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Homomorphisms of Graphs

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Abstract

Understanding the structure of graphs is fundamental to advances in many areas of graph theory, as well as in many applications. In many cases, an analysis of the structure of graphs follows one of two approaches; either many structural properties are considered over a restricted class of graphs, or a particular structural property is considered over many classes of graphs. Both approaches will be considered in this dissertation.

Graphs which do not contain a clique of size r, i.e., K_r-free graphs, are of fundamental importance in the area of extremal graph theory. Many results have been obtained about dense triangle-free graphs, but not much is known about dense K_r-free graphs when $r \geq 4$. Of particular interest are results pertaining to independent sets, colorings, and homomorphisms. Another method of describing the structure of a graph is through the concept of a role assignment. A role assignment is a mapping r from a graph G to a graph G_R, i.e., $r : G \to G_R$, by a surjective labeling of the vertices of G with the vertices of G_R, i.e., $r : V(G) \to V(G_R)$. For $S \subseteq V(G)$, we define $r(S) = \{r(s) : s \in S\}$. Each role assignment must satisfy the following condition:

$$\forall v \in V(G), \quad r(N(v)) = N(r(v)).$$

Not much is known about the classes of graphs for which meaningful results about role assignments can be obtained.

The goal of this dissertation is to address the following two issues. First, what results can be obtained about the structure of dense K_r-free graphs? Secondly, what additional properties of a graph are necessary in order to obtain results about role assignments? For which classes of graphs is the problem of determining role assignments tractable, and for which classes of graphs is it difficult?

In regard to the first issue, a structural theorem is obtained which allows many properties of dense K_r-free graphs to be described in terms of the properties of dense triangle-free graphs. In particular, we determine a minimum degree condition which will permit the vertices of a dense K_r-free graphs to be partitioned into sets $S_0, S_1, \ldots, S_{r-3}$, where each set S_i, $1 \leq i \leq r-3$, is independent, and the graph induced by the set S_0 is a dense triangle-free graph. Two results regarding the size of large independent sets in K_r-free graphs are obtained, for the cases $r = 4$ and $r \geq 5$ respectively. The binding number of dense K_r-free graphs is also considered. We improve previously published results regarding the binding number by giving a construction of K_4-free graphs with large binding number, and proving stricter upper bounds on the binding number of dense K_r-free graphs.

In regard to the second issue, role assignments are considered for the classes of chordal graphs, strongly chordal graphs, and trees. A necessary condition is given for a chordal graph to have a role assignment to K_r for some value of r. For chordal graphs with small clique size, as well as strongly chordal graphs, this necessary condition is shown to be sufficient. Additionally, role assignments of the d-dimensional hypercube, along with graphs of similar structure, are considered.

DEDICATION

This is dedicated to Emily Alice Lyle.

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Chapter 1

Introduction

1. Graph Colorings

For much of the last century, the most well known problem in graph theory was the Four Color Problem. The Four Color Problem originated in the field of cartography, and is the problem of determining the minimum number of colors needed to color the regions of a map such that regions which have a common boundary receive different colors. Every map can be represented by a graph such that each region of the map corresponds to a vertex, and two vertices are adjacent, i.e., $(u, v) \in E(G)$, if their respective regions have a common boundary. A coloring of the map then corresponds to a *proper coloring* of the graph G, a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V(G)$, where each set $V_i \subseteq V(G)$ is an *independent set*, i.e., if $u, v \in V_i$ then $(u, v) \notin E(G)$. Graphs which are associated with maps in this way are called *planar* graphs. The Four Color Problem can be restated as follows: Does every planar graph have a proper coloring with at most four colors? The Four Color Problem was finally resolved in 1976 by Appel and Haken (and Koch) [**5, 6**]. However, the more general problem of determining the minimum number of colors required to properly color any graph G, a quantity known as the *chromatic number* $\chi(G)$, remains an active area of research. This dissertation deals with generalizations of colorings and related structural questions in graphs.

The easiest lower bound on the chromatic number can be obtained by considering the cliques of a graph. A *clique* of size r, denoted K_r, is a set of r vertices, every pair of which are adjacent. The *clique number* of a graph G, denoted $\omega(G)$, is the maximum value of r such that the graph contains a K_r. It is trivial to see that $\chi(G) \geq \omega(G)$. However, the lack of a large clique does not imply a small chromatic number. A clique K_3 of size 3 is commonly referred to as a *triangle*. Mycielski in [**44**] gave a simple construction to obtain infinitely many relatively sparse graphs for which the maximum clique size is 2, i.e., triangle-free graphs, but which have arbitrarily large chromatic number.

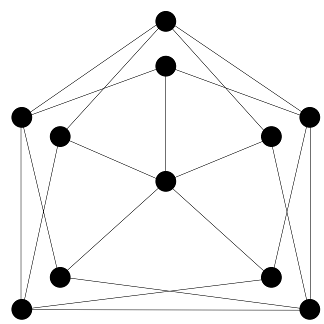

Figure 1.1. A triangle-free graph with chromatic number 4

The chromatic number is a very important parameter in describing the structure of a graph. As an example, consider an H-free graph G, that is, H is not a subgraph of G. In general, we will let n represent the cardinality of the vertex set, i.e., $n = |V(G)|$. The Erdős–Stone theorem [**20**] states that the maximum number of edges that such a graph can contain is dependent upon the chromatic number of H, namely $\left(\frac{r-2}{r-1} + o(1)\right)\binom{n}{2}$ edges, where $\chi(H) = r$.

This is a generalization of the seminal result of Turán [**57**], that any K_r-free graph can contain at most $\left(\frac{r-2}{r-1}\right)\binom{n}{2}$ edges. The special case for $r = 3$ was proved earlier by Mantel [**43**], and the bound is obtained uniquely by balanced complete bipartite graphs. The contrast between the unique and highly structured extremal triangle-free graph, and constructions of less dense triangle-free graphs of arbitrarily large chromatic number is interesting. This same contrast exists for K_r-free graphs for any value of r, and Chapters 2 and 3 examine this contrast. The *open neighborhood* of a vertex v is the set $N(v) = \{u : (u, v) \in E(G)\}$. The *degree* of a vertex, denoted $d(v)$, is the cardinality of its open neighborhood, $|N(v)|$. The *minimum degree* of a graph, denoted $\delta(G)$, is equal to the value of the minimum degree over all vertices in $V(G)$. In this thesis, we will be concerned with K_r-free graphs which have large minimum degree.

2. Graph Homomorphisms

Proper colorings of graphs can be generalized by considering the minimality of a coloring. A proper coloring $\Pi = \{V_1, \ldots, V_k\}$ is said to be *complete* if it is minimal in the sense that no two classes V_i and V_j can be combined to form a new partition which remains a proper coloring. Thus, a proper coloring is complete if there exists an edge between any two classes V_i and V_j.

A *homomorphism* of a graph G to a graph H is a mapping f from G to H, i.e., $f : G \rightarrow H$, by a surjective labeling of the vertices of G with the vertices of H, i.e., $f : V(G) \rightarrow V(H)$. Each graph homomorphism satisfies the following conditions, that $(u, v) \in E(G)$ implies $(f(u), f(v) \in E(H)$, and $(w, x) \in E(H)$ implies that there exists vertices $u, v \in V(G)$ such that $(u, v) \in E(G)$ with $f(u) = w$ and $f(v) = x$. A complete, proper k-coloring of a graph G is exactly a homomorphism from G to the complete graph K_k. For this reason, a homomorphism from G to H is alternatively described as an H-coloring of G. Graph homomorphisms allow for a description of the structure in terms of the independent sets of a graph.

3. Domination and Coloring

The basic ideas of graph colorings and graph homomorphisms (as related to independent sets) can be extended to other important sets, such as dominating sets. A *dominating set* in a graph G is a subset of the vertices, $S \subseteq V(G)$, such that any vertex which is not in S has a neighbor in S. A partition of the vertex set into k dominating sets is called a *domatic k-partition*, and was first considered by Cockayne and Hedetniemi in [**15**]. The *domatic number* of a graph, denoted by $dom(G)$, is the largest value of k for which G has a domatic k-partition. The domatic number of a graph without isolates is at least two, as every such graph has another dominating set in the complement of a minimal dominating set, as shown by Ore in [**45**]; the domatic number is clearly at most $\delta(G) + 1$.

Many types of dominating sets have been discussed in the literature (see Haynes, Hedetniemi, and Slater [**29, 28**]), but we consider only two: *total dominating sets* and *independent dominating sets*. A *total dominating set* is a dominating set S such that every vertex in $V(G)$ has a neighbor in S. In particular, every vertex in S has a neighbor in S. A partition of the vertex set into k total dominating sets is called a *total domatic k-partition*, and was first considered by Cockayne, Dawes, and Hedetniemi in [**13**]. The *total domatic number* of a graph G is the largest value of k for which G has a total domatic k-partition.

An *independent dominating set* is a dominating set which is also independent. A partition of the vertex set into k independent dominating sets is called an *idomatic k-partition*, and was first considered by Cockayne and Hedetniemi in [**14**]. The *idomatic number* of a graph is the largest value of k for which G has a idomatic k-partition. One can note that not all graphs can be partitioned into even two independent dominating sets.

In the same way that the concept of a proper coloring of a graph can be generalized by graph homomorphisms, *role assignments* generalize the concepts of idomatic partitions and total domatic partitions. Role assignments will be discussed further in Chapters 4 and 5.

4. Overview

A fundamental question of graph theory is the following: Given a class of graphs, what can we determine about their structure? Using the concepts of colorings, role assignments, homomorphisms, and independent sets, we examine structural results of several classes of graphs, as well as the tractability of these problems.

In Chapters 2 and 3, we consider questions in extremal graph theory. In particular, we consider dense graphs which contain no clique of size r. In the Chapter 2, we consider dense graphs with no cliques of size 4. We find conditions on the minimum degree which permit a partition of the vertices of a K_4-free graph into an independent set and a triangle-free graph. This allows many results from triangle-free graphs to be extended to K_4-free graphs. This includes improved bounds on the binding number of K_4-free graphs. In Chapter 3, we extend many of the results from Chapter 2 to K_r-free graphs.

In Chapter 4, we consider role assignments on several classes of graphs, namely chordal graphs, strongly chordal graphs, and trees. For chordal graphs with small minimum degree, we show that any $\chi(G)$ coloring can be transformed into a role assignment, and show that the question of determining if there is a role assignment to K_r for any value of r is NP-complete in general for chordal graphs. For strongly chordal graphs, we determine a polynomial-time algorithm for finding a role assignment to K_k. For any role graph G_R, we consider the question of determining whether a tree T has a role assignment to G_R.

In Chapter 5, we consider the problem of role assignments of cartesian products of graphs. In particular, we focus on the cartesian products of bipartite graphs, as we are interested in role assignments of the d-dimensional hypercube. Many results for cartesian products of bipartite graphs are obtained regarding the role assignments to K_3. Role assignments to the complete graphs K_k, $k > 3$, are also considered.

In Chapter 6, we review these results, as well as some of the remaining unanswered questions.

5. Definitions and Notations

We will assume a familiarity with the basic definitions of introductory graph theory, and we refer the reader to the textbook by West [**58**]. This section is intended to serve as a reference for the definitions and notation used in this dissertation. The well-informed reader is advised to skip this section, and refer back to it in the event that clarification is needed. Definitions for less common terms are additionally contained at the point at which that term is first encountered.

5.1 *Basic Terminology*

A *graph* $G = (V, E)$ is an ordered pair, comprised of a vertex set of the graph G, denoted $V(G)$, with $n \geq 1$ elements, and an edge set of the graph G, denoted $E(G)$, with $m \geq 1$ elements. An element of the vertex set is referred to as a *vertex*. An *undirected edge* $e \in E(G)$ is an unordered pair of vertices, denoted $e = (u, v) \in E(G)$ where u and v are vertices in $V(G)$. A *directed edge* $e \in E(G)$ is an ordered pair of vertices, denoted $e = (u, v) \in E(G)$ where u and v are vertices in $V(G)$. Unless otherwise noted, all edges will be assumed to be undirected.

We say that u is *adjacent* to v if the edge $(u, v) \in E(G)$, and also that u is a *neighbor* of v. Both u and v are referred to as *endpoints* of the edge $(u, v) \in E(G)$. We will restrain ourselves to graphs with no multiple edges, that is, for edges (unordered pairs) $(u, v), (w, x) \in E(G)$, $(u, v) \neq (w, x)$. A graph is said to have a *loop* on a vertex v if $E(G)$ contains the edge (v, v). In Chapters 2 and 3, we assume that all graphs do not contain multiple edges or loops. However, in Chapters 4 and 5, we allow the possibility that a graph may contain a loop.

The *neighborhood of a vertex* $v \in V(G)$, denoted $N(v)$, is the set defined by $N(v) = \{u : (u, v) \in E(G)\}$. We note that in the case of graphs with no loops, this is more commonly called the *open neighborhood* of a vertex. If a graph has a loop at a vertex v, then the neighborhood of v will contain v itself. The *nonneighborhood* of a vertex v is defined as the set $V(G) - N(v)$, and we denote this set as $H(v)$. We define the *degree* of a vertex v, denoted $d(v)$, as the cardinality of the neighborhood of v, i.e., $d(v) = |N(v)|$. Additionally, we define the degree of a vertex v *into* a set S as the cardinality of the set as $\{u : (u, v) \in E(G), u \in S\}$. A vertex with degree 0 is called an *isolate*. The *minimum degree* of a graph G, denoted $\delta(G)$, is the minimum degree over all vertices in the vertex set $V(G)$.

We define the *neighborhood of a set* $S \subseteq V(G)$, denoted $N(S)$, as the set $N(S) = \bigcup_{v \in S} N(v)$. The *binding number* of a graph G, denoted $bind(G)$, is given by

$$bind(G) = \min_{\substack{S \subseteq V(G) \\ N(S) \neq V(G)}} \frac{|N(S)|}{|S|}.$$

5.2 *Subgraphs and simple families of graphs*

An *isomorphism* from a graph G to a graph H is a bijection f which maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$, such that $(u, v) \in E(G)$ is mapped to $(f(u), f(v)) \in E(H)$. The graphs G and H are then said to be *isomorphic* if there is an isomorphism from G to H.

Consider a graph $G = (V, E)$. If H is a graph with vertex set $V(H) \subseteq V(G)$, and edge set $E(H) \subseteq E(G)$, such that $(u, v) \in E(H)$ implies $u, v \in V(H)$, then H is called a *subgraph* of G. If every edge in $E(G)$ which joins two vertices in $V(H)$ is also contained in $E(H)$, then H is an *induced subgraph*. For a graph G and a subset of the vertices, say $S \subseteq V(G)$, we say that the graph *induced* by S, denoted $\langle S \rangle$, is the induced subgraph H where $V(H) = S$. We say that a graph G *contains* a graph H if H is isomorphic to a subgraph of G. A graph G is *H-free* if G does not contain H. Additionally, a graph G which is H-free is *maximally H-free* if any graph G', where $V(G') = V(G)$ and $E(G') = E(G) \cup \{e\}$, for any edge e not contained in $E(G)$, necessarily contains H.

The *complete graph* on r vertices, is the graph G such that $|V(G)| = r$ and $E(G) = \{(u, v) : u, v \in V(G), u \neq v\}$. The *complete graph with loops* on r vertices is the graph G such that $|V(G)| = r$ and $E(G) = \{(u, v) : u, v \in V(G)\}$. A *clique* of size r in a graph G is a complete subgraph on r vertices, and is denoted as K_r. In particular, the clique of size 3, denoted K_3, is called a *triangle*. The *clique number* of a graph, denoted $\omega(G)$, is the size of the largest clique in the graph G.

The *path* on d vertices, denoted P_d, is a graph with vertex set $V(P_d) = \{v_1, v_2, \ldots, v_d\}$ and edge set $E(P_d) = \{(v_i, v_{i+1}) : 1 \leq i \leq d - 1\}$. A *cycle* on d vertices or d-cycle, denoted

C_d, is a graph with vertex set $V(C_d) = \{v_1, v_2, \ldots, v_d\}$ and edge set $E(C_d) = \{(v_i, v_{i+1}) : 1 \le i \le d-1\} \cup \{(v_1, v_d)\}$. The *wheel* on $d+1$ vertices or d-wheel, denoted W_d, is a graph with vertex set $V(W_d) = \{v_1, v_2, \ldots, v_{d+1}\}$ and edge set $E(W_d) = \{(v_i, v_{i+1}) : 1 \le i \le d-1\} \cup \{(v_1, v_d)\} \cup \{(v_i, v_{d+1}) : 1 \le i \le d\}$.

The *Möbius ladder* on d vertices (d even), denoted M_d, is the graph with vertex set $V(M_d) = \{v_1, v_2, \ldots, v_d\}$ and edge set $E(M_d) = \{(v_i, v_{i+1}) : 1 \le i \le d-1\} \cup \{(v_1, v_d)\} \cup \{(v_i, v_{i+d/2} : 1 \le i \le d/2\}$. The *$d$-dimensional hypercube*, denoted Q_d, is the graph with 2^d vertices corresponding to the set of binary strings of length d, where two vertices are adjacent if their corresponding strings differ in only one coordinate.

5.3 *Subsets of the vertices and edges*

An *independent set* in a graph G is a subset of the vertices $S \subseteq V(G)$ such that if $u, v \in S$, $(u, v) \notin E(G)$. A graph G is *bipartite* if two disjoint independent sets S_1 and S_2 can be specified, such that $S_1 \cup S_2 = V(G)$. The sets S_1 and S_2 are called the *partite* sets. In general, a graph G is *k-partite* if k disjoint independent sets $S_1, S_2, \ldots, S_k$ can be specified, such that $S_1 \cup S_2 \cup \cdots \cup S_k = V(G)$. A k-partite graph G is *complete* if for $1 \le i \le r$, $v \in V_i$ implies that $(u, v) \in E(G)$ if $u \notin V_i$. The complete bipartite graph, with partite sets S_1 and S_2, where $|S_1| = s$ and $|S_2| = t$, is denoted $K_{s,t}$. A k-partite graph G with partite sets $S_1, S_2 \ldots, S_r$ is *balanced* if $|S_1| = |S_2| = \cdots = |S_r|$.

A *dominating set* in a graph G is a subset of the vertices, $S \subseteq V(G)$, such that any vertex which is not in S has a neighbor in S. A partition of the vertex set into k dominating sets is called a *domatic k-partition*. The *domatic number* of a graph G, denoted by $dom(G)$ (to avoid confusion with the notation for the degree of a vertex), is the largest value of k for which G has a domatic k-partition. A *total dominating set* in a graph G is a dominating set S such that every vertex in $V(G)$ has a neighbor in S. A partition of the vertex set into k total dominating sets is called a *total domatic k-partition*. The *total domatic number* of a graph G is the largest value of k for which G has a total domatic k-partition. An *independent dominating set* in a graph G is a dominating set which is independent. A partition of the vertex set into k independent dominating sets is called an *idomatic k-partition*, . The *idomatic number* of a graph G is the largest value of k for which G has a idomatic k-partition.

A *cutset* of a graph G is a subset of the vertices $S \subseteq V(G)$ such that the graph induced by $V(G) - S$ has at least one more component then the graph G.

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that every endpoint of an edge in M is distinct. A *perfect matching* in a graph G is a set of edges $M^* \subseteq E(G)$ such that M^* is a matching, and every vertex in $V(G)$ appears as an endpoint of an edge in M^*.

