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DEVELOPMENT OF THE INTEGRATED CENSORED ROBUST-TOLERANCE ENGINEERING DESIGN SYSTEMS VIA IMPROVED RESPONSE SURFACE MODELING

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DEVELOPMENT OF THE INTEGRATED CENSORED ROBUST-TOLERANCE ENGINEERING DESIGN SYSTEMS VIA IMPROVED RESPONSE SURFACE MODELING

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ABSTRACT

In this dissertation, we intend to integrate censoring into robust-tolerance engineering using improved dual response surface modeling. This is perhaps the first attempt in the quality control literature. In the literature, response surface models have largely been restricted to second order (or quadratic) models and robust-tolerance designs have been restricted to situations involving complete observations. We intend to show that higher order response surface models can be more powerful in terms of prediction ability, and are therefore more reliable than the preferred quadratic models in the general context of robust design. We will also consider formulating robust and tolerance designs in the presence of censored data. This is especially important for lifetime studies, where experiments are designed to determine the expected lifetimes of products under a variety of conditions. It is most often necessary to terminate experiments of this nature before the failure of the all the elements in the sample. Thus, lifetimes are observed for failed items, but censored times are observed for surviving items only. Available robust design methodologies in the literature have paid very little or no attention to problems of this nature. The proposed study is naturally suited for larger-the-better type (L-type) quality characteristics. As a result to this, we intend to propose quality loss functions that properly model such quality characteristics in a very intuitive and practical way. At the conclusion of the study, we intend to develop a censored-robust tolerance design optimization procedure, which will integrate all the major concepts of this dissertation.

DEDICATION

This work is dedicated to:

My parents, Mallam Shaibu Alhassan and Mma Fuseina Muhammad. My graduation from their “college of wisdom” and being assured of their love, tremendous understanding, and support did so much to get me this far

My wife, Debbie Die (Ameenah) – the excellent lovely spouse of a graduate student, who did and continues to do special things that make me feel our home is home indeed.

Our children, Ayyoob and Amaanah, whose coming into my life not only enhanced my understanding of what it means to truly love, but has been a motivation to persevere.

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CHAPTER ONE

INTRODUCTION AND LITERATURE REVIEW

Introduction

Robust parameter design (RPD) or robust design (RD) is a cost-effective method of improving the quality of products and processes by optimizing the quality characteristic of interest (usually the mean) while minimizing the variability (usually the variance or standard deviation). Taguchi (Taguchi and Phadke (1984), Taguchi and Wu (1985), Taguchi (1986)) advocates for the need to simultaneously consider the mean and the variance of the quality characteristic of interest. Taguchi (1986) assumes that there two sets of factors that affect quality, namely controllable and uncontrollable (or noise) factors. The control factors, **x**, are variables that are easy to control and manipulate while the uncontrollable factors, **z**, are either difficult and expensive or impossible to control. The objective of RD is to choose or determine the settings of the controllable factors that simultaneously optimize a defined quality characteristic and minimize the effect of the noise factors (Taguchi (1986)). For the analyses of the studies, Taguchi proposed a technique of experimental design referred to as the orthogonal design, where orthogonal arrays of **x** and **z** are crossed. Using the responses at each setting of **x** from this design, he formulated a single performance measure called signal-to-noise ratio (SNR), which is a combination of the mean and variance of a predefined quality characteristic. Taguchi classified quality characteristics into three, and defined a different SNR for each class:

1. $SNR = 10\log_{10}(\bar{y}^2/s^2)$ for the nominal-the-best type (N-Type) quality characteristic, which has both upper and lower specification limits and the aim is for the quality characteristic to achieve the target value situated between the specification limits. Examples of such quality characteristics include the diameter of a bolt and the brightness of a television set.

2. $SNR = -10\log_{10}(1/n)\sum_{i=1}^{n} y_i^2$ for the smaller-the-better type (S-Type) quality characteristic, which has only an upper specification limit and the ideal value is zero. Impurity, shrinkage, and noise level are examples of an S-type quality characteristic.

3. $SNR = -10\log_{10}(1/n)\sum_{i=1}^{n} y_i^{-2}$ for the larger-the-better (L-Type) quality characteristic, which only has a lower specification limit and the objective is for the quality characteristic to achieve a value that is as large as possible. Examples include the strength of material and fuel efficiency.

Researchers have categorized Taguchi's contribution into three; his philosophy, his experimental design, and his method of data analysis. There is general agreement with his philosophy, but his experimental design and method of data analysis have attracted a lot of criticism in the literature, and have led to more developments in the robust design methodology. In the next three sections, we present a literature survey, the problem statement of this dissertation, and our proposed approach to solving the problems posed.

Literature Review

In this section, we present an overview of relevant literature organized into four subsections namely, performance measures and optimization, response surface in robust design, experimental designs in robust design, and quality loss functions. Under performance measures and optimization, we review developments that follow Taguchi's proposal of the SNR as a performance. We review the use of models for the mean and variability of quality characteristics in the RD methodology under response surface in robust design, developments following the introduction of the orthogonal designs by Taguchi are then reviewed and finally followed by developments in the area of quality loss functions in the last subsection.

Performance Measures and Optimization

Taguchi proposes that quality characteristics be chosen to minimize interaction of control factors (see Phadke and Taguchi (1987) and Phadke (1989) for guidelines and examples). He further assumes that the vector of control factors $(\mathbf{x})$ can be partitioned into $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1$ affects the mean and the variance, while $\mathbf{x}_2$ only affects the mean. He then proposes finding the control-factor settings that minimize the mean squared error, which is defined as

$$E(y-T)^2 = Var(y;\mathbf{x_1},\mathbf{z}) + [\mu(y;\mathbf{x_1},\mathbf{x_2}) - T]^2,$$

where $\mu(\cdot)$ and T denote the mean function and target value respectively. The following two-step algorithm is proposed as the solution process for this minimization problem:

1. Determine the values of $\mathbf{x}_1$ that minimizes the appropriate SNR (i.e. the objective function)
2. Adjust the values of $\mathbf{x}_2$ (i.e. the adjustment factors) so that $[\mu(y; \mathbf{x}_1, \mathbf{x}_2) - T]^2$ is minimized.

Leon et al (1987) observe that the SNR depends on values of the adjustment factors, $\mathbf{x}_2$, and hence, they suggest the use of what they call performance measures independent of adjustment (PerMIA), which they propose as a replacement for the SNR in the two-step algorithm above. The PerMIA are obtainable through transfer functions, which are postulated functions aimed at describing the values of a quality characteristic of interest (y) in terms of its mean. For example, with the transfer function

$$y = \mu(y; \mathbf{x}_1, \mathbf{x}_2) + \varepsilon(\mathbf{z}, \mathbf{x}_1), \tag{1}$$

where $E[\varepsilon(\mathbf{z}, \mathbf{x}_1)] = 0$, the variance of y is given by

$$Var(y) = E[(y - \mu(y; \mathbf{x}_1, \mathbf{x}_2))^2] = E[\varepsilon^2(\mathbf{z}, \mathbf{x}_1)]. \tag{2}$$

Clearly, $Var(y)$ is independent of the adjustment factors and is therefore a PerMIA. Additionally, they (Leon et al. (1987)) proposed a transfer function under some conditions to achieve the results of using Taguchi's SNR.

Nair and Pregibon (1988) observe variability is often a function of mean, which implies that the variance will often be a function $\mathbf{x}_2$. Therefore, they propose data transformation techniques to make the transformed data have variance that is independent of $\mathbf{x}_2$, and modified the original two-step algorithm into a three-step one, where the objective is to minimize the variance of the transformed data, while minimizing the deviation of the mean from the target. We observe that this procedure is one of achieving PerMIA through data transformation. In order to achieve the data transformation objectives of Nair and Pregibon (1988), Box (1988) proposes the use of 'lambda plots' – a method that considers a class of transformations, $\{ y^{\lambda} \}$, and selecting a value of the parameter λ that simplifies models (i.e. parsimony) and also partitions factors into $\mathbf{x}_1$ and $\mathbf{x}_2$ as explained above (i.e. separation). This method is based on plots of the F or t statistics versus λ (see Box (1988) for more details and examples). Box (1988) also addresses the use of the SNRs, graphically illustrating situations where they are inefficient as performance measures, and strongly advocates for the creative use of data analytic and statistical methods to "allow the data to speak for themselves" as opposed to rigidly sticking with "portmanteau criteria" (i.e. the SNRs).

The works cited in this section thus far assume the possibility of partitioning the control factors as assumed by Taguchi and are providing techniques for determining it. However, Robinson et al. (2004) observe that there are many situations in which the SNR depends on all the factors, and therefore makes the partitioning hard to achieve. Many authors have considered separate models for the mean and the variance of a system as a way of enhancing the understanding of the system. This led into the development of the response surface methodology in robust design, which we consider in the next subsection. It is however important to mention that Bartlett and Kendall (1946) were the first to consider variance modeling, but Box and Meyer (1986) introduced it in the RPD problem.

Response Surface Models and Optimization in RD

The understanding of any system is enhanced by the availability of fairly accurate mathematical relations connecting the input variables (or control factors) and outputs (or responses) of the system. The desire to obtain such mathematical relations led to the response surface methodology (RSM), which Montgomery (1997) defines as a collection of mathematical and statistical techniques that are useful for the modeling and analysis of problems in which a response of interest is influenced by several variables and the aim is to optimize the response. Box and Wilson (1951) are considered the pioneers of RSM when they developed methods for determining optimum conditions in chemical investigations. There are two main branches of RSM, namely the dual and single model approaches. The dual response approach was proposed by Vining and Myers (1990) exploring developments in Myers and Carter (1973), which is observed to be an extension of ridge analysis studied by numerous authors including Hoerl (1959), Draper (1963), Myers (1976), Box and Draper (1987), and Khuri and Cornell (1987). Basically, the dual response approach seeks to determine the settings of the control factors that optimize a primary response while maintaining a secondary response at a desired target. For N-Type quality characteristics, Vining and Myers (1990) proposed minimizing the variance while maintaining the mean on target. However, for L- and S-type quality characteristics, they set the mean as the primary response, the variance as the secondary response and then proposed the following algorithm:

1. Establish several possible values for the variance.
2. Using the values in step 1 as the constraints, find optimum values of the mean.

3. Select the best compromise solution.

The proposed response surfaces for the mean and variance that are used in formulating and solving these problems are respectively of the form

$$\hat{\mu}_y = \beta_0 + \mathbf{x}'\beta + \mathbf{x}'\mathbf{B}\mathbf{x} \tag{3}$$

and

$$\hat{\sigma}_y^2 = \gamma_0 + \mathbf{x}'\gamma + \mathbf{x}'\mathbf{C}\mathbf{x} \tag{4}$$

The parameters β, γ, and the matrices **B** and **C** are appropriately defined for the products to make sense. The method of Lagrange multipliers [See for example, Umland and Smith (1959) and Myers and Carter (1973)] is the solution method used in solving these problems.

Del Castillo and Montgomery (1993) observe that the proposal of Vining and Myers (1990) does not always yield local optima. Hence they proposed using the method of generalized reduced gradient with inequality constraints. Lin and Tu (1995) proposed minimizing the mean-squared error (MSE) arguing that allowing some bias in the primary response results in substantial reduction in variability (see also Cho (1994)). Lin and Tu (1995) demonstrated superiority in their methodology over the proposals of Vining and Myers (1990) and Del Castillo and Montgomery (1993). However, Lin and Tu (1995) are criticized for not placing bounds on the bias, and for not being applicable in situations where it is crucial for the mean to be on target. In reaction to this, Copeland and Nelson (1996) proposed minimizing the standard deviation subject to a constraint that bounds the bias (i.e. $(\hat{\mu}-\tau)^2 \leq \Delta^2$) and demonstrated that this methodology is equally as effective as that of Lin and Tu (1995). Several other approaches to solving the dual response problem have been proposed. For example, Kim and Lin (1998) proposed a fuzzy optimization methodology, Del Castillo et al. (1997) and Fan (2000) proposed computational methods for global optimization in a spherical region, Kim and Cho (2002) and Tang and Xu (2002) proposed the use of goal programming, and Köskoy and Dogamaksoy (2003) suggested treating the secondary response as another primary response and generating a string of optimal solutions called the Pareto optimal solutions.

The single response approach consists of using a single experimental design for both the control and noise variables and defining a single response surface, which is a function of the control and noise factors as well as the interaction between them. This method, proposed by Welch et al. (1990), addresses the two main criticisms of Taguchi's crossed array, namely

1. If there are numerous control and noise factors, Taguchi's design often requires too many runs to be of practical use.
2. Taguchi's design does not enable the study of control factor interactions

The proposed single response model is of the form

$$y(\mathbf{x},\mathbf{z}) = \beta_0 + \mathbf{x}'\beta + \mathbf{x}'\mathbf{B}\mathbf{x} + \mathbf{z}'\gamma + \mathbf{x}'\mathbf{D}\mathbf{z} + \varepsilon\,, \tag{5}$$

where the vectors β, γ, and the matrices **B** and **D** are appropriately defined for the products to make sense, and ε represents the errors assumed to be NID(0, σ^2). Myers et al. (1992) considered the model in (5) and showed that it could be used to formulate the dual response surfaces, where the unconditional expectation of (5) yields the mean response surface in equation (3) and the unconditional variance is

$$Var[y(\mathbf{x},\mathbf{z})] = (\gamma + \mathbf{x}'\mathbf{D})Var(\mathbf{z})(\gamma + \mathbf{x}'\mathbf{D})' + \sigma^2\,. \tag{6}$$

Thus, the difference between the single response surface and the dual response surface methodologies is that the single response surface methodology fits a single model of the form in equation (5) and the mean and variance are theoretically derived from the model, while the dual response surface methodology fits separate models for the mean and variance relying on replications. Robinson et al. (2004) clearly illustrate this with an example.

In their consideration of model (5), Myers et al. (1992) assume the levels of the noise factors to be fixed in the experiment, but random in the process, but Khuri (1992) used an example in the semiconductor industry to show that there are situations where the experimental fixed level assumption of Myers et al. (1992) is inapplicable, and therefore proposed a linear mixed model for the process mean and the robust design problem is approached by finding the settings of the control factors which optimize the mean, given constraints on the prediction variance.

Various authors have considered the semi-parametric approach, a method that combines parametric and non-parametric techniques has been used in modeling process mean and variance (see Einsporn and Birch (1993), Mays et al (2000), Robinson and Birch (2002), and Pickle et al (2008)). Physical programming, a method of multi-objective optimization has been employed as a tool for developing flexible design models in the presence of multiple quality characteristics (see for example Messac (1996, 2000), Messac and Gupta (1996), Messac and Ismail-Yahya (2001), and Kovach et al (2008)). Su and Chang (2000) proposed a two-phase method based on neural networks and simulated annealing to improving the Taguchi method of parameter design optimization.

Along the same lines, Chang (2008) incorporated data mining technique along with exponential desirability functions and proposed a four-stage approach to optimizing systems with multiple response involving continuous control factors. Naidu (2008) developed a mathematical cost model in terms of process capability index C_{pm} for N-type quality characteristics and used it as the objective function for tolerance and cost optimization.

So far, all the models considered assume constant residual variance. Considerations of variable residual variance led to the use of generalized linear models (GLMs) in robust design. Nelder and Lee (1991) and Myers et al. (1992) were the first to propose the application of GLMs to modeling in robust Parameter Design. Some relevant works in this direction include Lee and Nelder (1998, 2003), Brinkley et al. (1996), Paul and Khuri (2000), Myers and Montegomery (1997), Hamada and Nelder (1997), and Myers et al. (1997).

Experimental Designs in RD

Experimental design is the strategy of planning conducting tests with the aim of determining the influence of various combinations of input variables on one or more output variables. In the context of RD, the input variables are the control (or in some cases, the control and noise) variables and the output variables are responses of interest. As mentioned in the introduction, Taguchi (see Taguchi (1986, 1987), and Taguchi and Wu (1985) proposed the crossed array design, where an orthogonal array of the control factors is crossed with an orthogonal array of the noise factors (i.e. the Cartesian product of **x** and **z**). Table 1.1 shows a general layout of the design for r runs. For each run, the observations are combined into computing a value of the appropriate SNR, giving an array of $r \times 1$ values, which is utilized in the two-step algorithm mentioned in Section above. Many authors have addressed the weaknesses in this design, the two major of which are the prohibitive number of runs required when there are numerous factors and the inability to study the interactions.

Table 1.1: General Layout of Taguchi's Crossed Array Design

Run		Array of Noise Variables (**z**) (Outer Array)
1 2 3 ⋮ r	Array of Control (**x**) Factors (Inner Array)	Observations (y_i)

Alternative designs to the orthogonal array design are the *split-plot designs*, which are used in some multifactor designs involving randomized blocks, where experimenters are unable to completely randomize runs within the block. Each block in a split-plot design is divided into *whole plots* (i.e. the main treatments), and each whole plot is divided into parts called *subplots* or split-plots (see Montgomery (1997) for examples). Box and Jones (1992) consider *split-plot designs* in RD studies and demonstrate that they are easy to conduct, and also facilitate efficient estimation of the interaction between control and noise factors. As a way of reducing the size of the experiment, Bisgaard and Kulachi (2001) propose using split-plot confounding. Kowalski (2002) constructed 24-run designs using 16-run designs and a balanced incomplete designs. He demonstrated that using these designs, all main effects and most of two-factor interactions can be estimated for small numbers of control and noise factors. Bingham and Sitter (2003) propose a method for constructing split-plot fractional and fractional factorial designs for RD studies, and provided a catalog of designs for 16- and 32-run experiments.

Numerous authors suggest combining control and noise factors in a single design called a *combined array*. Examples of such works include Lucas (1989, 1994), Sacks et al. (1989, 1990), Box and Jones (1992), Montgomery (1990), Shoemaker et al. (1991), Myers et al. (1992), Myers and Montgomery (2002), Borkowski and Lucas (1997), and Montgomery (1999). We have discussed the method of analysis under such designs. Aggrawal and Kaul (1998) constructed non-orthogonal combined array designs, and showed that the design size reduces significantly if orthogonality is sacrificed. Another favored design by practitioners, especially in fitting second order response surfaces is the *central composite design* (CCD). This design has three parts namely, the factorial (or fractional factorial) points, the axial points, and the center points. The factorial portion of the design is used in the estimation of two-factor interaction terms, the axial portion contributes to the estimation of linear and quadratic terms, and the center points give information about curvature and also contributes in the estimation of quadratic terms (see Myers and Montgomery (1999) for details). Some constructions of the CCDs afford it the *rotatability* property, which ensures that are equidistant from the center of the design have the same predicted variability. Modifications of the CCD have been considered in the literature. For example, Hartley (1959) propose the *small composite designs*, Roquemore (1976) developed the class of hybrid designs, and Lucas (1989, 1994) discussed the construction and application of *mixed resolution designs*. Vining *et al*. (2005) considered industrial experiments involving hard-to-change factors in the context of split-plot design and modified the standard central composite design (CCD) to apply to such cases. Kowalski *et al*. (2006) proposed a further modification of the proposal by Vining *et al*. (2005) with the view to integrating the dual response methodology with the split-plot structure. Recently, Joseph (2007) highlighted the ambiguities in the selection of adjustment factors in Taguchi's approach to parameter design and advocated for a method a selection method that makes heavy use of the engineering knowledge of the system of interest, citing among other benefits of his proposed approach, reduced cost and time of experimentation. More recently, Frey and Sundarsanam (2008) proposed a five-step approach to parameter design, which involves combining an adaptive one-factor-at-a-time design with two-level resolution III fractional factorial outer noise arrays, and demonstrated the gains (with respect to system robustness) in their approach through four different case studies.

Quality Loss Functions

A manufactured item usually has some characteristics, which are required to have certain values for the proper functioning of the item. These characteristics are referred to as quality characteristics and the ideal value for each quality characteristic is called a target value. Usually, a range of values containing the target value are specified for each quality characteristic of the item, within which the item is classified as non-defective. Otherwise, the item is defective. A specified range of values for a quality characteristic is either bounded below by the value known as a lower specification limit and/or above by an upper specification limit. Three types of quality characteristics have been studied in the quality engineering literature. First, a nominal-the-best (N-Type) quality characteristic – this has both upper and lower specification limits and the aim is for the quality characteristic to achieve the target value situated between the specification limits. Examples of such quality characteristics include the diameter of a bolt and the brightness of a television set. Second, a smaller-the-better (S-Type) quality characteristic has an upper specification limit and the ideal value is zero. Impurity, shrinkage, and noise level are examples of an S-type quality characteristic. The third type is a larger-the-better (L-Type) quality characteristic with a lower specification limit and the objective is for the quality characteristic to achieve a value that is as large as possible. Examples include the strength of material, fuel efficiency.

Springer (1951) considers a normally distributed quality characteristic with upper and lower specification limits and proposes a simple method to determine the optimum position of the target value with the objective of minimizing total cost. Betes (1962) considers a similar problem with a lower specification limit and an arbitrary upper specification limit. He considers the situation where undersized and oversized items are reprocessed at a fixed cost. Hunter and Kartha (1977) propose a simple procedure for obtaining the optimal process mean in a situation where only a lower specification limit is considered and items meeting specification requirements are sold at a regular price, while out-of-specification items are sold at a reduced price at a secondary market. Bisgaard et al. (1984) extend this work to include the selection of the most favorable distribution of the quality characteristic. Carlsson (1984) then modifies Hunter and Kartha (1977) model to include both fixed and variable costs, and derives a revenue function assuming that the customer pays extra for high quality and is compensated for poor quality. Golhar (1987) considers a canning problem with a lower specification limit, where the under-filled cans are emptied and refilled, thereby incurring a reprocessing cost. Some further considerations of the canning problem are by Golhar and Pollock (1988), Schmidt and Pfeifer (1991), and Montegomery (1995) among others.

All the studies cited above assume that there is no cost associated with the quality characteristic of an item as long as it is within the specification limits. This assumption resulted in a step-loss function for a quality characteristic x, which may be expressed as follows:

$$L_s(x) = \begin{cases} 0 & \text{if } x \text{ is within the specification limits} \\ c_r & \text{if } x \text{ is outside the specification limits} \end{cases} \tag{6}$$

The out-of-specification cost c_r is generally referred to as the rejection cost, which may be the cost of scrapping or reworking the item, or the loss due to selling the item at a reduced price in a secondary market.

Taguchi (1981, 1986) argues that there is a loss associated with any deviation from the target value. As a result, he proposes a quadratic loss function to incorporate the loss due the deviation from the target value when the product performance falls within the specification limits. The loss $L(x)$ function is derived as follows, assuming that it is differentiable at the target τ:

$$L(x) = L(\tau) + L'(\tau)(x-\tau) + \frac{L''(\tau)}{2!}(x-\tau)^2 + \frac{L'''(\tau)}{3!}(x-\tau)^3 + \ldots \tag{7}$$

Assuming that the function achieves the minimum of zero at τ so that $L(\tau) = L'(\tau) = 0$, and that the deviations from τ are small enough to ignore higher powers of $(x-\tau)$,

$$L(x) \approx \frac{L''(\tau)}{2!}(x-\tau)^2 . \tag{8}$$

Hence for a quality characteristic with a lower specification limit *L* and an upper specification limit *U*, where the cost of falling below *L* is c_1 and the cost of exceeding *U* is c_2, Taguchi (1981) proposes a quadratic loss function, the general form of which is expressed as

$$L(x) = \begin{cases} K_1(x-\tau)^2 & \text{if} \quad L \le x \le \tau \\ K_2(x-\tau)^2 & \text{if} \quad \tau < x \le U \end{cases}, \tag{9}$$

where $K_1 = \frac{c_1}{(L-\tau)^2}$ and $K_2 = \frac{c_2}{(U-\tau)^2}$. Figure 1.1 illustrates the loss function along with the rejection costs in the case where $c_1 = c_2$. It can be observed from Figure 1.1, the loss function (9) is for an N-type quality characteristic. The loss functions for the L-type and S-type characteristics are found in Taguchi *et al*. (1989). It is however worth noting that the concept of the quadratic loss is not entirely new. It is the underlying concept of least square theory founded by Gauss in 1809.

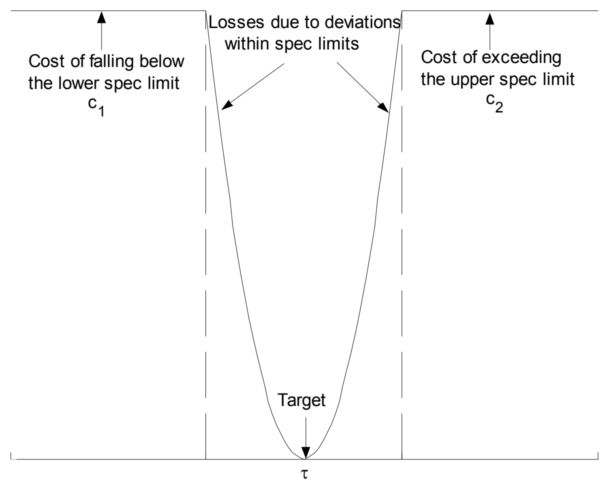

Figure 1.1 The Taguchi Loss Function and the Rejection Costs

Chan and Ibrahim (2005) state that deviations from the target value that are still within the specification limits do not cause a company to incur any internal cost, but may cause customer dissatisfaction, in which case the company may incur external costs that include repair, warranty, or loss of market share.

Many authors have modified the Taguchi loss function. For example, Spiring (1991) and Spiring and Yeung (1998) develop a class of functions based on inversions of probability density functions. Baker (1986) as well as Leon and Wu (1989) develop an asymmetric squared-error loss function when losses are not equivalent on each side of the target value. The quadratic loss function has been used in several research areas including optimal target value problems (Rahim and Shaibu (2000), Phillips and Cho (2000), Al-Fawzan and Rahim (2001), Teeravaraprug and Cho (2002)), economic design of control charts (Alexander et al. (1995), Chou et al. (2000), Kobayashi et al. (2003), Jiao and Helo (2008)), and capability process analysis (Ho and Quinino (2003)).

Recently, Pan (2007) proposed a methodology for assessing the likelihood and consequences of manufacturing and environmental risks using the relationship between process capability indices and loss functions. Khorramshahgol and Djavanshir (2008) proposed a six-step methodology based on the analytic hierarchy process (AHP) for determining the Taguchi loss constant. They also advocate their proposal as a prerequisite to design of experiment since it is meant to fish out only relevant quality characteristics. Cho and Cho (2008) pointed out a couple of problems with AHP, namely, inconsistent judgments and the handicap of pair wise comparison matrices to satisfy the inconsistency criterion. As a result, they proposed using loss functions in concert with inconsistency ratio for group evaluation quality.

All these studies consider the loss function as depicted in Figure 1.1, which is a piecewise continuous function with the within-specification-cost function (i.e. the quadratic part) intersecting the out-of-specification costs at the specification limits. This means that the within-specification-cost can get as close as possible to the out-of-specification costs. Obviously, this does not adequately model the practical situation in which the customer faced with the option of repairing a product or buying a new one resorts to buying a new one because the cost of repair is too close to the cost of buying a brand new replacement. Similarly from the manufacturer's perspective, the loss function shown in Figure 1.1 does not represent the situation where an item under warranty is replaced by the manufacturer because it is too time-consuming to fix or the cost of fixing is too close to the cost of replacing it.

In this work, we modify the Taguchi loss function to adequately represent the scenarios cited above. We also propose alternative loss functions for the N-type and L-type quality characteristics, and draw comparisons between all the loss functions we consider for each type of the two types of quality characteristics that we study.

Problem Statement

The main steps of the RD methodology are four, namely

1. Designing an experiment from which a set of data on a quality characteristic of interest is generated
2. Modeling the mean and variability responses using the data in step (1),
3. Formulating models to optimize the mean and variability response functions
4. Solving the optimizations models for optimum settings.

Obviously, the second step relies on the first since a good experimental design yields reliable data, which is essential in obtaining reliable models for the mean and variability response functions. Similarly, obtaining realistic solution(s) from the optimization models depends on whether the constituents of the optimization models (i.e. the mean and variability response functions) ‘accurately’ model the mean and variability of the quality characteristic of interest. In the RD literature, so much work has been done to develop sound experimental design, but not nearly as much has been done with regards to response surface modeling. The importance of sound response surface models has been highlighted, for example Lin and Tu (1995) reiterate the suggestion by Vining Myers (1990) that "The fitness or the prediction ability of the mean and variability models is an extremely important consideration when optimizing a dual response problem." This suggestion cannot be overemphasized since different models of the mean and variability of the same system can be used in solving the same optimization problem, all achieving the objective of the optimization model, but with different settings (or optimal solutions). Some of the settings may be more correct than others. Obviously, the most correct would be that which is based on the more powerful response surface models in terms of prediction. Therefore, step 2 above is worth spending time on to make sure that very good models are obtained before proceeding to the optimization step. In this work, we intend to select models based on statistical model selection techniques, and directly compare the models obtained to previous models. We will also compare the models by applying them to various optimization problems proposed in the context of RD.

In our literature study, we observe that traditional robust design principles have often been applied to situations in which the quality characteristics of interest are typically time-insensitive. When time-oriented quality characteristics are under study, censored data are often encountered, and current robust design models reported to the research community may not be effective in finding solutions based on such data. To address such practical needs for manufacturing industries, we intend to develop a censored robust design model. We will then integrate the censored robust design methodology with target value and tolerance problems.

Approach

This dissertation will be organized according to the following topics.

1. Dual Response Surface Modeling
2. Dual Response Surface Optimization
3. RD in the Presence of Censored Data
4. Process Target Value in the Presence of Censored Data
5. Tolerance Optimization in the Presence of Censored Data
6. Alternative Quality Loss Functions
7. Integrated Studies
   (a) Robust-Tolerance Design with Censoring
   (b) Process Target – Tolerance Design with Censoring

The relationship between the first five topics can be expressed in the chart in Figure 1.2, which shows that the study under (6) can be feasibly carried out.

