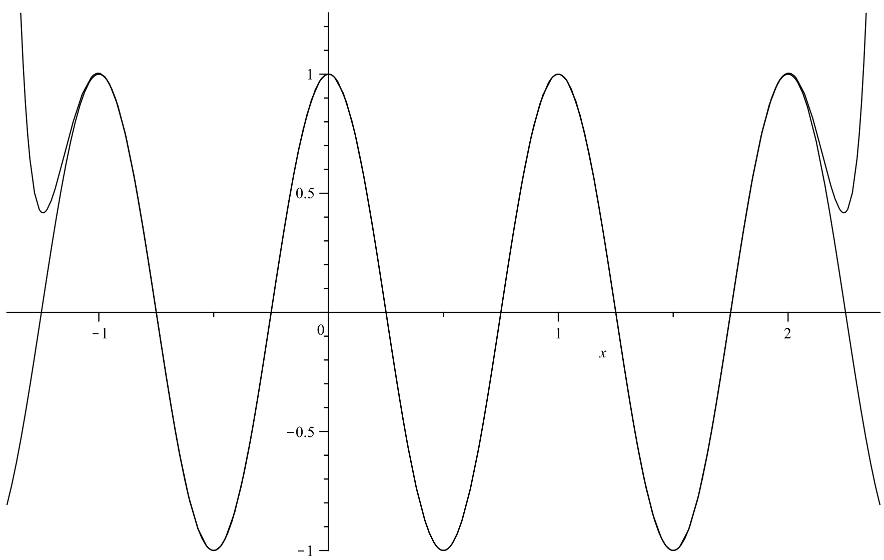

Figure 10: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 22$

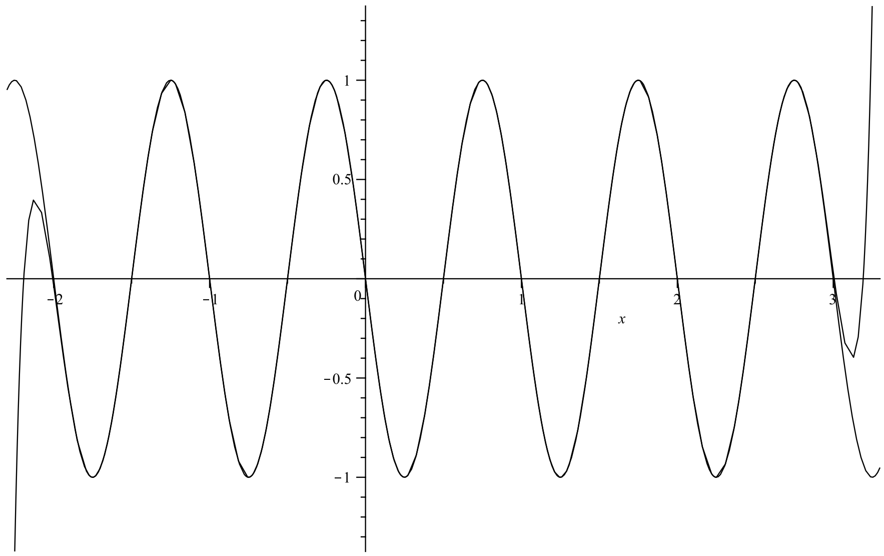

Figure 11: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 37$

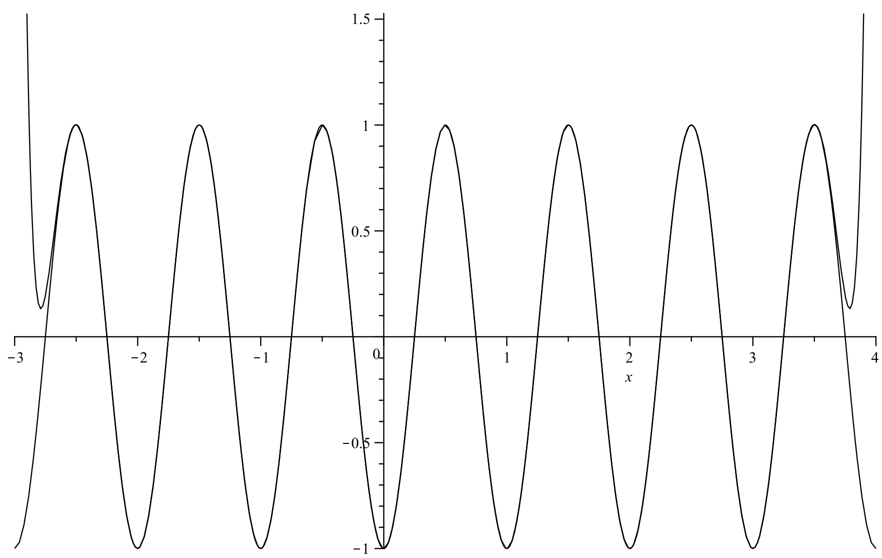

