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Galactic Chemical Evolution: Z Versus In(1/mu) Relationship

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GALACTIC CHEMICAL EVOLUTION: Z VERSUS $\ln(1/\mu)$ RELATIONSHIP

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ABSTRACT

By studying six different analytic models of the chemical evolution of galaxies in the presence of time-dependent continuous metal-poor infall, I show that the metallicity versus gas fraction relationship is well approximated by

$$Z(\mu) = \frac{y}{2}\left\{\ln\frac{1}{\mu} + \ln\left[\ln\left(\frac{1}{\mu}\right) + 1\right]\right\},$$

the analog of $Z = y \ln 1/\mu$ relation for closed evolution. This new result equals the old for gas-rich relatively unevolved systems ($\mu \to 1$) but grows more slowly during gas consumption ($\mu \to 0$). This approximation seems valid when infall rate is great enough that it is not negligible, but not so great that it dominates the gas budget. If the yield y can be calculated accurately, the difference from closed $Z(\mu)$ may allow an observational measure of the presence of infall in galactic evolution.

Subject headings: galaxies: evolution — nucleosynthesis — stars: abundances

I. INTRODUCTION

One of the objectives of abundance studies in galaxies is to ascertain whether they are reconcilable with conventional thinking on the chemical evolution of galaxies. It is natural to compare observations to the expectations of that theory expressed in the form of simplified analytic models, because the analytic models contain the essence of the numerical calculations in a transparent, readily graspable form. The most famous and often used comparison of this type is that introduced by Schmidt (1963) and by Searle and Sargent (1972), who treated the simple cycling of gas in a closed box to show that the gaseous concentration Z of a primary nucleosythesis product depends upon the fraction μ of the total mass remaining in gaseous form according to

$$Z = y \ln 1/\mu , \tag{1}$$

where the yield y is defined as the ratio of the mass rate of ejection of new nuclei of type Z to the mass rate of formation of stellar remnants. Although one can study the metallicities of stars to test this relation, a conceptually simpler idea is to study the gaseous concentrations directly via H II regions, as Shaver *et al.* (1983) have done for the Galaxy or as Garnett and Shields (1986) have recently done across the face of M81, and to compare them with the ratio of local gas mass to local total mass.

Equation (1) remains a popular point of reference even though the closed-box assumption is suspected of being incorrect. Infall of metal-poor gas onto the disk has been increasingly considered for both dynamical reasons and for reasons having to do with the smallness of the population of low-Z stars and to the relative flatness of the star formation rate. A natural question concerns the effect of infall on equation (1). What simple relation should one use in place of equation (1) to make a fast comparison of the theory with observation? In this paper I use analytic models of chemical evolution to discover the form

$$Z^*(\mu) = \frac{y}{2}\left\{\ln\left(\frac{1}{\mu}\right) + \ln\left[\ln\left(\frac{1}{\mu}\right) + 1\right]\right\} \tag{2}$$

and to show its wide applicability by comparing it to exact results from four different analytic models.

I wish to emphasize to the reader that the analytic models to be studied are fully time-dependent ones in which the mass of interstellar gas is determined by competition between time-varying infall rates and time-varying rates of star formation that are proportional to the mass of gas available. They qualify as true dynamic models, as opposed to models generated from arbitrarily constraining quantities in order to solve the differential equations much more easily. Many examples of the latter exist; e.g., star formation rate is constant, infall is constant, ratio of infall to gas mass is constant, etc. Such solutions based on constraints may, of course, be preferable if it can be shown that those arbitrary constraints are indeed true. In the absence of such demonstration, I find that the analytic time-dependent linear models are more physically believable. In any case they lead to more general physical comprehension of the system. The linearity need not be interpreted so narrowly as to require the star formation process to be proportional to the first power of the gas density. It is sufficient that star formation be a complicated nonlinear sequence of processes happening in clouds, but that the number of clouds be proportional in a given azimuthal volume to the mass of gas contained therein. So stated, the linear model seems very plausible and not a naive oversimplification of the admittedly complicated physics of star formation.

I make a clear distinction in my own mind between physical approximations that allow relatively simple differential equations to be written (e.g., instantaneous recycling, constant yield and constant return fraction, and linear star formation) and additional constraints, often arbitrary, that allow special solutions of those differential equations to be generated. The general solutions are much to be preferred unless a physical or observational argument suggests the constraint. For most of this paper I will deal in these general solutions, so that the $Z(\mu)$ relationships are also general; but in a final section I will illustrate the comparison of a treatment having severe constraints (Twarog 1980). The results achieved will be of interest partially in their primary purpose of identifying an appropriate $Z(\mu)$ relationship for infall models and partially in laying out some new analytic results of interest in their own right. Many readers will wish only to see the genesis of this $Z(\mu)$ relationship in §§ II and IV, whereas specialists in chemical evolution will also find the intermediate sections new and informative.

II. CLAYTON'S "STANDARD MODEL"

To construct analytic solutions one most easily uses the *linear model* (e.g., Clayton 1984, 1985) with the aid of two approximations, *instantaneous recycling* and *constant yield.* Clayton and Pantelaki (1986) have documented the adequacy of the first approximation except for the early very low Z growth and for the late very low gas domains, and they have also shown how analytic solutions are still routinely generated even if the yields vary smoothly with time. However, this study will keep things at their simplest by employing both approximations in order to achieve analytic evaluation of the Z, ln $(1/\mu)$ relationship.