5.4 *Colorings and Mappings of Graphs*

A *k-coloring* of a graph is a labeling of the vertices of a graph with the nonzero, positive integers less than k, i.e., $f : \{1, 2, \ldots, k\} \rightarrow V(G)$. Each integer is called a *color*, as the goal is a partition of the vertices, and the numerical value of each label is unimportant. This partitions the vertex set into *color classes*, and the partition can be written as $\Pi = \{V_1, V_2, \ldots, V_k\}$. A k-coloring f of a graph G is called *proper* if for any edge $(u, v) \in E(G)$, $f(u) \neq f(v)$. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum value of k for a proper k-coloring of G. A proper k-coloring is called *complete* if for $1 \leq i < j \leq k$, there exists vertices u and v such that $(u, v) \in E(G)$, where $f(u) = i$ and $f(v) = j$.

A *graph homomorphism* is a mapping f from a graph G (traditionally without loops) to a graph H, i.e., $f : G \rightarrow H$, by a surjective labeling of the vertices of G with the vertices of H, i.e., $f : V(G) \rightarrow V(H)$. Each graph homomorphism satisfies the following conditions, that $(u, v) \in E(G)$ implies $(f(u), f(v)) \in E(H)$, and $(w, x) \in E(H)$ implies that there exists vertices $u, v \in V(G)$ such that $(u, v) \in E(G)$ with $f(u) = w$ and $f(v) = x$. The graph G is said to be *homomorphic* to H if there is an homomorphism from G to H.

A *role assignment* is a mapping r from a graph G (allowing loops) to a graph G_R (allowing loops), i.e., $r : G \rightarrow G_R$, by a surjective labeling of the vertices of G with the vertices of G_R, i.e., $r : V(G) \to V(G_R)$. For $S \subseteq V(G)$, we define $r(S) = \{r(s) : s \in S\}$. Each role assignment must satisfy the following condition:

$$\forall v \in V(G), \quad r(N(v)) = N(r(v)).$$

The graph G_R is referred to as the *role graph*, and the vertices of G_R are referred to as *roles*.

5.5 *Subclasses of Graphs*

The *chordal graphs* are defined as the class of graphs which do not have any induced subgraph isomorphic to C_k, for any $k \geq 4$. Two important concepts for chordal graphs are those of simplicial vertices and perfect elimination orders. A *simplicial vertex* v is a vertex such that $\langle N(v) \rangle$ is a clique. A *simplicial elimination ordering*, also referred to as a *perfect elimination ordering* (PEO), is an ordering on the vertices of the graph, such that the vertex v_1 is a simplicial vertex, and upon the removal of the vertices $v_1, v_2, \ldots, v_{k-1}$, the vertex v_k is a simplicial vertex in the remaining graph.

A *k-sun*, or a *k-trampoline*, is the graph obtained by taking an cycle of length $2k$, where k is at least three, and adding edges to form a clique of size k of all the vertices of even index. *Strongly chordal graphs* are chordal graphs with the additional property that every even cycle of length six or larger has a *strong chord*, a chord where the distance along the cycle between the two endpoints of the chord is odd. A *strong elimination ordering* (SEO) is a perfect elimination ordering with the additional requirement that for $i < j < k < l$, if (v_i, v_k), (v_i, v_ℓ), $(v_j, v_k) \in E(G)$, then $(v_j, v_\ell) \in E(G)$.

A graph G is said to be *connected* if between any two vertices $u, v \in V(G)$ there exists a path. The *maximal connected subgraphs* of G are those subgraphs of G which are connected, and are not contained in any other connected subgraph. The maximal connected subgraphs of G are also referred to as the *components* of a graph G.

A graph G is a *tree* if it is connected and does not contain any cycle. A *rooted tree* is a tree in which one vertex has been designated the *root*. The root allows us to define other relationships between vertices. Let v_r be the root, and consider a vertex u, along with the unique induced path P which begins at v_r and terminates at u. The *parent* of u is its neighbor in P. The other neighbors of u in G are called its *children*. Let w be any vertex such that the unique path which begins at v_r and terminates at w contains the vertex u. Then w is called a *descendant* of u. The *subtree induced* by u, denoted $T_{\langle v \rangle}$ is defined to be the subgraph of G induced by the set of descendants of u.

In many cases, it is important to visit every vertex of a tree. The method or process in which this is accomplished is called a *traversal* of a tree. A *postorder labeling* is the order of traversal of a rooted tree in which the subtrees generated by the children of a vertex are recursively postorder labeled before the vertex is labeled.

5.6 *Graph Products*

For graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the *cartesian product* $G \Box H$ defines a new graph where $V(G \Box H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and $E(G \Box H)$ $= \{((g, h), (g', h'))\,|g = g', (h, h') \in E(H)$ or $h = h', (g, g') \in E(G)\}$.

The *lexicographic product* of the graph $G = (V(G), E(G))$ with $H = (V(H), E(H))$, denoted $G[H]$, is the graph with vertex set $V(G[H]) = \{(g, h) : g \in V(G), h \in V(H)\}$, where $((g, h), (g', h')) \in E(G[H])$ if and only if either $(g, g') \in E(G)$ or $g = g'$ and $(h, h') \in E(H)$.

The *tensor product* of the graph $G = (V(G), E(G))$ with $H = (V(H), E(H))$, denoted $G \otimes H$, is the graph with vertex set $V(G \otimes H) = \{(g, h) : g \in V(G), h \in V(H)\}$, where $((g, h), (g', h')) \in E(G \otimes H)$ if and only if $(g, g') \in E(G)$ and $(h, h') \in E(H)$.

The *corona product* of the graph $G = (V(G), E(G))$ with $H = (V(H), E(H))$, denoted $G \circ H$, is the graph with vertex set $V(G \circ H) = S_1 \cup S_2$, where $S_1 = V(G)$ and $S_2 = \{(g, h) : g \in V(G), h \in V(H)\}$. The edges of $G \circ H$ are such that the graph induced by S_1 is isomorphic to G. Similarly, fixing a vertex $g_0 \in V(G)$, the graph induced by the set of vertices of the form $\{(g_0, h) : h \in V(H)\}$ is isomorphic to H. Finally, all edges of the form $\{(g_0, (g_0, h)) : h \in H\}$ are contained in $E(G \circ H)$.

The *join* of the graph $G = (V(G), E(G))$ with $H = (V(H), E(H))$, denoted $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$, where $E(G \vee H) = E(G) \cup E(H) \cup \{(u, v) : u \in V(G), v \in V(H)\}$.

5.7 *Concluding notes*

A graph is loosely said to be *dense* if it contains relatively many edges. Also a graph is said to be *sparse* if it contains relatively few edges. One way to require a family of graphs to contain many edges is to insist that that every graph G satisfies $\delta(G) \geq C \cdot n$, for some constant C. As the number of vertices grows, this ensures that each graph still has relatively many edges. We purposely do not precisely define dense, as in Chapters 2 and 3, we would like a "dense" graph to be any graph which satisfies the minimum degree conditions given for each successive result.

Chapter 2

Dense K_4-free Graphs

1. Introduction

Triangle-free graphs with large minimum degree have been the focus of many papers. There are results about chromatic number [**56, 10**], about homomorphisms [**12, 26, 32, 40**], about similarity sets [**25**], and about the binding number [**54, 60**].

As regards dense K_r-free graphs, a famous old result due to Andrásfai, Erdős and Sós [**4**] is that a K_r-free graph with sufficiently large minimum degree is homomorphic to K_{r-1} and hence is $(r-1)$-colorable. However, apart from papers such as [**7, 61**], the structure of K_r-free graphs with minimum degree just below this has not been studied much.

In this chapter we consider results for dense K_4-free graphs, especially as regards independent sets, homomorphisms, chromatic number and binding number. We show that some of the results for triangle-free graphs carry over, but others do not. Also, while $n/3$ is a common minimum degree threshold for results on triangle-free graphs, the minimum degree conditions vary for some similar results on K_4-free graphs.

1.1 *Previous results*

The main result for the chromatic number of dense graphs in terms of the minimum degree is due to [**4**], and states the following.

Theorem 2.1 ([**4**]): *Let G be a K_r-free graph with $r \geq 3$, such that $\delta(G) > (3r-7)n/(3r-4)$. Then $\chi(G) \leq r-1$.*

Alon and Sudakov [**3**] bounded the number of edges that need to be removed to obtain a similar result for all sufficiently large H-free graphs.

THEOREM 2.2 ([**3**]): *Let H be a fixed graph on n_H vertices with chromatic number $r+1 \geq 3$, suppose $\varepsilon > 0$ and let G be an H-free graph of sufficiently large order n (depending only on n_H and ε) with minimum degree $\delta(G) \geq ((3r-7)/(3r-4)+\varepsilon)\,n$. Then one can delete at most $O(n^{2-1/(4r^{2/3}n_H)})$ edges to make G r-colorable.*

For triangle-free graphs in particular, Erdős and Simonovits [**19**] raised the question about the chromatic number of "dense" triangle-free graphs, with "dense" taken to mean $\delta(G) > n/3$. It is easy to show, and is well known, that every triangle-free graph G with $\delta(G) > 2n/5$ is bipartite. Thomassen [**56**] proved that the chromatic number of triangle-free graphs with $\delta(G) = cn$ with $c > 1/3$ is bounded as a function of c. Recently, Brandt and Thomassé significantly strengthened this:

THEOREM 2.3 ([**10**]): *Let G be a triangle-free graph such that $\delta(G) > n/3$. Then $\chi(G) \leq 4$.*

As regards homomorphisms, Häggkvist [**26**] showed that a triangle-free graph of minimum degree at least $3n/8$ is either bipartite or homomorphic to the 5-cycle. This result was extended by Jin [**32**] and Chen et al. [**12**], inter alia. Łuczak [**40**] provided a general result about the structure of triangle-free graphs with minimum degree more than $n/3$:

THEOREM 2.4 ([**40**]): *For every $\varepsilon > 0$, there exists a finite set of triangle-free graphs $\mathcal{G}_\varepsilon$ such that every triangle-free graph G with $\delta(G) > (1/3+\varepsilon)n$ is homomorphic to some graph $H \in \mathcal{G}_\varepsilon$.*

Earlier, Goddard and Kleitman [**25**] showed that a maximal triangle-free graph with minimum degree at least $(n+2)/3$ must have two similar vertices, that is, nonadjacent vertices with the same neighborhoods. Indeed:

THEOREM 2.5 ([**25**]): *Let G be a maximal triangle-free graph such that $\delta(G) > n/3$. Then there exists a set of vertices of size $3\delta(G) - n$ which are mutually similar.*

2. Preliminaries

To begin with, we develop a simple counting tool, which is used throughout this chapter and the next. Let X be a set and let $\mathcal{A} = \{A_i\}_{i=1}^{k}$ be a family of subsets of X. We define the *slack* of this set system, $\mathit{Slack}(\mathcal{A})$, such that

$$\mathit{Slack}(\mathcal{A}) = \sum_{x \in S} \left[(k-1) - r(x)\right],$$

where $S \subseteq X$ is the set of elements which are contained in at most $k-2$ members of $\mathcal{A}$, and $r(x)$ is the number of members of $\mathcal{A}$ containing an element x.

Lemma 2.1: *Let $\mathcal{A} = \{A_i\}_{i=1}^{k}$ be a family of subsets of some t-element set X such that $|A_i| \geq s$ for each $A_i \in \mathcal{A}$. Then the size of the intersection of all the sets in $\mathcal{A}$ is at least the following:*

$$\left| \bigcap_{A_i \in \mathcal{A}} A_i \right| \geq k \cdot s - (k-1) \cdot t + \mathit{Slack}(\mathcal{A}) \geq k \cdot s - (k-1) \cdot t.$$

Proof. Let $|\cap \mathcal{A}| = a$ and consider $\sum_{i=1}^{k} |A_i|$. Clearly $\sum_{i=1}^{k} |A_i| \geq ks$. By the definition,

$$\begin{aligned} \textstyle\sum_{i=1}^{k} |A_i| &= ak + (t-a)(k-1) - \mathit{Slack}(\mathcal{A}) \\ &= a - \mathit{Slack}(\mathcal{A}) + (k-1)t \end{aligned}$$

The lower bound on a follows. ■

In order to prove the main result about the structure of dense K_4-free graphs, we can apply this bound on the intersection of sets to observe a simple fact about nonadjacent vertices. Any two nonadjacent vertices in a maximal K_r-free graph can be extended to a large independent set, as noted by Kleitman [**35**], in a review of [**4**].

Observation 2.1: *In a maximal K_r-free graph G with $r \geq 3$, any two nonadjacent vertices are in an independent set of size $(r-2)\delta(G) - (r-3)n$.*

Proof. Let v and v' be nonadjacent vertices in a maximal K_r-free graph G. By the maximality of G, the addition of the edge (v, v') forms a K_r. That is, there is some set of $r-2$

vertices, say S, such that $S \subseteq N(v) \cap N(v')$ and $\langle S \rangle = K_{r-2}$. Let $I = \bigcap_{w \in S} N(w)$. Then I is an independent set, with $v, v' \in I$. Then apply Lemma 2.1 with $\mathcal{A} = \{N(s_i)\}_{s_i \in S}$ and $X = V(G)$, such that $|N(s_i)| \geq \delta(G)$ and $|X| = n$. Thus, $|I| \geq (r-2)\delta(G) - (r-3)n$. ■

The *lexicographic product* of the graph G with H, denoted $G[H]$, is the graph with vertex set $V(G[H]) = \{(g, h) : g \in V(G), h \in V(H)\}$, where $((g, h), (g', h')) \in E(G[H])$ if and only if either $(g, g') \in E(G)$ or $g = g'$ and $(h, h') \in E(H)$. We note that Observation 2.1 is sharp for the Turán graphs, $K_{r-1}\left[\overline{K_a}\right]$, the complete $(r-1)$-partite graphs with balanced parts of size a $(a \geq 1)$.

3. Structural Result for Dense K_4-free Graphs

The observation from above can now be used to determine a minimum degree requirement which will force structure on the nonneighborhood of a vertex. As the nonneighborhood of a vertex, $V(G) - N(u)$, is a central focus of several theorems, the notation $H(u) = V(G) - N(u)$ for any vertex $u \in V(G)$ is introduced. Additionally, the set I will always denote some maximum independent set in G.

Theorem 2.6: *Let G be a maximal K_4-free graph with $\delta(G) > \frac{3}{5}n$. Then there exists a vertex u such that the induced subgraph $\langle H(u) \rangle$ does not contain an edge ($H(u)$ is independent).*

Proof. Let I be a maximum independent set of G. We claim that the result holds for any vertex $u \in I$. We may assume for contradiction that there does exist some vertex $u \in I$ such that $H(u)$ is not independent.

That is $H(u) \supsetneq I$. Then any vertex $a \in H(u) - I$ has a neighbor $b \in I$ (See Figure 2.1). Then the independent set $I' = N(a) \cap N(b)$ will be considered, and in particular, $I' \cap N(u)$.

In order to find a lower bound for $I' \cap N(u)$, we consider the expressions $d(a) - |N(a) \cap N(u)|$ and $d(b) - |N(b) \cap N(u)|$. Since $b \in I$, $|N(b) \cap H(u)| \leq (n - d(u) - |I|)$. Since $a \in H(u) - I$, Observation 2.1 can be applied, giving $|N(a) \cap H(u)| \leq (n - d(u) - (2\delta(G) - n))$.

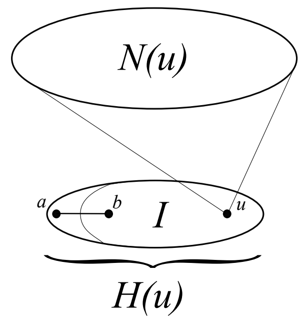

FIGURE 2.1. An illustration of the proof of Theorem 2.6

A lower bound for the size of $I' \cap N(u)$ can now be determined.

$$\begin{aligned}
|I' \cap N(u)| &\geq d(a) + d(b) - |N(b) \cap H(u)| - |N(a) \cap H(u)| - d(u) \\
&\geq d(a) + d(b) - [n - d(u) - |I|] - [n - d(u) - (2\delta(G) - n)] - d(u) \\
&\geq 5\delta(G) - 3n + |I|.
\end{aligned}$$

Since $\delta(G) > \frac{3}{5}n$, an independent set larger than I is formed, a contradiction. ■

The minimum degree conditions in the above theorem might not be best possible. For example, we suspect that if G is a maximal K_4-free graph of minimum degree $\delta(G) > 7n/12$, then G is the join of an independent set and a triangle-free graph. This would be best possible because of the graph of Figure 2.2.

Further information can be determined about the triangle-free graph induced by the neighborhood of any vertex in a K_4-free graph G with $\delta(G) > \frac{3}{5}n$.

LEMMA 2.2: *Let G be a K_4-free graph with $\delta(G) > \frac{3}{5}n$. Then the minimum degree of any vertex $v \in N(u)$ in $\langle N(u) \rangle$ is more than $\frac{1}{3}|N(u)|$.*

PROOF.

$$\frac{d_{\langle N(u)\rangle}(v)}{|\langle N(u)\rangle|} \geq \frac{\delta(G) - (n - d(u))}{d(u)} \geq 2 - \frac{n}{\delta(G)} > 2 - \frac{5}{3} = \frac{1}{3}$$ ■

These results can be stated more precisely for maximal K_4-free graphs.

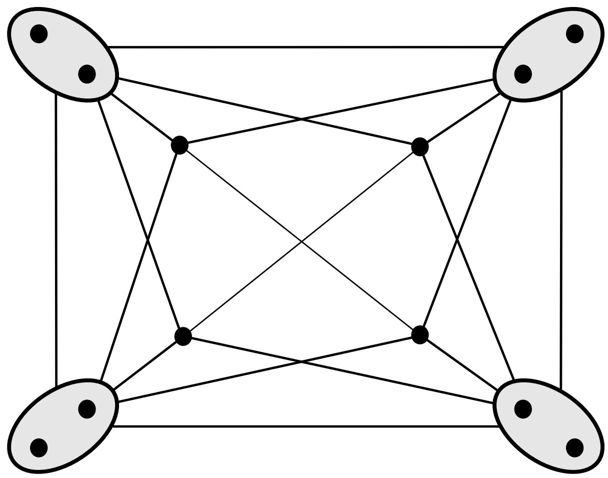

Figure 2.2. K_4-free graph with $\delta(G) = 7n/12$ such that every nonneighborhood is not independent. (Each pair of enclosed vertices are independent of each other, and have the same neighborhood.)

Corollary 2.1: *Let G be a maximal K_4-free graph with $\delta(G) > \frac{3}{5}n$. Then G is the join of an independent set and a triangle-free graph with minimum degree larger than $\frac{1}{3}$ its order.*

4. Application 1: The Chromatic Number and Homomorphisms of a Graph

For both of the below extensions, we simply apply the aforementioned theorems of Brandt and Thomassé [**10**] and Łuczak [**40**] to the triangle-free subgraph induced by the neighborhood of a vertex, as discussed in Lemma 2.2.

Corollary 2.2: *For every $\varepsilon > 0$, there exists a constant $M(\varepsilon)$ such that every K_4-free graph on n vertices with minimum degree at least $\left(\frac{3}{5} + \varepsilon\right) n$, is homomorphic to a K_4-free graph on at most $M(\varepsilon)$ vertices.*

Corollary 2.3: *Let G be a K_4-free graph with $\delta(G) > \frac{3}{5}n$. Then $\chi(G) \leq 5$.*

The degree bound for Corollary 2.3 is sharp, as a class of graphs can be defined that are K_4-free and have minimum degree approaching $\frac{3}{5}n$ such that this sequence of graphs has unbounded chromatic number, as follows.