Figure 12: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 48$

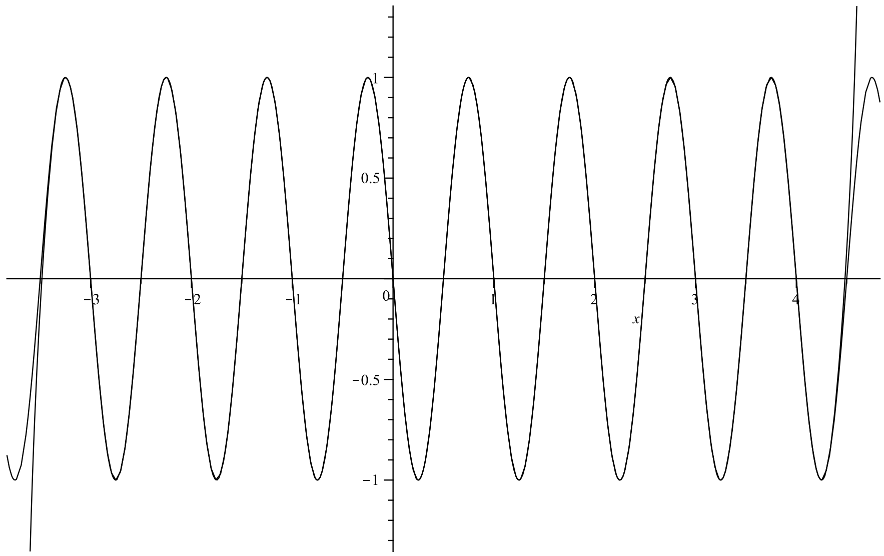

Figure 13: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 61$

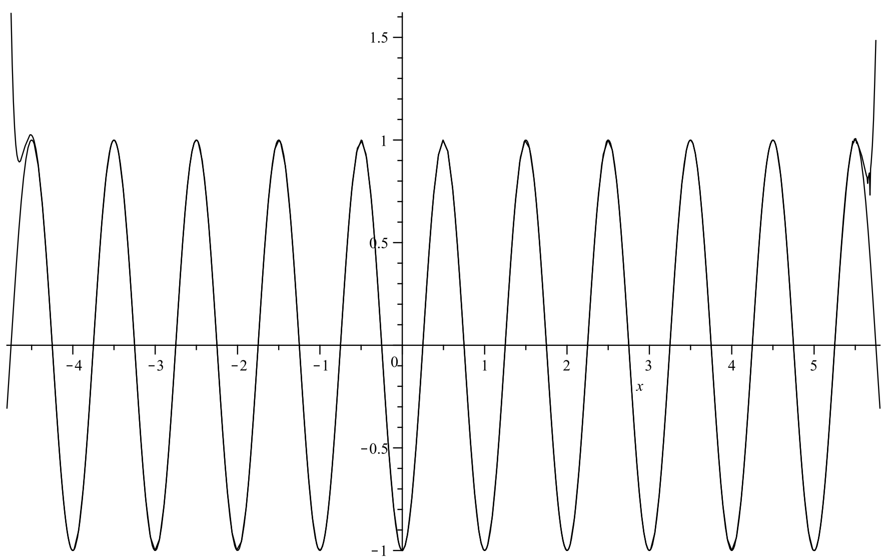

