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Donald D. Clayton

Clemson University, claydonald@gmail.com

Irani Pantelaki

Rice University

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SECONDARY METALLICITY IN ANALYTIC MODELS OF CHEMICAL EVOLUTION OF GALAXIES

DONALD D. CLAYTON AND IRINI PANTELAKI

Department of Space Physics and Astronomy, Rice University

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ABSTRACT

We present additional considerations and properties of analytic models of the chemical evolution of galactic regions that grow in mass owing to the continuous infall of matter. We emphasize the solutions for secondary nuclei, defined as those nuclei whose stellar yields are proportional to the abundance of a primary seed nucleus, in the analytic families of models described by Clayton in 1984 and 1985 publications. Wide variations in time dependence of both primary and secondary nuclei as well as in the ratio of secondary to primary are displayed by these model families, confirming again the usefulness of these families as interpretive guides if galaxies do in fact evolve with substantial infall. Additionally we present analytic solutions to two other possible interesting systems: (1) the evolution of abundances if the primary metallicity in the infall is increasing in time; (2) the evolution of abundances if the primary yield changes linearly with time owing to continuous changes in the stellar mass function, the opacity, or other astrophysical agents. Finally we conclude with test evaluations of the instantaneous recycling approximation on which these analytic models rely.

Subject headings: galaxies: evolution — galaxies: stellar content — nucleosynthesis

1. INTRODUCTION

Although some small differences in usage exist, it is common to define a primary nucleosynthesis product as one that would be substantially synthesized in stars formed from hydrogen and helium and to define a secondary nucleosynthesis product as one that requires in addition some other initial constituent. The astrophysical questions attendant to applications of this definition are great. The heavy isotopes of magnesium, for example, reveal an element of both. The yields of $^{25, 26}\text{Mg}$ are increased if the star initially possesses neutron enrichment needed for production of isotopes having more neutrons than protons (Arnett 1971), but electron capture during carbon burning produces considerable neutron enrichment, and therefore $^{25, 26}\text{Mg}$, in the carbon burning products of stars composed initially of hydrogen and helium (Arnett 1973). Similar problems complicate exclusive assignments of most secondary nuclei. But the problem is a significant one because it can illustrate the operation of chemical evolution (Arnett 1971).

In this work we will ignore astrophysical arguments over secondary assignment in order to display the analytic behavior of secondary metallicity in analytic models of chemical evolution. We define a fictional secondary metallicity Z_s as being one whose yield is proportional to the initial abundance Z_1 of a primary nucleus. Clayton (1984, 1985) has displayed whole families of analytic solutions $Z_1(t)$ to idealized models having the following properties:

1. Star formation rate $\psi(t)$ proportional to gas mass $M_G(t)$, (linear model).
2. Constant initial mass functions and yields.
3. Constant metallicity Z_f in infalling matter $f(t)$.
4. Instantaneous recycling with constant return fraction R .

None of these restrictions is believed to be strictly true. But the analytical models are nonetheless useful, both as standards against which relaxation of assumptions can be compared and as explicit analytic systems whose entire dependence on time and parameters is explicit. Such models have pedagogical value as well as value in motivating research. Because the

Clayton (1984, 1985) models are big improvements to the classes of analytically expressible models, we desire here to express the secondary metallicity as well. Relaxation of assumptions 2, 3, and 4 above will also be investigated later in this work, but for the moment we retain them.

The basic equations of this model in the same notation are as follows:

1. Mass Conservation:

$$\frac{dM_G}{dt} = -(1 - R)\psi(t) + f(t), \quad (1)$$

where the first term with constant instantaneous return fraction R can in the linear model be designated $(1 - R)\psi(t) \equiv \omega M_G(t)$, where ω is a constant measuring gas consumption.

2. Metallicity:

$$\frac{dZ}{dt} = y_Z \omega - (Z - Z_f) \frac{f(t)}{M_G(t)}, \quad (2)$$

where y_Z is the yield and Z_f the value in infalling matter.

3. Infall Cycle Number:

$$\theta(t) = \int_0^t f(t')/M_G(t') dt'. \quad (3)$$

With these, the time-dependent metallicity of a primary nucleus [here called $Z_1(t)$], which is for the moment defined as one for which y_Z is a constant independent of the current metallicity, is

$$Z_1 - Z_{1f} = y_1 \omega e^{-\theta(t)} \int_0^t e^{\theta(t')} dt' + e^{-\theta(t)} (Z_{10} - Z_{1f}). \quad (4)$$

What Clayton (1984, 1985) accomplished was a means of defining parameterized families of infall rates for which the equations are explicitly soluble. In this paper we show the corresponding solutions for a secondary nucleosynthesis product.