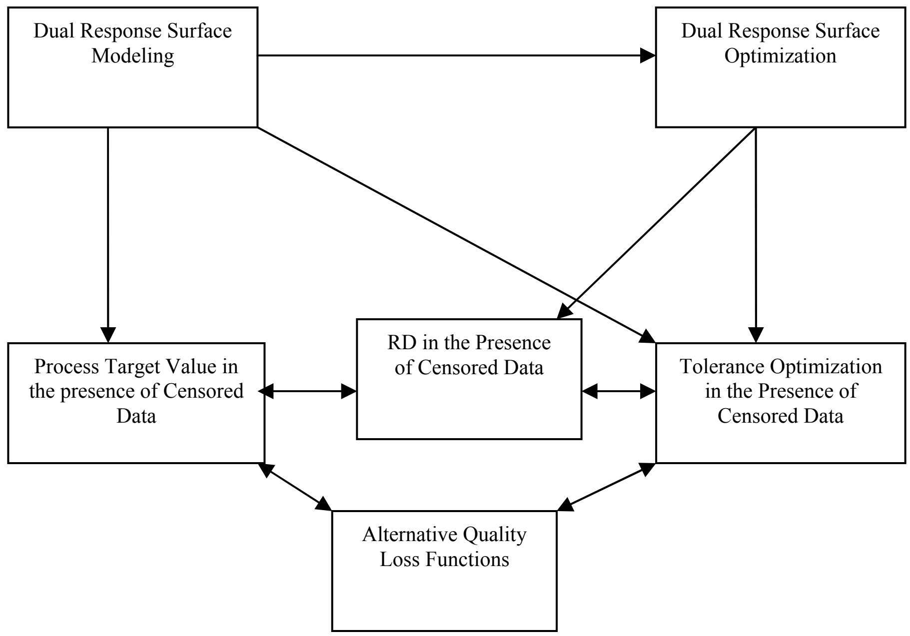

Figure 1.2 Relational Chart of the Main Topics of the Dissertation

Under dual response surface modeling, we will consider finding "best fitting" models for mean response and standard deviation for data from a designed experiment (e.g. factorial, CCD, etc). We intend to do so using statistical model selection techniques (e.g. R-Square, Adjusted R-Square, predicted R-square, Mallow's C_p, Mean-square error, etc.). As mentioned in the abstract, response surface modeling has largely been limited to second order (or quadratic) models, but in our modeling, we will consider models of higher orders and rely on the powerful statistical model selection techniques to aid us in choosing the best fitting models. This is particularly important for variability modeling. In modeling standard deviation for a given set of data from an experiment, Vining and Myers (1990) fit the full quadratic model to the standard deviation obtaining an R-square value of 45%. In an attempt to improve upon this model, Lin and Tu (1995) only obtained an R-square value of 48%. In fact, the predicted R-square for the two models are 0% and 22% respectively. We are confident of obtaining more acceptable values than this. We will use the statistical software SAS and MINITAB.

When we study dual response surface optimization, we will use the models we find in (1) and formulate and solve optimization problems for each of the three classes of quality characteristics (i.e. the S-, N- and L-type quality characteristics). We will utilize nonlinear optimization routines in MATLAB, such as *fmincon* for nonlinear constrained optimization, which is based on the method of Lagrange multipliers.

As mentioned earlier, the available RD methodologies in the literature have not considered time-dependent quality characteristics, which usually yield censored values in experiments. Besides yielding censored data, such experiments do not guarantee the same number of observations for all design points. Our objective is to formulate an RD methodology that would be suitable for censored data as well as unequal number of observations. This would be useful in incorporating RD with life-testing problems. The modeling and optimization techniques that would be developed in (1) and (2) will be integrated with the censored RD problem.

We observe that there is a logical link between what we intend to achieve under (1) through (3) on one hand, and (4) through (6) on the other. That is, the modeling and optimization techniques together with the censored RD methodology should be applicable to process and product target value problems and tolerance optimization problems.

Research Significance

If achieved, the objectives of this dissertation will be beneficial to practitioners and researchers in the RD community. We summarize the expected benefits of this work as follows:

1. It will provide a method of obtaining powerful response surface models in terms of prediction power, which is arguably one of the most important requirements for achieving reliable solutions to RD problems.

2. Another benefit of this dissertation is that it will provide practitioners and researchers with a methodology of applying RD to experiments that yield censored data with unequal number of observations per design point. As mentioned earlier, this is important in life-testing problems.

3. It will provide a way of setting up and solving target value and tolerance optimization problems via modeling. This is important because if we consider quality loss functions in the literature, Taguchi's (1981, 1986, 1989) quadratic loss function, which is based on Taylor series approximation is being heavily utilized, but we are hypothesizing that all losses in all processes may not necessary behave in accordance with one function type, even though they can all satisfy the basic assumption of the quadratic loss function. That is, zero cost at target and that loss increases with increasing deviation from target.

CHAPTER TWO

ANOTHER VIEW OF DUAL RESPONSE SURFACE MODELING AND OPTIMIZATION IN ROBUST PARAMETER DESIGN

Introduction

Robust parameter design (RPD) based on the concept of building quality into a design has received much attention from researchers and practitioners for years, and a number of methodologies have been studied in the research community. There have been many attempts to integrate RPD principles with well-established statistical techniques, such as response surface methodology, in order to model the response directly as a function of control factors. In this paper, we reinvestigate the dual response approach based on quadratic models (Vining and Myers, 1990), which is often referred to in the RPD literature and demonstrate that higher-order polynomial models may be more effective in finding better RPD solutions than the commonly-used quadratic model. We also propose optimization models for each of the three classes of quality characteristics (i.e. nominal-the-best, larger-the-better, and smaller-the-better). The optimal solutions obtained using the proposed models are compared with the solutions obtained using the RPD techniques in the current literature.

Research Motivation

There are four main steps in RPD methodology:

1. Design an experiment from which a set of data on a quality characteristic of interest is generated
2. Model the mean and variability responses using the data in Step (1)
3. Formulate models to optimize the mean and variability response functions
4. Obtain solutions to the optimization models

We observe that the second step relies on the first, since a good experimental design yields reliable data, which is essential in obtaining reliable models for the mean and variability response functions. Similarly, obtaining realistic solutions from the optimization models depends on whether the constituents of the optimization models (i.e. the mean and variability response functions) represent the mean and variability of the quality characteristics as accurately as possible. In the RPD literature, much work has been done to ensure sound experimental designs under various conditions, but not nearly as much has been done with regards to response surface modeling. However, the importance of response surface models has been highlighted by LT who reiterated the suggestion by VM that "The fitness or the prediction ability of the mean and variability models is an extremely important consideration when optimizing a dual response problem." This suggestion cannot be overemphasized since different models of the mean and variability of the same process or product characteristic can be used in solving the same optimization model, all achieving the objective of the optimization model (i.e. minimizing or maximizing the objective function), but with different optimal settings.

We believe that better RPD solutions can be found by obtaining accurate response surface models in terms of prediction. Therefore, the second step above deserves a great deal of attention in order to make sure that the most accurate models are obtained before proceeding to the optimization step. Achieving accurate response surface models will serve to reduce, as much as possible, the disparity between solutions to optimization models and the results obtained by actually applying those solutions to the processes or systems of interest. To this end, in this work, we select models based on statistical model selection techniques, and directly compare the models obtained to previous models. We will also compare the models by applying them to various optimization problems proposed in the context of RPD.

Proposed Model Development

The model development procedure we propose consists of three steps, namely *the experimental phase*, *the model selection phase*, and *the optimization phase*. We describe each of the phases in what follows.

Experimental Phase

In this phase, an experimental design (e.g. full or fractional factorial designs, central composite designs, etc) is selected and the response of interest is measured under the selected design. That is, for *n* factors, the measurements are taken at various design points, where each design point consists of a combination of the levels of the control variables $\{x_1, x_2, \ldots, x_n\}$ (see Table 1). We refer to the x_i's as the basic variables.

Model Selection Phase

When the response of interest (Y) is influenced by a set of factors $\{x_1, x_2, \ldots, x_n\}$, the functional relationship is often not known, but can be estimated to a reasonable degree of accuracy. If the relationship is polynomial in nature, then besides the linear terms in the basic variables, various powers as well as products of various forms also contain information about Y. Most RPD problems are analyzed by obtaining the second order estimated response surface functions. In this paper, we propose the use of higher-order polynomial functions in modeling the response. As we shall show in the numerical example, the prediction ability of the response surface models obtained using the proposed model is higher than that of the second order models.

The model selection phase consists of two stages. In the first stage, we form a set of variables (or factors) made up of the powers and cross-products of the basic variables and augment it with the set of basic variables to form a pool to choose from using statistical model selection techniques. The composition of the pool depends on the order of the model desired. For example, if we are interested in a third order model, we will construct a pool of the form

$$P = \left\{x_i, x_i^3, x_i^2, x_i x_j, x_i x_j x_k, x_i^2 x_j; i \neq j \neq k\right\}. \qquad (1)$$

It is easy to show that for n variables, this set will consist of $\frac{n}{6}\left(n^2 + 3n + 14\right)$ elements.

For example, if $n = 3$ basic variables, we will have a pool of 16 elements to choose from. The standard statistical techniques used include stepwise regression, all possible subset regression, the coefficient of determination (R^2), the adjusted R-square (R_a^2), the predicted R-square (R^2_{pred}), the prediction error sum of squares (PRESS), the root mean square error (RMSE), Mallows C_p, and the variance inflation factor (VIF). Intuitively, it is obvious that models obtained through this proposed procedure cannot perform any worse than the second order models being used presently. We assume that the functional relationship between the variables in the pool and the response variable of interest (**y**) is of the form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{2}$$

where **X** is a matrix with first column of ones, and the elements of the set *P* (in Equation (1)) in the rest of its columns. $\boldsymbol{\beta}$ is a column vector of parameters, and $\boldsymbol{\varepsilon}$ is a vector of random errors with constant variance and zero mean. The least squares estimator of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}, \tag{3}$$

and the least square predicted model is of the form

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}. \tag{4}$$

For a model with p parameters, R^2, R^2_a, and RMSE are respectively defined as

$$R^2 = \frac{\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X} - n\overline{\mathbf{y}}^2}{\mathbf{y}'\mathbf{y} - n\overline{\mathbf{y}}^2}, \tag{5}$$

$$R^2_a = 1 - (1 - R^2)\left(\frac{n-1}{n-p}\right), \tag{6}$$

and

$$RMSE = \sqrt{\frac{\mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}}{n-p}}. \tag{7}$$

Models with large values of R^2, and R_a^2, and small values of RMSE are sought. It is well known that R^2 is an increasing function of the number of predictors in the model. That is, it increases with additional predictor variables regardless of how significant or insignificant the variables are. On the contrary, R_a^2 may decrease if additional predictors do not contribute significantly to explaining the variability in the response. Thus, it is important to observe both statistics rather than R^2 alone.

The PRESS and R^2_{pred} are useful in assessing the prediction ability of models. If $e_i = y_i - \hat{y}_i$ represents the i^{th} residual, and h_{ii}, the ith diagonal of the hat matrix (see Montgomery and Peck, 1992), which is defined by $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, then

$$PRESS = \sum_{i=1}^{n}\left(\frac{e_i}{1-h_{ii}}\right)^2, \tag{8}$$

and

$$R^2_{pred} = 100\left[1 - \frac{PRESS}{\mathbf{y}'\mathbf{y} - n\bar{\mathbf{y}}^2}\right]\%. \tag{9}$$

Lower values of PRESS and higher values of R^2_{pred} indicate a model of high prediction ability.

Since the proposed model of this work recommends consideration of the possibility of adding more variables in modeling the response surfaces, we emphasize the inclusion of VIF and Mallows C_p in the selection criteria. This is because the VIF is an important diagnostic for multicollinearity, while the C_p criterion is used to diagnose bias or over-fit models, i.e. situations where more variables than necessary are added to the model. The VIF corresponding to the i^{th} variable is defined as

$$VIF_i = \frac{1}{1 - R_i^2}, \tag{10}$$

where R_i^2 is the coefficient of determination obtained by regressing the i^{th} variable against the rest of the variables in the model. In practice, VIF values not exceeding ten are tolerated. The Mallows C_p statistic (Mallows, 1964) is defined as

$$C_p = \frac{SSE_p}{\hat{\sigma}^2} - n + 2p, \tag{11}$$

where SSE_p is the residual sum of squares of the full model, and $\hat{\sigma}^2$ is an unbiased estimate of the error variance. Models with values of the C_p statistic that are close to *p* are considered as the least bias models. Detailed discussions on the use of all these selection criteria are available in Montgomery and Peck (1992).

Optimization Phase

In this phase, optimization models are formulated and solved for optimum values of the mean and standard deviation of the response of interest in terms of the control variables. We will briefly summarize the models by VM and LT, followed by our proposed models.

Models by VM and LT

As mentioned earlier, VM proposed optimization models for the *N*-Type, *L*-Type, and *S*-Type quality characteristics. These models are given in Equations (12) through (14), where T_m is the target mean response and T_s is the target standard deviation. The VM models for the three types of quality characteristics are

$$\begin{aligned} \min \quad & \hat{\sigma}(\mathbf{x}) \\ s.t. \quad & \hat{\mu}(\mathbf{x}) = T_m \end{aligned} \tag{12}$$

$$\begin{aligned} \max \quad & \hat{\mu}(\mathbf{x}) \\ s.t. \quad & \hat{\sigma}(\mathbf{x}) = T_s \end{aligned} \tag{13}$$

$$\begin{aligned} \min \quad & \hat{\mu}(\mathbf{x}) \\ s.t. \quad & \hat{\sigma}(\mathbf{x}) = T_s \end{aligned} \tag{14}$$

LT observed that $\hat{\mu}(\mathbf{x})$ and $\hat{\sigma}(\mathbf{x})$ "are only approximations of the 'true' responses (subject to certain random errors)", and that "restricting the optimization to equality constraints will inevitably exclude globally preferred values." Hence, they proposed minimizing the mean square error (MSE) instead (see Equation (15)), arguing that allowing some bias in the mean response results in greater reduction in variability.

$$\min \quad f(\mathbf{x}) = (\hat{\mu}(\mathbf{x}) - T)^2 + \hat{\sigma}^2(\mathbf{x}) \tag{15}$$

All the optimization models are constrained to the experimental region.

Proposed Optimization Models

The *L*-type optimization model of Equation (13) (proposed by VM) is equivalent to the model

$$\begin{aligned} &\min -\hat{\mu}(\mathbf{x}) \\ &s.t. \quad (\hat{\sigma}(\mathbf{x}) - T_S)^2 = 0 \end{aligned} \tag{16}$$

We observe that $(\hat{\sigma}(\mathbf{x}) - T_S)^2$ is positive for values of T_S not equal to $\hat{\sigma}(\mathbf{x})$, thus the equality constraint forces $\hat{\sigma}(\mathbf{x})$ to equal T_S. Also, assuming that the mean response is positive, the smallest value of $f(\mathbf{x}) = -\hat{\mu}(\mathbf{x}) + (\hat{\sigma}(\mathbf{x}) - T_S)^2$ is achieved when $-\hat{\mu}(\mathbf{x})$ is minimized and $(\hat{\sigma}(\mathbf{x}) - T_S)^2$ is as small as possible (i.e., zero at best). In other words, $\hat{\mu}(\mathbf{x})$ is maximized and $\hat{\sigma}(\mathbf{x})$ is as close to S as possible. Hence we propose the optimization model

$$\begin{aligned} &\min f(\mathbf{x}) = -\hat{\mu}(\mathbf{x}) + (\hat{\sigma}(\mathbf{x}) - T_S)^2 \\ &s.t. \quad \mathbf{x} \in \mathbf{\Omega} \end{aligned} \tag{17}$$

where $\mathbf{\Omega}$ denotes the experimental region of interest. This model relaxes the equality constraint in Equation (16) in the same way that the MSE optimization model in Equation (15) relaxes the equality constraint in Equation (14). Since the smallest variability is always desired, we can consider the target S as an upper bound and seek the solution to the problem

$$\begin{aligned} &\min f(\mathbf{x}) = -[\hat{\mu}(\mathbf{x}) + (\hat{\sigma}(\mathbf{x}) - T_S)^2] \\ &s.t. \quad \hat{\sigma}(\mathbf{x}) \le S \\ &\qquad \mathbf{x} \in \mathbf{\Omega} \end{aligned} \tag{18}$$

This model clearly seeks to maximize $\hat{\mu}(\mathbf{x})$ and simultaneously find the $\hat{\sigma}(\mathbf{x})$ that is at most equal to T_S.

Now we consider an S-type problem, where the objective is to minimize the mean response. In this case, the standard deviation is still the secondary response, and since smaller values are desired, we assume that an upper bound (T_S') is set for the standard deviation. Thus, an analogous optimization model to model (14) is

$$\begin{aligned} &\min f(\mathbf{x}) = \hat{\mu}(\mathbf{x}) - (\hat{\sigma}(\mathbf{x}) - T_S')^2 \\ &s.t. \quad \hat{\sigma}(\mathbf{x}) \le T_S' \\ &\qquad \mathbf{x} \in \mathbf{\Omega} \end{aligned} \tag{19}$$

However, if we only consider a target standard deviation (T_S) and not an upper bound, we will have the less restrictive model

$$\begin{aligned} &\min f(\mathbf{x}) = \hat{\mu}(\mathbf{x}) + (\hat{\sigma}(\mathbf{x}) - T_S)^2 \\ &s.t. \quad \mathbf{x} \in \mathbf{\Omega} \end{aligned} \tag{20}$$

Finally, we propose an *N*-Type optimization model, assuming target values T and T_S for the mean and standard deviation respectively. An appropriate optimization model to solve in this case is

$$\begin{aligned} &\min f(\mathbf{x}) = \left(\hat{\mu}(\mathbf{x}) - T\right)^2 + (\hat{\sigma}(\mathbf{x}) - T_S)^2 \\ &s.t. \quad \mathbf{x} \in \mathbf{\Omega} \end{aligned} \tag{21}$$

Numerical Example

Box and Draper (1987) describe an experiment that was conducted to determine the effect on the quality of a printing process of three control variables, namely speed (x_1), pressure (x_2), and distance (x_3). VM used the data of this experiment to illustrate their proposed dual response methodology. In order to have a fair basis for comparison with the results in VM, LT used the same data to illustrate their proposition of alternative optimization models and improved models of the mean and standard deviation. Similarly, for the purpose of fair comparison, we will use the same data here, which is given in

Table 2.1. We will first use our proposed methodology and find response surface models for the mean and standard deviation, and then compare the models obtained with the models of VM and LT. Next, we will consider solving the optimization models in VM and LT using their response surface models and the models obtained in this work. Finally, we will apply the three sets of response surface models (i.e. ours, VM's, and LT's) to each of our proposed optimization models and compare the optimal solutions we obtain.

Table 2.1 The Printing Process Data (Box and Draper, 1987)

Design Point u	Control Factors x_1	x_2	x_3	Observations y_{u1}	y_{u2}	y_{u3}	Mean $\bar{y}_u$	Std. Dev. s_u
1	-1	-1	-1	34	10	28	24.0	12.49
2	0	-1	-1	115	116	130	120.3	8.39
3	1	-1	-1	192	186	263	213.7	42.83
4	-1	0	-1	82	88	88	86.0	3.46
5	0	0	-1	44	188	188	140.0	83.14
6	1	0	-1	322	350	350	340.7	16.17
7	-1	1	-1	141	110	86	112.3	27.57
8	0	1	-1	259	251	259	256.3	4.62
9	1	1	-1	290	280	245	271.7	23.63
10	-1	-1	0	81	81	81	81.0	0.00
11	0	-1	0	90	122	93	101.7	17.67
12	1	-1	0	319	376	376	357.0	32.91
13	-1	0	0	180	180	154	171.3	15.01
14	0	0	0	372	372	372	372.0	0.00
15	1	0	0	541	568	396	501.7	92.50
16	-1	1	0	288	192	312	264.0	63.50
17	0	1	0	432	336	513	427.0	88.61
18	1	1	0	713	725	754	730.7	21.08
19	-1	-1	1	364	99	199	220.7	133.82
20	0	-1	1	232	221	266	239.7	23.46
21	1	-1	1	408	415	443	422.0	18.52
22	-1	0	1	182	233	182	199.0	29.44
23	0	0	1	507	515	434	485.3	44.64
24	1	0	1	846	535	640	673.7	158.21
25	-1	1	1	236	126	168	176.7	55.51
26	0	1	1	660	440	403	501.0	138.94
27	1	1	1	878	991	1161	1010.0	142.45

Model Selection

For the model selection, we will first create a pool of variables. Because of the coding system chosen for the factor levels (i.e. -1, 0, and 1),

$$x_i^3 = x_i \text{ and } x_i^4 = x_i^2; \quad i = 1, 2, 3. \tag{22}$$

Therefore, we exclude all x_i^3 and all x_i^4 from the pool of variables in the selection of predictor variables. Thus, the pool of variables is given by the set

$$\{ x_i,\ x_i^2,\ x_i x_j,\ x_i^2 x_j,\ x_i^2 x_j^2,\ x_i^2 x_j x_k,\ x_i^2 x_j^2 x_k,\ x_1^2 x_2^2 x_3^2;\ \ i \neq j \neq k;\ \ i, j, k = 1, 2, 3 \}.$$

This gives a total of thirty-one variables to consider in the model selection. Using the methods mentioned in the previous section, we obtain the response surfaces in Equations (23) and (24) for the mean and the standard deviation respectively. In Tables 2.2 and 2.3, we show the analysis of variance (ANOVA) results for the models as obtained using Minitab.

$$\begin{aligned}\hat{\mu}_{SC}(\mathbf{x}) = 339.47 + 177.0x_1 + 147.0x_2 + 115.53x_3 - 3.72x_1^2 - 58.11x_2^2 - 10.53x_3^2 + 47.67x_1x_2 + \\ 55.00x_1x_3 + 43.58x_2x_3 - 56.36x_2x_3^2 + 82.79x_1x_2x_3 + 80.53x_1^2x_2^2 + 30.71x_1x_2^2x_3 + \\ 27.54x_1x_2^2x_3^2 + 35.43x_1^2x_2^2x_3 - 41.26x_1^2x_2^2x_3^2\end{aligned} \tag{23}$$

Table 2.2 ANOVA for a Higher Order Mean Response Model

Predictor	Coef	SE Coef	T	P	VIF
Constant	339.47	25.90	13.11	0.000	
x1	177.000	8.936	19.81	0.000	1.0
x2	147.00	15.48	9.50	0.000	3.0
x3	115.53	11.99	9.64	0.000	1.8
x11	-3.72	26.81	-0.14	0.892	3.0
x22	-58.11	26.81	-2.17	0.055	3.0
x33	-10.53	20.77	-0.51	0.623	1.8
x12	47.67	18.96	2.51	0.031	3.0
x13	55.00	18.96	2.90	0.016	3.0
x23	43.58	10.94	3.98	0.003	1.0
x233	-56.36	18.96	-2.97	0.014	3.0
x123	82.79	13.40	6.18	0.000	1.0
x1122	80.53	38.85	2.07	0.065	7.0
x1223	30.71	23.22	1.32	0.215	3.0
x1233	27.54	23.22	1.19	0.263	3.0
x11223	35.43	17.98	1.97	0.077	1.8
x112233	-41.26	31.15	-1.32	0.215	3.8

S = 37.9142   R-Sq = 98.9%   R-Sq(adj) = 97.2%
PRESS = 78791.4   R-Sq(pred) = 94.14%

Source	DF	SS	MS	F	P
Regression	16	1331166	83198	57.88	0.000
Residual Error	10	14375	1437		
Total	26	1345541			

$$\begin{aligned}\hat{\sigma}_{SC}(\mathbf{x}) = {} & 34.208 + 36.49x_1 + 35.55x_2 - 19.25x_3 + 3.9x_1^2 + 16.93x_3^2 - 18.83x_1x_2 + 29.02x_1x_3 + \\ & 29.81x_2x_3 - 22.68x_1^2x_2 + 61.26x_1^2x_3 - 37.45x_1x_2^2 + 56.60x_2^2x_3 - 7.67x_2x_3^2 + \\ & 29.57x_1x_2x_3 - 23.59x_1^2x_2x_3 - 35.86x_1x_2^2x_3 + 39.83x_1x_2x_3^2 - 68.13x_1^2x_2^2x_3\end{aligned} \tag{24}$$

Table 2.3 ANOVA for a Higher Order Standard Deviation Model

Predictor	Coef	SE Coef	T	P	VIF
Constant	34.208	8.770	3.90	0.005	
x1	36.493	8.320	4.39	0.002	3.0
x2	35.55	10.74	3.31	0.011	5.0
x3	-19.25	14.41	-1.34	0.218	9.0
x11	3.900	8.320	0.47	0.652	1.0
x33	16.930	8.320	2.03	0.076	1.0
x12	-18.83	10.19	-1.85	0.102	3.0
x13	29.02	10.19	2.85	0.022	3.0
x23	29.81	10.19	2.93	0.019	3.0
x112	-22.68	10.19	-2.23	0.057	3.0
x113	61.26	17.65	3.47	0.008	9.0
x122	-37.45	10.19	-3.68	0.006	3.0
x223	56.60	17.65	3.21	0.012	9.0
x233	-7.67	10.19	-0.75	0.473	3.0
x123	29.566	7.205	4.10	0.003	1.0
x1123	-23.59	12.48	-1.89	0.095	3.0
x1223	-35.86	12.48	-2.87	0.021	3.0
x1233	39.83	12.48	3.19	0.013	3.0
x11223	-68.13	21.62	-3.15	0.014	9.0

S = 20.3790   R-Sq = 94.5%   R-Sq(adj) = 82.0%
PRESS = 21219.9   R-Sq(pred) = 64.64%

Source	DF	SS	MS	F	P
Regression	18	56682.3	3149.0	7.58	0.003
Residual Error	8	3322.4	415.3		
Total	26	60004.7			

Comparison with Previous Models

In modeling the mean response for the data of Table 2.1, VM and LT reported the models in Equations (25) and (26) respectively. Model (26) is an improvement upon (24), and was found using model selection criteria on the terms of the full third order model.

$$\hat{\mu}_{VM}(\mathbf{x}) = 327.6 + 177.0x_1 + 109.4x_2 + 131.5x_3 + 32.0x_1^2 - 22.4x_2^2 - 29.1x_3^2 + 66.0x_1x_2 + 75.5x_1x_3 + 43.6x_2x_3 \quad (25)$$

$$\hat{\mu}_{LT}(\mathbf{x}) = 314.67 + 177.0x_1 + 109.426x_2 + 131.463x_3 + 66.028x_1x_2 + 75.472x_1x_3 + 43.583x_2x_3 + 82.792x_1x_2x_3 \quad (26)$$

In Table 2.4, we compare these two models with the model we obtained in Equation (23) based on PRESS, RMSE, R^2, R_a^2, and R_{pred}^2. We observe in terms of all these criteria that the model $\hat{\mu}_{SC}(\mathbf{x})$ of Equation (23) is superior to the models of Equations (25) and (26). Thus, Equation (23) better describes the mean response and also has a greater ability to predict the mean response than the other previous models.

Table 2.4 Comparison of the Mean Response Models

Model	PRESS	RMSE	R^2	R_a^2	R_{pred}^2
$\hat{\mu}_{SC}(\mathbf{x})$	78791	37.9142	98.9	97.2	94.14
$\hat{\mu}_{VM}(\mathbf{x})$	337545	76.0429	92.7	88.8	74.91
$\hat{\mu}_{LT}(\mathbf{x})$	127072	55.0496	95.7	94.1	90.56

We also note that the decrease in the RMSE achieved by $\hat{\mu}_{SC}(\mathbf{x})$ is about 50% relative to $\hat{\mu}_{VM}(\mathbf{x})$ and about 31% relative to $\hat{\mu}_{LT}(\mathbf{x})$. The reductions achieved in the PRESS values are 77% and 38% relative to $\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\mu}_{LT}(\mathbf{x})$, respectively. In Figure 2.1, we present a graph of the observed mean values (μ) and the values obtained from the response surfaces $\hat{\mu}_{SC}(\mathbf{x})$, $\hat{\mu}_{VM}(\mathbf{x})$, and $\hat{\mu}_{LT}(\mathbf{x})$. The closeness of the values obtained from $\hat{\mu}_{SC}(\mathbf{x})$ to the observed mean values as compared to values obtained from $\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\mu}_{LT}(\mathbf{x})$ is evident from the graph, which indicates that the performance of $\hat{\mu}_{SC}(\mathbf{x})$ is indeed an improvement upon the performance of $\hat{\mu}_{VM}(\mathbf{x})$, and $\hat{\mu}_{LT}(\mathbf{x})$.

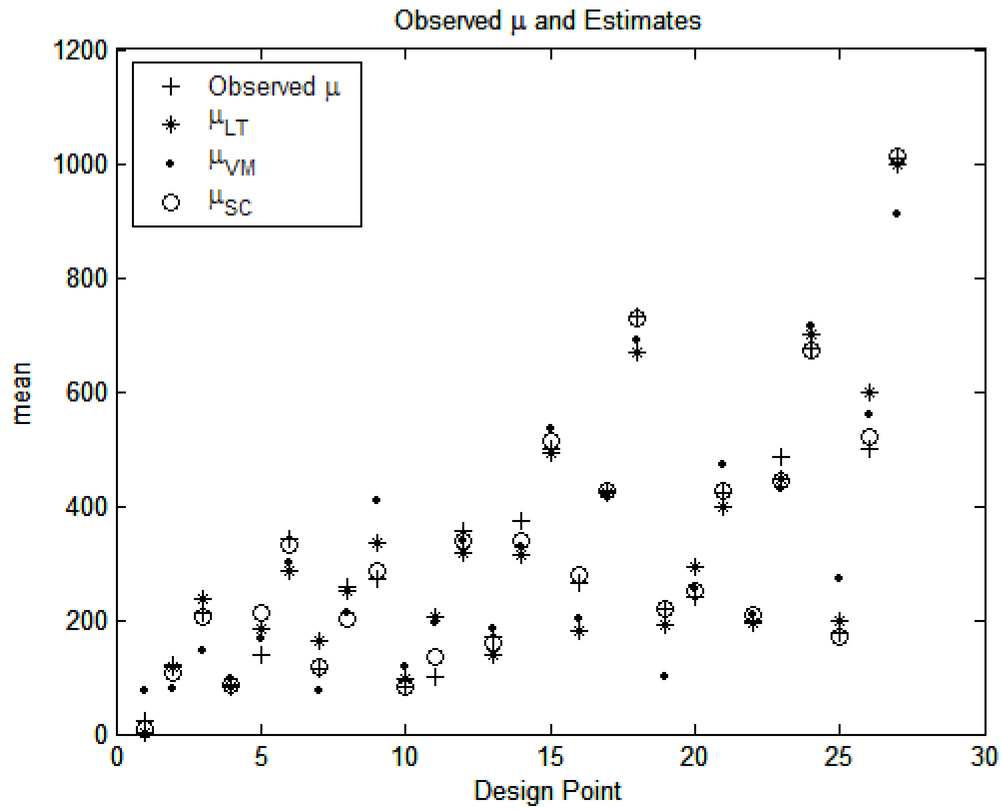

Figure 2.1 Comparing Models for the Mean Response

For the same data, VM modeled the standard deviation by the full quadratic model

$$\hat{\sigma}_{VM} = 34.9 + 11.5x_1 + 15.3x_2 + 29.2x_3 + 4.2x_1^2 - 1.3x_2^2 + 16.8x_3^2 + 7.7x_1x_2 + 5.1x_1x_3 + 14.1x_2x_3 \quad (27)$$

In an attempt to improve upon this model, LT considered the full third order model and obtained the model in Equation (28) via model selection techniques.

$$\hat{\sigma}_{LT} = 47.994 + 11.527x_1 + 15.323x_2 + 29.190x_3 + 29.566x_1x_2x_3 \quad (28)$$

Table 2.5 compares the standard deviation models of Equations (27) and (28) with the model $\hat{\sigma}_{SC}(\mathbf{x})$ in Equation (24) that we proposed in this paper. Again, we observe that our proposed model, $\hat{\sigma}_{SC}(\mathbf{x})$, gives the least values of PRESS and RMSE, and also the highest values of R^2, R_a^2, and R_{pred}^2. A graphical comparison of the three models is shown in Figure 2.2, and just as in the case of the means, we observe that the values obtained from $\hat{\sigma}_{SC}(\mathbf{x})$ are more likely to be closer to the observed standard deviation values than the values obtained from $\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\mu}_{VM}(\mathbf{x})$.

