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Formulation of Integral Equations Using Analytic Green’s Functions in Elliptic Cylindrical and Prolate Spheroidal Coordinates

Adam Schreiber
Clemson University, sadam@clemson.edu

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FORMULATION OF INTEGRAL EQUATIONS USING
ANALYTIC GREEN’S FUNCTIONS IN ELLIPTIC
CYLINDRICAL AND PROLATE SPHEROIDAL
COORDINATES

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Electrical Engineering

by
Adam W. Schreiber
December 2007

Accepted by:
Dr. Chalmers M. Butler, Committee Chair
Dr. L. Wilson Pearson
Dr. Xiao-Bang Xu
ABSTRACT

It is desired to develop and present techniques, using analytic Green’s functions in elliptic cylindrical coordinates and in prolate spheroidal coordinates, for formulating integral equations for physically practicable structures and sources. Equivalent models for two structures are introduced which serve as guides in the derivation of analytic Green’s functions, comprising special functions and satisfying required boundary conditions. Expressions for needed field components are represented in terms of integrals of Green’s functions times unknowns, and integral equations follow from the proper relationships among sources and field components needed to satisfy Maxwell’s equations.
DEDICATION

To my parents and educators who are each responsible for one third of my success.
ACKNOWLEDGMENTS

I gratefully acknowledge Clemson University’s Center for Research in Wireless Communication for financial support in performing this research.

My sincerest gratitude goes to Dr. Danilo Erricolo of the University of Illinois at Chicago for his kind assistance with Mathieu functions and in formulating an analytic Green’s function in elliptic cylindrical coordinates.
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CHAPTER I
INTRODUCTION

The goal of the study reported in this thesis is to develop and present techniques, using analytic Green’s functions in elliptic cylindrical coordinates and in prolate spheroidal coordinates, for formulating integral equations for physically practicable structures and sources. One structure selected for this purpose is a slotted conducting plane backed by a semi-elliptic channel containing a dielectric or conducting cylinder of general cross-section, excited by a wave that is invariant along and transverse electric to the slot axis. The other is a prolate spheroidal monopole driven by a coaxial waveguide with a TEM field in the coax incident upon the coax-monopole interface. In the case of the semi-elliptic channel, it is desired to develop a solution for the electromagnetic field in which the location and number of slots and the choice of materials is not constrained by the solution method [1, 2, 3, 4, 5]. In the case of the prolate spheroidal monopole, it is desired to develop a solution technique which is not dependent on an approximation for the driving source [6, 7, 8]. To this end, equivalent models are introduced which serve as guides in the derivation of analytic Greens functions, comprising special functions and satisfying required boundary conditions. Expressions for needed field components are represented in terms of integrals of Greens functions times unknowns, and integral equations follow from the proper relationships among sources and field components needed to satisfy Maxwell’s equations.
Equivalent models, field equations and integral equations concerning the slot-ted conducting plane with a semi-elliptic backing channel containing a cylinder are developed in Chapter II. Equivalent models, field equations and integral equations concerning the prolate spheroidal monopole are developed in Chapter III. Chapter IV builds on the models and field equations presented in Chapter III but the integral equation devised makes use of an alternate Green’s function in prolate spheroidal coordinates. The appendices contain derivations of the required Green’s functions and the prolate spheroidal angular and radial functions.
CHAPTER II
COUPLING THROUGH A SLOTTED SCREEN TO A CYLINDER IN A SEMI-ELLiptIC BACKING CHANNEL

Determining the electromagnetic field due to a general TE source in the presence of a slotted perfectly conducting screen backed by a semi-elliptical backing channel with a general cylinder in the channel is a pseudo canonical problem in electromagnetics due to its conformance to constant surfaces in elliptic cylindrical coordinates. The conformance leads to simplified analysis.

Analysis pertaining to the upper half-space functions is performed in circular cylindrical coordinates and involves the use of the familiar Hankel function of the second kind, to meet the radiation condition. However, the analysis in the semi-elliptic channel requires an understanding of elliptic cylindrical coordinates and Mathieu functions. Elliptic cylindrical coordinates is one of the eleven curvilinear coordinate systems in which the homogeneous scalar Helmholtz equation is separable. The three orthogonal surfaces in elliptic cylindrical coordinates, as depicted in Figure II.1, are elliptic cylinders, hyperbolic cylinders (both branches) and planes that correspond to surfaces of constant $u, v$ and $z$, respectively. The elliptic and hyperbolic cylinders have foci located at $\pm d/2$ along the $x$-axis. The products of angular and radial Mathieu functions are solutions of the $z$-independent homogeneous scalar Helmholtz equation [9, 10] (Appendix B).

An extensive survey of apertures in conducting surfaces is presented in [11]. Ad-
Figure II.1: Elliptic cylindrical coordinates
ditional work on narrow slots can be found in [12]. Elliptic cavities are used in the creation of dual-mode narrow-band filters [13]. Elliptical waveguides are studied in [14, 15, 16, 17]. Scattering from general cylinders in free space is explored in [18, 19]. Scattering from elliptic cylinders is explored in [20, 21, 22]. Extensive study of the geometry in question has been performed. Investigation of an empty backing channel with isorefractive material in the channel occurs in [1, 2, 3]. The authors of [4, 5] examine the coupling through a slotted conducting screen to a cylinder in the channel with isorefractive materials. Examination of an empty channel, without the restriction to isorefractive materials, was presented in [23]. It is desired to build on the method presented in [23] to develop coupled integral equations for the tangential-to-the-ground plane electric field in the slot and the field ($E$ or $H_z$) tangential to the surface of the cylinder in the channel.

II.1 Structure and source

The structure of interest is depicted in cross-section in Figure II.2. It consists of an infinite PEC screen with a uniform-width slot which is backed by a semi-elliptic channel with a PEC wall. A dielectric or conducting cylinder of general cross-section is contained in the channel. The screen resides in the $xz$-plane and the axes of the slot, the cylinder and the channel are parallel. The structure is separated into three regions: region a, the upper half-space, region b, the region in the semi-elliptic channel, and region c, the region inside the cylinder. The materials in the regions are characterized by $(\mu_a, \epsilon_a), (\mu_b, \epsilon_b)$ and $(\mu_c, \epsilon_c)$. Alternatively, the cylinder could
Figure II.2: Slotted screen with cylinder in semi-elliptic backing channel with source be a perfect conductor. The slot from $x = a$ to $x = b$ is shown off center and is not constrained in size or location above the backing channel. In fact, the integral equation derived herein is easily adaptable to the case of multiple uniform width slots in the conducting screen. A source in the upper half-space causes an excitation which is invariant with respect to displacement along the slot axis and whose electric field is entirely transverse to the slot axis.

II.2 Equivalent model of structure and source

In Figure II.3 one finds an illustration of the original structure and source with the $x$ component of the total electric field in the slot denoted $E^A_x(x)$ to emphasize that this is the electric field in the plane of the screen in the slot. Figure II.4 depicts a model electromagnetically equivalent to the original structure and source. The equivalent model is arrived at by shorting the slot and placing an equivalent surface magnetic
current \[ [E_A^x(x)\hat{x}] \times \hat{y} = E_A^x(x)\hat{z} = M_z(x)\hat{z} \] over the upper side of the short and \[ [E_A^x(x)\hat{x}] \times -\hat{y} = -E_A^x(x)\hat{z} = -M_z(x)\hat{z} \] over the underside of the short. These equal but oppositely directed surface currents cause the tangential-to-the-screen component of the electric field to approach \( E_A^x(x)\hat{x}, \) for \( x \in (a, b), \) in the limit as the point of observation approaches the \( xz \)-plane from either side.

II.3 General description of integral equations and governing principles

The coupled integral equations are derived under the constraints that the tangential electric field is zero on all conducting surfaces and that the tangential electric and magnetic fields are continuous through slots and across any dielectric-to-dielectric interface. The magnetic field in region a is \( H_z^a(\rho) \) and is the sum of the short circuit magnetic field \( H_z^{sc}(\rho) \) and the contribution from the tangential slot electric field \( E_A^x. \) The magnetic field in region b outside the cylinder is \( H_z^b(\rho) \) and is made up of con-
tributions from $E_x^a$ and the tangential field ($E$ or $H_z$) at the surface of the cylinder. The magnetic field in region c is $H_z^c(\rho)$ and has only a contribution from the field ($E$ or $H_z$) at the surface of the cylinder. These field components are expressed in such a way that the corresponding tangential-to-conducting-surface electric field components are zero at these surfaces. The final integral equations enforcement of the above conditions is expressed as

$$\lim_{\rho \downarrow t} H_a^z(\rho) = \lim_{\rho \uparrow t} H_b^z(\rho)$$

(II.1)

in which $\rho$ approaches the contour of the slot(s) $t$ in the limit from above the contour in region a and from below the contour in region b and

$$\lim_{\rho \downarrow \hat{C}} H_b^z(\rho) = \lim_{\rho \uparrow \hat{C}} H_c^z(\rho)$$

(II.2)

$$\lim_{\rho \downarrow \hat{C}} \hat{l} \cdot E_b^b(\rho) = \lim_{\rho \uparrow \hat{C}} \hat{l} \cdot E_c^c(\rho)$$

(II.3)
in which $\rho$ approaches the contour of the cylinder $C$ in the limit from outside the contour in region b and from inside the contour in region c and $\hat{l}$ is the unit vector tangential to $C$ at $\rho$.

II.4 Region a equivalent model

A simple equivalent model for region a is achieved by applying image theory to region a of the model in Figure II.4. The equivalent magnetic current and source are mirrored in the lower half-space and the entire region is filled with material characterized by $(\mu_a, \epsilon_a)$. This new model is depicted in Figure II.5

II.5 Field in region a

A single unity-strength, $z$-directed magnetic line current located at $x = x'$ on the $xz$-plane in free space characterized by $(\mu_a, \epsilon_a)$ radiates a $z$-directed magnetic field of value

$$-\frac{k_a}{4\eta_a} H_0^{(2)}(k_a|\rho - x'|\hat{x}|)$$

(II.4)
in which \( k_a = \omega \sqrt{\mu_a / \varepsilon_a} \) and \( \eta_a = \sqrt{\mu_a / \varepsilon_a} \). Therefore, the magnetic current \( 2M_z(x)\hat{z} \) on the \( xz \)-plane in Figure II.5 radiates a \( z \)-directed magnetic field given by

\[
-\int_a^b M_z(x') \frac{k_a}{2\eta_a} H_0^{(2)}(k_a |\rho - x'\hat{x}|) dx'
\] (II.5)

Thus, the total magnetic field in region a is seen to be

\[
H^a_z(\rho) = H^{sc}_z(\rho) - \int_a^b M_z(x') \frac{k_a}{2\eta_a} H_0^{(2)}(k_a |\rho - x'\hat{x}|) dx'
\] (II.6)

in which \( H^{sc}_z \) is the short-circuit magnetic field due to the original source radiating in the presence of the shorted slot.