II. SECONDARY METALLICITY

For the secondary nucleus whose mass fraction is designated by Z_s the yield is defined as being time-dependent in proportion to the primary metallicity $Z_1(t)$:

$$y_s(t) \equiv \beta Z_1(t), \quad (5)$$

so that

$$\frac{dZ_s}{dt} = y_s \omega - (Z_s - Z_{sf}) \frac{f(t)}{M_G(t)} \quad (6)$$

$$= \beta \omega Z_1(t) - (Z_s - Z_{sf}) \frac{d\theta}{dt} \quad (7)$$

whose solution when equation (4) is substituted for $Z_1(t)$ is

$$\begin{aligned} Z_s(t) - Z_{sf} = e^{-\theta(t)} & \left[\beta \omega Z_{1f} \int_0^t e^{\theta(t')} dt' \right. \\ & + \beta y_1 \omega^2 \int_0^t \int_0^{t'} e^{\theta(t'')} dt'' dt' \\ & \left. + \beta \omega (Z_{10} - Z_{1f}) t + Z_{s0} - Z_{sf} \right]. \quad (8) \end{aligned}$$

In equations (4) and (8) the final terms show only how mixing through θ cycles of infall gradually changes Z in the absence of any nucleosynthesis from its initial value Z_0 to the value Z_f carried in the infalling matter. The next-to-last term of equation (8) has a related interpretation, showing how the secondary nucleosynthesis is altered in the absence of primary nucleosynthesis by the shift of Z_1 from Z_{10} to Z_{1f} . These terms are interesting only if one believes both that the infalling abundance differs from the initial abundance in the disk and that at least one of them is large enough to be significant to observable abundances. The latter supposition is commonly discussed as a means of avoiding the unobserved existence of a large number of low- Z dwarfs. Whether the smallness of that stellar population is attributable to a large Z_0 ($\sim 0.1 Z_\odot$) or to galactic infall $f(t)$ is a much debated matter (e.g., Tinsley 1980). The analytic infall models under discussion here are able, through infall, to avoid this problem without requiring significant initial metallicity (Clayton 1984, 1985). We believe this issue will be settled more by a self-consistent model of galaxy formation than by star counts, but in any case our purpose here is to display the mathematical solutions.

$$a) Z_0 = Z_f$$

The most physically transparent form of these results occurs if the infalling concentration Z_f is equal to the initial disk concentration Z_0 , for then the terms involving their difference vanish, leaving

$$Z_1(t) = Z_{1f} + y_1 \omega e^{-\theta(t)} \int_0^t e^{\theta(t')} dt' \quad (9)$$

and

$$\begin{aligned} Z_s(t) = Z_{sf} + e^{-\theta(t)} & \left[\beta \omega Z_{1f} \int_0^t e^{\theta(t')} dt' \right. \\ & \left. + \beta y_1 \omega^2 \int_0^t \int_0^{t'} e^{\theta(t'')} dt'' dt' \right]. \quad (10) \end{aligned}$$

It is evident by comparing the first term of equation (10) with equation (9) that the effect of initial primary abundance $Z_{10} =$

Z_{1f} is to create a term in Z_s that looks like primary production with yield $y_s = \beta Z_{1f}$. That is, the time dependence of the terms in Z_s are a primary-like term and an intrinsically secondary term. We will for the most part use these simpler equations because they generate the essence of our results. Moreover, it makes good physical sense that the infalling matter have the same composition as the initial disk if it is primitive gas rather than ejecta from halo stars.

It is instructive to examine the expansion at early times to clarify the nature of these two terms. The expansion of equation (3) at early time is $\theta(t) \rightarrow [f(0)/M_G(0)]t$ and $e^{\theta(t)} \rightarrow 1 + \theta(t)$, so that

$$Z_s(t \rightarrow 0) \rightarrow Z_{sf} + \beta Z_{1f} \omega t + \beta \omega [y_1 \omega - Z_{1f} f(0)/M_G(0)] t^2 / 2 \quad (11)$$

showing again an initially linear increase (as for a primary) with yield $y_s = \beta Z_{1f}$ and a quadratic increase (intrinsically secondary). This early behavior is except for the $f(0)$ term equal to the temporal behaviour of the linear model without infall and with zero initial abundance for which $Z_1 \propto t$ and $Z_s \propto t^2$ for all times. Notice that if $Z_0 = Z_f = 0$, then

$$Z_1(t) = y_1 \omega e^{-\theta(t)} \int_0^t e^{\theta(t')} dt'$$

and

$$Z_s(t) = \beta y_1 \omega^2 e^{-\theta(t)} \int_0^t \int_0^{t'} e^{\theta(t'')} dt'' dt'$$

so that without infall [$\theta(t) = 0$] these quantities are exactly linear and quadratic, respectively. But in the face of metal-free infall, the cycle number $\theta(t)$ increases monotonically from zero, so that the relationship between primary and secondary nucleosynthesis becomes moderated.

$$b) Z_f = Z_f(t)$$

These equations admit analytic solutions even when the infalling material does not have constant abundances. In that case we write $Z_f = Z_f(t)$. We find the solution to equation (2) for the primary abundance to be

$$\begin{aligned} Z_1(t) = e^{-\theta(t)} & \left[Z_{10} + y_1 \omega \int_0^t e^{\theta(t')} dt' \right. \\ & \left. + \int_0^t e^{\theta(t')} Z_{1f}(t') \frac{d\theta}{dt'} dt' \right]. \quad (12) \end{aligned}$$