Lemma 2.3: *For any $k > 3$ and $\varepsilon > 0$, there exists a K_4-free graph $H(k, \varepsilon)$ with $n_{k,\varepsilon}$ vertices, such that $\chi(H(k, \varepsilon)) \geq k$ and $\delta(H(k, \varepsilon)) = (3/5 - \varepsilon)n_{k,\varepsilon}$.*

Proof. This is achieved by exploiting the construction of Erdős, Hajnal, and Simonovits [**19**]. The *Kneser graph* $KG(a, b)$ is the graph whose vertices correspond to the b-element subsets of a set of a elements, where two vertices are connected if and only if the two corresponding sets are disjoint. Taking the Kneser graph $KG(2m + k, m)$, and a large complete bipartite graph $K_{\ell,2\ell}$ where $(2m+k)$ divides 2ℓ, one can partition the independent set of size 2ℓ in $K_{\ell,2\ell}$ into $2m + k$ sets, each representing an element of the $2m + k$ base set of $KG(2m + k, m)$. Then, a vertex $v \in KG(2m + k, m)$ is adjacent to each of the sets representing the elements in its m element subset. This graph is then triangle-free and for ℓ much larger than m much larger than k, has minimum degree approaching $n/3$. In [**39**], Lovász established the chromatic number of the Kneser graph $KG(a, b)$ as $a - 2b + 2$, so $\chi\left(KG(2m + k, m)\right) = k$. Thus, for any k and $\varepsilon > 0$, one can construct a triangle-free graph $G(k, \varepsilon)$ such that $\chi(G(k, \varepsilon)) \geq k$ and $\delta(G(k, \varepsilon)) = (1/3 - \varepsilon)|V(G(k, \varepsilon))|$.

We can then define the graph $H(k, \varepsilon) = G(k, \varepsilon) \vee \overline{K_\ell}$, where $\ell = (2/3)|V(G(k, \varepsilon))|$. This graph has $n_{k,\varepsilon} = (5/3)|V(G(k, \varepsilon))|$ vertices, chromatic number at least $k + 1$, clique number 3, and minimum degree larger than $(3/5 - \varepsilon)n_{k,\varepsilon}$. ■

As in the case of triangle-free graphs, it is unclear what happens to the chromatic number when the minimum degree is exactly $3n/5$.

Additionally, by the same idea as in the proof of Lemma 2.2, we can bound from below the minimum degree of the triangle-free portion of the K_4-free graph, giving specific homomorphism results. For example, recall that Häggkvist [**26**] proved that any triangle-free graph G such that $\delta(G) > 3n/8$ is homomorphic to a 5-cycle or is bipartite. The 5-wheel W_5 is the graph formed by adding a vertex to a 5-cycle, and adjoining it to every vertex in the 5-cycle.

Extending the result of Häggkvist, we get

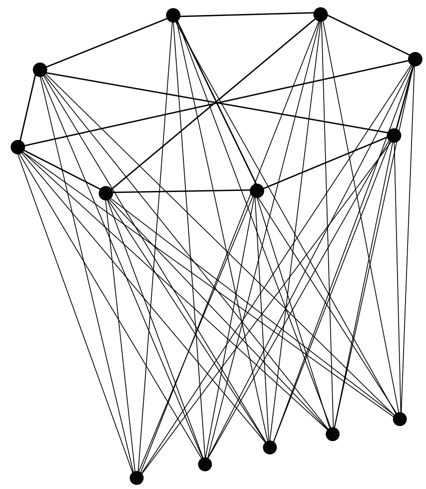

FIGURE 2.3. A Möbius ladder on 8 vertices joined to an independent set of five vertices

COROLLARY 2.4: *Any K_4-free graph G with minimum degree $\delta(G) > 8n/13$ is homomorphic to W_5 or K_3.*

PROOF. We may assume G is a maximal K_4-free graph. Since $\delta(G) > 8n/13 > 3n/5$, by Theorem 2.6 there is some vertex u such that the subgraph induced by $H(u) = V - N(u)$ is independent. Again, $\langle N(u) \rangle$ is triangle-free, and for any vertex $v \in \langle N(u) \rangle$ we have

$$\frac{d_{\langle N(u) \rangle}(v)}{|\langle N(u) \rangle|} \geq \frac{\delta - (n - d(u))}{d(u)} \geq 2 - \frac{n}{\delta(G)} > 2 - \frac{13}{8} = \frac{3}{8}.$$

By the result of Häggkvist, this implies that $\langle N(u) \rangle$ is homomorphic to the 5-cycle or is bipartite. Thus G is homomorphic to a 5-wheel or a triangle. ■

This result is sharp, as can be seen taking the join of an independent set of five vertices with the Möbius ladder on 8 vertices (See Figure 2.3).

In the same way, other results about homomorphisms of dense triangle-free graphs can easily be extended to K_4-free graphs.

5. Application 2: The Independence Number of a Graph

The next area to consider for K_4-free graphs is finding bounds for the independence number. Observation 2.1 gives a trivial bound, and this is sharp for $K_4[\overline{K_a}]$. However, for dense K_4-free graphs with lower degree, a different approach is more effective in determining a large bound for the size of the largest independent set. To begin with, we prove an easy analogue to the result of Andrásfai, Erdős and Sós, in terms of the sum of the degrees of endpoints of an edge. Let $\delta_2(G)$ denote $\min_{(u,v)\in E(G)} \{d(u)+d(v)\}$. The conditions of Theorem 2.1 for $r=3$ can be relaxed to the following.

Theorem 2.7: *Let G be a triangle-free graph such that $\delta_2(G) > \frac{4}{5}n$. Then G is bipartite.*

Proof. The proof of this relaxation does not depart far from the original proof of Andrásfai, Erdős and Sós. Assume for contradiction that G is not bipartite. Then G must contain an odd cycle. Let C_{2k+1} $(k \geq 2)$ be some shortest odd cycle in G. Since the sum of degrees across any edge is at least $\delta_2(G)$, we can choose some vertex with degree at least $(1/2)\delta_2(G)$. If this vertex is removed from the cycle, the remaining path has a perfect matching, and so the sum of degrees in the cycle is at least $\left(k+\frac{1}{2}\right)\delta_2(G)$. On the other hand, no vertex in the remainder of the graph can be adjacent to more than two vertices in the cycle without creating a smaller odd cycle. Thus, we get $\frac{2k+1}{2}\left(\delta_2(G)-2\right) \leq 2(n-(2k+1))$, which implies that $\delta_2(G) \leq \frac{4}{5}n$, a contradiction. ■

In order to prove that there is a large independent set in a K_4-free graph, we require a particular case of a result by Brandt [**9**].

Theorem 2.8 ([**9**]): *Let G be a maximal K_4-free graph with $\delta(G) \geq \frac{4}{7}n$. Then G is 3-colorable, or G contains the 5-wheel W_5.*

Note that containing a wheel is a much weaker condition than being homomorphic to it. In the event that the graph is three colorable, the graph has an independent set of size $\frac{n}{3}$. If the graph is not three colorable, there is a W_5 contained in the graph, which implies that some neighborhood is not bipartite.

THEOREM 2.9: *Let G be a maximal K_4-free graph with $\delta(G) \geq \frac{4}{7}n$. Then G has an independent set I such that*

$$|I| \geq \min\left\{\frac{n}{3}, \frac{11}{5}\delta(G) - n\right\}.$$

PROOF. By Theorem 2.8, either the graph is three colorable, or the graph has a vertex u whose neighborhood is not bipartite. In this case, by Theorem 2.7 there is some edge $(a, b) \in E(G)$ such that the sum of degrees of the endpoints restricted to $\langle N(u)\rangle$ is at most $\frac{4}{5}|N(u)|$. If we let $X = V(G)$ and $\mathcal{A} = \{N(a), N(b)\}$, then we have shown that $Slack(A) \geq \frac{1}{5}|N(u)| \geq \frac{1}{5}\delta(G)$. We can apply Lemma 2.1 to find that the independent set $I' = N(a) \cap N(b)$ is at least of size

$$|I'| \geq 2\delta(G) - n + \frac{1}{5}|N(u)| \geq \frac{11}{5}\delta(G) - n. \quad \blacksquare$$

There is one particular value for the minimum degree to evaluate this theorem which provides an interesting result. This is a minimum degree condition which implies that there is an independent set of size at least $\frac{n}{3}$.

COROLLARY 2.5: *Let G be a K_4-free graph with $\delta(G) \geq \frac{20}{33}n$. Then G has an independent set of size $\frac{n}{3}$.*

Interestingly, $\frac{20}{33} < \frac{5}{8}$, the minimum degree condition from Theorem 2.1 which forces the graph to be 3-colorable. In the case of triangle-free graphs, a minimum degree condition of $\frac{2}{5}n$ implies an independent set of size at least $\frac{n}{2}$, but this result is sharp, as the graph $G = C_5\left[\overline{K_a}\right]$ illustrates. We compare the various minimum degree thresholds in Table 2.1.

6. Application 3: The Binding Number and Cliques of a Graph

Lastly, we can use the insight about the structure of dense K_4-free graphs, as well as our results about the independence number of these graphs, to consider the binding number of a graph.

Structural Property	Minimum Degree Threshold
Similarity sets	Possibly $\frac{1}{2}n$, at most $\frac{3}{5}n$,
Join of an independent set and a triangle-free graph	At least $\frac{7}{12}n$, at most $\frac{3}{5}n$
Bounded chromatic number	$\frac{3}{5}n$
Homomorphic to a finite set of triangle-free graphs	$\frac{3}{5}n$
Independent set of size $\frac{n}{3}$	$\frac{20}{33}n$
3-colorable or homomorphic to W_5	$\frac{8}{13}n$
3-colorable	$\frac{5}{8}n$

Table 2.1. Thresholds on the minimum degree for structural properties of dense K_4-free graphs.

6.1 *Binding number*

Woodall [**60**] defined the binding number of a graph as follows. If $S \subseteq V(G)$, then we write the open neighborhood of the set S as $N(S) = \bigcup_{v \in S} N(v)$. The binding number of G, denoted $bind(G)$, is given by

$$bind(G) = \min_{\substack{S \subseteq V(G) \\ N(S) \neq V(G)}} \frac{|N(S)|}{|S|}.$$

In [**60**], Woodall showed that a binding number of at least 3/2 implies that the graph must be hamiltonian. Additionally, Shi Ronghua showed in [**54**] that a binding number of 3/2 also implies that the graph contains a triangle, and furthermore, the graph is pancyclic [**53**] (contains a cycle of every intermediate length). (See [**25**] for a short proof for a triangle.)

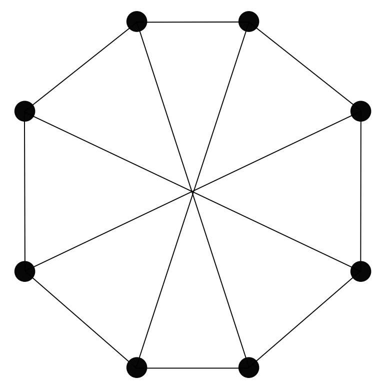

Figure 2.4. The graph Γ_3, the Möbius ladder on 8 vertices

The binding number 3/2 is best possible for the existence of a triangle, because of the following family. For $i \geq 2$, let $\Gamma_i = \overline{C_{3i-1}^{i-1}}$, the complement of the $i-1$-th power of the cycle C_{3i-1}. For example, Γ_2 is the 5-cycle and Γ_3 is the Möbius ladder on 8 vertices (See Figure 2.4).

This family is well known. We state the following lemma to summarize some known properties of these graphs.

Lemma 2.4: *The graph Γ_i is a 3-colorable, i-regular, triangle-free graph, and has the property that any set of b vertices has an open neighborhood of size at least* $\min\{b+i-1, n\}$.

Proof. For a vertex v on the cycle C_{3i-1}, there are $i-1$ vertices at a distance between 1 and $i-1$ from v moving clockwise about the cycle, and a similar number moving in a counterclockwise direction. There are $(3i-2)-2(i-1)=i$ vertices in $V(C_{3i-1})-\{v\}$ at a distance of i or more from v. Thus, every vertex in Γ_i has degree i.

To show that Γ_i is triangle-free and 3-colorable, we consider the vertices of Γ_i arranged as on the cycle C_{3i-1}. If there is any triangle in Γ_i, we consider the two vertices a and b which are closest in distance along the cycle C_{3i-1}. Then vertex a must be at a distance of at least i along the cycle from the vertex b. The last vertex c of the triangle must be one of the remaining vertices which is not on the shortest path along the cycle from a to b. Moving along the cycle, the vertex c must be distance i from both a and b, however this would imply that the cycle is of size at least $3i$, a contradiction. Additionally, if we split the cycle into paths of length i, i, and $i-1$, then in Γ_i, these sets correspond to a partition of the vertex set into three independent sets.

To show the property about the open neighborhood of sets, we use induction. This is certainly true for a singleton set, $b=1$, as Γ_i is i-regular. We then assume for induction that any set of $b-1$ vertices has an open neighborhood of size at least $\min\{b+i-1, n\}$. We then consider a set S of size b, such that the neighborhood of S is not the entire set of vertices. Therefore, there is one vertex v which is not a neighbor of any vertex in the set S, and thus if the vertices of Γ_i are aligned along the cycle C_{3i-1}, all vertices in the set S must be within distance $i-1$ of this vertex. Without loss of generality, we travel around the cycle in a clockwise direction to the vertex farthest from v, say u. If we now travel around the cycle in counterclockwise direction from v, the first vertex encountered which is adjacent to u cannot be adjacent to any other vertex in S. Thus, consider the set $S - \{u\}$. This set has $b-1$ vertices, and satisfies the induction hypothesis. The addition of u adds at least one vertex to $N(S)$, so that $|N(S)| \geq \min\{((b-1)+i-1)+1), n\}$. ∎

In light of these properties, it is not hard to show that the binding number of this family of graphs tends to $3/2$ from below, as noted by Woodall [**60**]. This family figures in other results. For example, Chen, Jin, and Koh [**12**] showed that a 3-colorable triangle-free graph with minimum degree $\delta(G) > n/3$ must be homomorphic to some Γ_i. (The definition of their family looks different but it is the same family of graphs.)

Kane and Mohanty [**33**] observed the following theorem, which follows from Theorem 2.1:

Theorem 2.10 ([**33**]): *For any K_r-free graph G, bind$(G) \geq r - 4/3$.*

In addition, they observed that there are K_r-free graphs with binding number $r-2$, such as the tensor product $K_{r-1} \otimes K_m$ where $m \geq (r-2)^2+1$.

6.2 *Better bounds*

We improve on both the lower bound and upper bound of Kane and Mohanty, for K_4-free graphs. First we construct examples to give K_4-free graphs with larger binding number. Let $\Gamma_i = \overline{C_{3i-1}^{i-1}}$ as before. Define the graph $G(i,a) = \Gamma_i \vee \overline{K_a}$, the join of the triangle-free graph Γ_i and an independent set of size a. This graph is K_4-free.

LEMMA 2.5: *For $i \geq 2$ and $a \geq 1$, the binding number of $G(i,a)$ is given by*

$$bind(G(i,a)) = \min\left\{\frac{3i-2+a}{2i-1}, \frac{3i-1}{a}\right\}.$$

PROOF. For a binding set S, we can take vertices either only from Γ_i or from $\overline{K_a}$.

From Lemma 2.4, we know that Γ_i has the property that any set of b vertices has an open neighborhood of size at least $b+i-1$, and choosing b consecutive vertices achieves the bound. Therefore, choosing the set S from Γ_i, we get the following bound on the binding number:

$$bind(G(i,a)) \leq \min_b \frac{b+i-1+a}{b} = \frac{3i-2+a}{2i-1},$$

since the middle expression is decreasing in b, and therefore achieves its minimum at $b = 2i-1$.

On the other hand, letting S be all of $\overline{K_a}$ yields

$$bind(G(i,a)) \leq \frac{3i-1}{a}.$$

It follows that $bind(G(i,a))$ is the minimum of the above two bounds. ■

Since one expression in the formula of Lemma 2.5 is increasing in a and the other expression decreasing in a, we let $a(i)$ be the nearest integer to the value of a where the two expressions are equal. It can be calculated that

$$\lim_{i\to\infty} bind(G(i,a(i))) = \frac{\sqrt{33}-3}{7-\sqrt{33}} \approx 2.186.$$

We next provide an upper bound on the binding number of all K_4-free graphs.

THEOREM 2.11: *For any K_4-free graph G,*

$$bind(G) \leq \frac{\sqrt{91}-6}{11-\sqrt{91}} \approx 2.423.$$

Proof. If i is the size of the largest independent set, as before we have $bind(G) \leq (n-i)/i = n/i-1$. By Theorem 2.9, $i \geq \min\{\frac{n}{3}, \frac{11}{5}\delta(G)-n\}$. If $i \geq \frac{n}{3}$, then we have $bind(G) \leq 2$. So assume that $i \geq (11/5)\delta(G) - n$. Then we get $bind(G) \leq (10n - 11\delta(G))/(11\delta(G) - 5n)$. This decreases with $\delta(G)$. Also, again we note that we can choose the nonneighborhood of a minimum degree vertex, and get $bind(G) \leq (n-1)/(n-\delta(G)) < n/(n-\delta(G))$. This increases with $\delta(G)$. Then, at $\delta(G) = ((16-\sqrt{91})/11)n$, these two values are equal, and give a maximum binding number of $(\sqrt{91}-6)/(11-\sqrt{91}) \approx 2.423$. ■

We conjecture that the lower bound $(2.186\ldots)$ is the correct threshold.

7. A Counterexample to a Conjecture of Caro

The same family of graphs disproves an open conjecture. A conjecture regarding the size of the neighborhoods of independent sets was given by Yair Caro, in a postscript to [**49**]:

Conjecture 2.1 (Caro): *Let $r \geq 2$ be an integer and let G be a graph of order n such that $|N(X)| > (r-2)(n+|X|)/r$ for every independent subset X of $V(G)$. Then G contains a copy of K_r.*

The result is trivial for $r = 2$, and the main result of [**49**] showed the conjecture true for $r = 3$. However, the conjecture is false for $r \geq 4$, and the graphs can be constructed in the following way. Let $\Gamma_i = \overline{C_{3i-1}^{i-1}}$, as before. Now, define the graph $G(i,r,a) = \Gamma_i \vee \left(\vee_{j=1}^{r-3}\overline{K_a}\right)$, the join of the triangle-free graph Γ_i and $r-3$ independent sets of size a.

Consider $G(i,r,a)$ with $r \geq 4$. Since any independent set in a join of subgraphs is contained in only one of the subgraphs, there are two cases to consider. First, an independent set X contained in a subgraph of the form $\overline{K_a}$ has $|N(X)| = n - a$ and $|X| \leq a$. It follows by some algebra that the hypothesis of the conjecture holds for all such X if and only if

$$2a < 3i - 1.$$

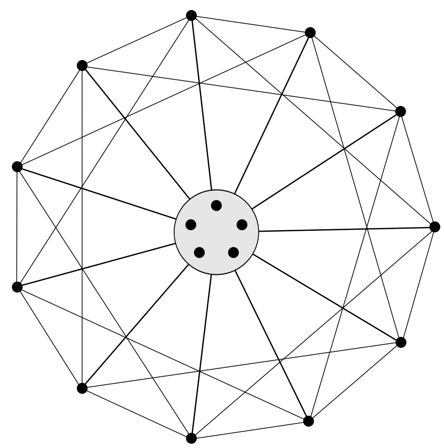

Figure 2.5. A counterexample to Caro's conjecture

Second, an independent set X contained in the subgraph Γ_i has $|N(X)| = |X|+i-1+(r-3)a$ and $|X| \le i$. It follows by some algebra that the hypothesis of the conjecture holds for all such X ($|X| = 1$ is the hardest) if and only if

$$a > i.$$

So if $i < a < 3i-1$, then we have a K_r-free graph which satisfies the hypothesis of the conjecture. Thus, for example, $G(4, r, 5)$ is a counterexample. See Figure 2.5 for $G(4, 4, 5)$.

Chapter 3

Dense K_r-free Graphs

1. Introduction

In this chapter, the structure of K_r-free graphs, $r > 3$, is considered. Most of the results proven in this chapter provide an extension of the results for K_4-free graphs. The previous chapter can be used as a guide, even though most of these results were greatly simplified for the case of $r = 4$. However, some of the techniques for $r \geq 5$ differ from those for $r = 4$, especially in the case of finding large independent sets.