II.6 Region b equivalent model

A simpler equivalent model for region b is achieved by applying image theory to region b of the model shown in Figure II.6. The equivalent magnetic current, cylinder and channel wall are mirrored in the upper half-space. This intermediate step is shown in Figure II.7. The final equivalent model, shown in Figure II.8, is arrived at by placing equivalent magnetic surface currents \( M_b^z \) on the cylinder surface defined by contour \( C \) and \( M_{bi}^z \) on its image defined by contour \( C^i \) and replacing the material inside the cylinder and its image with \((\mu_b, \varepsilon_b)\).

II.7 Field in region b

At this stage, it is convenient to introduce a Green’s function to simplify the task of determining an expression for \( H^b_z(\rho) \). The Green’s function of interest is for the electric vector potential \( \mathbf{F} = f(\rho; \rho')\hat{z} \) due to a unity-strength \( z \)-directed line current
Figure II.6: Semi-elliptic channel with magnetic current and cylinder

Figure II.7: Elliptic cavity with magnetic current, cylinder and its image
Figure II.8: Region b equivalent model

at $\mathbf{\rho}'$ in the empty elliptic cavity of Figure II.8. The electromagnetic field in region b is determined from this Green’s function weighted by $-2M, M^b$ and $M^{bi}$. Because the wall of the cavity is a perfect conductor, the tangential component of the electric field evaluated at the wall must be zero. Hence, $f$ of the Green’s function satisfies

$$\nabla_t^2 f + k_b^2 f = -\epsilon_b \delta(\mathbf{\rho} - \mathbf{\rho}')$$  \phantom{.} (II.7)

in which $\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$ is the transverse Laplacian operator, and it must satisfy the boundary condition

$$\frac{\partial f}{\partial n} = 0 \quad \text{at cylinder wall}$$  \phantom{.} (II.8)

It is convenient to express the Green’s function in the form

$$f(\mathbf{\rho}; \mathbf{\rho}') = \frac{\epsilon_b}{4j} H_0^{(2)}(k_b|\mathbf{\rho} - \mathbf{\rho}'|) + f^h(\mathbf{\rho}; \mathbf{\rho}')$$  \phantom{.} (II.9)
in which the first term on the right is the particular solution of (II.7) and the second is the homogeneous solution that causes \( f \) to satisfy (II.8). Now it can be seen that the magnetic and electric fields caused by a unity-strength magnetic line current are

\[
h_z(\rho; \rho') = -\frac{k_b}{4\eta_b} H_0^{(2)}(k_b|\rho - \rho'|) - j\omega f^h(\rho; \rho') \tag{II.10}
\]

\[
e(\rho; \rho') = \frac{1}{\epsilon_b} \hat{z} \times \nabla_t f
\]

\[= -\frac{k_b}{4j} \hat{z} \times \hat{u} H_1^{(2)}(k_b|\rho - \rho'|) + \frac{1}{\epsilon_b} \hat{z} \times \nabla_t f^h(\rho; \rho') \tag{II.11}
\]

where \( \hat{u} \) is the unit vector in the direction of \( \rho - \rho' \) as shown in Figure [II.9]. For simplicity, define the homogeneous components of (II.10) and (II.11) as \( h_z^h(\rho; \rho') = -j\omega f^h(\rho; \rho') \) and \( e^h(\rho; \rho') = \frac{1}{\epsilon_b} \hat{z} \times \nabla_t f^h(\rho; \rho') \). The tangential electric field due to a unity-strength magnetic line current can now be written as

\[
\hat{l} \cdot e(\rho; \rho') = -\frac{k_b}{4j} \hat{l} \cdot (\hat{z} \times \hat{u}) H_1^{(2)}(k_b|\rho - \rho'|) + \hat{l} \cdot e^h(\rho; \rho') \tag{II.12}
\]

From Figure [II.9] it can be seen that \( \hat{l} \cdot (\hat{z} \times \hat{u}) = \cos \theta \), where \( \theta \) is the angle between \( \hat{u} \) and the normal vector at \( \rho \). In terms of \( h_z^h \), the magnetic field in the cavity is

\[
H_z^b(\rho) = \int_a^b M_z(x') \left\{ \frac{k_b}{2\eta_b} H_0^{(2)}(k_b|\rho - x'|) - 2h_z^h(\rho; x') \right\} dx' - \int_C M_z^b(\rho') \left\{ \frac{k_b}{4\eta_b} H_0^{(2)}(k_b|\rho - \rho'|) - h_z^h(\rho; \rho') \right\} dl' - \int_{C'} M_z^b(\rho') \left\{ \frac{k_b}{4\eta_b} H_0^{(2)}(k_b|\rho - \rho'|) - h_z^h(\rho; \rho') \right\} dl' \tag{II.13}
\]
where $C$ and $C^i$ are the contours of the cylinder and its image. In terms of $e^h$, the tangential electric field in the cavity is

$$\hat{l} \cdot E^b(\rho) = \int_{a}^{b} M_z(x') \left[ \cos \theta \frac{k_b}{2j} H_1^{(2)}(k_b|\rho - x'|) - 2\hat{l} \cdot e^h(\rho; x' \hat{x}) \right] dx'$$

$$- \int_{C} M_z^b(\rho') \left[ \frac{k_b}{4j} \cos \theta H_1^{(2)}(k_b|\rho - \rho'|) - \hat{l} \cdot e^h(\rho; \rho') \right] dl'$$

$$- \int_{C^i} M_z^{bi}(\rho') \left[ \frac{k_b}{4j} \cos \theta H_1^{(2)}(k_b|\rho - \rho'|) - \hat{l} \cdot e^h(\rho; \rho') \right] dl'$$  \hspace{1cm} (II.14)

The relation between the cylinder and its image’s location is recognized to be $\rho^i = \rho - 2y'\hat{y}$ as depicted in Figure II.10 which allows (II.13) and (II.14) to be rewritten as

$$H^b_z(\rho) = \int_{a}^{b} M_z(x') \left\{ \frac{k_b}{2\eta_b} H_0^{(2)}(k_b|\rho - x'|) - 2h_z^b(\rho; x') \right\} dx'$$

$$- \int_{C} M_z^b(\rho') \left\{ H_0^{(2)}(k_b|\rho - \rho'|) + H_0^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}) \right\} dl'$$

$$+ \int_{C} M_z^{bi}(\rho') \left\{ h_z^b(\rho; \rho') + h_z^b(\rho; \rho' - 2(\hat{y} \cdot \rho')\hat{y}) \right\} dl'$$  \hspace{1cm} (II.15)

and

$$\hat{l} \cdot E^b(\rho) = \int_{a}^{b} M_z(x') \left[ \cos \theta \frac{k_b}{2j} H_1^{(2)}(k_b|\rho - x'|) - 2\hat{l} \cdot e^h(\rho; x' \hat{x}) \right] dx'$$

$$- \int_{C} M_z^b(\rho') \left[ \frac{k_b}{4j} \cos \theta H_1^{(2)}(k_b|\rho - \rho'|) \right.$$ \hspace{1cm} \left. + \frac{k_b}{4j} \cos \theta h_1^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}) \right] dl'$$

$$+ \int_{C} M_z^{bi}(\rho') \left[ \hat{l} \cdot e^h(\rho; \rho) + \hat{l} \cdot e^h(\rho; \rho - 2(\hat{y} \cdot \rho')) \right] dl'$$  \hspace{1cm} (II.16)

where $\theta^i$ is the angle between the $\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}$ directed unit vector and the normal vector, $\hat{n}$.  

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Figure II.9: Vectors for E field integral equation

Figure II.10: Vector relationship between the cylinder and its image
II.8 Region c equivalent model

An equivalent model in region c is achieved by placing $M_c^z$ on the interior side of the surface of the cylinder. The entirety of space is filled with $(\mu_c, \epsilon_c)$ to complete the model shown in Figure II.11.

II.9 Field in region c

The expression for the magnetic field in region c requires the same Green’s function as used in region a. Making use of this fact and the derivation for the electric field in section II.7, the magnetic and tangential electric fields in region c are simply

$$H_c^z(\rho) = - \int_C M_c^z(\rho') \frac{k_c}{4\eta_c} H_0^{(2)}(k_c |\rho - \rho'|) dl'$$ (II.17)

and

$$\hat{\mathbf{l}} \cdot E^c(\rho) = - \int_C M_c^e(\rho') \frac{k_c}{4j} \cos \theta H_1^{(2)}(k_c |\rho - \rho'|) dl'$$ (II.18)
II.10 Integral equations for slot E-field and equivalent magnetic current on
dielectric cylinder

When (II.1) is enforced, the result which ensures "continuity" of the \( z \)-component of the magnetic field in the slot is

\[
\int_a^b M_z(x') \left\{ \frac{k_a}{2\eta_a} H_0^{(2)}(k_a|x - x'|) + \frac{k_b}{2\eta_b} H_0^{(2)}(k_b|x - x'|) - 2h_z^b(x\hat{x}; x'\hat{x}) \right\} dx' \\
- \int_C M_z^b(\rho') \frac{k_b}{4\eta_b} \left\{ H_0^{(2)}(k_b|x\hat{x} - \rho'|) + H_0^{(2)}(k_b|x\hat{x} - \rho' + 2(\hat{y} \cdot \rho')\hat{y}|) \right\} dl'' \\
+ \int_C M_z^b(\rho') \left\{ h_z^b(x\hat{x}; \rho') + h_z^b(x\hat{x}; \rho' - 2(\hat{y} \cdot \rho')\hat{y}) \right\} dl'' = H^e_z(\rho), \quad x \in (a,b)
\]

(II.19)

When (II.2) is enforced, the result that ensures the "continuity" of the \( z \)-component of the magnetic field at the surface of the cylinder is

\[
\int_a^b M_z(x') \left\{ \frac{k_b}{2\eta_b} H_0^{(2)}(k_b|\rho - x'|) - 2h_z^b(\rho; x') \right\} dx' \\
- \int_C M_z^b(\rho') \frac{k_b}{4\eta_b} \left\{ H_0^{(2)}(k_b|\rho - \rho'|) + H_0^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}|) \right\} dl'' \\
+ \int_C M_z^b(\rho') \left\{ h_z^b(\rho; \rho') + h_z^b(\rho; \rho' - 2(\hat{y} \cdot \rho')\hat{y}) \right\} dl'' \\
+ \int_C M_z^c(\rho') \frac{k_c}{4\eta_c} H_0^{(2)}(k_c|\rho - \rho'|) dl'' = 0, \quad \rho \in C
\]

(II.20)
When (II.3) is enforced, the result which ensures the "continuity" of the tangential-to-the-cylinder electric field at the surface of the cylinder is

\[
\int_{a}^{b} M_z(x') \left\{ \cos \theta \frac{k_b}{2j} H_1^{(2)}(k_b|\rho - x'|) - 2\hat{l} \cdot e^h(\rho; x') \right\} dx' 
- \frac{1}{2} M_z^b(\rho) - \int_{C} M_z^b(\rho') \left[ \frac{k_b}{4j} \cos \theta H_1^{(2)}(k_b|\rho - \rho'|) \right.
\left. + \frac{k_b}{4j} \cos \theta^i H_1^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}) \right] dl' 
+ \int_{C} M_z^b(\rho') \left[ \hat{l} \cdot e^h(\rho; \rho) + \hat{l} \cdot e^h(\rho; \rho - 2(\hat{y} \cdot \rho')) \right] dl'
- \frac{1}{2} M_z^c(\rho) + \int_{C} M_z^c(\rho') \cos \theta H_1^{(2)}(k_c|\rho - \rho'|) dl' = 0, \quad \rho \in C
\] (II.21)

(II.19), (II.20) and (II.21) form a coupled set of equations for the unknown electric field in the slot and the equivalent magnetic surface current on the cylinder.