With those physical approximations, the two basic equations (in Clayton's notation) to be satisfied are that of mass conservation

$$\frac{dM_G}{dt} = -\omega M_G + f\,, \tag{1}$$

where $M_G(t)$ is the gas mass and $f(t)$ the infall rate, and the growth of the gas concentration $Z \equiv X(Z)$,

$$\frac{dZ}{dt} = y_Z\,\omega - \lambda Z - (Z - Z_f)\,\frac{f(t)}{M_G(t)}\,, \tag{2}$$

where the linear model approximation defines ω by

$$(1 - R)\psi(t) \equiv \omega M_G(t)\,. \tag{3}$$

For the present purposes I consider stable primary nuclei having constant yield y_Z, so the radioactive decay rate $\lambda = 0$. Clayton (1984, 1985) showed analytic time-dependent solutions of these equations. The simplest representation is the one advocated by Clayton (1985) wherein equation (1) is satisfied by

$$f(t) = \frac{kM_G(0)}{\Delta}\left(\frac{t+\Delta}{\Delta}\right)^{k-1} e^{-\omega t} \tag{4}$$

and

$$M_G(t) = M_G(0)\left(\frac{t+\Delta}{\Delta}\right)^{k} e^{-\omega t}\,, \tag{5}$$

with Δ an arbitrary constant and k an integer. Choice of k and Δ approximates $f(t)$ by a smooth function that rises to a maximum early and eventually declines to zero. The stable metallicity exceeds the value Z_f carried in the infalling matter by

$$(Z - Z_f) = \frac{y_Z\,\omega\Delta}{k+1}\left[\left(\frac{t+\Delta}{\Delta}\right) - \left(\frac{t+\Delta}{\Delta}\right)^{-k}\right]. \tag{6}$$

In using this and other models to explore $Z(\mu)$, I will let $Z_f = 0$, requiring only that Z subsequently mean "the growth of Z" rather than its total value.

a) Simple Closed Model

The so-called "simple model," having no infall, takes $k = 0$. It is immediately evident from equation (6) that with $k = 0$

$$Z = y\omega t = y \ln\,[M_G(0)/M_G(t)] = y \ln\,(1/\mu)\,, \tag{7}$$

the historically influential relationship that I hope to generalize.

b) $k = 1$ Standard Model

With $k = 1$ the infall is pure exponential:

$$f_{k=1} = \frac{M_G(0)}{\Delta} \exp\,(-\omega t)\,, \tag{8}$$

where the decay rate ω is the same constant as the gas depletion rate ω owing to stellar birth. Because the total mass $M(t)$ needed to evaluate μ is

$$M(t) = M_G(0) + \int_0^t f(t')dt'\,, \tag{9}$$

one easily finds

$$\mu_{k=1} = \frac{\omega(t+\Delta)e^{-\omega t}}{1 - e^{-\omega t} + \Delta\omega} \tag{10}$$

and

$$Z_{k=1} = \frac{y\omega t}{2}\left(\frac{t+2\Delta}{t+\Delta}\right). \tag{11}$$

To choose an appropriate $Z(\mu)$ relation, it is useful to examine first these results at early time, when $\omega t \ll 1$ and $t < \Delta$. Then in that limit, $t \to 0$,

$$\mu \to e^{\omega t}, \qquad Z \to y\omega t\,, \tag{12}$$

exactly the same results that are true for all time in the $k = 0$ closed model. Thus $Z(\mu)$ begins as $\ln(1/\mu)$.

On the other hand, for $t \gg \Delta$ (not difficult, since Clayton 1985 showed that $\Delta = 10^8$–10^9 yr are interesting, whereas $t \approx 15 \times 10^9$ yr today), solutions (10) and (11) become essentially

$$Z \to \frac{y\omega t}{2}, \qquad \mu \to \frac{\omega t e^{-\omega t}}{1 + \Delta\omega}. \tag{13}$$

Before combining these, note that a large galactic abundance gradient, say a factor of 10 from center to extremity, cannot be modeled at all at constant y, constant $t = T$, and constant star formation rate per unit mass ω. Under those conditions, Z has the same value everywhere. To obtain an abundance gradient one must either take different galactic age T (not likely) or different yield $y(r)$, as Güsten and Mezger (1982) have advocated, or different star formation per unit gas mass ω. That is, in the last option one says that outer galaxies are metal poor because star formation has been less efficient. I will follow that option, although the argument at $y(r)$ decreases is also quite interesting. This is exactly what is done in the $k = 0$ closed azimuthal zone model, where in order to obtain a gradient one says ωT decreases with radius. But it will be noted from equation (8) that, in Clayton's (1985) "standard model," a radially varying $\omega(r)$ also introduces a radially varying infall rate $f(t)$. If $\omega(r)$ decreases radially to obtain an abundance gradient, the infall rate per initial unit mass of disk increases in this representation. Thus in this analytic model, the radial gradient is a perhaps not implausible mix of decreasing star formation efficiency plus increased importance of late-time infall. Having said that we can return to equation (13) and simply treat $\omega(r)$ as a radial variable. Equation (13) can be solved by iteration, letting $\omega t = \phi > 1$:

$$\phi = \ln(1/\mu) + \ln\phi - ln(1+\Delta\omega)\,, \qquad Z/y = \phi/2\,. \tag{14}$$

Now, rather than generate a useless unending iteration of the first of these, I guess for the first iteration that $\ln[\phi/(1+\Delta\omega)] \approx 1$, which is not far off if $T \approx 15$, $\omega \approx 0.3$, and $\Delta \approx 1$ (Clayton 1985). and otherwise ignore $\ln(1+\Delta\omega)$, giving for large t

$$Z^*(\mu) \approx \frac{y}{2}\left\{\ln\frac{1}{\mu} + \ln\left[\ln\left(\frac{1}{\mu}\right) + 1\right]\right\}. \tag{15}$$