2. Structural Results for Dense K_r-free Graphs

We start with an observation about dense K_r-free graphs. We give the degree requirements to force any clique of size k to be extended to a clique of size $k + 1$.

Observation 3.1: *If $\delta(G) > \frac{k-1}{k}n$, any K_k in G can be extended to a K_{k+1}.*

Proof. Suppose for contradiction that some K_k cannot be extended. Let these vertices be the set S. Then the sum of the degrees from vertices in S into $V(G) - S$ is at least $k\delta(G) - k(k-1)$ and $|N(v) \cap S| \leq k-1$ for any other $v \in V(G) - S$. Therefore, $(n-k)(k-1) \geq k\delta(G) - k(k-1)$, which implies $n(k-1) \geq k\delta(G)$, a contradiction. ■

2.1 *K_k-free nonneighborhoods*

Our main goal in this section is to determine minimum degree requirements which force structure on the nonneighborhood of a vertex in a maximum independent set. The notation $H(u) = V(G) - N(u)$ from the previous chapter will again be used. In general, we prove results of the form that a maximal K_r-free graph with $\delta(G) > C_{r,k} \cdot n$ has a vertex u such that the induced subgraph $\langle H(u) \rangle$ is K_k-free.

THEOREM 3.1: *Let G be a maximal K_r-free graph with $r \geq 4$ and $\delta(G) > \left(\frac{k(r-3)+1}{k(r-2)+1}\right) n$ for some $k < r-1$. Then there exists a vertex u such that the induced subgraph $\langle H(u) \rangle$ is K_k-free.*

PROOF. Let I be a maximum independent set of G. We claim that the result holds for any vertex $u \in I$. Note that $I \subseteq H(u)$.

If $k < r-2$, we will use induction on k. That is, since the expression $(k(r-3)+1)/(k(r-2)+1)$ increases as k grows smaller, we may assume if $k < r-3$ that for all $u \in I$ that $H(u)$ is K_{k+1}-free, and we want to show that it is additionally K_k-free.

We assume for contradiction that there does exist some $K_k \subseteq H(u)$, say given by $\langle T \rangle$. From our minimum degree condition, we can apply Observation 3.1 and extend this clique (if needed) to a set of vertices S such that $\langle S \rangle = K_{r-2}$.

Then, we consider the set $I' = \left(\bigcap_{s_i \in S} N(s_i)\right) \cap N(u)$. This is an independent set. By Lemma 2.1, the size of this set, $|I'|$, is bounded below by the expression $(r-2)\delta(G) - \left(\sum_{s_i \in S} |N(s_i) \cap H(u)|\right) - (r-3)d(u)$. We then consider bounding the size of the sum $\sum_{s_i \in S} |N(s_i) \cap H(u)|$ in two cases.

(1) *A set $T \subseteq H(u)$ exists such that $\langle T \rangle = K_k$ and $T \cap I \neq \emptyset$.*

Every vertex $v \in T$ is nonadjacent to u, so by Observation 2.1 we have $|N(v) \cap H(u)| \leq (n - d(u)) - ((r-2)\delta(G) - (r-3)n)$. If we let v^* be the vertex in $T \cap I$, we also note that $|N(v^*) \cap H(u)| \leq (n - d(u)) - |I|$. For every other vertex $s'_j \in S - T$, it trivially holds that $|N(s'_j) \cap H(u)| \leq |H(u)| = n - d(u)$. We can then use these upper bounds and Lemma 2.1 to determine a lower bound on $|I'|$.

$$\begin{aligned}
|I'| &\geq (r-2)\delta(G) - \left[\textstyle\sum_{s_i \in S} |N(s_i) \cap H(u)|\right] - (r-3)d(u) \\
&\geq (r-2)\delta(G) - \left[(r-2)(n-d(u)) - |I| - (k-1)\left((r-2)\delta(G) - (r-3)n\right)\right] \\
&\qquad -(r-3)d(u) \\
&\geq (r-2)\delta(G) + \delta(G) - (r-2)n + (k-1)\left((r-2)\delta(G) - (r-3)n\right) + |I| \\
&= \left[k(r-2)+1\right]\delta(G) - \left[k(r-3)+1\right]n + |I|.
\end{aligned}$$

Since $\delta(G) > \left(\frac{k(r-3)+1}{k(r-2)+1}\right)n$, this would imply $|I'| > |I|$, a contradiction.

(2) *No set $T \subseteq H(u)$ exists such that $\langle T \rangle = K_k$ and $T \cap I \neq \emptyset$.*

Under the conditions of this case, each vertex in I can be adjacent to at most $k-2$ of the vertices of any $K_k \subseteq \langle H(u) \rangle$. There are two subcases.

(a) $k = r-2$

In this case, the base case for the induction, first assume for contradiction that $\langle H(u) \rangle$ is not K_{r-1}-free, i.e., there exists some set $U \subseteq H(u)$ such that $\langle U \rangle = K_{r-1}$. Removing some vertex v from the set U, we consider the independent set $I'' = \bigcap_{s_i \in U - \{v\}} N(s_i)$. If we let $X = V(G)$ and $\mathcal{A} = \{N(s_i)\}_{s_i \in U-\{v\}}$, then *Slack*$(\mathcal{A}) \geq |I|$, and applying Lemma 2.1 yields the following bound on $|I''|$:

$$|I''| \geq (r-2)\delta(G) - (r-3)n + |I|.$$

This is a contradiction, since $(r-2)\delta(G) - (r-3)n > 0$ from the minimum degree condition. That is, $\langle H(u) \rangle$ is K_{r-1}-free.

Therefore, any vertex of $H(u)$ not in $I \cup T$ can only be adjacent to at most $r-3$ of the vertices in T, and as noted previously, any vertex in I can be adjacent to at most $r-4$ of the vertices of T. Therefore,

$$\begin{aligned}
\textstyle\sum_{s_i \in S} |N(s_i) \cap H(u)| &\leq (r-4)|I| + (r-3)(n-d(u)-|I|) \\
&= (r-3)\left(n-d(u)\right) - |I|.
\end{aligned}$$

We can use these upper bounds to determine a lower bound on $|I'|$.

$$\begin{aligned} |I'| &\geq (r-2)\delta(G) - (r-3)\left[n - d(u)\right] + |I| - (r-3)d(u) \\ &= (r-2)\delta(G) - (r-3)n + |I|. \end{aligned}$$

Again, since $(r-2)\delta(G) - (r-3)n > 0$, this would imply $|I'| > |I|$, a contradiction.

(b) $k < r-2$

For $k < r-2$, every vertex of $H(u)$ not in $I \cup T$ can only be adjacent to at most $k-1$ of the vertices in S, since $\langle H(u) \rangle$ is K_k-free by applying our induction hypothesis. Therefore,

$$\begin{aligned} \textstyle\sum_{s_i \in T} |N(s_i) \cap H(u)| &\leq (k-2)|I| + (k-1)(n - d(u) - |I| - k) \\ &= (k-1)\left(n - d(u)\right) - |I|. \end{aligned}$$

For any other vertex $s'_j \in S - T$, $|N(s'_j) \cap H(u)| \leq |H(u)| = n - d(u)$, which gives us $\sum_{s_i \in S} |N(s_i) \cap H(u)| \leq (r-3)(n-d(u)) - |I|$. We can then use these upper bounds and Lemma 2.1 to determine a lower bound on $|I'|$.

$$\begin{aligned} |I'| &\geq (r-2)\delta(G) - \left[(r-3)(n-d(u)) - |I|\right] - (r-3)d(u) \\ &\geq (r-2)\delta(G) - (r-3)n + |I|. \end{aligned}$$

Since $(r-2)\delta(G) - (r-3)n > 0$, this would imply $|I'| > |I|$, a contradiction. Therefore, we have shown that for any vertex $u \in I$, the subgraph induced by its nonneighborhood $\langle H(u) \rangle$ is K_k-free. ■

3. Inductive Result

Now, Theorem 3.1 can be used inductively to prove the main structural result about K_r-free graphs.

Theorem 3.2: *Let $r \geq 3$, and let G be a K_r-free graph with $\delta(G) > \left(\frac{k(r-3)+1}{k(r-2)+1}\right) n$, for some $k \geq r-1$. Then $V(G)$ can be partitioned into $\{S_0, S_1, \ldots, S_{r-(k+1)}\}$ where each $\langle S_i \rangle$, $i \neq 0$, is K_k-free and $\langle S_0 \rangle$ is K_{k+1}-free with minimum degree larger than $\left(\frac{k^2-2k+1}{k^2-k+1}\right) |S_0|$.*

Proof. For a fixed value for k, we proceed by induction on increasing r, and note that the case where $r = k+1$ is trivial. So we assume $r \geq k+2$, and that the result holds for K_{r-1}-free graphs. Consider a K_r-free graph G, where $\delta(G) > \left(\frac{k(r-3)+1}{k(r-2)+1}\right) n$. By applying Theorem 3.1 (to G made maximally K_r-free), we know that there exists some vertex u such that $\langle H(u) \rangle$ is K_k-free.

Recall that the subgraph $\langle N(u) \rangle$ is K_{r-1}-free. Furthermore, as in Lemma 1.3 of [**4**], for any vertex $v \in \langle N(u) \rangle$ we have

$$\frac{d_{\langle N(u) \rangle}(v)}{|\langle N(u) \rangle|} \geq \frac{\delta(G) - (n - d(u))}{d(u)} \geq 2 - \frac{n}{\delta(G)} > 2 - \frac{k(r-2)+1}{k(r-3)+1} = \frac{k(r-4)+1}{k(r-3)+1}.$$

Thus we can apply the inductive hypothesis to $\langle N(u) \rangle$ to obtain the desired partition. ■

In the particular case of $k = 2$, Theorem 3.2 reduces to the following useful corollary.

Corollary 3.1: *Let $r \geq 3$, and let G be a K_r-free graph with $\delta(G) > \left(\frac{2r-5}{2r-3}\right) n$. Then $V(G)$ can be partitioned into $\{\Gamma, \overline{K_{a_1}}, \ldots, \overline{K_{a_{r-3}}}\}$ where $\langle \Gamma \rangle$ is a triangle-free graph with minimum degree larger than $|\Gamma|/3$.*

In the case of maximal K_r-free graphs, we get the following result which generalizes Corollary 2.1.

Corollary 3.2: *Let $r \geq 3$, and let G be a maximal K_r-free graph with $\delta(G) > \left(\frac{2r-5}{2r-3}\right) n$. Then $G = \langle \Gamma \rangle \vee_{j=1}^{r-3} \overline{K_{a_j}}$, where $\langle \Gamma \rangle$ is a triangle-free graph with minimum degree larger than $|\Gamma|/3$.*

Proof. Applying the previous corollary, we get a partition of G into independent sets $\overline{K_{a_j}}$ and a triangle-free graph Γ. Since the graph $\Gamma \vee_{j=1}^{r-3} \overline{K_{a_j}}$ is a maximal K_r-free graph containing G, then $G = \Gamma \vee_{j=1}^{r-3} \overline{K_{a_j}}$. ■

We note at this point that a somewhat similar structural result to Corollary 3.2 was obtained in [**7**] for K_r-free graphs, though under different restrictions than the minimum degree. Their result is that any K_r-free graph which has the property that every induced K_{r-1}-free subgraph is contained in the neighborhood of a vertex can be expressed as the join of independent sets and certain dense triangle-free graphs.

4. Application 1: The Chromatic Number and Homomorphisms of a Graph

Again, for both extensions below, the theorems of Brandt and Thomassé [**10**] and Łuczak [**40**] can be applied to the triangle-free subgraph $\langle \Gamma \rangle$ obtained in Corollary 3.2.

COROLLARY 3.3: *For $r \geq 3$, and every $\varepsilon > 0$, there exists a constant $M(\varepsilon)$ such that every K_r-free graph on n vertices with minimum degree at least $\left(\frac{2r-5}{2r-3} + \varepsilon\right) n$, is homomorphic to a K_r-free graph on at most $M(\varepsilon)$ vertices.*

COROLLARY 3.4: *For $r \geq 3$, let G be a K_r-free graph with $\delta(G) > \frac{2r-5}{2r-3} n$. Then $\chi(G) \leq r+1$.*

Using the construction of Erdős, Hajnal, and Simonovits [**19**], the sharpness of these results can be established for all values of r. Let $G(k,\varepsilon)$ be the graphs described in the previous chapter, such that $\chi(G(k,\varepsilon)) \geq k$ and $\delta(G(k,\varepsilon)) = (1/3-\varepsilon)n_{k,\varepsilon}$, where $n_{k,\varepsilon}$ is the number of vertices in $G(k,\varepsilon)$. We can then define the graph $H(r,k,\varepsilon) = G(k,\varepsilon) \vee_{i=1}^{r-3} \overline{K_\ell}$, where $\ell = (2/3)n_{k,\varepsilon}$. This graph has $n = ((2/3)r-1)n_{k,\varepsilon}$ vertices, chromatic number at least $k+r-3$, and minimum degree larger than $((2r-5)/(2r-3)-\varepsilon)n$. We conjecture that the above approach is in some sense the only way to produce such dense graphs with high chromatic number, in that there will always be a vertex such that the subgraph induced by its neighborhood has very high chromatic number.

Lastly, as in the case of K_4-free graphs, any other result about homomorphisms of triangle-free graphs can be extended to a result about homomorphisms of dense K_r-free graphs.

5. Application 2: The Independence Number of a Graph

The next area where we can apply these ideas is in computing the independence number of a dense K_r-free graph. For K_r-free graphs where $r \geq 5$, new techniques are developed to find large independent sets. In particular, for graphs which cannot be partitioned into an independent set and a K_{r-1}-free set, a large independent set can be found.

Theorem 3.3: *Let G be a maximal K_r-free graph, $r \geq 5$, with $\delta(G) > \frac{r-3}{r-2}n$. Then either G has a maximum independent set I, such that*

$$|I| \geq 2\left[(r-2)\delta(G) - (r-3)n\right]$$

or G is the join of an independent set and a K_{r-1}-free graph.

Proof. Let I be a maximum independent set in G. If every other vertex is adjacent to all of I, then G is the join of an independent set and a K_{r-1}-free set, so we are done. Thus we may assume that there exists a vertex $w \in V(G) - I$ such that w is not adjacent to all of I (but by the maximality of I is adjacent to some of I).

Consider some vertex $x \in I$ so that x is nonadjacent to w. Then, by the maximality of G, there must exist some K_{r-2} in $N(x) \cap N(w)$. We let the set S be those $r-2$ vertices along with w, so that $\langle S \rangle = K_{r-1}$, but $S \cap I = \emptyset$. For each $s_i \in S$, we define $k_i = |I - N(s_i)|$.

Now, we will consider three cases, based on the values of the k_i. In each case, we will build a set T, so that the graph induced by T is K_{r-2}. In anticipation of applying Lemma 2.1, we define $\mathcal{A} = \{N(s_i)\}_{s_i \in T}$ and consider bounds for *Slack*$(\mathcal{A})$.

(1) *There exists some i such that $k_i \geq |I|/2$.*

In this case, we take any edge from s_i to a vertex y in I, and using Observation 3.1, extend the pair to a set of vertices T such that the graph induced by T is K_{r-2}. Then at least k_i vertices in I are nonadjacent to 2 vertices in T (namely s_i and y), which implies *Slack*$(\mathcal{A}) \geq |I|/2$.

For the remaining two cases, we may assume that any two vertices in S share a common neighbor in I.

(2) *There exists some i, j $(i \neq j)$ such that $k_i + k_j \geq |I|/2$.*

In this case, let y be any vertex in I which is adjacent to both s_i and s_j. Using Observation 3.1, we can extend (if needed) $\{s_i, s_j, y\}$ to a set of vertices T such that the graph induced by T is K_{r-2}. Since all vertices in I are nonadjacent to y, and s_i and s_j have at least $k_i + k_j$ nonadjacencies in I, then this implies that $\mathit{Slack}(\mathcal{A}) \geq k_i + k_j \geq |I|/2$.

(3) *There is no pair i, j such that $k_i + k_j \geq |I|/2$.*

Let $s_i = w$ and let s_j be any other vertex of S. Then let $T = (S - \{s_i, s_j\}) \cup \{x\}$. Every vertex $z \in I$ that is adjacent to both w and s_j must be nonadjacent to some vertex $s_i \in S - \{w, s_j\}$, otherwise we have a K_r. And there are at least $|I| - k_i - k_j$ choices for z. It follows that $\mathit{Slack}(\mathcal{A}) \geq |I|/2$.

We can then consider the independent set I', where $I' = \bigcap_{s'_i \in T} N(s'_i)$. Applying Lemma 2.1 with the bounds obtained on the quantity $\mathit{Slack}(\mathcal{A})$, we get $|I| \geq |I'| \geq (r-2)\delta(G) - (r-3)n + |I|/2$. Solving for $|I|$, we get $|I| \geq 2\left[(r-2)\delta(G) - (r-3)n\right]$. ■

We note that this bound on the size of an independent set is sharp. Let $\Gamma_{i,r} = \overline{C_{ri-1}^{i-1}}$, the complement of the $(i-1)$-th power of the cycle C_{ri-1}. The bound obtained by this theorem will be sharp for the lexicographic product $\Gamma_{2,r}\left[\overline{K_a}\right]$ for any a, and every value of $r \geq 5$.

For $r \geq 5$, we can then determine a general formula for bounds on the minimum degree which will imply that a K_r-free graph has an independent set of size at least $\frac{n}{r-1}$.

THEOREM 3.4: *Let G be a maximal K_r-free graph, $r \geq 5$, with $\delta(G) > \frac{2r^2-8r+7}{2r^2-6r+4}n$. Then G has an independent set of size at least $\frac{n}{r-1}$.*

PROOF. First, suppose that G cannot be partitioned into an independent set and the neighborhood of a vertex. Then by Theorem 3.3, G must contain an independent set of size $2\left[(r-2)\delta(G) - (r-3)n\right]$. Plugging in a minimum degree of $\delta(G) > \frac{2r^2-8r+7}{2r^2-6r+4}n$ yields an independent set of size at least $n/(r-1)$.

In the case that G can be partitioned into an independent set and the neighborhood of a vertex, we proceed by induction on r. The degree of a vertex, say u, in the independent set is at least $\frac{r-2}{r-1}n$, or the result is trivial. Considering the minimum degree of the K_{r-1}-free graph induced by this neighborhood, we get that

$$\frac{d_{\langle N(u)\rangle}(v)}{|\langle N(u)\rangle|} \geq \frac{\delta(G)-(n-d(u))}{d(u)} \geq \frac{r-1}{r-2}\frac{\delta(G)}{n} - \frac{1}{r-2}.$$

For the case $r = 5$, then

$$\frac{r-1}{r-2}\frac{\delta(G)}{n} - \frac{1}{r-2} = \frac{4}{3}\left(\frac{17}{24}\right) - \frac{1}{3} = \frac{11}{18} > \frac{20}{33}.$$

From Corollary 2.5, there is an independent set of size at least $|N(u)|/3$ contained in the K_4-free induced graph formed from the vertices in $N(u)$ (where $|N(u)| \geq \frac{3}{4}n$), and this proves the base case.

For the case $r > 5$, the following general expression is obtained:

$$\frac{r-1}{r-2}\frac{\delta(G)}{n} - \frac{1}{r-2} = \frac{2r^2-8r+7}{2(r-2)^2} - \frac{2r-4}{2(r-2)^2} = \frac{2r^2-10r+11}{2(r-2)^2} > \frac{2(r-1)^2-8(r-1)+7}{2(r-1)^2-6(r-1)+4}.$$

Therefore, the induction hypothesis can be applied to prove the result. ■

It should be noted at this point that $\frac{2r^2-8r+7}{2r^2-6r+4}n < \frac{2r-5}{2r-3}n$ (the bound of Corollary 3.1) for $r > 5$.

