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GALACTIC CHEMICAL EVOLUTION AND NUCLEOCOSMOCHRONOLOGY: STANDARD MODEL WITH TERMINATED INFALL

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ABSTRACT

I present techniques for generating exact solutions to the differential equations for the chemical evolution of galactic gas $M_G(t)$, the star mass $S(t)$, the metallicity $Z(t)$, and the abundances of the radioactive chronometers $Z_i(t)$ in the face of galactic infall $f(t)$. I assume instantaneous recycling and a linear dependence of star formation rate upon gas mass. Because some of these solutions are simple analytic functions, they can serve as standard models for the usual tests of galactic evolution. This paper emphasizes cases where infall $f(t)$ terminates at some time t_0 and is zero thereafter. The solutions are displayed for two parameterizations of the infall rate:

$$(1) \quad f(t) = A \cos(\omega_1 t + \Delta) \sin^{k-1}(\omega_1 t + \Delta) e^{-\omega t},$$

where k is a positive integer and t_0 is defined by the other two parameters according to $\omega_1 t_0 + \Delta = \pi/2$; and

$$(2) \quad f(t) = A(t + \Delta)^{k-1}(t_0 - t)e^{-\omega t}.$$

An especially simple model of the first class is recommended as a standard model of galaxies with truncated infall not because it is necessarily physically correct but rather because it is internally consistent and displays the physical effects of the parameters on the observables in a compact form. For $k = 1$ these infall rates decline monotonically to zero at $t = t_0$, but for $k > 1$ may experience a single maximum before doing so. Many other solutions can be generated in the same way.

Subject headings: galaxies: evolution — nucleosynthesis

1. INTRODUCTION

In this work I present a new analytic model of the chemical evolution of the Galaxy. I even suggest that it makes a good “standard model” for galactic evolution, because known properties of chemical evolution are well modeled by it. In setting out upon this task I held it as desirable that the abundances of radioactive chronometers be expressible analytically, so that the cosmochronometers could display explicit dependence upon the properties of the total model. The gas mass $M_G(t)$, the star mass $S(t)$, and the metallicity $Z(t)$ are also expressed analytically. With such a class of models, the major tests of chemical evolution are quickly evaluated, so that their dependence upon the model histories can be more instructive than are computer printouts. In this sense a good “standard model” of chemical evolution can play the same useful role as does a standard analytic model of a star, or of other astronomical objects—namely, in providing an easily understandable frame of reference within which more specific and special results can be comprehended.

In constructing this model I have been guided by a few astrophysical biases in addition to the need to see the nuclear cosmochronology explicitly. Despite much discussion of more complicated models, I find it most natural to assume that the initial mass function is time independent, and that the star formation rate $\psi(t)$ depends linearly upon the gas mass $M_G(t)$. It is also natural to assume that the disk component of the Galaxy grew as matter fell upon it during the Galaxy’s maturation, with the total mass of the disk growing to many times its initial value (defined as the value where star formation began in the disk). Because the infall, which I call $f(t)$ following Tinsley’s (1980) excellent description of galactic evolution, has probably decreased in time, and may practically have ceased today, I choose an infall rate having that property. In Paper I (Clayton 1984a) of this series I solved a simple model in which infall still continues, whereas in this work I describe mathematical representations in which infall decreased continuously to zero at some time now past. The gas mass $M_G(t)$, and hence the star formation rate $\psi(t)$, first grows as matter falls onto the disk, and later decreases as the infall abates. I will also employ the instantaneous recycling approximation, namely, that short-lived massive stars so dominate the mass recycling through stars that the time delay between star birth and ejection can be neglected. Many studies have confirmed the general validity of this approximation (e.g., Tinsley 1980, and many references therein). This simplification allows the ejecta to be written $E(t) = R\psi(t)$, where R is the return fraction.

Before turning to a class of solvable models incorporating these astrophysical biases, I must add that my reconsideration of this old and important problem, on which so much has been done by so many (e.g., Tinsley 1980, and references therein), was stimulated directly by the discovery (Mahoney *et al.* 1982) of live ^{26}Al in the interstellar medium and by uncertainty about its implications for models of the chemical evolution of the Galaxy. My analysis (Clayton 1984b) of that situation has led me to reconsider the allowable expectations for the abundances of all radioactive nuclei within the interstellar medium; and I suspect that the family of models which I have discovered probably maps the range of physically realistic models.

There is ample evidence (Fabian *et al.* 1984) and theoretical interpretation (Binney and Cowie 1981) that M87 in the Virgo Cluster

is accreting today about $10 M_{\odot} \text{ yr}^{-1}$ of cooling atmosphere, but it is uncertain whether this is primordial low- Z material or Population I gas that has previously been ejected by M87. In the Virgo Cluster, M87 is a special galaxy, and may be accreting far more than is typical for galaxies in general, and that accretion may be the result of a different situation than the accretion that initially formed the galaxies. In any case, the analytic models to be described here will be valid for any *constant* metallicity in the infalling matter.

II. MATHEMATICAL LINEAR MODEL

By the linear model I mean that the star formation rate $\psi(t)$ is taken to be proportional to the gas mass $M_G(t)$, which may itself be thought of either as the total disk or as a solar annulus. So with a mass fraction R returned instantaneously, the equation (following Tinsley's notation)

$$\frac{dM_G}{dt} = -\psi(1 - R) + f \quad (1)$$

can be simplified greatly by setting

$$\psi(1 - R) = \omega M_G, \quad (2)$$

where ω is a constant by the assumption of linearity. Were there no infall ($f = 0$), ω would equal the rate of exponential decrease in the gas mass as it turns into stars. That simple closed exponential model is unsatisfactory, so with $f(t) \neq 0$ the function M_G will not be exponential; nonetheless, the constancy of ω remains reasonable as long as one expects the rate of star formation to be proportional to the mass of gas available for forming stars. I find this to be a very natural assumption even though many have speculated, following Schmidt (1959, 1963), that a steeper nonlinear dependence upon M_G is called for. Keeping the linear assumption, the solution of (1) can be written either

$$M_G(t) = M_{G0} \exp \left[-\omega t + \int_0^t \frac{f(t')}{M_G(t')} dt' \right] \quad (3)$$

or

$$M_G(t) = e^{-\omega t} \left[M_{G0} + \int_0^t f(t') e^{\omega t'} dt' \right], \quad (4)$$

which shows explicitly how the gas mass is rendered nonexponential by the infall. The ratio $f(t)/M_G(t) \equiv \omega_f(t)$ is the instantaneous rate at which the current $M_G(t)$ is being replenished by the current infall $f(t)$. Because it appears so often in the mathematical analysis, it is convenient to define a dimensionless function of time

$$\theta(t) = \int_0^t \omega_f(t') dt' = \int_0^t f(t')/M_G(t') dt', \quad (5)$$

where the angular mnemonic $\theta = \bar{\omega}_f t$ allows us to understand θ as being the number of times that the interstellar medium has been cycled by infall. I even suggest naming $\theta(t)$ the "infall cycle number," because it is so instrumental in measuring the extent of interstellar dilution by infalling matter. Then $M_G(t) = M_{G0} \exp [-\omega t + \theta(t)]$ gives the time dependence of the gas. For later applications it will also be useful to invert equation (3) to read $\exp \theta(t) = M_G(t) \exp (\omega t)/M_{G0}$.

Lynden-Bell (1975) has emphasized the importance of having a nonmonotonic function $M_G(t)$. Under heavy infall $M_G(t)$ grows at first, but because it is initially small the number of low- Z stars is not severe since $Z(t)$ grows quickly, so that the G-dwarf problem is circumvented. Larson (1972) had first made the same physical point by advancing a simple model in which both $M_G(t)$ and $f(t)$ were constants in time. Equation (3) shows clearly that $M_G(t)$ first increases if $\theta(t)$ initially increases faster than ωt . The model that I will construct can have that property.

The mass of stars $S(t)$ is given by the integral over time of $\psi(1 - R)$, which by (2) becomes

$$S(t) = \omega M_{G0} \int_0^t \exp [-\omega t' + \theta(t')] dt'. \quad (6)$$

Observationally, this function $S(t)$ assumes importance in studying the distribution of stars with age, or more easily, with metallicity.

The equation for the mass fraction $Z(t)$ of a primary nucleosynthesis product is approached through the total mass of element Z in the gas (e.g., Tinsley 1980, eq. [3.18]):

$$\frac{d}{dt} (Z M_G) = -Z(1 - R)\psi + y_z(1 - R)\psi + Z_f f - \lambda Z M_G, \quad (7)$$

where λ is the radioactive decay rate of species Z , and where y_z is the "yield" of element Z per unit increase of the mass of stars [remembering that $\psi(1 - R)dt$ is the net increase in star mass during dt], and where Z_f is the mass fraction of element Z in the infalling matter. For later evaluation, I will recommend taking $Z_f = 0$, at least until that time that astrophysical arguments demand

a significant Z_f in the infalling matter. Using (1) in the identity $M_G dZ/dt = d(ZM_G)/dt - ZdM_G/dt$ yields the equation for the gas abundance

$$M_G dZ/dt = y_z(1 - R)\psi - (Z - Z_f)f - \lambda M_G Z \quad (8)$$

$$= y_z \omega M_G - (Z - Z_f)f - \lambda M_G Z, \quad (9)$$

where (8) is general and (9) applies to the linear model to be developed herein. Clearly (9) can also be written

$$dZ/dt = (y_z \omega - \lambda Z) - (Z - Z_f)f/M_G \quad (10)$$

$$= (y_z \omega - \lambda Z) - (Z - Z_f)d\theta/dt. \quad (11)$$

This equation has a very simple solution that appears to have not been previously described, although Tinsley (1975) showed the way with a similar result restricted to $Z_f = 0$:

$$(Z - Z_f) = e^{-\lambda t - \theta} \left[(y_z \omega - \lambda Z_f) \int_0^t e^{\lambda t' + \theta(t')} dt' + Z_0 - Z_f \right]. \quad (12)$$

The infall Z_f is restricted to a constant in this solution, which may or may not be reasonable, depending on the history of nucleosynthesis in the infalling matter. It is at least a valid assumption in the special case when it is so small that it essentially vanishes. For this reason I will take metal-poor infall for an attempt at a simple standard model. Setting $Z_f = 0$ is physically reasonable if halo stars were not able to appreciably enrich the infalling matter. The initial abundance in the disk Z_0 will also be set to zero in the first attempt, although equation (12) is valid for any choice of Z_0 . In particular these solutions are applicable to "the second era" as defined by Tinsley (1981) and generalize her restrictive assumption of constant infall rate. And to determine the growth of stable metallicity one simply sets the radioactive decay rate $\lambda = 0$. Because I will later make use of the point, I note that the integral in equation (12) is the same integral as in equation (6), except that the parameter $-\omega$ has been replaced by the parameter λ . Simple analytic solutions for this key function of time will therefore be desirable.