Another reason for guessing this form is that it has the correct low-t limit, equation (12), as $\mu \to 1$, namely,

$$Z^*(\mu) \to y\ln(1/\mu) \text{ as } \mu \to 1\,, \tag{16}$$

assuring that equation (15) has appropriate limits. Now this hardly qualifies as a "derivation," so I next show that equation (15) works well by examining the exact solutions parametrically in several different analytic models. Figure 1 shows for this $k = 1$ model the exact curves $Z(\omega)$ and $\mu(\omega)$ at $t = 15$ Gyr from equations (10) and (11), along with both $Z^*(\mu)$ from equation (15) and $\ln(1/\mu)$. One sees that $Z^*(\mu)$ is an adequate first approximation to $Z(\mu)$ for this model, whereas $\ln(1/\mu)$ is correct only for small Z (large μ).

c) $k = 2$ *Standard Model*

The infall $f(t)$ rate equation (4) rises to a maximum at $t(\max f) = 1/\omega - \Delta$ for $k = 2$ models. At values $\omega < 0.1$ Gyr^{-1} needed to achieve high μ and low Z at $T = 15$ Gyr, the infall *grows* for the first 10^{10} yr. This is a sharp physical contrast to $k = 1$ models, wherein the maximum $f(t)$ is at $t = 0$ for all values of ω. It is in light of this large physical difference instructive to see how well $Z^*(\mu)$ works for $k = 2$ models. The exact solutions can be written

$$Z_{k=2} = \frac{y\omega t}{3}\left(\frac{t^2 + 3\Delta t + 3\Delta^2}{t^2 + 2\Delta t + \Delta^2}\right), \tag{17}$$

and

$$\mu_{k=2} = \frac{\omega^2(t+\Delta)^2 e^{-\omega t}}{\Delta^2\omega^2 + 2[1 + \Delta\omega - e^{-\omega t}(1 + \Delta\omega + \omega t)]}, \tag{18}$$

which also approaches equation (16) in the limit $t \to 0(\mu \to 1)$. By carrying through a similar approximate numerical analysis to that

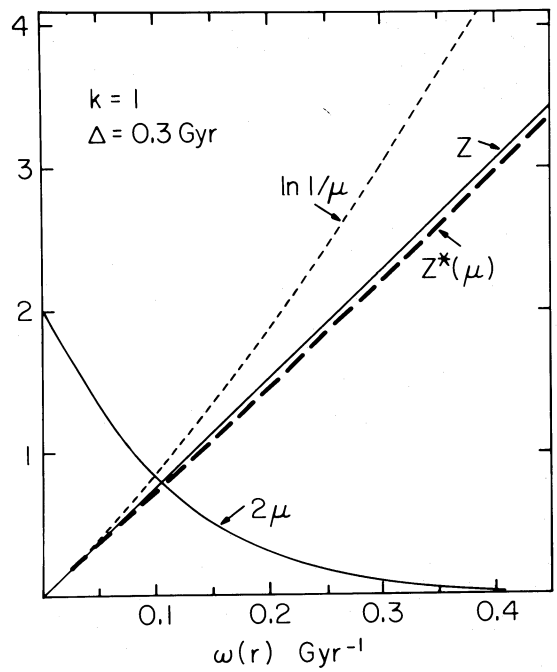

FIG. 1

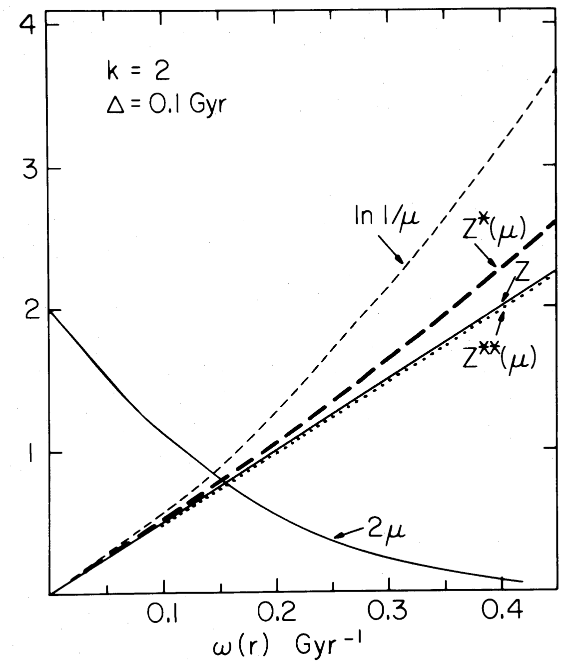

FIG. 2

FIG. 1.—Clayton's (1985) $k = 1$ "standard analytic model" with parameter $\Delta = 0.3$ Gyr and galactic age $t = 15$ Gyr. The net star formation rate per unit mass $\omega(r)$ and the rate of decline of the exponential infall are both equal to $\omega(r)$ in this model, so the abundance gradient is a combination of the two effects. The approximate $Z^*(\mu)$ shown in heavy dash is a better approximation to the exact $Z(\mu)$, shown as solid line, than is the dashed approximation $\ln(1/\mu)$. Note that the Z ordinate is actually Z/y_Z, whereas for the gas fraction it is 2μ.