In general this expression will not be analytically soluble, as will not therefore the secondary metallicity. But if one is willing to specify a simple time dependence for $Z_f(t)$, the standard analytic models will be soluble. For example, it may be plausible for the infalling mass fraction to increase linearly with time:

$$Z_{1f}(t) = Z_{1f}(0)(1 + \alpha t). \quad (13)$$

In that case the primary metallicity

$$\begin{aligned} Z_1(t) = e^{-\theta(t)} & \left\{ Z_{10} + y_1 \omega \int_0^t e^{\theta(t')} dt' + Z_{1f}(0) \right. \\ & \left. \times \left[e^{\theta(t)} - 1 + \alpha \int_0^t t' e^{\theta(t')} \frac{d\theta}{dt'} dt' \right] \right\} \quad (14) \end{aligned}$$

can be analytically solved for Clayton's (1984, 1985) standard

models. So, furthermore, can the secondary metallicity. We will clarify this by example later for one specific standard model (the $k = 1$ model of Clayton 1985).

III. STANDARD MODEL NUMBER 1

In the first in a series of papers on analytic models of chemical evolution, Clayton (1985) laid out what he called a standard model—one in which infall $f(t)$ occurs continuously, cycling the gas at a rate

$$\frac{d\theta}{dt} = \frac{f(t)}{M_G(t)} = \frac{k}{t + \Delta}, \quad \theta = \ln \left(\frac{t + \Delta}{\Delta} \right)^k \quad (15)$$

that decreases continuously with time ($k = \text{integer}$ and $\Delta = \text{time constant}$). He showed that virtually every observable of interest (within the idealized constraints) is analytically expressible in that family of models, which corresponds to an infall rate

$$f(t) = \frac{kM_G(0)}{\Delta} \left(\frac{t + \Delta}{\Delta} \right)^{k-1} e^{-\omega t} \quad (16)$$

and to a gas mass

$$M_G(t) = M_G(0) \left(\frac{t + \Delta}{\Delta} \right)^k e^{-\omega t}. \quad (17)$$

In particular the primary metallicity is

$$Z_1(t) - Z_{1f} = \frac{y_1 \omega \Delta}{k+1} (x - x^{-k}) + x^{-k} (Z_{10} - Z_{1f}), \quad (18)$$

where for compactness here and in what follows the timelike variable $x \equiv (t + \Delta)/\Delta$. The secondary metallicity in this model is from equation (8) given by

$$\begin{aligned} Z_s(t) - Z_{sf} = & \frac{\beta \omega Z_{1f} \Delta}{k+1} (x - x^{-k}) \\ & + \frac{\beta y_1 \omega^2 \Delta}{k+1} \left[\frac{\Delta}{k+2} (x^2 - x^{-k}) - t x^{-k} \right] \\ & + x^{-k} [\beta \omega (Z_{10} - Z_{1f}) + Z_{s0} - Z_{sf}]. \quad (19) \end{aligned}$$

If the initial disk abundances Z_0 are equal to the infall abundances Z_f as described in § IIa, the final terms (multiplying x^{-k}) in expressions (18) and (19) vanish, leaving a more transparent dependence. The first term in expression (19) behaves temporally as primary production, whereas the second term is the intrinsically secondary time dependence for exclusively disk production. For numerical examples we will use that simplifying assumption about the initial metallicities.

An interesting point about the primary metallicity was not explicitly remarked upon by Clayton (1985). If the parameter $\Delta \ll \text{galactic age today}$, so that $x \gg 1$, as it must be if low- k models are to accrete considerably more gas over the galactic age than the amount $M_G(0)$ that was initially present, then equation (18) shows that $Z_1(t) = y_1 \omega t / (k+1) + \text{smaller terms}$. This is the same linear dependence upon time that one obtains exactly in the infall-free model except that its slope is smaller by the factor $(k+1)^{-1}$. That is, given physically a yield y , the metal-poor infall of this standard model dilutes the disk continuously such that the primary metallicity $Z_1(t)$ grows almost linearly but more slowly than in the closed linear model, where the result $Z_1(t; k=0) = y_1 \omega t + Z_{10}$ is exact for all times. Thus this standard model does not invalidate the linearity of the time dependence of growth of metallicity. It rather dilutes its

rate of growth. This is physically important when one compares the theoretical yield of stars with the observed metallicity. And in exactly the same vein, equation (19) shows that the second term remains roughly proportional to t^2 when $t \gg \Delta$, not only when t is small as in equation (11), but the curvature of its parabolic growth is moderated by the factor $2[(k+1)(k+2)]^{-1}$ in comparison with the closed model. This means that the secondary production will grow less rapidly in comparison with the primary production than in the closed model. This is a general feature of infall models. It is made explicit by the analytic solutions.