Armed with these results of the linear theory, we can search for physically motivated analytic solutions. They require that $\theta(t)$ have a functional form that is both physically meaningful and manageable in equations (6) and (12).

Especially useful for the generation of simple analytic solutions is the substitution $f(t) = F(t)e^{-\omega t}$, whereupon equation (4) gives $M_G(t) = [M_{G0} + G(t)]e^{-\omega t}$, with $G(t) \equiv \int_0^t F(t')dt'$. The function $F(t)$ may be any convenient function of time. Especially manageable is a polynomial, in which case $G(t)$ is also a polynomial, with boundary condition $G(0) = 0$. It is noteworthy of this substitution that the infall cycle number will then satisfy the relation

$$e^{\theta(t)} = \frac{M_{G0} + G(t)}{M_{G0}} \quad (13)$$

so that its appearance in equations (6) and (12) will present no obstacles to evaluation if $G(t)$ is chosen to be a polynomial. Clayton (1984a) recommended the especially compact polynomial $F(t) = kM_{G0}/\Delta[(t + \Delta)/\Delta]^{k-1}$ as a useful "standard model" because the continuous but decreasing (at large times) infall then gives very elementary expressions for all observables of interest, inasmuch as $e^{\theta(t)} = [(t + \Delta)/\Delta]^k$. A polynomial chosen to vanish at $t = t_0$ would be a good way of generating solutions in which the infall terminates smoothly at that time; and indeed I will outline an uncomplicated form for this in § IV. But I will choose instead for purposes of evaluation an even simpler choice having that property. I will myself call it "standard model number 2," to distinguish its easy workability from the infinity of models having terminated infall.

III. THE STANDARD MODEL NUMBER 2

The infall function $f(t)$ is free, unrestricted by the equations of chemical evolution, although, of course, it will be determined by the correct dynamic theory of galaxy formation. I here suggest, as a parameterized model, the form

$$\begin{aligned} \frac{d\theta}{dt} &\equiv \omega_f \equiv f(t)/M_G(t) = k\omega_1 \cot(\omega_1 t + \Delta) \\ &= 0 \quad \text{for} \quad \omega_1 t + \Delta > \pi/2, \end{aligned} \quad (14)$$

where k is an arbitrary positive integer (for mathematical simplicity) and ω_1 and Δ are arbitrary parameters. My reasons for choosing this form to define a model are really several: (1) this rate ω_f to replace M_G by infall decreases with time; (2) the absolute infall rate $f(t)$ peaks at an earlier time than does $M_G(t)$; (3) the free choice of three parameters, k , ω_1 , and Δ , allows adequate form-fitting to ω_f ; and (4) the solutions can be evaluated analytically. In order that the infall cycle rate be nonnegative, i.e., $f(t) \geq 0$, equation (14) must have two branches, remaining $f = 0$ when $t > t_0$, where $\omega_1 t_0 + \Delta = \pi/2$. Thus a physical parameter of this representation is the time t_0 at which galactic infall ceased:

$$t_0 = (\pi/2 - \Delta)/\omega_1. \quad (15)$$

It is really reason (4) that motivates me, because a "standard model" must, if the name is to have useful meaning, be subject to quick interpretation. With that in mind, notice the simplicity of the results. From equation (5) the infall cycle number is

$$\begin{aligned}\theta(t) &= \int_0^t f(t')/M_G(t') dt' = \ln \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^k, & t < t_0, \\ &= \ln \sin^{-k} \Delta, & t > t_0,\end{aligned}\quad (16)$$

whereupon the gas mass becomes

$$\begin{aligned}M_G(t) &= M_{G0} \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^k e^{-\omega t}, & t < t_0, \\ &= M_G(t_0) \exp[-\omega(t - t_0)] = \frac{M_{G0}}{\sin^k \Delta} e^{-\omega t}, & t > t_0.\end{aligned}\quad (17)$$

The final equality shows that after infall ceases, the gas mass is at all times greater than it would have been in the absence of infall by the factor $\sin^{-k} \Delta$. The infall rate is

$$\begin{aligned}f(t) &= k\omega_1 \cot(\omega_1 t + \Delta) M_G(t) \\ &= \frac{k\omega_1 M_{G0}}{\sin \Delta} \cos(\omega_1 t + \Delta) \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^{k-1} e^{-\omega t}, & t < t_0, \\ &= 0, & t > t_0;\end{aligned}\quad (18)$$

and the total mass of stars from (6) is

$$\begin{aligned}S(t) &= \omega M_{G0} I_k(t, -\omega), & t < t_0, \\ &= S(t_0) + M_{G0} \sin^{-k} \Delta (e^{-\omega t_0} - e^{-\omega t}), & t > t_0,\end{aligned}\quad (19)$$

where we define a simple integral that is of repeated use in this model:

$$I_k(t, -\omega) \equiv \int_0^t \left[\frac{\sin(\omega_1 t' + \Delta)}{\sin \Delta} \right]^k e^{-\omega t'} dt', \quad t < t_0, \quad (20)$$

is a simple function of t that depends explicitly upon the parameters k , Δ , ω_1 , and ω . I evaluate it below. Furthermore, the mass fraction of a primary nucleosynthesis product can also be evaluated from equation (12) as

$$\begin{aligned}(Z - Z_f) &= (y_z \omega - \lambda Z_f) e^{-\lambda t} \left[\frac{\sin \Delta}{\sin(\omega_1 t + \Delta)} \right]^k \int_0^t \left[\frac{\sin(\omega_1 t' + \Delta)}{\sin \Delta} \right]^k e^{\lambda t'} dt' \\ &= (y_z \omega - \lambda Z_f) e^{-\lambda t} \left[\frac{\sin \Delta}{\sin(\omega_1 t + \Delta)} \right]^k I_k(t, \lambda), & t < t_0, \\ Z &= Z(t_0) e^{-\lambda(t-t_0)} + y_z \omega (1 - e^{-\lambda(t-t_0)})/\lambda, & t > t_0,\end{aligned}\quad (21)$$

where the integral $I_k(t, \lambda)$ is the same function as the integral $I_k(t, -\omega)$ that occurs in $S(t)$ except that the parameter $(-\omega)$ has been replaced by the parameter $(+\lambda)$; hence the notation showing explicitly the parameter.

For the case of a stable nucleus, the metallicity reduces to

$$\begin{aligned}(Z - Z_f) &= y_z \omega \left[\frac{\sin \Delta}{\sin(\omega_1 t + \Delta)} \right]^k I_k(t, 0), & t < t_0, \\ Z &= Z(t_0) + y_z \omega (t - t_0) = y_z \omega [\sin^k \Delta I_k(t_0, 0) + t - t_0], & t > t_0.\end{aligned}\quad (22)$$

TABLE 1
STABLE METALLICITY^a $Z_k(t)$ FOR $k = 0, 1, 2, 3$

k	$t < t_0$	$t > t_0$
$k = 0 \dots\dots$	$y_z \omega t$	$y_z \omega t$
$k = 1 \dots\dots$	$y_z \omega \frac{\cos \Delta - \cos(\omega_1 t + \Delta)}{\omega_1 \sin(\omega_1 t + \Delta)}$	$y_z \omega \left(\frac{\cos \Delta}{\omega_1} + t - t_0 \right)$
$k = 2 \dots\dots$	$\frac{y_z \omega}{\omega_1 \sin^2(\omega_1 t + \Delta)} \left[\frac{\omega_1 t}{2} - \frac{\sin 2(\omega_1 t + \Delta)}{4} + \frac{\sin 2\Delta}{4} \right]$	$y_z \omega \left[\frac{1}{\omega_1} \left(\frac{\pi}{4} - \frac{\Delta}{2} + \frac{\sin 2\Delta}{4} \right) + t - t_0 \right]$
$k = 3 \dots\dots$	$\frac{y_z \omega}{\omega_1 \sin^3(\omega_1 t + \Delta)} \left[\frac{\cos^3(\omega_1 t + \Delta) - \cos^3 \Delta}{3} - \cos(\omega_1 t + \Delta) + \cos \Delta \right]$	$y_z \omega \left[\frac{\cos \Delta}{\omega_1} - \frac{\cos^3 \Delta}{3\omega_1} + t - t_0 \right]$

^a Calculated from eq. (22) with infall $Z_f = 0$.