6. Application 3: The Binding Number and Cliques of a Graph

Insight about the structure of dense K_r-free graphs, as well as the results about the independence number of these graphs can be applied to consider the binding number of a graph. Recall the result of Kane and Mohanty [**33**], that for any K_r-free graph G, $bind(G) \geq r - 4/3$. For values of $r \geq 5$, results about the independence number of a graph can be used to decrease this bound.

THEOREM 3.5: *For $r \geq 5$, any K_r-free graph G,*

$$bind(G) < r - 3/2 - \frac{1}{4r-6}.$$

PROOF. If $\delta(G) < \frac{2r^2-8r+7}{2r^2-6r+4}n$, then taking S to be the set of vertices nonadjacent to a vertex of minimum degree, we get

$$bind(G) < \frac{n}{n-\delta(G)} \leq \frac{1}{1-\frac{2r^2-8r+7}{2r^2-6r+4}} = r - \frac{3}{2} - \frac{1}{4r-6}.$$

Otherwise, we can apply Theorem 3.4, and get an independent set of size at least $i = \frac{n}{r-1}$. Taking the independent set as the set S, the binding number is at most the following.

$$bind(G) < \frac{n-i}{i} = \frac{n}{i} - 1 = (r-1) - 1 = r - 2. \qquad \blacksquare$$

Note that the bound of this proof is not sharp, as further work can be done to optimize the tradeoff between the minimum degree and the size of the largest independent set.

7. Conclusion

In conclusion, we have extended many of the results from dense triangle-free graphs to dense K_r-free graphs. Several of the results and bounds seem a natural extension of their triangle-free counterparts, but there are still many questions which arise, especially as regards sharpness.

Finally, it is very annoying that we are unable to prove an equivalent of Theorem 2.5 even for K_4-free graphs. For example, we were unable to resolve whether it is true that a maximal K_4-free graph of minimum degree at least $n/2+1$ must have two similar vertices.

Chapter 4

Role Assignments

1. Introduction

The concept of a role assignment takes its roots from both the application of social network theory and a refinement of the concept of a graph homomorphism. Everett and Borgatti [**21**] developed role assignments as a way to map a social network so that all individuals of a similar role interact with individuals of different roles in a similar manner. Role assignments also follow naturally as refinements of graph homomorphisms, as evidenced by the definition.

A *role assignment* is a mapping r from an input graph or network, G_I, to a graph G_R (where G_R is referred to as a role graph, and the vertices of G_R as roles), i.e., $r : G_I \to G_R$, by a surjective labeling of the vertices of G with the vertices of G_R, i.e., $r : V(G) \to V(G_R)$. For $S \subseteq V(G_I)$, we define $r(S) = \{r(s) : s \in S\}$. Each role assignment must satisfy the following condition:

$$\forall v \in V(G_I), \quad r(N(v)) = N(r(v)). \tag{1}$$

Figure 4.1 gives an example of a role assignment. At this point, we allow graphs to contain loops, and both the input graph G_I and the role graph G_R may have loops. In the event that both G_I and G_R are loopless, then a role assignment from G_I to G_R implies that G_I is homomorphic to G_R. For $v \in V(G_R)$, let $r^{-1}(v) = \{u : r(u) = v\} \subseteq V(G_I)$. If a vertex $v \in V(G_R)$ has a loop, then each vertex $u \in r^{-1}(v)$ must be adjacent to another vertex in $r^{-1}(v)$. If a vertex $v \in V(G_R)$ does not have a loop, $r^{-1}(v)$ is an independent set in G_I.

If G_R is a complete graph (with no loops), then a role assignment is a partition of G_I into independent dominating sets.

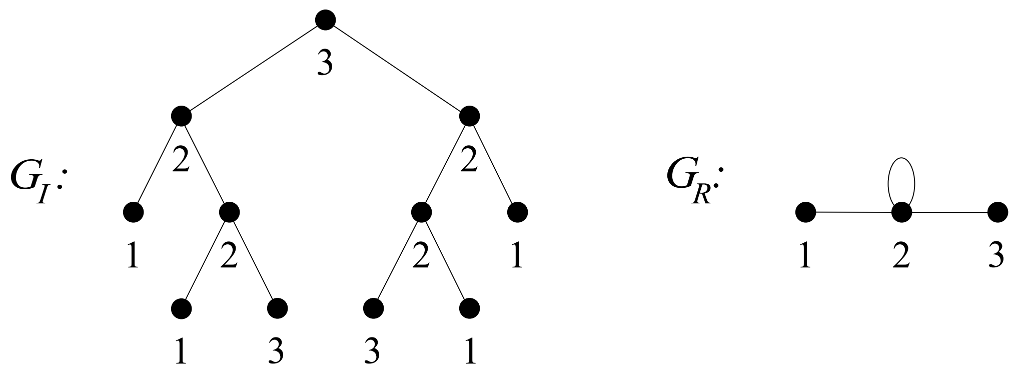

FIGURE 4.1. A sample role assignment of the vertices of G_I to the roles 1, 2, and 3 (the vertices of G_R).

Role assignments first appeared in work by Sailer [**51**] and by White and Reitz [**59**]. They have reappeared in many places, surfacing again as a particular variety of the generalized H-colorings of Kristiansen and Telle [**36**], and as a generalization of fall colorings, partitions of the vertex set of a graph into independent dominating sets, which were first studied by Dunbar et al. [**18**].

In general, the task of determining if a given graph G have a role assignment to a fixed role graph G_R is a very difficult one. In [**48**], Roberts and Sheng considered the complexity of determining whether a graph has a role assignment to one of the 6 distinct graphs on two vertices (allowing loops), and showed that for two cases the question is NP-complete, and in fact, asking if there is a role assignment to some graph on two vertices is an NP-complete question. For graph homomorphisms, a strict dichotomy of the complexity of the problem was determined by Hell and Nešetřil [**31**]. Fiala and Paulusma [**23**] obtained a similar result for role assignments, showing that for any connected fixed role graph G_R which is not K_2, the question of whether an arbitrary graph G has a role assignment to G_R is NP-complete.

In the remaining sections of this chapter, we first consider basic properties of role assignments and then consider subclasses of graphs. Chordal graphs are considered first, followed by strongly chordal graphs and then trees.

2. Properties of a Role Assignment

The first things to consider are the basic properties of role assignments. Every graph with no isolates has two trivial role assignments. The first is the identity mapping, given by $r(u) = u$ for all $u \in V(G)$, i.e., G always has a role assignment to itself. The second is the mapping of a graph to the role graph having a single vertex v with a loop, given by $r(u) = v$ for all $u \in V(G)$. It can easily be seen that both of these mappings satisfy the definition of a role assignment.

In [**23**], the following observation was made about compositions of role assignments.

Observation 4.1 ([**23**]): *Let G_1, G_2, and G_3 be graphs. If there exist role assignments $r_1 : G_1 \to G_2$ and $r_2 : G_2 \to G_3$, then there exists a role assignment $r_3 : G_1 \to G_3$ given by the composition of r_1 and r_2.*

Proof. If $r_3(v) = r_2(r_1(v))$ for $v \in V(G)$, then this gives the desired role assignment $r_3 : G_1 \to G_3$. ■

An additional observation in [**23**], especially relevant to Chapter 4, is presented here in a strengthened form. For graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the *cartesian product* $G \square H$ defines a new graph where $V(G \square H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and $E(G \square H) = \{((g, h), (g', h'))\, | g = g', (h, h') \in E(H) \text{ or } h = h', (g, g') \in E(G)\}$.

Observation 4.2 ([**23**]): *Let G_1 and G_2, as well as H_1 and H_2, be graphs. If there exist role assignments $r_1 : G_1 \to H_1$ and $r_2 : G_2 \to H_2$, then there exists a role assignment $r_3 : G_1 \square G_2 \to H_1 \square H_2$, given by the product of r_1 and r_2.*

Proof. For a vertex $(g_1, g_2) \in G_1 \square G_2$, we define $r_3((g_1, g_2)) = (r_1(g_1), r_2(g_2)) \in V(H_1 \square H_2)$. This gives the desired role assignment $r_3 : G_1 \square G_2 \to H_1 \square H_2$. ■

The last observation is that the presence of a cycle in a role graph corresponds to the presence of a cycle in the input graph.

Observation 4.3: *Let G_R be a role graph, such that there is a role assignment $r : G \to G_R$, where G is finite. If G_R contains a cycle of length ℓ, then G contains a cycle of length $k \cdot \ell$ for some integer $k > 0$.*

Proof. Suppose G_R contains the cycle C_ℓ. Then we start with any vertex $v_1 \in V(G_R)$ in C_ℓ. Take a vertex $u_1 \in r^{-1}(v_1)$. If v_2 is a neighbor of v_1 in the cycle in G_R, then there must be a vertex $u_2 \in r^{-1}(v_2)$ such that $(u_1, u_2) \in E(G)$. In this way, we can construct a path of ℓ distinct vertices in G. If the vertex u_ℓ is adjacent to u_1 in G, then we have a cycle of length ℓ. Otherwise, we have a distinct vertex in $r^{-1}(v_1)$ and we can continue the process, building the path. Since G is finite, eventually there must be some repeated vertex in this path, forming a cycle. In the event that the first vertex which is repeated is u_1, we have formed a cycle of the form $C_{k \cdot \ell}$, where k represents the number of distinct vertices from the set $r^{-1}(v_1)$ in this path. If the first repeated vertex is not u_1, we can simply restart this process choosing the first repeated vertex to be u_1. Therefore, we construct a cycle of length $C_{k \cdot \ell}$. ■

Corollary 4.1: *Let G be any tree, and r be a role assignment, such that $r : G \to G_R$, for some role graph G_R. Then G_R has no cycles (excluding loops).*

This observation gives some indication of what subclasses of graphs may be of interest to examine. In the following sections of this chapter, role assignments of chordal graphs, strongly chordal graphs, and trees are examined. In Chapter 5, the role assignments of cartesian products of graphs are discussed, with particular attention to the d-dimensional hypercube and other bipartite graphs.

2.1 *Role assignments to K_k*

A role assignment to a role graph G_R which has no loops represents a proper coloring, and in particular, a role assignment to K_k represents a partition of the vertices into independent dominating sets. In this instance, the terminology used in [**18**] is more convenient. We say that a vertex is *colorful* in a proper coloring if it has a neighbor in each color class which it is not contained in. As noted in [**18**], a coloring in which every vertex is colorful is a partition of the vertex set into independent dominating sets, so this coloring is again a role assignment to the complete graph on k vertices.

Starting with any proper coloring, it is possible in some situations to make another coloring such that a vertex v, which is not colorful becomes "closer" to being colorful by adjusting the previous coloring. We can define "closer" by requiring that in the new coloring, on the same number of colors, the number of distinct colors that appear in the neighborhood of v increases by one. In order to do this, we need to consider a *cutset* of a graph. Suppose that a graph G is a connected graph. A subset of the vertices $S \subseteq V(G)$ is a cutset if the removal of S breaks G into disconnected components.

Theorem 4.1 (Cutset Theorem): *Consider a graph $G = (V, E)$ with a k-coloring f, and a vertex v, where v is not colorful, with $x, y \in N(v)$ such that $f(x) = f(y)$. If there is a cutset $S \subseteq N(x) \cap N(y)$ which separates x and y, then a k-coloring f' can be obtained such that $\left|f'|_{N(v)}\right| = \left|f|_{N(v)}\right| + 1$, and the vertices that are colorful in f are colorful in f', where $f'|_{N(v)}$ denotes the set of colors of f' restricted to $N(v)$ and similar notation for $f|_{N(v)}$.*

Proof. Since v is not colorful, there exists a color, say b, which does not appear in $N(v)$. If $f(x) = f(y) = a$, then no vertex in S has the color a.

We define C_1 as the connected component that would contain x if S were removed, and $C_2 = V(G) - (S \cup C_1)$. Now in C_1, any vertex with the color a can be recolored with the color b, and any vertex previously colored b can be recolored with the color a. Since vertices labeled a and b do not appear in S, this defines a new proper k-coloring f'.

Vertices labeled a and b do not appear in S, and there are no edges from C_1 to C_2, so every vertex that was colorful in C_1 remains colorful. Again, no edges go from C_1 to C_2 and vertex colors in S were not changed, so any vertices that were colorful in C_2 remain colorful. Lastly, considering S, all vertices in S are adjacent to x and y with the colors a

and b, so any vertex that was colorful in S remains colorful. Finally, we can see that adding the color b to $N(v)$ increases the number of colors that in the neighborhood of v. ■

If the graph satisfies certain restrictions, we may be able to apply this method repeatedly to make vertices colorful. If every vertex can be made colorful in a proper k-coloring, then this coloring gives a role assignment to K_k. We consider applying this result to chordal graphs.

3. Chordal Graphs

The first subclass of graphs we will consider are *chordal graphs.* The class of chordal graphs is defined as the class of graphs which do not contain any C_k, $k \geq 4$, as an induced subgraph. Chordal graphs were first considered in [**27**], and have practical application in modeling the process of Gaussian elimination with minimal fill-in [**47, 50**].

One advantage in dealing with chordal graphs is that there are many alternative characterizations of a chordal graph. A *simplicial vertex* v is a vertex such that $\langle N(v) \rangle$ forms a clique. A *simplicial elimination ordering*, also referred to as a *perfect elimination ordering* (PEO), is an ordering on the vertices of the graph, such that the vertex v_1 is a simplicial vertex, and upon the removal of the vertices $v_1, v_2, \ldots, v_{k-1}$, the vertex v_k is a simplicial vertex in the remaining graph. One characteristic of chordal graphs can be given in terms of a simplicial elimination ordering.

Chordal Characterization 1 ([**16**]): *Chordal graphs are exactly those graphs for which there is a simplicial elimination ordering.*

This characterization can be used to provide a fast method for determining $\omega(G)$, the size of the largest clique, for the class of chordal graphs. Additionally, in [**8**], it was shown that $\chi(H) = \omega(H)$ for all induced subgraphs H of a chordal graph G. Therefore, the chromatic number of chordal graphs can be efficiently computed.

Using this characterization of chordal graphs, we can determine a necessary condition for a chordal graph to have a role assignment to K_k.

Lemma 4.1: *Let G be a chordal graph. If there exists some role assignment from G to K_k, then $\delta(G) + 1 = \omega(G) = k$.*

Proof. Suppose that there exists some role assignment from G to K_k. Consider a simplicial vertex v of G. The neighbors of v, along with v, form a clique of size at least $\delta(G) + 1$, so $\omega(G) \geq \delta(G) + 1$.

It must be the case that $k \leq \delta(G) + 1$ as every vertex must have an edge to a vertex of each color apart from its own, and also $k \geq \chi(G) = \omega(G)$, since this is a proper coloring. Thus, if $\omega(G) > \delta(G) + 1$, there can be no role assignment to K_k. There can only be a role assignment to K_k if $\delta(G) + 1 = \omega(G) = k$. ■

Since $\chi(G) = \omega(G)$ for chordal graphs, if there is a role assignment from G to K_k then $k = \chi(G)$. For this reason, we consider modifying a proper coloring of the chordal graph G which has $\chi(G)$ colors.

3.1 *Role assignments on chordal graphs where $\omega(G) \leq 4$*

For chordal graphs where $\delta(G) + 1 = \omega(G) = k$ for small values of k, we show that the Cutset Theorem can be used repeatedly on any vertex that is not colorful in a $\chi(G)$ coloring to generate a role assignment of the graph G to K_k. We note that in any proper k-coloring, a vertex in a clique of size k is colorful already. To begin with, we note that the case $k = 2$ is trivial.

Theorem 4.2: *Let G be a chordal graph G with $\delta(G) + 1 = \omega(G) = 2$. Then there exists a role assignment $r : G \to K_2$.*

Proof. Here, the only chordal graphs which satisfy $\delta(G)+1 = \omega(G) = 2$ are collections of trees. Since these graphs are bipartite with no isolates, there is a role assignment to K_2. ■

Therefore, the first value for which we apply Theorem 4.1 is $k = 3$.

Theorem 4.3: *Let G be a chordal graph G with $\delta(G) + 1 = \omega(G) = 3$. Then there exists a role assignment $r : G \to K_3$.*

Proof. We consider a proper 3-coloring which has the maximum number of colorful vertices. If every vertex is colorful, then this gives a role assignment $r : G \to K_3$. Therefore, assume for contradiction that there exists a vertex v which is not colorful in this coloring.

Since v is not colorful, it sees only one color, so $N(v)$ is independent. Since v has at least two neighbors, then there exists $x, y \in N(v)$ such that x and y have the same color. Since G is chordal and xvy is a path, any vertex u in a vertex disjoint path from x to y must be adjacent to v, or a chordless 4-cycle will exist. Since x, v, u would form a clique, we can conclude that the vertex v is a cutset, and v satisfies the conditions of Theorem 4.1. Therefore, we have a contradiction, as there is a proper 3-coloring with a larger number of colorful vertices. ■

For the case where $k = 4$, the neighborhoods of a vertex which is not colorful can become more complex, but the same result can still be achieved.

Theorem 4.4: *Let G be a chordal graph G with $\delta(G) + 1 = \omega(G) = 4$. Then there exists a role assignment $r : G \to K_4$.*

Proof. We consider a proper 4-coloring which has the maximum number of colorful vertices. If every vertex is colorful, then this gives a role assignment $r : G \to K_4$. Therefore, assume for contradiction that there exists a vertex v which is not colorful in this coloring.

The graph induced by $N(v)$ has at most two colors in the given coloring, and since G is chordal, must contain no cycles. Thus the graph induced by $N(v)$ must be a collection of trees and/or isolated vertices, and must be 2-colored in the given coloring.

If $N(v)$ is disconnected, the removal of v disconnects the graph, as any path between vertices $x, y \in N(v)$ must be contained in $N(v)$ or else a chordless cycle of length four would be present. In this case, we can choose vertices x and y from two distinct components which have the same color, and using the vertex v as a cutset, apply Theorem 4.1.

If $N(v)$ is connected, the graph induced by it is a tree with at least 3 vertices (since $\delta(G) \geq 3$). Therefore, there is a path $xwy \in N(v)$ such that x and y have the same color.

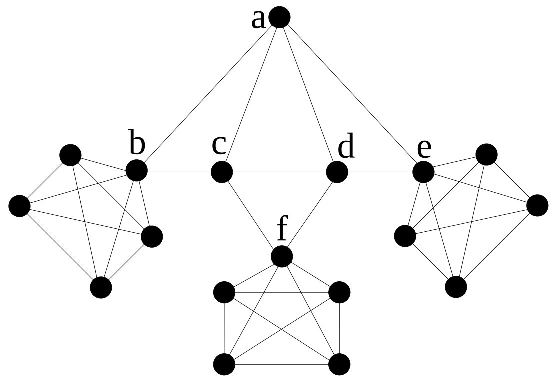

Figure 4.2. A chordal graph with no role assignment to K_5.

Removal of the set $\{v, w\}$ disconnects the graph since there are no other vertex disjoint paths between x and y. Thus, we can use $\{v, w\}$ as the cutset and apply Theorem 4.1.

In this manner, the vertex v can be made colorful. Therefore, there is a contradiction, as a proper 4-coloring can be found with more colorful vertices. ■

These results combine to give the following theorem.

Theorem 4.5: *For a chordal graph G with $k \leq 4$, $r : G \to K_k$ if and only if $\delta(G) + 1 = \omega(G) = k$.*

3.2 *Role assignments on chordal graphs where $\omega(G) = 5$*

At this point, we have shown that all chordal graphs that satisfy $\delta(G) + 1 = \omega(G) = k$ have role assignments to a complete graph for $k \leq 4$. However, for $k = 5$, we introduce the first example of a graph which satisfies $\delta(G) + 1 = \omega(G) = k$ and yet has no role assignment to K_5.