Figure 14: Illustration of Theorem 4.2.1 for $m = 6$ and $n = 80$

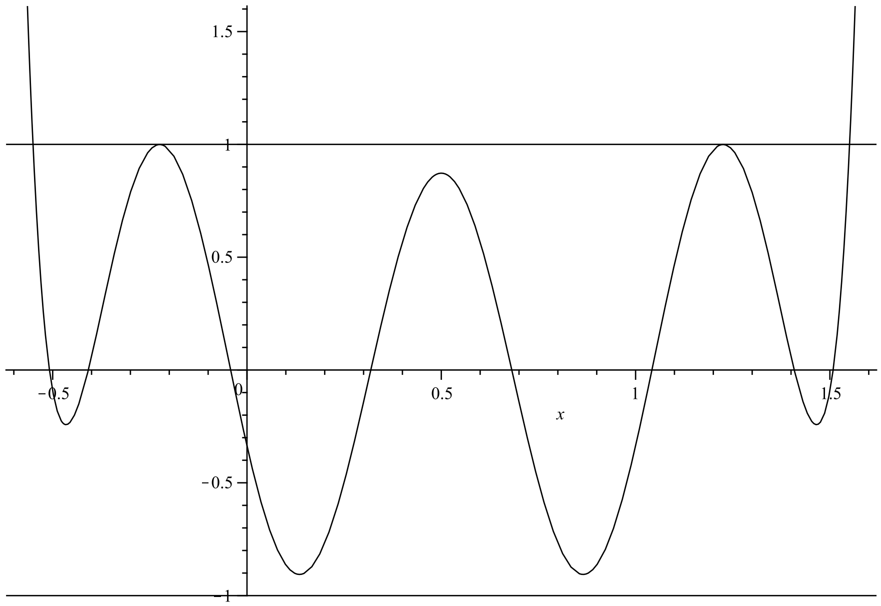

Figure 15: $P^*_{12}(x)$ and lines at $y = \pm 1$ for comparison

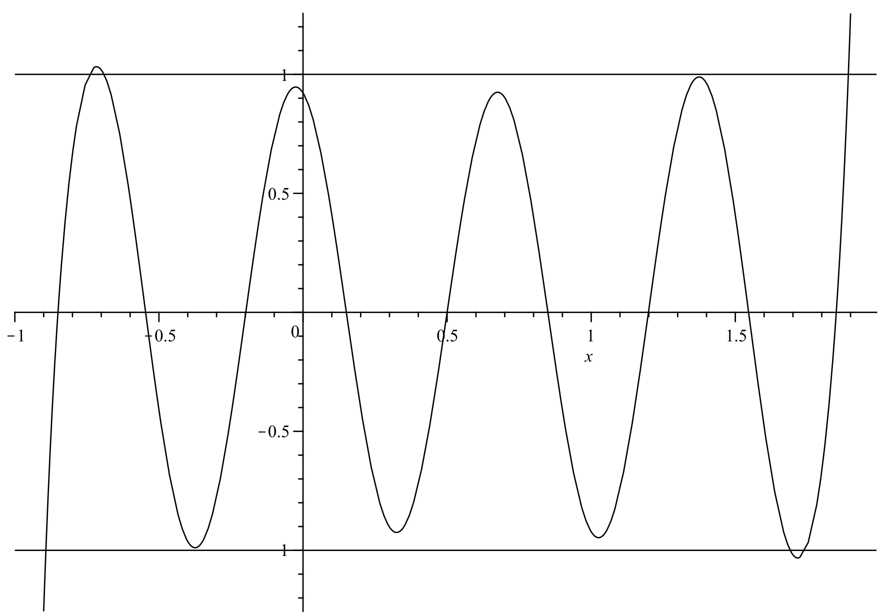

Figure 16: $P^*_{21}(x)$ and lines at $y = \pm 1$ for comparison