These assume simple forms for specific values of k as shown in Table 1 for $k = 0, 1, 2$, and 3 . The total mass of the galactic disk at any time is

$$M_{\text{tot}} = M_G + S = M_{G0} \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^k e^{-\omega t} + \omega I_k(t, -\omega), \quad t < t_0, \\ = M_G(t_0) + S(t_0), \quad t > t_0. \quad (23)$$

These results specify virtually everything of interest to such models as simple functions of time that can even be evaluated on a hand calculator. The integrals $I_k(t, \lambda)$ are easily evaluated as

$$k = 0: I_0 = (e^{\lambda t} - 1)/\lambda;$$

$$k = 1: I_1 = \frac{\omega_1}{\lambda^2 + \omega_1^2} \frac{1}{\sin \Delta} \left\{ e^{\lambda t} \left[\frac{\lambda}{\omega_1} \sin(\omega_1 t + \Delta) - \cos(\omega_1 t + \Delta) \right] - \frac{\lambda}{\omega_1} \sin \Delta + \cos \Delta \right\};$$

$$k = 2: I_2 = \frac{\omega_1}{\lambda^2 + 4\omega_1^2} \frac{1}{\sin^2 \Delta} \left\{ e^{\lambda t} \left[\frac{\lambda}{\omega_1} \sin^2(\omega_1 t + \Delta) - \sin 2(\omega_1 t + \Delta) + \frac{2\omega_1}{\lambda} \right] - \frac{\lambda}{\omega_1} \sin^2 \Delta + \sin 2\Delta - \frac{2\omega_1}{\lambda} \right\};$$

$$k = 3: I_3 = \frac{\omega_1}{\lambda^2 + 9\omega_1^2} \frac{1}{\sin^3 \Delta} \left\{ e^{\lambda t} \left[\frac{\lambda}{\omega_1} \sin^3(\omega_1 t + \Delta) - 3 \sin^2(\omega_1 t + \Delta) \cos(\omega_1 t + \Delta) \right] \right. \\ \left. + \frac{6[(\lambda/\omega_1) \sin(\omega_1 t + \Delta) - \cos(\omega_1 t + \Delta)]}{(\lambda/\omega_1)^2 + 1} \right\} - \frac{\lambda}{\omega_1} \sin^3 \Delta + 3 \sin^2 \Delta \cos \Delta - \frac{6[(\lambda/\omega_1) \sin \Delta - \cos \Delta]}{(\lambda/\omega_1)^2 + 1} \Bigg\};$$

and in general I_k can be calculated from the recursion relation

$$I_n = \frac{\omega_1}{\lambda^2 + n^2 \omega_1^2} \frac{1}{\sin^n \Delta} \left\{ e^{\lambda t} \sin^{n-1}(\omega_1 t + \Delta) \left[\frac{\lambda}{\omega_1} \sin(\omega_1 t + \Delta) - n \cos(\omega_1 t + \Delta) \right] \right. \\ \left. - \sin^{n-1} \Delta \left(\frac{\lambda}{\omega_1} \sin \Delta - n \cos \Delta \right) \right\} + \frac{n(n-1)\omega_1^2}{(\lambda^2 + n^2 \omega_1^2) \sin^2 \Delta} I_{n-2}.$$

For the similar integral $I_k(t, -\omega)$ occurring in (19) and elsewhere one need only replace λ by $-\omega$. Because these functions are products of simple polynomials and exponentials, their evaluation is numerically easy. In presenting the first results of this model, I have been content to use $k = 1, 2$, or 3 as a contrast to the simple $k = 0$ model, which has no infall. For the display and discussion of numerical results I will express time in units of 10^9 years.

These results bear strong formal similarity to standard model number 1 (Clayton 1984a) in that the factor $[\sin(\omega_1 t + \Delta)/\sin \Delta]^k$ in equations (16), (17), and (20) has replaced the factor $[(t + \Delta)/\Delta]^k$ in model number 1. Physically the distinction is that model number 1 applies to situations where infall continues up to the present time, although it asymptotically approaches zero; this model number 2, on the other hand, describes infall that decreases to zero at time t_0 and remains so thereafter. Of course, t_0 can be chosen to exceed the galactic age sufficiently by taking a small value of ω_1 so that infall continues also in model number 2. And, as Clayton (1984a) remarked, model number one remains analytically applicable if the infall is arbitrarily set to zero in that model for $t > t_0$, in which case the solutions at that time need to be joined as in the present work to those without infall for $t > t_0$. Thus the classes of models are not all that different. The nicety of the present model is that the infall approaches zero smoothly at t_0 rather than discontinuously.

a) Gas Mass $M_G(t)$

As a function of time the gas mass $M_G(t)$ reaches a maximum at $t(\max M_G)$ satisfying the relation $\omega_1 t(\max M_G) + \Delta = \tan^{-1} k\omega_1/\omega$, as is easily seen by differentiating (17). At earlier times the rate of star formation is also increasing, because by assumption it occurs in proportion to the disk gas mass. This feature has the useful result that the number of stars born very early is small. In Figure 1 I display two examples of the function $M_G(t)$ for the particular choice $k = 1$. For $\Delta = 0.1$ and $\omega_1 = 0.3$ the gas $M_G(t)$ rises to a pronounced maximum (at $t = 3.0$ for the case shown $\omega = 0.2$) whereas for $\Delta = 0.3$ the maximum is near $t = 0$. Even these two examples make it evident that a very wide range of shapes for $M_G(t)$ is possible, although they all have a single maximum. The following rules of thumb for $M_G(t)$ are almost self-evident: (1) increasing k moves $t(\max M_G)$ to larger t ; (2) increasing ω moves $t(\max M_G)$ to smaller t ; (3) increasing Δ moves $t(\max M_G)$ to smaller t , being at $t = 0$ if $\Delta > \tan^{-1} k\omega_1/\omega$; (4) for a chosen $t(\max M_G)$, the relative height $M_G(t_{\max})/M_{G0}$ of the maximum increases as Δ is made smaller and also decreases with ω . These are the same rules of thumb that characterized $M_G(t)$ in standard model number 1.

These features of $M_G(t)$ are of course not arbitrary, because M_G is the result of a physical model, the linear model with infall. To understand them, therefore, one must observe the nature of the corresponding infall rates $f(t)$. From equation (18) it is evident that $f(t)$ may also have a maximum at $t > 0$. If so, it occurs earlier than the maximum in the gas mass. This may often be inapplicable, of course. In the case $k = 1$, the infall rate is

$$k = 1: f_1(t) = \frac{M_{G0} \omega_1}{\sin \Delta} \cos(\omega_1 t + \Delta) e^{-\omega t}, \quad t < t_0, \\ = 0, \quad t > t_0, \quad (24)$$

FIG. 1.—Two contrasting examples of the behavior of the gas mass $M_G(t)$ for models having $k = 1$ are displayed (with scale changes as noted) as multiples of the initial mass M_{G0} . Models with small Δ (solid lines) have a maximum before declining because the early infall is large, whereas those having large Δ (dashed lines) decrease monotonically owing to the small early infall. The infall terminates at times $t_0 = 4.9$ and 12.7 Gyr in the two cases shown. Also shown are the total star masses in units of M_{G0} , showing that the small Δ cases have a greater growth factor. Models having $k > 1$ are qualitatively similar except that f can first rise to its maximum before declining. The time dependence of M_G/M_{tot} is also shown for those times late enough that it has fallen below 20%.

so that $M_{G0} \omega_1 \cot \Delta$ becomes the parameterized initial infall rate. That is, the physical meaning of $(\omega_1 \cot \Delta)^{-1}$ for $k = 1$ is that it is the time required for the initial infall to increase the disk mass to twice its initial value. For larger values of k the initial infall $f_k(0) = k M_{G0} \omega_1 \cot \Delta$, so that the initial infall would double the disk mass in a time shorter by a factor $1/k$. The infall will at first increase with time if

$$\omega_1 t(\max f) = \tan^{-1} \left\{ \frac{(\omega^2 + 4[k-1]\omega_1^2)^{1/2} - \omega}{2\omega_1} \right\} - \Delta \quad (25)$$

is greater than zero. If $t(\max f) < 0$, on the other hand, the infall decreases monotonically.

Because of the infall the total mass $M_{\text{tot}} = M_G(t) + S(t)$ increases with time, reaching an asymptotic value as $t \rightarrow t_0$. For the case $k = 1$, equation (24) shows that the total asymptotic infall is simply

$$\int_0^\infty f_1 dt = \frac{M_{G0} \omega_1^2}{\sin \Delta (\omega^2 + \omega_1^2)} \left(\frac{\omega}{\omega_1} \cos \Delta - \sin \Delta + e^{-\omega t_0} \right). \quad (26)$$

Analogous results exist for larger k . In Figure 1, the quantities $M_G(t)$ and $S(t)$ were displayed as a function of time. Their sum in the case $\Delta = 0.1$, $\omega = 0.2$, and $\omega_1 = 0.3$ rises to 7.51 times greater than M_{G0} , which is the same sum as $M_{G0} + \int f dt$. The case $\Delta = 0.3$, $\omega = 0.3$, $\omega_1 = 0.1$ reaches the value 1.88 at $t = 15$, again in agreement with equation (26). But the major distinction between these two cases is clear: in the first parameterization the infall greatly increases the initial mass, whereas in the second parameterization it is increased by only 88%.

It is not clear physically how the infall should go at short times—does it increase before decreasing, etc.? But it is perhaps useful that for $k > 1$ the infall in this mathematical model can be made to have a maximum at $t > 0$, as displayed in equation (25). Larson's (1976) dynamic infall models showed, for example, a peak in the star formation rate (at $r = 7.5$ kpc) near 5×10^9 yr when about 20–50% of the final star mass was already accomplished (see also Tinsley and Larson 1978). Such features are easily parameterized by the standard mathematical model advanced here.

Before going on I would observe that it is possible to reduce this standard model to certain limiting cases that have provided the traditional analytic solutions.

1. A value of $k = 0$ corresponds to no infall, whereupon $M_G(t)$ is itself a simple exponential decrease. From equation (22) the stable concentration increases linearly: $(Z - Z_f) = y_z \omega t$. The radioactive chronologies also have simple solutions. Thoroughly studied, this model in its simplest form fails, primarily because $S(Z)$ has far too many low- Z stars (unless a high initial abundance Z_0 is postulated) and because astrophysical dynamics suggest that the disk mass should be continuously grown by infall.