II.11 Integral equations for slot E-field and equivalent magnetic current on conducting cylinder

If the cylinder is perfectly conducting, (II.2) and (II.3) would be supplanted by

\[
\lim_{\rho \to C} \hat{l} \cdot E^b(\rho) = 0, \quad \rho \in C
\] (II.22)

The integral equations that enforce (II.1) and (II.22) can now be written as

\[
\int_{a}^{b} M_z(x') \left\{ \frac{k_b}{2\eta_b} H_0^{(2)}(k_b|\rho - x'|) - 2h_z^b(\rho; x') \right\} dx' 
- \int_{C} M_z^b(\rho') \frac{k_b}{4\eta_b} \left[ H_0^{(2)}(k_b|\rho - \rho'|) + H_0^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}) \right] dl'
+ \int_{C} M_z^b(\rho') \left\{ h_z^h(\rho; \rho') + h_z^b(\rho; \rho' - 2(\hat{y} \cdot \rho')\hat{y}) \right\} dl' = 0, \quad \rho \in C
\] (II.23)
and

\[
\int_a^b M_z(x') \left[ \cos \theta \frac{k_b}{2j} H_1^{(2)}(k_b|\rho - x'|) - 2\hat{l} \cdot e^h(\rho; x'\hat{x}) \right] dx' \\
- \frac{1}{2} M_z^b(\rho) - \int_C M_z^b(\rho') \frac{k_b}{4j} \left[ \cos \theta H_1^{(2)}(k_b|\rho - \rho'|) \right. \\
+ \cos \theta H_1^{(2)}(k_b|\rho - \rho' + 2(\hat{y} \cdot \rho')\hat{y}) \left. \right] dl' \\
+ \int_C M_z^b(\rho') \hat{l} \cdot \left[ e^h(\rho; \rho) + e^h(\rho; \rho - 2(\hat{y} \cdot \rho')\hat{y}) \right] dl' = 0, \quad \rho \in C
\]  

(II.24)

(II.23) and (II.24) form a coupled set of integral equations for the unknown slot electric field and the equivalent magnetic surface current on the cylinder. The surface electric current on the cylinder can be computed from \( H_z^b \).
CHAPTER III
PROLATE SPHEROIDAL MONOPOLE ANTENNA DRIVEN BY COAXIAL WAVEGUIDE

Determining the electromagnetic field in the presence of a prolate spheroidal monopole antenna with a coaxial feed is a pseudo canonical problem in electromagnetics due to its conformance to constant surfaces in both prolate spheroidal and cylindrical coordinates. The conformance and its \( \phi \) symmetry lead to simplified analysis.

Analysis in the coaxial feed is performed in cylindrical coordinates and is characterized by the familiar exponential and Bessel functions. However, the analysis in the upper half-space requires an understanding of prolate spheroidal coordinates and functions. Prolate spheroidal coordinates is one of the eleven curvilinear coordinate systems in which the homogeneous scalar Helmholtz equation is separable. The three orthogonal surfaces in prolate spheroidal coordinates, as depicted in Figure III.1 are a prolate spheroid, a hyperboloid of two sheets and a half plane that correspond to surfaces of constant \( \xi, \eta \) and \( \phi \), respectively. The spheroid and hyperboloid have foci located at \( \pm f \) along the \( z \)-axis. The products of angular and radial prolate spheroidal functions are solutions of the \( \phi \)-independent homogeneous scalar Helmholtz equation, and are represented as infinite series of associated Legendre and spherical Bessel functions, respectively [24, 25] (Appendix D).

In the past, the finite feed gap and magnetic frill models have been used in methods
Figure III.1: Prolate Spheroidal Coordinates
to approximate the field in the coaxial aperture of the antenna \([6, 7, 8]\). These methods have been used to solve for the electric current on the surface of the monopole, from which the electromagnetic field could be computed at all points above the ground plane. However, it is possible to solve for the exact tangential-to-ground plane field distribution on the aperture \([26, 27]\), thus increasing the accuracy of the calculation of the electromagnetic field in any region. To this end, a single fully coupled integral equation with one unknown, the electric field in the coaxial aperture, is developed for the given system. The excitation is an incident \(\phi\)-symmetric TEM mode in the coaxial feed line. Maxwell’s equations and electromagnetic boundary conditions are satisfied by an eigen-series kernel in the coaxial feed and prolate spheroidal angular and radial functions in the upper half-space.

III.1 Structure and source

The prolate spheroidal monopole being analyzed is shown in Figure III.2. The inner and outer radii of the coaxial transmission line are \(a\) and \(b\). The material in the coax where \(z < -L\) is characterized by \((\mu_g, \epsilon_g)\). The dielectric "bead" for physical support, \((-L \leq z \leq 0)\), is characterized by \((\mu_c, \epsilon_c)\). The monopole is a constant surface of \(\xi\) denoted by \(\xi_0\). \(\xi_0\) in prolate spheroidal coordinates, corresponds to a radius \(a_0\) in cylindrical coordinates. While \(a_0\) and \(a\) are selected equal in Figure III.2 for illustrative purposes, \(a_0\) can in fact take on any value between \(0^+\) and \(b\). The material in the upper half-space is characterized by \((\mu, \epsilon)\). The excitation under consideration is a TEM wave launched towards the aperture with a known magnitude.
Figure III.2: Prolate spheroidal monopole antenna above ground plane fed by TEM wave in coaxial guide incident upon the aperture at the aperture \((z = 0), E_0\).

III.2 Equivalent model of structure and source

An equivalent model is created by shorting the aperture and placing equal but oppositely directed surface magnetic currents on the sides of the short. \(M_\phi(\rho)\) is placed on the upper side and \(-M_\phi(\rho)\) is placed on the lower side of the short as
Figure III.3: Equivalent Model

depicted in Figure III.3. The magnetic currents, of proper value, cause the electric field tangential to the $z = 0$ plane to be "continuous" through the aperture. Image theory is used to form upper half-space and coaxial region models as outlined in [20].

III.3 General description of integral equations

It is desired to derive an integral equation for the unknown $\rho$-directed electric field $E_\rho^A$ in the aperture of the antenna. This integral equation is derived under the requirement that the $\phi$ component of the H-field is continuous in the limit as the observation point approaches the aperture from each side and the electric field
tangential to the surface of the monopole and the ground plane is zero.

\[
\lim_{z \to 0} H_{\phi}^o(\rho, z) = \lim_{z \to 0} H_{\phi}^u(\rho, z), \rho \in (a, b) \quad (\text{III.1})
\]

The magnetic field in the coaxial region can be written \[27\]

\[
H_{\phi}^c(\rho, z) = H_{\phi}^s(\rho, z) + H_{\phi}^e(\rho, z), z \in (-L, 0) \quad (\text{III.2})
\]

where \( H_{\phi}^s \) is the contribution from the TEM incident wave, with the aperture shorted, and \( H_{\phi}^e \) is the contribution from the aperture electric field. The total magnetic field in the upper half-space can be written

\[
H_{\phi}^u(\rho, z) = H_{\phi}^r(\rho, z) + H_{\phi}^s(\rho, z) \quad (\text{III.3})
\]

where \( H_{\phi}^r \) is the field due to the aperture radiating in the upper half-space with the monopole removed and \( H_{\phi}^s \) represents the companion magnetic field to the electric field that sets the total tangential electric field to be zero on the surface of the monopole.

III.4 Equivalent model in the coaxial region

To determine the magnetic field in the coaxial region, the equivalent model introduced in section \[\text{III.2} \] must be elaborated on. In Figure \[\text{III.4} \] the aperture is shorted and an equivalent surface magnetic current of density \(-M = -M_{\phi} \hat{\phi}\), where

\(-M_{\phi} = \hat{\phi} \cdot [E_{\rho}^A \hat{\rho} \times \hat{z}] \) or \(-M_{\phi} = E_{\rho}^A \), is placed on the bottom side of the shorted annulus. In the next section, the magnetic field in the coaxial region will be considered as a superposition due to the impinging TEM wave and the magnetic field caused by the equivalent magnetic current on the short.

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III.5 Magnetic field in the coaxial region

Since \(-M_\phi(\rho) = E^A_\rho(\rho)\), the magnetic field due to the aperture electric field can be written as

\[
H^c_\phi(\rho, z) = \int_a^b E^A_\rho(\rho')G^c(\rho, z; \rho')d\rho', \quad z \in (-L, 0) \tag{III.4}
\]

where the Green’s function \(G^c\) is given in Butler et al. [27] which they determine by standard methods (Appendix C):

\[
G^c(\rho, z; \rho') = -\frac{1}{\eta_c \rho \ln \frac{a}{a}} e^{jk_cz} \left[ \frac{1 - \Gamma e^{-j2k_c(z+L)}}{1 + \Gamma e^{-j2k_cL}} \right] - \frac{k_c}{\eta_c \rho'} \sum_{q=1}^{\infty} \frac{1}{k_{zq} N_q^2} \frac{d}{d\rho'} \Phi_q(\rho') \frac{d}{d\rho} \Phi_q(\rho) e^{j k_{zq} z} \tag{III.5}
\]

where \(k_c = \omega \sqrt{\mu_c \epsilon_c}\) and \(\eta_c = \sqrt{\mu_c / \epsilon_c}\), where

\[
\Gamma = \frac{\eta_b - \eta_c}{\eta_b + \eta_c} \tag{III.6}
\]
where \( \eta_g = \sqrt{\mu_g/\epsilon_g} \), where

\[
\Phi_q(\rho) = N_0(k_{t_q}a)J_0(kt_q\rho) - J_0(kt_qa)N_0(kt_q\rho), \tag{III.7}
\]

and where

\[
k_{z_q} = \begin{cases} 
\sqrt{k_c^2 - k_{t_q}^2} & k_{t_q}^2 < k_c^2 \\
-j\sqrt{k_{t_q}^2 - k_c^2} & k_{t_q}^2 > k_c^2
\end{cases} \tag{III.8}
\]

in which \( k_{t_q}, q = 1, 2, \ldots \), is the \( q^{th} \) root of \( \Phi_q(b) = 0 \). In (III.7), \( J_0 \) and \( N_0 \) are the 0\( ^{th} \) order Bessel and Neumann functions and the norm \( N^2_q \) in (III.5) is

\[
N^2_q = \frac{b^2}{2} \left( \left[ \frac{d}{d\rho} \Phi_q(\rho) \right] \right)^2_{\rho=b} - \frac{2}{\pi^2} \tag{III.9}
\]