FIG. 2.—Clayton.s (1985) $k = 2$ "standard analytic model" has infall rising to a maximum at $t(\max f) = 2/\omega - 0.1$ for $\Delta = 0.1$, the case shown here. Format is as in Fig. 1. A still better approximation $Z^{**}(\mu)$ was derived in eq. (19) and is shown here, but $Z^*(\mu)$ remains a fine approximation despite the widely varying shape of the infall $f(t)$ for varying $\omega(r)$.

for $k = 1$ one can easily obtain the analogous result,

$$Z^{**}(\mu) \approx \frac{y}{3}\left\{\ln\frac{1}{\mu} + 2\left[\ln\left(\frac{1}{\mu}\right) + 1\right]\right\}. \tag{19}$$

Both approximate expressions, $Z^*(\mu)$ and $Z^{**}(\mu)$, are displayed along with the exact $Z(\omega)$ and $\mu(\omega)$ in Figure 2. One sees that $Z^*(\mu)$ fits quite nicely, although $Z^{**}(\mu)$ is more accurate. It can be surmised, although I have not demonstrated it, that for general k, for which $Z \approx y\omega t/(k+1)$ for large t, that the approximate relationship is

$$Z^{*(k)}(\mu) \approx \frac{y}{k+1}\left\{\ln\frac{1}{\mu} + k\left[\ln\left(\frac{1}{\mu}\right) + 1\right]\right\}. \tag{20}$$

But it is the adequacy of $Z^*(\mu)$ from equation (15) that I intend to emphasize from its adequate fit to several analytic models. As a final point here we notice that Figure 2 does not differ greatly from Figure 1 *except* in the appropriate value of the star formation efficiency ω. The value of ω for a given μ is greater in the $k = 2$ model by a factor of ~ 1.3.

III. OTHER ANALYTIC MODELS

a) $k = 1$ with Radial Variation of $f(0)/M_G(0)$

Equations (4) and (5) show that for Clayton's (1985) "standard model" the ratio $f(0)/M_G(0) = k/\Delta$ is a constant (independent of radius). In considering a radial gradient in the representation of that model, therefore, the radial variations in the importance of infall today are contained entirely in radial variations of $\omega(r)$. But it is easy to generalize the $k = 1$ models to include also a gradient in $f(0)/M_G(0) \equiv \omega_f(0)$, letting that initial replenishment rate be a function of radius. In this way one can examine whether the $Z(\mu)$ relationship is thereby altered, for example by letting $M_G(0)$ be large and $f(0)$ small in the central portions of the galaxy (after the initial collapse), but letting $f(0)/M_G(0)$ increase radially so that infall is increasingly more important at larger radius. The generation of these solutions is made (Clayton 1984) by noting that if the function $F(t)$ is defined by

$$f(t) \equiv F(t)e^{-\omega t}, \tag{21}$$

and if

$$G(t) \equiv \int_0^t F(t')dt', \tag{22}$$

then

$$M_G(t) = e^{-\omega t}[M_G(0) + G(t)] . \quad (23)$$

and

$$Z - Z_f = \frac{M_G(0)}{M_G(0) + G(t)} \left[y\omega \int_0^t \frac{M_G(0) + G(t')}{M_G(0)} dt' + Z_0 - Z_f \right] . \quad (24)$$

For present purposes we wish as a generalization of equation (4) to let $F(t) = f(0)[t + \Delta)/\Delta]^{k-1}$, where initial infall $f(0)$ may nonetheless be a function of radius. For then, if we let the timelike variable $(t + \Delta)/\Delta \equiv x$, we have

$$G(t) = \frac{f(0)\Delta}{k} \left[\left(\frac{t + \Delta}{\Delta} \right)^k - 1 \right] = \frac{f(0)\Delta}{k} (x^k - 1) , \quad (25)$$

leading to

$$\mu_k(t) = \frac{M_G(t)}{M_{\rm tot}(t)} = \frac{\{M_G(0) + [f(0)\Delta/k](x^k - 1)\}e^{-\omega t}}{M_G(0) + f(0)I_{k-1}(t, -\omega)} , \quad (26)$$

where the simple function $I_k(t, -\omega)$ is explicitly givey by Clayton (1985), and if $Z(0) = Z_f$

$$Z - Z_f = \frac{y\omega [\![t\{1 - [f(0)\Delta/kM_G(0)] + [f(0)\Delta^2/k(k+1)M_G(0)](x^{k+1} - 1)]\!]}{1 + [f(0)\Delta/kM_G(0)](x^k - 1)} . \quad (27)$$

For the case $k = 1$ having exponential infall, these reduce to

$$\mu_{k=1}(t) = \frac{[M_G(0) + f(0)t]e^{-\omega t}}{M_G(0) + [f(0)/\omega](1 - e^{-\omega t})} , \quad (28)$$

so that not only $\omega(r)$ is a function of radius, but also $\omega_f(0) = f(0)/M_G(0)$ can be as well. Note that equation (28) reduces to equation (10) for the particular choice $f(0) = kM_G(0)/\Delta$ that defines Clayton's "standard model." Likewise

$$Z_{k=1} = \frac{y\omega t}{2} \left[\frac{f(0)t + 2M_G(0)}{f(0)t + M_G(0)} \right] , \quad (29)$$

which reduces to equation (11) for the "standard model." Clearly equations (28) and (29) are in principle more versatile, because they allow radial variations in $f(0)/M_G(0)$. However, it is immediately evident that $\omega(r)$ must still bear the brunt of galactic gradients as great as a factor of 10, such as the Galaxy or M81, because the second factor in equation (29) can vary no more than a factor of 2, for any ratio $f(0)/M_G(0)$. Just as in the $k = 1$ "standard model," this $k = 1$ generalized model describes large galactic gradients as primarily resulting from a gradient in $\omega(r)$, the stellar remnant formation rate per unit mass of gas. Analysis shows that this $Z(\mu)$ relationship is not noticeably different from the one displayed in Figure 1 as long as infall is indeed important over most of the disk. Only the $\omega(r)$ scale in Figure 1 is slightly contracted if $f(0)/M_G(0)$ increases radially. If $M_G(0) \gg f(0)t$ throughout the disk, of course, the model is nearly a closed model, leading to $Z \approx y \ln 1/\mu$. Thus equation (15) for $Z^*(\mu)$ remains an excellent approximation as long as infall in $k = 1$ models is significant. I do not display a figure because it is so like Figure 1 in terms of $Z(\mu)$. But many of these same features will be evident in the next section, which considers yet another analytic model.