As a numerical example we consider a set of parameters that gives a gas fraction $\mu = 9.1\%$ at $t = 15$ Gyr under the influence of purely exponential infall $f(t) = M_G(0)e^{-\omega t}/\Delta$. These particular parameter values are $k = 1$, $\Delta = 0.1$ Gyr, and $\omega = 0.248$ Gyr $^{-1}$. The initial disk abundances and the infall abundances are set equal to each other so that the simpler forms of the equations (without terms in $Z_0 - Z_f$) apply. The results for both primary abundance Z_1 and for secondary abundance Z_s computed with three distinct values of the initial metallicity Z_{10} are shown in Figure 1. For each curve the ordinate is normalized by dividing the metallicity growth by its total eventual growth up to $t = 15$, so that all curves pass through unity at $t = 15$ Gyr, and by dividing each normalized metallicity growth by $(t/15 \text{ Gyr})$, so that a linear growth rate would appear as a constant in Figure 1. This ordinate so defined has a simple physical interpretation: it is the average rate of growth of metallicity over the interval 0 to t divided by its average rate of growth over 15 Gyr. Quite evidently the primary metallicity equation (18) does grow almost linearly in this example, because the curve in Figure 1 has a value near unity. This linear growth begins early whenever Δ is a time much smaller than galactic age. In that case $x = (t + \Delta)/\Delta$ becomes large and the terms proportional to x^{-k} in equation (18) are essentially negligible. Then $Z_1 - Z_{1f} \propto x$ is linearly increasing with time.

FIG. 1.—Metallicity increase $Z(t) - Z_f$ for both primary and secondary nucleus is normalized by dividing by its ultimate increase $Z(15) - Z_f$ up to $t = 15$ Gyr and also by dividing by $(t/15)$. The physical interpretation of this ordinate is the ratio of the average rate of increase of Z over the interval 0 to t divided by the average rate over the first 15 Gyr. This figure is a $k = 1$ standard model 1 with exponential infall as described by parameters $\omega = 0.248$ Gyr $^{-1}$ and $\Delta = 0.1$ Gyr. For $t \gg \Delta$ these models have an almost constant rate of increase for primary metallicity and an almost linearly increasing rate of increase of secondary metallicity. Three different values $Z_1(0)$ for the initial disk metallicity are shown.

That this class of standard models admits of an exactly linear $Z_1(t)$ if the initial conditions are chosen to be just right can be seen directly from the differential equation (2). Taking the special case of metal-free infall ($Z_f = 0$), and remembering that in this class of models $f/M_G = d\theta/dt = k/(t + \Delta)$, equation (2) can be written

$$\frac{dZ}{dt} + Z \frac{k}{t + \Delta} = y_Z \omega, \quad (20)$$

so that, if one assumes a linear Z of the form $Z = \alpha(t + \Delta)$, one sees that equation (20) is satisfied if $\alpha = y_Z \omega / (1 + k)$. Thus

$$Z = \frac{y_Z \omega}{1 + k} (t + \Delta)$$

is an exact solution to this standard model with metal-free infall and with initial abundance $Z(0) = y_Z \omega \Delta / (1 + k)$. That these initial conditions lead to such an exact solution is also verifiable by evaluating equation (18) for that initial metallicity. This slope $\alpha = y_Z \omega / (1 + k)$ becomes the asymptotic slope of a linear primary metallicity when $t \gg \Delta$, as can also easily be seen from the differential equation (20). This linearity is revealed in Figure 1, where the metallicity is divided by t for visual convenience, by noting that the average rate of growth for $Z_1(t)$ approaches unity for $t \gg \Delta = 0.1$ Gyr.

This discussion has yielded an important new insight into Clayton's (1985) standard model. For small Δ , the primary metallicity quickly becomes linear in time, but with the slope

$$\frac{dZ_1}{dt} \rightarrow \frac{y_Z \omega}{(1 + k)} \text{ (standard model 1).} \quad (21)$$

Increasing order k in the standard family 1 dilutes by factor $(1 + k)$ the slope $y_Z \omega$ that would obtain in the infall-free ($k = 0$) model. That is, increasing k decreases the quasi-linear growth rate of the primary metallicity. This should be regarded as the physical significance chemically of the index k denumerating this family of standard models. Because y_Z is to be separately computed from stellar evolution theory and ω is to be set for each k to get the gas fraction μ (today) equal to its observed value, an observed growth rate $Z_1(t)$ is in principle adequate to determine k . In the case of constant infall balancing star formation (Larson 1972), $Z_1(t)$ saturates to a constant value. Thus parameter k spans the growth rates for $Z_1(t)$ between the infall-free model and the saturated constant-infall constant-star-formation limits. This reaffirms Clayton's (1985, p. 74, paragraph 2) conclusion that the Larson limit corresponds to $k \rightarrow \infty$ simultaneous with $\Delta \rightarrow \infty$ in such a way that $f/M_G = k/(t + \Delta) \rightarrow \text{constant} = \omega$.