The short-circuit magnetic field in the bead supported region of the coax, \( -L < z < 0 \), is determined from simple transmission line theory under the constraint that higher order modes vanish after traveling less than \( L \) in the guide to be

\[
H_{sc}^\phi(\rho, z) = 2\frac{E_0}{\eta_g} \rho e^{jk_cL} \cos k_cL + j\eta_c \sin k_cL \cos k_cz, \quad z \in (-L, 0) \tag{III.10}
\]

### III.6 Equivalent model in the upper half-space

To determine the magnetic field in the upper half-space, the equivalent model introduced in section III.2 must be elaborated on. In Figure III.5 the aperture is shorted and an equivalent surface magnetic current of density \( \mathbf{M} = M_\phi \hat{\phi} \), where \( M_\phi = \hat{\phi} \cdot [E^A_\rho \hat{\rho} \times \hat{z}] \) or \( M_\phi = -E^A_\rho \), is placed on the upper side of the shorted annulus. Image theory is then used to remove the ground plane and arrive at the final equivalent model for the upper half-space depicted in Figure III.6.
**Figure III.5:** Model equivalent to coax-driven prolate spheroidal monopole in the upper half-space

**Figure III.6:** Prolate spheroid with annulus of magnetic current - upper half-space equivalent model
III.7 Magnetic field in the upper half-space

From the equivalent model, the total magnetic field can be written as the integral over the aperture of a dense set of loops of magnetic current in the presence of the prolate spheroid. For a loop of magnetic current of strength $K$ in free space, $\mathbf{F} = F_\phi \hat{\phi}$ where

$$F_\phi = \epsilon \rho' K \int_{-\pi}^{\pi} \cos(\phi') \frac{e^{-jkR}}{4\pi R} d\phi' \quad (III.11)$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Because the divergence of $\mathbf{F}$ is zero, the magnetic field has only a $\phi$ component which can be written as

$$H_\phi = -j\omega F_\phi \quad (III.12)$$

The electric field can also be determined from $F_\phi$ via

$$\mathbf{E} = -\frac{1}{\epsilon} \nabla \times (F_\phi \hat{\phi}) \quad (III.13)$$

From (III.13), the $\xi$ and $\eta$ components of the electric field are derived to be

$$E_\xi = -\frac{1}{\epsilon h_\xi h_\phi} \left[ F_\phi \frac{\partial h_\phi}{\partial \eta} + h_\phi \frac{\partial F_\phi}{\partial \eta} \right] \quad (III.14)$$

$$E_\eta = -\frac{1}{\epsilon h_\xi h_\phi} \left[ F_\phi \frac{\partial h_\phi}{\partial \xi} + h_\phi \frac{\partial F_\phi}{\partial \xi} \right] \quad (III.15)$$

where $h_\xi$ and $h_\phi$ are prolate spheroidal metric coefficients.

To set $E_\eta$ on the surface of the spheroid to zero, $F_\phi$ must be expressed in prolate spheroidal coordinates. To this end, the free space Green’s function in prolate
spheroidal coordinates is \[25\] (Appendix E)

\[
G(\xi, \eta, \phi; \xi', \eta', \phi') = \frac{e^{-jkR}}{4\pi R} = -\frac{jk}{2\pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2 - \delta_{0m}}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') 
\cdot R^{(1)}_{mn}(c, \xi_<) R^{(4)}_{mn}(c, \xi_<) \cos m(\phi - \phi')
\]

(III.16)

where \(S_{mn}, R^{(1)}_{mn}\) and \(R^{(4)}_{mn}\) are the prolate spheroidal angular function of the first kind and the prolate spheroidal radial functions of the first and fourth kinds respectively and where

\[
N_{mn} = 2 \sum_{r=0,1}^{\infty} \frac{(r + 2m)! (d^{mn}_r)^2}{(2r + 2m + 1)r!}
\]

(III.17)

in which \(d^{mn}_r\) are the expansion coefficients of the prolate spheroidal functions (Appendix D). To satisfy the boundary and radiation conditions to be met (III.16) must be modified to be of the form

\[
G_1(\xi, \eta, \phi; \xi', \eta', \phi') = -\frac{jk}{2\pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2 - \delta_{0m}}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi') 
\cdot \left[ R^{(1)}_{mn}(c, \xi_<) R^{(4)}_{mn}(c, \xi_<) + A_{mn} R^{(4)}_{mn}(c, \xi') R^{(4)}_{mn}(c, \xi) \right]
\]

(III.18)

\(A_{mn}\) is determined by evaluating (III.15) at the surface of a spheroid, \(\xi = \xi_0 < \rho'\), using (III.11) with (III.18) as the Green’s function instead of (III.16), to be

\[
A_{mn} = -\frac{\xi_0 R^{(1)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(4)}_{mn}(c, \xi_0)}{\xi_0 R^{(4)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(4)}_{mn}(c, \xi_0)}
\]

(III.19)
This result allows (III.18) to be written as

\[ G_1(\xi, \eta, \phi; \xi', \eta', \phi') = -\frac{j k}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2 - \delta_{0m}}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi') \]

\[ \cdot \left[ R^{(1)}_{mn}(c, \xi_0) R^{(4)}_{mn}(c, \xi) - \frac{\xi_0 R^{(1)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(1)'}_{mn}(c, \xi_0)}{\xi_0 R^{(4)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(4)'}_{mn}(c, \xi_0)} R^{(4)}_{mn}(c, \xi) R^{(4)}_{mn}(c, \xi) \right] \]

(III.20)

or alternatively as

\[ G_1(\rho, \phi, z; \rho', \phi', z') = e^{-j k R} \frac{4\pi R}{4\pi R'} \]

\[ + \frac{j k}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2 - \delta_{0m}}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi') \]

\[ \cdot \frac{\xi_0 R^{(1)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(1)'}_{mn}(c, \xi_0)}{\xi_0 R^{(4)}_{mn}(c, \xi_0) + (\xi_0^2 - 1) R^{(4)'}_{mn}(c, \xi_0)} R^{(4)}_{mn}(c, \xi) R^{(4)}_{mn}(c, \xi) \]

(III.21)

where \( \xi, \eta, \xi' \) and \( \eta' \) are expressed in terms of the cylindrical coordinates.

From (III.12), one can write the total magnetic field due to a loop of magnetic current of strength \( K \) in the presence of a prolate spheroid as

\[ H_\phi(\rho, z) = -\frac{k}{\eta m} \rho' K \int_{-\pi}^{\pi} \cos(\phi') e^{-j k R} \frac{4\pi}{4\pi R'} d\phi' \]

\[ - k^2 \sqrt{\frac{c}{\mu}} \rho' K \sum_{n=1}^{\infty} \frac{1}{N_{1n}} S_{1n}(c, \eta) S_{1n}(c, \eta') \]

\[ \frac{\xi_0 R^{(1)}_{1n}(c, \xi_0) + (\xi_0^2 - 1) R^{(1)'}_{1n}(c, \xi_0)}{\xi_0 R^{(4)}_{1n}(c, \xi_0) + (\xi_0^2 - 1) R^{(4)'}_{1n}(c, \xi_0)} R^{(4)}_{1n}(c, \xi) R^{(4)}_{1n}(c, \xi) \]

(III.22)

Integrating (III.22) over the dimensions of the annulus with \( K \) replaced by \( 2M_\phi(\rho') \), from the equivalent model, makes it possible to write the magnetic field in the upper
half-space as

\[ H_\phi^u(\rho, z) = -j \frac{k}{\eta_m} \frac{1}{2\pi} \int_a^b \rho' M_\phi(\rho') \int_{-\pi}^\pi \cos(\phi') \frac{e^{-jkR}}{R} d\phi' d\rho' \]

\[ - \frac{2k^2}{\eta_m} \int_a^b \rho' M_\phi(\rho') \sum_{n=1}^\infty \frac{1}{N_{1n}} S_{1n}(c, \eta) S_{1n}(c, \eta') \]

\[ \cdot \left( \frac{\xi_0 R_{1n}^{(1)}(c, \xi_0)}{\xi_0 R_{1n}^{(4)}(c, \xi_0)} + (\xi_0^2 - 1) R_{1n}^{(4)}(c, \xi_0) R_{1n}^{(4)}(c, \xi) d\rho' \right) \quad (\text{III.23}) \]

Note that when the integration is performed, \( \xi, \xi', \eta \) and \( \eta' \) must be treated as functions of \( \rho, \rho', z \) and \( z' \). Writing the above field equation in terms of the unknown aperture electric field yields

\[ H_\phi^u(\rho, z) = j \frac{k}{\eta_m} \frac{1}{2\pi} \int_a^b \rho' E_\rho^A(\rho') \int_{-\pi}^\pi \cos(\phi') \frac{e^{-jkR}}{R} d\phi' d\rho' \]

\[ - \frac{2k^2}{\eta_m} \int_a^b \rho' E_\rho^A(\rho') \sum_{n=1}^\infty \frac{1}{N_{1n}} S_{1n}(c, \eta) S_{1n}(c, \eta') \]

\[ \cdot \left( \frac{\xi_0 R_{1n}^{(1)}(c, \xi_0)}{\xi_0 R_{1n}^{(4)}(c, \xi_0)} + (\xi_0^2 - 1) R_{1n}^{(4)}(c, \xi_0) R_{1n}^{(4)}(c, \xi) d\rho' \right) \quad (\text{III.24}) \]

The first term of (\text{III.24}) is recognized to be \( H_\rho^s \) and the second to be \( H_\phi^s \).