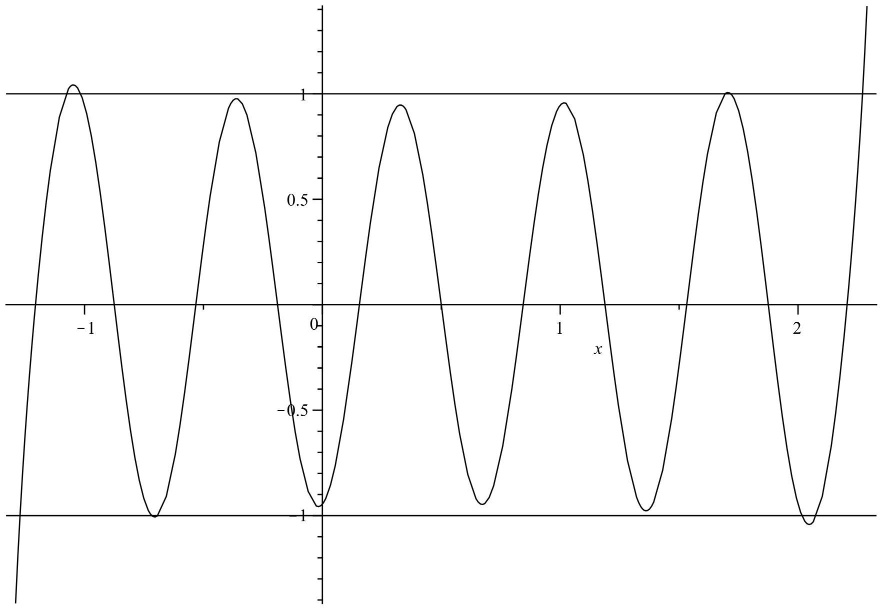

Figure 17: $P_{31}^{*}(x)$ and lines at $y = \pm 1$ for comparison

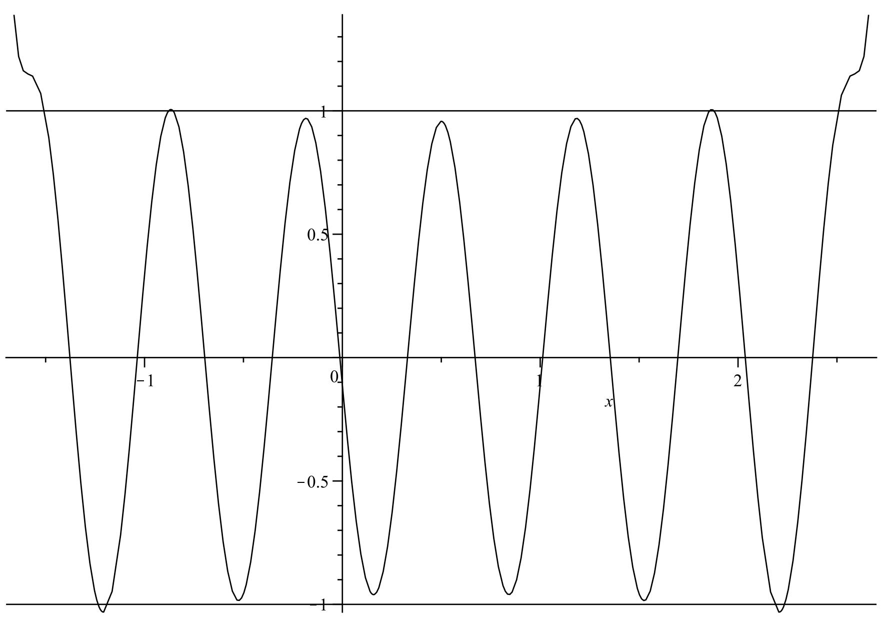

Figure 18: $P_{40}^{*}(x)$ and lines at $y = \pm 1$ for comparison

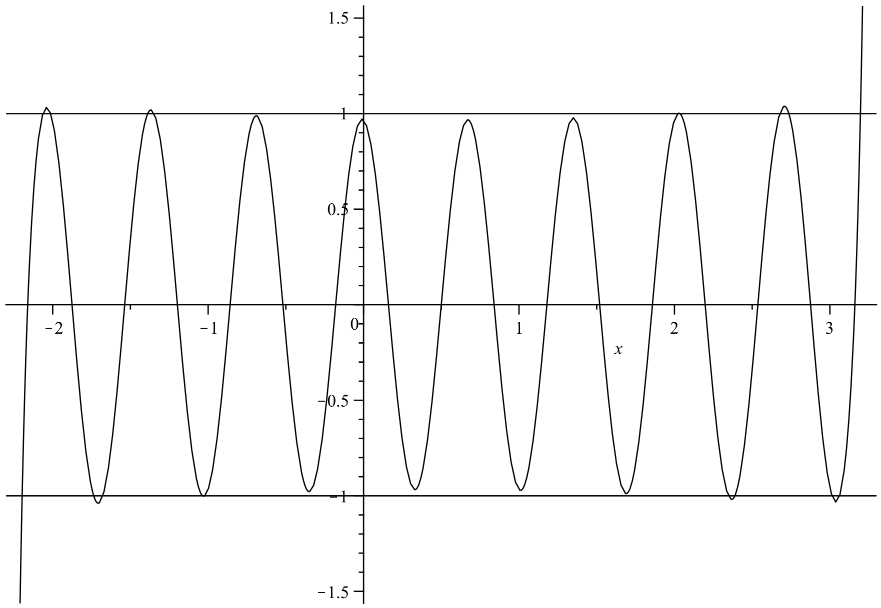