Figure 1 shows that the secondary metallicity also has a simple time dependence if $\Delta \ll$ Galactic age. Since it is there evident that $(Z_s - Z_{sf})/t$ increases linearly with time (except for the initial adjustments), it is clear that $Z_s \propto t^2$. This result follows from the exact equation (19) but can also be seen directly from the differential equation (7) whenever $Z_1(t)$ is linear. For then the approximate differential equation for $t \gg \Delta$ and for metal-free infall,

$$\frac{dZ_s}{dt} + \frac{k}{t} Z_s \approx \beta \omega Z_1(t) \quad (22)$$

is always approximately solved by

$$Z_s(t) = \frac{\beta \omega}{k + 2} t Z_1(t) \quad (23)$$

whenever $Z_1(t)$ is linear in time. Therefore the linear increases in $Z_s(t)/t$ in Figure 1 are simply understood. Variations of the initial disk abundance Z_{10} cause only small changes in the slope and intercept of the secondary metallicity. These are easily understood by examining the leading terms of equation (19) for $t \gg \Delta$. The three initial disk primary metallicities shown in Figure 1 are chosen to be 1/6, 1/11, and zero times the final disk metallicity $Z_1(15)$. If the disk begins with primary metallicity, the secondary metallicity at early times is a larger fraction of the final secondary metallicity than in the case of zero $Z_1(0)$.

These results are not surprising, easily matching our intuition. The useful feature is the explicit representation of this standard model, whose solutions will always closely resemble Figure 1 if $\Delta \ll$ age today. To alter these behaviors requires taking the standard model toward the Larson limit, letting both k and Δ become large.

a) Toward the Larson Limit

An example partway toward the Larson limit is seen in Figure 2, where $k = 4$, $\Delta = 4$ Gyr, and $\omega = 0.5 \text{ Gyr}^{-1}$. The peak of the gas mass M_G at $t_{\text{max}} = 4$ Gyr is now only 2.16 times greater than $M_G(0)$ and declines gradually by a factor of 7.4 over the next 11 Gyr. By contrast, the peak M_G corresponding to Figure 1 is $15.2 M_G(0)$ at about the same $t_{\text{max}} = 3.93$ Gyr. See Figure 6.1 of Clayton (1985). This shift to larger k and Δ shows itself in the differences between Figures 2 and 1. In Figure 2 the

FIG. 2.—Same format as Fig. 1. The standard model 1 with $k = 4$ and $\Delta = 4$ Gyr and $\omega = 0.5 \text{ Gyr}^{-1}$. Maximum infall rate is at $t = 2$ Gyr and $M_G(t)$ peaks at 4 Gyr. The average rate of primary metallicity increase over 0 to t declines continuously owing to the tendency of $Z_1(t)$ to saturate in the face of the continuing infall, which declines less rapidly than in Fig. 1. Secondary metallicity shows correspondingly a larger fraction of its increase in the first few Gyr. Letting k and Δ increase still further approaches the Larson limit.

primary metallicity $Z_1(t)$ no longer grows linearly, which would be constant on the Figure 2 ordinate. Instead its rate of growth has decreased with time, so that $Z_1(2 \text{ Gyr})$ is about twice as great relative to $Z_1(15)$ as it would have been for a linear growth. Or restated, the average rate of growth of Z over the first 2 Gyr is twice as great as its average rate of growth over 15 Gyr. Thus stars born at $t = 2 \text{ Gyr}$ in that model are less metal-poor by a factor of 2. The secondary metallicity $Z_s(t)$ also shows a larger early growth than in Figure 1. This comparison illustrates once again the usefulness of this family of analytic models for covering a wide range of galactic histories.

b) Linearly Increasing $Z_f(t)$

In § IIb we displayed the disk metallicity for time-dependent infall, remarking that a linear increase in the infall metallicity would also be soluble in this family of models. To briefly quote one specific result we consider the example of a $k = 1$ model in which $Z_{1f}(t)$ increases linearly, possibly because an increasingly higher fraction of the infalling matter has been ejected from halo stars. The result takes on its simplest representation by defining

$$Z_{1f}(t) = Z_{10} + \gamma y_1 \omega t \quad (24)$$

so that the parameter γ spans the range from constant infall abundance ($\gamma = 0$) to one that increases as rapidly as in a closed linear system ($\gamma = 1$). For the $\theta(t)$ defining this $k = 1$ infall model, equation (14) can then be solved:

$$Z_1(t) = Z_{10} + \frac{y_1 \omega \Delta}{2} \left[(1 + \gamma)x - 2\gamma - \frac{(1 - \gamma)}{x} \right]. \quad (25)$$

Two interesting limits serve to establish the physical results. If $\gamma = 0$, equation (25) immediately reduces to equation (18) with $k = 1$ for constant infall metallicity. If $\gamma = 1$ on the other hand, equation (25) reduces to $Z_1(t) = Z_{10} + y_1 \omega t$, exactly the result of the closed model. This last result is clear, because if the infall metallicity is itself the same as in a closed model, the disk metallicity must do likewise.

