1-15-1985

Galactic Chemical Evolution and Nucleocosmochronology: Analytic Quadratic Models

Donald D. Clayton
Clemson University, claydonald@gmail.com

THE ASTROPHYSICAL JOURNAL, **288**:569–574, 1985 January 15

GALACTIC CHEMICAL EVOLUTION AND NUCLEOCOSMOCHRONOLOGY: ANALYTIC QUADRATIC MODELS

DONALD D. CLAYTON
Department of Space Physics and Astronomy, Rice University
Received 1984 June 11; accepted 1984 August 1

ABSTRACT

This paper studies models of the chemical evolution of the Galaxy for a star formation rate proportional to the square of the mass of gas. The first objective is to find analytic solutions to the gas mass and star mass for time-dependent rates of gaseous infall onto the disk. The second objective is to compare these quadratic models to models having linear star formation rates. I do this by comparing $m(t)$, $S(t)$, $Z(t)$, $S(<Z)$, and $r(^{235}\mathrm{U})/r(^{238}\mathrm{U})$ for comparison models that have the same initial disk mass, the same infall rate, and the same final gas fraction.

Subject headings: abundances — galaxies: Milky Way — nucleosynthesis — stars: formation

I. INTRODUCTION

This study is motivated in part by the desire to find analytic solutions to simplified models of the chemical evolution of galaxies. Clayton (1984*a*, *b*) has discussed this general objective and has provided many families of analytic solutions for the physical case in which the rate of star formation is proportional to the amount of gas available in the disk for star formation (the linear model) while the total mass is being continuously increased by metal-poor infall onto the disk. Those models, constructed with the simplifying assumptions of constant initial mass function and instantaneous recycling, provide a useful parameter space for explicit evaluation of the common tests of chemical evolution, which are described thoroughly by Tinsley (1980) and selectively by Clayton (1984*a*, *b*).

The present study attempts the same thing for quadratic star formation—a star formation rate proportional to the square of the gas mass. This has been a popular point of discussion ever since Schmidt's (1959) study suggesting it as a result of his analysis of the variation of star formation with height above the galactic midplane. Although one cannot really argue that the complicated star forming processes depend upon any single power of the gas mass, a contrast between linear and quadratic models at least increases understanding of the effect of nonlinearity upon the usual tests of chemical evolution. This comparison is another of the objectives of this work.

The search for analytic solutions to the quadratic case (as well as higher powers) is only partially successful. I will describe a successful generator of exact solutions to the gas mass for selected families of infall rates. I find that one of these families of analytic solutions also allows an analytic expression for the mass of stars (or of total mass), so that the gas/star ratio can be evaluated analytically. The parameter space of these solutions can describe many physically different infall rates; so this objective is achieved. However the metallicity $Z(t)$ and the concentrations of radioactive nuclei do not seem to also admit of easy solutions. At least I have not found them. But their evaluation by numerical integration is made easier by having the star formation rate $\psi(t)$ and the gas mass $M_G(t)$ both explicit known functions of time.

To compare these solutions with those of the linear model I chose a linear star formation rate coefficient ω_1 such that, with identically the same infall $f(t)$ and with the same initial disk mass, the gas mass reaches the same final fraction. In this way I compare exactly the same physical problem and boundary conditions, except for the change of the star formation prescription.

II. QUADRATIC MODELS

By *quadratic models* I imply a star formation rate that is proportional to the square of the gas mass $M_G(t)$, which may itself be thought of either as the gas mass of the total system or as the gas in a solar annulus of a disk. Thus the stellar birthrate $\psi(t) = cM_G^2$, where c is some constant. If the rate of mass infall into the defined system is $f(t)$, and if the return fraction has the value R, following Tinsley's (1980) notations, then the gas mass satisfies

$$\frac{dM_G}{dt} = -(1 - R)\psi(t) + f(t) , \qquad (1)$$

where I have assumed a constant initial mass function so that R is a constant, and I have assumed instantaneous recycling so that the fraction R is given back instantaneously. Letting the constant $k = c(1 - R)$ simplifies equation (1) to