Figure 19: $P^*_{53}(x)$ and lines at $y = \pm 1$ for comparison

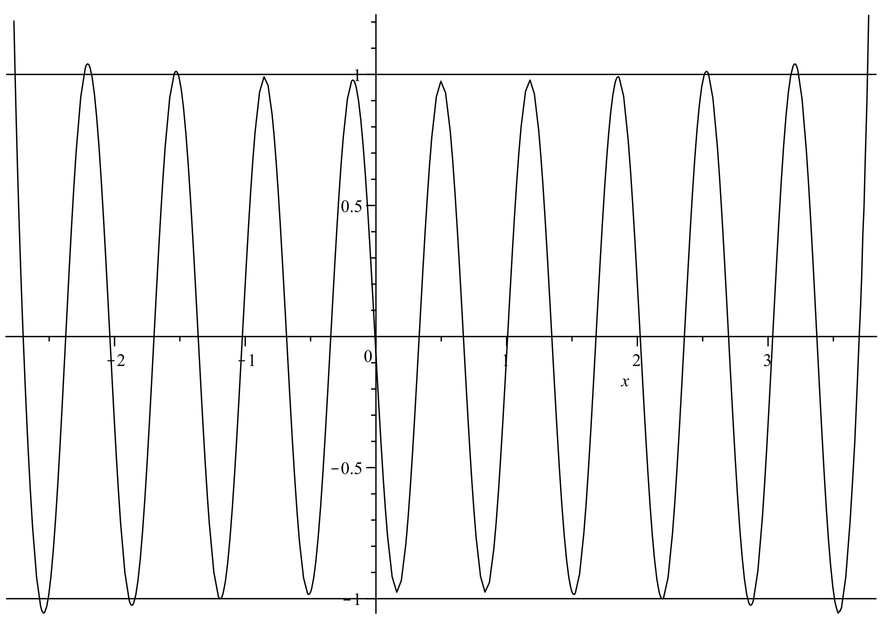

Figure 20: $P^*_{68}(x)$ and lines at $y = \pm 1$ for comparison

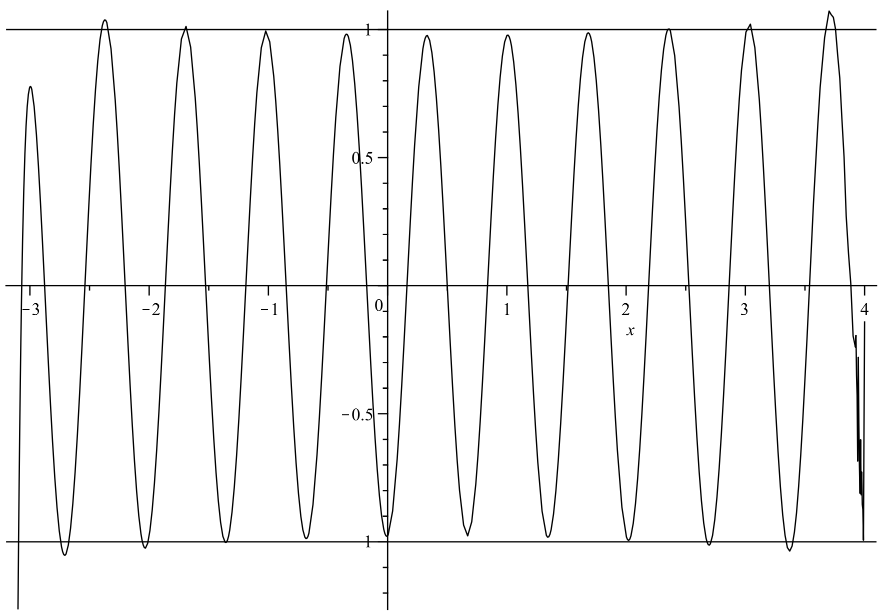

Figure 21: $P^*_{75}(x)$ and lines at $y = \pm 1$ for comparison