### III.8 Integral equation

Because the use of image theory and the modified Green’s function in the formulation of the magnetic field in the upper half-space satisfies the conditions that the tangential electric field be zero on the ground plane and on the surface of the spheroid, the only remaining condition to be satisfied is that the H-field be continuous in the limit as the observation point approaches a point in the aperture from each side. This
condition is met by enforcing (III.1). The resulting integral equation for $E^A_{\rho}$ is

$$
- \int_a^b E^A_{\rho}(\rho')G^c(\rho,0;\rho')d\rho' + j \frac{k}{\eta_m} \int_a^b \rho'E^A_{\rho}(\rho')G^a(\rho,0;\rho')d\rho' \\
- \frac{2k^2}{\eta_m} \int_a^b \rho'E^A_{\rho}(\rho') \sum_{n=1}^{\infty} \frac{1}{N_{1n}}S_{1n}(c,0)S_{1n}(c,0) \\
\frac{\xi_0R^{(1)}_{1n}(c,\xi_0) + (\xi_0^2 - 1)R^{(1)'}_{1n}(c,\xi_0)}{\xi_0R^{(4)}_{1n}(c,\xi_0) + (\xi_0^2 - 1)R^{(4)'}_{1n}(c,\xi_0)} R^{(4)}_{1n}(c,\xi)R^{(4)}_{1n}(c,\xi')d\rho' = H^sc \quad (III.25)
$$

$$
\rho \in (a, b), \ z = 0 \quad (III.26)
$$

in which

$$
G^a(\rho, z; \rho') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\phi')\frac{e^{-jkR/R}}{R}d\phi'. \quad (III.27)
$$

$\xi$ and $\xi'$ are related to $\rho$ and $\rho'$ by the relation

$$
\xi = \sqrt{1 + \left(\frac{\rho}{\eta_c}\right)^2}.
$$

Observe that $E^A_{\rho}$ can be obtained by solving a single integral equation. Knowledge of $E^A_{\rho}$ allows one to determine all field components in the upper half-space and in the coaxial region.
CHAPTER IV
ALTERNATE INTEGRAL EQUATION

Because Flammer constrains $m \geq 0$ and captures the $\phi$ variation with only a cosine function, his free space Green’s function, (III.16), can only model $\phi$-symmetric, even function distributed fields. However, if $m$ is allowed to take on all values in the set of integers and a complex exponential is used to capture the $\phi$ variation, then any general field can be modeled with the resulting Green’s function. The development of such a Green’s function and the resulting integral equation for the prolate spheroidal monopole follows.

IV.1 Derivation of free space Green’s function

In the inhomogeneous differential equation

$$\nabla^2 G + k^2 G = -\delta(r - r')$$  \hspace{1cm} (IV.1)

the 3 dimensional delta function in prolate spheroidal coordinates is

$$\delta(r - r') = (h_\xi h_\eta h_\phi)^{-1}\delta(\xi - \xi')\delta(\eta - \eta')\delta(\phi - \phi')$$

$$= \frac{1}{f^3(\xi^2 - \eta^2)}\delta(\xi - \xi')\delta(\eta - \eta')\delta(\phi - \phi')$$  \hspace{1cm} (IV.2)

and the scalar Laplacian is expanded in prolate spheroidal coordinates to arrive at

$$\frac{1}{f^2(\xi^2 - \eta^2)} \left\{ \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial G}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial G}{\partial \eta} \right] + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 G}{\partial \phi^2} \right\} + k^2 G$$

$$= -\frac{1}{f^3(\xi^2 - \eta^2)}\delta(\xi - \xi')\delta(\eta - \eta')\delta(\phi - \phi')$$  \hspace{1cm} (IV.3)
$G$ is expanded in a Fourier-Prolate Spheroidal series as

$$G(\xi, \eta, \phi; \xi', \eta', \phi') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn}(\xi; \xi', \eta', \phi') S_{mn}(c, \eta) e^{jm\phi} \quad (IV.4)$$

where

$$a_{mn}(\xi) = \frac{1}{N_{mn}} \int_{-1}^{1} \int_{-\pi}^{\pi} G(\xi, \eta, \phi; \xi, \eta, \phi) S_{mn}(c, \eta) e^{-jm\phi} d\phi d\eta \quad (IV.5)$$

After multiplying both sides of (IV.3) by \(\frac{1}{N_{mn}} S_{mn}(c, \eta) e^{-jm\phi}\) and integrating over \(\eta \in (-1, 1)\) and \(\phi \in (-\pi, \pi)\), one arrives at

$$\frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial}{\partial \xi} a_{mn}(\xi) \right] + \frac{1}{N_{mn}} J^n_{mn}(\xi) + c^2 \xi^2 a_{mn}(\xi) = \left[ \lambda_{mn}(c) + c^2 \xi^2 - \frac{m^2}{\xi^2 - 1} \right] a_{mn}(\xi)$$

$$= -\frac{1}{N_{mn}} S_{mn}(c, \eta') e^{-jm\phi'} \delta(\xi - \xi') \quad (IV.6)$$

where

$$J^n_{mn}(\xi) = \int_{-1}^{1} \int_{-\pi}^{\pi} \frac{1}{(1 - \eta^2)} \frac{\partial}{\partial \eta} \left[ (\xi^2 - 1) \frac{\partial}{\partial \eta} \right] S_{mn}(c, \eta) e^{-jm\phi} d\phi d\eta \quad (IV.7)$$

and

$$J^\phi_{mn}(\xi) = \int_{-1}^{1} \int_{-\pi}^{\pi} \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2}{\partial \phi^2} S_{mn}(c, \eta) e^{-jm\phi} d\phi d\eta \quad (IV.8)$$

(IV.6) can be further simplified, by performing integration by parts twice on both \(J^n_{mn}(\xi)\) and \(J^\phi_{mn}(\xi)\), to

$$\frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial}{\partial \xi} a_{mn}(\xi) \right] - \left[ \lambda_{mn}(c) a_{mn}(\xi) - c^2 \xi^2 + \frac{m^2}{\xi^2 - 1} \right] a_{mn}(\xi)$$

$$= -\frac{1}{fN_{mn}} S_{mn}(c, \eta') e^{-jm\phi'} \delta(\xi - \xi') \quad (IV.9)$$

which is recognized as an inhomogeneous form of (D.6). Solutions to this equation are constructed from appropriate linear combinations of prolate spheroidal radial
functions: $R_{mn}^{(1)}(c, \xi), R_{mn}^{(2)}(c, \xi), R_{mn}^{(3)}(c, \xi)$ and $R_{mn}^{(4)}(c, \xi)$. To determine $a_{mn}(\xi)$ such that $G$ is continuous at $\xi = \xi'$ and the radiation condition is satisfied, one constructs a homogeneous solution of the form

$$a_{mn}(\xi) = \begin{cases} 
C_{mn}R_{mn}^{(4)}(c, \xi')R_{mn}^{(1)}(c, \xi) & \xi < \xi' \\
C_{mn}R_{mn}^{(1)}(c, \xi')R_{mn}^{(4)}(c, \xi) & \xi' < \xi
\end{cases}$$  \hspace{1cm} (IV.10)

or alternatively

$$a_{mn}(\xi) = C_{mn}R_{mn}^{(1)}(c, \xi < \xi')R_{mn}^{(4)}(c, \xi > \xi')$$  \hspace{1cm} (IV.11)

where the subscripts $<$ and $>$ indicate the lesser and greater of $\xi$ and $\xi'$. The constant, $C_{mn}$, is determined by enforcing the jump condition on (IV.9) by integrating both sides over $\xi \in (\xi' - \Delta, \xi' + \Delta)$ in the limit as $\Delta \to 0$

$$C_{mn}(\xi^2 - 1) \left[ R_{mn}^{(1)}(c, \xi')R_{mn}^{(4)'}(c, \xi') - R_{mn}^{(4)}(c, \xi')R_{mn}^{(1)'}(c, \xi') \right] = -\frac{1}{fN_{mn}(c, \eta')e^{-jm\phi'}}$$

and making use of the Wronskian

$$W[R_{mn}^{(1)}(c, \xi), R_{mn}^{(2)}(c, \xi)] = \frac{1}{c(\xi^2 - 1)}$$  \hspace{1cm} (IV.12)

to arrive at

$$C_{mn} = -j \frac{k}{N_{mn}}S_{mn}(c, \eta')e^{-jm\phi'}$$  \hspace{1cm} (IV.13)

The solution for $a_{mn}(\xi)$ is substituted into (IV.4) to find the free space Green’s function in prolate spheroidal coordinates

$$G(\xi, \eta, \phi; \xi', \eta', \phi') = -jk \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{N_{mn}}S_{mn}(c, \eta)S_{mn}(c, \eta') \cdot R_{mn}^{(1)}(c, \xi < \xi')R_{mn}^{(4)}(c, \xi > \xi')e^{jm(\phi - \phi')}$$  \hspace{1cm} (IV.14)

$$= e^{-jkR/4\pi R}$$

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IV.2 Satisfaction of boundary conditions

The same procedure as in section III.7 to satisfy the Dirichlet condition on the surface of the monopole yields the same constant as (III.19). This result matches intuition as the $\xi$ dependent portions of (III.16) and (IV.14) are the same. The new Green’s function that satisfies the boundary conditions is

$$G_1(\xi, \eta; \xi', \eta') = -j k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') e^{jm(\phi - \phi')} \cdot \left[ R_{mn}^{(1)}(c, \xi, \eta) R_{mn}^{(4)}(c, \xi') \right]$$

$$= \frac{e^{-jkR}}{4\pi R}$$

$$+ j k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') e^{jm(\phi - \phi')} \cdot \left[ \frac{\xi_0 R_{mn}^{(1)}(c, \xi_0) + (\xi'^2 - 1) R_{mn}^{(1)'}(c, \xi_0) R_{mn}^{(4)'}(c, \xi') R_{mn}^{(4)}(c, \xi)}{\xi_0 R_{mn}^{(4)}(c, \xi_0) + (\xi_0^2 - 1) R_{mn}^{(4)'}(c, \xi)} \right]$$

(IV.15)

IV.3 Magnetic field in the upper half-space

An expression for the magnetic field in the upper half-space (III.3) is found by substituting the two parts of the right-hand side of (IV.15) into (III.12), replacing $K$ with $2M_\phi(\rho)$ and integrating over the radius of the aperture to arrive at

$$H_{\phi}^n(\rho, z) = -j \frac{k}{\eta_{mn}} \frac{1}{2\pi} \int_a^b \rho' M_\phi(\rho') \int_{-\pi}^{\pi} \frac{e^{-jkR}}{R} d\phi' d\rho'$$

$$= -j \frac{k}{\eta_{mn}} \frac{1}{2\pi} \int_a^b \rho' M_\phi(\rho') \int_{-\pi}^{\pi} \frac{e^{-jkR}}{R} d\phi' d\rho'$$

(IV.16)
and

\[ H^s_\phi(\rho, z) = \frac{2k^2}{\eta_m} \int_a^b \rho' M_\phi(\rho') \int_{-\pi}^{\pi} \cos(\phi - \phi') \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \]

\[ \cdot \frac{\xi_0 R_{mn}^{(1)}(c, \xi_0) + (\xi_0^2 - 1) R_{mn}^{(1)'}(c, \xi_0)}{\xi_0 R_{mn}^{(4)}(c, \xi_0) + (\xi_0^2 - 1) R_{mn}^{(4)'}(c, \xi_0)} R_{mn}^{(4)}(c, \xi) R_{mn}^{(4)}(c, \xi') \]

\[ \cdot e^{im(\phi - \phi')} d\phi' d\rho' \quad \text{(IV.17)} \]

The \( \phi' \) dependent portion of (IV.17) is simplified by performing the integration in \( \phi' \)

\[ \int_{-\pi}^{\pi} \cos(\phi - \phi') e^{im(\phi - \phi')} d\phi' = \begin{cases} 0 & m \neq -1, 1 \\ \pi & m = -1, 1 \end{cases} \]