$$\frac{dM_G}{dt} = -kM_G^2 + f . \qquad (2)$$

It is additionally useful to define a constant $\omega = kM_0$ having units of inverse time, where M_0 is an arbitrary measure of mass which will, however, be taken to be the initial gas mass $M_0 = M_G(0)$ for this formulation. Then equation (1) reads

$$\frac{dM_G}{dt} = -\frac{\omega}{M_0} M_G^2 + f \qquad (3)$$

which can be made more pleasing by measuring the gas mass in units of the initial disk mass: $m = M_G(t)/M_0$. For then

$$\frac{dm}{dt} = -\omega m^2 + \frac{f(t)}{M_0} \qquad (4)$$

is the general version of equation (1) in which all masses are measured in terms of the initial mass. That is, $m(0) = 1$. The constant ω is, physically interpreted, the initial rate for converting gas to stellar remnants. The time scale for converting gas to remnants is defined

$$\frac{1}{\tau_*} = -\frac{1}{m}\left(\frac{dm}{dt}\right)_* = \omega m\ , \tag{5}$$

which has the value ω initially, but is at later times moderated by the factor m owing to the quadratic dependence of star formation upon mass.

a) Generator of Analytic Solutions

There appears to be no general way of solving equation (4) for arbitrary infall rate $f(t)$, in contrast to the case of the linear model (e.g., eq. [4] of Clayton 1984*b*). However I have found a way to generate useful analytic solutions. To do this one gives up total arbitrariness of $f(t)$ and instead asks for simply parameterized families of functions $f(t)$ that do admit of analytic solutions. The solutions are then useful if the families of exact solutions can, through the parameters scaling $f(t)$, match $f(t)$ to physically meaningful forms. I have previously (Clayton 1984*a*, *b*) exploited this same philosophy in generating exact solutions to the linear model.

The *Ansatz* that accomplishes this aim is to define

$$f(t)/M_0 \equiv m^2(t)g(t)\ , \tag{6}$$

where $g(t)$ is an arbitrary function of time, because then equation (4) is separable

$$\frac{dm}{m^2} = [-\omega + g(t)]dt \tag{7}$$

with solutions

$$\frac{1}{m} = 1 + \omega t - \int_0^t g(t')dt'\ . \tag{8}$$

It will be clear that higher mass dependences than quadratic can be solved in the same way.

Any form of $g(t)$ that is analytically integrable will yield an analytic form for $m(t)$ as well. Not all of these will be interesting, of course. It is necessary to choose $g(t)$ such that equation (8) does not vanish ($m = \infty$). Such a runaway is easily possible, as we will see for a specific useful choice, because equation (6) warns us that the *Ansatz* chosen requires that when m increases, the infall increases in proportion to m^2 unless $g(t)$ is such as to cut it down. Furthermore, the interesting choices for $g(t)$ will lead to functional forms $m(t)$ and $f(t)$ that decline smoothly toward zero after first experiencing a single maximum.

A form for $g(t)$ that accomplishes these objectives is

$$g(t) = \omega_f\left(\frac{\Delta}{t+\Delta}\right)^n\ , \tag{9}$$

where n is a positive integer and Δ is an arbitrary time parameter. The notation ω_f for a parameter that measures the strength of the infall at $t = 0$ comes from noting that at $t = 0$, when $g = \omega_f$ and when $m = 1$, equations (4) and (6) lead to the limit

$$\frac{1}{m}\frac{dm}{dt} \to -\omega + \omega_f \quad (\text{at } t = 0)\ , \tag{10}$$

so that ω represents the initial time scale for remnant formation and ω_f represents the initial time scale for increasing the mass of the system by infall. For the gas mass to grow initially, one wants $\omega_f > \omega$. For $n = 1, 2, 3, \ldots$ the solutions of equation (8) are:

$$\frac{1}{m} = 1 + \omega t - \omega_f \Delta \ln\frac{t+\Delta}{\Delta} \quad (n = 1) \tag{11}$$

and

$$\frac{1}{m} = 1 + \omega t - \frac{\omega_f \Delta}{n-1}\left[1 - \left(\frac{\Delta}{t+\Delta}\right)^{n-1}\right] \quad (n > 1)\ . \tag{12}$$

These solutions $m(t)$ also yield the infall rates for which $m(t)$ is the exact solution: namely,

$$\frac{f(t)}{M_0} = m^2(t)\omega_f\left(\frac{\Delta}{t+\Delta}\right)^n\ . \tag{13}$$