For the secondary metallicity it may be more physically interesting to imagine the infall Z_{sf} to be constant, as if no secondary star formation occurred in the halo population. In that case the differential equation (7) can also be easily solved with $Z_1(t)$ given by equation (25):

$$Z_s(t) - Z_{sf} = \frac{\beta \omega t}{2x} \left[y_1 \omega \Delta \left(\frac{1 + \gamma}{3} x^2 + \frac{1 - 2\gamma}{3} x - \frac{2 - \gamma}{3} \right) + Z_{10}(x + 1) \right]. \quad (26)$$

If $\gamma = 0$ this reduces immediately (but with some rearranging) to equation (19) with $k = 1$ for constant infall metallicity. For $\gamma = 1$ for the rate of growth of infall metallicity, this result approaches for $t \gg \Delta$ the behavior $Z_s \rightarrow \beta y_1 \omega^2 t^2/3$, exactly $\frac{2}{3}$ of the closed box result. That is, the secondary metallicity does not grow as fast as in the closed linear model even though the primary metallicity does. That strange result reflects the dilution of secondary nuclei but not of primary ones by infall. The point here is the demonstration by this example that many analytic solutions to time-dependent infall are possible.

IV. STANDARD MODEL NUMBER 2

Standard model 1 experiences continuous infall, declining monotonically after a single maximum. In a second paper on

analytic models Clayton (1984) described another analytic family, one having the property that the infall declines continuously to zero at some time t_0 and remains so thereafter. Therefore the analytic solutions take different forms on opposite sides of t_0 even though the differential equations remain the same. Specifically this family is defined by

$$\begin{aligned} \frac{d\theta}{dt} = \frac{f(t)}{M_G(t)} &= k \omega_1 \cot(\omega_1 t + \Delta) \quad t < t_0 \\ &= 0, \quad t > t_0 \end{aligned} \quad (27)$$

where the transition time t_0 is defined in terms of the arbitrary parameters ω_1 and Δ by the relation $\omega_1 t_0 + \Delta = \pi/2$. Clayton (1984) describes the mathematical properties in detail, showing of particular use to this present work that the gas mass is

$$\begin{aligned} M_G(t) &= M_G(0) \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^k e^{-\omega t} \quad t < t_0 \\ &= M_G(0) \sin^{-k} \Delta e^{-\omega t} \quad t > t_0, \end{aligned} \quad (28)$$

and that the primary metallicity $Z_1(t)$ in the case of constant infall metallicity is given for $t < t_0$ by

$$Z_1(t) - Z_{1f} = y_1 \omega \left[\frac{\sin \Delta}{\sin(\omega_1 t + \Delta)} \right]^k I_k(t) \quad (29)$$

and grows linearly with slope $y_1 \omega$ for $t > t_0$. The explicit forms of this function for $k = 0, 1, 2, 3$ are tabulated in Table 1 of Clayton (1984). And with the form $\theta(t)$ implied by equation (27), the secondary metallicity in equation (10) can be easily evaluated.

When considering the secondary metallicity for any functional form of truncated infall, it is instructive to take note of a simple relationship for $t > t_0$, after the infall has ceased. This relationship comes from the simple form of the differential equations for $t > t_0$, namely

$$\frac{dZ}{dt} = y_1 \omega, \quad Z_1(t) = Z_1(t_0) + y_1 \omega(t - t_0), \quad (30)$$

and

$$\frac{dZ_s}{dt} = \beta \omega Z_1(t), \quad t > t_0. \quad (31)$$

Equation (31) is immediately expressible with the aid of equation (30):

$$Z_s(t > t_0) - Z_s(t_0) = \beta \omega Z_1(t_0)(t - t_0) + \beta y_1 \omega^2 (t - t_0)^2/2. \quad (32)$$

Both primary and secondary metallicities grow for $t > t_0$ in the manner long known to apply to the closed-box model having initial abundances. Thus the relationship of Z_s and Z_1 for $t > t_0$ to the forms of the infall for $t < t_0$ is entirely contained in the magnitudes of Z_s and Z_1 at $t = t_0$. Differing infall histories have differing values for t_0 and, because of their differing forms, differing relationships of $Z_s(t_0)$ to $Z_1(t_0)$.

Because it is a bit cumbersome to display a general result for this standard family of models even though it is easy to generate them, we are content to display only the $k = 1$ solutions and to compare their behavior to the $k = 1$ solutions of model 1 shown in Figure 1. For $k = 1$ the infall in this family is (Clayton 1984)

$$f(t) = \omega_1 M_G(0) \frac{\cos(\omega_1 t + \Delta)}{\sin \Delta} e^{-\omega t}, \quad (33)$$

and the explicit form of the primary metallicity is (for $k = 1$)

$$Z_1(t) - Z_{1f} = y_1 \omega \frac{\cos \Delta - \cos(\omega_1 t + \Delta)}{\omega_1 \sin(\omega_1 t + \Delta)}, \quad t < t_0$$

$$= y_1 \omega \left(\frac{\cos \Delta}{\omega_1} + t - t_0 \right), \quad t > t_0. \quad (34)$$