Table 2.5: Comparing the Standard Deviation Models

Model	PRESS	RMSE	R^2	R_a^2	R_{pred}^2
$\hat{\sigma}_{SC}(\mathbf{x})$	21219.9	20.3790	94.5	82.0	64.64
$\hat{\sigma}_{VM}(\mathbf{x})$	93691.4	44.0414	45.0	16.0	0.00
$\hat{\sigma}_{LT}(\mathbf{x})$	46809.5	37.6679	48.0	38.5	21.99

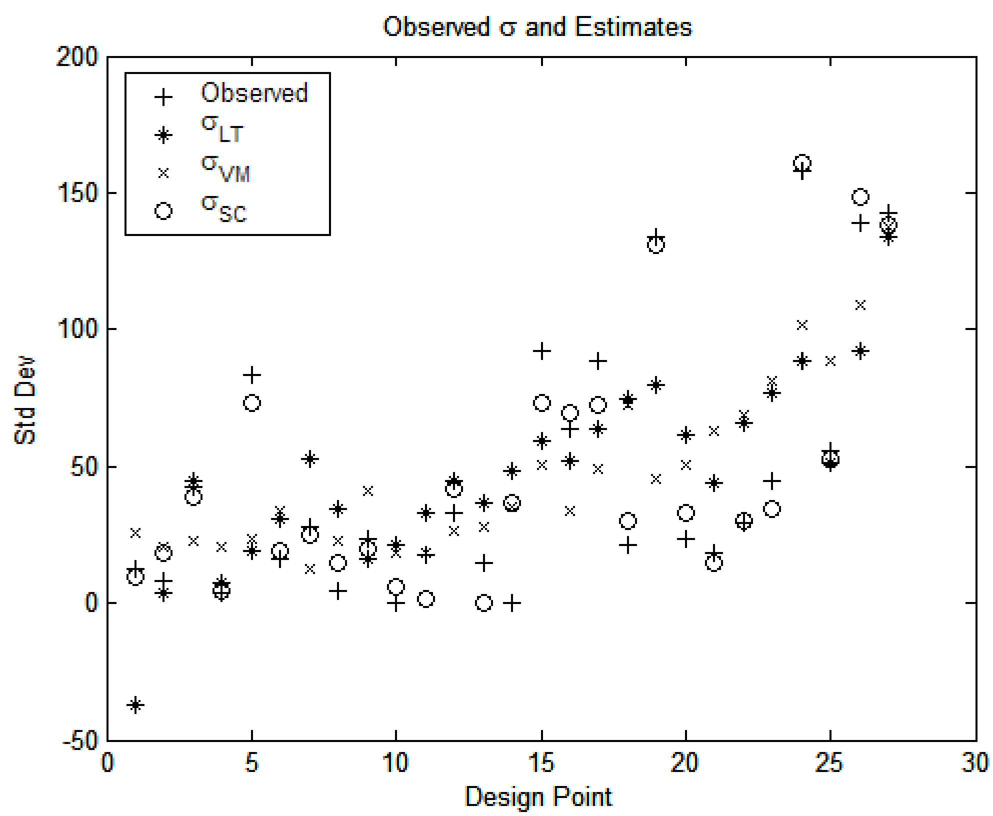

Figure 2.2 Comparing Models for the Standard Deviation

Optimization

In this part of the example, we will solve the various optimization models presented above, based on the response surface models obtained through our proposed methodology and those of VM and LT. We solve the optimization models of Equations (12), (13) and (15) using the *fmincon* routine in MATLAB, and compare the optimum solutions.

Table 2.6 shows the solutions to the MSE optimization model (i.e. Equation (15)) with a target value of 500 for the mean response. We observe that the smallest MSE value is obtained from using the higher order models of this paper (i.e. $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$). Also, $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ give the smallest value of standard deviation and the closest mean value to the target mean. The observations here support the observations made in the previous section when the various response surface models were compared.

Table 2.6 Comparing Solutions to the MSE Optimization Problem (15) with *T*=500

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$	$f(\mathbf{x}^*)$ (MSE)
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(1.000, 1.000, -0.525)	492.231	44.136	2008.309
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(1.000, 0.0599, -0.242)	494.651	44.599	2017.668
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(1.000, 1.000, -0.561)	499.884	12.106	146.557

In Table 2.7, we display the solutions to the optimization problem in Equation (12) for the various response surface models of mean and standard deviation. Again, the results based on $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ give the least values of MSE and standard deviation. By examining Tables 2.6 and 2.7 together, we observe that $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ are relatively more robust in the sense that the MSE value at optimality is about the same for both optimization methods.

Table 2.7 Comparing Solutions to Optimization Problem (12) (VM) with T=500

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$	$f(\mathbf{x}^*)$ (MSE)
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(1.000, 1.000, -0.502)	500	45.503	2070.529
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(1.000, 0.104, -0.250)	500	45.241	2046.718
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(1.000, 0.105, -0.561)	500	12.107	146.570

VM used the method of Lagrange multipliers and solved the L-type optimization problem in Equation (13) for the data of Table 2.1, assuming target standard deviation values (T_S) of 60, 75, and 90. For T_S = 60, we solve the same problem for the three sets of response surface models. Table 2.8 compares the solutions obtained. The solution based on $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ gives a larger mean response value (at the target standard deviation) than the solutions based on the response surface models of MV and LT.

Table 2.8 Comparing Solutions to the VM's L-Type Optimization Problem (13) with S=60

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(1.000, 1.000, -0.254)	582.355	60
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(1.000, 1.000, -0.278)	616.994	60
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(1.000, 1.000, 0.384)	856.962	60

In Table 2.9, we show the solutions to VM's S-type optimization problem in (14). Again, the solution based on $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ yields the most desired result, i.e. the smallest mean response value.

Table 2.9 Comparing Solutions to the VM's *S*-Type Optimization Problem (14) with S=60

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(-1.0000, -1.0000, 0.6604)	157.5408	60
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(-1.0000, -0.3810, 1.0000)	172.8955	60
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(-1.0000, -1.0000, 0.5582)	149.9620	60

It is worth noting from Tables 2.7, 2.8, and 2.9 that all the response surfaces considered gave optimal solutions that achieved the desired target in each case. However, the optimal settings in each case are not exactly the same. This accentuates the need for obtaining more effective response surface models in terms of prediction ability.

In what follows, we will solve the proposed optimization models of this work for all the response surface models we have been considering, and compare the results in a similar manner. We solved our proposed *L*-Type model in Equation (17) for the data shown in Table 2.1 and obtained exactly the same solution set in Table 2.8. However, we recommend that the model in Equation (18) be considered, since it has the flexibility of considering smaller values of the standard deviation in the optimization process. Table 2.10 compares the solutions to Equation (17) obtained by using the various response surface models. Clearly, by allowing a little bias in the standard deviation, larger mean response values are obtained as compared to the solution of VM's model in Equation (13), which allows no bias in the standard deviation. We again observe that $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ give the smallest optimum objective function value, the smallest optimum standard deviation, and the largest optimum mean response value.

Table 2.10 Comparing Solutions to the Proposed *L*-Type Optimization Model (17) with $T_S = 60$

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$	$f(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(1.000, 1.000, -0.206)	598.491	62.840	-590.423
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(1.000, 1.000, -0.200)	637.747	63.156	-627.790
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(1.000, 1.000, 0.399)	861.624	61.518	-859.320

For our proposed *S*-Type model in (19), we use the target standard deviation T_S = 60 and solve it obtaining exactly the same solutions for VM's *S*-Type model in Equation (14) shown in Table 2.9. Table 2.11 shows the optimal solutions to model (20) (i.e the alternative *S*-type model to (19)) for the same data when T_S = 60. We observed here that by relaxing the equality constraint, the standard deviation actually dropped slightly and the mean response is further decreased. Also in the case also, the solutions based on $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ are more desirable in terms of the objectives, i.e. smaller standard deviation and smaller mean.

Table 2.11 Comparing Solutions to the Proposed *S*-Type Optimization Model (20) with T_S=60

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$	$f(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(-1.0000, -1.0000, 0.6466)	156.8879	59.1920	156.8879
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(-1.0000, -0.4822, 1.0000)	162.1393	59.4685	167.5959
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(-1.0000, -1.0000, 0.5542)	149.3975	57.6641	149.6801

Finally, we solve our proposed *N*-Type problem in Equation (21) for the data of Table 2.1 using the sets of target values (T = 500, T_S = 60) and (T = 600, T_S = 20). Table 2.12 shows the optimum solutions of this model for the first set of target values, where all the response surfaces used achieved the desired targets.

Table 2.12 Comparing Solutions to the Proposed *N*-Type Optimization Model (21) with T=500 and T_S=60

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(0.575, 0.900, -0.192)	500.000	60.000
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(0.344, 0.504, 0.274)	500.000	60.000
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(0.477, 0.545, -0.017)	500.000	60.000

In Table 2.13, we display the optimum solutions for the second set of target values. We observe in this case that only the solutions based on $\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$ achieved the desired targets, and therefore give the smallest possible objective function value (to three decimals). The solutions based on the response surfaces of VM and LT give much larger standard deviation values and smaller means than T.

Table 2.13 Comparing Solutions to the Proposed *N*-Type Optimization Model (16) with T=600 and T_S=20

Models of Mean and Standard Deviation	Optimal Settings $\mathbf{x}^*$	$\hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}(\mathbf{x}^*)$	$f(\mathbf{x}^*)$
$\hat{\mu}_{LT}(\mathbf{x})$ and $\hat{\sigma}_{LT}(\mathbf{x})$	(1.000, 1.000, -0.223)	592.640	61.811	1802.29
$\hat{\mu}_{VM}(\mathbf{x})$ and $\hat{\sigma}_{VM}(\mathbf{x})$	(0.344, 0.504, 0.274)	595.085	56.947	1389.24
$\hat{\mu}_{SC}(\mathbf{x})$ and $\hat{\sigma}_{SC}(\mathbf{x})$	(0.943, 0.997, -0.289)	600.000	20.000	0.00

Concluding Remarks

In the context of robust parameter design optimization problems, we have addressed the need to consider higher order polynomial response surface models for the mean and standard deviation of quality characteristics as a way of increasing the predictive ability of the response surface models. A numerical example was used to illustrate the increased accuracy of the response surface models obtained in this work relative to existing response surface models for the same example. Significant increases in *R*-square, adjusted *R*-square, and predicted *R*-square were achieved by the models of this work, as well as significant decreases in root mean square error, and predicted error sum of squares. For example, relative to the best of the existing response surface models for the example considered, the mean response model obtained in this paper is shown to be higher by 3.95% in predicted R-square, while the standard deviation model is higher by 193.95%. The improvement in the modeling of the standard deviation is particularly important since modeling variability has generally been problematic, as observed in the literature. Optimization models were proposed and solved for the larger-the-better type, the smaller-the-better type, and the nominal-the-best type quality characteristics assuming in each case that there is a target value for the standard deviation.

We believe that considering higher order response surface models and using the well-known statistical model selection techniques properly will enhance the quality of solutions to robust parameter design problems. In fact, where the most powerful models are of lower order, such models will be sought out by the model selection procedure. Therefore, the proposed procedure of this paper is very likely to yield response surface models that are at least as powerful as the existing lower order models in the literature.

A natural extension of this work would be in the consideration of multiple quality characteristics, and modeling involving both controllable and noise factors. Finally, we recommend a great deal of caution when using the kinds of response surface models proposed in this paper and in VM and LT, as it is possible for such models to give results of no practical meaning or significance. For example, LT's standard deviation model, $\hat{\sigma}_{LT}(\mathbf{x})$ in Equation (7), gives a negative standard deviation (-37.3) when $\mathbf{x}$ = [-1, -1, -1].

Therefore, even with very powerful response surfaces, we recommend including constraints in optimization models that will serve to prevent such results from occurring (e.g. $0 \leq \sigma(\mathbf{x}) \leq T_S'$, where T_S' is the upper bound value for the standard deviation). Another option is to consider functional forms such as exponential and logistic functions, which do not allow negative output values.

CHAPTER THREE

DEVELOPMENT OF REALISTIC QUALITY LOSS FUNCTIONS FOR INDUSTRIAL APPLICATIONS

Introduction

A number of quality loss functions, most recently the Taguchi loss function, have been developed to quantify the loss due to the deviation of product performance from the desired target value. All these loss functions assume the same loss at the specified specification limits. In many real life industrial applications, however, the losses at the two different specifications limits are often not the same. Further, current loss functions assume a product should be reworked or scrapped if product performance falls outside the specification limits. It is a common practice in many industries to replace a defective item rather than spending resources to repair it, especially if considerable amount of time is required. To rectify these two potential problems, this Chapter proposes more realistic quality loss functions for proper applications to real-world industrial problems. We also conduct comparison studies of all the loss functions it considers. We organize the rest of the Chapter under the following Section headings:

- Notation and Assumptions
- Modification of the Quadratic Loss Function
- Exponential Loss Functions
- Numerical Example
- Conclusions

Notation and Assumptions

The notation and assumptions we employ throughout this paper are as follows:

Notation

X a quality characteristic of interest

x a value assumed by the quality characteristic

L lower specification limit of the quality characteristic

U upper specification limit of the quality characteristic

τ target value of the quality characteristic

$\Delta_l = \tau - L$ and $\Delta_u = U - \tau$

μ mean of quality characteristic

σ^2 variance of quality characteristic

$L_z = \frac{L-\mu}{\sigma}$, $U_z = \frac{U-\mu}{\sigma}$, and $\tau_z = \frac{\tau-\mu}{\sigma}$

c_l the cost of the measure of the quality characteristic falling below the lower specification limit L

c_u the cost of the measure of the quality characteristic falling above the upper specification limit U

$cmax_l$ maximum allowable cost for deviations to the left of the target value

$cmax_u$ maximum allowable cost for deviations to the right of the target value

K_l Loss function constant for deviations to the left of the target value

K_u Loss function constant for deviations to the right of the target value

K_l and K_u will be computed differently for each loss function.

Assumptions

1. On each side of the target value, the cost due to a deviation from the target value within specification limits is less than the out-of-specification cost, but is zero at the target value while the within-specification cost increases as product performance deviates from the target value.
2. For quality characteristic values within the specification limits, the out-of-specifications costs are incurred, when the cost of deviations from the target value are 'significantly' close to the out-of-specification costs. For reasonability of this assumption, refer to the scenarios outlined in the introduction.
3. On each side of the target value, a cost $cmax_i$ is set, which is less than the out-of-specification cost in that direction. Any costs greater than $cmax_i$ are declared as significantly close to the out-of-specification cost.

Note that the first assumption above embodies the assertions put forth in Taguchi (1981, 1986) with regard to the within-specification loss due to deviations from the target value. The second and third assumptions capture the practical scenarios cited in the introduction.

Quadratic Loss Function

In this section, we consider the Taguchi loss function and modify it to satisfy the assumptions outlined above. We will first consider the situation where the maximum allowable within-specification costs are achieved at the specification limits. Secondly, we consider the situation where setting the maximum allowable within-specification costs entail narrowing the tolerance.

Consider the situation where the within-specification costs vary according a quadratic model satisfying the assumptions outlined above, where the maximum allowable within-specification costs are incurred at the specification limits. Then the within-specification deviation costs are given by the function

$$L(x) = \begin{cases} K_l (x-\tau)^2 & \text{if } \mathrm{L} \le x \le \tau \\ K_u (x-\tau)^2 & \text{if } \tau < x \le U \end{cases}, \qquad (1)$$

where $K_l = \frac{c\max_l}{(L-\tau)^2} = \frac{c\max_l}{\Delta_l^2}$ and $K_u = \frac{c\max_u}{(U-\tau)^2} = \frac{c\max_u}{\Delta_u^2}$. Figure 3.1 illustrates this function along with the out-of-specification costs. Note that when $c\max_l = c_l$ and $c\max_u = c_U$, this loss function becomes the traditional Taguchi loss function defined in Chapter 1.

It is observed that the maximum allowable costs may not be incurred at the specification limits. In fact,-the costs at the specification limits may be greater than the maximum allowable costs. An example of such a case would be when the loss function shown in Figure 3.1 adequately models the losses. In such cases, if company policy sets maximum allowable costs, it would imply computing new specification limits, L_1 and U_1 with $L < L_1$ and $U_1 < U$, thereby creating tighter tolerance. In this paper, we propose the following lemmas on the values of L_1 and U_1 which can be used for a general form of the quadratic loss function in order to make its results practically applicable.

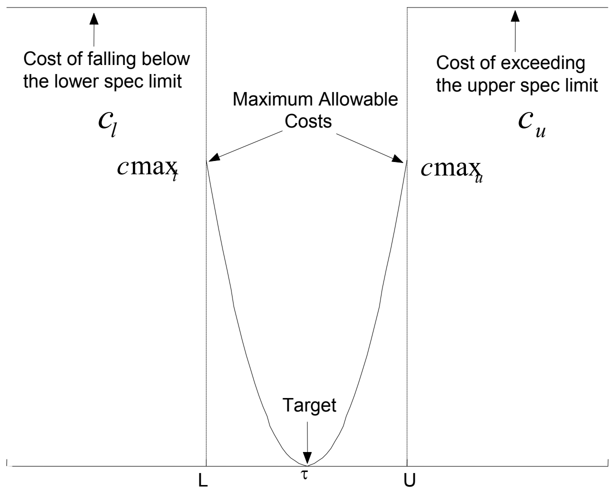

Figure 3.1 Modified Taguchi Loss Function

Lemma 1: *Suppose the loss due to a deviation from its target value is modeled by the function*

$$L(x) = \begin{cases} \gamma_1 (x-\tau)^2 & if \quad L \le x \le \tau \\ \gamma_2 (x-\tau)^2 & if \quad \tau < x \le U \end{cases} \tag{2}$$

where γ_1 and γ_2 are constants, and L and U are the upper and lower specification limits, respectively. Also suppose $c\max_l < \gamma_1 (L-\tau)^2$ and $\mathrm{cmax}_u < \gamma_2 (U-\tau)^2$. Then the new specification limits L_1 and U_1 that give the maximum allowable costs of cmax_l *and* cmax_u *under (6) are given by*

$$L_1 = \tau - \sqrt{\frac{\mathrm{cmax}_l}{\gamma_1}} \text{ and } U_1 = \tau + \sqrt{\frac{\mathrm{cmax}_u}{\gamma_2}}.$$

Proof:

Clearly, $L_1 < \tau$ and so

$$L(L_1) = C_1\left(\tau - \sqrt{\frac{\text{cmax}_l}{\gamma_1}} - \tau\right)^2 = \text{cmax}_l$$

Similarly,

$$L(U_1) = \gamma_2\left(\tau + \sqrt{\frac{\text{cmax}_u}{\gamma_2}} - \tau\right)^2 = \text{cmax}_u .$$

The loss function that is obtained from applying the proposed specification limits L_1 and U_1 specified in Lemma 1 to any loss function in the form of Equation (2) is identical to the loss function in Equation (1), which is illustrated in Figure 3.1, and satisfies the assumptions outlined in Section 2. In the following Lemma, we present the expected loss under the loss function (2) when the quality characteristic follows a normal distribution with the mean μ and variance σ^2.

Lemma 2: *Suppose a quality characteristic follows the normal distribution with mean μ and variance σ^2. Then under the loss function shown in (1), the expected loss for deviations within the specification limits is given by*

$$\begin{aligned} E[L(x)] = \frac{\sigma^2}{\sqrt{2\pi}} &\left\{ K_l\left(L_z - 2\tau_z\right)e^{-\frac{1}{2}L_z^2} - K_u\left(U_z - 2\tau_z\right)e^{-\frac{1}{2}U_z^2} + (K_l - K_u)\tau_z e^{-\frac{1}{2}\tau_z^2} \right\} \\ &+ \sigma^2(1+\tau_z^2)\left\{K_u\Phi(U_z) - K_l\Phi(L_z) + \left(K_l - K_u\right)\Phi\left(\tau_z\right)\right\} \end{aligned} \quad (3)$$

where L_z, τ_z, and U_z are the standardized values (see notations) of L, τ, and U, respectively.

Proof: See the Appendix.

The result of Lemma 2 is quite general for quality characteristics following the normal distribution with mean μ and variance σ^2. If K_l and K_u coincide with the constants of the K_1 and K_2 defined in Chapter 1, then the result gives the expected loss under the original Taguchi loss function. A number of special cases of (3) are possible, namely $K_l = K_u$ or $\tau = \mu$, or both. If $K_l = K_u$, the expected loss function is as shown in Equation (4) below.

$$E[L(x)] = \frac{K_l \sigma^2}{\sqrt{2\pi}} \left\{ (L_z - 2\tau_z) e^{-\frac{1}{2}L_z^2} - (U_z - 2\tau_z) e^{-\frac{1}{2}U_z^2} \right\} + K_l \sigma^2 (1 + \tau_z^2) \{\Phi(U_z) - \Phi(L_z)\} \tag{4}$$

If the process mean coincides with the target value, i.e. $\tau = \mu$, then $\tau_z = 0$ and the expected loss is

$$E[L(x)] = \frac{\sigma^2}{\sqrt{2\pi}} \left\{ K_l L_z e^{-\frac{1}{2}L_z^2} - K_u U_z e^{-\frac{1}{2}U_z^2} \right\} + \sigma^2 \{K_u \Phi(U_z) - K_l \Phi(L_z) + 0.5(K_l - K_u)\} \tag{5}$$

If $K_l = K_u$ and $\tau = \mu$, the expected loss is then given by

$$E[L(x)] = \frac{K_l \sigma^2}{\sqrt{2\pi}} \left\{ L_z e^{-\frac{1}{2}L_z^2} - U_z e^{-\frac{1}{2}U_z^2} \right\} + K_l \sigma^2 \{\Phi(U_z) - \Phi(L_z)\} \tag{6}$$

Proposed Loss Functions for N-type Quality Characteristics

In this section, we propose two alternatives to the quadratic loss function, which also satisfy the assumptions of this Chapter as well as the basic postulate of the Taguchi loss function. That is, the loss is zero at the target value and increases with increasing deviation from the target. For each loss function, the expected loss is obtained when the quality characteristic follows a normal distribution with mean μ and variance σ^2.

Type I Exponential Loss Function

Let a quality characteristic have a target value τ, lower and upper specification limits L and U respectively, and the maximum allowable losses for deviations within the specification limits $c\max_l$ and $c\max_u$. The first proposed quality loss function is given by

$$L(x) = \begin{cases} e^{-K_l(x-\tau)}\text{-1} & \text{if } L \le x \le \tau \\ e^{K_u(x-\tau)}\text{-1} & \text{if } \tau < x \le U \end{cases}, \tag{7}$$

where $K_l = \frac{1}{\Delta_l}\ln(1 + c\max_l)$ and $K_u = \frac{1}{\Delta_u}\ln(1 + c\max_u)$. In this paper, this function is referred to as a type I Exponential loss function which is depicted in Figure 3.2. We observe that Taguchi's basic postulate is satisfied by our proposed loss function – the loss increases with increasing deviation from the target value. Also, all the assumptions outlined in this Chapter are satisfied. In Lemma 3, we give the expected loss under this function when the quality characteristic follows a normal distribution with mean μ and variance σ^2.

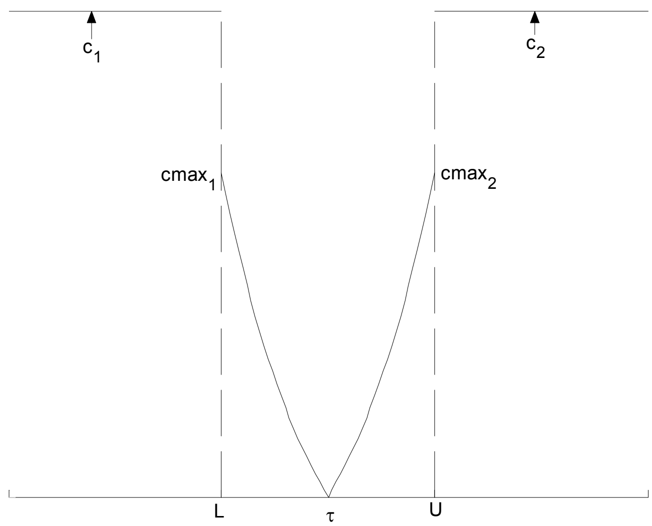

Figure 3.2 Type I Exponential Loss Function

Lemma 3: *Suppose a quality characteristic follows the normal distribution with mean μ and variance σ^2. Then under the loss function in Equation (7), the expected loss for a deviation within the specification limits is given by*

$$\begin{aligned} E[L(x)] = &\{\Phi(\tau_z + K_l\sigma) - \Phi(L_z + K_l\sigma)\}\exp\left\{\frac{K_l\sigma(K_l\sigma + 2\tau_z)}{2}\right\} + \Phi(L_z) + \\ &\{\Phi(U_z - K_u\sigma) - \Phi(\tau_z - K_u\sigma)\}\exp\left\{-\frac{K_u\sigma(2\tau_z - K_u\sigma)}{2}\right\} - \Phi(U_z) \end{aligned} \quad (8)$$

Proof: The method of proof of this Lemma is similar to the method used in the Appendix to prove Lemma 2.

Again the result shown in Equation (8) is general, where the target value does not necessarily coincide with the mean. However, if $\tau = \mu$, then $\tau_z = 0$ and the expected loss function in Equation (8) reduces to Equation (9) below.

$$E\left[L(x)\right]=\left\{\Phi\left(K_l\sigma\right)-\Phi\left(L_z+K_l\sigma\right)\right\}\exp\left\{\frac{K_l^2\sigma^2)}{2}\right\}+\Phi\left(L_z\right)+$$
$$\left\{\Phi\left(U_z-K_u\sigma\right)-\Phi\left(-K_u\sigma\right)\right\}\exp\left\{\frac{K_u^2\sigma^2}{2}\right\}-\Phi\left(U_z\right) \quad (9)$$

Type II Exponential Loss Function

For the quality characteristic described above, we propose another quality loss function, referred to as a type II exponential loss function, as follows:

$$L(x)=\begin{cases} c_l\left(1-\mathrm{e}^{-K_l(x-\tau)^2}\right) & \text{if } \mathrm{L}\le x\le\tau \\ \mathrm{c_u}\left(1-\mathrm{e}^{-K_u(x-\tau)^2}\right) & \text{if } \tau< x\le U \end{cases}, \quad (10)$$

where $K_l=-\frac{1}{\Delta_l^2}\ln(1-\frac{\mathrm{c}\max_l}{c_l})$ and $K_u=-\frac{1}{\Delta_u^2}\ln(1-\frac{\mathrm{c}\max_u}{c_u})$. Note that this function does not allow $c\max_l=c_l$ and $c\max_{ul}=c_u$, as K_l and K_u do not exist under these cases. For K_l and K_u to exist, we must have $c\max_l<c_l$ and $c\max_{ul}<c_u$. A schematic graph of this loss function is shown in Figure 3.3 below.

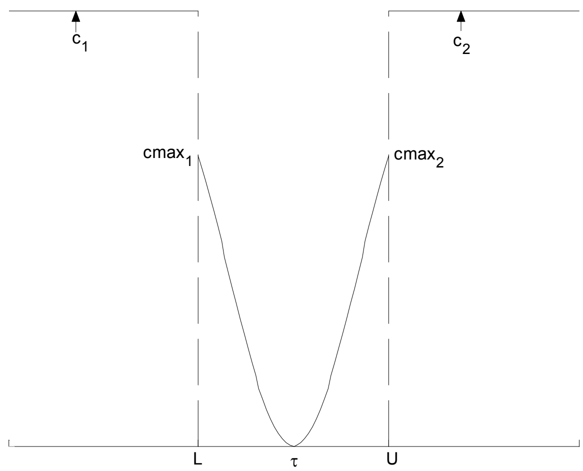

Figure 3.3 Type II Exponential Loss Function

The following lemma, which can be proved by following the method of the Appendix, gives the expected loss under the type II exponential loss function for a normally distributed quality characteristic.