Many other forms for $g(t)$ will generate simple exact solutions [e.g., $g(t) = \omega_f \exp(-t/\tau_f)$ or a polynomial in t that decreases to zero and remains zero thereafter); however, I find equation (9) to be adequate for a study of the observable differences between quadratic and linear star formation. In fact, in what follows I make the specific choice $n = 2$, not only because it still allows two free parameters but also because it will allow analytic solution of the remnant star mass $S(t)$.

b) Case n = 2

With the choice $n = 2$ we have $g(t) = \omega_f[\Delta/(t + \Delta)]^2$ and equation (12) can be written more simply

$$\frac{1}{m} = 1 + \omega t - \omega_f \Delta\left(\frac{t}{t+\Delta}\right) \quad (n = 2) \tag{14}$$

and

$$\frac{f(t)}{M_0} = \omega_f\left(\frac{\Delta}{t+\Delta}\right)^2\left[1 + \omega t - \omega_f\Delta\left(\frac{t}{t+\Delta}\right)\right]^{-2}\ . \tag{15}$$

The function $m(t)$ begins at $m(0) = 1$ and increases to a maximum at $t_{\max} = \Delta[(\omega_f/\omega)^{1/2} - 1]$, whereafter it declines smoothly toward zero. This physically desired behavior requires that the parameters not be chosen such that the right-hand side of equation (14) can vanish. From equation (14) it follows that

$$(1 + \omega t)(t + \Delta) - \omega_f \Delta \neq 0 \tag{16}$$

is a condition that ensures that m does not run away to infinity. Because the vanishing of equation (16) would require infinite mass at t equal to

$$t_\infty = (2\omega)^{-1}\{-(1 + \omega\Delta - \omega_f\Delta) \pm [(1 + \omega\Delta - \omega_f\Delta)^2 - 4\Delta\omega]^{1/2}\}\ ,$$

we see that t_∞ cannot lie on the positive real axis unless both

$$1 + \omega\Delta - \omega_f\Delta < 0 \tag{17a}$$

and

$$(1 + \omega\Delta - \omega_f\Delta)^2 > 4\Delta\omega\ . \tag{17b}$$

Thus ω, ω_f, and Δ should be chosen such that the conditions of equation (17) are *not both* satisfied.

Figure 1 shows two examples from the large space spanned by these parameters. Both $m(t)$ and $f(t)$ are displayed there for the same choices, $\omega = 0.3$ and $\Delta = 4$ (in units 10^9 yr), but for two different values of $\omega_f = 0.8$ and 1.0. For both choices of ω_f

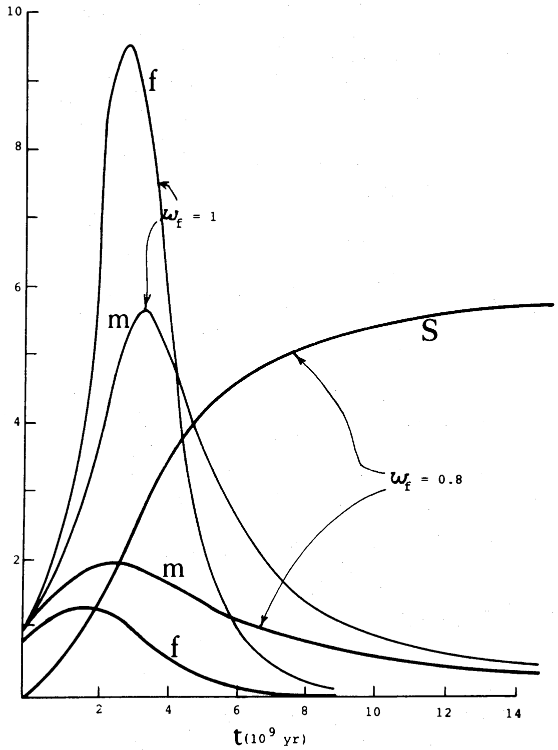

FIG. 1.—The functions $f(t)$ and $m(t)$ shown for two different values of the initial infall rate, $\omega_f = 0.8$ and 1.0, computed for the parameters $\omega = 0.3$ and $\Delta = 4$ Gyr. Both f and m rise, for these choices of parameters, to a single maximum and then decline smoothly toward zero. For the case $\omega_f = 0.8$ the star mass $S(t)$ is also shown. This model is the one used for comparison with the linear star formation model.