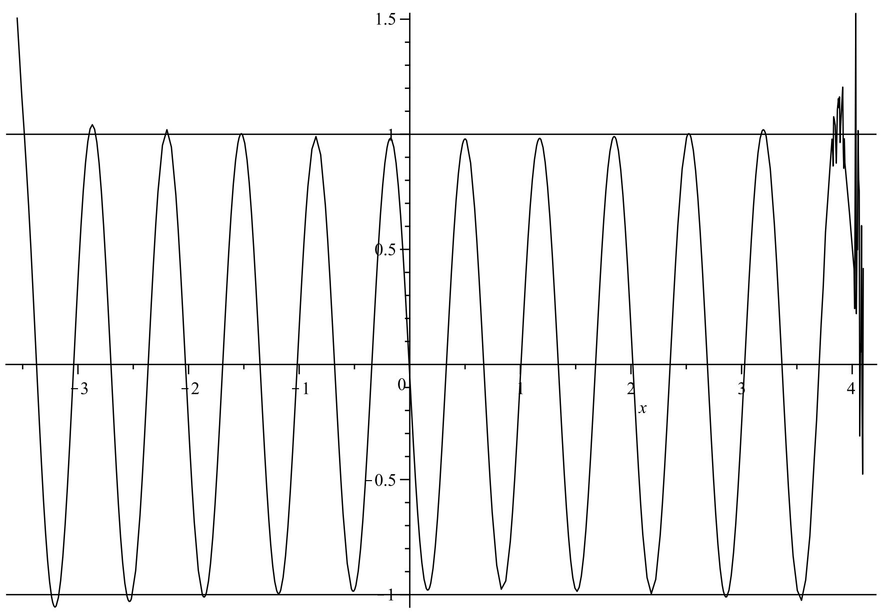

Figure 22: $P^*_{88}(x)$ and lines at $y = \pm 1$ for comparison

Appendix C More Strodt Polynomials

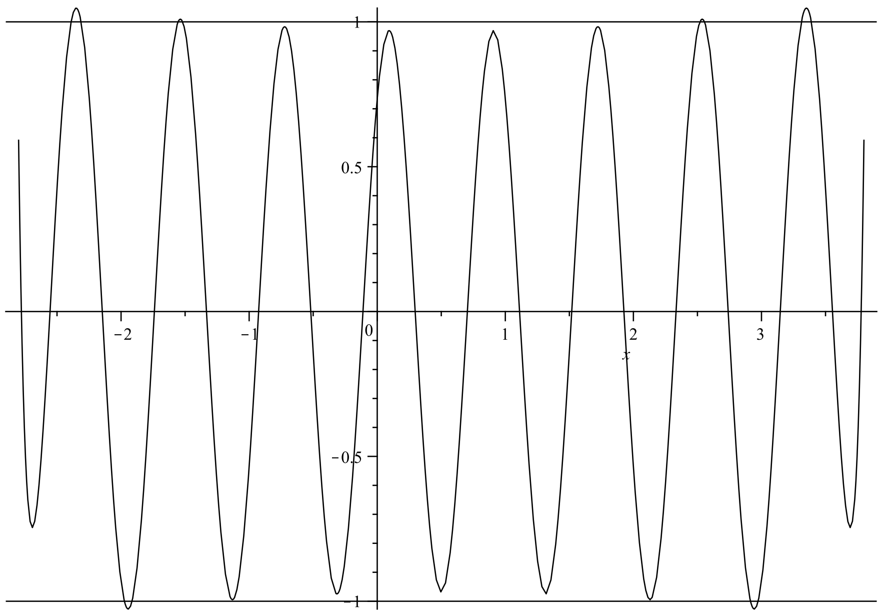

Figure 23: $P_{54}^{**}(x)$, as defined