We have assumed for this result the simpler and physically reasonable case that infall $Z_{1f} = \text{constant} = Z_{10}$, the initial disk metallicity. For $t < t_0$, the secondary metallicity is evaluated from equation (10) to be (for $k = 1$)

$$Z_s(t < t_0) - Z_{sf} = \beta \omega Z_{1f} \frac{\cos \Delta - \cos(\omega_1 t + \Delta)}{\omega_1 \sin(\omega_1 t + \Delta)} + \frac{\beta y_1 \omega^2}{\omega_1^2} \left[\frac{\omega_1 t \cos \Delta + \sin \Delta}{\sin(\omega_1 t + \Delta)} - 1 \right], \quad (35)$$

whereas for $t > t_0$ the secondary metallicity is more easily evaluated from equation (32):

$$Z_s(t > t_0) - Z_{sf} = \beta \omega Z_{1f} \left(\frac{\cos \Delta}{\omega_1} + t - t_0 \right) + \beta y_1 \omega^2 \left[\frac{\omega_1 t \cos \Delta + \sin \Delta - 1}{\omega_1^2} + \frac{(t - t_0)^2}{2} \right]. \quad (36)$$

We show these $k = 1$ results in Figure 3 in the same format as Figures 1 and 2. The cessation of infall is chosen to be $t_0 = 8.5$ Gyr by taking the parameters $\omega_1 = 0.17 \text{ Gyr}^{-1}$, $\Delta = 0.12 \text{ Gyr}$, and $\omega = 0.1 \text{ Gyr}^{-1}$. These parameters correspond to a total infall over the first $t_0 = 8.5$ Gyr that increases the total disk mass by a factor of 6.525.

The physical difference between Figure 3 and the preceding ones results from the fact that all metallicities increase faster after the infall has stopped. There the primary ordinate shows a mean growth rate that grows toward its final value following its initial decline. The secondary metallicity shows a continuing upward curvature.

V. BROADER ANALYTIC MODELS

Although the two families of analytic models described in the preceding two sections are probably adequate for most

FIG. 3.—Same format as Fig. 1. The standard model 2 with $k = 1$ has continuous declining infall until $t = 8.5$ Gyr when infall stops. Because metallicities grow faster after termination of infall, the average rate of Z_1 increase is increasing with age, in stark contrast with Fig. 2.

survey models, a broader class was also described by Clayton (1984). Defining the infall $f(t) = F(t)e^{-\omega t}$ results in

$$M_G(t) = [M_G(0) + G(t)]e^{-\omega t} \quad (37)$$

and

$$e^{\theta(t)} = 1 + G(t)/M_G(0) \quad (38)$$

where

$$G(t) = \int_0^t F(t') dt'. \quad (39)$$

In particular it seems that almost any form-fitting to $f(t)$ can be described by a polynomial for $F(t)$, in which case $G(t)$ is also polynomial with $G(0) = 0$. Considering again only the case of constant infall metallicity equal to initial disk metallicity (§ IIa), equation (9) can be written

$$Z_1(t) - Z_{1f} = \frac{y_1 \omega}{M_G(0) + G(t)} \int_0^t [M_G(0) + G(t')] dt' \quad (40)$$

as a ratio of two polynomials, inasmuch as $\int G(t') dt'$ is also a polynomial. It is noteworthy that not only stable primary nuclei but also radioactive ones admit of exact solutions in these families, because $\int e^{\lambda t'} G(t') dt'$, which occurs in equation (12) of Clayton (1984), is an elementary product of polynomial and exponential.

In exactly like manner the secondary metallicity in equation (10) can be expressed as a ratio of polynomials. Letting the polynomial $G_1(t)$ be the first integral of $M_G(0) + G(t)$ occurring in equation (40) and letting the polynomial $G_2(t)$ be its second integral we have

$$Z_s(t) - Z_{sf} = [M_G(0) + G(t)]^{-1} [\beta \omega Z_{1f} G_1(t) + \beta y_1 \omega^2 G_2(t)]. \quad (41)$$

We show this result for thoroughness, to display the power of the analytic models, but find no need to illustrate specific choices graphically.

VI. DISCUSSION

We have presented these solutions because they are pedagogically interesting and because they can provide motivation for understanding and explaining numerical research results. It seems to us that these classes of standard models may have interesting interpretive roles to play in future studies of the chemical evolution of galaxies. In comparing with astronomical data the difficult astrophysical questions of nucleosynthesis parentage need addressing. Considering just as an example the production of ^{14}N , one must independently determine both its primary and secondary yields from theoretical and observational guidelines, after which one may write for its yield

$$y(^{14}\text{N}) = y_{14} + \beta_{14} Z_{\text{CNO}}(t) \quad (42)$$

to reflect its double nature. Similar equations may be true for many species. Given that, one would in these classes of models describe $Z_{14}(t)$ as a sum of the primary and secondary analytic solutions that we have described.

a) Time-dependent Primary Yield

A second mathematical application of our results may be possible if one of our assumptions, the constancy of the birth-rate mass function is violated. The primary nucleus Z_1 has a constant yield y_1 only if that mass function is constant in time

FIG. 4.—Comparison of gas mass $M_G(t)$ calculated with instantaneous recycling approximation (IRA) to that calculated with standard mass function and stellar delays. The analytic IRA calculation is a standard model 1 having $k = 1$, $\omega = 0.181 \text{ Gyr}^{-1}$, $\Delta = 0.1 \text{ Gyr}$, which has exponential infall rate. Same infall rate used for numerical calculation. IRA is reasonable approximation.