b) $f = f_0(r)e^{-\omega' t}, \omega' \neq \omega$

In seeking a general $Z(\mu)$ approximation one might question using the same value of $\omega(r)$ in the star formation rate and in the infall rate, especially for $k = 1$ models where the infall is pure exponential. To test this suspicion I derive yet another analytic model, this one defined by

$$f(t) = f_0(r)e^{-\omega' t} , \quad (30)$$

where ω' is a new constant that is to be the same for every radius. This model is easily generated from equations (21–(24):

$$\mu = \frac{e^{-\omega t}\{M_G(0) + [f_0/(\omega - \omega')][e^{(\omega - \omega')t} - 1]\}}{M_G(0) - (f_0/\omega')(e^{-\omega' t} - 1)} , \quad (31)$$

and

$$(Z - Z_f) = y\omega \frac{\{M_G(0) - [f_0/(\omega - \omega')]\}t + [f_0/(\omega - \omega')^2][e^{(\omega - \omega')t} - 1]}{M_G(0) + [f_0/(\omega - \omega')][e^{(\omega - \omega')t} - 1]} , \quad (32)$$

where now $\omega(r)$, $M_G(0, r)$, and $f_0(r)$ are all functions of radius only, whereas ω' is a strict constant. To my knowledge, this very nice analytic model has not appeared before. Its features are illustrated in Figure 3 for the arbitrary choices $\omega' = 0.21$ Gyr^{-1} and today $= t = 15$ Gyr. The gradient in this case is established by two independent features: (1) $\omega(r)$ varies between 0–0.4 Gyr^{-1}; (2) $f_0(r)/M_G(0, r) \equiv \omega_f(0)$ is chosen to increase in the opposite direction [specifically $\omega_f(0) = 1$–2.5ω] so that decrease in birth rate is reinforced by increase in infall, both decreasing Z. One again finds that $Z^*(\mu)$ is a very useful approximation to $Z(\mu)$ except as

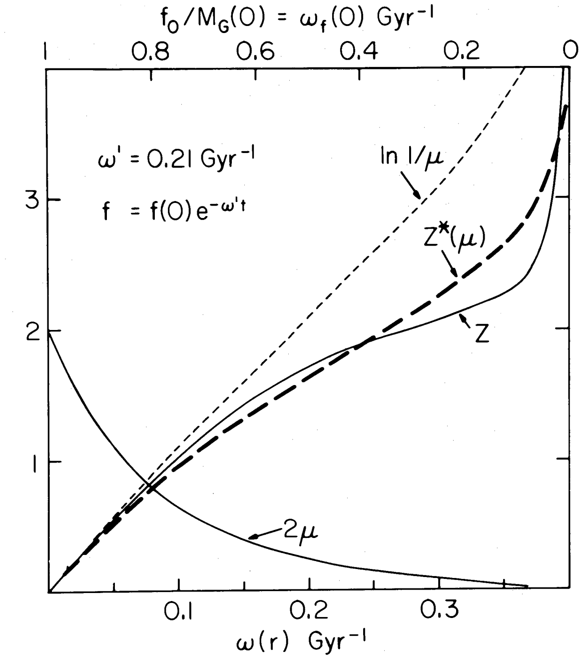

FIG. 3.—Another analytic model having invariant time dependence at every radius for the infall $f = f_0(r)e^{-\omega' t}$. Again $Z^*(\mu)$ is a good approximation to exact $Z(\mu)$ except for very small values of infall $f_0/M_G(0)$, which is here arbitrarily correlated antilinearly with $\omega(r)$ so that both effect the gradient in the same direction. But any other choice for varying $\omega_f(0)$ gives a very similar result, showing that its variation does not invalidate the $Z^*(\mu)$ approximation unless f is so small that the model is effectively closed or remains large for so long that the Larson limit (eq. [37]) is approached.

$\omega \to 0.4$. When this occurs, infall $f_0 \to 0$ by the prescription illustrated so that zones in that neighborhood behave as closed systems. Accordingly, $Z(\mu)$ turns abruptly upward, crossing $Z^*(\mu)$, as it moves asymptotically toward ln $1/\mu$, the correct behavior for the infall-free zone. With that exception, $Z^*(\mu)$ is clearly a much better general approximation than is $y \ln 1/\mu$.

The upturn in Figure 3 as $\omega_f(0) \to 0$ is not there if one takes $f_0/M_G(0)$ to instead be constant from zone to zone. In that case the ratio $f(t)/M_G(0)$ is invariant to radius, so that the abundance gradient must be entirely due to $\omega(r)$. The corresponding curves for constant $\omega_f(0)$ look otherwise similar to Figure 3, with $Z^*(\mu)$ being a good match to Z. Although a better approximation than is ln $(1/\mu)$, $Z^*(\mu)$ also fails if $w_f(0)$ is made so large that infall rates today exceed the rate of mass return from stars, however, for then $Z/y \to 1$ (Larson 1972), although ln $(1/\mu)$ becomes increasingly larger.