in Section 4.3, and lines at $y = \pm 1$ for comparison

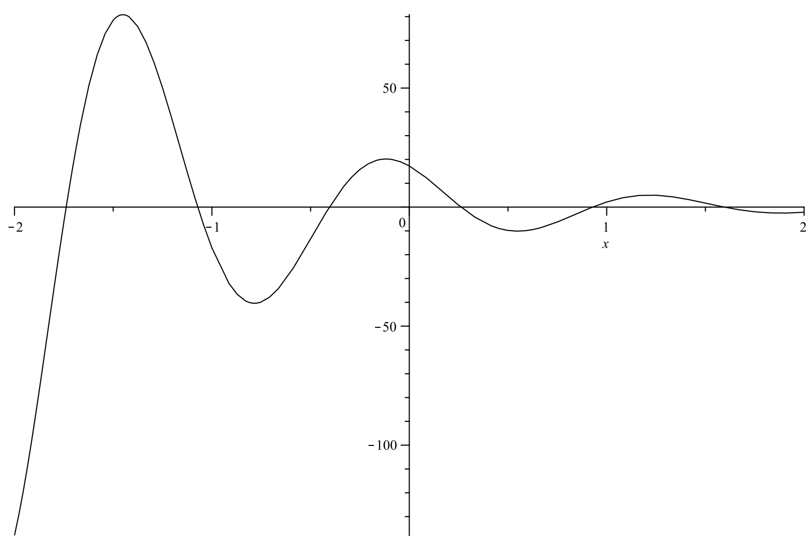

Figure 24: The (scaled) polynomial resulting from the point $\frac{1}{6}$ with weight one-third and the point $\frac{5}{6}$ with weight two-thirds, $n = 30$