After this simplification, (IV.17) becomes

\[ H^s_\phi(\rho, z) = \frac{2\pi k^2}{\eta_m} \int_a^b \left\{ \rho' M_\phi(\rho') \sum_{n=1}^{\infty} G_{-1n}(\xi, \xi', \eta, \eta') + G_{1n}(\xi, \xi', \eta, \eta') \right\} d\rho' \quad \text{(IV.18)} \]

where

\[ G_{mn}(\xi, \xi', \eta, \eta') = \frac{1}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \]

\[ \frac{\xi_0 R_{mn}^{(1)}(c, \xi_0) + (\xi_0^2 - 1) R_{mn}^{(1)'}(c, \xi_0)}{\xi_0 R_{mn}^{(4)}(c, \xi_0) + (\xi_0^2 - 1) R_{mn}^{(4)'}(c, \xi_0)} R_{mn}^{(4)}(c, \xi) R_{mn}^{(4)}(c, \xi') \quad \text{(IV.19)} \]

**IV.4 Integral equation**

The derivation of the integral equation is completed by substituting \( E^A_\rho \) for \(-M_\phi\) and enforcing (III.1) to arrive at

\[ - \int_a^b E^A_\rho(\rho') G^c(\rho, 0; \rho') d\rho' + j \frac{k}{\eta_m} \int_a^b \rho' E^A_\rho(\rho') G^a(\rho, 0; \rho') d\rho' \]

\[ - \frac{2\pi k^2}{\eta_m} \int_a^b \left\{ \rho' M_\phi(\rho') \sum_{n=0}^{\infty} G_{-1n}(\xi, \xi', \eta, \eta') + G_{1n}(\xi, \xi', \eta, \eta') \right\} d\rho' \]

\[ = H^s_\phi, \rho \in (a, b), z = 0 \quad \text{(IV.20)} \]
where $G^c$ and $G^a$ are the same as in (III.5) and (III.27). Evaluation of the prolate spheroidal radial functions in this integral equation requires negative values of $m$. Flammer [25] provides a correction factor to the expansion coefficients $a_{mn}^r(c)$ for negative values of $m$, but no tabulated values are available for this case.
CHAPTER V
CONCLUSIONS AND OBSERVATIONS

Analytic Green’s functions in elliptic cylindrical coordinates and in prolate spheroidal coordinates are used in developing and presenting techniques for formulating integral equations for physically practicable structures and sources. For the case of the semi-elliptic channel, a solution for the electromagnetic field in which the location and number of slots and the choice of materials is not constrained by the solution method is developed. For the case of the prolate spheroidal monopole, a solution technique which is not dependent on an approximation for the driving source is developed. Equivalent models are introduced, analytic Greens functions, comprising special functions and satisfying required boundary conditions are derived, integral equations are formulated and expressions for needed field components are represented in terms of integrals of Greens functions times unknowns.

It is believed that the techniques developed and demonstrated herein are applicable to other orthogonal curvilinear coordinate systems where the scalar Helmholtz equation is separable \([28]\).
APPENDICES
APPENDIX A
GREEN’S FUNCTION FOR 2-DIMENSIONAL MAGNETIC SOURCE

For a $z$-directed unity-strength filament of magnetic current located along the $z$-axis with infinite length and radius $a$, the electric vector potential has a $z$ component and satisfies

$$\nabla^2 F_z + k^2 F_z = 0, \quad \rho \neq 0 \quad (A.1)$$

$F_z$ is invariant with respect to $\phi$ and $z$ because the source is invariant with respect to $\phi$ and $z$. Expanding the Laplacian in cylindrical coordinates and taking into account the properties of $F_z$ yields

$$\frac{d^2 F_z}{d\rho^2} + \frac{1}{\rho} \frac{dF_z}{d\rho} + k^2 F_z = 0 \quad (A.2)$$

which is recognized to be the zero order Bessel’s equation of variable $k\rho$. To satisfy the radiation condition, $F_z$ takes the form of

$$F_z = AH^{(2)}_0(k\rho) \quad (A.3)$$

The constant $A$ is found by making use of Ampere’s Law and duality to find that

$$\oint_C \frac{1}{\epsilon} \nabla \times (F_z \hat{z}) \cdot dl = 1 \quad (A.4)$$

$C$ is chosen to be a circular path of radius $a$ about the $z$-axis, which allows the above equation to be written as

$$\int_0^{2\pi} dF_z \frac{d\phi}{d\rho} ad\phi = -\epsilon \quad (A.5)$$
which can be rewritten as

\[ \frac{dF_z}{d\rho} = -\frac{\epsilon}{2\pi a} \]  \hspace{1cm} (A.6)

Because \( \frac{dF_z}{d\rho} = -AkH_1^{(2)}(ka) \) it is found that

\[ A = \frac{\epsilon}{2\pi ka} \frac{1}{H_1^{(2)}(ka)} \]  \hspace{1cm} (A.7)

In the limit as \( ka \to 0 \)

\[ \lim_{ka \to 0} A = -\frac{j\epsilon}{4} \]  \hspace{1cm} (A.8)

Thus,

\[ F_z(\rho) = -\frac{j\epsilon}{4} H_0^{(2)}(k\rho) \]  \hspace{1cm} (A.9)

This Green’s function can be generalized to a source located at \( \rho' \) by substituting \( |\rho - \rho'| \) for \( \rho \) in (A.9).

\[ F_z(\rho, \rho') = -\frac{j\epsilon}{4} H_0^{(2)}(k|\rho - \rho'|) \]  \hspace{1cm} (A.10)

This 2-D Green’s function is equivalent to that given in Harrington [29].
APPENDIX B
GREEN’S FUNCTION FOR 2-DIMENSIONAL SOURCE INSIDE OF AN ELLIPTIC CYLINDRICAL CAVITY

It is convenient to analyze the field in region b in Figure II.2 in elliptic cylindrical coordinates as depicted in Figure II.1. The conversion between rectangular and elliptic cylindrical coordinates is

\[ x = \frac{d}{2} \cosh u \cos v \quad y = \frac{d}{2} \sinh u \sin v \]  \hspace{1cm} (B.1)

and the parameter c is defined by \( c = \frac{kd}{2} \).

From Chapter II it is known that the total magnetic field due to a unity-strength \( z \)-directed magnetic line source in an elliptic cavity is given by

\[ h(\rho; \rho') = -\frac{k}{4\eta} H_0^{(2)}(k|\rho - \rho'|) + h^b(\rho; \rho') \] \hspace{1cm} (B.2)

If the Hankel function is expanded in elliptic cylindrical radial and angular functions with Stratton’s notation [9], the incident field is given by

\[ h^i(\rho; \rho') = -\frac{k}{\eta} \sum_{m=0}^{\infty} \left[ \frac{1}{N_m^{(e)}} R e_m^{(1)}(c, u_<) R e_m^{(4)}(c, u_>) S e_m(c, v') S e_m(c, v) 
+ \frac{1}{N_m^{(e)}} R o_m^{(1)}(c, u_<) R o_m^{(4)}(c, u_>) S o_m(c, v') S o_m(c, v) \right] \] \hspace{1cm} (B.3)

where \( u, v, u' \) and \( v' \) are located by \( \rho \) and \( \rho' \) and where \( u_< \) (\( u_> \)) is the smaller (larger) of \( u \) and \( u' \) [30, 10]. The electric field is given by

\[ e(\rho; \rho') = \frac{jk}{c\sqrt{\cosh^2 u - \cos^2 v}} \left( -\frac{\partial h}{\partial v} \hat{u} + \frac{\partial h}{\partial u} \hat{v} \right) \] \hspace{1cm} (B.4)
To satisfy the boundary condition that the tangential E field is zero on the wall it is required that

$$\frac{\partial h}{\partial u} = 0|_{u=u_1} \quad (B.5)$$

where $u_1$ is the radial coordinate corresponding to the wall of the cavity. Because of this condition and the requirement that $h^h$ is non-singular in the interior of the cavity, the homogeneous part of the magnetic field must be of the form

$$h^h_z(\rho; \rho^\prime) = -\frac{k}{\eta} \sum_{m=0}^{\infty} \left[ \frac{a_m}{N_{m}^{(e)}} \Re_{m}^{(1)}(c, u) S_{m}(c, v) + \frac{b_m}{N_{m}^{(o)}} \Ro_{m}^{(1)}(c, u) S_{o}(c, v) \right] \quad (B.6)$$

If the boundary condition is enforced and the jump condition is satisfied the constants are found to be [31]

$$a_m = - \frac{\Re_{m}^{(4)}(c, u_1)}{\Re_{m}^{(4)}(c, u_1)} \Re_{m}^{(1)}(c, u^\prime) S_{m}(c, v^\prime) \quad (B.7)$$

$$b_m = - \frac{\Ro_{m}^{(4)}(c, u_1)}{\Ro_{m}^{(4)}(c, u_1)} \Ro_{m}^{(1)}(c, u^\prime) S_{o}(c, v^\prime) \quad (B.8)$$

The components of the homogeneous electric field are determined from [B.4] to be [31]

$$e^h_u = \frac{jk}{c \sqrt{\cosh^2 u - \cos^2 v}} \sum_{m=0}^{\infty} \left[ \frac{a_m}{N_{m}^{(e)}} \Re_{m}^{(1)}(c, u) S_{m}^\prime(c, v) + \frac{b_m}{N_{m}^{(o)}} \Ro_{m}^{(1)}(c, u) S_{o}^\prime(c, v) \right] \quad (B.9)$$

$$e^h_v = \frac{-jk}{c \sqrt{\cosh^2 u - \cos^2 v}} \sum_{m=0}^{\infty} \left[ \frac{a_m}{N_{m}^{(e)}} \Re_{m}^{(1)}(c, u) S_{m}(c, v) + \frac{b_m}{N_{m}^{(o)}} \Ro_{m}^{(1)}(c, u) S_{o}(c, v) \right] \quad (B.10)$$

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APPENDIX C

COAXIAL REGION GREEN’S FUNCTION

An infinite coaxial waveguide with inner radius \(a\) and outer radius \(b\) is depicted in Figure C.1. The material in the guide for \(z < -L\) is characterized by \((\mu_g, \epsilon_g)\) and elsewhere by \((\mu_c, \epsilon_c)\). The unknown electric field \(E^A_\rho(\rho)\) is located at \(z = 0\). The distance \(L\) is selected such that any higher order modes generated at \(z = 0\) will decay to "zero" before traveling a distance \(L\). Under the constraint on \(L\) at \(z = L\), there are only be positive and negative traveling TEM modes. There is a matching interface at \(z = L\) that is not shown in the figure but does not need to be considered because \(E^A_\rho(\rho)\) contains all of the required information about the electromagnetic field in the region where \(z \in (-L, 0)\).