and if the subsequent evolution of stars is not much influenced by their initial metallicity (through the opacity, for example). But let us suppose that the mass function evolves gradually with time, as do also structural effects owing to opacity and other metallicity dependent factors. Then it may be possible to write

$$y_1(t) = y_1 + \left(\frac{dy_1}{dt}\right)t = y_1 + y_1' t \quad (43)$$

in which case a primary metallicity shows both primary and secondary mathematical characteristics. We point out that the mathematical solutions of equation (2) can be then constructed in exact analogy to the secondary solutions that we have described. We express this in the notation of the broad class of models of § V; namely

$$Z_1(t) - Z_{1f} = \frac{\omega}{M_G(0) + G(t)} \left[y_1 \int_0^t [M_G(0) + G(t')] dt' + y_1' \int_0^t t' [M_G(0) + G(t')] dt' \right]. \quad (44)$$

Because both standard models 1 and 2 are generated by specific choices of $G(t)$, this result allows any of the analytic models to be evaluated for a linearly changing primary yield. Indeed, more complicated (e.g., polynomial) expressions for $y_1(t)$ can be handled in the same way.

b) Instantaneous Recycling Approximation

A very real criticism of the usefulness of all of these analytic models is their dependence on the instantaneous recycling approximation. That approximation cannot be valid at short times, before the first generation of relevant stars has had time to evolve. Nor can it be valid near gas exhaustion, when star formation has practically ceased but when low-mass stars continue to return their specialized yields. Pantelaki (1984) has studied the use of this approximation by comparing numerical integrations having stellar delays with the exact instantaneous-recycling solutions of standard model 1.

In Figure 4 we compare the gas mass $M_G(t)$ for a $k = 1$, $\Delta = 0.1 \text{ Gyr}$, $\omega = 0.181 \text{ Gyr}^{-1}$ standard model 1, which has exponential infall rate leading to $\mu = 0.19$ at $t = 15 \text{ Gyr}$, with numerical calculations (Pantelaki 1984) based on a standard

FIG. 5.—Hypothetical Z_3 synthesized only in $M_3 = 7.0 M_\odot$. Same comparison as in Fig. 4. Numerical value falls below IRA because much of M_G comes back from dwarfs that formed earlier when Z_3 was actually smaller. IRA is reasonable approximation.

FIG. 6.—Hypothetical Z_2 synthesized only in $M_2 = 1.38 M_\odot$. Same comparison as in Figs. 4 and 5. IRA is a rather poor approximation at most times, although it happens in this particular model to give almost correct answer today ($t \approx 15$ Gyr). A possible example of such a nucleus is ^3He .

mass function and standard stellar delays. Infall rates $f(t) = 10 M_G(0)e^{-0.181t}$ are identical. Near $t = 5$ Gyr the M_G is 8% smaller than the exact result based on instantaneous recycling approximation (IRA) because abundant dwarfs lock up much matter before returning it later. Beyond $t = 13$ Gyr the M_G exceeds that with IRA because the return from dwarfs is becoming more important as the gas mass declines toward zero. Note that in the linear models these curves also give the star formation rate, which is growing before $t = 6$ Gyr and declining thereafter. But it is clear for this sample model that IRA is not bad for these dynamic quantities.

Figure 5 compares for the same two calculations the hypothetical primary metallicity Z_3 produced only in a star having $M_3 = 7 M_\odot$. On the scale of this figure one can hardly see the delay of almost 10^8 year before the numerical Z_3 begins rising. The slightly growing overestimate of the IRA for all massive stars has a simple explanation. In the numerical case a substantial fraction of M_G is being returned from dwarfs that formed earlier when Z_3 was smaller, thereby lowering Z_3 from the exact IRA value. But nonetheless one sees that IRA is accurate to about 7% for this result. The inadequacy of IRA shows up in Figure 6 which compares the hypothetical primary metallicity Z_2 produced only in a star having $M_2 = 1.38 M_\odot$. The

IRA overestimates Z_2 for most of the galactic lifetime, but eventually underestimates it in old gas-poor galaxies.

These comparisons confirm a reasonable degree of accuracy of the approximation for most quantities after the initial adjustment for the evolution of the first stars. The approximation cannot be used for the study of very old metal-poor stars. The most severe problem is associated with high yield from low-mass stars (e.g., ^3He), which not only take a long time to produce any yield at all, but which also continue to produce after star formation has shrunk to a near halt. But for yields dominated by stars having $M > 2 M_\odot$ instantaneous recycling is a quite useful approximation over almost the entire model lifetime. This seems to imply that data surveys based on analytic models are a useful way to narrow down parameters. The real problem of course, is the herculean task of obtaining the quantitatively good data for such surveys. This old and sometimes vexing science continues to receive fresh input, however, and we can be hopeful of continuing improvements for some time to come.

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DONALD D. CLAYTON and IRINI PANTELAKI: Department of Space Physics and Astronomy, Rice University, Houston, TX 77251-1892