Lemma 4: *Suppose a quality characteristic follows the normal distribution with mean μ and variance σ^2. Then under the loss function in (10), the expected loss for a deviation within the specification limits is given by*

$$E[L(x)] = c_u\Phi\left(U_z\right) - c_l\Phi\left(L_z\right) + (c_l - c_u)\Phi\left(\tau_z\right) -$$
$$\frac{c_l\sigma}{\sqrt{1+2K_l\sigma^2}}\left\{\Phi\left(\frac{\tau_z}{\sqrt{1+2K_l\sigma^2}}\right) - \Phi\left(\frac{2K_l\,\sigma(L-\tau)+L_z}{\sqrt{1+2K_l\sigma^2}}\right)\right\}\exp\left\{\frac{-K_l\sigma^2\tau_z}{1+2K_l\sigma^2}\right\} - \quad (11)$$
$$\frac{c_u\sigma}{\sqrt{1+2K_u\sigma^2}}\left\{\Phi\left(\frac{2K_u\sigma(U-\tau)+U_z}{\sqrt{1+2K_u\sigma^2}}\right) - \Phi\left(\frac{\tau_z}{\sqrt{1+2K_u\sigma^2}}\right)\right\}\exp\left\{\frac{-K_u\sigma^2\tau_z}{1+2K_u\sigma^2}\right\}$$

Under the special case of $\tau = \mu$, we have $\tau_z = 0$ and the expected loss in (11) becomes

$$E[L(x)] = c_u\Phi(U_z) - c_l\Phi(L_z) + 0.5(c_l - c_u) - \frac{c_l\sigma}{\sqrt{1+2K_l\sigma^2}}\left\{0.5 - \Phi\left(\frac{2K_l\sigma(L-\tau)+L_z}{\sqrt{1+2K_l\sigma^2}}\right)\right\} - \frac{c_u\sigma}{\sqrt{1+2K_u\sigma^2}}\left\{\Phi\left(\frac{2K_u\sigma(U-\tau)+U_z}{\sqrt{1+2K_u\sigma^2}}\right) - 0.5\right\} \quad (12)$$

In the next section, we present a numerical example as an illustration of the results of this work. Through the example, we compare all the loss functions considered in this paper.

<u>Numerical Example 1</u>

Consider the manufacturing case of electronic equipment where the distance between two pins of a memory chip is known to follow the normal distribution with mean 5*mm* and standard deviation 0.0002*mm*. The target value is adjusted to coincide with the mean and a product is functional if the distance between the pins is in the interval 1.5 ± 0.01. The cost of the distance falling out of the specification interval on either side is $100, which is the cost of replacing a defective chip. Company policy requires that the cost of repairs for distances within the specification limits must not exceed $80.

Using the notation of this chapter, the parameter values for this problem are:

L = 1.49, U = 1.51, τ = 1.5, μ =1.5, σ =0.0002, c_l = 100, c_u =100, $cmax_l$ = 80, $cmax_u$ = 80.

In the following subsection, we present all the loss functions considered in this study for this problem, and the expected loss within the specification limits under each of the loss functions.

Loss Functions and Expected Losses

For the data of this example, Figure 3.4 shows the traditional quadratic loss function, its proposed modification, and the two exponential loss functions.

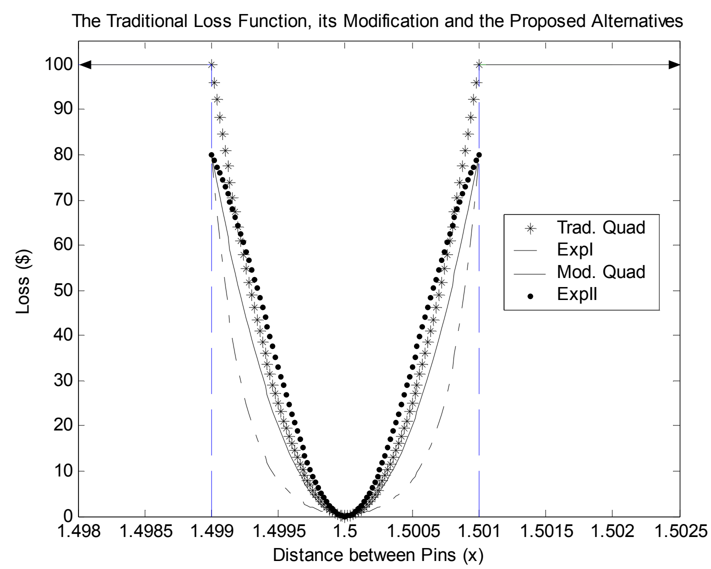

Figure 3.4 The Traditional Loss Function, its Modification and the Proposed Alternatives

Clearly, the traditional quadratic loss function does not adequately represent the situation of this problem as it has no logical way of taking the company policy into consideration. That is, the traditional loss function cannot put a cap on the losses incurred by the deviations within the specification limits. Equation (3) gives the expected loss for the deviations within the specification limits under the modified quadratic loss function.

Using the traditional Taguchi loss constants K_1 and K_2 as defined in Chapter 1 in place of K_l and K_u respectively in Equation (3), we obtain the expected loss under the traditional quadratic loss function. Equations (8) and (11) give the expected loss under the type I and type II exponential loss functions, respectively. Table 3.1 below shows the expected loss within the specification limits under each of the loss functions.

Table 3.1 Expected Loss under Each Loss Function

Function	Traditional Quadratic	Modified Quadratic	Type I Exponential	Type II Exponential
Expected Loss ($)	0.0400	0.0320	0.0742	0.0643

We observe that the loss function with the highest penalty is the type II exponential loss function, while the type I exponential loss function is the loss function with the lowest penalty. The modified quadratic loss function, which we propose in this paper, to incorporate the company policy of imposing a cap on the loss within specification gives lowest expected loss, compared to the loss resulting from the traditional loss function. These results corroborate the trends that can be observed in Figure 3.4.

Loss Functions for L-Type Quality Characteristics

The loss functions discussed thus far are for N-type quality characteristics. In this section, we propose two loss functions for L-type quality characteristics, which also satisfy the realistic properties proposed above for N-type quality characteristics, namely, there is a maximum allowable loss for quality characteristic values that are within specification but deviate from the desired target. As discussed earlier, L-type quality characteristics only have lower specification limits and the larger the value of the quality characteristic, the better. In other words, the target value for an L-type quality characteristic is at infinity. This property of the L-type quality characteristics will also be exhibited in the loss functions we propose.

Type I Exponential Loss Function for L-Type Quality Characteristics

For an L-type quality characteristic with a lower limit L, we propose the loss function

$$L(x) = \begin{cases} c_l & \text{if } x < L \\ k_1 e^{-k_2(x-L)} & \text{if } x \geq L \end{cases}, \tag{13}$$

where $k_1 = cmax_l$ is the maximum allowable loss for quality characteristic values within specification limits, which occurs at the lower specification limit when $x = L$. We call (13) a type I exponential loss function, which we illustrate in Figure 3.5.

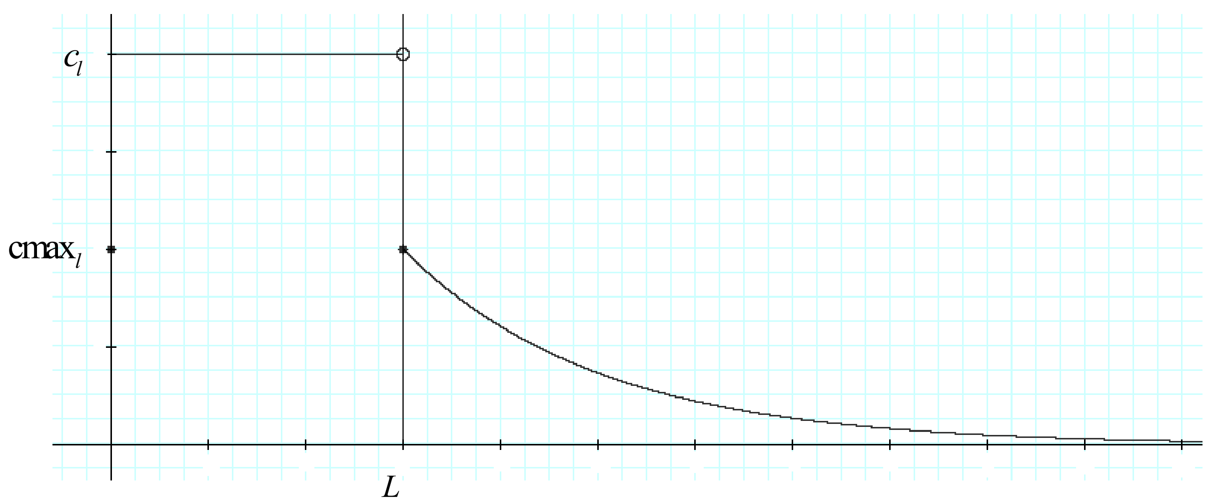

Figure 3.5 Type I Exponential Loss Function for L-Type Quality Characteristics

The constant k_2 can be determined in two ways:

(a) Suppose a loss value is known for a certain value of the quality characteristic, say $L(x_0) = c$. Then, $k_1 e^{-k_2(x_0 - L)} = c$ and

$$k_2 = -\frac{1}{x_0 - L}\ln\frac{c}{k_1} = -\frac{1}{x_0 - L}\ln\frac{c}{\mathrm{cmax}_l}. \tag{14}$$

(b) For $x \geq L$, taking logarithm of the loss function, we get

$$\ln L(x) = \ln k_1 - k_2(x - L)$$

Therefore, if loss values are known for several values of the quality characteristic, we can determine k_2 by finding a linear regression model of $\ln L(x)$ versus $(x - L)$, in which case $-k_2$, is given by the slope of the regression line.

The total expected loss under the loss function in (13) for a normally distributed L-type quality characteristic with a lower limit L is given by

$$E[L(x)] = c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{\frac{k_2^2\sigma^2 + 2k_2(L-\mu)}{2}\right\}\cdot\left\{1 - \Phi\left(\frac{L - (\mu - k_2\sigma^2)}{\sigma}\right)\right\} \tag{15}$$

Type II Exponential Loss Function for L-Type Quality Characteristics

For an L-Type quality characteristic with a lower limit L, we propose the loss function

$$L(x) = \begin{cases} c_l & \text{if } x < L \\ k_1 e^{-k_2(x-L)^2} & \text{if } x \geq L \end{cases} \tag{16}$$

where $k_1 = \text{cmax}_l$, the maximum allowable loss, which is realized at the lower specification limit when $x = L$. We refer to this function as a type II exponential loss function for L-type quality characteristics. Figure 3.6 illustrates the intuitiveness of this loss function is illustrated in Figure 3.1 – the loss decreases with increasing values of the quality characteristic.

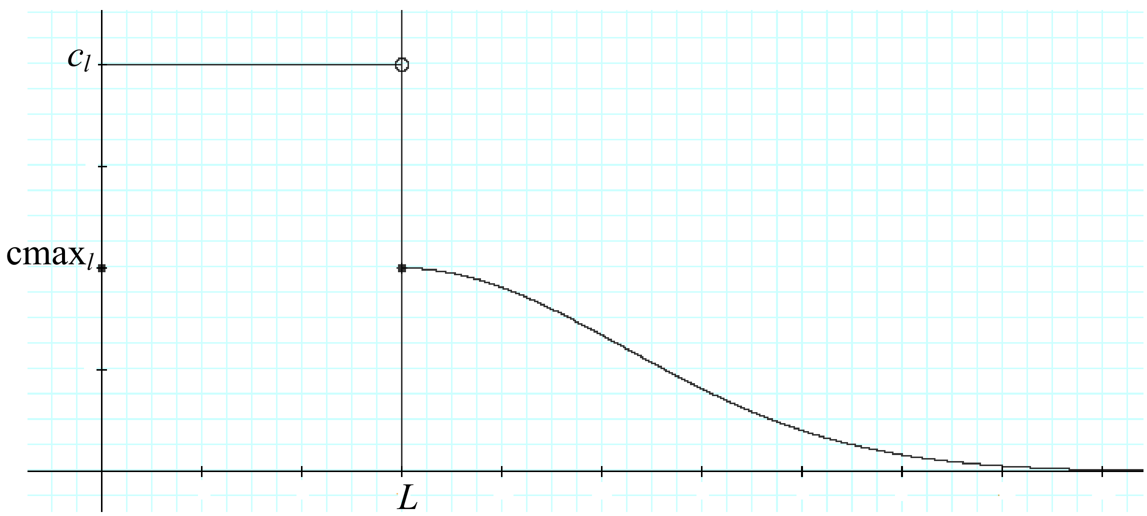

Figure 3.6 Type II Exponential Loss Function for L-Type Quality Characteristics

Similar to the case with the type I loss function, we propose two ways of finding the constant k_2 depending on available data:

(a) If a value of the loss function is know for a value of the quality characteristic, say $L(x_0) = c$, where $x_0 \neq L$, then $k_1 e^{-k_2 (x_0 - L)^2} = c$ and

$$k_2 = -\frac{1}{(x_0 - L)^2} \ln \frac{c}{k_1} = -\frac{1}{(x_0 - L)^2} \ln \frac{c}{\text{cmax}_l} \tag{17}$$

(c) For $x \geq L$, the logarithm of the loss function is

$$\ln L(x) = \ln k_1 - k_2 (x - L)^2 \tag{18}$$

Therefore, by considering the regression of $\ln L(x)$ versus $(x-L)^2$, the slope is found as -k_2. Under normal distribution, the total expected loss when the type II exponential function applies is given by

$$E[T(x)] = c_l\Phi\left(\frac{L-\mu}{\sigma}\right) + \frac{k_1}{\sqrt{2k_2\sigma^2+1}}\exp\left[-\frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}\right]\left[1-\Phi\left(\frac{L-\mu}{\sigma\sqrt{2k_2\sigma^2+1}}\right)\right] \quad (19)$$

<u>Numerical Example 2</u>

Consider an L-Type quality characteristic that is normally distributed with mean 650 units, standard deviation 15 units, and a lower specification limit of 600 units. The out-of-specification cost is \$20 and company policy limits with-in-specification costs to a maximum of \$12, and the loss experienced on a typical item with the quality characteristic at 630 units is \$4. For this scenario, the given parameters are as follows:

L = 600, μ = 650, σ = 15, c_l = 20, $cmax_l$ = 12 = k_1, x_0 = 630, c = 4

For the type I exponential loss function, $k_2 = -\frac{1}{x_0 - L}\ln\frac{c}{k_1} \approx 0.0366$ and so the loss function is of the form

$$L(x) = \begin{cases} 20 & \text{if } x < 600 \\ 12e^{-0.0366(x-L)} & \text{if } x \geq 600 \end{cases}$$

Similarly, for the type II exponential loss function, $k_2 = -\frac{1}{(x_0 - L)^2}\ln\frac{c}{k_1} \approx 0.0012$ and the loss function is given by

$$L(x) = \begin{cases} 20 & \text{if } x < 600 \\ 12e^{-0.0012(x-L)^2} & \text{if } x \geq 600 \end{cases}$$

Figure 3.7 displays the exponential loss functions and the distribution of the quality characteristic (not to scale) for values of the quality characteristic that exceed the lower specification limit.

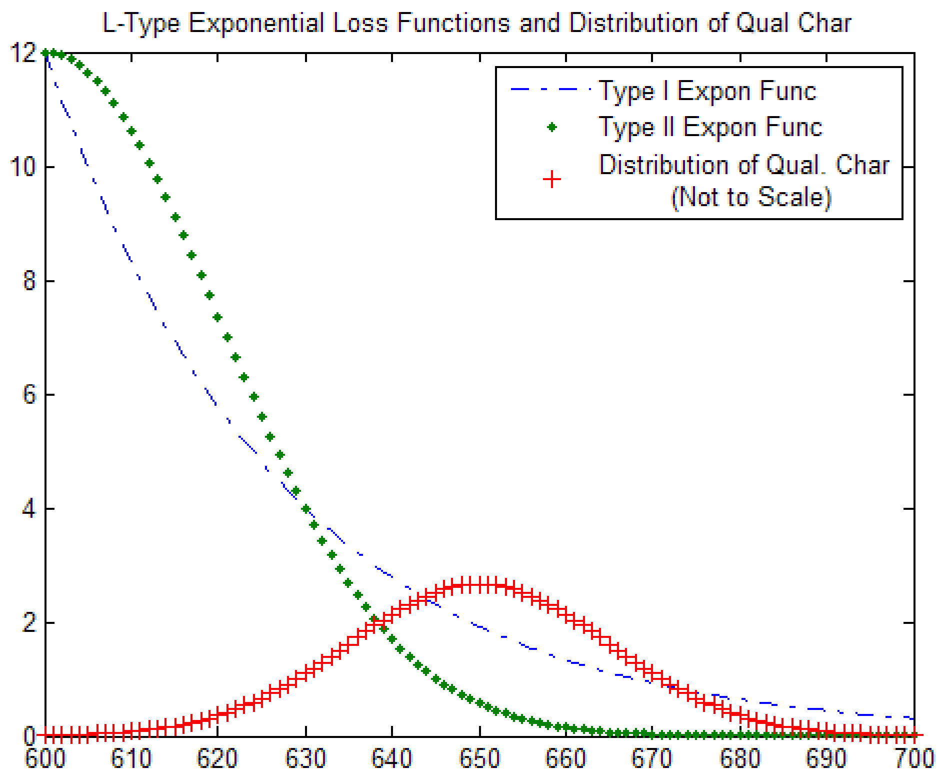

Figure 3. 7 L-Type Loss Functions and Distribution of Quality Characteristic

The total expected loss computed using Equations (15) and (19) for the type I and type II loss functions respectively are \$2.2387 and \$1.3485, which indicates that the type I loss function is more penalizing, on average, as compared to the type II loss function.

Conclusion

Based on the assumption that the there is a cap on the losses due to deviations from the target value of a quality characteristic, which is 'significantly' different from the out-of-specification costs, the Taguchi loss function has been modified to adequately model the losses within the specification limits of N-type quality characteristics. Two alternatives to the quadratic loss function were proposed for the same type of quality characteristics. For a normally distributed quality characteristic, the expected loss is derived for each of the loss functions considered. Two loss functions were proposed for L-type quality characteristics based on the same assumption.

Besides modeling losses under the given assumptions given in this paper, it is also important to be able to model losses under other functions other than the quadratic loss function, as all losses cannot be expected to behave similarly under different situations. It is, however, worth noting from the derivation of the quadratic loss function that it closely approximates functions that are smooth at the target, but if the deviations are not small enough or if the loss functions are not smooth at the target, seeking the actual function will surely improve upon the accuracy of the cost analysis. The behavior of the type I exponential loss function supports this point. The points of this paper are well made for a single quality characteristic. An interesting extension would be to consider multiple quality characteristics as a future study.

CHAPTER FOUR

DEVELOPMENT OF CENSORED ROBUST DESIGN MODEL FOR TIME-ORIENTED QUALITY CHARACTERISTICS

Introduction

Robust design techniques, which are based on the concept of building quality into products or process, are increasingly popular in industry primarily because of their practicality. However, traditional robust design principles have often been applied to situations in which the quality characteristics of interest are time-insensitive. When time-oriented quality characteristics are studied, censored data often occur. As a result, current robust design models reported in the literature may not be effective in finding solutions based on such data. To address such practical needs, this paper develops a censored robust design model. We also propose an estimation method that is closely related to the expectation-maximization algorithm and compare it with the method of maximum likelihood estimation via a numerical example. Model validation is conducted, and a comparative study is conducted for model verification. The rest of this Chapter will be sectioned under the following headings:

- Expectation-Maximization (EM) Algorithm
- Proposed Censored Robust Design Model
- Optimization Models
- Numerical Experience
- Conclusion

Expectation-Maximization (EM) Algorithm

In this section, we briefly discuss the EM algorithm as an alternative to the method of maximum likelihood (ML) in situations where ML is either difficult or impossible to apply.

The EM Algorithm versus the Method of ML Estimation

In some statistical parameter estimation problems, it is easy to find closed-form ML estimates, which have a number of desirable properties, namely, asymptotic normality, consistency, and asymptotic un-biasness. However, where closed-form solutions are unattainable, numerical methods (e.g. Newton-Raphson gradient method, Gauss-Siedel, etc.) are used to obtain the required estimates. For complicated likelihoods, these methods may be ineffective, and can be sensitive to initial values used. In instances where the likelihood is almost flat near its maximum, the methods may terminate before reaching the actual optimum solution. One of the issues that complicate the method of maximum likelihood estimation is the presence of incomplete or censored data. As we shall see, the EM algorithm works round this complication by including a step where incomplete data are estimated, thereby making it easy for the method of ML to apply. Thus, the EM algorithm is described as a powerful but simple iterative algorithm based on the use of the method of ML estimation under conditions where the direct use of the ML is difficult.

EM Procedures

Though the algorithm was formalized by Dempster et al. (1977), some earlier EM-type methods and ideas cited in the literature include Fisher (1925), McKendrick (1926), Hartley (1958), Baum et al. (1970), and Sundberg (1974, 1976). The origin of the algorithm is based on situations where there are observations with missing, incomplete, or censored data. Though this is the most apparent context of the algorithm, it is quite useful in other problems such as finite mixture models, where the mixing proportions and/or the parameters of the component distributions are of interest, and multinomial models with pooled cells. Such conditions may arise due to censoring or missing data, which are common occurrences in practice. For example, in lifetime studies, censored data arise when experiments are terminated before the failure of some of the items under study. In such situations, the method of ML estimation is more difficult to apply than in situations where all the failure times are known. The EM algorithm overcomes this difficulty through the use of an iterative process, which may be summarized as follows:

1. Given an incomplete data set, $\mathbf{x}$, augment it in a way to form a data set, $\mathbf{y}$ and form a likelihood $g(\mathbf{y}|\boldsymbol{\theta})$. Starting from $\boldsymbol{\theta}^{(0)}$, generate a sequence of iterates $\{\boldsymbol{\theta}^{(m)}\}$ as outlined in steps 2 and 3 below.
2. Form $E[\log g(\mathrm{y}|\theta) \mid \mathrm{x}, \theta^{(m)}] = Q(\theta, \theta^{(m)})$. This is the E-step of the algorithm.
3. Find the value of $\boldsymbol{\theta}$, $\boldsymbol{\theta}^{(m+1)}$, which maximizes the expectation in step 2. This is the M-step of the algorithm. Steps 2 and 3 are continued until convergence is achieved.

McLachlan and Krishnan (1997) have shown that the likelihood function L is monotonically increasing with respect to the iterates $\{\boldsymbol{\theta}^{(m)}\}$, i.e. $L(\boldsymbol{\theta}^{(m+1)}) \geq L(\boldsymbol{\theta}^{(m)})$, which ensures convergence if the likelihood function is bounded above. The algorithm is known to converge in most cases to a stationary value of the likelihood surface. However, it is recommended to repeat runs several times with different starting points to ensure that the optimum solution obtained is actually a global maximum. In the next section, we propose an estimation algorithm involving censored data that is closely related to the EM algorithm described here.

Proposed Censored Robust Design Model

In this Section, we describe the proposed experimental procedure, the methods of parameter estimation, and then propose some optimization models that are solved with the objective of simultaneously achieving small variability and a large mean response.

Experimental Phase

The general methodology used in this work is the dual response approach, which may be summarized as follows:

- Collect data from a designed experiment
- Estimate the mean and standard deviation for each design point
- Estimate response surfaces for the mean, $\mu(\mathbf{x})$ and standard deviation, $\sigma(\mathbf{x})$
- Formulate optimization models
- Obtain optimum settings for the control factors

Table 4.1 displays the general framework of the RD methodology, where the y_i 's are the observed lifetimes and the T_i's are the censored times.

Table 4.1 General Layout of the Proposed Methodology

Design Point	Control Factors	Experimental Observations	Estimates	
	$x_1, x_2, \ldots, x_f$		Means	Variances
1	Experimental Design	$y_1, \ldots, y_{n_1}, T_1, \ldots, T_{m_1}$	$\hat{\mu}_1$	$\hat{\sigma}_1$
2		$y_1, \ldots, y_{n_2}, T_1, \ldots, T_{m_2}$	$\hat{\mu}_2$	$\hat{\sigma}_2$
$\vdots$		$\vdots$	$\vdots$	$\vdots$
i		$y_1, \ldots, y_{n_i}, T_1, \ldots, T_{m_i}$	$\hat{\mu}_i$	$\hat{\sigma}_i$
$\vdots$		$\vdots$	$\vdots$	$\vdots$
d		$y_1, \ldots, y_{n_d}, T_1, \ldots, T_{m_d}$	$\hat{\mu}_d$	$\hat{\sigma}_d$

The mean and variance for each design point are estimated using the observed lifetimes and the censored observations as outlined in the next subsection.

Estimation Phase

In this subsection, we present a description of the estimation methods used in this chapter, namely, the method of ML estimation and our proposed algorithm. Results from both estimation procedures will be compared in the context of a numerical example.

ML Estimation Approach to Censored RD Data

From the general layout of our proposed methodology in Table 1, the observation vector for the i^{th} design point is $y_1, \ldots, y_{n_i}, T_1, \ldots, T_{m_i}$, where the y's are the observed lifetimes and the T's are the censored observations. Suppose that the underlying distribution of the lifetimes has a probability density function $f(x, \boldsymbol{\theta})$ and a cumulative distribution function $F(x,\ \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of parameters of the distribution. Then the likelihood function for the i^{th} design point is

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_i} f(y_j, \boldsymbol{\theta}) \prod_{k=1}^{n_i} [1 - F(T_k, \boldsymbol{\theta})], \tag{1}$$

and the loglikelihood function is

$$l(\boldsymbol{\theta}) = \sum_{j=1}^{n_i} \ln f(y_j, \boldsymbol{\theta}) + \sum_{k=1}^{m_i} \ln[1 - F(T_k, \boldsymbol{\theta})] \tag{2}$$

By setting to zero each of the partial derivatives of $l(\boldsymbol{\theta})$ with respect to each of the components of $\boldsymbol{\theta}$, we obtain a system of equations that are solved for the unknown parameters. For example, if p_c is a component of $\boldsymbol{\theta}$, then the system of equations to be solved will constitute equations of the form

$$\sum_{j=1}^{n_i} \frac{\frac{\partial}{\partial p_k} f(y_j, \boldsymbol{\theta})}{f(y_j, \boldsymbol{\theta})} + \sum_{k=1}^{m_i} \frac{\frac{\partial}{\partial p_k}[1 - F(T_k, \boldsymbol{\theta})]}{1 - F(T_k, \boldsymbol{\theta})} = 0\,. \tag{3}$$

Proposed Estimation Algorithm for Censored Data

Suppose we have the observation vector $y_1, \ldots, y_{n_i}, T_1, \ldots, T_{m_i}$ from a distribution as described earlier. Let $y = [y_1, y_2, \ldots, y_{n_i}]$ and $T = [T_1, T_2, \ldots, T_{m_i}]$. In order to find the estimates of the unknown parameters of the distribution, we propose the following algorithm:

1. Find the loglikelihood function

$$l(\boldsymbol{\theta}|y,T) = \sum_{j=1}^{n_i} \ln f(y_j, \boldsymbol{\theta}) + \sum_{k=1}^{m_i} \ln[1 - F(T_k, \boldsymbol{\theta})] \tag{4}$$

2. With an initial guess $\boldsymbol{\theta}^{(0)}$, compute $E[x_k | x_k \geq T_k]$ for each T_k

3. Replace the T_k 's in step 1 with the conditional expectations in step 2.

4. Find the values of $\boldsymbol{\theta} = \boldsymbol{\theta}^{(1)}$ that maximize the likelihood function in Equation (4).

5. Replace $\boldsymbol{\theta}^{(0)}$ with $\boldsymbol{\theta}^{(1)}$ in step 2, and continue to iterate between steps 2, 3, and 4 until convergence is achieved for the sequence $\{\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots\}$.

The difference between this algorithm and the EM algorithm is that we are not finding $E[\log g(\mathrm{y} \mid \boldsymbol{\theta}) \mid \mathrm{x}, \boldsymbol{\theta}^{(m)}] = Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$, but rather the conditional expectation $E[x_k | x_k \geq T_k]$, which is simpler and a reasonable replacement for the censored value T_k.

Illustration: To illustrate this algorithm, suppose we have the uncensored sample vector

x = [8.7, 5.0, 10.4, 10.9, 6.6, 13.6, 13.6, 9.9, 11.0, 10.5],

and the censored sample vector

T = [9.4, 12.2, 8.2, 16.5, 10.0]

from a normal distribution with mean μ and variance σ^2.

For a normally distributed random variable X, it is easy to show (Shaibu et. al, 2005) that

$$E[X \mid X \geq x\ ] = \mu + \frac{\sigma \exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}}{\sqrt{2\pi}\left[1 - \Phi\left(\dfrac{x-\mu}{\sigma}\right)\right]} \tag{5}$$

Table 4.2 displays ten iterations of the algorithm in MATLAB with initial guesses $\mu^{(0)} = \sigma^{(0)} = 5$. We observe that all estimates converged in the sixth iteration.

Table 4.2 Results of Implementing the Proposed Algorithm for $N(\mu, \sigma^2)$

Iteration	Parameter Estimates		$E[x_k \mid x_k \geq T_k]$				
	$\hat{\mu}$	$\hat{\sigma}$					
0	5	5					
1	11.2564	3.0854	12.1495	14.439	11.2252	18.2072	12.6257
2	11.3412	3.0473	12.6706	14.3487	12.1546	17.7664	12.9781
3	11.3437	3.0465	12.6861	14.3444	12.1833	17.7534	12.988
4	11.3438	3.0465	12.6868	14.3445	12.1843	17.7532	12.9885
5	11.3439	3.0466	12.6869	14.3446	12.1844	17.7532	12.9886
6	11.3439	3.0466	12.6869	14.3446	12.1845	17.7533	12.9887
7	11.3439	3.0466	12.6869	14.3446	12.1845	17.7533	12.9887
8	11.3439	3.0466	12.6869	14.3446	12.1845	17.7533	12.9887
9	11.3439	3.0466	12.6869	14.3446	12.1845	17.7533	12.9887
10	11.3439	3.0466	12.6869	14.3446	12.1845	17.7533	12.9887

For a normally distributed random variable X, it is easy to show (Shaibu et. al, 2005) that

$$E[X \mid X \geq x\ ] = \mu + \frac{\sigma \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}{\sqrt{2\pi}\left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]} \tag{5}$$

Table 2 displays ten iterations of the algorithm in MATLAB with initial guesses $\mu^{(0)} = \sigma^{(0)} = 5$. We observe that all estimates converged in the sixth iteration.