Theorem 4.6: *Consider the chordal graph G shown in Figure 4.2 with $\delta(G)+1 = \omega(G) = 5$. There does not exists a role assignment $r : G \to K_5$.*

Proof. First, we can easily see that G is indeed chordal with $\delta(G) + 1 = \omega(G) = 5$. Vertices a, c, and d are all minimum degree vertices. Thus, for a to be colorful, the vertices b, c, d, and e must all have distinct colors. Then for c to be colorful, f must have the same color as e. Also, for d to be colorful, f must have the same color as b. Since we have already stated that b and e must have different colors, G does not have a role assignment to K_5. ■

We note that the vertices a, b, c, d, e, and f form a sun graph, a forbidden subgraph in the class of strongly chordal graphs, which will be discussed in Section 4. This leads to the consideration of strongly chordal graphs as the next class of graphs to examine. However, first we consider the complexity of the problem of determining when a chordal graph has a role assignment to a complete graph.

3.3 *Complexity of Role Assignments of Chordal Graphs*

Now that we have examples of chordal graphs which have a role assignment to a complete graph, as well as an example of a chordal graph which does not, we consider the complexity of taking a chordal graph and determining which of those two categories it falls into. Since the question of determining $\delta(G)$ is simple for any graph, and determining $\omega(G)$ for the class of chordal graphs can be done in polynomial time, the question of the complexity of this problem lies with the chordal graphs which satisfy $\delta(G) + 1 = \omega(G)$. To begin with, we consider the following problem.

CLIQUE ROLE ASSIGNMENT (CRA)
INSTANCE: A graph $G = (V, E)$
QUESTION: Does there exist some k for which there is a role assignment from G to K_k ?

This problem was determined to be NP-complete for general graphs [**18**]. This result can be further refined such that Problem CRA is determined to be NP-complete even when restricted to the class of chordal graphs.

Theorem 4.7: *Problem CRA is NP-complete, even when restricted to chordal graphs where* $\delta(G) + 1 = \omega(G)$.

Proof. First, for some mapping $f : V \to \mathbb{Z}^+$ by looking at each vertex, and scanning all of its neighbors, we can verify whether or not f is a role assignment mapping G to K_k. Since this takes polynomial time, Problem CRA is in the class NP.

To show that Problem CRA is NP-complete, we use a polynomial transformation from NOT-ALL-EQUAL-3SAT. An instance ϕ of NOT-ALL-EQUAL-3SAT is composed of two things: a set of variables, $X = \{x_i\}_{i=1}^m$ and a set of clauses, $C = \{c_k\}_{k=1}^n$ for some $m, n > 0$, where each clause consists of three literals, where a literal is either a variable x_i or its negation $\overline{x_i}$. The instance ϕ is a member of NOT-ALL-EQUAL-3SAT if there exists an assignment of truth values such that this assignment, as well as its complement, satisfy ϕ, which means that each clause must contain at least one true literal and one false literal.

We can transform ϕ to a graph G_ϕ by creating a vertex to represent each clause, which we denote v_{c_k}, $\forall k \leq n$, and a vertex for each literal, v_{x_i} and $v_{\bar{x}_i}$, $\forall i \leq m$. We also add one additional vertex for each variable, denoted v_i, $\forall i \leq m$, which we will refer to as the variable clause vertex. Lastly, add a set of $m + n + 1$ vertices for each v_{x_i} and $v_{\bar{x}_i}$, which we denote as V_{x_i} and $V_{\bar{x}_i}$. We see that the total number of vertices is $n + 2m + m + 2m(m + n + 1) = 2m^2 + 2mn + 5m + n$, where $m < 3n$.

Now we add edges to G_ϕ by first making all the clause vertices v_{c_k}, $\forall k \leq n$, and all the variable clause vertices v_i, $\forall i \leq m$, into one large clique. Each clause vertex v_{c_k} is joined to the 3 vertices which correspond to the literal values of the clause c_k (v_{x_i} and $v_{\bar{x}_i}$ vertices). For each $i \leq m$, the variable clause vertex v_i is joined to both v_{x_i} and $v_{\bar{x}_i}$. We also note that for each clause vertex v_{c_k}, $d(v_{c_k}) = (m+n-1)+3 = m+n+2$, and for each variable clause vertex v_i, $d(v_i) = (m + n - 1) + 2 = m + n + 1$. Finally, each vertex set V_{x_i} should form a clique along with v_{x_i}, and each vertex set $V_{\bar{x}_i}$ should form a clique along with $v_{\bar{x}_i}$. Then we see that both sets $\{V_{x_i} + v_{x_i}\}$ and $\{V_{\bar{x}_i} + v_{\bar{x}_i}\}$ form cliques of size $m + n + 2$. Also, we can see that every vertex has degree greater than or equal to $m + n + 1$.

This forms the graph G_ϕ, shown in Figure 4.3, by a polynomial construction. The graph G_ϕ has been constructed such that $\delta(G_\phi) = m+n+1$ and $\omega(G_\phi) = m+n+2$. We can also show that G_ϕ is chordal by defining a perfect elimination ordering. If we remove the vertices by first taking all vertices that occur in some V_{x_i} or $V_{\bar{x}_i}$, then their neighborhoods form cliques. Then we can remove the vertices v_{x_i} and $v_{\bar{x}_i}$, $\forall i \leq m$, because their only remaining neighbors are vertices v_{c_k} and v_i for some values of i and k, and vertices of that

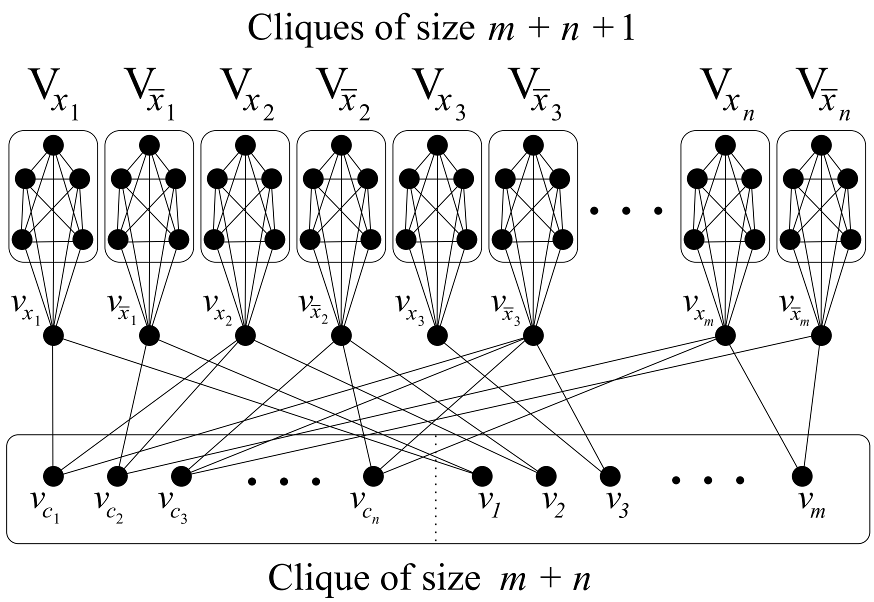

FIGURE 4.3. The graph G_ϕ

type form a large clique. Once those vertices are removed, we only have v_{c_k} and v_i vertices left for all values of i and k, and since these form a large clique, we can remove them in any order. Therefore, G_ϕ is chordal.

If we assume $\phi \in$ NOT-ALL-EQUAL-3SAT, then there is a truth assignment to each of the literals, so that for x_i and $\bar{x}_i$, one is true and one is false, and for each clause, there is one false value, and one true value. We can let the assignment of the literals of ϕ be interpreted as the colors T and F in G_ϕ on the vertices x_i and $\bar{x}_i$. The clause and variable clause clique in G_ϕ has $m+n$ vertices, and each of those can be assigned a distinct color, not T and not F. Then there are $m+n+2$ colors on $V(G_\phi)$, and we can see that each clause and variable clause will be colorful, since they are adjacent to $m+n$ colors in their clause, and also T and F. Now in V_{x_i} and $V_{\bar{x}_i}$, we can assign one vertex to have the color T or F that its associated literal does not have. Then we can color the other $m+n$ vertices with the distinct $m+n$ colors used in the clause and variable clause clique. Then every vertex in $\{V_{x_i} + v_{x_i}\}$ and $\{V_{\bar{x}_i} + v_{\bar{x}_i}\}$ is colorful, and so G_ϕ has a role assignment to K_k.

If we assume G_ϕ has a role assignment to K_k, we can then show that $\phi \in$ NOT-ALL-EQUAL-3SAT. Since G_ϕ is chordal, and $\delta(G_\phi)+1 = \omega(G_\phi) = m+n+2$, then $k = m+n+2$. Therefore each variable clause vertex v_i sees $m+n$ colors in the clique of clause and variable clause vertices, and two additional colors on the literal vertices, v_{x_i} and $v_{\bar{x}_i}$. Thus, for a given i, v_{x_i} and $v_{\bar{x}_i}$ have different colors, and all pairs v_{x_i} and $v_{\bar{x}_i}$ must have the same two colors, since every variable clause vertex is missing the same two colors. If we let those two colors represent TRUE and FALSE, we can see that each clause vertex is colorful if and only if it is adjacent to one literal vertex with the color TRUE and one literal vertex with the color FALSE. Then we can see that the colors TRUE and FALSE are a NOT-ALL-EQUAL-3SAT assignment on the literals, $X = \{x_i\}_{i=1}^m$. Therefore, we have shown that $\phi \in$ NOT-ALL-EQUAL-3SAT if and only if $G_\phi \in$ CLIQUE ROLE ASSIGNMENT. ■

In [**1**], Acharya and Walikar showed that no finite characterization by forbidden subgraphs exists for determining if a general graph G has a role assignment to K_k for some value of k. Using the same technique, with a different construction, we can restrict the result to chordal graphs where $\delta(G) + 1 = \omega(G)$.

Theorem 4.8: *For any k, every chordal graph G where $\delta(G) + 1 = \omega(G) = k$ can be embedded in another chordal graph G' where $\delta(G') + 1 = \omega(G') = k$ and G' has a role assignment to K_k.*

Proof. First, if G has a role assignment to K_k, then the theorem is trivially true. If G does not have a role assignment to K_k, we can take any minimum coloring f, with $\chi(G) = \omega(G) = k$ colors. To form the graph G', we consider every vertex in G. If in f, a vertex, say v, is missing a color from its neighborhood, we can add a clique of size k, and add an edge from one vertex in the clique to v. This forms a graph G' where G is a subgraph, and every vertex has an attachment for each missing color. Each clique can be colored with k colors, so that the missing color is on the adjacent vertex. Furthermore, in a PEO, all nonadjacent vertices in the added cliques can be removed, then all the adjacent vertices, and then the original vertices can be removed. Thus, G' is chordal, with $\delta(G') + 1 = \omega(G') = k$. ■

COROLLARY 4.2: *The class of chordal graphs which have a role assignment $r : G \to K_k$ cannot be characterized by a finite family of forbidden subgraphs, for any value k.*

Note that this result implies the result of Acharya and Walikar. In light of the NP-completeness result, as well as the fact that there is no forbidden subgraph characterization for any particular value of k, the question of determining if a chordal graph has a role assignment to K_k seems to be difficult for larger values of k. Therefore, we restrict our attention to a subclass of chordal graphs, the class of strongly chordal graphs.

4. Strongly Chordal Graphs

Similarly to chordal graphs, a *strongly chordal graph* can be defined in terms of its induced subgraphs. Chordal graphs are defined as graphs which have the property that every cycle of length four or larger has a chord. Strongly chordal graphs are chordal graphs with the additional property that every even cycle of length six or larger has a *strong chord*, a chord where the distance along the cycle between the two endpoints of the chord is odd. This class of graphs was introduced by Farber in [**22**], which included several alternative characterizations of a strongly chordal graph. One characterization is that the vertices of a strongly chordal graph can be ordered in a *strong elimination ordering* (SEO). A strong elimination ordering is a perfect elimination ordering with the additional requirement that for $i < j < k < l$, if (v_i, v_k), (v_i, v_ℓ), $(v_j, v_k) \in E(G)$, then $(v_j, v_\ell) \in E(G)$.

STRONGLY CHORDAL CHARACTERIZATION 1 ([**22**]): *Strongly chordal graphs are exactly those graphs for which there is a strong elimination ordering.*

A *k-sun*, or a *k-trampoline*, is the graph obtained by taking an cycle of length $2k$, where k is at least three, and adding edges to form a clique of size k of all the vertices of even index (See Figure 4.4). Another characterization of strongly chordal graphs can be given by considering a forbidden subgraph characterization within the class of chordal graphs.

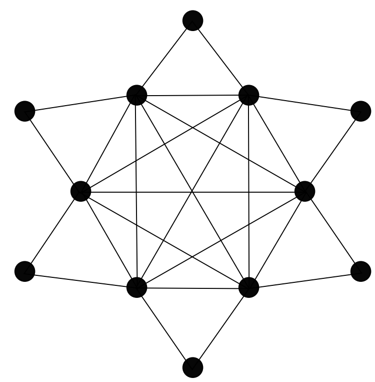

Figure 4.4. A 6-sun graph

Strongly Chordal Characterization 2 ([**22**], [**11**]): *Strongly chordal graphs are exactly those chordal graphs which contain no sun as an induced graph.*

The class of strongly chordal graphs was defined in order to consider a subclass of chordal graphs where the weighted dominating set problem becomes tractable. This class seems to provide the "right" amount of structure, such that many problems become tractable when considered on strongly chordal graphs. Notable subclasses of strongly chordal graphs include interval graphs and trees.

4.1 *Description of Role Assignment Algorithm*

Strongly chordal graphs are a subclass of chordal graphs, so the question of determining whether a strongly chordal graph G has a role assignment $r : G \to K_k$ can again be restricted to those graphs for which $\delta(G) + 1 = \omega(G)$. In this case, however, the problem is more tractable. In particular, we present a simple greedy algorithm, Algorithm **SCRA**, which processes each vertex in the reverse order of a strong elimination ordering and forces it to be colorful in a $\delta(G) + 1$ coloring. We will then show that this algorithm correctly determines a role assignment from a strongly chordal graph where $\delta(G) + 1 = \omega(G) = k$ to the complete graph K_k. The results of this section form the basis of [**42**].

Algorithm 1 SCRA (Strongly Chordal Role Assignments)

Require: A strongly chordal graph $G = (V, E)$ with $\delta(G) + 1 = \omega(G)$, and a SEO with adjacency lists of the highest ordered $\delta(G)+1$ ($\delta(G)$ for minimum degree vertices) sorted adjacencies for each vertex
Ensure: A role assignment $r : G \rightarrow K_k$, or message that no such assignment is possible.
  for $i = n$ to 1 **do**
    Let S_i be the sorted adjacency list for the vertex v_i.
    if Vertex v_i has not been assigned a color **then**
      Check S_i for unused color.
      if No color is possible **then**
        Exit - No role assignment possible
      else
        Assign v_i an unused color
    Check S_i and v_i for colors needed
    Assign necessary colors to uncolored vertices in S_i from highest order to lowest, in order to make v_i colorful.

4.2 *Correctness*

Now we prove the correctness of Algorithm **SCRA**. To begin with, we prove a lemma which shows that at every step the previously colored vertices adjacent to the vertex being processed are colored distinctly.

Lemma 4.2: *At each execution step of **SCRA**, any colored vertices adjacent to a vertex being processed have distinct colors and **SCRA** retains a proper coloring.*

Proof. If we consider the base case, the first vertex is not adjacent to any colored vertices and **SCRA** will label its neighbors 2,3, $\ldots$, $\delta(G) + 1$, all distinct. Now we can inductively assume that when processing vertex v_r, a proper coloring is present and show that the coloring is extended and a distinct coloring is present. First we show that all the colored vertices adjacent to v_r have distinct colors. This is done by considering vertices v_i and v_j adjacent to v_r such that for the coloring f, $f(v_i) = f(v_j) = \ell$ for some color ℓ. We consider three cases for contradiction.

(1) *For vertex v_r, $r < i < j$.*

The vertices are arranged in a reverse perfect elimination ordering, which is additionally a strong elimination ordering. Thus, if (v_i, v_r), $(v_j, v_r) \in E(G)$ and

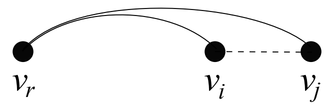

Figure 4.5. Illustration of case 1

$r < i < j$, then $(v_i, v_j) \in E(G)$. By induction, we may assume the algorithm has maintained a proper coloring to this point, hence v_i and v_j could not have both been assigned the color ℓ.

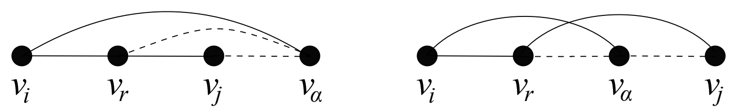

Figure 4.6. Illustration of case 2

(2) *For vertex v_r, $i < r < j$.*

For v_i to have a color, it must have been colored by some vertex v_α where $\alpha > r$. Since (v_r, v_i), $(v_\alpha, v_i) \in E(G)$, and $r, \alpha > i$, then $(v_\alpha, v_r) \in E(G)$. Now if (v_α, v_r), $(v_j, v_r) \in E(G)$, then $(v_\alpha, v_j) \in E(G)$. Thus the vertex v_α was already adjacent to a lower indexed vertex with the color ℓ and so would not have colored v_i with the color ℓ. Hence, v_i and v_j could not have both been assigned the color ℓ.

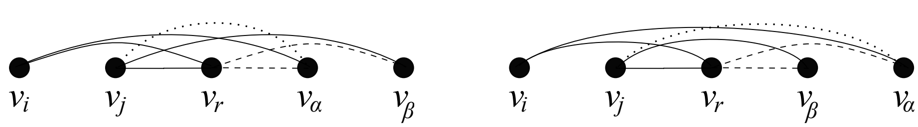

Figure 4.7. Illustration of case 3

(3) *For vertex v_r, $i < j < r$.*

The vertex v_i must then have been colored by some vertex v_α where $\alpha > r$, and also v_j was colored by some vertex v_β where $\beta > r$. From the perfect elimination ordering, if (v_r, v_i), $(v_\alpha, v_i) \in E(G)$ then $(v_r, v_\alpha) \in E(G)$, and also, if (v_r, v_j), $(v_\beta, v_j) \in E(G)$ then $(v_r, v_\beta) \in E(G)$. Now using the extra condition of a strong elimination ordering, if $i < j < r < \alpha$ and (v_i, v_r),(v_i, v_α),$(v_j, v_r) \in E(G)$, then

$(v_j, v_\alpha) \in E(G)$. If v_α is adjacent to a lower indexed vertex v_j, colored ℓ, it would not have colored v_i. Hence, v_i and v_j could not have both been assigned the color ℓ.

This proves that the colored vertices adjacent to v_r all have distinct colors. Finally, we want to prove that coloring the uncolored vertices adjacent to v_r with the unused colors preserves a proper coloring. If v_r is coloring its neighbor v_k with color k, then we assume for contradiction that there is another vertex v'_k such that v'_k already has color k and v'_k is adjacent to v_k. If $k' > k$, then v_r is adjacent to v'_k and **SCRA** will not attempt to color v_k with the same color. If $k < k'$, we can see that v'_k was colored by some v_α, which implies that v_α is adjacent to v_k, since $\alpha, k < k'$. Thus v_α would have colored the vertex v_k before v'_k.