This new analytic model seems a very useful one for analytic studies.

c) *Lynden-Bell's "Best Accretion Model"*

Lynden-Bell (1975) has outlined a different technique for generating analytic models of chemical evolution. Considering the star mass $S(t)$ as the independent variable (rather than time), he generates analytic families by choosing $M_G(S)$ to have certain functional forms capable of analytic integration. His approach has considerable power. By showing that analytic solutions require the integration of both $dS/M_G(S)$ and $\exp(\int dS'/M_G)dS$, he chooses

$$\frac{1}{M_G(S)} = \frac{1}{S+\Gamma} + \frac{1}{M_\infty - S} \tag{33}$$

as a quadratic polynomial satisfying those requirements, where M_∞ is the asymptotic star mass when $M_G \to 0$ and where Γ is determined by the initial gas mass $M_G(0)$ when star formation begins; $\Gamma = M_\infty M_G(0)/[M_\infty - M_G(0)]$. Unfortunately, his paper has been made difficult to read by printer errors: his equation (4.4) defining $M_G(S)$ is not consistent with equation (33) above unless the mass M appearing there is replaced by Γ, with the same error recurring in his equation (4.7). Interested readers will follow his development more readily if those misprints are first corrected. (For example, Güsten and Mezger 1982 use this model for their interpretation of the radial gradient in terms of a gradient in the yield $y[r]$; however, they also make an error, not serious for their physical model, in replacing M in Lynden-Bell's eq. [4.4], which they realized could not be correct as printed, by the initial gas mass M_0 instead of by Γ. If the accretion is large enough for M_∞ to greatly exceed M_0, then $\Gamma \approx M_0$, but for no accretion $\Gamma \to \infty$.)

Lynden-Bell then chooses a "best accretion model" by taking an initial gas mass in the disk $M_G(0) = M_\infty/6$ (leading to $\gamma = \Gamma/M_\infty = \frac{1}{5}$). By defining $s = S/M_\infty$ as the star mass normalized to unit asymptotic value, he shows that

$$Z(s) = y_Z\left(\frac{6}{5s+1}\right)^2\left[-\ln(1-s) - \frac{5s}{6}\right], \tag{34}$$

which he notes has the following asymptotic behavior as $t \to \infty$, $s \to 1$:

$$Z_\infty(s) \to y_Z[\ln(1/\mu) - 5/6]\ . \tag{35}$$

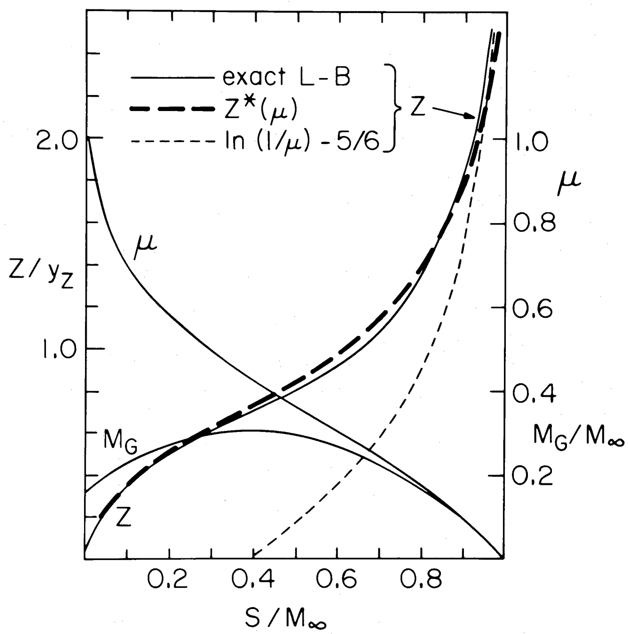

FIG. 4.—Lynden-Bell's (1975) "best accretion model." an exact time-dependent solution, is compared with $Z^*(\mu)$. The asymptotic form ln $1/\mu - 5/6$ derived by him is not useful, but $Z^*(\mu)$ is a good match.

One easily shows that in the same model the gas fraction

$$\mu = \frac{M_G}{M} = \frac{1 + 4s - 5s^2}{1 + 10s - 5s^2}, \quad (36)$$

enabling one to plot the metallicity $Z(\mu)/y_Z$ of a primary element along with $\mu(s)$ in Figure 4. Lynden-Bell's exact solution $Z(s)$ is shown as the solid rising curve, and equation (35) for his asymptotic fit $Z_\infty(s)$ is shown as the steep short-dash curve. From the latter one sees that the asymptotic form is of no use, since it agrees with Z only when μ is very small (<0.1) and is therefore of no applicability to the problem of galactic abundance gradients. It differs in this regard from the closed model $Z = y \ln \mu^{-1}$ result, which is valid for all values of μ. On the other hand, the approximation $Z^*(\mu)$ derived in § II is seen to fit the exact $Z(\mu)$ very well. It is once again the heavy long-dash rising curve in Figure 4. Lynden-Bell's "best accretion model" is thereby seen as another independent analytic model for which $Z^*(\mu)$ is a very good approximation. It is a power of Lynden-Bell's approach that my conclusion in this regard is entirely independent of temporal assumptions, because they have not been specified to this point. One introduces time with another condition relating star formation rate to $M_G(t)$. Lynden-Bell has done this for both linear and quadratic star formation rates.