2. When $k \rightarrow \infty$ and $\omega_1 \rightarrow 0$ such that $f/M_G = k\omega_1 \cot \Delta \rightarrow \text{constant}$, then the simplest solution is $M_G = \text{constant}$, making stars at the constant rate f . In this case (Larson 1972), $Z = y_z(1 - e^{-v}) \rightarrow y_z$, where $v = M_{\text{tot}}/M_G - 1$. This model is too simple in the opposite sense to the above: almost all stars have the same value of Z , namely $Z = y_z$.

It is precisely to parameterize the large space between these two limits that an exactly soluble standard model is being advanced in the present work.

Figure 1 shows, for the specific case $k = 1$, how the gas fraction $M_G(t)/M_{\text{tot}}$ declines with time for two different models. I have shown the large t and small gas fraction portion of these curves. They are mostly unremarkable since the trends are so evident. For the case $k = 2$, a very similar suite of curves obtains, except that (approximately) the labels on ω must be increased by about 0.02.

Figure 2 displays a related consideration. Suppose one requires instead a set of parameters that make today's gas fraction about 5%. This figure shows the locus of ω versus galactic age T for which the gas mass $M_G(T)$ is exactly 5% of the star mass $S(T)$. It does, in that sense, delineate the parameter space (for the case $k = 1$ and $k = 2$) appropriate to our own Galaxy. The case $k = 3$ would demand still larger values of ω for each of the choices.

If the parameters are chosen such that $t_0 = (\pi/2 - \Delta)/\omega_1$ is less than the present galactic age, which is consistent with the assumption that infall has stopped, it is possible to write an explicit expression for the age T_β at which $M_G(T_\beta) = \beta S(T_\beta)$:

$$T_\beta(k=1) = -\frac{1}{\omega} \ln \left\{ \frac{\beta\omega}{1+\beta} \left[\frac{\omega_1}{\omega^2 + \omega_1^2} \left(\frac{\omega_1}{\omega} \sin \Delta + \cos \Delta - \frac{\omega}{\omega_1} e^{-\omega t_0} \right) + e^{-\omega t_0/\omega} \right] \right\},$$

$$T_\beta(k=2) = -\frac{1}{\omega} \ln \left\{ \frac{\beta\omega}{1+\beta} \left[\frac{\omega_1}{\omega^2 + 4\omega_1^2} \left[\frac{\omega}{\omega_1} \sin^2 \Delta + \sin 2\Delta + \frac{2\omega_1}{\omega} - e^{-\omega t_0} \left(\frac{\omega}{\omega_1} + 2 \frac{\omega_1}{\omega} \right) \right] + e^{-\omega t_0/\omega} \right] \right\}. \quad (27)$$

These are the loci ω versus T_β that are plotted in Figure 2, with $\beta = 0.05$. For a given model they enable selection of the value of ω that will yield 5% gas at a desired age. Those curves are, in fact, very accurately hyperbolae. Look, for example, at the pair $k = 1$ and $k = 2$ belonging to the parameters $\omega_1 = 0.3$ and $\Delta = 0.1$ (for which $t_0 = 4.9$). They satisfy to three significant figures the simpler relationships $\omega[T_\beta(1) - 1.2] = 3.095$ and $\omega[T_\beta(2) - 1.9] = 3.10$. Thus at chosen ω , $T_\beta(2)$ occurs 0.7 Gyr later than $T_\beta(1)$. Similarly, among the curves belonging to $k = 1$ and $\Delta = 0.1$, lowering ω_1 from 0.3 to 0.2 changes the hyperbolic locus to $\omega[T_\beta(1) - 1.4] = 3.20$. What has been done in Figure 2 is to label the curves within different parameter families, so that the effect of changing one parameter is visible. The third grouping, belonging to $k = 1$ and $\omega_1 = 0.3$, shows that increasing Δ from 0.1 to 1.0 decreases $T_\beta(1)$ at fixed ω by 1.6 Gyr. It is also clear that choosing $\beta = 10\%$ rather than 5% would result in a systematic shift toward lower ω of the families of curves. These results make it very easy to choose parameters to achieve a desired gas fraction. The reader needs only to remember that equation (27) is inapplicable if infall continues today ($t_0 > T$).

When a reliable description of galactic accretion has been established, it will be instructive to attempt to fit that $f(t)$ with appropriate choices of k , Δ , and ω_1 . Until then we can only hope that other observational constraints can narrow the description. It is in fact with that intent that I have sought an exactly soluble parameterization in which the nuclear cosmochronologies can also be evaluated.

b) Age-Metallicity Relation

The metallicity $Z(t)$ of newly forming stars increases with time according to equation (22) for these models. One test of chemical evolution is the metallicity of stars according to their age. Although the ages of stars are not easy to obtain with assurance, the

FIG. 2.—The locus of models (ω vs. T_β) resulting in a gas/star ratio $M_G(T_\beta)/S(T_\beta) = \beta = 0.05$. Here T_β is the galactic age rather than time. Loci for $\Delta = 0.1$, $\omega_1 = 0.3$ are shown for each of two values of $k = 1, 2$. Such a relation, calculated from eq. (27), allows a quick choice of parameters suitable to our Galaxy. For example, the $\omega_1 = 0.3$, $\Delta = 0.1$ locus is fitted accurately for $k = 1$ by the hyperbola $\omega(T_\beta - 1.2) = 3.095$, with analogous hyperbolae for other parameters. This result is valid only if infall has ceased, however ($T > t_0$). The shift of the locus with changes of ω_1 and Δ is also shown.

FIG. 3

FIG. 4

FIG. 3.—The age-metallicity relation for stars in the Galaxy taken from Twarog (1980) is displayed as points with error bars. The curve is $\log(Z/Z_{15})$, the calculated $Z(t)$ normalized to its value at $t = 15$, when $[\text{Fe}/\text{H}] = 0.1$, for the model $k = 1$, $\omega = 0.25$, $\omega_1 = 0.11$, and $\Delta = 0.01$. The absence of lower observed metallicity can be attributed, in this model, to the dearth of low- Z stars.

FIG. 4.—The fraction $S(<Z)$ of stars that formed with metallicity less than Z displayed as a function of $\log(Z/Z_{15})$. The linear model without infall ($k = 0$) has far too many stars having $[Z] < -0.7$ to fit the observations, but the $k = 1$ and $k = 2$ models chosen to achieve 5% gas mass at $t = 15$ do very well. The data would seem to lie roughly between the curves $k = 1$ and $k = 2$. Nonzero values for the initial abundance ($Z_0 \neq 0$) or infall abundance ($Z_f \neq 0$) would make each curve fall even more steeply toward low metallicity.

problem has been studied many times. In Figure 3 the $[\text{Fe}/\text{H}] \equiv \log(\text{Fe}/\text{H})/(\text{Fe}/\text{H})_\odot$ is displayed against age using Twarog's (1980) data. For comparison I have plotted equation (22) setting $Z_f = 0$ on the right-hand ordinate, where $[Z/Z_{15}] \equiv \log(Z/Z_{15})$ and $Z_{15} = Z(15)$ is the metallicity at $T = 15$, when the gas has fallen to 6% in the model $k = 1$. I have arbitrarily set $[Z/Z_{15}] = 0$ at $[\text{Fe}/\text{H}] = 0.1$. Although the resulting fit is moderately good, it is not strong evidence for the specific parameters ($k = 1$, $\omega = 0.25$, $\omega_1 = 0.11$, and $\Delta = 0.01$), because many other sets of parameters also give good fits. Nonetheless one sees that this model is suitable for describing these data and for eliminating that portion of parameter space that does not fit it.

The physical consequence of infall is that $Z(t)$ grows less rapidly with time than in the closed model ($k = 0$), for which $Z(t)$ grows linearly. Table 1 shows the growth equation for four values of k . For all values of k it is linear after t_0 , but the growth prior to t_0 is considerably less. Quantitatively this is easily seen from the ratio of $Z(t_0)$ to the value it would have had for $k = 0$. For $k = 1$ and $k = 2$ the results are easily shown to be $Z_1(t_0)/Z_0(t_0) = \cos \Delta/\omega_1 t_0 = \cos \Delta/(\pi/2 - \Delta) \approx 0.67$ for small Δ and $Z_2(t_0)/Z_0(t_0) = 0.5 + \sin 2\Delta/(2\pi - 4\Delta) \approx 0.50$ for small Δ . Thus if t_0 is a significant fraction of the galactic age, the overall spread of Z is reduced, up to a factor 2 for $k = 2$.

c) $S(Z)$ versus Z

Because $Z(t)$ grows quasi-linearly with time, one might fear the same excess of low- Z dwarfs that plagues the simple closed $k = 0$ model, where $Z(t)$ is linear in time. But that problem does not occur because the early rate of star formation can be small in these infall models. Star formation at first increases with time in those models for which $M_G(t)$ has a maximum at $t > 0$, such as the $\Delta = 0.1$ case in Figure 1. The way to make this test exactly is, of course, to consider equations (19) and (22) as parametric expressions for the normalized function $S(<Z)$ as a function of $\log Z$. This function is displayed in Figure 4 for three different values of k with parameters chosen to give 6% gas at $t = 15$. Case $k = 0$ is the simple closed exponential model for which too many low- Z stars result (assuming $Z \approx 0$ initially), and is independent of the parameters Δ and ω_1 (which do not exist for $k = 0$). These three k -values are compared for the case $Z_f = 0$, which is the hardest case to make right. The curve $k = 1$ is calculated for two specific choices ($\omega = 0.2$, $\omega_1 = 0.3$, $\Delta = 0.1$ and $\omega = 0.25$, $\omega_1 = 0.11$, $\Delta = 0.01$) in terms of $\log Z/Z_{15}$, where Z_{15} is the concentration at $t = 15$, when the gas mass is 6% in both examples. The fraction of stars with $S(<Z)$ is satisfactorily smaller for small metallicity if ω_1 is small enough that infall does not cease too early. The case $k = 2$, calculated with $\omega = 0.25$, $\omega_1 = 0.11$, and $\Delta = 0.01$ to obtain 10% final mass fraction in gas, is even more deficient in low- Z stars. Thus, there is no "G-dwarf problem" in this model. That infall is capable of solving that problem has been known since Larson's (1972) original argument. What is being put forward here is an exactly soluble realistic model for evaluating the data. The data themselves, as far as I can tell in a preliminary study, fall intermediate to the $k = 1$ and $k = 2$ cases in Figure 4. Of course, if $Z_f > 0$, the curves are compressed toward higher Z/Z_{15} , also alleviating the problem (e.g., Tinsley 1980).

d) Age Distribution and Mean Age of Dwarfs

In a linear model, where the birthrate of dwarfs is proportional to $M_G(t)$, the age distribution of dwarfs has the same functional shape as M_G . Simple closed models therefore have most of the dwarfs born at the beginning, because M_G decreases monotonically in time in those models. The present mathematical model has more realistic properties, since $M_G(t)$ may increase to its maximum at $t(\max M_G)$, such as in Figure 1. It need not be displayed further. On the other hand, the age distribution is a notoriously difficult observational problem.