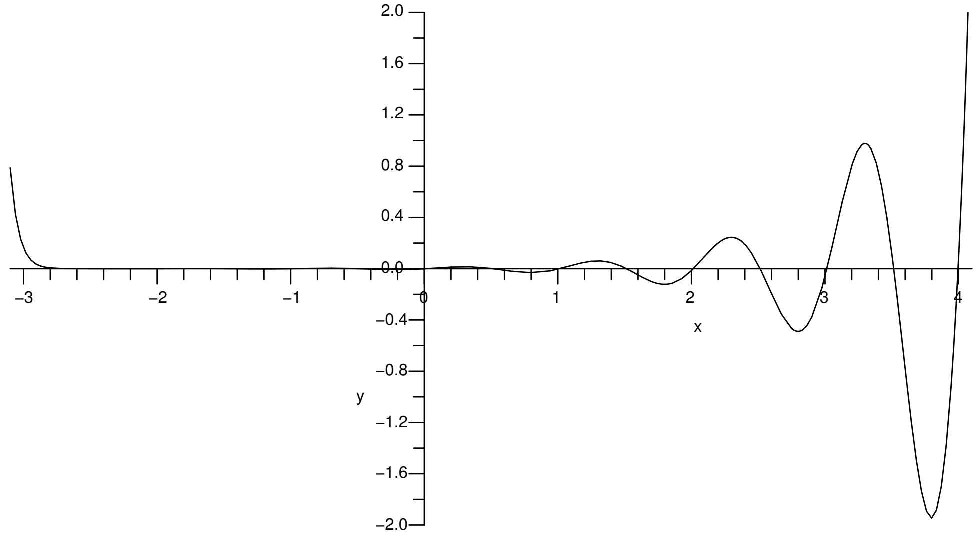

Figure 25: The (scaled) polynomial resulting from the points $\frac{1}{8}$ and $\frac{3}{8}$ with weight one-third and the points $\frac{5}{8}$ and $\frac{7}{8}$ with weight two-thirds, $n = 50$

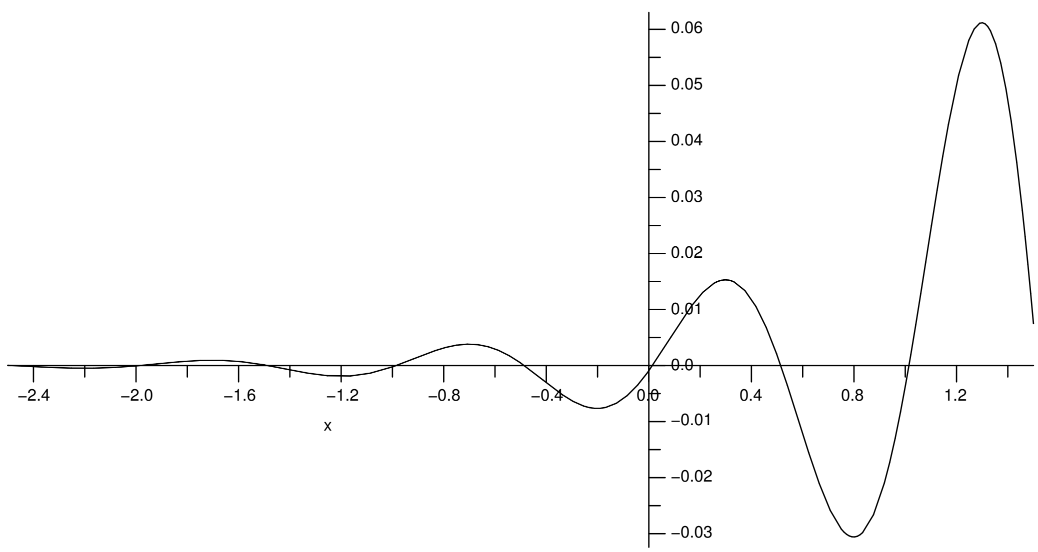

Figure 26: The same polynomial as in Figure 25 above, zoomed in to show the middle range of x values

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