At first glance it might seem curious that Lynden-Bell's best accretion model yields $Z(\mu)$ that is independent of the temporal rate of star formation. For example, $\psi(t) \propto M_G$, M_G^2, or even sinusoidal all give the same $Z(\mu)$, shown in Figure 4 to be very accurately also equal to the approximation $Z^*(\mu)$. It might therefore seem that his approach is in some sense more general than the time-dependent ones that I have advocated. That is not the case, for what actually happens in his model is that the infall rate $f(t)$ is forced to accept a specific time dependence that compensates for the time dependence of $\psi(t)$. As an extreme example, if star formation suddenly stops, the infall must stop. In Lynden-Bell's approach the infall $f(t)$ is calculated as a result of the model rather than a precondition of the model. The precondition in Lynden-Bell's model is equation (33) specifying $M_G(S)$. Differing star formation rates $\psi(t)$ then yield differing $f(t)$ in order to meet the required $M_G(S)$. One might seek refinement in $f(t)$ by changing the prescription $M_G(S)$. But one thing is now very clear; if one adopts Lynden-Bell's model, one is assured that $Z^*(\mu)$ represents that relationship well.

d) Twarog's (1980) Constraints

Each of the analytic models described qualifies as being a dynamic model. As was emphasized in the introduction, I mean by this statement that the exact solutions are generated from the differential equations by plausible physical relationships; e.g., "the rate of star formation increases as some positive power of the gas mass," and "the gas mass responds to the rate of infall," etc. These causal relations make the analytic models more believable than more arbitrary models, but that does not necessarily render them more correct. Among the many attempts to make more arbitrary sense of galactic data, I find Twarog's (1980) especially noteworthy because of its clear application of stellar data to the history of star formation in the solar neighborhood. It is similar to Pagel and Patchett's (1975) earlier study. It is therefore of interest in the present work to identify the special nature of Twarog's constraints, which are also similar to Pagel and Patchett's. Having demonstrated that the average past star formation rate is probably no more than twice the present rate, Twarog then postulates for the sake of model comparison that the star formation rate is a constant *independent of the gass mass* $M_G(t)$, and that the infall rate $f(t)$ also remains constant. The gas mass $M_G(t)$ declines linearly with time and is today almost gone ($\mu \approx 0.05$). It is rather strange to image the star formation rate to hold steady while the mass of disk gas decreases by a factor of 20, but Twarog suggests that it may nonetheless be true even though requiring star formation to be decoupled entirely from the gas mass. He shows that these simplifying constraints result in (with the correction of a sign misprint in his equation [13]) in this $Z(\mu)$ relationship:

$$\frac{Z}{y} = \frac{1}{Q}\left\{1 - \left[\frac{\mu}{1 - Q(1-\mu)}\right]^{Q/(1+Q)}\right\}, \tag{37}$$

where $Q = f/(1-R)\psi$ is the constant ratio of the constant infall rate to the constant rate of growth of star mass. One easily shows that equation (37) behaves as $\ln 1/\mu$ for $\mu \to 1$, and in Figure 5 I have compared it for two values of $Q = 0.2$ and 0.8 over the full range of μ with $Z^*(\mu)$ from equation (15). One sees that this $Z(\mu)$ relation is only weakly dependent upon Q, and is also well matched by $Z^*(\mu)$ except near $\mu = 0$. In fact, equation (37) with $Q = 0.4$ matches $Z^*(\mu)$ almost perfectly for $\mu > 0.04$. As $\mu \to 0$ the infall becomes so great in comparison with M_G that the growth of Z saturates, reaching an asymptotic limit $Z \to y/Q$ (and reaching it at earlier μ for higher Q). This behavior is similar to the Larson (1972) limit, where $M_G(t)$ is also constant and $Z \to y$. The reader will be reminded that *all* of these analytic models fail for $\mu < 0.05$, however, because the instantaneous recycling approximation on which they are all based fails for $\mu < 0.05$ (Clayton and Pantelaki 1986).

The fact that $Z^*(\mu)$ well approximates equation (37) for $\mu > 0.1$ despite the lack of immediate physical justification for Twarog's constraints and despite the difference of a factor 4 in the two ratios f/ψ shown in Figure 5 may be taken as another evidence that the $Z^*(\mu)$ relation is *not* a sensitive way to distinguish between galactic histories as long as the yield y is constant. A changing yield will, of course, alter $Z(\mu)$. Having said that, I also note that the $Z^*(\mu)$ relationship advanced in equation (15) seems once again to be an appropriate approximation for significant but not dominant infall.

IV. DISCUSSION

What I have attempted to do is to reexamine the ways in which a radial abundance gradient in a disk galaxy may be interpreted, and to ask how metallicity Z depends upon gas mass fraction μ in analytic models characterized by a temporally varying history of metal-poor infall. I presented results for a large number of models that show that equation (15).

$$Z^*(\mu) = \frac{y}{2}\left\{\ln\left(\frac{1}{\mu}\right) + \ln\left[\left(\frac{1}{\mu}\right) + 1\right]\right\}$$

characterizes most of them nicely. It therefore may be thought of as a replacement of the classical $Z = y \ln 1/\mu$ result for closed systems. For that to be so it seems to be necessary that infall be significant for the galaxy, but not dominant. That is, this approximation underestimates Z if the system is indeed closed, but it overestimates Z if continuing infall and the current metallicity are so great that

$$f(t)Z(t) \to y_Z\,\omega M_G(t) = y_Z \times (\text{net rate of star formation})\,. \tag{37}$$