It is interesting that the mean age of the dwarfs is also easily evaluated in this model. The mean formation time is

$$\begin{aligned}\langle t \rangle &= \frac{\int_0^{t_0} t dS/dt dt}{S(t)} = \frac{\int_0^{t_0} t M_G dt}{\int_0^{t_0} M_G dt} = t - \frac{\int_0^{t_0} I_k(t') dt'}{I_k(t)}, & t < t_0, \\ &= \frac{t_0 I_k(t_0) - \int_0^{t_0} I_k(t') dt' + \sin^{-k} \Delta [e^{-\omega t_0} (t_0/\omega + 1/\omega^2) - e^{-\omega t} (t/\omega + 1/\omega^2)]}{I_k(t_0) + \sin^{-k} \Delta (e^{-\omega t_0} - e^{-\omega t})/\omega}, & t > t_0,\end{aligned}\quad (28)$$

whereas the mean age $\langle \text{age} \rangle = t - \langle t \rangle$. Again one needs only integrate the simple functions I_k stated earlier for parameter $\lambda = -\omega$. The results are, for this particular quantity, too lengthy for explicit display, but they are easily obtained. If t_0 occurs early, the stars will have a high average age, because a large fraction of them form before t_0 if Δ is also small (see Fig. 1, for example). Only if infall was important over most of the galactic lifetime does one achieve average ages that are significantly less than the galactic age. And the higher the value of the integer k , the younger is the average star. For example, with $k = 2$ and $t_0 > 15$, the average age can be less than half the galactic age, as also was found in model number 1 (Clayton 1984a).

A related concept is the ratio of the average prior rate of star formation to the present rate:

$$\begin{aligned}\frac{\langle \psi(0-t) \rangle}{\psi(t)} &= \frac{S(t)/t}{\omega M_G(t)} = t^{-1} \left[\frac{\sin \Delta}{\sin(\omega_1 t + \Delta)} \right]^k e^{\omega t} I_k(t, -\omega), & t < t_0, \\ &= t^{-1} [\omega \sin^k \Delta I_k(t_0, -\omega) + e^{\omega(t-t_0)} - 1], & t > t_0,\end{aligned}\quad (29)$$

where the constant factor $(1 - R)$ has been divided from numerator and denominator in the first equality and the second equality comes from the ratio of (19) to (17). For example, in the model $k = 1$, $\omega = 0.2$, $\omega_1 = 0.3$, and $\Delta = 0.1$ for which $t_0 = 4.9$ the ratio at $t = 15$ is $\langle \psi \rangle / \psi = 4.7$, the ratio S/M_G being 14.06 at $t = 15$ in that model. By contrast the closed ($k = 0$) model giving the same final gas fraction (5%) at $t = 15$ has $\langle \psi \rangle / \psi = (e^{\omega t} - 1)/\omega t = 6.4$. A wide range of ratios $\langle \psi \rangle / \psi$ is available to the parameter space of these models.

e) BASIC Program for M_G, f, S, Z

Because this model has such potential for a standard model, I wish to make its initial implementation easier for other workers. To this end I present in Table 2 a short program written in MICROSOFT BASIC 80 suitable for most microcomputers. It calculates explicitly only $M_G(t), f(t), S(t)$, and stable metallicity $Z(t)$ and $\log(Z/Z_{15})$ for $k = 1$ and $k = 2$; but ample space exists for inserting the corresponding expressions for larger k or for adding radioactive concentrations or other quantities. The parenthetical remarks on the program are not part of it, but are noted to ease reading by workers not familiar with BASIC and to clarify notation. The results of running this program with the indicated input DATA are contained in Table 3, with units of mass, time, and concentration being, respectively, M_{G0} , 10^9 yr, and y_z . The program will print Table 3 in a slightly different format (lines 330 to 360), but I have reorganized the table for journal publication and added $t = 0$ to the list of results. The actual model tabulated is an interesting one for galactic studies.

f) Metallicity Gradients

I have presented this "standard model" as if the parameters k, Δ, ω_1 , and ω are fixed for the entire Galaxy. That is likely to not be reasonable (e.g., Tinsley and Larson 1978). In that case we may choose them for studies of the solar neighborhood to be the values that were appropriate to infall and star formation in our annular zone. During considerations of an entire galaxy, however, whose gradients in both metallicity and gas fraction are known to exist, we need only choose the parameters to vary over the disk in a sensible way. One problem with this is that the parameter k is not continuous unless numerical complications in the evaluation of I_k are acceptable. My point is only to state that the abundance gradients can be modeled parametrically, although I will not do so here.

One might at first argue that imagining the parameter ω to be a function $\omega(r)$ of radius physically violates the simple assumption of the linear model, that $\psi(t) \propto \omega M_G(t)$. That is not necessarily the case, because the scale height of the disk also varies with r . And therefore, the mass density varies with r even if the surface density were constant. For the linear model to be physically meaningful it is sufficient that at fixed r , where the scale height $h(r)$ might be considered fixed, the star formation rate be proportional to $M_G(t)$ at that value of radius. The increase of gas fraction with radius ensures the existence of a metallicity gradient. Tinsley and Larson's (1978) discussions of their numerical models are very relevant and interesting for the question of trying to establish an analytic standard model.

g) Nuclear Cosmochronology

A major asset of this mathematical model is that the cosmochronologies can be evaluated, at least in the case of metal-poor infall. Although equation (21) is an exact solution for the concentration of a radioactive species, it is not physically plausible to assume that Z_f is constant for a radioactive species in the infall—unless, of course, $Z_f = 0$. So in this work I restrict myself to infall of metal-poor gas only, such that $Z_f \ll Z$, in which case $Z_f = 0$ may not be unphysical. This is a problem on which more knowledge is needed. How much nucleosynthesis, metallicity, etc., resided in the halo gas that later accreted to the disk? In any case, for numerical example take equation (21) with $Z_f = 0$. Likewise take $Z_0 = 0$ (no initial spike).

The real importance of nuclear cosmochronology is that time is an essential ingredient. Others have shown that for a wide class of models time can be eliminated in considering the interrelations between metallicity, star mass, gas mass, the distribution $S(Z)$, etc. But not so nuclear cosmochronology. It is for that very reason that I have sought a realistic standard model in which their concentrations can be evaluated explicitly. As far as I know, only the simple closed exponential model for the gas and the model of

TABLE 2
BASIC PROGRAM FOR $M_G(t), f(t), S(t), Z(t)$ FOR $k = 1, 2^a$

```

10 READ w, w1, del                      (names for  $\omega, \omega_1, \Delta$ )
20 DATA 0.25, 0.11, 0.01              (change data for different model)
30  $t_0 = (1.571 - \text{del})/w1$              (infall duration  $t_0$ ;  $\emptyset \equiv \text{zero}$ )
40 LPRINT "for k = 1, 2, w = "; w; ", w1 = "; w1; ", Del = "; del; "t0 = ";
   t0
50  $e\emptyset = \exp(-w*t\emptyset)$                 (define for writing ease)
60 sd = sin(del)                        (define for writing ease)
70 cd = cos(del)                        (define for writing ease)
80  $wr1 = w*w1/(w^2 + w1^2)$             (define for writing ease) (for k = 1)
90  $wr2 = w*w1/(w^2 + 4*w1^2)$           (define for writing ease) (for k = 2)
100 FOR t = 15 TO 1 STEP -1              (choose range of t)
110 e = exp(-w*t)
120 st = sin(w1*t + del)
130 ct = cos(w1*t + del)
140 IF t > t0 GOTO 240                  (branch in solutions)
150 z1 = w/w1*(cd - ct)/st              (for k = 1, t < t0)
160 z2 = 0.5*w/w1*(w1*t - st*ct + sd*cd)/st^2 (k = 2, t < t0)
170 m1 = st*e/sd                        (let m1 =  $M_G$  for k = 1)
180 m2 = (st/sd)^2*e                    ( $M_G$  for k = 2)
190 f1 = w1*ct*e/sd                     (infall for k = 1)
200 f2 = 2*w1*ct*st*e/s^2
210 s1 = wr1/sd*(w*sd/w1 + cd - e*(w*st/w1 + ct)) (s1 = S(t), k = 1)
220 s2 = wr2/sd^2*(w*sd^2/w1 + 2*sd*cd + 2*w1/w - e*(w/w1*st^2 + 2*st*ct + 2*w1/w)) (S(t) for k = 2)
230 GOTO 320                             (since t < t0)
240 m1 = e/sd                           ( $M_G(t)$  for k = 1 and t > t0)
250 m2 = e/sd^2
260 f1 = 0                               (no infall for t > t0)
270 f2 = 0
280 z1 = w*cd/w1 + w*(t - t0)
290 z2 = 0.5*w/w1*(w1*t0 + sd*cd) + w*(t - t0)
300 s1 = wr1/sd*(w*sd/w1 + cd - w*e0/w1) + (e0 - e)/sd
310 s2 = wr2/ds^2*[w/w1*sd^2 + 2*sd*cd + 2*w1/w - e0*(w/w1 + 2*w1/w)]
   + (e0 - e)/sd^2
320 IF t = 15 THEN z115 = z1             (record Z(15))
325 IF t = 15 THEN z215 = z2
330 LPRINT "t = "; t; "M1 = "; m1; "f1 = "; f1; "S1 = "; s1
340 LPRINT "Z1 = "; z1; "log (Z1/Z1(15)) = "; log (Z1/Z115)/2.3
350 LPRINT; "(k = 2)", "M2 = "; m2; "f2 = "; f2; "S2 = "; s2
360 LPRINT; "Z2 = "; z2; "log (Z2/Z2(15)) = "; log (Z2/Z215)/2.3
370 LPRINT
380 NEXT t
390 LPRINT
400 END

```

^a Microsoft BASIC 80. $Z(t)$ = stables only. Masses are in units of M_{G0} , time is in units of 10^9 yr, and Z is in units of y_z .