The required field equations from [26] are specialized to the above conditions leading to

\[
E_\rho(\rho, z) = B^+_0 \frac{e^{-jk_c(z+L)}}{\rho} + B^-_0 \frac{e^{jk_cz}}{\rho} + \sum_{q=1}^{\infty} B^-_q e^{jk_{zq}z} \frac{d\Phi_q(\rho)}{d\rho}, z \in (-L, 0) \tag{C.1}
\]

\[
H_\phi(\rho, z) = B^+_0 \frac{e^{-jk_c(z+L)}}{\eta_c \rho} - B^-_0 \frac{e^{jk_cz}}{\eta_c \rho} + \frac{k_c}{\eta_c} \sum_{n=q}^{\infty} \frac{1}{k_{zq}} B^-_q e^{jk_{zq}z} \frac{d\Phi_q(\rho)}{d\rho}, z \in (-L, 0) \tag{C.2}
\]

which defines \(B^+_0\) as the coefficient of the positive propagating TEM wave at \(z = -L\) and \(B^-_0\) and \(B^-_q\) as the coefficients of the negative propagating TEM wave and evanescent higher order modes at \(z = 0\). The wave numbers \(k_c\) and \(k_{zq}\) are the same as defined in section III.5. \(B^+_0\) can be rewritten in terms of \(B^-_0\) as

\[
B^+_0 = \Gamma B^-_0 e^{-jk_cL} \tag{C.3}
\]
because the negative TEM wave travels a distance $L$ and reflects off the interface at $z = -L$. $\Gamma = \frac{\eta_g - \eta_c}{\eta_g + \eta_c}$ is the reflection coefficient at the interface at $z = -L$, where $\eta_g$ and $\eta_c$ are defined as in section III.5. Substituting (C.3) into (C.1) yields

$$E_\rho(\rho, z) = B_0^- \rho \left[1 + \Gamma e^{-jk_c(z+2L)} + e^{jk_cz} \right] + \sum_{q=1}^{\infty} B_q^- e^{jk_qz} d\Phi_q(\rho)$$  \hspace{1cm} (C.4)$$

To determine the constant $B_0^-$ one evaluates $E_\rho$ at $z = 0$ and integrates over the radius of the guide

$$E_A^\rho(\rho) = \int_a^b \left\{ \frac{B_0^-}{\rho} \left[1 + \Gamma e^{-jk_cL} \right] + \sum_{q=1}^{\infty} B_q^- e^{jk_qz} d\Phi_q(\rho) \right\} \, d\rho$$  \hspace{1cm} (C.5)$$

Because $\Phi(\rho) = 0$ for $\rho = a, b$ the summation vanishes and only

$$\int_a^b E_A^\rho(\rho) \, d\rho = \ln \frac{b}{a} B_0^- \left[1 + \Gamma e^{-jk_cL} \right]$$  \hspace{1cm} (C.6)$$

remains. Thus the negative TEM constant is found to be

$$B_0^- = \frac{1}{\ln \frac{b}{a}} \frac{1}{1 + \Gamma e^{-jk_cL}} \int_a^b E_A^\rho(\rho) \, d\rho$$  \hspace{1cm} (C.7)$$

To find the higher order mode constants $B_q^-$, both sides of (C.5) are multiplied by $\rho \frac{d\Phi_m}{d\rho}$ and integrated over the radius of the guide. Similarly to above, the TEM portion

Figure C.1: Infinite Coaxial Guide with Interface and Impressed Field
vanishes and because of the orthogonality relationship between \( \frac{d\Phi_q}{d\rho} \) and \( \frac{d\Phi_m}{d\rho} \) it is found that

\[
B_q = \frac{1}{N_q^2} \int_a^b E^A_{\rho} (\rho) \rho \frac{d\Phi_q}{d\rho} d\rho \tag{C.8}
\]

where

\[
N_q^2 = \int_a^b \rho \left( \frac{d\Phi_q}{d\rho} \right)^2 d\rho \tag{C.9}
\]

The values of the constants can now be substituted into (C.2) to find that

\[
H_\phi (\rho, z) = \int_a^b E^A_{\rho} (\rho') \left\{ -\frac{1}{\eta_c \rho \ln \frac{b}{a}} e^{jk_c z} \left[ \frac{1 - \Gamma e^{-j2k_c (z+L)}}{1 + \Gamma e^{-j2k_c L}} \right] \right.
- \rho' \frac{k_c}{\eta_c} \sum_{q=1}^{\infty} 1 \frac{d\Phi_q (\rho)}{d\rho'} \frac{d\Phi_q (\rho)}{d\rho} e^{j k_{cq} z} \right\} d\rho' \tag{C.10}
\]

Thus, the Green’s function in the coaxial region is

\[
G^c (\rho, z; \rho') = -\frac{1}{\eta_c \rho \ln \frac{b}{a}} e^{jk_c z} \left[ \frac{1 - \Gamma e^{-j2k_c (z+L)}}{1 + \Gamma e^{-j2k_c L}} \right]
- \rho' \frac{k_c}{\eta_c} \sum_{q=1}^{\infty} 1 \frac{d\Phi_q (\rho)}{d\rho'} \frac{d\Phi_q (\rho)}{d\rho} e^{j k_{cq} z} \tag{C.11}
\]

\( z \in (-L, 0) \)
APPENDIX D
PROLATE SPHEROIDAL FUNCTIONS

The prolate spheroidal functions are solutions to the homogeneous scalar Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0$$

which in prolate spheroidal coordinates is

$$\frac{1}{f^2(\xi^2 - \eta^2)} \left\{ \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial \psi}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial \psi}{\partial \eta} \right] + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 \psi}{\partial \phi^2} \right\} + k^2 \psi = 0 \quad (D.1)$$

Multiplying both sides of (D.1) by $f^2(\xi^2 - \eta^2)$ yields

$$\frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial \psi}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial \psi}{\partial \eta} \right] + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 \psi}{\partial \phi^2} + c^2(\xi^2 - \eta^2) \psi = 0 \quad (D.2)$$

where

$$c = fk$$

Assume that $\psi$ is separable and takes the form of $\psi(\xi, \eta, \phi) = \Xi(\xi)H(\eta)\Phi(\phi)$. This product is substituted for $\psi$ in (D.2) and $m^2$, an eigenvalue that is equal to $-\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}$, is added and subtracted to the result to yield

$$\frac{(\xi^2 - 1)(1 - \eta^2)}{\xi^2 - \eta^2} \left\{ \frac{1}{\Xi} \frac{d}{d\xi} \left[ (\xi^2 - 1) \frac{d\Xi}{d\xi} \right] + \frac{1}{H} \frac{d}{d\eta} \left[ (1 - \eta^2) \frac{dH}{d\eta} \right] + c^2(\xi^2 - \eta^2) \right\} + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + m^2 - m^2 = 0 \quad (D.3)$$

which leads to

$$\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0 \quad (D.4)$$
and has the solution

\[ \Phi(\phi) = \cos m\phi. \]

because \( \psi \) must be an even function in \( \phi \). \( m \) is not chosen to be zero identically because the solution to the scalar Helmholtz equation would be generalized associated Legendre functions \[28\] instead of prolate spheroidal angular and radial functions.

The eigenvalue \( \lambda_{mn}(c) \) is added and subtracted from the remaining equation to yield

\[
\frac{1}{\Xi} \frac{d}{d\xi} \left[ (\xi^2 - 1) \frac{d\Xi}{d\xi} \right] + \frac{1}{H} \frac{d}{d\eta} \left[ (1 - \eta^2) \frac{d\Phi}{d\eta} \right] + c^2(\xi^2 - \eta^2) - m^2 \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} + \lambda_{mn}(c) - \lambda_{mn}(c) = 0 \quad (D.5)
\]

which can be separated into

\[
\frac{d}{d\xi} \left[ (\xi^2 - 1) \frac{dR_{mn}}{d\xi} \right] - \left[ \lambda_{mn}(c) - c^2\xi^2 + \frac{m^2}{\xi^2 - 1} \right] R_{mn} = 0 \quad (D.6)
\]

\[
\frac{d}{d\eta} \left[ (1 - \eta^2) \frac{dS_{mn}}{d\eta} \right] + \left[ \lambda_{mn}(c) - c^2\eta^2 - \frac{m^2}{1 - \eta^2} \right] S_{mn} = 0 \quad (D.7)
\]

where \( R_{mn} \) and \( S_{mn} \) are the radial and angular prolate spheroidal functions which are defined as

\[
S_{mn}^{(1)}(c, \eta) = \sum_{r=0,1}^{\infty} d_{r}^{mn}(c) P_{m+r}^{m}(\eta) \quad (D.8)
\]

\[
S_{mn}^{(2)}(c, \eta) = \sum_{r=-\infty}^{\infty} d_{r}^{mn}(c) Q_{m+r}^{m}(\eta) \quad (D.9)
\]

\[
P_{mn}^{(p)}(c, \xi) = \left\{ \sum_{r=0,1}^{\infty} \frac{(2m+r)!}{r!} a_{r}^{mn}(c) \right\}^{-1} \left( \frac{\xi^2 - 1}{\xi^2} \right)^{m/2}
\]

\[
\cdot \sum_{r=0,1}^{\infty} \frac{(2m+r)!}{r!} a_{r}^{mn}(c) Z_{m+r}^{(p)}(c\xi) \quad (D.10)
\]
where \( P \) and \( Q \) are the first and second kind associated Legendre functions and

\[
Z_r^{(p)}(z) = \begin{cases} 
  j_r(z) & p = 1 \\
  n_r(z) & p = 2 \\
  h_r^{(1)}(z) & p = 3 \\
  h_r^{(2)}(z) & p = 4 
\end{cases}
\]

, the spherical Bessel functions.

### D.1 Determination of eigenvalues and expansion coefficients

The expansion coefficients \( d_r^{mn}(c) \) in the prolate spheroidal functions are determined by substituting (D.8) into (D.7) and applying various recursion formulas for the associated Legendre functions yields a recursion relationship for the coefficients \( d_r^{mn}(c) \).