In the former case the old $Z \to y \ln 1/\mu$ applies, whereas if equation (37) approaches satisfaction, Z becomes independent of μ. But for a large variety of intermediate histories that seem more likely to be relevant to disk galaxies, $Z^*(\mu)$ suggests itself as a trial function to ascertain whether such dynamic models are indeed physically relevant to the problem of galaxies. Galactic data on metallicity versus gas fraction may be fitted to both $Z^*(\mu)$ and to $y \ln 1/\mu$ to ascertain which fits better. If $Z^*(\mu)$ is consistently better, it may suggest that galactic infall has consistently occurred. It is important to notice that both functions may in some cases fit the

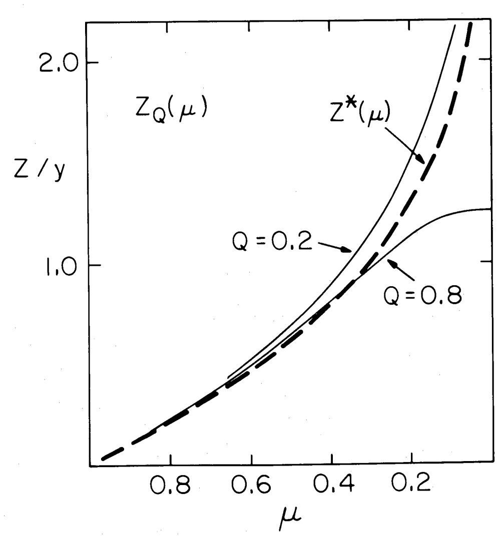

FIG. 5.—Twarog's (1980) constrained solutions for $Q = 0.2$ and $Q = 0.8$ are seen to bracket $Z^*(\mu)$, which is well matched to the $Q = 0.4$ solution. One sees that disconnecting the star formation rate from the gas mass does not seriously effect the approximate validity of the $Z^*(\mu)$ relationship.

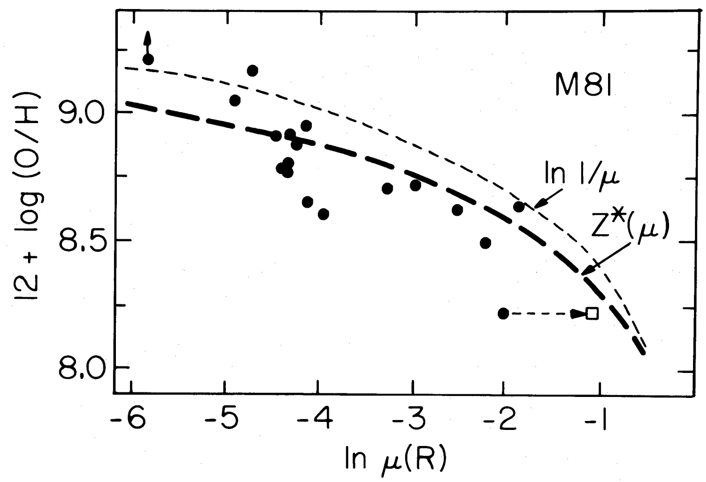

FIG. 6.—Garnett and Shields (1986) data on the abundance gradient in M81 are compared with both $y \ln 1/\mu$ and with $Z^*(\mu)$. These theoretical mass fractions were converted to log (O/H) before plotting. The "best" yield $y_0 = 3 \times 10^{-3}$ was chosen from a grid of values calculated by Chiosi and Matteucci (1984). One sees that very accurate data and a very accurate knowledge of the oxygen yield will be needed to ascertain an importance for infall from the $Z(\mu)$ data.

data to the accuracy obtainable, but *even then*, the fit will occur for significantly different values of the yield y_Z. In such cases, determination of y_Z from stellar evolution theory may indicate whether the model has been closed or subjected to infall.

These points are illustrated in Figure 6 with Garnett and Shields' (1986) data on giant H II regions across the face of M81. I have chosen $y_0 = 0.003$ for this comparison for the following reason. Chiosi and Matteucci (1984) calculated oxygen and carbon yields for 11 different assumptions about IMF shape, upper and lower mass limits, strength of mass loss, convective overshoot, etc., and they find y_0 values that vary by more than a power of 10—highly uncertain. However, if one selects those combinations for which $y_0/y_C \approx 2$, their rough relative abundances, then one finds that for those cases (sets 1, 3, 5, 8) the value $y_0 = 0.003$ is allowed, whereas the cases with $y_0 > 0.01$ and the single case with $y_0 < 0.002$ are not allowed. These may not be compelling, but nonetheless are my reasons for this choice of y_0. Although $Z^*(\mu)$ fits better (marginally) for this value of y_0, it clearly would not for a smaller value. Thus little could be concluded unless the data and yields can be made very precise.

Even more disconcerting is the physical sameness of $Z^*(\mu)$ and $\ln 1/(\mu)$ when plotted in a log-log display of data carrying natural error. The difference in shapes is hardly discernible. From this I make two obvious conclusions: (1) a rough agreement $Z \approx y \ln (1/\mu)$ is no evidence for truth of the assumptions of the closed model that led to it; (2) one cannot be optimistic for abundance data and yields so accurate that a distinction between closed models and infall models can be made on the basis of the $Z(\mu)$ relation, even though it is possible to do so in principle. My justification for this careful presentation lies both in making those points and in presenting several new exact analytic results for several physically distinct models of chemical evolution, which are of utility in their own right.

I have concentrated on interpretations of abundance gradients in terms of radial gradients in $\omega(r)$ and in $f(r, t)$. If Güsten and Mezger's (1982) interesting model of radial variations in yield $y(r)$ is indeed correct, as I think it may be, all of the present results are clearly still applicable except that the radial function $y(r)$ now multiplies each expression.

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