Thus we can see that at each execution step, the colors surrounding v_r are distinct, and a proper coloring is propagated. ■

Theorem 4.9: *Algorithm **SCRA** produces a role assignment $r : G \to K_k$ on strongly chordal graphs where $\delta(G) + 1 = \omega(G) = k$.*

Proof. Three things must be considered in order to prove the correctness of Algorithm **SCRA**; The first step executes correctly, each additional step executes correctly, and the last step terminates correctly. To begin with, we can easily see that the first vertex processed can choose 1, and label its first $\delta(G)$ neighbors 2,3, ..., $\delta(G)+1$. Now we need to show that each step of the algorithm can be completed, and completion yields a colorful vertex. To begin with, we consider if a vertex v_r being processed is uncolored by any previous vertex.

Here it is necessary to show that there exists a color for v_r to take. If v_r is adjacent to all of the $\delta(G) + 1$ colors, we first assume that there is a v_α, such that $\alpha < r$, that is colored. This was colored by some v_β where $\beta > r$, and therefore we know that since (v_β, v_α) and (v_r, v_α) are edges, then the edge $(v_r, v_\beta) \in E$, and then v_β would have colored v_r before v_α. We can then assume that there is no adjacent colored vertex with index smaller than r. Therefore, v_r is adjacent to all $\delta(G)+1$ colors by vertices ordered below it, which means there are $\delta(G)+1$ vertices adjacent to v_r ordered below it. Since in a perfect elimination ordering, a vertex is simplicial in the vertices labeled after it, this implies that there is a clique of size at least $\delta(G)+2$, which is a contradiction. Therefore, a color always exists for an uncolored vertex.

Thus we can assign a color to an uncolored vertex. Now we scan for available colors, and using the lemma above, since all the vertices adjacent to a v_r have a distinct color, and since v_r has degree at least $\delta(G)$, it is possible to make v_r colorful by coloring its remaining uncolored neighbors.

The coloring at this point is proper, and each vertex processed is colorful. The last step, useful in the complexity analysis, is proving that only the highest indexed (in the SEO) $\delta(G)$ neighbors to v_r need to be considered in coloring.

Again for vertex v_r, we assume that there is an adjacent vertex v_k that has a color and is not one of the highest $\delta(G)$ vertices adjacent to v_r. Then there has to be some v_α that colors v_k. Now since v_r and v_α are adjacent to v_k, and $k < r, \alpha$, then $(v_r, v_\alpha) \in E$. Now we can split the highest indexed $\delta(G)$ neighbors of v_r into two sets, letting V_L represent the vertices with lower index than v_r, and V_H represent the vertices with higher index. Now since $r < h, \alpha$ for any $v_h \in V_H$, we have $(v_h, v_\alpha) \in E$ for every vertex $v_h \in V_H$. Also we have that $k < \ell < r < \alpha$ for every vertex $v_\ell \in V_L$, and we have the edges (v_k, v_r), (v_r, v_ℓ), (v_k, v_α), so we know that from the strong elimination ordering, that the edge (v_ℓ, v_α), for all $v_\ell \in V_L$. Thus v_α is adjacent to all of the $\delta(G)$ neighbors of v_r (minus itself) and v_r, which all have index higher than v_k. Thus v_α would never color the vertex v_k, and v_r has no colored neighbors that are beyond the highest indexed $\delta(G)$.

The last step is to prove the algorithm terminates correctly. Since the last vertex processed is simplicial in G, its closed neighborhood forms a clique of size $\delta(G)+1$, and all of them have distinct colors. ■

We have shown that this algorithm works for strongly chordal graphs where $\delta(G)+1=\omega(G)$. We can write the following theorem that characterizes the class of strongly chordal graphs which have a role assignment to K_k.

THEOREM 4.10: *For a strongly chordal graph G, there exists a role assignment from G to K_k if and only if $\delta(G) + 1 = \omega(G) = k$.*

PROOF. This follows from the more general result for chordal graphs, and the correctness of the algorithm. ■

Lastly, we note the complexity of implementing this algorithm.

THEOREM 4.11: *Algorithm* ***SCRA*** *runs in $O(\delta(G) \cdot n)$ time*

PROOF. This is easy to see, as each vertex v_i is required to scan S_i, where $|S_i| = \delta(G) + 1$. ■

However, we note that algorithm **SCRA** takes as input a strong elimination ordering of the vertices. Currently, the best known algorithms for finding a strong elimination ordering run in time $O(\min \{n^2, m \log n\})$. The $O(m \log n)$ algorithm for more sparse graphs is due to Paige and Tarjan [**46**], and the $O(n^2)$ algorithm is due to Spinrad [**55**].

5. Role Assignments of Trees

We can further narrow our class of graphs, looking at the set of all trees. In this case, we have the trivial role assignment to K_2, and there is no assignment to K_k for $k \geq 3$. Therefore, we want to consider the more general question. Given a tree T and a role graph G_R, can we determine if there is a role assignment $r : T \to G_R$?

In this section, we present an algorithm which, when given an input tree T and a role graph G_R, determines if there is a role assignment r, such that $r : T \to G_R$. We let n_T be the number of vertices in T and n_R be the number of vertices in G_R, and we note that $|E(T)| = n_T - 1$, and $|E(G_R)| \leq 2n_R - 1$ by Corollary 4.1. The results of this section form a basis for [**41**].

5.1 *Description of Algorithm*

For the graph G_R, create an indexed set by taking each loop edge, and considering each non-loop edge as two directed edges, and define this set as $\overrightarrow{E}(G_R)$ (See Figure 4.8

for an example). Then for each vertex $u_j \in V(G_R)$, we construct the set $A_j = \{a_i : e_{a_i} = (u_k, u_j) \in \overrightarrow{E}(G_R)\}$. This set holds the indices of all directed edges whose head is the vertex u_j. A *postorder traversal* is a traversal of the vertices in a tree, such that each subtree is recursively processed, and then the root is finally processed. We can root the tree T at any vertex, and then order the vertices by a postorder traversal of the tree. For each vertex, v_i, we construct a boolean array of size $|\overrightarrow{E}(G_R)|$, which we will denote P_i, an array representing the possibility that the edge from a vertex to its parent can be assigned to each directed edge $e_j \in \overrightarrow{E}(G_R)$. Additionally, we form the sets C_i, which hold the indices of the children of vertex v_i in T.

The algorithm processes vertices by working up from the leaves of the tree T. At each step, we consider the edge between a vertex v_i and its parent, say v_p, and determine the possible edges in G_R that this edge can be mapped to. In order to aid our discussion, we define a *partial role assignment* on a subtree of T. We say that the subtree *generated* by a vertex v_i, denoted $T\langle v_i \rangle$, is the subtree induced by the descendants of v_i. We say that a mapping $r_{i,j}$ is a partial role assignment on $T\langle v_i \rangle$, extendable with edge $e_j = (u_a, u_b) \in \overrightarrow{E}(G_R)$, if

$$
\begin{aligned}
&r_{i,j}(N(v)) = N(r_{i,j}(v)) && \text{for } v \in T\langle v_i \rangle - \{v_i\}; \text{ and} \\
&N(u_a) - \{u_b\} \subseteq r_{i,j}(N(v_i) - \{v_p\}) \subseteq N(u_a) && \text{where } r_{i,j}(v_i) = u_a
\end{aligned}
\tag{2}
$$

Now the key point is that if there exists a partial role assignment on $T\langle v_i \rangle$, extendable with edge $e_j = (u_a, u_b) \in \overrightarrow{E}(G_R)$, then there must be some mapping of the children of v_i to partial role assignments extendable on each of the adjacencies into u_a (with the possible exception of edge e_j), and each child must be mapped. The task of constructing and solving this bipartite matching problem is handled by the subroutine MATCHING.

At vertex v_{n_T}, the root of T, we must determine whether there is some role $u_a \in V(G_R)$ which we can assign v_{n_T} so that partial role assignments on the child subtrees of v_{n_R} can be found where one child is assigned to each of the adjacencies of u_a in G_R, accomplished by a final matching problem. If a suitable matching is found, then there

Algorithm 2 MATCHING (C', A')

$V_A = \{s_i : i \le |A'|\}$, $V_B = \{t_i : i \le |C'|\}$, $E = \emptyset$, $M^* = \emptyset$
if $|A'| \le |C'|$ **then**
  for $k = 1$ to $|V_A|$, $l = 1$ to $|V_B|$ **do**
    if $P_{c_l}[a_k] = 1$ **then**
      Add edge (s_k, t_l) to E.
  Find a max matching M^* of $G = (V_A \cup V_B, E)$

Algorithm 3 TRA (Tree Role Assignments)

INITIALIZATION
for $i = 1$ to $n_T - 1$ **do**
  for $j = 1$ to $|\overrightarrow{E}(G_R)|$ **do**
    For $e_j = (u_{j_1}, u_{j_2})$, let $e_{r_j} = (u_{j_2}, u_{j_1})$
    if every $v_{c_\alpha} \in C_i$, has some $a_j \in A_{j_2}$ such that $P_\alpha[a_j] = 1$ **then**
      $M^* = \text{MATCHING}(C_i, A_{j_2} - \{r_j\})$.
      if $|M^*| = |A_{j_2} - \{r_j\}|$ **then**
        $P_i[j] = 1$.
for $j = 1$ to n_R **do**
  $M^* = \text{MATCHING}(C_i, A_j)$.
  if $|M^*| = |A_j|$ **then**
    There is a valid role assignment.

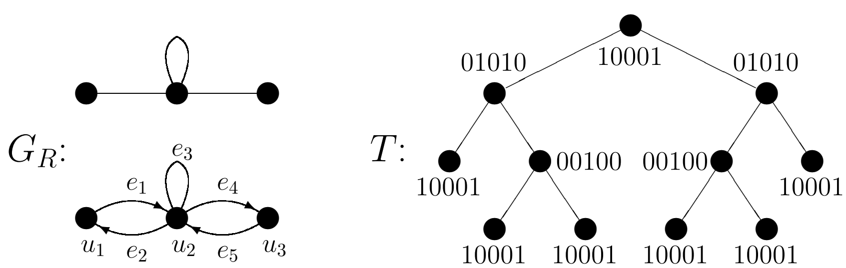

Figure 4.8. Algorithm execution, with P_i vectors shown for each vertex in T, where $A_1 = \{2\}$, $A_2 = \{1, 3, 5\}$, and $A_3 = \{4\}$.

exists a role assignment from T to G_R, and if no matching is found, there is no such role assignment.

5.2 *Correctness*

In order to prove the correctness of the Algorithm **TRA**, we need to show that the array P_i for each vertex in T correctly reflects the fact that a partial role assignment either does or does not exist.

Lemma 4.3: *For a processed vertex $v_i \in V(T)$ and $e_j = (u_a, u_b) \in \overrightarrow{E}(G_R)$, then $P_i[j] = 1$ if and only if there exists a partial role assignment on $T\langle v_i \rangle$, extendable with edge e_j.*

Proof. We proceed by induction on i. First, consider the case where v_i is a leaf. If $P_i[j] = 1$, then $|A_j| - 1 \leq |C_i| = 0$, implying u_a is a leaf in G_R, and we can define a partial role assignment which simply takes v_i to u_a. On the other hand, if there is a partial role assignment, then $|N(v_i)| - 1 = 0 \geq |N(u_a)| - 1$, so $|N(u_a)| = 1$, which implies u_a is a root, and the conditions to set $P_i[j] = 1$ are trivially satisfied.

Now, we assume that $P_k[j] = 1$ if and only if there exists a partial role assignment on $T\langle v_k \rangle$, extendable with edge e_j for all vertices v_k such that $k < i$. If $P_i[j] = 1$, then a matching exists, which implies that there is a matched assignment of edges e_{a_j} into u_a (possibly excluding e_j) to children of v_i, say v_{c_i} such that $P_{c_i}[a_j] = 1$. Using our induction assumption, we can then define the partial role assignment $r_{i,j}$ to be $r_{i,j}(v) = r_{c_i,a_j}(v)$ where $v \in T\langle v_{c_i} \rangle$, and $r_{i,j}(v_i) = u_a$. On the other hand, if a partial role assignment exists, then there must be some partial role assignments on each of the subtrees generated by the children, and by the inductive assumption, they must have $P_{c_i}[a_j] = 1$. This then creates a matching of the correct cardinality, and $P_i[j] = 1$. ■

Finally, we just need to prove that the final step in the algorithm correctly determines a role assignment for the entire tree.

Theorem 4.12: *Algorithm* **TRA** *determines whether a role assignment exists from an input tree T to a role graph G_R.*

Proof. From Lemma 4.3, each child of the root contains a list of partial role assignments. A suitable matching implies that we can extend these partial role assignments to a full role assignment, where every mapping of each vertex satisfies the condition of a role assignment, and furthermore, since G_R is connected, every role is assigned. Additionally, a possible role assignment implies there are partial role assignments on each of the children of the root, which will yield a suitable matching. ■

5.3 *Complexity analysis*

The dominant factor involved in the Algorithm **TRA** is the work involved in solving a bipartite matching problem for every combination of a directed edge of G_R and a vertex of T. At each step, we consider the graph $G = (V_A \cup V_B, E)$, where $|V_A| = |A_j|$ or $|A_j| - 1$ and $|V_B| = |C_i|$, with $|A_j| - 1 \leq |C_i|$. This allows us to consider this problem as an *unbalanced* bipartite graph, in which one bipartition is larger, first considered in [**2**]. In particular, Kao et al. in [**34**] give a simple adaptation of a bipartite matching algorithm by Gabow and Tarjan [**24**], which yields a running time of $O(\sqrt{n_s} m \log n_s)$ where n_s is the cardinality of the smallest bipartition for the bipartite matching problem, and m is the number of edges.

THEOREM 4.13: *For any role graph G_R with n_R vertices and any tree T with n_T vertices, Algorithm* **TRA** *runs in* $O\left(n_T n_R^{2.5} \log n_R\right)$

PROOF. The work involved in the initialization for the algorithm is at most $O(n_T n_R)$, the initialization of each array P_i, and this is overshadowed by the rest of the of the algorithm, solving a matching problem for every combination of an edge in G_R and a vertex in T. Therefore, we get that the running time $RT\,(\mathbf{TRA})$ of the algorithm is given by

$$\begin{aligned} RT\,(\mathbf{TRA}) &= O\left(\sum_{i}^{n_T} \sum_{\substack{e_j=(u_{j_1},u_{j_2}) \\ e_j \in \overrightarrow{E}(G_R)}} RT\,(\text{MATCHING}(C_i, A_{j_2}))\right) \\ &= O\left(\sum_{i}^{n_T} \sum_{j}^{n_R} d(u_j) RT\,(\text{MATCHING}(C_i, A_j))\right). \end{aligned}$$

From [**34**], the unbalanced bipartite matching problem can be solved in $O(\sqrt{n_s} m \log n_s)$ time, where $n_s \leq |A_j| \leq 2d(u_j)$, and $m \leq |A_j||C_j| \leq 2d(u_j)d(v_i)$. Since constructing the bipartite graph takes only $O(d(v_i)d(u_j))$ time, we get that $RT(\text{MATCHING}(C_i, A_j)) = O(d(v_i)d(u_j)\sqrt{d(u_j)} \log d(u_j))$. Substituting this into the running time of Algorithm **TRA**, we get

$$
\begin{aligned}
RT\,(\mathbf{TRA}) &= O\left(\sum_{i}^{n_T}\sum_{j}^{n_R} d(u_j)\left(d(v_i)d(u_j)^{1.5}\log d(u_j)\right)\right) \\
&= O\left(\sum_{i}^{n_T}\left(d(v_i)n_R^{1.5}\log n_R \sum_{j}^{n_R} d(u_j)\right)\right)
\end{aligned}
$$

From Corollary 4.1, $|E(G_R)| \leq 2n_R - 1$ yielding

$$
RT\,(\mathbf{TRA}) = O\left(\sum_{i}^{n_T} d(v_i)\left(n_R^{2.5}\log n_R\right)\right) = O\left(n_T n_R^{2.5}\log n_R\right). \qquad \blacksquare
$$

Thus, for a fixed graph G_R, Algorithm **TRA** runs in linear time.

Chapter 5

Role Assignments and Cartesian Products

1. Introduction

This chapter will focus on determining whether a graph G has a role assignment $r : G \to K_k$, where G is a cartesian product of graphs. In [**17**] and [**18**], this particular question has been considered, mainly for the products of paths and cycles. The d-dimensional hypercube, denoted Q_d, can be defined as the graph on 2^d vertices corresponding to the set of binary strings of length d, where two strings are adjacent if they differ in only one coordinate. We note that $Q_d = \square_{i=1}^{d} K_2$. The related question of determining the domatic number of the d-dimensional hypercube has received considerable attention. (Recall that the domatic number of a graph is the largest value k for which the vertex set $V(G)$ can be partitioned into k dominating sets.)

In [**62**] the domatic number of the d-dimensional hypercube was first considered, and it was shown that hypercubes Q_d and Q_{d-1} where $d = 2^k$ for some value k have domatic number d. Additionally, it was conjectured that if $d + 1$ was not a power of 2, then $dom(Q_d) = d$. In [**37**], a simplified version of the previous result was given, where the dominating sets are translates of the Hamming code. Additionally, it was shown that $dom(Q_5) = 4$, disproving the conjecture in [**62**].

In this chapter, we use role assignments in order to accomplish two main goals. The first goal is to simplify and generalize some results given in [**17**] and [**18**] which involve role assignments of product graphs to K_3. We derive these results by determining graphs G such that there exists a role assignment from the d-dimensional hypercube Q_d to G. The second goal is to consider role assignments of the hypercube to K_k, for $k > 3$. This answers an open question given in [**18**]. The contents of this chapter form the basis of [**38**].

2. Partitioning into 3 Independent Dominating Sets

To begin with, we present an alternate proof of a result in [**18**], using role assignments. On an $s \times t$ chessboard, rooks move either horizontally or vertically. Thus, the Rook's graph can be represented as $K_s \Box K_t$. The following result is easily derived.

Theorem 5.1: *For $s \leq t$, the Rook's graph on an $s \times t$ board, $K_s \Box K_t$, has a role assignment to K_t.*

Proof. Color the vertices in the first row as $1, 2, \ldots, t$, and similarly color the vertices on the second row as $2, 3, \ldots, t, 1$, each time shifting to the right. Since $s \leq t$, these color classes are independent, and it is easy to see that for every vertex, there is a neighbor that is colored with each of the $t - 1$ other colors. Thus, the vertices of $K_s \Box K_t$ can be partitioned into t independent dominating sets. ■

The following corollary, noted here as Corollary 5.1, is actually Theorem 4 in [**18**] and is easily deducible from Theorem 5.1 using role assignments.

Corollary 5.1: *For $s \leq t$, if G has a role assignment to K_s and H has a role assignment to K_t, then $G \Box H$ has a role assignment to K_t.*

Proof. Since there exist role assignments $r_1 : G \to K_s$ and $r_2 : H \to K_t$, then by Observation 4.2 there exists a role assignment $r_3 : G \Box H \to K_s \Box K_t$ and since $K_s \Box K_t$ has an assignment to K_t, then $G \Box H$ has an assignment to K_t. ■

2.1 *Role assignments of $K_2 \Box G$ to K_3*

In this section, we provide more results about the role assignments of cartesian products of graphs, and in particular, we look at role assignments of cartesian products of some graphs to a triangle K_3. First, we prove a result about the cartesian product $G\Box K_2$, where we let $V(K_2) = \{a, b\}$. For any two graphs, G_1 and G_2, we loosely say $G_1\Box G_2 = G_2\Box G_1$.

THEOREM 5.2: *For any graph G, $K_2\Box G$ has a role assignment to K_3 if and only if G has a role assignment to K_3.*

PROOF. Suppose G has a role assignment to K_3. Since K_2 is 2-colorable, by Corollary 5.1 $K_2\Box G$ has a role assignment to K_3.