TABLE 3
RESULTS OF TABLE 2 PROGRAM^a (units: $M_{G0}, 10^9$ yr, y_z)

t	$M1$	$f1$	$S1$	$Z1$	$\log \frac{Z1}{Z(15)}$	$M2$	$f2$	$S2$	$Z2$	$\log \frac{Z2}{Z2(15)}$
15.....	2.35	0	35.8	2.47	0	235	0	2060	1.99	0
14.....	3.02	6.9(-3)	35.1	2.23	-0.046	302	1.38	1993	1.74	-0.058
13.....	3.84	0.056	34.3	1.99	-0.094	381	11.0	1908	1.52	-0.118
12.....	4.84	0.131	33.2	1.78	-0.143	470	25.4	1802	1.32	-0.177
11.....	6.00	0.242	31.9	1.59	-0.193	563	45.4	1673	1.16	-0.236
10.....	7.35	0.401	30.2	1.41	-0.245	659	71.9	1520	1.01	-0.295
9.....	8.87	0.626	28.2	1.24	-0.300	746	105	1344	0.875	-0.357
8.....	10.52	0.937	25.7	1.08	-0.359	817	146	1148	0.754	-0.421
7.....	12.22	1.359	22.9	0.934	-0.424	860	191	938	0.643	-0.490
6.....	13.86	1.924	19.6	0.791	-0.496	860	239	722	0.540	-0.567
5.....	15.22	2.670	16.0	0.653	-0.579	808	283	512	0.443	-0.653
4.....	16.00	3.644	12.1	0.520	-0.678	696	317	323	0.350	-0.755
3.....	15.75	4.899	8.09	0.390	-0.804	525	326	169	0.261	-0.882
2.....	13.83	6.496	4.34	0.262	-0.976	315	296	63.4	0.175	-1.055
1.....	9.32	8.505	1.39	0.136	-1.263	111	204	10.9	0.091	-1.341
0.....	1	11	0	0	...	1	22	0	0	...

^a For $k = 1, 2, w = 0.25, w1 = 0.11, \text{Del} = 0.01, t_0 = 14.19$.

constant gas and star formation fed with constant infall have been solved exactly. These new models span the space in between in a specific useful parametric representation. To avoid from the outset a recurring confusion, one must distinguish between the “exponential models of cosmochronology” popularized by Fowler (1972) and the exponential gas of the closed linear model. In the latter $k = 0$ model the concentration $Z(t)$ is linear in t , and the radioactive concentrations also have relations that Fowler (1972) would term “constant nucleosynthesis.” Fowler’s point was quite different. Stated very simply, it was that if the age distribution of the elements in the solar system (not in the Galaxy as a whole) is monotonic, why not approximate that monotonic function by an exponential with a suitably chosen parameter? By doing so Fowler delineated much of the range of possibilities in nuclear cosmochronology. But that parameterization is not so easily and directly tied to other aspects of galactic evolution as is the model I advance here, where everything is physically motivated and exactly soluble.

The essence of nuclear cosmochronology is not simply the gas concentrations $Z(t)$, for they depend upon nuclear yields y_z that are difficult to calculate. It is rather in the ratio of the concentration $Z(t)$ to the concentration it would have had if the nucleus were stable that the time information lies. I therefore define the “remainder”

$$r(t) = \frac{Z(t, \lambda)}{Z(t, \lambda = 0)} = e^{-\lambda t} I_k(t, \lambda) / I_k(t, 0), \quad t < t_0,$$

$$= \frac{\sin^k \Delta I_k(t_0, \lambda) e^{-\lambda t} + (1 - e^{-\lambda(t-t_0)})/\lambda}{\sin^k \Delta I_k(t_0, 0) + t - t_0}, \quad t > t_0, \quad (30)$$

where the second equality comes from the ratio of (21) to (22) with infall Z_f set to zero. This remainder r may as a mnemonic be thought of as “the fraction remaining,” with the understanding that it refers to the gas at that time, not to the total galactic content.

There is a significant and fascinating feature to equation (30); namely, it does not depend upon ω . This result is known historically only for the simple $k = 0$ model without infall, wherein $r_0 = (1 - e^{-\lambda t})/\lambda t$ independent of the exponential rate of decrease of the gas (true in that model). But we here see that the same result is true for this entire class of linear models provided that the infall is metal free ($Z_f = 0$) and the initial gas is metal free ($Z_0 = 0$). This rather surprising result is best seen from the general solution, equation (12), which shows that, if $Z_0 = Z_f = 0$, the ratio $r \equiv Z(t, \lambda)/Z(t, 0)$ is independent of the rate ω for the gas mass to convert to stars. As a result one finds that the nuclear cosmochronologies depend only upon the parameters k , ω_1 , and Δ in this class of standard models. Thus, a correct nucleocosmochronological fit determines constraints on k , ω_1 , and Δ , whereas the other astronomical evidence constrains also ω .

Because of remaining uncertainties on the production ratios for the radioactive chronometers, I will not try to isolate a “best model” here, preferring instead to illustrate the behavior and mathematics. I will show two formats whose meanings are easily understood. In the first one I arbitrarily choose an interesting model and display the behavior of the various remainders as a function of time, illustrating for interest the time when the ratios fit those inferred for the early solar system. The decay rates per 10^9 yr of the species to be considered here are, in increasing order: $\lambda(^{232}\text{Th}) = 0.0499$, $\lambda(^{238}\text{U}) = 0.154$, $\lambda(^{235}\text{U}) = 0.972$, and $\lambda(^{244}\text{Pu}) = 8.48$. So in Figure 5 I take as given the parameters $\omega_1 = 0.2$ and $\Delta = 0.1$ and plot $r(t)$. For $\omega = 0.2$ these parameters yield 7.5% gas for $k = 1$ and 9.3% gas for $k = 2$ after $T = 15$, so it is an interesting pair of parameters. The remainders for $k = 1$ are plotted as solid curves for ^{232}Th , ^{238}U , and ^{235}U , whereas the ratios $r(238)/r(232)$ and $r(235)/r(238)$ are plotted as dashed curves for $k = 0, 1, 2$. Each remainder decreases monotonically in time, with ^{235}U of course the fastest of the three. That behavior is expected and is familiar from the simple closed exponential gas models, for which the remainder is simply $r_0 = (1 - e^{-\lambda t})/\lambda t$. The several $r(t)$ do not decrease as rapidly as in the simple closed ($k = 0$) model, however. The remainders r_1 for $k = 1$ and r_0 for the simple closed $k = 0$ case both fall from unity at $t = 0$, with r_1 falling less rapidly, followed by a reconvergence after infall ceases, until by $t = 15$ the remainder r_1 is, for the case ^{235}U , a factor 1.08 times greater than the value r_0 (not shown) for the closed model. This final ratio is only slightly greater than unity because, for $\omega_1 = 0.2$ and $\Delta = 0.1$, the infall ceased at $t_0 = 7.35$, about half the galactic age. For

FIG. 5.—The remainder $r(t) \equiv Z(t, \lambda)/Z(t, 0)$ compares the radioactive abundance in the gas to the value it would have had were it stable. The time dependence for $\omega_1 = 0.2$ and $\Delta = 0.1$ is illustrated for ^{235}U , ^{238}U and ^{232}Th . Galactic cosmochronology depends strongly on the ratios of remainders $r(235)/r(238)$ and $r(238)/r(232)$, which are shown as dashed curves. Boxes on the right-hand ordinate are the observed ratios of remainders if Fowler’s (1972, 1978) production ratios are adopted. The specific $k = 1$ model shown is, therefore, consistent with observations if the Sun were born at $9 < t_\odot < 12$, which would require a 13–17 Gyr old Galaxy. Model comparisons for different ω_1 are in Figs. 6 and 7.

$t_0 = 14.7$ on the other hand ($\omega_1 = 0.1$) the ratio is $r_1/r_0 = 1.53$ at $t = 15$. This would be the most convenient way to visually scale the results of the calculations: to display them as $r_k(t)/r_0(t)$, so that they are of order unity independent of the decay rate λ . This ratio is analogous in spirit to Fowler's (1972) exponential parameterization, wherein the deviations from r_0 are labeled by a second time scale. And I again note, to dispel confusion, that the value r_0 is sometimes called the value for "constant nucleosynthesis" not because galactic nucleosynthesis occurs at a constant rate, for it does not, but rather because *the age spectrum of the elements in the gas is flat*. But there is a big difference between these models and Fowler's parameterization—a difference in sign. Because these models have infall, the remainders r_k/r_0 are greater than unity; i.e., elements look "younger" than in the closed-model case. But in Fowler's parameterization the elements look "older" than the closed case because $r/r_0 < 1$ for those models.