Optimization Models

The objective of this paper is to formulate methods of maximizing mean response, while simultaneously minimizing variability. Thus, in order to achieve this through one objective function, and as a solution to a minimization problem, the chosen objective function must vary directly as the variability and inversely as the mean of the system of interest. That is, if $f(\mathbf{x})$ is the objective function, then we propose that it satisfies

$$f(\mathbf{x}) = k\frac{\sigma(\mathbf{x})}{\mu(\mathbf{x})} \ , \tag{6}$$

where k is a constant. Obviously, the values of **x** that maximize $\mu(\mathbf{x})$ and minimize $\sigma(\mathbf{x})$ will minimize $f(\mathbf{x})$. Hence, assuming a unit constant of proportionality, the desired solution will be the solution to the problem

$$\min_{\mathbf{x}\in\Omega}\left(\frac{\sigma(\mathbf{x})}{\mu(\mathbf{x})}\right), \tag{7}$$

where Ω is the region of feasibility. Similarly, a comparable solution can be achieved by solving the problem

$$\min_{\mathbf{x}\in\Omega}\left(\sigma(\mathbf{x}) + \frac{1}{\mu(\mathbf{x})}\right) \tag{8}$$

Part of the feasibility requirements for these proposed objective functions is that the mean response is nonzero, i.e. $\mu(\mathbf{x}) \neq 0$, which is naturally satisfied by the nature of the problem at hand – a larger-the-better-type (L-Type) problem, where the objective is to get $\mu(\mathbf{x})$ as large as possible. We will demonstrate that all these functions yield optimal solutions that do not vary significantly.

Using Taylor series expansion, Kapur and Cho (1994) derived an approximation to Taguchi's expected loss function for the L-Type quality characteristic as

$$\frac{1}{\mu^2(\mathbf{x})}\left(1 + 3\frac{\sigma^2(\mathbf{x})}{\mu^2(\mathbf{x})}\right) \tag{9}$$

By inspection, we notice that this function decreases as $\mu(\mathbf{x})$ increases and as $\sigma(\mathbf{x})$ decreases. Hence, we will also consider this function as an objective function to be minimized in the experimental region, which implies solving the problem

$$\min_{\mathbf{x}\in\Omega} \frac{1}{\mu^2(\mathbf{x})}\left(1+3\frac{\sigma^2(\mathbf{x})}{\mu^2(\mathbf{x})}\right) \tag{10}$$

In the next section, we will illustrate the proposals of this work with a numerical example.

Numerical Experience

Consider an experiment on the lifetimes of light bulbs where the factors of concern are the filament resistance (x_1), filament melting temperature (x_2), and the amount of argon gas in the light bulb (x_3). Thus, the vector of control factors is $\mathbf{x}=[x_1, x_2, x_3]$. The objective of the experiment is to determine the settings of the control factors, $\mathbf{x}^*=[x_1^*, x_2^*, x_3^*]$, that give the longest possible lifetimes and the smallest possible variability (or standard deviation). The chosen design is a spherical central composite design (CCD) consisting of 8 factorial points, 6 axial points ($\pm\alpha=\pm\sqrt{3}$), and 5 center, i.e., all the factorial and axial points are located on the sphere of radius $\sqrt{3}$ (see Montgomery, 1997). Suppose 20 light bulbs are subjected to each design point and the experiment is run for 500 hours. The manufacturer believes that the lifetimes are normally distributed. Notice that this is a censoring problem because the experiment is terminated after 500 hours, and the observed lifetimes are for the bulbs that failed before the termination of the experiment. For the rest of the bulbs, we only know that their lifetimes exceed 500 hours but we do not know their actual lifetimes.

For this problem, the probability density function at each design point is the normal density function. Thus, the density and distribution functions at each design point with mean, μ, and standard deviation, σ, are respectively given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right] \tag{11}$$

and

$$F(x) = \int_{-\infty}^{x} f(y)dy = \Phi\left[\frac{x-\mu}{\sigma} \right] \tag{12}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, and μ and σ are the mean and standard deviation of the distribution. We note that the parameters of the distribution may vary from one design point to another. Using these functions in the methods of MLE and the proposed algorithm above, estimates of the means and standard deviations are obtained for each design point.

Table 3 shows the experimental design together with the observed lifetimes (or failure times) for bulbs that failed. The observed lifetimes per design point range from five (5) to thirteen (13), thereby giving rise to an experiment where the number of observations per design point is not the same throughout the experiment. In Table 4.3, we show the estimates of the mean and standard deviation obtained by our proposed algorithm, and by the method of ML estimation.

Table 4.3 Observed Lifetimes from a Designed Experiment

Design Point	Control Factors			Observed Lifetimes												
	x1	x3	x3													
1	-1	-1	-1	488.77	484.37	490.76	491.34	486.22	494.56	494.56	490.18	491.48	490.94	489.65	492.90	488.21
2	-1	-1	1	490.04	491.24	495.78	490.98	490.24	486.72							
3	-1	1	-1	491.43	481.27	494.05	499.71	485.29	494.94	497.41	479.66	480.62	493.16			
4	-1	1	1	487.54	493.10	493.74	493.21	496.16	492.99	495.65	483.45	489.48				
5	1	-1	-1	487.89	478.91	490.46	482.31	497.64	483.87	492.14	490.22	483.14	475.39	488.49	482.59	
6	1	-1	1	494.30	493.38	494.10	483.43	492.28	480.28	488.82	488.58	488.99	486.25			
7	1	1	-1	497.25	476.03	492.48	495.82	494.64	493.55	489.71	494.26	493.49				
8	1	1	1	487.30	486.42	487.01	478.47	487.46								
9	$-\sqrt{3}$	0	0	490.77	491.81	497.81	488.27	493.45	494.39	495.14	484.87	491.27	491.40	484.78	486.20	
10	$\sqrt{3}$	0	0	498.66	487.33	492.20	489.38	482.63	483.34	492.70	479.70					
11	0	$-\sqrt{3}$	0	494.54	493.15	484.01	487.66	481.60	474.94	495.89	486.00	491.56	490.95			
12	0	$\sqrt{3}$	0	489.11	481.27	481.71	486.08	479.88	480.88	489.37	479.64	488.63	480.32	478.63		
13	0	0	$-\sqrt{3}$	489.23	493.16	490.49	492.20	486.29	488.58	489.22	486.15	485.80	493.09	490.11	487.99	492.68
14	0	0	$\sqrt{3}$	491.17	485.89	495.96	491.42	496.56	495.09	487.21	484.58	489.85	488.74	486.52		
15	0	0	0	491.77	497.10	486.16	484.54	487.77	492.66							
16	0	0	0	484.50	482.63	488.45	488.74	491.71	485.89	488.29	486.76					
17	0	0	0	488.67	497.35	485.83	481.85	491.62	484.24	489.29	485.72	491.98	492.07	488.00	478.05	489.81
18	0	0	0	497.66	494.57	480.59	488.67	485.43	483.58	482.46	490.65	486.79	489.40	487.12	486.59	491.09
19	0	0	0	493.01	481.79	483.60	494.68	487.00	481.67	490.79						

Table 4.4 Estimates from Observed Lifetimes from a Designed Experiment

Design Point	Control Factors			Estimates			
				Proposed Algorithm		MLE	
	x_1	x_2	x_3	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}_{ML}$	$\hat{\sigma}_{ML}$
1	-1	-1	-1	495.142	6.9830	495.298	7.5193
2	-1	-1	1	501.899	7.3896	505.507	11.8991
3	-1	1	-1	498.768	10.2961	499.748	12.3244
4	-1	1	1	499.426	7.4588	500.477	9.3782
5	1	-1	-1	494.611	11.4029	495.045	12.6385
6	1	-1	1	497.931	9.4337	498.803	11.2639
7	1	1	-1	500.041	8.3662	501.238	10.5286
8	1	1	1	504.252	11.0593	512.645	20.3113
9	$-\sqrt{3}$	0	0	496.466	7.5355	496.753	8.3529
10	$\sqrt{3}$	0	0	500.541	10.7034	502.813	14.3400
11	0	$-\sqrt{3}$	0	498.034	10.9227	499.055	13.0545
12	0	$\sqrt{3}$	0	494.665	12.9847	495.423	14.8462
13	0	0	$-\sqrt{3}$	494.739	7.2761	494.899	7.8292
14	0	0	$\sqrt{3}$	497.167	8.1778	497.657	9.3684
15	0	0	0	502.357	8.4223	506.444	13.5196
16	0	0	0	499.862	10.5345	502.090	14.1310
17	0	0	0	494.187	9.2314	494.401	9.9585
18	0	0	0	494.158	9.1060	494.368	9.8200
19	0	0	0	501.603	10.7585	505.042	15.6233

Estimated Response Surfaces

Using the estimates in Table 4.4, we find second order response surfaces with interactions for the mean and standard deviation for each method of estimation. For the proposed estimation algorithm, the response surfaces for the mean and standard deviation are respectively found as

$$\hat{\mu}(\mathbf{x}) = 498.433 + 0.619x_1 + 0.505x_2 + 1.368x_3 + 0.418x_1^2 - 0.300x_2^2 - 0.432x_3^2 + 1.325x_1x_2 + 0.015x_1x_3 - 0.651x_2x_3 \quad (13)$$

and

$$\hat{\sigma}(\mathbf{x}) = 9.6105 + 0.9730\,x_1 + 0.3959\,x_2 - 0.0104\,x_3 - 0.2687\,x_1^2 + 0.6760\,x_2^2 \\ - 0.7329\,x_3^2 - 0.5992\,x_1 x_2 + 0.3943\,x_1 x_3 - 0.1773\,x_2 x_3 \tag{14}$$

For the method of maximum likelihood, the response surfaces for the mean and the standard deviation are respectively given by

$$\hat{\mu}_{ML}(\mathbf{x}) = 500.469 + 1.228x_1 + 0.940x_2 + 2.206x_3 + 0.405x_1^2 - 0.443x_2^2 \\ - 0.763x_3^2 + 2.577x_1x_2 + 0.528x_1x_3 - 0.229x_2x_3 \tag{15}$$

and

$$\hat{\sigma}_{ML}(\mathrm{x}) = 12.611 + 0.989x_1 + 0.508x_2 + 0.516x_3 - 0.907x_1^2 + 0.192x_2^2 \\ - 0.402x_3^2 + 0.194x_1x_2 + 0.291x_1x_3 + 0.160x_2x_3 \tag{16}$$

Optimization Models and Solutions

For the proposed estimation method, we solve the minimization problems listed below, each subject to the constraints

$$\sum_{i=1}^{3} x_i^2 \le 3 \tag{17}$$

(1) Minimize $\dfrac{\hat{\sigma}(\mathbf{x})}{\hat{\mu}(\mathbf{x})}$,

(2) Minimize $\hat{\sigma}(\mathbf{x}) + \dfrac{1}{\hat{\mu}(\mathbf{x})}$,

(3) Minimize $\dfrac{1}{\hat{\mu}^2(\mathbf{x})}\left(1 + 3\dfrac{\hat{\sigma}^2(\mathbf{x})}{\hat{\mu}^2(\mathbf{x})}\right)$ (approximation of Kapur and Cho (1994))

For comparison purposes, we find the settings that minimize only the standard deviation ($\sigma(\mathbf{x})$), and also the settings that maximize the only mean response ($\mu(\mathbf{x})$) in the feasible region. Note that maximizing the mean response is equivalent to minimizing its negative.

The same problems are also solved using the estimated response surfaces for the method of ML, $\hat{\mu}_{ML}(\mathbf{x})$ and $\hat{\sigma}_{ML}(\mathbf{x})$.

Table 4.5 shows the results obtained for each method of estimation. For each method, all of the proposed optimization problems resulted in approximately the same optimal value of standard deviation, which is the same as the value obtained by minimizing the standard deviation in the feasible region. This is an indication of the appropriateness of the models for minimizing the standard deviation. Even though the model based on the approximation of Kapur and Cho (1994) gives that greatest optimal mean, it also gives values of standard deviation that are much greater than the values given by our proposed models. Therefore, for this example, the best model in terms of acceptable optimum results is the first model. Another interesting observation is that the two proposed models give results that are identical to the results obtained by simply minimizing the standard deviation. However, it is better to use the proposed models since they are constructed to simultaneously minimize the variability and maximize the mean response while the method of simply minimizing the standard deviation does not necessarily guarantee the achievement of such optimization objective.

In comparing the two estimation methods, we observe that our proposed estimation method tends to result in higher optimal means and lower standard deviations than the method of ML. This observation necessitates a comparison study between these two methods of estimation, which we consider in the next section.

Table 4.5 Solutions to our Proposed Optimization Problems

Objective Function for Minimization	Proposed Algorithm			ML Estimation		
	x*	$\hat{\mu}*$	$\hat{\sigma}*$	x*	$\hat{\mu}^*_{ML}$	$\hat{\sigma}^*_{ML}$
$\frac{\sigma(\mathbf{x})}{\mu(\mathbf{x})}$	[-1.2027 -0.4079 1.1777]	499.9908	6.0362	[-0.0904 0.0043 -1.7297]	494.3501	7.4334
$\sigma(\mathbf{x})+\frac{1}{\mu(\mathbf{x})}$	[-1.2012 -0.4030 1.1810]	499.9833	6.0361	[-0.0893 -0.0000 -1.7297]	494.3454	7.4334
$\frac{1}{\mu^2(\mathbf{x})}\left(1+3\frac{\sigma^2(\mathbf{x})}{\mu^2(\mathbf{x})}\right)$	[1.5667 0.5222 0.5222]	502.1261	10.5419	[1.1921 1.1921 0.3974]	507.5592	17.7402
$\sigma(\mathbf{x})$	[-1.2012 -0.4030 1.1810]	499.9833	6.0361	[-0.0893 -0.0000 -1.7297]	494.3454	7.4334
$-\mu(\mathbf{x})$	[0.8628 0.2859 -502.3538]	502.3538	10.6906	[1.3582 0.9418 0.5178]	507.8704	17.4722

Comparison Study

In this section, we generate data from known normal distributions, and use it as a means of comparing our proposed method of estimation and the method of ML estimation. We do this by estimating the parameters under each method, and then computing the absolute percentage errors in the estimates, which quantifies of how good the estimates are. The absolute percentage error in $\hat{\mu}$ as an estimator of μ is defined as

$$\left|100\frac{(\hat{\mu}-\mu)}{\mu}\right|\% . \qquad (18)$$

The absolute percentage errors in the other estimators are similarly defined. Table 4.6 shows the generated data along with the means and standard deviations of the normal distributions that they are generated from. The censoring time is set at 500 time units. In Table 4.7, we show the estimates obtained using the method of ML and our proposed estimation method along with the absolute percentage errors in the estimates. Figure 4.1 (page 88) shows line graphs of the estimates of the means under each method of estimation along with the actual mean values. Figure 4.2 (page 89) shows similar graphs for the estimates of the standard deviation and the actual values of the standard deviation. The percentage errors in the estimates are illustrated in Figure 4.3 for the standard deviations and in Figure 4.4 for the means. From Table 4.7 and Figure 4.4, we observe that the estimates of the mean obtained using our proposed algorithm yields smaller absolute percentage errors than estimates from the method of ML in 6 samples. However, in majority of the samples, the two methods appear to give estimates that are fairly close (See Figures 4.1 and 4.4). We also observe from Table 4.6 and Figure 4.3 that the estimates of the standard deviation obtained from the proposed algorithm gave smaller values of absolute percentage error than the maximum likelihood estimates in 3 samples. In general, the method of ML appears to perform better than the proposed algorithm. Based on these observations, we recommended abiding by the results obtained via the method of ML.

Table 4.6 Generated Data from Normal Distributions with Censoring Time of 500

Sample	Mean, μ	Std Dev. σ	Observations														
1	505.5564	8.6745	484.53	492.43	495.79	498.49	489.40	499.90	499.19								
2	510.3492	8.9694	498.04	498.98	490.82	494.39											
3	504.6075	5.5731	498.09	497.36	497.16	498.46	489.26	499.83									
4	503.5091	7.6843	495.17	488.90	497.38	494.37	497.56	492.28	498.79	499.35	492.01						
5	498.4188	5.9393	498.77	493.04	495.72	497.12	499.34	493.04	492.35	495.83	498.70						
6	494.8655	5.7466	499.83	492.17	495.97	497.69	496.40	489.81	490.64	492.64	490.84	485.84	499.21	497.91	491.43	494.53	492.91
7	502.393	5.0101	495.71	499.11	499.48	499.69	496.63	499.56	498.65								
8	498.7699	7.0545	498.67	494.40	486.30	499.68	492.58	498.41	495.90	493.59	499.06	498.56					
9	496.8241	7.5276	481.11	499.49	496.56	494.11	489.32	491.89	487.06	486.59	491.05	496.10	492.17	491.41	487.21	489.28	
10	502.3779	8.0832	497.80	494.54	498.55	483.12	499.80	496.25	496.54	496.44	498.57	498.39					
11	495.2319	8.687	485.14	484.71	490.82	498.18	499.91	489.28	499.22	496.50	479.75	495.47	496.05				
12	498.8475	8.1454	497.16	496.63	490.41	486.24	497.70	498.01	496.94	496.42	498.42	492.31	497.89	494.34			
13	494.4866	9.7611	486.82	485.98	498.93	494.56	495.25	489.78	493.53	489.84	489.53	493.66	490.07	479.28	496.78	487.17	498.25
14	504.8757	9.0665	494.57	492.92	498.81	498.52	490.91	499.36	499.99								
15	502.4554	6.7103	497.68	497.54	497.90	498.31	495.97	497.38	495.79	491.59	496.24	496.47	488.46	494.20	482.98		
16	500.0909	7.9244	492.79	498.90	497.00	487.45	499.60	495.50									
17	503.0287	5.6699	492.44	496.17	497.20	499.51	494.75										
18	504.0574	5.2414	498.16	495.11	491.30	498.19	498.65	497.90									
19	499.3115	9.839	494.72	480.51	491.04	477.17	499.79	475.76	496.87	497.82	485.87	475.24	496.91				
20	501.5214	5.8619	488.41	495.25	495.34	498.68	497.09	492.25	494.66	486.79	499.15	493.92					

Table 4.7 Parameters, their Estimates, and the Absolute Percentage Errors in the Estimates

Sample	Parameters		Estimates				Absolute Percentage Errors			
			MLE		Proposed Method		Estimates of μ		Estimates of σ	
	μ	σ	$\hat{\mu}_{ML}$	$\hat{\sigma}_{ML}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}_{ML}$	$\hat{\mu}$	$\hat{\sigma}_{ML}$	$\hat{\sigma}$
1	505.5564	8.6745	503.6816	9.0871	501.6854	6.3008	0.37%	0.77%	4.76%	27.36%
2	510.3492	8.9694	506.3699	7.6462	502.1725	3.6091	0.78%	1.60%	14.75%	59.76%
3	504.6075	5.5731	503.1864	5.7657	501.4656	3.6456	0.28%	0.62%	3.46%	34.59%
4	503.5091	7.6843	500.6373	6.1871	499.936	4.9174	0.57%	0.71%	19.48%	36.01%
5	498.4188	5.9393	500.4558	4.9373	499.8974	3.9247	0.41%	0.30%	16.87%	33.92%
6	494.8655	5.7466	496.1795	5.3684	496.1356	5.1487	0.27%	0.26%	6.58%	10.40%
7	502.393	5.0101	501.0186	2.5179	500.4655	1.7458	0.27%	0.38%	49.74%	65.15%
8	498.7699	7.0545	500.2708	5.9254	499.7903	4.9402	0.30%	0.20%	16.01%	29.97%
9	496.8241	7.5276	495.0445	7.615	494.9417	7.1962	0.36%	0.38%	1.16%	4.40%
10	502.3779	8.0832	500.5997	6.2374	500.0863	5.192	0.35%	0.46%	22.84%	35.77%
11	495.2319	8.687	499.0984	9.7225	498.5572	8.4404	0.78%	0.67%	11.92%	2.84%
12	498.8475	8.1454	498.6797	5.4322	498.4793	4.8763	0.03%	0.07%	33.31%	40.13%
13	494.4866	9.7611	494.5437	7.3771	494.4844	7.0791	0.01%	0.00%	24.42%	27.48%
14	504.8757	9.0665	502.299	5.6527	501.0573	3.9199	0.51%	0.76%	37.65%	56.77%
15	502.4554	6.7103	497.977	6.0309	497.8305	5.5522	0.89%	0.92%	10.12%	17.26%
16	500.0909	7.9244	504.0999	7.7234	501.7868	4.8644	0.80%	0.34%	2.54%	38.61%
17	503.0287	5.6699	503.9819	6.1128	501.4826	3.372	0.19%	0.31%	7.81%	40.53%
18	504.0574	5.2414	502.753	5.3166	501.1574	3.3403	0.26%	0.58%	1.43%	36.27%
19	499.3115	9.839	498.4609	14.2583	497.672	12.3848	0.17%	0.33%	44.92%	25.87%
20	501.5214	5.8619	499.7977	6.9126	499.2498	5.7765	0.34%	0.45%	17.92%	1.46%

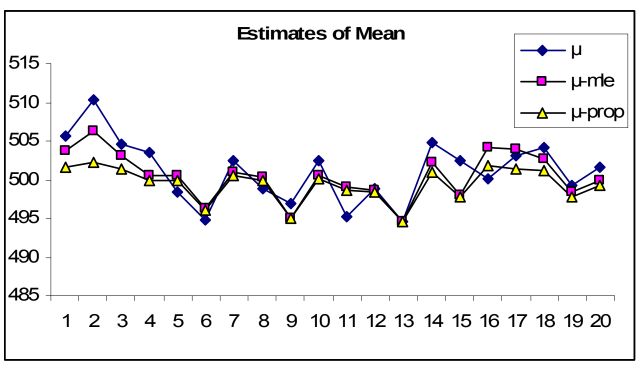

Figure 4.1 Comparing the Estimates of the Mean μ

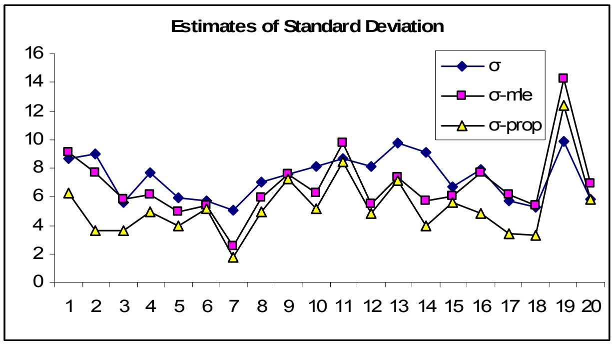

Figure 4.2 Comparing the Estimates of σ

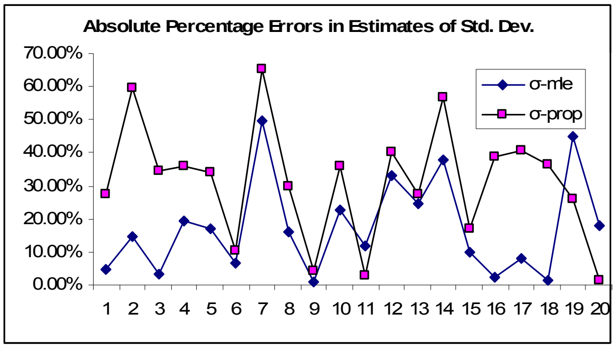

Figure 4.3 Absolute Percentage Errors in Estimates of σ

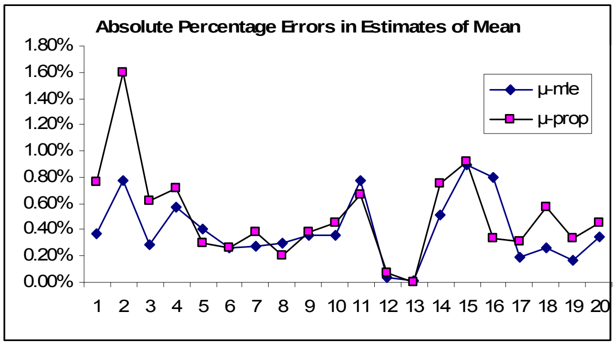

Figure 4.4 Absolute Percentage Errors in Estimates of μ

Conclusion

In this work, we demonstrated how to apply the method of RD to experimental results involving censored data, specifically, censored lifetimes. The proposed methodology is particularly useful for experiments in life-testing and reliability, where it is often practically difficult or infeasible to observe lifetimes of all specimens, thereby yielding censored lifetimes and unequal number of observations per design point. Two optimization models have been proposed that yield identical optimal standard deviations, and quite close optimal mean values. The best optimization model in terms of 'acceptable' optimal values for the mean and standard deviation is the model proposed in equation (8).

In addition to the method of ML estimation, another estimation method closely related to the EM algorithm is proposed. Comparison studies of the two methods show the method of ML to be superior in performance to the proposed algorithm. Nevertheless, it will be interesting to pursue studies aimed at improving upon the proposed estimation algorithm.

CHAPTER FIVE

TOLERANCE OPTIMIZATION FOR L-TYPE QUALITY CHARACTERISTICS IN THE PRESENCE OF CENSORED DATA

Introduction

In this chapter, we consider tolerance design and optimization involving larger-the-better-type (L-type) quality characteristics via response surface models for mean response and standard deviation in the presence of right-censored data. The optimization objective we consider is similar to that in Chapter 4, namely, simultaneously maximizing the mean response and minimizing the standard deviation. However, additional interests of this chapter include a lower specification limit and proportion of defective products, which we include in the formulation of optimization models. Two formulations of optimization models are considered here – one is based on the relationship between the lower specification limit, the proportion of defective products, the mean response, and the standard deviation while the other formulation is based on expected loss, which implicitly considers the relationship between the parameters in the first formulation. This chapter demonstrates the integration of the methods of the previous chapters, a fact we illustrated using a numerical example. The next two sections will address the model formulations, following which we will showcase the integration of the concepts of this dissertation through a numerical example.

Model Formulation Based on the Direct Relationship between L, μ, and σ

For L-type quality characteristics, only a lower limit, L, is of interest and the objective is to maximize the mean response (as much as possible) and to minimize the variability (usually, the standard deviation or variance) at the same time. We may connect L and μ by a relationship of the form

$$L = \mu - \kappa\sigma\,, \tag{1}$$

where κ is a real positive parameter indicating the distance (in number of standard deviations) between the mean and the lower specification limit. Its value may be fixed as a control on the proportion of defective items produced. Obviously, larger values of κ favor the L-type objective. The proportion of defective products is given by

$$\delta = P(X < L) = F(L) = F(\mu - \kappa\sigma)\,, \tag{2}$$

where F is the distribution function of the quality characteristic of interest. Thus, with specified proportion of defective products, δ, the parameter κ can be determined as

$$\kappa = \frac{\mu - F^{-1}(\delta)}{\sigma} \tag{3}$$

By considering Equation (1) and noting in general that the higher the rate (or level) of satisfaction of the specification limits, the higher the quality of products, then we observe that the manufacturers who work to satisfy $\mu \geq L + \kappa\sigma$ are more likely to produce fewer defective items, and will therefore be more competitive than those who work to satisfy $\mu = L + \kappa\sigma$. Hence, for a more competitive set up, it is desirable for the mean response and standard deviation response surfaces to satisfy the inequality

$$\frac{L + k\sigma(\mathbf{x})}{\mu(\mathbf{x})} \leq 1\,, \tag{4}$$

where smaller values of the ratio on the left hand side of the inequality are preferred. Therefore, for solving tolerance problems for L-type quality characteristics, we propose the optimization problem

$$\begin{aligned} &\min_{\mathbf{x}\in\Omega} \frac{L + k\sigma(\mathbf{x})}{\mu(\mathbf{x})} \\ &s.t. \quad \sigma(\mathbf{x}) \geq 0 \\ &\qquad\ \mu(\mathbf{x}) - (L + \kappa\sigma(\mathbf{x})) \geq 0 \end{aligned} \tag{5}$$

Figures 5.1 and 5.2 show a surface plot and a contour plot respectively of the objective function for fixed *L* and *k*. It is clear that the function has numerous local minima. Therefore in solving model (5), we propose generating many solutions in order to ensure that the solution obtained is global.

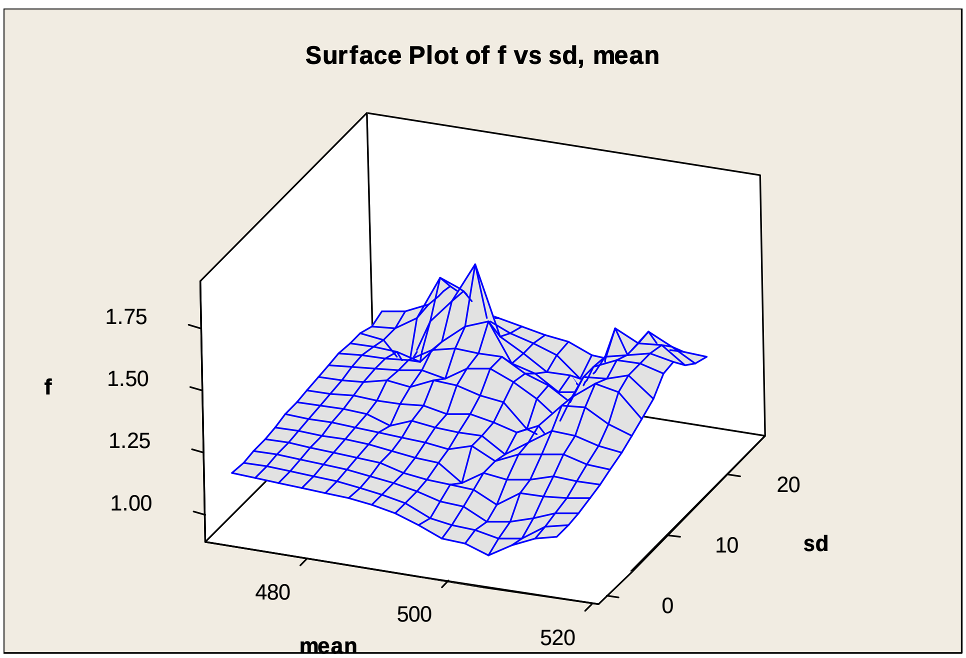

Figure 5.1 Objective function (5) for fixed L and k

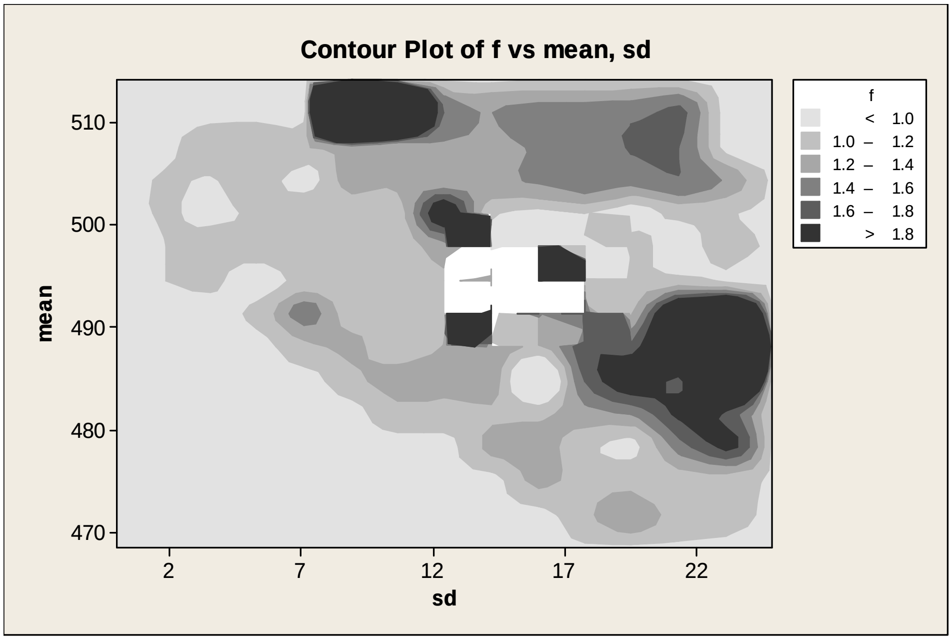

Figure 5.2 Contour plot of objective function (5)

The optimization problem in (5) may be solved under each of the following conditions:

1. The lower limit *L* is specified together with a desired upper bound for the proportion of defective products, and it is desired to find the settings that simultaneously maximize the mean response and minimize the standard deviation. This may be a situation with existing systems.
2. Nothing is specified, but the settings that maximize the mean and minimize the standard deviation are being sought together with an idea about a lower specification limit for the system. This may be a situation with prototypes of completely new systems, about which not much is known.