The derivation of the recursion relationship follows.

\[
\begin{align*}
\frac{d}{d\eta} \left[ (1 - \eta^2) \sum_{r=0,1}^{\infty}' d_r^{mn}(c) P_{m+r}^m(\eta) \right] & \quad + \left[ \lambda_{mn}(c) - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right] \sum_{r=0,1}^{\infty}' d_r^{mn}(c) P_{m+r}^m(\eta) = 0 \\
\sum_{r=0,1}^{\infty}' d_r^{mn}(c) \left[ (1 - \eta^2) \frac{d^2}{d\eta^2} P_{m+r}^m(\eta) - 2 \eta P_{m+r}^m(\eta) - \frac{m^2}{1 - \eta^2} P_{m+r}^m(\eta) \\
+ \left( \lambda_{mn}(c) - c^2 \eta^2 \right) P_{m+r}^m(\eta) \right] & = 0
\end{align*}
\]

(D.11)
Because the first three terms are components of the associated Legendre's equation, the summation can be further simplified to

\[
\sum_{r=0,1}^{\infty} d_{rn}^m(c) (m + r)(m + r + 1) P^m_{m+r}(\eta)
- \lambda_{mn} \sum_{r=0,1}^{\infty} d_{rn}^m(c) P^m_{m+r}(\eta)
+ c^2 \sum_{r=0,1}^{\infty} d_{rn}^m(c) \eta^2 P^m_{m+r}(\eta) = 0
\] (D.12)

\(\eta^2 P^m_{m+r}(\eta)\), from the third term of (D.12), is expanded, using equation 8.5.3 from \[32\], as

\[
\eta^2 P^m_{m+r}(\eta) = \frac{\eta}{2m + 2r + 1} \left[ (r + 1) P^m_{m+r+1}(\eta) + (2m + r) P^m_{m+r-1}(\eta) \right]
\]

\[
= \frac{(r + 1)}{(2m + 2r + 1)(2m + 2r + 3)} \left[ (r + 2) P^m_{m+2}(\eta) + (2m + r + 1) P^m_{m+r}(\eta) \right]
\]

\[
+ \frac{(2m + r)}{(2m + 2r + 1)(2m + 2r - 1)} \left[ r P^m_{m+r}(\eta) + (2m + r - 1) P^m_{m+r-2}(\eta) \right]
\]

\[
= \frac{(r + 1)(r + 2)}{(2m + 2r + 1)(2m + 2r + 3)} P^m_{m+2}(\eta)
+ \frac{(2m + r)(m + r + 1) - 2m^2 - 1}{(2m + 2r - 1)(2m + 2r + 3)} P^m_{m+r}(\eta)
+ \frac{(2m + r)(2m + r - 1)}{(2m + 2r + 1)(2m + 2r - 1)} P^m_{m+r-2}(\eta)
\]

(D.13)
Substituting (D.13) into (D.12) and combining terms with similar indices of summation lead to

\[
\sum_{r=0,1}^{\infty} d_{r}^{mn}(c) \left( \frac{(2m+r)(2m+r-1)c^2}{(2m+2r+1)(2m+2r-1)} P_{m+r-2}^m(\eta) \right) \\
+ \sum_{r=0,1}^{\infty} d_{r}^{mn}(c) \left[ (m+r)(m+r+1) - \lambda_{mn}(c) \right] \\
+ \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} c^2 \right] P_{m+r}^m(\eta) \\
+ \sum_{r=2,3}^{\infty} d_{r-2}^{mn}(c) \left( \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)} P_{m+r}^m(\eta) \right) = 0
\]

(After combining the terms with similar indices of summation in the above equation and neglecting the trivial solutions \( P_{m+r}^m(\eta) = 0 \) and \( d_{r}^{mn}(c) = 0 \) the recursion formula is)

\[
\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)} d_{r+2}^{mn}(c) \\
+ \left[ (m+r)(m+r+1) - \lambda_{mn}(c) \right] \\
+ \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} c^2 \right] d_{r}^{mn}(c) \\
+ \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)} d_{r-2}^{mn}(c) = 0
\]

(D.14)
The limit as $r$ approaches infinity of $\frac{d_{mn}^r}{d_{r+2}^{mn}}$ is found to either increase as $-4r^2/c^2$ or decrease as $-c^2/4r^2$. The second solution is chosen, under which condition (D.8) converges absolutely for all finite $\eta$.

To determine the eigenvalues, $\lambda_{mn}(c)$, a transcendental equation is formed from (D.15). The process begins by introducing some abbreviations

$$\alpha_r = \frac{(2m + r + 2)(2m + r + 1)}{(2m + 2r + 5)(2m + 2r + 3)}c^2$$  \hspace{1cm} (D.16)

$$\beta_r = (m + r)(m + r + 1) + \frac{2(m + r)(m + r + 1) - 1}{(2m + 2r - 1)(2m + 2r + 3)}c^2$$  \hspace{1cm} (D.17)

$$\gamma_r = \frac{r(r - 1)}{(2m + 2r - 3)(2m + 2r - 1)}c^2$$  \hspace{1cm} (D.18)

Thus, (D.15) can be rewritten as

$$\alpha_0 d_{2}^{mn}(c) + [\beta_0 - \lambda_{mn}(c)] d_0^{mn}(c) = 0$$

$$\alpha_1 d_{3}^{mn}(c) + [\beta_1 - \lambda_{mn}(c)] d_1^{mn}(c) = 0$$

$$\alpha_r d_{r+2}^{mn}(c) + [\beta_r - \lambda_{mn}(c)] d_r^{mn}(c) + \gamma_r d_{r-2}^{mn}(c) = 0 \hspace{1cm} (r \geq 2) \hspace{1cm} (D.19)$$

Dividing the third term in (D.19) by $d_r^{mn}(c)$ leads to the following

$$\alpha_k \frac{d_{r+2}^{mn}(c)}{d_r^{mn}(c)} + \beta_r - \lambda_{mn}(c) + \gamma_r \frac{d_{r-2}^{mn}(c)}{d_r^{mn}(c)} = 0$$  \hspace{1cm} (D.20)

Introducing the convenient substitutions

$$N_r^m = -\alpha_{r-2} \frac{d_{r}^{mn}(c)}{d_{r-2}^{mn}(c)} \hspace{1cm} \gamma_r^m = \beta_r \hspace{1cm} \beta_r^m = \gamma_r \alpha_{r-2}$$  \hspace{1cm} (D.21)

allows (D.20) to be written as

$$N_{r+2}^m = \gamma_r - \lambda_{mn}(c) - \frac{\beta_r^m}{N_r^m} \hspace{1cm} (r \geq 2) \hspace{1cm} (D.22)$$
Inverting (D.22) to solve for \( N_r \) yields

\[
N_r^m = \frac{\beta_r}{\gamma_r - \lambda_{mn}(c) - N_{r+2}}
\] (D.23)

Note that the above equation for \( N_r^m \) contains a minus sign not present in (3.1.7) of [25]. It is required to achieve the same recursion relations given by Flammer and is independently present in [33].

By iteration of (D.22) and introducing the condition \( d_{mn}^r = 0 \) for \( r < 0 \) one obtains the finite continued fraction

\[
U_1(\lambda_{mn}) = N_{r+2}^m = \gamma_r^m - \lambda_{mn} - \frac{\beta_r^m}{\gamma_{r-2}^m - \lambda_{mn} - \gamma_{r-4}^m - \lambda_{mn} - \cdots}
\] (D.24)

On the other hand, iteration of (D.23) and introducing the condition \( \lim_{r \to \infty} N_r^m = 0 \), one obtains the convergent infinite continued fraction

\[
U_2(\lambda_{mn}) = N_{r+2}^m = \frac{\beta_{r+2}^m}{\gamma_{r+2}^m - \lambda_{mn}(c) - \gamma_{r+4}^m - \lambda_{mn}(c) - \gamma_{r+6}^m - \lambda_{mn}(c) - \cdots}
\] (D.25)

Setting (D.24) and (D.25) equal and bringing (D.25) to the left side results in the transcendental equation

\[
U_1(\lambda_{mn}) - U_2(\lambda_{mn}) = 0
\] (D.26)

which can be solved for the eigenvalues \( \lambda_{mn} \).

The relation between the expansion coefficients \( d_{mn}^r(c) \) is determined from (D.19) and \( \lambda_{mn}(c) \). Stratton and Chu’s normalization, that the prolate spheroidal angular functions reduce to the corresponding associated Legendre functions when \( c = 0 \), specifies the values of the expansion coefficients completely. Details of this normalization can be found in [24] and [25].
APPENDIX E

FLAMMER’S PROLATE SPHEROIDAL GREEN’S FUNCTION

The initial procedure in deriving Flammer’s Prolate Spheroidal Green’s Function is the same as in section [IV.1] Thus, beginning with [IV.3], reproduced here,

\[
\frac{1}{f^2(\xi^2 - \eta^2)} \left\{ \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1) \frac{\partial G}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial G}{\partial \eta} \right] + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 G}{\partial \phi^2} \right\} + k^2 G = -\frac{1}{f^3(\xi^2 - \eta^2)} \delta(\xi - \xi') \delta(\eta - \eta') \delta(\phi - \phi')
\]

expand \(G\) as

\[
G(\xi, \eta, \phi, \xi', \eta', \phi') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi')
\]

Substituting (E.1) into (IV.3), integrating with respect to \(\xi\) over \((\xi' - \epsilon, \xi' + \epsilon)\) and taking the limit as \(\epsilon \to 0\) yields

\[
(\xi^2 - 1) \frac{\partial G}{\partial \xi} \bigg|_{\xi=\xi'}^{\xi=\xi'} = -\frac{1}{f} \delta(\eta - \eta') \delta(\phi - \phi')
\]

Performing the derivative of \(G\) with respect to \(\xi\) and evaluating at the limits of integration leads to

\[
(\xi^2 - 1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi') \cdot \left[ R_{mn}^{(1)}(c, \xi') R_{mn}^{(4)}(c, \xi') - R_{mn}^{(1)}(c, \xi') R_{mn}^{(4)}(c, \xi') \right]
\]

\[
= -\frac{1}{f} \delta(\eta - \eta') \delta(\phi - \phi')
\]
The Wronskian, (IV.12), allows the above to be simplified to

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} A_{mn} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi') = -jk\delta(\eta - \eta')\delta(\phi - \phi') \quad (E.4)$$

Both sides are multiplied by $S_{mn}(c, \eta) \cos m\phi$ and integrated with respect to $\eta$ and $\phi$ over $(-1, 1)$ and $(-\pi, \pi)$ respectively. Because the cosines vanish when $m = 0$, the integrals need to be analyzed separately for $m = 0$ and $m \neq 0$. When $m = 0$,

$$A_{mn} = -\frac{jk}{2\pi N_{mn}} \quad (E.5)$$

In the case that $m \neq 0$,

$$A_{mn} = -\frac{jk}{\pi N_{mn}} \quad (E.6)$$

Combining these two cases yields

$$A_{mn} = -jk \frac{2 - \delta_{0m}}{2\pi N_{mn}} \quad (E.7)$$

where $\delta_{0m}$ is the Kronecker delta. Thus,

$$\frac{e^{-jk|r-r'|}}{4\pi |r-r'|} = \frac{jk}{2\pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2 - \delta_{0m}}{N_{mn}} S_{mn}(c, \eta) S_{mn}(c, \eta') \cos m(\phi - \phi')$$

$$\cdot R_{mn}^{(1)}(c, \xi_<) R_{mn}^{(4)}(c, \xi_> ) \quad (E.8)$$

Note that the derivation above is different from that found in [25] in the usage of $R_{mn}^{(4)}$ instead of $R_{mn}^{(3)}$ to satisfy the radiation condition. This is because Flammer has chosen to express his time variation with $e^{-i\omega t}$ instead of $e^{j\omega t}$.
REFERENCES