It is of interest to compare these remainders with estimates of them, and I will concentrate on Fowler's (1972) estimates because they are so well reasoned and have been so influential to the field of nuclear cosmochronology. The uranium isotope ratio in the gas at any time is

$$\frac{Z(^{235}\text{U})}{Z(^{238}\text{U})} = \frac{y(^{235}\text{U})}{y(^{238}\text{U})} \frac{r(^{235}\text{U})}{r(^{238}\text{U})}, \quad (31)$$

a ratio that is known to have had the value 0.313 at 4.6 Gyr ago. Division of that ratio by the production ratio, which Fowler has long maintained should be $y(^{235})/y(^{238}) = 1.42 \pm 0.19$, gives the required ratio of remainders when the solar system formed: $r_{\odot}(^{235})/r_{\odot}(^{238}) = 0.22 \pm 0.3$. That band is illustrated on Figure 5 as boxes on the right ordinate. One sees that the gas ratio $r(^{235})/r(^{238})$ falls to a value that low in that particular ($k = 1$, $\omega_1 = 0.2$, $\Delta = 0.1$) model between about 9 and 13 Gyr, suitable times for solar birth. Similarly shown is the required value $r_{\odot}(^{238})/r_{\odot}(^{232}) = 0.72 \pm 0.07$ obtained from an abundance ratio (now) $\text{Th}/\text{U} = 4.0 \pm 0.2$ and Fowler's revised (1978) production ratio $y(^{232})/y(^{238}) = 1.8 \pm 0.1$, somewhat higher than his 1972 value 1.65 ± 0.15 . The particular $k = 1$ model displayed in Figure 5 passes into that band for $t > 6$ Gyr and out of it for $t > 12$. Thus this $k = 1$ model would be formally concordant to Fowler's production ratios for $9 < t_{\odot} < 12$. Models with continuing infall do not lead to easy concordance (Clayton 1984a).

The concordance behavior of the class of $k = 1$ models is shown for different values of $t_0 = (\pi/2 - \Delta)/\omega_1$ in Figures 6 and 7 for $\Delta = 0.1$. It should be said right away that these ratios of remainders are very insensitive to Δ , although one can see that they do depend upon t_0 through ω_1 . Note that infall has ceased for models having $\omega_1 > 0.1$. Although all models pass into the concordant band, notice that the model having continuing infall ($\omega_1 = 0.05$) does so only for $t \gtrsim 12$, and never does reach the most probable value $r(^{235})/r(^{238}) = 0.22$. This confirms the properties of standard model number 1 (Clayton 1984a), which have continuing but small infall and which allow concordance only for galaxies older than $t \gtrsim 15$. The curves are raised still higher for $k > 1$, as was

FIG. 6

FIG. 7

FIG. 6.—The ratio of remainders for ^{235}U and ^{238}U are shown as a function of time for several different values of ω_1 , which is of importance in determining the end $t_0 = (\pi/2 - \Delta)/\omega_1$ of galactic infall. Larger t_0 requires an older birth date for the Sun, only passing into the traditionally expected band for $t_{\odot} > 12$ if $\omega_1 = 0.05$, which would require the galactic age today to be $T > 16$. Models with early termination ($t_0 < 7.3$ if $\omega_1 \geq 0.2$) of infall, on the other hand, have no difficulty striking concordance for birth dates $t_{\odot} > 9$. One concludes that metal-free infall ceased before the Sun was born or the Galaxy is older than $T = 15$ Gyr or the production ratio $y(^{235})/y(^{238}) = 1.4 \pm 0.2$ (Fowler 1972, 1978) is too large. Although these curves are calculated for $\Delta = 0.1$, their sensitivity to Δ is very weak.

FIG. 7.—Same as Figure 6 but for the ratio for remainders for ^{238}U and ^{232}Th . The main conclusion is that increasing the duration of infall (lower ω_1) increases the allowable upper limit to the solar birth date (up to $t_{\odot} = 14$ if $\omega_1 = 0.05$).

shown for $k = 2$ in Figure 5, because $M_G(t)$ peaks later for $k > 1$ and hence the elements look younger for $k > 1$. What Figure 7 shows is that $r(238)/r(232)$ also allows an older birth date for the Sun if infall continues, up to $t_\odot = 14$ if $\omega_1 < 0.1$. Thus a model with $\omega_1 = 0.05$ would require $12 < t_\odot < 14$, or $16 < \text{galactic age} < 19$.

We must be careful, however, before asserting that younger galaxies are inconsistent with continuing infall, because the production ratio $y(235)/y(238)$ may actually be smaller than 1.4 ± 0.2 , as was concluded on grounds related to beta-delayed fission by Wene and Johansson (1976) who obtained $y(235)/y(238) = 0.89$, and then again by Thielemann, Metzinger, and Klapdor (1983) who obtain $y(235)/y(238) = 1.24$. Because my purpose here is to advance a convenient model, I will not enter into this nuclear question at this time; however, the reader may study Thielemann *et al.*'s thorough but readable discussion of this point. However, I must note that their calculated $^{232}\text{Th}/^{238}\text{U} = 1.39$ is also much smaller than Fowler's 1.8, reducing the required value of $r(238)/r(232)$ to about 0.55 ± 0.05 , and demanding an older galaxy.

h) Extinct Radioactivities

Several important nuclei (i.e., ^{26}Al , ^{129}I , ^{244}Pu) are now extinct, and the issue has been often raised of their applicability to nuclear cosmochronology. In the present work I address only the issue of their expected *average* abundance in the interstellar gas. A last nucleosynthesis "spike" may be easily added to the models presented, but it is instead the mathematics of the continuous (average) nucleosynthesis that is our immediate concern.

The limit $Z(t)$ as $\lambda \rightarrow \infty$ (short-lived) is easily obtained, either from setting $dZ/dt = 0$ in equation (10) or by evaluating (12) for very large λ :

$$\begin{aligned} Z(t) \rightarrow \frac{y_z \omega}{\lambda + (f/M_G)} &= \frac{y_z \omega}{\lambda + k\omega_1 \cot(\omega_1 t + \Delta)}, & t < t_0 \text{ as } \lambda \rightarrow \infty, \\ &= y_z \omega / \lambda, & t > t_0, \end{aligned} \quad (32)$$

where the second equality for $t < t_0$ is expressed for the models of this paper. The remainder for large λ becomes

$$\begin{aligned} r = \frac{Z(t, \lambda)}{Z(t, 0)} &\rightarrow \left[\frac{\sin(\omega_1 t + \Delta)}{\sin \Delta} \right]^k \{I_k(t, 0)[\lambda + k\omega_1 \cot(\omega_1 t + \Delta)]\}^{-1}, & t < t_0, \\ &\rightarrow \{\lambda[t - t_0 + \sin^k \Delta I_k(t_0, 0)]\}^{-1}, & t > t_0, \end{aligned} \quad (33)$$

where the approximation is true if $t \gg \tau \equiv 1/\lambda$. For the case $k = 0$ equation (33) goes to $r = \tau/T$, the oft-used estimate for the fraction remaining of a radioactive species. For $k > 0$, however, that estimate is too small by a modest factor in these exact models. For those models belonging to $k = 1$ the remainder is greater than τ/T by the factors 1.2 ($\omega_1 = 0.2$), 1.5 ($\omega_1 = 0.1$), and 1.9 ($\omega_1 = 0.05$), increasing roughly linearly with the infall duration t_0 . For larger $k > 1$, the enhancement factor is still greater, having the value $k + 1$ in standard model 1 (Clayton 1984a), which is characterized by continuing infall. The present models can also characterize continuing infall if $\omega_1 < 0.1$. The reason for this enhancement of the remainder for extinct radioactivities lies in a smaller stable abundance, rather than in the radioactivities themselves, for if we compare the remainders of two short-lived activities we obtain $r(\tau_1)/r(\tau_2) = \tau_1/\tau_2$. Thus, the enhancement factor enters in the expected average $^{129}\text{I}/^{127}\text{I}$ ratio, but not in the $^{107}\text{Pd}/^{129}\text{I}$ ratio, for example. This is of importance in relating meteoritic concentrations to the interstellar medium, becoming more important for $^{129}\text{I}/^{127}\text{I}$ as k becomes a large integer and as ω_1 becomes small.

The use of extinct radioactivities in cosmochronology is notoriously complicated by issues of the last "spike," if in fact it occurred at noticeable levels, and by the so-called free-decay interval. These issues will not be addressed here except to recommend the mixing model of interstellar phases (Clayton 1983) as a better model of the free-decay interval, some model of which must be used to relate the average interstellar radioactive concentrations to those actually found in the forming solar system. On top of that, I think that an exact model from the family that I have here introduced, chosen to satisfy the various astronomical constraints, is a more realistic framework for nuclear cosmochronology than are more arbitrary parameterizations.

Returning to the ^{26}Al concentration in today's interstellar medium, I would add that the fraction remaining from supernova nucleosynthesis is changed by the infall only by the modest enhancement factor $\lesssim (k + 1)$ from the simpler closed-model arguments given by Clayton (1984b). My initial impression is that the range $k = 1$ to 3 will be adequate to find a good overall model, so that the factor $(k + 1)$ is nowhere near large enough to alter my previous conclusion that continuous galactic supernova nucleosynthesis is not the origin of the observed ^{26}Al γ -ray line. All of the conclusions of Clayton (1984b) remain valid in models of this type.