the first condition of equation (17) is satisfied but the second is not; therefore, the mass does not diverge. Figure 1 shows that $m(t)$ and $f(t)$ both rise to a maximum and then decline, a reasonable possibility for galactic growth. Although $\omega_f = 1.0$ only barely exceeds $\omega_f = 0.8$, the curves m and f show much more pronounced peaks. The choice $\Delta = 4$ causes $t_{max} = \Delta[(\omega_f/\omega)^{1/2} - 1] = 3.3$ for $\omega_f = 1$. It is immediately evident that other choices of these free parameters would enable one to shape $f(t)$ according to one's wish, so that the family of solutions $n = 2$ is not very restrictive.

Another interesting feature of $n = 2$ families is that the mass $S(t)$ of stellar remnants can also be expressed analytically. It is

$$S(t) = \int_0^t (1 - R)\psi(t')dt'$$

$$= \int_0^t \omega m^2(t')dt' \quad \text{(in units of } M_0\text{)}$$

$$= \omega\Delta \int_1^x \frac{x^2 dx}{[\omega\Delta x^2 + (1 - \omega\Delta - \omega_f \Delta)x + \omega_f \Delta]^2}\,,$$

$$x = \frac{t + \Delta}{\Delta}\,. \quad (18)$$

This integral is found in standard tables, but its form depends upon the discriminant of the quadratic polynomial $X = ax^2 + bx + c$ in the denominator. For those choices leading to large integrated infall, the discriminant $b^2 - 4ac < 0$, where $b = 1 - \omega\Delta - \omega_f \Delta$, $a = \omega\Delta$, and $c = \omega_f \Delta$. This discriminant is, for example, negative in both cases shown in Figure 1. If $b^2 - 4ac < 0$ the solution is, with $x = (t + \Delta)/\Delta$,

$$S(t) = \frac{(b^2 - 2ac)x + bc}{(4ac - b^2)X} - \frac{b^2 - 2ac + bc}{(4ac - b^2)} + \frac{4ac}{(4ac - b^2)^{3/2}} \times \left[\tan^{-1}\frac{2ax + b}{(4ac - b^2)^{1/2}} - \tan^{-1}\frac{2a + b}{(4ac - b^2)^{1/2}}\right]. \quad (19)$$

This solution is also shown in Figure 1 for the case $\omega_f = 0.8$. The value of S at $t = 15$ has risen to 5.73 (in units of M_0), whereas m has fallen to $m = 0.336$, giving a total galactic mass of 6.06. The gas is 5.56% of the total mass at that time. I will later use this as a model of the solar neighborhood today. For the case $\omega_f = 1$, on the other hand, S rises to 30.14 at $t = 15$, and m falls to 0.427, so that the gas is only 1.40% of the total mass at $t = 15$ in that model. For both of these examples, however, the infall has increased the total mass by a large factor. For those interested in numerically exploring this parameter space, I list in Table 1 a program in Microsoft Basic to calculate these quantities for $t = 1$ to 15 for either sign of the discriminant $b^2 - 4ac$ and for any choices of the DATA ω, ω_f, Δ. One need only be sure that both of conditions (eq. [17]) are not satisfied by the choices ω, ω_f, Δ.

For linear models Clayton (1984*a*, *b*) was able to find exact analytic solutions not only for $f(t)$, $m(t)$, $S(t)$, but also for the metallicity $Z(t)$ and for the radioactive "remainders" $r(t) = Z_\lambda(t)/Z(t)$. The differential equation is in general

$$\frac{dZ}{dt} = y_Z(1 - R)\frac{\psi(t)}{M_G(t)} - \lambda Z - (Z - Z_f)\frac{f(t)}{M_G(t)}\,, \quad (20)$$