Solution Algorithm

As observed above, the surface plots of the objective function has several local extreme points, and so it is advisable to solve the optimization problem with several initial guesses to make sure that the final solution is actually a global solution. Figure 5.3

shows the algorithm we use in solving the problem using several initial guesses that are uniformly generated.

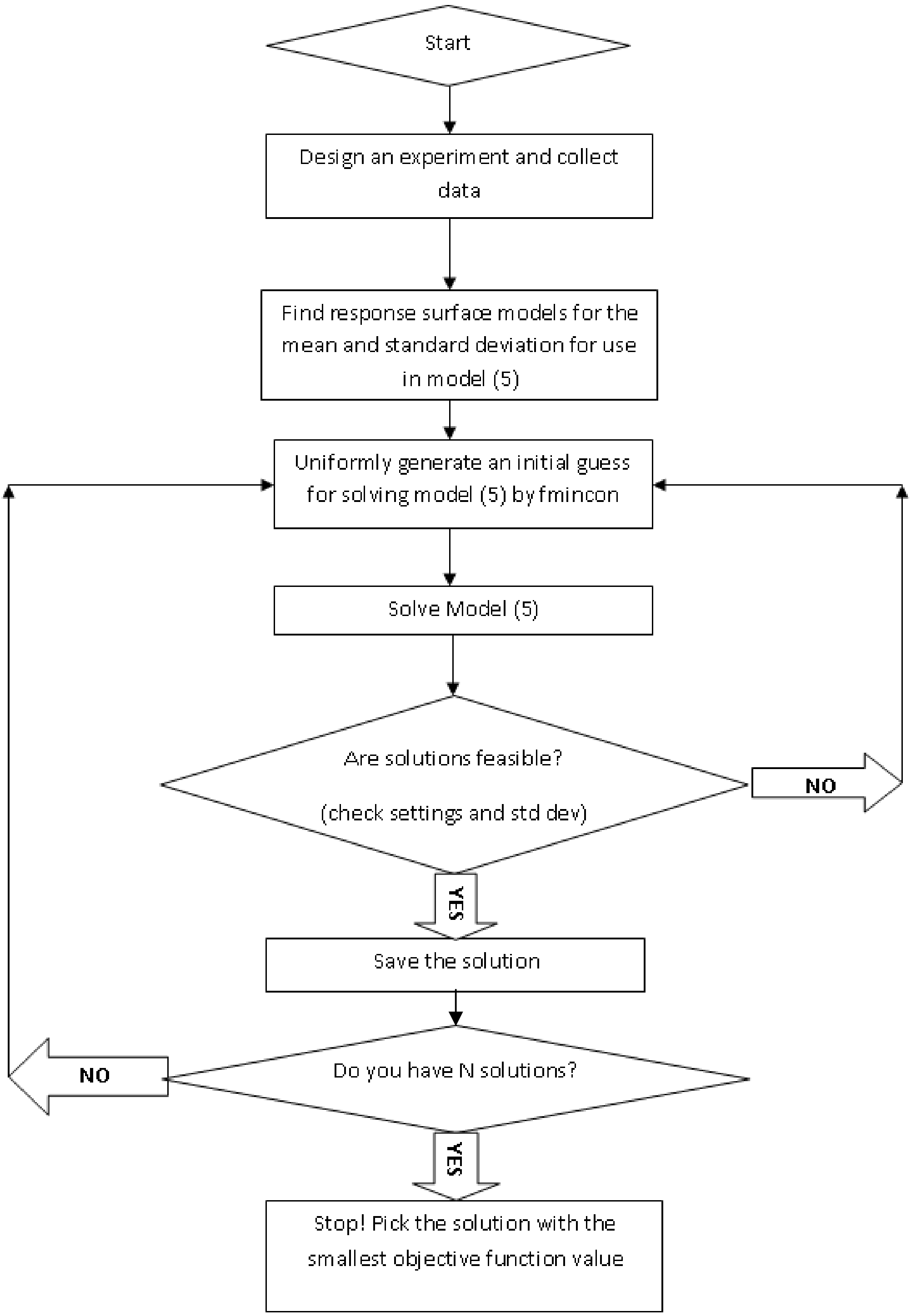

Figure 5.3 Flow Chart of Solution Algorithm

Model Formulation Based on Expected Loss

In the formulation of the previous section (i.e. model (5)), cost was not incorporated into the model. Even though that formulation makes practical sense, it will be much better in terms of interpretation of results if a model can be formulated, which directly minimizes cost. Therefore, in this section, we propose a model based on expected loss, where the objective function is an expected loss function found in the manner of Chapter 3. This allows for direct minimization of cost, thereby eliminating the possibility of generating solutions that may not be optimum or even tenable in terms of cost. If the appropriate expected loss function is $E[T(x)]$, we propose the following optimization model:

$$\begin{aligned} &\min_{\mathbf{x}\in\Omega} E[T(\mathbf{x})] \\ &s.t. \quad \sigma(\mathbf{x}) \geq 0 \\ &\qquad\ \ \mu(\mathbf{x}) - (L + \kappa\sigma(\mathbf{x})) \geq 0 \end{aligned} \tag{6}$$

The constraints are also set up to avoid the possibility of obtaining solution that yield negative values of standard deviation (i.e. the first constraint), and also to ensure the desirable relationship between parameters for an L-type system. In general, suppose an L-type quality characteristic with a lower specification limit L follows a distribution with a cumulative distribution function F and a probability density function f. If the out of specification cost is c_L and the within-specification cost is given by a function l(x), then the total expected cost is computed as

$$E[T(x)] = c_L F(L) + \int_L^\infty l(x) f(x) dx \tag{7}$$

We note that the expected loss functions as we have seen in Chapter 3 are functions of μ and σ, which are related to the settings of the control factors, **x** through the response surface models. Therefore, finding the values of **x** that minimize the expected loss functions is equivalent to finding the values of the mean response and standard deviation that minimize the expected loss functions. Additionally, using the constraints of model (5) in model (6) ensures obtaining feasible solutions in a competitive manner as described in the previous section.

In the solution procedure, we propose using the algorithm of the previous section. Figures 5.4 and 5.5 show the types I and II exponential expected loss functions under normal distribution (Equations (15) and (19) from Chapter 3) as functions of process standard deviation (σ) and process mean (μ). From both figures, we observe that the smallest expected loss function values call for small values of the standard deviation and large values of the mean. This observation supports the suitability of the expected loss functions as models for L-type quality characteristics, where the primary objective is to simultaneously maximize mean response and minimize variability.

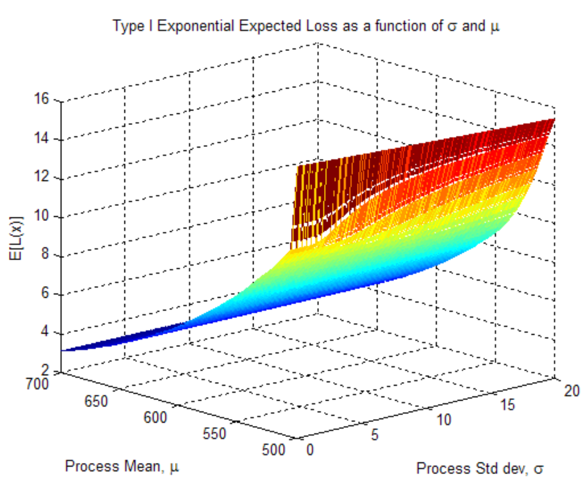

Figure 5.4 Response Surface Plot of Expected Total Loss for Type I Exponential Loss

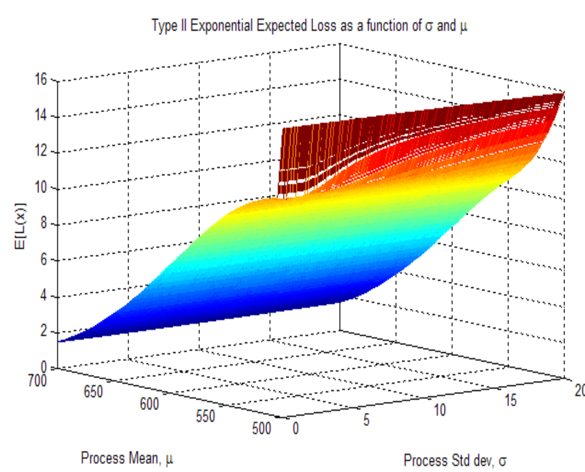

Figure 5.5 Response Surface Plot of Expected Total Loss for Type II Exponential Loss

Numerical Example

In this section, we present a numerical example by means of which we illustrate the integration of the concepts in Chapters 2, 3, and 4. That is, with censored samples from a designed experiment, we find more powerful response surface models than the usual second order models. With the models obtained, we formulate optimization models for solving tolerance problems. The optimization models may involve expected losses for L-type quality characteristics as derived in Chapter 3.

Consider an experiment involving three control factors run on a central composite design with 6 axial points and 5 center points as shown in Table 5.1. Twenty specimens are subjected to each design point and the observations are censored at 500 time units. Thus, the observations not shown are known to exceed 500 time units, but their exact values are unknown. The observations are believed to be normally distributed. The objective of the experiment is to determine control factor settings $\mathbf{x}^* = [x_1^*, x_2^*, x_3^*]$ that give the largest possible mean and the smallest possible standard deviation with consideration for tolerance specifications such as desired lower specification limit and proportion of defective products. A practical application of this example could be as illustrated in Chapter 4 – the lifetimes of light bulbs, where the control factors of interest are the filament resistance (x_1), filament melting temperature (x_2), and the amount of argon gas in the bulb (x_3). Using the method of maximum likelihood estimation (MLE) and the proposed estimation algorithm in Chapter 4, we obtained the mean and standard deviation estimates as shown in Table 5.2. Through this example, we will do the following:

1. Find second order response surface models and use them in solving the optimization problems.
2. Find higher order response surface models and compare them with the second order models in order to justify the need for them, and then apply them in solving the optimization models.

Table 5.1 Observations from a CCD Censored at 500

Design Point	Levels of Control Factors			Observations (Censoring Point at 500)														
	x_1	x_2	x_3															
1	-1	-1	-1	497.9022	498.3152	493.3232	495.27	494.5075	471.0686	494.0787	492.4907	489.8381	492.3599	485.7602				
2	-1	-1	1	487.1475	483.7946	494.3957	496.9277	491.5743	483.4863	485.4163	497.9412	473.8951	496.7536	485.039	485.492			
3	-1	1	-1	496.6207	484.3385	494.9387	487.7014	490.0905	499.9218	493.5718	494.2378	482.9191	485.6409	494.7121				
4	-1	1	1	492.64	496.3845	490.0919	496.397	494.2302	492.0504	496.13	495.7074	494.3256	498.7554	490.4928				
5	1	-1	-1	499.0758	498.4897	492.5969	495.9484	486.5981	489.8201	494.5571	482.758	498.6196						
6	1	-1	1	494.2913	499.4643	489.592	484.4937	479.0861	504.4383	494.206	482.718							
7	1	1	-1	474.4152	495.5384	497.3025	491.6318	479.8603	491.6294	498.2359	497.9011	492.7706	480.0253	496.861	489.3977	487.2194	492.2825	496.2312
8	1	1	1	496.2905	495.4059	487.4363	477.1122	493.2308	487.4781	499.7242	494.9123	490.7401	496.341	493.8122	499.7202			
9	-1.7321	0	0	497.2685	494.4964	482.6597	496.5538	465.5205	493.6998	488.6409	486.6966	492.6535	491.904	479.8228	477.9386	499.1839		
10	1.7321	0	0	498.8078	494.9368	486.4925	493.6911	492.1945	495.7651	485.1349	484.3596	469.9788	492.652	497.9886	499.5171	483.5709		
11	0	-1.7321	0	498.943	494.7508	493.1543	492.6021	499.3383	498.7012	489.2208	499.1526							
12	0	1.7321	0	490.9889	484.602	496.7084	490.1958	496.4905	496.2622	491.7294	491.1753	490.2009	496.8286	484.8011	491.534	492.125		
13	0	0	-1.7321	489.3218	492.1967	492.3287	489.8772	491.3791	499.0921	485.145	495.8856	483.784	474.0171					
14	0	0	1.7321	492.5879	487.3753	491.2781	499.3688	480.3538										
15	0	0	0	499.3968	493.7899	476.1107	495.6713	492.7366	492.8928	489.8946	489.2101	490.8123	497.7759	491.9352	483.8338			
16	0	0	0	495.4788	491.52	492.9095	496.2601	493.2554	484.858	494.2331	499.4445	496.7262	494.9186					
17	0	0	0	492.9384	492.4374	495.1995	493.6595	499.6611	497.4936	497.0399	495.5272	495.7443	480.8977	499.0689	498.8495	491.2038	499.7283	
18	0	0	0	496.7811	479.0982	488.8783	487.7824	480.633	498.6195	491.94	497.3491							
19	0	0	0	486.5472	493.8829	498.57	498.1957	478.0298	493.5794	482.8966	495.4852	495.3617	497.1138	493.8498				

Table 5.2 Parameter Estimates

Design Point	Levels of Control Factors			Estimates			
				MLE		Proposed Method	
	x_1	x_2	x_3	muHat_ML	sigHat_ML	muHat	sigHat
1	-1	-1	-1	498.99	10.88	498.39	9.44
2	-1	-1	1	496.35	11.73	495.94	10.56
3	-1	1	-1	498.36	9.42	497.85	8.20
4	-1	1	1	498.68	5.64	498.39	4.92
5	1	-1	-1	501.21	9.19	500.16	7.30
6	1	-1	1	503.71	13.40	501.57	10.02
7	1	1	-1	494.49	9.24	494.41	8.84
8	1	1	1	498.15	8.76	497.83	7.85
9	-1.7321	0	0	495.47	13.03	495.16	12.00
10	1.7321	0	0	496.42	11.01	496.15	10.13
11	0	-1.7321	0	501.17	5.71	500.51	4.53
12	0	1.7321	0	496.29	7.20	496.13	6.66
13	0	0	-1.7321	499.50	12.38	498.53	10.35
14	0	0	1.7321	510.02	15.29	503.78	8.45
15	0	0	0	497.40	9.53	497.05	8.57
16	0	0	0	499.70	6.95	499.15	5.81
17	0	0	0	497.68	6.00	497.59	5.64
18	0	0	0	503.18	13.32	501.06	9.95
19	0	0	0	499.02	9.74	498.48	8.46

Second Order Response Surface Models

The second order response surface models (Vining and Myers (1995)) for the mean and standard deviation using the maximum likelihood estimates are given by

$$\begin{aligned}\hat{\mu}_{ML}(\mathbf{x}) = {} & 499.394 + 0.842x_1 - 2.356x_2 + 2.731x_3 - 4.062x_1^2 - 1.274x_2^2 + 4.759x_3^2 \\ & - 5.243x_1x_2 + 3.179x_1x_3 + 1.546x_2x_3\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{\sigma}_{ML}(\mathbf{x}) = {} & 9.109 - 0.070x_1 - 1.183x_2 + 0.721x_3 + 2.345x_1^2 - 3.220x_2^2 + 4.162x_3^2 \\ & + 1.107x_1x_2 + 2.494x_1x_3 - 3.498x_2x_3\end{aligned} \tag{9}$$

Similarly, the second order response surfaces for the estimates from the proposed method of Chapter 4 are

$$\begin{aligned}\hat{\mu}(\mathbf{x}) = {} & 498.665 + 0.633x_1 - 1.877x_2 + 1.487x_3 - 3.191x_1^2 - 0.520x_2^2 + 2.313x_3^2 \\ & - 4.283x_1x_2 + 2.529x_1x_3 + 1.873x_2x_3\end{aligned} \tag{10}$$

$$\hat{\sigma}(\mathbf{x}) = 7.684 - 0.289x_1 - 0.471x_2 - 0.463x_3 + 3.212x_1^2 - 2.257x_2^2 + 1.548x_3^2 + 2.344x_1x_2 + 1.460x_1x_3 - 3.037x_2x_3 \quad (11)$$

Solutions to Optimization Models Based on Second Order RSMs

Table 5.3 shows solutions to the integrated model in (5) using the estimated response surface models, where a lower specification limit is given (L = 480), and a desired maximum proportion of defective products is specified as one in a million (i.e. δ=1/1000000).

Table 5.3 Optimization Results to model (5) using models:
Lower spec limit L = 480, prop. defective = 1/1 000 000 (k = 4.7534)

Meth. Of Estimation	$\mathbf{x}^* = [x_1^* \quad x_2^* \quad x_3^*]$	$\hat{\mu}^* = \hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}^* = \hat{\sigma}(\mathbf{x}^*)$	Objective function value
Prop Alg	[0.1271 -1.5094 -1.7321]	509.8675	3.4550e-013	0.9414
MLE	[-0.3543 1.6648 1.7321]	515.7393	7.1054e-015	0.9307

For the same values of L and δ, we solve the expected loss optimization model proposed in (6) using each of the expected exponential loss functions and the estimated response surface models above. That is, we solve the problem

$$\begin{aligned} &\min_{\mathbf{x}\in\Omega} E[T(\mathbf{x})] \\ &s.t. \quad \sigma(\mathbf{x}) \geq 0 \\ &\qquad \mu(\mathbf{x}) - (L + \kappa\sigma(\mathbf{x})) \geq 0 \end{aligned} \quad (12)$$

where $E[T(\mathbf{x})]$ is the expected total loss either under the type I exponential loss function or under the type II exponential loss function as derived in Chapter 3. That is, in the case of the type I exponential loss function, we have

$$E[T(x)] = c_l\Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{\frac{k_2\sigma^2 + 2k_2(L-\mu)}{2}\right\}\cdot\left\{1-\Phi\left(\frac{L-(\mu-k_2\sigma^2)}{\sigma}\right)\right\}, \quad (13)$$

while in the case of the type II exponential loss function, we have

$$E[T(x)] = c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + \frac{k_1}{\sqrt{2k_2\sigma^2+1}} \exp\left[-\frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}\right]\left[1-\Phi\left(\frac{L-\mu}{\sigma\sqrt{2k_2\sigma^2+1}}\right)\right] \quad (14)$$

In solving these problems, we suggest the use of the search algorithm proposed in the second section of this Chapter for the same reasons therein. In Table 5.4 below, we show the optimization results for model (12) based on 5000 different initial points that were uniformly generated from the experimental region. The optimal settings obtained here are quite close to the solutions of model (5), which are displayed in Table 5.3. This observation seems to suggest that the minimization of the expected total loss may be achieved through the minimization of the relatively simple model in (5).

Table 5.4 Optimization Results for Model (12)

Loss Function	Estimation Method	Optimal Settings ($\mathbf{x}^*$)	μ^*	σ^*	$E[T(\mathbf{x}^*)]$
Type I Expon	Prop Alg.	[0.1271 -1.5094 -1.7321]	509.8675	3.1860 X 10^{-8}	4.6827
	MLE	[-0.1293 1.2724 1.7321]	516.7219	7.7254	3.8866
Type II Expon	Prop Alg.	[0.1270 -1.5095 -1.7321]	509.8675	5.8515 X 10^{-8}	3.3446
	MLE	[-0.3543 1.6648 1.7321]	515.7394	8.5535 X 10^{-10}	1.9264

In what follows, we will find higher order response surface models, compare them with the second order models above, and then apply them in solving models (5) and (12).

Higher Order Response Surface Models

Optimization results from robust tolerance design problems are meant to be recommendations about how real systems should run in order to yield desired optimum results. The problems are formulated using response surface models, usually of the mean and standard deviation. Hence, as emphasized in Chapter 2, it is important that the response surface models used are powerful, especially in terms of their predictive ability, otherwise solutions from the formulated optimization models may be misleading and of no real practical significance. Using the method of stepwise regression with a maximum order of 6 and 83 terms (Table 5.5), we found the following response surface models for $\hat{\mu}(\mathbf{x})$, $\hat{\sigma}(\mathbf{x})$, $\hat{\mu}_{ML}(\mathbf{x})$, and $\hat{\sigma}_{ML}(\mathbf{x})$.

$$\begin{aligned}\hat{\mu}(\mathbf{x}) = {} & 498.630 - 1.084x_2 - 0.949x_1^2 - 1.428x_1x_2 + 0.843x_1x_3 + \\ & 0.624x_2x_3 + 0.488x_3^3 + 0.285x_3^4\end{aligned} \tag{15}$$

$$\begin{aligned}\hat{\sigma}(\mathbf{x}) = {} & 8.196 + 0.981x_1^2 - 0.842x_2^2 + 0.781x_1x_2 - 1.012x_2x_3 - \\ & 0.938x_1^2x_2\end{aligned} \tag{16}$$

$$\begin{aligned}\hat{\mu}_{ML}(\mathbf{x}) = {} & 499.048 - 1.360x_2 - 1.748x_1x_2 + 1.060x_1x_3 - 0.353x_1^4 + \\ & 0.340x_3^5 + 0.211x_3^6\end{aligned} \tag{17}$$

$$\begin{aligned}\hat{\sigma}_{ML}(\mathbf{x}) = {} & 9.121 + 0.980x_1^2 - 0.875x_2^2 - 1.166x_2x_3 - 1.518x_1^2x_2 + \\ & 0.525x_3^4\end{aligned} \tag{18}$$

Table 5.5 Terms Used in Stepwise Model Selection

Order	Terms
First	x_1, x_2, x_3
Second	x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3
Third	x_1^3, x_2^3, x_3^3, $x_1^2x_2$, $x_1^2x_3$, $x_2^2x_3$, $x_1x_2^2$, $x_1x_3^2$, $x_2x_3^2$, $x_1x_2x_3$
Fourth	x_1^4, x_2^4, x_3^4, $x_1^3x_2$, $x_1^3x_3$, $x_1^2x_2^2$, $x_1^2x_2x_3$, $x_1^2x_3^2$, $x_1x_2^3$, $x_1x_2^2x_3$, $x_1x_2x_3^2$, $x_1x_3^3$, $x_2^3x_3$, $x_2^2x_3^2$, $x_2x_3^3$
Fifth	x_1^5, x_2^5, x_3^5, $x_1^4x_2$, $x_1^4x_3$, $x_1^3x_2^2$, $x_1^3x_2x_3$, $x_1^3x_3^2$, $x_1^2x_2^3$, $x_1^2x_2^2x_3$, $x_1^2x_2x_3^2$, $x_1^2x_3^3$, $x_1x_2^4$, $x_1x_3^4$, $x_1x_2^3x_3$, $x_1x_2^2x_3^2$, $x_1x_2x_3^3$, $x_2^4x_3$, $x_2^3x_3^2$, $x_2^2x_3^3$, $x_2x_3^4$
Sixth	x_1^6, x_2^6, x_3^6, $x_1^5x_2$, $x_1^5x_3$, $x_1^4x_2^2$, $x_1^4x_3^2$, $x_1^4x_2x_3$, $x_1^3x_2^3$, $x_1^3x_3^3$, $x_1^3x_2^2x_3$, $x_1^3x_2x_3^2$, $x_1^3x_3^3$, $x_1^2x_2^4$, $x_1^2x_2^3x_3$, $x_1^2x_2^2x_3^2$, $x_1^2x_2x_3^3$, $x_1^2x_3^4$, $x_1x_2^5$, $x_1x_2^4x_3$, $x_1x_2^3x_3^2$, $x_1x_2^2x_3^3$, $x_1x_2x_3^4$, $x_2^5x_3$, $x_2^4x_3^2$, $x_2^3x_3^3$, $x_2^2x_3^4$, $x_2x_3^5$

The outputs for the stepwise regression are shown in Tables A2 through A5, the regression outputs for the second order models are displayed in Tables A6 through A9, and the regression outputs for the higher order models are in Tables A10 through A13 in Appendix A.

As with the work in Chapter 2, the low variance inflation factors for these models indicate no problems with multicollinearity. However, as shown in Table 5.6, all the higher order models are improvements upon the corresponding second order models, even though the higher order models include fewer terms than the second order models.

Table 5.6 Comparing Second and Higher Order Response Surface Models

Statistic	$\hat{\mu}$		$\hat{\sigma}$		$\hat{\mu}_{ML}$		$\hat{\sigma}_{ML}$	
	Quad	Higher	Quad	Higher	Quad	Higher	Quad	Higher
No. of terms	9	7	9	**5**	9	**6**	9	**5**
R^2	84.8	**87.4**	67.1	**65.5**	78.8	**87.4**	62.600	**66.800**
R^2_{adj}	69.6	**79.4**	34.3	**52.3**	57.6	**81.1**	25.100	**54.100**
R^2_{pred}	43.04	**73.34**	0.00	**41.86**	0.000	**62.53**	0.000	**36.810**
RMSE	1.299	**1.071**	1.672	**1.425**	2.325	**1.554**	2.448	**1.917**
PRESS	56.976	**26.671**	106.034	**44.508**	242.394	**86.058**	214.120	**90.986**
ANOVA p-value	0.009	**0.000**	0.151	**0.009**	0.032	**0.000**	0.228	**0.007**

Solutions to Optimization Models Based on High Order RSMs

We now present solutions to the optimization models (5) and (6), applying the higher order and more powerful response surface models found in the previous section. Table 5.7 shows the optimal results obtained from solving model (5) while Table 5.8 displays the optimal solutions from solving model (6).

Table 5.7 Optimization Results for Model (5) Using Higher Order Models

Meth. Of Estimation	$\mathbf{x}^* = [x_1^* \quad x_2^* \quad x_3^*]$	$\hat{\mu}^* = \hat{\mu}(\mathbf{x}^*)$	$\hat{\sigma}^* = \hat{\sigma}(\mathbf{x}^*)$	Objective function value
Prop Alg.	[-1.2288 1.7321 1.7321]	503.5374	8.3056×10^{-9}	0.9533
MLE	[-1.7326 1.7326 0.9972]	502.1820	0.0457	0.9563

Again, looking at the two results together, we observe as in the case of the second order models, the closeness of the results (especially, the optimal settings) from the two optimization models, which further supports the claim that the relatively simper model in (5) is nearly as effective as the more complicated model in (6) in maximizing the mean response on one hand, while minimizing the standard deviation and the total expected loss on the other.

Table 5.8 Optimization Results for Model (6) Using Higher Order Models

Loss Function	Estimation Method	Optimal Settings ($\mathbf{x}^*$)	μ^*	σ^*	$E[T(\mathbf{x}^*)]$
Type I Expon	Prop Alg.	[-1.2288 1.7321 1.7321]	503.5374	8.8818 X 10^{-16}	5.7163
	MLE	[-0.6756 1.7321 1.7321]	513.1322	6.9701	4.3281
Type II Expon	Prop Alg.	[-1.2288 1.7321 1.7321]	503.5374	8.8818 X 10^{-16}	5.4276
	MLE	[-0.6756 1.7321 1.7321]	513.1322	6.9702	2.8284

Conclusion

In this chapter, we have considered tolerance optimization for L-type quality characteristics, and important concept that can find application in lifetime studies. This study is a way of integrating the main concepts of this dissertation, namely, dual response surface modeling and optimization, quality loss functions, and censored robust design modeling. In the presence of censored data, we found more powerful response surface models for the mean response and standard deviation than the usual second order models found in the literature. We proposed two optimization models for solving L-type tolerance problems, and used a numerical example to show how such problems can be solved. The proposed optimization models gave results that are nearly identical even though one of the objective functions is much more complicated (i.e. the expected loss model in (6)) than the other model shown in (5). The illustration was done using expected loss function of the proposed exponential loss function found in Chapter 3. The proposed quadratic loss function for L-Type quality characteristics can also be used in the same manner. For further studies along the same lines it will be interesting to consider different censoring methods than the right censoring we used, and also extend the ideas here to cases involving the S- and N-Type quality characteristics.

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

Introduction

In this chapter, we discuss the contribution of this dissertation to the studies in robust parameter design, and then make suggestions for further research that can be carried out based on the results of this work. This dissertation mainly focused on dual response surface modeling and optimization, quality loss functions, robust parameter design, and tolerance design and optimization. As shown throughout the work, especially in Chapter 5, all these concepts can be integrated into a procedure for determining optimum mean response and variability (i.e. standard deviation), and hence tolerance for systems and products. In the next section, we will specify the unique contributions of this dissertation and then make recommendations for future studies in the final section.

Contributions

We will organize the contributions of this work by topic in this section, and similarly present recommendations for further work in the next section.

Dual Response Surface Modeling and Optimization

In the literature, the most popular dual response surface models have been second order or quadratic models (Vining and Myers, 1995). In this work, we proposed the proper use of statistical model selection techniques, and demonstrated through a numerical example that much better models (in terms of statistical indicators) than the existing ones are obtainable that way.