IV. GENERAL REMARKS ON OTHER ANALYTIC SOLUTIONS

Having examined standard model number 2, we can usefully compare its formulae with other models in which the infall also goes smoothly to zero at $t = t_0$. Writing as in the first section the substitution $f(t) = F(t)e^{-\omega t}$, we see that any polynomial $F(t)$ having its smallest positive root at $t = t_0$ would suffice. The most interesting and simple of such forms would seem to be

$$F(t) = f_0 \left(\frac{t + \Delta}{\Delta} \right)^{k-1} \left(\frac{t_0 - t}{t_0} \right), \quad (34)$$

where k is a positive integer and f_0 is an arbitrary initial infall rate. The factor $(t + \Delta)^{k-1}$ still allows $f(t)$ to increase at small times if $k > 1$ before turning quasi-exponential, as in the parameterization of § III and in the analogous models having continuing infall (Paper I), but the factor $t_0 - t$ makes it vanish smoothly at a parametric time t_0 . The initial infall f_0 is allowed to be arbitrary, in contrast to Paper I where it was chosen to be $f_0 = kM_{G0}/\Delta$ in order to achieve the simplest of all representations and in contrast to standard model number 2 where it is chosen to be $f_0 = k\omega_1 \cot \Delta$ for exactly the same reason.

Noting that $t_0 - t$ can be written $t_0 + \Delta - (t + \Delta)$, it follows that equation (34) leads

$$G(t) = f_0 \left\{ \frac{(t_0 + \Delta)\Delta}{t_0 k} \left[\left(\frac{t + \Delta}{\Delta} \right)^k - 1 \right] - \frac{\Delta^2}{t_0(k+1)} \left[\left(\frac{t + \Delta}{\Delta} \right)^{k+1} - 1 \right] \right\}, \quad t < t_0,$$

$$= G(t_0), \quad t > t_0. \quad (35)$$

The advantage of expressing G as a polynomial in $(t + \Delta)/\Delta$ is that the integrals occurring in equations (6) and (12) then become the same ones $I_k^{(0)} \equiv \int_0^t [(t' + \Delta)/\Delta]^k e^{-\omega t'} dt'$ that were defined in Paper I (designed here by superscript I to avoid confusion with eq. [20]).

Routine algebra then shows that for this parameterization of truncated infall the star mass $S(t)$ and the concentration $Z(t)$ both depend upon an identical function $\phi(t)$:

$$S(t) = \omega \phi(t, -\omega), \quad t < t_0, \quad (36)$$

$$Z(t) = \frac{y_z \omega e^{-\omega t}}{M_0 + G(t)} \phi(t, \lambda), \quad t < t_0, \quad (37)$$

where the function $\phi(t)$ is, for the truncated polynomial above,

$$\phi(t) = \left[M_0 - \frac{f_0 \Delta(t_0 + \Delta)}{t_0 k} + \frac{f_0 \Delta^2}{t_0(k+1)} \right] I_0^{(0)}(t) + \frac{f_0 \Delta(t_0 + \Delta)}{t_0 k} I_k^{(0)}(t) - \frac{f_0 \Delta^2}{t_0(k+1)} I_{k+1}^{(0)}(t) \quad (38)$$

expressed in terms of the functions $I_k^{(0)}(t)$ defined in Paper I. Note that $\phi(t, \lambda)$ depends upon $I_k^{(0)}(t, \lambda)$ and $\phi(t, -\omega)$ depends upon $I_k^{(0)}(t, -\omega)$, which are the same functions with λ replaced by $-\omega$. In linear star formation models, all infall rates admit a function ϕ satisfying (36) and (37). Because this particular form of ϕ consists of elementary known functions of time, the evaluation of these results for any choice of parameters is easy. The equations just appear to be more complicated because they have more terms. Because of the recursion relation between I_k and I_{k+1} (Clayton 1984a), the function $\phi(t)$ can in fact be written in terms of a single function I_k as follows:

$$\phi(t, \omega) = \left[M_0 - \frac{f_0 \Delta(t_0 + \Delta)}{t_0 k} + \frac{f_0 \Delta^2}{t_0(k+1)} \right] \left(\frac{1 - e^{-\omega t}}{\omega} \right) - \frac{f_0 \Delta^2}{\omega t_0(k+1)} \left[1 - \left(\frac{t + \Delta}{\Delta} \right)^{k+1} e^{-\omega t} \right] + \frac{f_0 \Delta}{t_0} \left(\frac{t_0 + \Delta}{k} - \frac{1}{\omega} \right) I_k^{(0)}(t, -\omega) \quad (39)$$

with $-\omega \rightarrow \lambda$ for the function $\phi(t, \lambda)$ appearing in $Z(t)$. This form is easier than equation (38) to evaluate because only $I_k^{(0)}$ must be evaluated. It is a rapid calculator program to do so.

Just as in the standard model number 2 of § III, the solutions (36) and (37) are valid for $t \leq t_0$, while infall still occurs. For $t > t_0$ they must be continued in the same way as in § III; e.g., $M_G(t) = M_G(t_0) \exp[-\omega(t - t_0)]$, $Z(t) = Z(t_0) + y_z \omega(t - t_0)$, etc. Thus this model is also easy to implement, although in this work I have been content to use the standard choice number 2 which is easier to evaluate because it is so compact. But a few words about a trial comparison will be meaningful. Consider only the simplest case, $k = 1$, characterizing monotonically declining infall truncated in both cases at $t = t_0$. The standard model number 2 with $\omega = 0.25$, $\omega_1 = 0.11$, and $\Delta = 0.01$ was tabulated for illustrative purpose in Table 3, where we see that the total mass is increased from unity to 38.167 by the time $t_0 = 14.189$ when infall ceases. The initial infall was $f(0) = \omega_1 \cot \Delta = 11$, all in units of M_{G0} . Let us compare that to the truncated polynomial infall for $k = 1$, namely $f(t) = f_0(1 - t/t_0)e^{-\omega t}$, for which $G(t) = f_0(t - t^2/2t_0)$. Note that parameter Δ disappears for $k = 1$. If we take $t_0 = 14.189$ to match, we easily find that the total mass increases to the same $M_{\text{tot}} = M_{G0} + \omega \phi(t_0, -\omega) = 38.167 M_{G0}$ if the initial infall is chosen to be $f_0 = 12.05$, not greatly different from standard model number 2. Furthermore, the ratio of gas to stars at $t = 15$ would then be $M_G/S = 0.056$, whereas in the Table 3 model at $t = 15$ it is $M_G/S = 2.35/35.8 = 0.066$, so these two $k = 1$ models with truncated infall at $t_0 = 14.189$ are very little different. Choosing between them would be largely a matter of preference.

Of course, it is mathematically more general that the initial infall f_0 be a free parameter, as it is in this truncated polynomial result. With some sacrifice of compactness, this is easily done also for the standard model of § III. Multiply that initial infall by a constant A_f , so that $f(t) = A_f k \omega_1 \cot(\omega_1 t + \Delta) [\sin(\omega_1 t + \Delta)/\sin \Delta]^{k-1} e^{-\omega t}$, in which case the function $G(t) = A_f [(\sin(\omega_1 t + \Delta)/\sin \Delta)^k - 1]$, and $M_G(t) = [M_{G0} + G(t)]e^{-\omega t}$. All other observables are evaluated in the same way as before, resulting merely in more terms because of the arbitrary A_f . If A_f is chosen to equal M_{G0} , of course, the solutions contract to the more compact ones displayed in § III. I emphasize again, however, that it is easy to keep A_f as a free parameter if one prefers to select a specific form for $f(t)$ and then vary its magnitude.

The standard model number 1 (Clayton 1984a) with continuing infall would be generalized in exactly the same way if one wishes the initial infall to remain a free parameter, i.e., writing

$$f(t) = f_0 \left(\frac{t + \Delta}{\Delta} \right)^{k-1} e^{-\omega t}.$$

For then

$$G(t) = f_0 \Delta / k \left[\left(\frac{t + \Delta}{\Delta} \right)^k - 1 \right]$$

and the evaluation proceeds as before, reducing exactly to the simple standard model if $f_0 = k M_{G0}/\Delta$.

It will also be evident that a wide number of other exact solutions can be generated by choosing a useful form for $F(t)$. The function $\phi(t)$ appearing in equations (36) and (37) is in general $\phi(t, -\omega) = \int_0^t [M_{G0} + G(t')]e^{-\omega t'} dt'$. But it hardly seems justifiable to present other solutions explicitly. I have instead presented here and in Clayton (1984a) a few simple ones that are likely to be adequate for simple models of galactic evolution with infall.

V. CONCLUSION

I have advanced some exactly soluble families of models for the chemical evolution of the galaxy; and I suggest adopting them as a "standard model" for evaluating the classic lines of evidence on the subject. They are almost as simple mathematically as the simple closed model so often used, but allow great freedom for parameterizing galactic infall and the history of the star formation rate. These models have the physical advantage of being solutions of the linear dependence of star formation rate upon gas mass, so that the parameterizations become a class of physical models in addition to a class of merely useful parameterizations.

In subsequent papers I plan to treat these and many additional results, subjecting them where possible to the known data. My primary concerns about the usefulness of this model are:

1. Perhaps the infall abundance X_f is not constant in time, as it surely is not for radioactive nuclei, and as it is not expected to be if halo nucleosynthesis continued enriching the infalling gas for an appreciable time.
2. Perhaps the linear model is physically misleading, and a higher dependence of star formation rate upon M_G is true. This is being examined in a third paper in this series of exact solutions. The referee of this paper has cautioned that a single appropriate power may not be definable, because star formation is a multistep process of unusual complexity with a large number of key parameters. Key features of chemical evolution may be overlooked by any specific choice of exponent. This uncertainty will be reduced only by continuing interplay with differing perspectives.
3. Perhaps radial transport of matter in the disk is of such importance that the concept of a historically identifiable solar neighborhood is not meaningful. In this case the basic differential equations must be altered. A similar problem exists if galactic fountains store enriched gas in a halo that falls back to the disk in a different location.
4. How shall specific elements be treated when they do not arise equally from the same portions of the initial mass function, and is it itself constant in time?

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