where λ is the radioactive decay rate and Z_f is the value of Z in

TABLE 1

BASIC PROGRAM FOR f, m, S

```
10 READ W,WF,DEL
20 DATA .3,.8,4
21 A=W*DEL
22 B=1-W*DEL-WF*DEL
23 C=WF*DEL
24 D=B^2-4*A*C
25 P=(-B+(ABS(D))^.5)/(2*A)
26 Q=(-B-(ABS(D))^.5)/(2*A)
27 TMAX=DEL*((WF/W)^.5-1)
30 LPRINT "for w=";W;",wf=";WF;"Del=";DEL;",t(max)=";TMAX
35 LPRINT "a=";A,"b=";B,"c=";C,"b^2-4ac=";D,"p=";P,"q=";Q
40 FOR T=1 TO 15
45 X=(T+DEL)/DEL
47 Y=A*X^2+B*X+C
50 M=(1+W*T+C*(1/X-1))^-1
55 F=WF*(M/X)^2
60 S1=(2*A*C-B^2)/D*(X/Y-1)-B*C/D*(1/Y-1)
61 IF D<0 GOTO 67
65 S2=-2*A*C/D^1.5*LOG(ABS((X-P)/(1-P)/(X-Q)*(1-Q)))
66 GOTO 70
67 PHI=ATN((2*A+B)/SQR(-D))
68 S2=4*A*C/(-D)^1.5*(ATN((2*A*X+B)/SQR(-D))-PHI)
70 S=S1+S2
75 LPRINT "t=";T,"m=";M,"f=";F,"S=";S,"M tot=";S+M
77 LPRINT
80 NEXT T
85 LPRINT
90 END
```

the infalling matter (which I again take to be $Z_f = 0$ in this study). For the quadratic models this becomes

$$\frac{dZ}{dt} = y_Z \omega m(t) - \lambda Z - (Z - Z_f)m(t)g(t) , \qquad (21)$$

an equation that I have been unable to solve, either for the $n = 2$ family discussed here or for other functional forms for $g(t)$. In making the usual tests of galactic evolution, therefore, I have integrated this equation numerically, which is a trivial task since $m(t)$ and $g(t)$ are known functions of t.

III. COMPARISON WITH LINEAR MODEL

The next purpose of this paper is to compare observable quantities of galactic evolution for the contrasting cases of quadratic and linear star formation. At first sight this seems, to some astrophysicists, hopeless. For one thing the form of $f(t)$ is not known for our Galaxy; and if it were, one would simply integrate everything numerically. For another thing, the assumptions of these models—instantaneous recycling, constant initial mass function, no radial mixing—may be violated. And on top of this, star formation is a complicated physical process involving different phases of the ISM and perturbations of those phases, so that its rate probably cannot be well represented by any fixed power of the gas mass. But I argue that these caveats do not devalue the physical insights that result from comparing two models

Because so many quantities depend strongly on the relative masses of stars and of gas, the most meaningful comparison would seem to be between linear and quadratic models that today have identical gas fractions by mass. Furthermore, because the infall rate $f(t)$ is a given of the circumstances of galaxy formation, even though it is not known, the comparison should be between linear and quadratic models subjected to identically the same infall. For the quadratic model of the comparison I take the ω, ω_f, $\Delta = 0.3, 0.8, 4$ solution that was shown in Figure 1. The function $f(t)$ shown there and expressed by equation (15) is also taken to be the infall rate for a linear comparison model of galactic evolution. For that linear model $(1 - R)\psi(t) = \omega_1 m_1$ rather than ωm^2, so that I found by numerical integration of equation (1) the linear conversion rate ω_1 and the gas mass $m_1(t)$ that also produces $m_1 = 0.336$, or 5.6% of the total galactic mass, at $t = 15$. That value is $\omega_1 = 0.2385$ (in contrast to $\omega = 0.3$ to achieve a final 5.6% for m in the quadratic model). These then are the comparison models, one quadratic with $\omega = 0.3$ and one linear with $\omega_1 = 0.2385$, but both yielding $m = 0.336$ at $t = 15$ after evolving from $m(0) = 1$ with the same infall rate $f(t)$. Figure 2 shows $f(t)$ on an expanded scale [the same $f(t)$ as in Fig. 1], as well as the rate for forming stellar remnants in the comparison models. The star formation rate ωm^2 in the quadratic model has a peak value almost twice as great as the corresponding rate, $\omega_1 m_1$, in the linear model; but in the latter half of the galactic lifetime, the star formation rate in the quadratic model is only about half of the star formation rate in the linear model. (Note that because the final star mass at $t = 15$ is $S = 5.73$ for both models, the areas under the two star formation curves are equal.) Thus a first conclusion of the comparison is that *if the solar neighborhood evolved via quadratic star formation the stellar birthday spectrum is older than if it evolved by linear star formation.* Or stated somewhat differently, *the fraction of dwarf stars that are young is greater if the star formation was linear.* Figure 2 shows the differences to be appreciable. However, this