For example, in the case with uncensored data in Chapter 2, gains in the coefficient of determination (R^2) with respect to the best existing second order models ranged from about 4% to about 194%. In the case of the censored data, Table 6.1 shows the percentage change in values (from quadratic to higher order models) of some of the statistics used in the model selection in Chapter 5. Positive values indicate percentage increase and negatives indicate decreases. The blank cells show that we could not compute the percentage change because the corresponding values of the statistic for the second order model are too small (i.e. almost zero, see Table 5.6).

Table 6.1 Percentage Change from Second Order Models to Higher Order

Statistic	Proposed Est. Meth.		Maximum Likelihood	
	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}_{ML}$	$\hat{\sigma}_{ML}$
R^2	3.07%	-2.38%	10.91%	6.71%
R^2_{adj}	14.08%	52.48%	40.80%	115.54%
R^2_{pred}	70.40%			
RMSE	-17.55%	-14.77%	-33.16%	-21.69%
PRESS	-53.19%	-58.02%	-64.50%	-57.51%

The general trends we observe, indicating improvements, are increases in the values of *R*-square, adjusted *R*-square, and prediction *R*-square, and decreases in the values of root mean-squared error. Particularly significant are the improvements achieved in the modeling of the standard deviation, since that has generally been problematic. Also, one of the problems overcame by this dissertation is that the existing second order models are sometimes either statistically insignificant (see ANOVA p-values in Table 5.6) or marginally significant, but the models we find through our proposal are statistically significant.

In the formulation of optimization models, we considered the main objective of the problem at hand, and applied methods of mathematical variation to come up with plausible models. Many of the optimizations models of this dissertation are perhaps appearing in the literature for the first time.

Quality Loss Functions

In our study of loss functions, we modified the traditional loss function and proposed new ones based on an innovated assumption aimed at making the loss functions more realistic in terms of a warranty provider, for example, deciding whether to repair or replace a product, or an individual deciding whether to incur the expense of repairing a faulty product or to junk it for a new one. This innovation resulted in loss functions with jump discontinuities at the specification limits of the quality characteristics of interest. Expectations were derived for all the loss functions, which we later used as objective functions in setting up optimization problems. Again, this is perhaps the first such consideration of loss functions in the quality control literature.

Censored Robust Parameter Design

Traditional robust parameter design (RPD) procedures have always assumed the availability of complete data, thus leaving out the very practical situations such as life-testing and reliability studies, which give rise to incomplete data in the form of right-censored data. Also, in the traditional RPD set up, the number of observations per design point is constant. In this dissertation, we proposed how to use RPD principles and take care of these two important situations and illustrated our proposal with numerical examples involving randomly generated data. We use the method of maximum likelihood estimation in the presence of censored data and also proposed and used a modification of the expectation-maximization (EM) algorithm, which is new to the quality control literature. In this part of the work, we demonstrated how the principles of variation can be used to set up plausible optimization models, several of which we proposed and solved.

Robust-Tolerance Design and Optimization with Censored Data

Under this topic, we solved tolerance design and optimization problems for L-type quality characteristics, where the available data are right-censored. This part of the study served as an integration of the major concepts of this dissertation. In the solution process, we estimated response surfaces for the mean and standard deviation (i.e. dual response surface modeling), formulated and solved optimization models to give means that are as large as possible (L-type) and values of standard deviation that are as small as possible (i.e. RPD). The expectation of the L-type loss functions that were derived in Chapter 3 were used as objective functions in optimization models, which enabled us to interpret optimal solutions in terms of minimum loss.

Process Target Values

In general, process target value problems are set up to find settings of processes that yield desired values (usually mean and/or standard deviation) of quality characteristics in the output that minimize cost (or maximize profit). Therefore, Chapter 5 is explicitly solving a process target problem in the cases where the objective functions were the expected loss functions. Considering the work in Chapters 2 and 4, we realize that we were working out a design to put majority of products within specification with minimum possible variability, and therefore, as few as possible out of specification, and hence, minimum cost due to variability and out-of-specification products. Thus, the works of Chapters 2 and 4 are implicitly solving target value problems.

Recommendations for Further Studies

In the last section, we outlined the significant research contributions of this dissertation. However, it is possible to pose many questions in relation to the general concepts studied here that will be unanswered in the entire work. These unanswered questions should form an obvious basis for further research that can stem from this work. We make recommendations in the following subsections.

Response Surface Models

The most commonly used response surface models in the literature, including those proposed here, are polynomial in nature. As pointed out in Chapter 2, these response surface models can sometimes yield values of no practical significance such as negative standard deviation. In this dissertation we proposed the use constraints in optimization models as a precaution against such occurrences. Another plausible solution that will be interesting to consider is the use of non-polynomial functions like exponential and logistic, whose ranges exclude negative values. The logistic function will particularly be interesting in cases where there is some idea about the bounds of the quantity being modeled.

Multiple Quality Characteristics

A natural extension of the entire dissertation would be in the consideration of multiple quality characteristics, and modeling involving both controllable and noise factors. For such studies, it may be useful to integrate the methods of generalize linear models (GLMs) that can be found in the literature. Extensions of this nature will involve modeling multiple (two or more) mean responses and variability simultaneously, and proceeding along the lines already outlined in this function, namely, formulating and solving appropriate optimization models. This extension is particularly important when it is necessary to design for systems where several critical quality characteristics either vary independently or otherwise. In situations where the quality characteristics are dependent, it will be interesting to model the nature of the dependence.

Nonparametric Methods

The work in this dissertation is parametric in nature, where partial knowledge of the sampled populations is assumed. This assumption may not hold in some instances. Therefore, we recommend the use of nonparametric methods as a way of avoiding problems arising from lack of knowledge of the underlying population.

Censoring

This dissertation only considered right-censored data because it is targeted at time dependent quality characteristics in application areas such as reliability and life-testing. However, the proposals of this situation can be modified to apply to other censoring procedures, namely, left-censoring, interval censoring, random censoring, etc. That forms another direction for further study.

Loss Functions

We have shown how losses can be modeled using exponential functions, and proposed an assumption that is meant to make all loss functions as practicable as possible (Chapter 3). Under these assumptions, we also modified the usual quadratic loss function. When modeling loss functions, we recommend being flexible about the appropriate function that may be applicable. It may be possible to have quadratic, exponential, or a different kind of function altogether. Perhaps using model selection techniques without restricting the possibilities to exponential and quadratic functions will be an interesting way to find more accurate loss functions than the functions presently being used.

Estimation Method

In our work on censored robust design, we used two estimation methods in the parameter estimation phase - the method of maximum likelihood and a method we proposed based on the expectation maximization algorithm. The results based on the proposed estimation method are encouraging. However, based the comparison study (Chapter 4), further work at improving the method is worth considering.

Censored Robust Design with other Quality Characteristics

The area of application targeted by this dissertation, namely, lifetime and reliability studies, has skewed considerations towards L-type quality characteristics. We however believe that it is possible to find application areas involving S- and N-type quality characteristics, where some kind of censoring features. Finding such application areas and modifying the methods of this dissertation to apply will be an interesting study undertake.

APPENDICES

Appendix A

Expectation of Loss Functions for Normally Distributed Quality Characteristics

A1. The Expected Loss of the Modified Quadratic Loss Function for Normally Distributed N-Type Quality Characteristics

For the loss function in Equation (1) of Chapter 1, if the quality characteristic follows a normal distribution with mean μ and variance σ^2, the expected loss within the specification limits is given by

$$E[L(x)] = \int_L^{\tau} K_l (x-\tau)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx + \int_{\tau}^{U} K_u (x-\tau)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \qquad \text{(A1)}$$

We find the integral of the form $\int_a^b K(x-\tau)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$ and then apply the result to (A1).

$$(x-\tau)^2 = (x-\mu)^2 - 2(\tau-\mu)(x-\mu) + (\tau-\mu)^2$$

Therefore,

$$I = \int_a^b K(x-\tau)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{K}{\sqrt{2\pi}\sigma}\left\{\int_a^b (x-\mu)^2 \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx - 2(\tau-\mu)\int_a^b (x-\mu) \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx + (\tau-\mu)^2 \int_a^b \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx\right\}$$

Let $z = \frac{x-\mu}{\sigma}$, $a_z = \frac{a-\mu}{\sigma}$, and $b_z = \frac{b-\mu}{\sigma}$. Then

$$I = \frac{K}{\sqrt{2\pi}}\left\{\int_{a_z}^{b_z} \sigma^2 z^2 e^{-\frac{z^2}{2}} dz - (\tau-\mu)\int_{a_z}^{b_z} \sigma z e^{-\frac{z^2}{2}} dz + (\tau-\mu)^2 \int_{a_z}^{b_z} e^{-\frac{z^2}{2}} dz\right\} \qquad \text{(A2)}$$

Let us find each of the integrals in (A2) separately.

$$\int_{a_z}^{b_z} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}\left[\Phi(b_z^2) - \Phi(a_z^2)\right], \qquad \text{(A3)}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

$$\int_{a_z}^{b_z} z e^{-\frac{z^2}{2}} dz = \int_{a_z}^{b_z} e^{-\frac{z^2}{2}} d\left(\frac{z^2}{2}\right) = \frac{1}{2}\int_{a_z}^{b_z} e^{-\frac{z^2}{2}} d\left(z^2\right) = \left[-e^{-\frac{z^2}{2}}\right]_{a_z}^{b_z} = e^{-\frac{a_z^2}{2}} - e^{-\frac{b_z^2}{2}} \qquad \text{(A4)}$$

$$\int_{a_z}^{b_z} z^2 e^{-\frac{z^2}{2}} dz = \int_{a_z}^{b_z} z \cdot z e^{-\frac{z^2}{2}} dz = \left[-z e^{-\frac{z^2}{2}}\right]_{a_z}^{b_z} + \int_{a_z}^{b_z} e^{-\frac{z^2}{2}} dz$$

$$= \left[a_z e^{-\frac{a_z^2}{2}} - b_z e^{-\frac{b_z^2}{2}}\right] + \sqrt{2\pi}\left[\Phi(b_z^2) - \Phi(a_z^2)\right] \qquad \text{(A5)}$$

Thus putting (A3) through (A5) in (A2), we get

$$I = \frac{K}{\sqrt{2\pi}}\left\{\sigma^2\left[a_z e^{-\frac{a_z^2}{2}} - b_z e^{-\frac{b_z^2}{2}} + \sqrt{2\pi}\left[\Phi(b_z) - \Phi(a_z)\right]\right] - 2(\tau-\mu)\sigma\left(e^{-\frac{a_z^2}{2}} - e^{-\frac{b_z^2}{2}}\right) + (\tau-\mu)^2\sqrt{2\pi}\left[\Phi(b_z) - \Phi(a_z)\right]\right\}$$

$$= \frac{K}{\sqrt{2\pi}}\left\{\sqrt{2\pi}\left[\sigma^2 + (\tau-\mu)^2\right]\left[\Phi(b_z) - \Phi(a_z)\right] + \left[\sigma^2 a_z - 2\sigma(\tau-\mu)\right]e^{-\frac{a_z^2}{2}} - \left[\sigma^2 b_z - 2\sigma(\tau-\mu)\right]e^{-\frac{b_z^2}{2}}\right\}$$

Using $\tau_z = \frac{\tau-\mu}{\sigma}$, we have $\sigma\tau_z = \tau - \mu$ and

$$I = \frac{K\sigma^2}{\sqrt{2\pi}}\left\{\sqrt{2\pi}(1+\tau_z^2)\left[\Phi(b_z^2) - \Phi(a_z^2)\right] + (a_z - 2\tau_z)e^{-\frac{a_z^2}{2}} - (b_z - 2\tau_z)e^{-\frac{b_z^2}{2}}\right\} \tag{A6}$$

For the first integral in (A1), use the result of (A6) with $K = K_l$, $a_z = L_z = \frac{L-\mu}{\sigma}$, and $b_z = \tau_z = \frac{\tau-\mu}{\sigma}$. Similarly for the second integral, use the result of (A6) with $K = K_u$, $a_z = \tau_z = \frac{\tau-\mu}{\sigma}$, and $b_z = U_z = \frac{U-\mu}{\sigma}$. Thus, the expected loss is

$$\begin{aligned} E[L(x)] = &\frac{K_l\sigma^2}{\sqrt{2\pi}}\left\{\sqrt{2\pi}(1+\tau_z^2)\left[\Phi(\tau_z) - \Phi(L_z)\right] + (L_z - 2\tau_z)e^{-\frac{L_z^2}{2}} + \tau_z e^{-\frac{\tau_z^2}{2}}\right\} + \\ &\frac{K_u\sigma^2}{\sqrt{2\pi}}\left\{\sqrt{2\pi}(1+\tau_z^2)\left[\Phi(U_z) - \Phi(\tau_z)\right] - \tau_z e^{-\frac{\tau_z^2}{2}} - (U_z - 2\tau_z)e^{-\frac{U_z^2}{2}}\right\} \end{aligned} \tag{A7}$$

Rearranging the terms in (A7) gives the expected loss in equation (7) of Lemma 2 □

A2. The Expected Loss of the Type I Exponential Loss Function for Normally Distributed N-Type Quality Characteristics

Consider the type I exponential loss function in quation (11), and suppose that the quality characteristic is normally distributed with mean μ and variance σ^2. Then the expected loss for deviations within the specification limits is

$$E[L(x)] = \int_L^\tau \left(e^{-K_l(x-\tau)} - 1\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx + \int_\tau^U \left(e^{K_u(x-\tau)} - 1\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\left\{\int_L^\tau \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - K_l(x-\tau)\right\}dx + \int_\tau^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + K_u(x-\tau)\right\}dx - \int_L^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx\right\} \quad \text{(B1)}$$

Consider the exponent in the first integral, and complete squares in terms of x as follows

$$-\frac{(x-\mu)^2}{2\sigma^2} - K_l(x-\tau) = -\left[\frac{(x-\mu)^2 + 2\sigma^2 K_l(x-\tau)}{2\sigma^2}\right]$$
$$= -\left[\frac{x^2 - 2(\mu-\sigma^2K_l)x - 2\sigma^2K_l\tau + \mu^2}{2\sigma^2}\right] = -\left[\frac{[x-(\mu-\sigma^2K_l)]^2 - \sigma^2K_l[\sigma^2K_l + 2\tau - 2\mu]}{2\sigma^2}\right]$$
$$= -\frac{[x-(\mu-\sigma^2K_l)]^2}{2\sigma^2} + \frac{K_l[\sigma^2K_l + 2\tau - 2\mu]}{2}$$

Thus,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_L^\tau \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - K_l(x-\tau)\right\}dx = \exp\left\{\frac{K_l[\sigma^2K_l + 2\tau - 2\mu]}{2}\right\}\frac{1}{\sqrt{2\pi}\sigma}\int_L^\tau \exp\left\{-\frac{[x-(\mu-\sigma^2K_l)]^2}{2\sigma^2}\right\}dx$$
$$= \left\{\Phi\left(\frac{\tau-(\mu-\sigma^2K_l)}{\sigma}\right) - \Phi\left(\frac{L-(\mu-\sigma^2K_l)}{\sigma}\right)\right\}\exp\left\{\frac{K_l[\sigma^2K_l + 2\tau - 2\mu]}{2}\right\} \quad \text{(B2)}$$

Similarly, the second integral in (B1) can expressed as

$$\frac{1}{\sqrt{2\pi}\sigma}\int_\tau^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + K_u(x-\tau)\right\}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_\tau^U \exp\left\{-\frac{[x-(\mu+K_u\sigma^2)]^2}{2\sigma^2} - \frac{K_u(2\tau-2\mu-K_u\sigma^2)}{2}\right\}dx$$
$$= \exp\left\{-\frac{K_u(2\tau-2\mu-K_u\sigma^2)}{2}\right\}\frac{1}{\sqrt{2\pi}\sigma}\int_\tau^U \exp\left\{-\frac{[x-(\mu+K_u\sigma^2)]^2}{2\sigma^2}\right\}dx$$
$$= \left\{\Phi\left(\frac{U-(\mu+\sigma^2K_u)}{\sigma}\right) - \Phi\left(\frac{\tau-(\mu+\sigma^2K_u)}{\sigma}\right)\right\}\exp\left\{-\frac{K_u(2\tau-2\mu-K_u\sigma^2)}{2}\right\} \quad \text{(B3)}$$

Finally, if we let $z = \frac{x-\mu}{\sigma}$, $L_z = \frac{L-\mu}{\sigma}$ and $U_z = \frac{U-\mu}{\sigma}$, then the third integral in (B1) can be computed as follows

$$\frac{1}{\sqrt{2\pi}\sigma}\int_L^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx = \Phi(U_z) - \Phi(L_z) \quad \text{(B4)}$$

Put (B2) through (B4) in (B1) to get

$$E[L(x)] = \left\{\Phi\left(\frac{\tau-(\mu-\sigma^2 K_l)}{\sigma}\right) - \Phi\left(\frac{L-(\mu-\sigma^2 K_l)}{\sigma}\right)\right\} \exp\left\{\frac{K_l[\sigma^2 K_l + 2\tau - 2\mu]}{2}\right\} + \left\{\Phi\left(\frac{U-(\mu+\sigma^2 K_u)}{\sigma}\right) - \Phi\left(\frac{\tau-(\mu+\sigma^2 K_u)}{\sigma}\right)\right\} \exp\left\{-\frac{K_u(2\tau-2\mu-K_u\sigma^2)}{2}\right\} - \left(\Phi(U_z) - \Phi(L_z)\right) \tag{B5}$$

Substituting $\sigma U_z = U - \mu$, $\sigma L_z = L - \mu$, and $\sigma\tau_z = \tau - \mu$, we obtain the result of Lemma 3 as follows

$$E[L(x)] = \{\Phi(\tau_z + K_l\sigma) - \Phi(L_z + K_l\sigma)\} \exp\left\{\frac{K_l\sigma(K_l\sigma + 2\tau_z)}{2}\right\} + \{\Phi(U_z - K_u\sigma) - \Phi(\tau_z - K_u\sigma)\} \exp\left\{-\frac{K_u\sigma(2\tau_z - K_u\sigma)}{2}\right\} + \Phi(L_z) - \Phi(U_z)$$

A3. The Expected Loss of the Type II Exponential Loss Function for Normally Distributed N-Type Quality Characteristics

$$E[L(x)] = \int_L^\tau c_l\left(1 - e^{-K_l(x-\tau)^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx + \int_\tau^U c_u\left(1 - e^{-K_u(x-\tau)^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx$$

(C1)

Rewrite as

$$E[L(x)] = \frac{1}{\sqrt{2\pi}\sigma}\left\{c_l\int_L^\tau \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx + c_u\int_\tau^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx\right\} - \frac{c_l}{\sqrt{2\pi}\sigma}\int_L^\tau \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - K_l(x-\tau)^2\right\}dx - \frac{c_u}{\sqrt{2\pi}\sigma}\int_\tau^u \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - K_u(x-\tau)^2\right\}dx$$

(C2)

The first two integrals are easy to compute using the normal cumulative distribution function as follows:

$$\frac{1}{\sqrt{2\pi}\sigma}\left\{c_l\int_L^\tau \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx + c_u\int_\tau^U \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx\right\} = c_u\Phi\left(\frac{U-\mu}{\sigma}\right) - c_l\Phi\left(\frac{L-\mu}{\sigma}\right) + (c_l - c_u)\Phi\left(\frac{\tau-\mu}{\sigma}\right)$$

(C3)

By expanding and completing squares in x, it is easy to show that

$$-\frac{(x-\mu)^2}{2\sigma^2} - K(x-\tau)^2 = -\frac{1+2K\sigma^2}{2\sigma^2}\left(x - \frac{\mu+2K\tau\sigma^2}{1+2K\sigma^2}\right)^2 - \frac{K(\tau-\mu)^2}{1+2K\sigma^2} = -\frac{(x-\gamma)^2}{2\lambda^2} - \frac{K(\tau-\mu)^2}{1+2K\sigma^2},$$

(C4)

where $\gamma = \frac{\mu+2K\tau\sigma^2}{1+2K\sigma^2}$ and $\lambda = \frac{\sigma}{\sqrt{1+2K\sigma^2}}$. Therefore using (C4), we can write

$$\frac{1}{\sqrt{2\pi}\sigma}\int_a^b \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - K(x-\tau)^2\right\}dx = \exp\left\{-\frac{K(\tau-\mu)^2}{1+2K\sigma^2}\right\}\frac{1}{\sqrt{2\pi}\sigma}\int_a^b \exp\left\{-\frac{(x-\gamma)^2}{2\lambda^2}\right\}dx$$
$$= \exp\left\{-\frac{K(\tau-\mu)^2}{1+2K\sigma^2}\right\}\frac{\lambda}{\sigma}\frac{1}{\sqrt{2\pi}\lambda}\int_a^b \exp\left\{-\frac{(x-\gamma)^2}{2\lambda^2}\right\}dx = \frac{\lambda}{\sigma}\exp\left\{-\frac{K(\tau-\mu)^2}{1+2K\sigma^2}\right\}\cdot\left\{\Phi\left(\frac{b-\gamma}{\lambda}\right) - \Phi\left(\frac{a-\gamma}{\lambda}\right)\right\} \quad \text{(C5)}$$

$$\frac{a-\gamma}{\lambda} = \frac{\sqrt{1+2K\sigma^2}}{\sigma}\left(a - \frac{\mu-2K\tau\sigma^2}{1+2K\sigma^2}\right) = \frac{2K\sigma^2(a-\tau)+a-\mu}{\sigma\sqrt{1+2K\sigma^2}} \quad \text{(C6)}$$

Using (C6) in (C5), we get that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_a^b \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}-K(x-\tau)^2\right\}dx = \exp\left\{-\frac{K(\tau-\mu)^2}{1+2K\sigma^2}\right\}\cdot\left\{\Phi\left(\frac{2K\sigma^2(b-\tau)+b-\mu}{\sigma\sqrt{1+2K\sigma^2}}\right)-\Phi\left(\frac{2K\sigma^2(a-\tau)+a-\mu}{\sigma\sqrt{1+2K\sigma^2}}\right)\right\} \quad \text{(C7)}$$

when $K=K_l$, $a=L$, and $b=\tau$, equation (C7) gives the third integral in (C2). On the other hand, if $K=K_u$, $a=\tau$, and $b=U$, (C7) gives the fourth integral in (C2). Thus, the expected loss function is obtained as

$$E[L(x)] = c_u\Phi\left(\frac{U-\mu}{\sigma}\right)-c_l\Phi\left(\frac{L-\mu}{\sigma}\right)+(c_l-c_u)\Phi\left(\frac{\tau-\mu}{\sigma}\right)-$$
$$\frac{c_l\sigma}{\sqrt{1+2K_l\sigma^2}}\exp\left\{-\frac{K_l(\tau-\mu)^2}{1+2K_l\sigma^2}\right\}\cdot\left\{\Phi\left(\frac{\tau-\mu}{\sigma\sqrt{1+2K_l\sigma^2}}\right)-\Phi\left(\frac{2K_l\sigma^2(L-\tau)+L-\mu}{\sigma\sqrt{1+2K_l\sigma^2}}\right)\right\}-$$
$$\frac{c_u\sigma}{\sqrt{1+2K_u\sigma^2}}\exp\left\{-\frac{K_u(\tau-\mu)^2}{1+2K_u\sigma^2}\right\}\cdot\left\{\Phi\left(\frac{2K_u\sigma^2(U-\tau)+U-\mu}{\sigma\sqrt{1+2K_u\sigma^2}}\right)-\Phi\left(\frac{\tau-\mu}{\sigma\sqrt{1+2K_u\sigma^2}}\right)\right\}$$

Substitute $\sigma U_z = U-\mu$, $\sigma L_z = L-\mu$, and $\sigma\tau_z = \tau-\mu$ to get

$$E[L(x)] = c_u\Phi(U_z)-c_l\Phi(L_z)+(c_l-c_u)\Phi(\tau_z)-$$
$$\frac{c_l\sigma}{\sqrt{1+2K_l\sigma^2}}\left\{\Phi\left(\frac{\tau_z}{\sqrt{1+2K_l\sigma^2}}\right)-\Phi\left(\frac{2K_l\sigma(L-\tau)+L_z}{\sqrt{1+2K_l\sigma^2}}\right)\right\}\exp\left\{\frac{-K_l\sigma^2\tau_z}{1+2K_l\sigma^2}\right\}-$$
$$\frac{c_u\sigma}{\sqrt{1+2K_u\sigma^2}}\left\{\Phi\left(\frac{2K_u\sigma(U-\tau)+U_z}{\sqrt{1+2K_u\sigma^2}}\right)-\Phi\left(\frac{\tau_z}{\sqrt{1+2K_u\sigma^2}}\right)\right\}\exp\left\{\frac{-K_u\sigma^2\tau_z}{1+2K_u\sigma^2}\right\}$$

A4. The Expected Loss of the Type I Exponential Loss Function for Normally Distributed L-Type Quality Characteristics

$$E[T(x)] = c_l \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{L}^{\infty} k_1 e^{-k_2(x-L)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + \frac{k_1}{\sqrt{2\pi}\sigma} \int_{L}^{\infty} \exp\left\{-k_2(x-L) - \frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$-k_2(x-L) - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\left[2\sigma^2 k_2(x-L) + (x-\mu)^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu - k_2\sigma^2)x + \mu^2 - 2k_2\sigma^2 L\right]$$

$$= -\frac{1}{2\sigma^2}\left[\left(x - (\mu - k_2\sigma^2)\right)^2 - (\mu - k_2\sigma^2)^2 + \mu^2 - 2k_2\sigma^2 L\right]$$

$$= -\frac{1}{2\sigma^2}\left[\left(x - (\mu - k_2\sigma^2)\right)^2 + k_2\sigma^2\left[2(\mu - L) - k_2\sigma^2\right]\right]$$

$$= -\frac{1}{2\sigma^2}\left(x - (\mu - k_2\sigma^2)\right)^2 + \frac{k_2\left[2(L-\mu) + k_2\sigma^2\right]}{2}$$

Therefore,

$$E[T(x)] = c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{\frac{k_2\left[2(L-\mu) + k_2\sigma^2\right]}{2}\right\} \frac{1}{\sqrt{2\pi}\sigma} \int_{L}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x - (\mu - k_2\sigma^2)\right)^2\right\} dx$$

$$= c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{\frac{k_2\left[2(L-\mu) + k_2\sigma^2\right]}{2}\right\} \left\{1 - \Phi\left(\frac{L - (\mu - k_2\sigma^2)}{\sigma}\right)\right\} dx$$

A5. The Expected Loss of the Type II Exponential Loss Function for Normally Distributed L-Type Quality Characteristics

$$E[T(x)] = c_l \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{L}^{\infty} k_1 e^{-k_2(x-L)^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + \frac{k_1}{\sqrt{2\pi}\sigma} \int_{L}^{\infty} \exp\left\{-k_2(x-L)^2 - \frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$-k_2(x-L)^2 - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\left[2\sigma^2 k_2 (x-L)^2 + (x-\mu)^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[(2k_2\sigma^2+1)x^2 - 2(2k_2L\sigma^2+\mu)x + \mu^2 + 2k_2L^2\sigma^2\right]$$

$$= -\frac{2k_2\sigma^2+1}{2\sigma^2}\left[x^2 - 2\frac{(2k_2L\sigma^2+\mu)}{(2k_2\sigma^2+1)}x + \frac{\mu^2+2k_2L^2\sigma^2}{2k_2\sigma^2+1}\right]$$

$$= -\frac{2k_2\sigma^2+1}{2\sigma^2}\left[\left(x - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2 - \left(\frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2 + \frac{\mu^2+2k_2L^2\sigma^2}{2k_2\sigma^2+1}\right]$$

$$= -\frac{2k_2\sigma^2+1}{2\sigma^2}\left[\left(x - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2 + \frac{2k_2\sigma^2(L^2+\mu^2-2L\mu)}{\left(2k_2\sigma^2+1\right)^2}\right]$$

$$= -\frac{2k_2\sigma^2+1}{2\sigma^2}\left[\left(x - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2 + \frac{2k_2\sigma^2(\mu-L)^2}{\left(2k_2\sigma^2+1\right)^2}\right]$$

$$= -\frac{2k_2\sigma^2+1}{2\sigma^2}\left(x - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2 - \frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}$$

Thus,

$$E[T(x)] = c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{-\frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}\right\} \frac{1}{\sqrt{2\pi}\sigma} \int_{L}^{\infty} \exp\left\{-\frac{2k_2\sigma^2+1}{2\sigma^2}\left(x - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}\right)^2\right\} dx$$

$$= c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{-\frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}\right\} \frac{1}{\sqrt{2k_2\sigma^2+1}} \left\{1 - \Phi\left(\frac{L - \frac{2k_2L\sigma^2+\mu}{2k_2\sigma^2+1}}{\sigma\big/\sqrt{2k_2\sigma^2+1}}\right)\right\}$$

$$= c_l \Phi\left(\frac{L-\mu}{\sigma}\right) + k_1 \exp\left\{-\frac{k_2(\mu-L)^2}{2k_2\sigma^2+1}\right\} \frac{1}{\sqrt{2k_2\sigma^2+1}} \left\{1 - \Phi\left(\frac{L-\mu}{\sigma\sqrt{2k_2\sigma^2+1}}\right)\right\}$$

Appendix B

Minitab Outputs for Stepwise Regression

Table B- 1 Results of Stepwise Regression for $\hat{\mu}(\mathbf{x})$

Stepwise Regression: muHat versus x1, x2, ...

```
  Alpha-to-Enter: 0.15  Alpha-to-Remove: 0.15


Response is muHat on 80 predictors, with N = 19


Step               1        2        3        4        5        6        7
Constant       499.2    499.2    499.2    499.2    498.6    498.6    498.6

x11            -1.15    -1.15    -1.15    -1.15    -0.95    -0.95    -0.95
T-Value        -2.11    -2.30    -2.59    -3.03    -2.84    -3.18    -3.40
P-Value        0.050    0.035    0.020    0.009    0.014    0.008    0.006

x2                      -1.08    -1.08    -1.08    -1.08    -1.08    -1.08
T-Value                 -2.05    -2.30    -2.69    -3.16    -3.54    -3.79
P-Value                 0.057    0.036    0.017    0.008    0.004    0.003

x12                              -1.43    -1.43    -1.43    -1.43    -1.43
T-Value                          -2.29    -2.68    -3.15    -3.53    -3.77
P-Value                          0.037    0.018    0.008    0.004    0.003

x333                                       0.49     0.49     0.49    0.49
T-Value                                    2.55     2.99     3.35    3.59
P-Value                                   0.023    0.010    0.006    0.004

x3333                                               0.29     0.29    0.285
T-Value                                             2.50     2.80    3.00
P-Value                                            0.027    0.016    0.012

x13                                                          0.84    0.84
T-Value                                                      2.08    2.23
P-Value                                                     0.059    0.048

x23                                                                  0.62
T-Value                                                              1.65
P-Value                                                              0.127

S               2.16     1.98     1.76     1.51     1.28     1.15    1.07
R-Sq           20.80    37.24    53.54    68.29    78.59    84.27    87.39
R-Sq(adj)      16.15    29.40    44.25    59.22    70.36    76.41    79.36
PRESS        93.1418  80.6823  64.7534  72.5420  38.4630  32.1053    26.6706
R-Sq(pred)      6.89    19.34    35.27    27.48    61.55    67.90    73.34
```

Table B- 2 Results of Stepwise Regression for $\hat{\sigma}(\mathbf{x})$

Stepwise Regression: sigHat versus x1, x2, ...