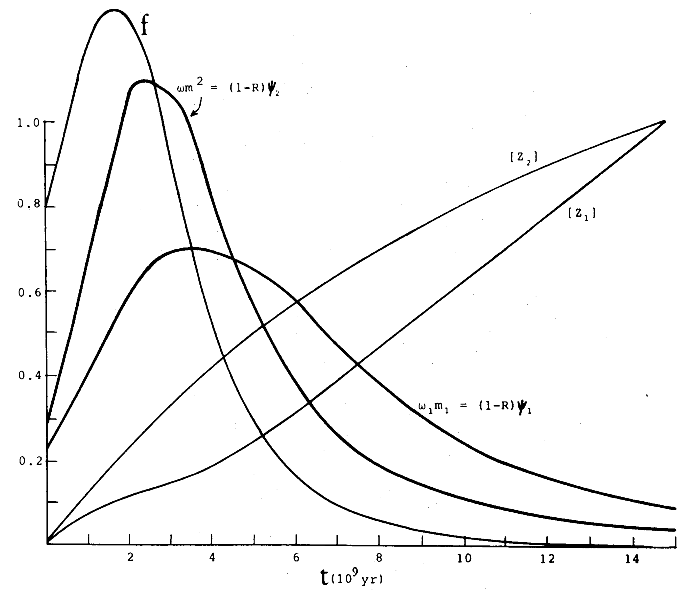

FIG. 2.—The infall $f(t)$ is the same as in Fig. 1. The rates for forming stellar remnants are shown for linear and quadratic models that both achieve 5.6% final gas mass and a final total mass that is 6.06 times greater than the initial disk mass. The quadratic star formation rate is earlier. Shown also is $[Z] \equiv Z(t)/Z(15)$ for both models, showing that the quadratic model growth is faster at early times.

result does depend upon properties of $f(t)$—that it peaks relatively early and has declined to a low value (relatively) today. To see that this is the case, one need only consider the limiting case of constant infall matched by constant star formation; for in this case the distribution of dwarf ages would be flat, whether the star formation were linear or quadratic. It is only when $m(t)$ is in fact time-dependent that the dwarf ages must concentrate more toward the peak of $m(t)$ in the quadratic case.

To compare the metallicity $Z(t)$ with age between the two models, I numerically integrated equation (20) for the linear model, with $(1 - R)\psi_1(t)/M_G(t) = \omega_1$, and equation (21) for the quadratic model. These results are also displayed in Figure 2. Here we see that $[Z_2]$, defined as $Z_2(t)/Z_2(15)$, initially grows more rapidly for the quadratic model than does the corresponding solution for the linear model. This again is no surprise and can be stated as a second conclusion of the comparisons; namely, *because nucleosynthesis has a larger early/late proportion for the quadratic model, that model also has a larger early/late proportion in the growth of* $Z(t)$ *than does the linear model.* However, one again sees by considering the constant infall and star formation counterexample that this conclusion depends on the shape of $f(t)$, because in that example with constant $f(t)$ the functions $Z(t)$ grow at exactly the same rate in the two cases.

The number of stars $S(<Z)$ having metallicity $<Z$ is not so obvious, however. Figure 3 shows that *the linear model has a greater percentage of low-Z stars than does the quadratic model*; even though the star formation occurs earlier in the quadratic model, the larger early metallicity of the quadratic model more than compensates in the function $S(<Z)$. The difference is, moreover, potentially observable, because in the range near $[Z] = Z/Z(15) \approx 0.2$, near the lower end of the Population I distribution, the percentages $S(<0.2)$ differ by a factor of 2. Both models, by the way, have many fewer low-Z stars than does the closed model with no infall. So either model helps