Alpha-to-Enter: 0.15 Alpha-to-Remove: 0.15

Response is sigHat on 80 predictors, with N = 19

Step	1	2	3	4	5
Constant	7.484	8.196	8.196	8.196	8.196
x11	1.11	0.98	0.98	0.98	0.98
T-Value	2.38	2.25	2.41	2.57	2.70
P-Value	0.029	0.039	0.030	0.022	0.018
x22		-0.84	-0.84	-0.84	-0.84
T-Value		-1.93	-2.06	-2.21	-2.31
P-Value		0.071	0.057	0.045	0.038
x23			-1.01	-1.01	-1.01
T-Value			-1.79	-1.92	-2.01
P-Value			0.093	0.076	0.066
x112				-0.94	-0.94
T-Value				-1.77	-1.86
P-Value				0.098	0.085
x12					0.78
T-Value					1.55
P-Value					0.145
S	1.84	1.70	1.60	1.49	1.42
R-Sq	25.05	39.24	49.95	59.14	65.52
R-Sq(adj)	20.64	31.64	39.94	47.47	52.26
PRESS	69.7858	63.1558	59.1129	52.6459	44.5075
R-Sq(pred)	8.84	17.50	22.78	31.23	41.86

Table B- 3 Results of Stepwise Regression for $\hat{\mu}_{ML}(\mathbf{x})$

Stepwise Regression: muHat_ML versus x1, x2, ...

Alpha-to-Enter: 0.15 Alpha-to-Remove: 0.15

Response is muHat_ML on 80 predictors, with N = 19

Step	1	2	3	4	5	6
Constant	498.5	498.5	498.5	498.5	499.0	499.0
x333333	0.232	0.232	0.232	0.232	0.211	0.211
T-Value	2.68	3.23	3.61	4.20	4.31	4.74
P-Value	0.016	0.005	0.003	0.001	0.001	0.000
x33333		0.340	0.340	0.340	0.340	0.340
T-Value		2.96	3.30	3.84	4.42	4.86
P-Value		0.009	0.005	0.002	0.001	0.000
x2			-1.36	-1.36	-1.36	-1.36
T-Value			-2.22	-2.59	-2.98	-3.27
P-Value			0.042	0.021	0.011	0.007
x12				-1.75	-1.75	-1.75
T-Value				-2.51	-2.89	-3.18
P-Value				0.025	0.013	0.008
x1111					-0.35	-0.35
T-Value					-2.35	-2.59
P-Value					0.035	0.024
x13						1.06
T-Value						1.93
P-Value						0.078
S	3.08	2.56	2.29	1.97	1.71	1.55
R-Sq	29.69	54.52	65.79	76.43	83.47	87.38
R-Sq(adj)	25.55	48.83	58.95	69.70	77.12	81.08
PRESS	339.888	147.012	125.427	100.363	97.2956	86.0578
R-Sq(pred)	0.00	35.99	45.39	56.30	57.64	62.53

Table B- 4 Results of Stepwise Regression for $\hat{\sigma}_{ML}(\mathbf{x})$

Stepwise Regression: sigHat_ML versus x1, x2, ...

Alpha-to-Enter: 0.15 Alpha-to-Remove: 0.15

Response is sigHat_ML on 80 predictors, with N = 19

Step	1	2	3	4	5
Constant	9.210	8.213	8.213	8.213	9.121
x3333	0.52	0.61	0.61	0.61	0.53
T-Value	2.38	2.95	3.21	3.38	2.95
P-Value	0.030	0.009	0.006	0.005	0.011
x11		1.17	1.17	1.17	0.98
T-Value		1.92	2.09	2.20	1.92
P-Value		0.073	0.054	0.045	0.078
x112			-1.52	-1.52	-1.52
T-Value			-2.00	-2.10	-2.24
P-Value			0.064	0.054	0.043
x23				-1.17	-1.17
T-Value				-1.61	-1.72
P-Value				0.129	0.109
x22					-0.88
T-Value					-1.71
P-Value					0.111
S	2.52	2.34	2.15	2.04	1.92
R-Sq	24.92	39.00	51.81	59.36	66.83
R-Sq(adj)	20.50	31.38	42.17	47.75	54.07
PRESS	134.024	120.095	106.392	94.9184	90.9860
R-Sq(pred)	6.92	16.60	26.11	34.08	36.81

Appendix C

Minitab Outputs for Regression Analyses

Table C- 1 Second Order Regression Output for $\hat{\mu}_{ML}(\mathbf{x})$

Regression Analysis: muHat_ML versus x1, x2, ...

```
The regression equation is
muHat_ML = 499 + 0.486 x1 - 1.36 x2 + 1.58 x3 - 1.35 x11 - 0.425 x22 + 1.59 x33
           - 1.75 x12 + 1.06 x13 + 0.515 x23


Predictor     Coef  SE Coef       T      P    VIF
Constant   499.394    1.040  480.22  0.000
x1          0.4862   0.6215    0.78  0.454  1.000
x2         -1.3601   0.6215   -2.19  0.056  1.000
x3          1.5767   0.6215    2.54  0.032  1.000
x11        -1.3541   0.6028   -2.25  0.051  1.054
x22        -0.4247   0.6028   -0.70  0.499  1.054
x33         1.5863   0.6028    2.63  0.027  1.054
x12        -1.7475   0.8221   -2.13  0.062  1.000
x13         1.0598   0.8221    1.29  0.230  1.000
x23         0.5153   0.8221    0.63  0.546  1.000


S = 2.32533   R-Sq = 78.8%   R-Sq(adj) = 57.6%

PRESS = 242.394   R-Sq(pred) = 0.00%


Analysis of Variance

Source          DF       SS      MS     F      P
Regression       9  181.017  20.113  3.72  0.032
Residual Error   9   48.665   5.407
Total           18  229.681
```

Table C- 2 Second Order Regression Output for $\hat{\sigma}_{ML}(\mathbf{x})$

Regression Analysis: sigHat_ML versus x1, x2, ...

```
The regression equation is
sigHat_ML = 9.11 - 0.041 x1 - 0.683 x2 + 0.416 x3 + 0.782 x11 - 1.07 x22
            + 1.39 x33 + 0.369 x12 + 0.831 x13 - 1.17 x23


Predictor      Coef  SE Coef      T      P    VIF
Constant      9.109    1.095   8.32  0.000
x1          -0.0406   0.6541  -0.06  0.952  1.000
x2          -0.6829   0.6541  -1.04  0.324  1.000
x3           0.4163   0.6541   0.64  0.540  1.000
x11          0.7817   0.6345   1.23  0.249  1.054
x22         -1.0733   0.6345  -1.69  0.125  1.054
x33          1.3873   0.6345   2.19  0.057  1.054
x12          0.3690   0.8654   0.43  0.680  1.000
x13          0.8314   0.8654   0.96  0.362  1.000
x23         -1.1660   0.8654  -1.35  0.211  1.000


S = 2.44763   R-Sq = 62.6%   R-Sq(adj) = 25.1%

PRESS = 214.120   R-Sq(pred) = 0.00%


Analysis of Variance

Source          DF       SS      MS     F      P
Regression       9   90.077  10.009  1.67  0.228
Residual Error   9   53.918   5.991
Total           18  143.995
```

Table C- 3 Second Order Regression Output for $\hat{\mu}(\mathbf{x})$

Regression Analysis: muHat versus x1, x2, ...

```
The regression equation is
muHat = 499 + 0.365 x1 - 1.08 x2 + 0.858 x3 - 1.06 x11 - 0.173 x22 + 0.771 x33
        - 1.43 x12 + 0.843 x13 + 0.624 x23


Predictor      Coef  SE Coef       T      P    VIF
Constant   498.665    0.581  858.32  0.000
x1          0.3654   0.3472    1.05  0.320  1.000
x2         -1.0837   0.3472   -3.12  0.012  1.000
x3          0.8585   0.3472    2.47  0.035  1.000
x11        -1.0636   0.3367   -3.16  0.012  1.054
x22        -0.1735   0.3367   -0.52  0.619  1.054
x33         0.7710   0.3367    2.29  0.048  1.054
x12        -1.4276   0.4593   -3.11  0.013  1.000
x13         0.8428   0.4593    1.83  0.100  1.000
x23         0.6242   0.4593    1.36  0.207  1.000


S = 1.29910   R-Sq = 84.8%   R-Sq(adj) = 69.6%

PRESS = 56.9764   R-Sq(pred) = 43.04%


Analysis of Variance

Source          DF       SS     MS     F      P
Regression       9   84.841  9.427  5.59  0.009
Residual Error   9   15.189  1.688
Total           18  100.029
```

Table C- 4 Second Order Regression Output for $\hat{\sigma}(\mathbf{x})$

Regression Analysis: sigHat versus x1, x2, ...

```
The regression equation is
sigHat = 7.68 - 0.167 x1 - 0.272 x2 - 0.267 x3 + 1.07 x11 - 0.752 x22
         + 0.516 x33 + 0.781 x12 + 0.487 x13 - 1.01 x23


Predictor      Coef  SE Coef      T      P    VIF
Constant     7.6839   0.7477  10.28  0.000
x1          -0.1667   0.4468  -0.37  0.718  1.000
x2          -0.2717   0.4468  -0.61  0.558  1.000
x3          -0.2671   0.4468  -0.60  0.565  1.000
x11          1.0707   0.4334   2.47  0.036  1.054
x22         -0.7521   0.4334  -1.74  0.117  1.054
x33          0.5158   0.4334   1.19  0.264  1.054
x12          0.7812   0.5911   1.32  0.219  1.000
x13          0.4867   0.5911   0.82  0.432  1.000
x23         -1.0124   0.5911  -1.71  0.121  1.000


S = 1.67184   R-Sq = 67.1%   R-Sq(adj) = 34.3%

PRESS = 106.034   R-Sq(pred) = 0.00%


Analysis of Variance

Source          DF      SS     MS     F      P
Regression       9  51.395  5.711  2.04  0.151
Residual Error   9  25.156  2.795
Total           18  76.550
```

Table C- 5 Higher Order Regression output for $\hat{\mu}(\mathbf{x})$

Regression Analysis: muHat versus x2, x11, x12, x13, x23, x333, x3333

```
The regression equation is
muHat = 499 - 1.08 x2 - 0.949 x11 - 1.43 x12 + 0.843 x13 + 0.624 x23
        + 0.488 x333 + 0.285 x3333


Predictor     Coef  SE Coef        T      P    VIF
Constant   498.630    0.364  1369.61  0.000
x2         -1.0837   0.2862    -3.79  0.003  1.000
x11        -0.9487   0.2788    -3.40  0.006  1.063
x12        -1.4276   0.3786    -3.77  0.003  1.000
x13         0.8428   0.3786     2.23  0.048  1.000
x23         0.6242   0.3786     1.65  0.127  1.000
x333        0.4877   0.1360     3.59  0.004  1.000
x3333      0.28545  0.09521     3.00  0.012  1.063


S = 1.07094   R-Sq = 87.4%   R-Sq(adj) = 79.4%

PRESS = 26.6706   R-Sq(pred) = 73.34%


Analysis of Variance

Source          DF       SS      MS      F      P
Regression       7   87.413  12.488  10.89  0.000
Residual Error  11   12.616   1.147
Total           18  100.029
```

$$\hat{\mu}(\mathbf{x}) = 498.630 - 1.084x_2 - 0.949x_1^2 - 1.428x_1x_2 + 0.843x_1x_3 + 0.624x_2x_3 + 0.488x_3^3 + 0.285x_3^4$$

Table C- 6 Higher Order Regression output for $\hat{\sigma}(\mathbf{x})$

Regression Analysis: sigHat versus x11, x22, x12, x23, x112

The regression equation is
sigHat = 8.20 + 0.981 x11 - 0.841 x22 + 0.781 x12 - 1.01 x23 - 0.938 x112

Predictor	Coef	SE Coef	T	P	VIF
Constant	8.1957	0.5213	15.72	0.000	
x11	0.9813	0.3638	2.70	0.018	1.022
x22	-0.8415	0.3638	-2.31	0.038	1.022
x12	0.7812	0.5038	1.55	0.145	1.000
x23	-1.0124	0.5038	-2.01	0.066	1.000
x112	-0.9379	0.5038	-1.86	0.085	1.000

S = 1.42486   R-Sq = 65.5%   R-Sq(adj) = 52.3%

PRESS = 44.5075   R-Sq(pred) = 41.86%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	5	50.157	10.031	4.94	0.009
Residual Error	13	26.393	2.030		
Total	18	76.550			

$$\hat{\sigma}(\mathbf{x}) = 8.196 + 0.981x_1^2 - 0.842x_2^2 + 0.781x_1x_2 - 1.012x_2x_3 - 0.938x_1^2x_2$$

Table C- 7 Order Regression output for $\hat{\mu}_{ML}(\mathbf{x})$

```
Regression Analysis: muHat_ML versus x2, x12, ...

The regression equation is
muHat_ML = 499 - 1.36 x2 - 1.75 x12 + 1.06 x13 - 0.353 x1111 + 0.340 x33333
           + 0.211 x333333


Predictor      Coef   SE Coef        T      P    VIF
Constant   499.048     0.439  1135.60  0.000
x2         -1.3601    0.4153    -3.27  0.007  1.000
x12        -1.7475    0.5494    -3.18  0.008  1.000
x13         1.0598    0.5494     1.93  0.078  1.000
x1111      -0.3530    0.1364    -2.59  0.024  1.036
x33333     0.33974   0.06991     4.86  0.000  1.000
x333333    0.21079   0.04449     4.74  0.000  1.036


S = 1.55394   R-Sq = 87.4%   R-Sq(adj) = 81.1%

PRESS = 86.0578   R-Sq(pred) = 62.53%


Analysis of Variance

Source          DF       SS      MS      F      P
Regression       6  200.705  33.451  13.85  0.000
Residual Error  12   28.977   2.415
Total           18  229.681
```

$$\hat{\mu}_{ML}(\mathbf{x}) = 499.048 - 1.360x_2 - 1.748x_1x_2 + 1.060x_1x_3 - 0.353x_1^4 + 0.340x_3^5 + 0.211x_3^6$$

Table C- 8 Higher Order Regression output for $\hat{\sigma}_{ML}(\mathbf{x})$

Regression Analysis: sigHat_ML versus x11, x22, x23, x112, x3333

```
The regression equation is
sigHat_ML = 9.12 + 0.980 x11 - 0.875 x22 - 1.17 x23 - 1.52 x112 + 0.525 x3333


Predictor      Coef  SE Coef      T      P    VIF
Constant     9.1207   0.8403  10.85  0.000
x11          0.9798   0.5114   1.92  0.078  1.117
x22         -0.8751   0.5114  -1.71  0.111  1.117
x23         -1.1660   0.6777  -1.72  0.109  1.000
x112        -1.5181   0.6777  -2.24  0.043  1.000
x3333        0.5253   0.1781   2.95  0.011  1.161


S = 1.91681   R-Sq = 66.8%   R-Sq(adj) = 54.1%

PRESS = 90.9860   R-Sq(pred) = 36.81%


Analysis of Variance

Source          DF       SS      MS     F      P
Regression       5   96.231  19.246  5.24  0.007
Residual Error  13   47.764   3.674
Total           18  143.995
```

$$\hat{\sigma}_{ML}(\mathbf{x}) = 9.121 + 0.980x_1^2 - 0.875x_2^2 - 1.166x_2x_3 - 1.518x_1^2x_2 + 0.525x_3^4$$

Appendix D

Computer Programs

D1. A Program for Solving the MSE Optimization Models and the Models by Vining and Myers.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This code solves the MSE optimization model by Lin and Tu, and the %%
% N-Type Optimization problems by Vining and Myers.                   %%
% LTOrig is the MSE model built with the RSMs of Lin and Tu           %%
% LT1 is the MSE model built the improved RSMs                        %%
% ObjVMmse is the MSE model built with the RSMs of Vining and Myers  %%
% stdevLT is the std dev model by Lin and Tu                          %%
% stdevSC is the std dev model of this work
% stdevVM is the std dev model of Vining and Myers
% AnMod is a matrix of all the response surface models
% objFunc is a matrix of all objective functions.
% sphConstrLT is the constraint to the N-Type problem based on RSMs by
% Lin and Tu
% sphConstrVM is the constraint to the N-Type problem based on RSMs by
% Vining and Myers
% sphConst is the constraint to the N-Type problem based on RSMs of %
% this work
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
close all
global T c S;
c=1;
T=500; % Target Value for Mean
S=10; % Target Value for Standard Deviation
x0=[1 1 1]; %% Initilizing for Numerical Solution

% Optimization in Cuboidal Region
% solving mse with models of LT
[X1 V1]=fmincon('LTOrig',x0,[],[],[],[],[-1 -1 -1],[1 1 1],[]);
% solving mse with our models
[X2 V2]=fmincon('LT1',x0,[],[],[],[],[-1 -1 -1],[1 1 1],[]);
% solving mse with models of VM
[X3 V3]=fmincon('objVMmse',x0,[],[],[],[],[-1 -1 -1],[1 1 1],[]);

% Solving the VM problem - minimizing sigma with mu on target %%%
[X4 V4]=fmincon('stdevLT',x0,[],[],[],[],[-1 -1 -1],[1 1
1],'sphConstrLT');
[X5 V5]=fmincon('stdevSC',x0,[],[],[],[],[-1 -1 -1],[1 1
1],'sphConstr');
[X6 V6]=fmincon('stdevVM',x0,[],[],[],[],[-1 -1 -1],[1 1
1],'sphConstrVM');

% Evaluating Optimal MSEs
R1=AnMod(X1); MS1=objFunc(X1);MSE1=MS1(1,1);
R2=AnMod(X2); MS2=objFunc(X2);MSE2=MS2(1,5);
R3=AnMod(X3); MS3=objFunc(X3);MSE3=MS3(3,1);
R4=AnMod(X4);MS4=objFunc(X4);MSE4=MS4(1,1);
R5=AnMod(X5);MS5=objFunc(X5);MSE5=MS5(1,5);
```

D1continued

```
R6=AnMod(X6);MS6=objFunc(X6);MSE6=MS6(3,1);
% Displaying Results
disp('Results for MSE - Cuboidal Region')
disp([X1 R1(1,3) (R1(2,3)) MSE1;X2 R2(1,1) R2(3,1) MSE2;X3 R3(1,4)
R3(2,4) MSE3]);
disp(' ')
disp('Results for VM N-Type Problem - Cuboidal Region')
disp([X4 R4(1,3) (R4(2,3)) MSE4;X5 R5(1,1) R5(3,1) MSE5;X6 R6(1,4)
R6(2,4) MSE6]);
```

D1.1 A Program for AnMod

```
function y=AnMod(x)
x1=x(1);
x2=x(2);
x3=x(3);
x11=x1^2;
x22=x2^2;
x33=x3^2;
x12=x1*x2;
x13=x1*x3;
x23=x2*x3;
x113=x11*x3;
x112=x11*x2;
x122=x1*x22;
x133=x1*x33;
x223=x22*x3;
x233=x2*x3^2;
x123=x1*x2*x3;
x1122=x112*x2;
x1133=x113*x3;
x2233=x223*x3;
x1123=x112*x3;
x1223=x122*x3;
x1233=x123*x3;
x11223=x1122*x3;
x112233=x1122*x33;
%%%%%%% MEANS %%%%%%%%%%%%%%%%%%%%%
y(1,1)=339 + 177 *x1 + 147 *x2 + 116 *x3 - 3.7 *x11 - 58.1 *x22 - 10.5
*x33 + 47.7 *x12 + 55.0 *x13 + 43.6 *x23 - 56.4 *x233 + 82.8 *x123 +
80.5 *x1122 + 30.7 *x1223 + 27.5 *x1233 - 41.3 *x112233 + 35.4
*x11223;%% 98.9 M1
y(1,2) = 344 + 177 *x1 + 147 *x2 + 131 *x3 - 70.1 *x22 + 47.7 *x12 +
55.0 *x13 + 43.6 *x23 - 56.4 *x233 + 82.8 *x123 + 70.9 *x1122 - 32.4
*x1133 + 30.7 *x1223 + 27.5 *x1233;%98.3 M2
y(1,3) = 315 + 177 *x1 + 109 *x2 + 131 *x3 + 66.0 *x12 + 75.5 *x13 +
43.6 *x23 + 82.8 *x123;% Lin and Tu 95.7
y(1,4) = 328 + 177 *x1 + 109 *x2 + 131 *x3 + 66.0 *x12 + 75.5 *x13 +
43.6 *x23 + 31.6 *x11 - 22.8 *x22 - 28.9 *x33; % VM 92.7
%%%%%%%%%%% STANDARD DEVIATIONS %%%%%%%%%%%%%%%%%%%%%%%%%
```

D1.1 Continued

```
y(2,1) = 36.8 + 11.5 *x1 + 30.4 *x2 + 29.0 *x3 + 16.9 *x33 - 18.8 *x12
+ 29.0 *x13 + 29.8 *x23 - 22.7 *x112 + 29.6 *x123 - 23.6 *x1123 - 35.9
*x1223 + 39.8 *x1233; % 75.2 S1
y(2,2) = 48.1 + 15.3 *x2 + 29.0 *x3 + 29.6 *x123 + 21.0 *x1233;%49.99
S2
y(2,3) = 48.1 + 11.5 *x1 + 15.3 *x2 + 29.0 *x3 + 29.6 *x123; % LT 48%
y(2,4) = 35.3 + 11.5 *x1 + 15.3 *x2 + 29.0 *x3 + 3.9 *x11 - 1.6 *x22 +
16.9 *x33 + 5.1 *x13 + 7.7 *x12 + 14.1 *x23; % VM 45.0%
%%%%%%%%%%%%%%
y(3,1)=36.8 + 36.5 *x1 + 35.6 *x2 - 19.3 *x3 + 16.9 *x33 - 18.8 *x12 +
29.0 *x13 + 29.8 *x23 - 22.7 *x112 + 61.3 *x113 - 37.4 *x122 + 56.6
*x223 - 7.67 *x233 + 29.6 *x123 - 23.6 *x1123 - 35.9 *x1223 + 39.8
*x1233 - 68.1 *x11223; %%94.3 S3
y(3,2)= 48.1 + 36.5 *x1 + 15.3 *x2 + 34.3 *x113 - 37.4 *x122 + 29.6
*x123 + 21.0 *x1233;%%61.6 S4
```

D1.2 A Program for objFunc

```
%% FILE OF OBJECTIVE FUNCTIONS
function y=objFunc(x)
global T S;
% T=500; S=10;
F=AnMod(x);
M1=F(1,1);
LTm=F(1,3);
VMm=F(1,4);
S3=F(3,1);
LTs=F(2,3);
VMs=F(2,4);
yLT=(LTs)^2 + (LTm - T)^2;%% LT's MSE Objective Function

%%%%%% OUR OWN MODELS FOR N-TYPE PROBLEMS%%%%%%%%%%%%%
y1=S3*(M1-T)^2;
y2=((S3)^2)*(M1-T)^2;
y3=(exp(S3))*(M1-T)^2;
y4=(S3)^2 + (M1-T)^2 ;%% MSE Based on our Models of mean and std
%%%%%% OUR OWN MODELS FOR L-TYPE PROBLEMS%%%%%%%%%%%%%
y5 = S3 - M1;
y6=S3/M1;
y7=exp(S3)/M1;
y8=VMs^2 + (VMm-T)^2; %% MSE Based on models of VM
y=[yLT y1 y2 y3 y4;y5 y6 y7 M1 S3;y8 0 0 0 0];
```

D1.3 A Program for LTOrig

```
function y=LTOrig(x)
F=objFunc(x);
y=F(1,1);
```

D1.4 A Program for LT1

```
function y=LT1(x)
F=objFunc(x);
y=F(1,5);
```

D1.5 A Program for ObjVMmse

```
function y=objVMmse(x)
F=objFunc(x);
y=F(3,1);
```

D1.6 A Program for SphConsrLT

```
%% Positivity constraint for standard deviation
function [Ceq C]=sphConstrLT(x)
global c;
global S T;
M=AnMod(x);
Ceq=(M(2,3)-S);
C=[];
```

D1.7 A Program for SphConsrVM

```
%% Positivity constraint for standard deviation
function [Ceq C]=sphContrVM(x)
global T S c;
M=AnMod(x);
Ceq=(M(2,4)-S);
C=[];
```

D2. Functions Called in D1

```
%% FILE OF OBJECTIVE FUNCTIONS
function y=objFunc(x)
global T S;
F=AnMod(x);
M1=F(1,1);
LTm=F(1,3);
VMm=F(1,4);
S3=F(3,1);
LTs=F(2,3);
VMs=F(2,4);
yLT=(LTs)^2 + (LTm - T)^2;%% LT's MSE Objective Function

%%%%%% OUR OWN MODELS FOR N-TYPE PROBLEMS%%%%%%%%%%%%%
y1=S3*(M1-T)^2;
y2=((S3)^2)*(M1-T)^2;
y3=(exp(S3))*(M1-T)^2;
y4=(S3)^2 + (M1-T)^2 ;%% MSE Based on our Models of mean and std
%%%%%% OUR OWN MODELS FOR L-TYPE PROBLEMS%%%%%%%%%%%%%
y5 = S3 - M1;
y6=S3/M1;
y7=exp(S3)/M1;
y8=VMs^2 + (VMm-T)^2; %% MSE Based on models of VM
y=[yLT y1 y2 y3 y4;y5 y6 y7 M1 S3;y8 0 0 0 0];
```

D3. A Code for Tolerance Optimization for L-Type Quality Characteristics

```
close all
clear all
global k L ind objf
warning off all
% ind = 1 indicates proposed estimation method;
% ind = 2 indicates method of MLE
ind=1;

% objf = i indicates ith objective function
objf=1;

L=50;
k=6;%norminv(1-del);
% x0=[1 1 1 300]; %% Initilization
n=1000; %n1=20;

   RR=[];
   for i=1:n
      x0=unifrnd(-sqrt(3), sqrt(3), 1, 3);
      [X V]=fmincon('tolF1L',x0,[],[],[],[],sqrt(3)*[-1 -1 -1],sqrt(3)*[1 1 1],'DissTolConstrL');
      R=DissRSModels(X(1:3));
      m=R(1,ind);
      s=R(2,ind);
      if (s > 0)
         RR=[RR;X m s m-k*s V];
      end

   end
RR=sortrows(RR,-7);
RR=sortrows(RR,-7);
disp(RR(length(RR),1:7))
disp(RR(length(RR),5))
```

D3.1 A Code for the Objective Functions Used in the Tolerance Optimization

```
function y=tolF1L(x)
global k L objf ind
x1=x(1); x2=x(2); x3=x(3);%lam=x(4);

R=DissRSModels([x1 x2 x3]);

% Mean and Std Dev: ind=1 for EM, and ind=2 for MLE
```

D3.1 Continued

```
m=R(1,ind);
s=R(2,ind);

% Objective for work with tolerance, L given:
% maximizing the diff btn L and m with m>=(L+k*s)
% if objf==1
%     y=s/(m) - ((L+k*s)- m)^2;
% elseif objf==2
%     y=(s + 1/m)-((L+k*s)- m)^2;
% elseif objf==3
%     y=(1/m^2)*(1+3*s^2/m^2)-((L+k*s)- m)^2;
% end

% Objective for work with tolerance, L given:
% minimizing the diff btn (L+k*s) and m with m>=(L+k*s)
if objf==1
    y=((L+k*s)/m
elseif objf==2
    y=(s + 1/m)+((L+k*s)- m)^2;
elseif objf==3
    y=(1/m^2)*(1+3*s^2/m^2)+((L+k*s)- m)^2;
end
```

D3.2 A Code for the Expected Type I Exponential Loss

```
function y=TypeIEL(x)
global L cL cmaxL x0 c ind
x1=x(1); x2=x(2); x3=x(3); L=x(4);%k=x(5);

% R=DissRSModels([x1 x2 x3]);
R=DissRSModelsImp([x1 x2 x3]);

% Mean and Std Dev: ind=1 for EM, and ind=2 for MLE
m=R(1,ind);
s=R(2,ind);

k1=cmaxL;
k2=-(1/(x0 - L))*log(c/k1);
E1=cL*normcdf((L-m)./s);
E2=k1*exp(0.5*k2*(k2*s^2+2*(L-m)));
E3=1-normcdf((L-(m-k2*s^2))./s);
y=E1 + E2*E3;
```

D3.3 A Code for the Expected Type II Exponential Loss

```
function y=TypeIIEL(x)
global L cL cmaxL x0 c ind
x1=x(1); x2=x(2); x3=x(3);%L=x(4);%k=x(5);

% R=DissRSModels([x1 x2 x3]);
R=DissRSModelsImp([x1 x2 x3]);

% Mean and Std Dev: ind=1 for EM, and ind=2 for MLE
m=R(1,ind);
s=R(2,ind);
% m=x(1);
% s=x(2);
k1=cmaxL;
k2=-(1/(x0 - L)^2)*log(c/k1);
den=2*k2*s^2+1;
den1=s*sqrt(den);
E0=cL*normcdf((L-m)/s);
E1=k1/sqrt(den);
E2=exp(-k2*((m-L)^2)/den);
E3=1-normcdf((L-m)/den1);
y=E0+E1*E2*E3;
```

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