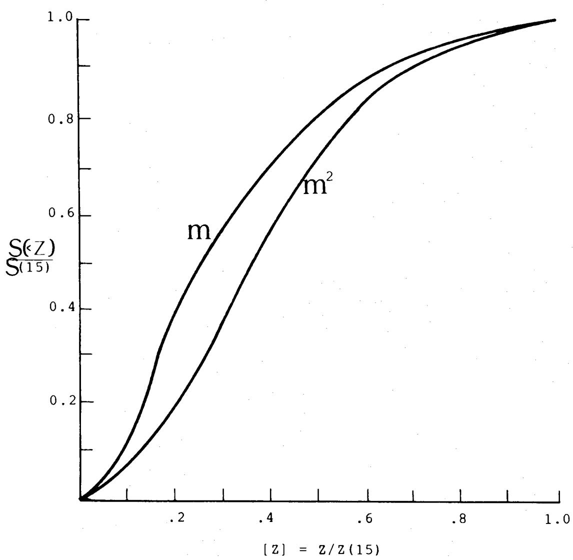

FIG. 3.—The fraction $S(<Z)$ of stars that formed with metallicity $<Z$ is shown for both comparison models as a function of $[Z] = Z/Z(15)$. The quadratic model has a smaller percentage of low-Z stars.

solve "the G-dwarf problem," but the quadratic model does it more effectively. Before this test can be used, however, it will be necessary to know with conviction the exact primary cause for the paucity of low-Z dwarfs—initial metallicity, infall, variable IMF, metal enhanced star formation, or some complicated combination of these or other causes.

Another possible test of the difference between linear and quadratic star formation lies in nuclear cosmochronology. I will here concentrate on ^{235}U ($\lambda = 0.972$) and ^{238}U ($\lambda = 0.154$). Equations (20) and (21) were integrated numerically for both values of λ. The straightforward difference can be seen in the ^{235}U "remainder" at $t = 15$, defined as the ratio of its actual concentration to the concentration it would have had were it stable: $r(t) \equiv Z_\lambda(t)/Z_{\lambda=0}(t)$. For the quadratic model of Figure 1 the final remainder is $r_{235}(15) = 0.0508$. In contrast, the corresponding ($\omega_1 = 0.2385$) linear model reaches $r_{235}(15) = 0.0913$, a value almost twice as great. That difference is, in principle, measurable. The chronometric information must usefully be expressed as the ratio of the ^{235}U remainder to that of ^{238}U, however, because the absolute production of ^{235}U is too difficult to calculate with sufficient confidence to predict the abundance ^{235}U would have were it stable. The relative production rates can be calculated with greater confidence, although even that ratio remains too uncertain to fix the galactic chronology absolutely. But Figure 4 shows the remainder ratio $r_{235}(t)/r_{238}(t)$ for both linear and quadratic models. Also shown is the horizontal band $r_{235}(t_\odot)/r_{238}(t_\odot) = 0.22 \pm 0.03$ that is believed (see Clayton 1984*a*, *b* for discussion of this ratio) to have been applicable to the solar neighborhood at the time $t_\odot$ of solar formation. The linear model passes through the center of that band at $t_\odot = 12.5$, whereas the quadratic model does so at $t_\odot = 9.0$. This may be stated as another conclusion for this $f(t)$: *the Galaxy appears to be 3.5 billion years older in the linear model than in the quadratic model.* It matters little, stated this way, that the correct value for r_{235}/r_{238} is unknown because the production ratio y_{235}/y_{238} is uncertain, because, whatever the correct production ratio, whatever the correct horizontal value for the ratio of remainders, the quadratic model passes through it several billion years earlier than the linear model does.

A related remark applies to the extinct radioactivities (e.g., ^{107}Pd, ^{129}I, ^{244}Pu). Their remainders measure the ratios of their average concentrations in the solar annulus to the concentrations they would have had were they stable. These remainders are commonly estimated as the ratio τ/T of the radioactive lifetime to the age of the Galaxy, an exact result for a closed linear model. Clayton (1984*a*) showed the ratio τ/T to be too small by a factor $(k + 1)$ in his analytic standard linear models with continuing infall; but Clayton (1984*b*) showed that if infall ceased at some distant past time, the underestimate is not as great. It is of interest in this context to see what the present comparison models yield. Evaluated at $T = 15$ they give

$$\frac{r_\lambda}{r_{\lambda=0}} = 1.35\tau/T \quad \text{(linear model)}$$

$$= 0.726\tau/T \quad \text{(quadratic model)}\,.$$

This comparison, important to such astrophysical issues as the ^{129}I/^{127}I ratio, yields a final conclusion: *the concentrations of extinct radioactivities are roughly half as large if star formation has been quadratic than they are if it has been linear.*

IV. CONCLUSION

The comparisons between linear and quadratic star formation have been performed for the identical infall rate $f(t)$ given by equation (15) with ω, ω_f, $\Delta = 0.3$, 0.8, 4. The new exact solutions of quadratic models presented in this work allow this infall to be easily identified with analytic forms of $m(t)$ and $S(t)$ that show 5.6% gas at $t = 15$ billion years. The linear star formation model having rate $\omega_1 = 0.2385$ also gives 5.6% gas at $t = 15$ with the same infall rate. Both models started with a gaseous disk that was 16.5% of the total final mass at $t = 15$. The models thus provide exact physical comparisons between star formation rates proportional to first and second powers of the gas mass.

The comparison has been made for an infall $f(t)$ that is

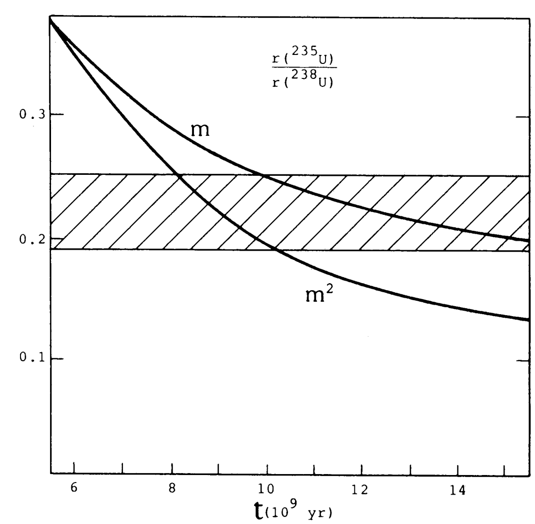

FIG. 4.—Ratios of residuals for uranium isotopes. The band 0.22 ± 0.3 is the most likely value at the time of solar formation (Clayton 1984*a*, *b*). One sees that whatever the exact ratio, the quadratic galaxy is several billion years younger than the linear model.

concentrated into the first half of the galactic lifetime, fully 80% of the infall having occurred by $t = 4$. Infall still occurs today in this example, but only at a rate so slow that 84 billion years would be required at that rate to add an amount equal to today's gas mass. So the infall has almost ceased in the example comparison. I emphasize this because the conclusions drawn are valid only for an infall rate having these general features. A model at the other extreme, infall occurring at a constant rate that just balances a constant rate of star formation, would show no distinctions between linear and quadratic star formation. Nonetheless, the conclusions drawn from the comparison are of physical interest, because a sizable augmentation of the disk mass after star formation has begun in the disk but in the first several billion years is quite plausible. With that restriction upon $f(t)$, the major results of the comparison are as follows:

1. The average dwarf age is greater in the quadratic model.
2. The metallicity $Z(t)$ grows initially faster in the quadratic model.
3. The quadratic model has a smaller percentage of low-Z dwarfs.
4. The $^{235}U/^{238}U$ isotopic ratio indicates a younger quadratic model.
5. The concentrations of extinct radioactivities are smaller in a quadratic model.

This research was supported in part by NASA grant NSG-7361 and in part by the Robert A. Welch Foundation.

REFERENCES

Clayton, D. D. 1984*a*, in *Challenges and New Developments in Nucleosynthesis*, ed. W. D. Arnett, W. Hillebrandt, and J. W. Truran (Chicago: University of Chicago Press), in press.
Clayton, D. D. 1984*b*, *Ap. J.*, **285**, 411.
Schmidt, M. 1959, *Ap. J.*, **129**, 243.
Tinsley, B. 1980, *Fund. Cosmic Phys.*, **5**, 287.

DONALD D. CLAYTON: Department of Space Physics and Astronomy, Rice University, Houston, TX 77251