Now assume that $K_2\Box G$ has a role assignment to K_3; let $r : K_2\Box G \to K_3$, with roles $\{1, 2, 3\}$, be this assignment. Using the mapping r, we can construct a labeling $g(v)$ of the graph G, where $g(v) = (r((a, v)), r((b, v)))$ for every vertex $v \in V(G)$. Using the labels of $g(v)$, we can partition the vertices of G into sets V_1, V_2, and V_3, where we let $V_1 = \{v \in G : g(v) = (1, 2) \text{ or } g(v) = (1, 3)\}$, $V_2 = \{v \in G : g(v) = (2, 1) \text{ or } g(v) = (2, 3)\}$, and $V_3 = \{v \in G : g(v) = (3, 1) \text{ or } g(v) = (3, 2)\}$. This completely partitions the vertices of G.

Furthermore, we claim that each set V_i is independent in G. Let $v, v' \in V_i$ and suppose for contradiction that $(v, v') \in E(G)$. Both $g(v) = (r(a, v), r(b, v))$ and $g(v') = (r(a, v'), r(b, v'))$ have the same first coordinate $c_i \in \{1, 2, 3\}$. In other words, $r(a, v) = r(a, v')$. But this implies that (a, v) and (a, v') are adjacent in $K_2\Box G$, and since r is a proper coloring, $r(a, v) \neq r(a, v')$, a contradiction. Thus we have a proper 3-coloring for G.

Now we show that each vertex of G is colorful in this partition. If for a vertex v, $g(v) = (r(a, v), r(b, v)) = (c_1, c_2)$ for some $c_1, c_2 \in \{1, 2, 3\}$ such that $c_1 \neq c_2$, then (a, v) must be adjacent to a vertex labeled c_3 by r, and since no adjacent vertices can have labels with an element in common, v must be adjacent to a vertex labeled (c_3, c_1). Also, (b, v) must be adjacent to a vertex labeled c_3 by the mapping r. Again since no adjacent vertices can have labels with an element in common, v must be adjacent to a vertex labeled (c_2, c_3). Therefore, v is colorful. Thus we have a role assignment of the graph G to K_3, and hence the result. ■

The following corollaries allow us to expand the theorem above.

Corollary 5.2: *For any graph G and $d \geq 1$, $Q_d \Box G$ has a role assignment to K_3 if and only if G has a role assignment to K_3.*

Proof. We represent Q_d as $\Box_{i=1}^{d} K_2$. We proceed with induction on d. For the base case, we know by Theorem 5.2 that G has a role assignment to K_3 if and only if $K_2 \Box G = Q_1 \Box G$ has a role assignment to K_3. Now, we assume G has a role assignment to K_3 if and only if $Q_d \Box G$ has a role assignment to K_3. By applying Theorem 5.2, $Q_d \Box G$ has a role assignment to K_3 if and only if $K_2 \Box (Q_d \Box G) = Q_{d+1} \Box G$ has a role assignment to K_3. Therefore, G has a role assignment to K_3 if and only if $Q_{d+1} \Box G$ has a role assignment to K_3, and this proves the result. ■

Corollary 5.3: *Q_d does not have a role assignment to K_3 for any value of d.*

Proof. Let $G = K_2$ in Corollary 5.2. Since K_2 does not have a role assignment to K_3, neither does $Q_{d-1} \Box K_2 = Q_d$. ■

Now we consider the set of graphs which are role graphs for some hypercube, which we will denote by $\mathcal{R}(Q)$. Since the set of hypercubes is closed under the cartesian product operation, i.e., $Q_s \Box Q_t = Q_{s+t}$, we also know that by Observation 4.2 $\mathcal{R}(Q)$ is closed under the cartesian product operation.

To conclude this section, we show that some elementary graphs, such as paths and stars, are in the set $\mathcal{R}(Q)$. Recall that the hypercube Q_d is defined as the set of binary strings of length d, where two strings are adjacent if they differ in only one coordinate. With this definition, we define the weight of a vertex $v \in V(Q_d)$ to be the number of ones in its binary string. Each vertex v of Q_d with weight k, $0 < k < d$, is adjacent to vertices of weight $k-1$ or $k+1$ which agree in $d-1$ coordinates. For the path, we define a role assignment which acts on a vertex according to its weight.

Proposition 5.1: *The path on d vertices P_d is in $\mathcal{R}(Q)$, for all d.*

Proof. We define a mapping $f : Q_d \to P_d$ where $V(P_d) = \{v_0, v_1, \ldots, v_d\}$, such that for $u \in V(Q_d)$ with weight k, then $f(u) = v_k \in P_d$. Every vertex of weight k has only neighbors of weight $k-1$ or $k+1$, and at least one neighbor of each weight, unless $k=0$ or d. Since the vertices of weight 0 and d have neighbors of weight 1 and $d-1$ respectively, then f is a role assignment from Q_d to P_d. ■

A *t-star*, denoted $K_{1,t}$, is the complete bipartite graph in which one bipartition contains only one vertex, and the other bipartition contains t vertices.

PROPOSITION 5.2: *The star $K_{1,t}$ is in $\mathcal{R}(Q)$, for all t.*

PROOF. We proceed by induction. From above, $P_3 = K_{1,2} \in \mathcal{R}(Q)$. Now assuming that $K_{1,t} \in \mathcal{R}(Q)$, we want to show $K_{1,t+1} \in \mathcal{R}(Q)$. Since $K_{1,t} \in \mathcal{R}(Q)$, there is a role assignment from Q_r to $K_{1,t}$ for some r. Thus, there exists a role assignment from $Q_r \Box Q_r$ to $K_{1,t} \Box K_{1,t}$. Now, for $K_{1,t} \Box K_{1,t}$, we label the leaves of $K_{1,t}$ as $\{0, 1, \ldots, t-1\}$, letting $\ell(v)$ denote their labels. Now for $(u,v) \in V(K_{1,t} \Box K_{1,t})$ where u and v are leaf vertices, we assign (u,v) the label $\ell(u) + \ell(v) \mod t$. The vertex (u,v), where u and v are the center vertices, is assigned the label t. Every other vertex in $K_{1,t} \Box K_{1,t}$ is of the form (u,v) where u or v is a center, but not both. These we assign the label $t+1$. Using these labels as an assignment of the vertices, we can see that all vertices labeled 0,1 $\ldots$, t have a neighbor labeled $t+1$ (and only neighbors labeled $t+1$), and all vertices labeled $t+1$ have a neighbor with every label from $\{0, 1, \ldots, t\}$. Therefore, this labeling is a role assignment from $K_{1,t} \Box K_{1,t}$ to $K_{1,t+1}$. This implies that $K_{1,t+1} \in \mathcal{R}(Q)$, and so every star is an element of $\mathcal{R}(Q)$. ■

To summarize some of these results, we give the following corollary.

COROLLARY 5.4: *For any graph $G \in \mathcal{R}(Q)$, G does not have a role assignment to K_3.*

PROOF. If G has a role assignment to K_3, that implies Q_d (for some d) has a role assignment to K_3 by Observation 4.1, a contradiction to Corollary 5.3. ■

This implies that any graph of the form $G = \Box_{i=1}^{n_1}(\Box_{j=1}^{a_i} P_i)\Box_{k=1}^{n_2}(\Box_{\ell=1}^{b_k} K_{1,k})$ has a role assignment to K_2, but not to K_3.

2.2 *The Product of two trees*

Here, we present a proof that is similar to Theorem 5.2; however, we place more restrictions in order to get a theorem about the cartesian product of two trees.

Theorem 5.3: *Let G be a graph such that $\delta(G) = 1$ and let T be any tree. Then $G \Box T$ does not have a role assignment to K_3.*

Proof. Let a be a vertex of degree one in G, and b be the vertex in G such that $(a, b) \in E(G)$. Now we assume for contradiction that $G \Box T$ has a role assignment to K_3. Consider any leaf in T, which we will denote v_0. If the vertex $(a, v_0) \in V(G \Box T)$ has the color c_1, it is adjacent to a vertex $(a, v_1) \in V(G \Box T)$ with a color c_2, where $(v_0, v_1) \in E(T)$. Therefore, (b, v_0) must have the third color, c_3. Then, (b, v_1) is adjacent to vertices with the color c_2 and c_3 and so must have the same color c_1 as (a, v_0). Since (a, v_1) is colorful, there must be another vertex (a, v_2) such that (a, v_2) has the color c_3 and $((a, v_1), (a, v_2)) \in E(G \Box T)$.

For $i = 1$, we can see that this shows (a, v_{i-1}) and (b, v_i) have the same color, forcing (a, v_i) to be adjacent to a vertex (a, v_{i+1}) with a different color than (a, v_i) and (a, v_{i-1}). Now if we assume that this is true for $i = k$, we can see that (b, v_{k+1}) must have a color different from (a, v_{k+1}) and (b, v_k). Since (b, v_k) has the same color as (a, v_{k-1}), then (b, v_{k+1}) must have the third color, the color of (a, v_k). Furthermore, this means that (a, v_{k+1}) must be adjacent to another vertex that does not have the same color as (a, v_{k+1}) or (a, v_k), which we can denote as (a, v_{k+2}). This proves our induction hypothesis, which implies that we create an infinite sequence of vertices in T, where each vertex is different from the preceding one. Since T has no cycles, we have a contradiction and therefore, $G \Box T$ does not have a role assignment to K_3. ■

We can then let G be another tree, and we get the following corollary.

Corollary 5.5: *Let T_1, T_2 be any two trees. Then $T_1 \Box T_2$ has no role assignment to K_3.*

One can also note that $T_1 \Box T_2$ has no role assignment to K_k for $k > 3$, since $\delta(T_1 \Box T_2) = 2$, which implies that $\psi_f(T_1 \Box T_2) \leq 3$ (in the notation of [**18**] where $\psi_f(T_1 \Box T_2)$

represents the largest value k for which $T_1 \Box T_2$ can be partitioned into k independent sets), using the observation from [**18**] that $\psi_f(T) \leq \delta(T) + 1$.

2.3 *Complexity results for bipartite graphs*

We consider the complexity of role assignments of bipartite graphs to a complete graph K_k in order to illustrate the relationship between a sequence of decision problems and our results on hypercubes. As mentioned before, the problem of partitioning a general graph into k independent dominating sets has been shown to be NP-complete ([**18, 30**]). Complexity results for homomorphisms, or H-colorings, of graphs have been rather remarkable, with a sharp delineation of which problems are NP-complete (when H is not bipartite), and where polynomial-time algorithms are known (H is bipartite) [**31**]. The corresponding class of problems for role assignments is given below.

H-ROLE ASSIGNABLE (H-RA)
INSTANCE: A graph $G = (V, E)$
QUESTION: Is there a role assignment from G to H?

Recall that Fiala and Paulusma were able to prove a dichotomy result for role assignments, determining which graphs H cause this problem to be NP-complete.

Theorem 5.4 ([**23**]): *For any simple, connected graph H on at least three vertices, problem H-RA is NP-complete.*

In the preceding work, we have shown many bipartite graphs which do not have a role assignment to K_3. Note that the 6-cycle C_6 is an example of a graph which has a role assignment to K_3. If we let G be the 6-cycle with a pendant vertex, we get that $G \Box G$ has a role assignment to K_3, whereas G does not. We can consider the following general problem, determining when a bipartite graph can be partitioned into r independent dominating sets.

K_r-ROLE-ASSIGNMENT(BIPARTITE)(K_r-RA(B))
INSTANCE: A bipartite graph $G = (V, E)$ (with no isolates)
QUESTION: Does G have a role assignment to K_r?

First, we define a graph G_r for some r, where $V(G_r) = \{a_i : 0 \leq i \leq r-1\} \cup \{b_i : 0 \leq i \leq r-1\}$, and $E(G_r) = \{(a_i, b_j) : i \neq j\}$. Thus, G_r is a complete bipartite graph on $2r$ vertices minus a perfect matching.

Lemma 5.1: *A graph G is bipartite and has a role assignment to K_r, for $r > 2$, if and only if there exists a role assignment from G to G_r.*

Proof. First, suppose G is a bipartite graph with bipartite sets A and B, and there is a role assignment $r_1 : G \to K_r$, where $r_1^{-1}(v_i) = V_i$. Since no independent dominating set of a graph can be strictly contained in another, each of the r independent dominating sets $V_i \subseteq V(G)$ can be partitioned into $V_i \cap A \neq \emptyset$ and $V_i \cap B \neq \emptyset$.

Since A and B are independent, and each set V_i is an independent dominating set, every vertex $v \in V_i \cap A$ must be adjacent to a vertex in $V_j \cap B$ for all $j \neq i$, and be nonadjacent with every vertex in $A = \bigcup_j (V_j \cap A)$ and $V_i \cap B$. Thus the mapping $r(V_i \cap A) = a_i$ and $r(V_i \cap B) = b_i$ is a role assignment $r : G \to G_r$, where G_r is the graph defined above.

Now suppose that G is a graph such that there exists a role assignment from G to G_r. Because G_r is bipartite, there is a role assignment from G_r to K_2. Observation 4.1 then implies there is a role assignment from G to K_2, so G is bipartite. Similarly, the mapping r_3, where $r_3(a_j) = v_j$ and $r_3(b_j) = v_j$, defines a role assignment from G_r to K_r. Again by Observation 4.1, there must exist a role assignment from G to K_r, ■

For each value of r, Problem K_r-RA(B) can now be viewed as a particular instance of Problem H-RA.

Corollary 5.6: *Problem K_r-RA(B) is NP-complete for all values of $r > 2$.*

Proof. From Lemma 5.1, a bipartite graph G has a role assignment to K_r, for $r > 2$, if and only if there exists a role assignment from G to G_r. Thus, Problem K_r-RA(B) is equivalent to Problem H-RA, where $H = G_r$. By Theorem 5.4, Problem H-RA is NP-complete for $H = G_r$. ■

Since K_3-RA(B) is NP-complete, we can extend the results to the following problem. For a product graph H resulting from a cartesian product of G_1 and G_2, i.e., $H = G_1 \square G_2$, the graphs G_1 and G_2 are referred to as the *factors* of H. Since every graph can be written as $G = G\Box K_1$, we define a nontrivial factor as any factor which is not K_1.

> K_3-ROLE-ASSIGNMENT ON L-FACTORS (L, K_3-RA)
> INSTANCE: A bipartite product graph $G = (V, E)$ with at least ℓ nontrivial bipartite factors
> QUESTION: Does G have a role assignment to K_3?

THEOREM 5.5: *(L, K_3-RA) is NP-complete for any value ℓ.*

PROOF. We proceed by induction on ℓ. For the base case, we note that (1, K_3-RA) is NP-complete, since it is equivalent to Problem K_3-RA(B). Then we may assume that (L, K_3-RA) is NP-complete. Thus, we can take any instance of G from (L, K_3-RA) and consider $G\Box K_2$. By Theorem 5.2, G has a role assignment to K_3 if and only if $G\Box K_2$ has a role assignment to K_3. Therefore, (L+1, K_3-RA) is NP-complete. ■

This result is the complexity version of the nonexistence result for role assignments to K_3 from hypercubes.

3. Partitioning into k independent dominating sets, $k > 3$

In this section, we consider creating new role assignments of graphs from role assignments of iterated cartesian products. We define the graph H_r for some r, where $V(H_r) = \{a_i : 0 \leq i \leq r-1\} \cup \{b_i : 0 \leq i \leq r-1\}$ and $E(H_r) = \{(a_i, b_j) : i \neq j\} \cup \{(b_i, b_j) : i \neq j\} \cup \{(a_i, a_j) : i \neq j\}$. Here, H_r is just a complete graph on $2r$ vertices minus a perfect matching. Let G_r be the graph defined as in the previous section. We then show that for every value of $r > 2$, there exists a role assignment from $G_r\Box K_r$ to H_r.

LEMMA 5.2: *For $r > 2$, there exists a role assignment from $G_r\Box K_r$ to H_r.*

PROOF. Each vertex in $G_r\Box K_r$ can be written in the form $\{(a_i, v_j) : a_i \in G_r, v_j \in K_r\}$ or $\{(b_i, v_j) : b_i \in G_r, v_j \in K_r\}$ where a_i and b_i are from the definition of G_r. Then we can define sets A_i, B_i by letting $A_i = \{(a_k, v_{(k+i) \bmod r}) : 0 \leq k \leq r-1\}$ and similarly, $B_i = \{(b_k, v_{(k+i) \bmod r}) : 0 \leq k \leq r-1\}$, for $0 \leq i \leq r-1$. Figure 5.1 demonstrates the set A_0 for the graph $G_4\Box K_4$.

Consider a vertex $(a_k, v_j) \in A_i$ where $j \equiv (k+i) \bmod r$. In each set A_ℓ, there is a vertex $(a_k, v_{j'})$, where $j' \equiv (k+\ell) \bmod r$, such that $((a_k, v_j), (a_k, v'_j)) \in E(G_r \square K_r)$. Also, for each set B_ℓ where $\ell \neq i$, there is a k' such that $j \equiv (k'+\ell) \bmod r$ and $k \neq k'$, which implies that $((a_k, v_j), (b'_k, v_j)) \in E(G_r \square K_r)$. Furthermore, if two vertices $(a_k, v_j), (a_{k'}, v_{j'}) \in A_i$, then $k \neq k'$ and $j \neq j'$, which implies each set A_i is independent. Similarly, this holds for each set B_i for any $0 \leq i \leq r-1$. Finally, for vertices $(a_k, v_j) \in A_i$ and $(b_{k'}, v_{j'}) \in B_i$, if $k \neq k'$ then $j \neq j'$, and if $k = k'$ then $(a_k, b_k) \notin E(G_r)$. Therefore, if we define a map $f : V(G_r \square K_r) \to H_r$ such that for $v \in A_i$, $f(v) = a_i \in V(H_r)$ and for $v \in B_i$, $f(v) = b_i$, then $f : G_r \square K_r \to H_r$ is a role assignment. ■

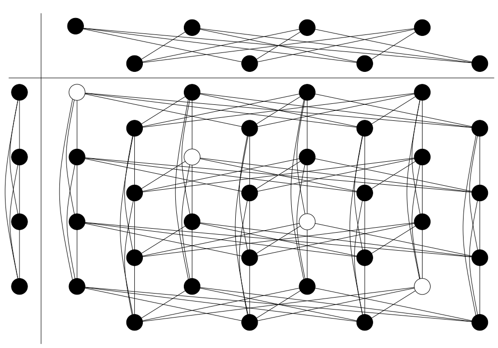

Figure 5.1. $G_4 \square K_4$, with the white vertices representing the set A_0

Corollary 5.7: *Suppose graphs G and G' have a role assignment to K_r, and G is bipartite. Then there is a role assignment from $G \square G'$ to H_r.*

Proof. The graph G is bipartite and has a role assignment to K_r, so by Lemma 5.1, there is a role assignment from G to G_r. Since G' has a role assignment to K_r, Observation 4.2 implies there is a role assignment from $G \square G'$ to $G_r \square K_r$. By Lemma 5.2, there is a role assignment from $G_r \square K_r$ to H_r. Thus, by Observation 4.1, there exists a role assignment from $G \square G'$ to H_r. ■

We end this section by noting that if G is a bipartite graph which has a role assignment to K_r, then there is a role assignment from $G \square G$ to H_r.

3.1 *Generating new colorings of $H_r \square K_2$*

In this section, we determine for which values of k it is possible to decompose the cartesian product $H_r \square K_2$ into independent dominating sets for a role assignment to K_k. Then, using Corollary 5.7, we apply those results to repeated cartesian products, and specifically, the d-dimensional hypercube. In the results below, we let $V(K_2) = \{a, b\}$, and let the vertices of H_r be defined as in the previous section.

To begin with, we consider three methods of partitioning some of the vertices of H_r into independent dominating sets, so that we can combine different methods to eventually create role assignments of H_r.

Lemma 5.3 (**Method 1**): *The sets $V_{(j,1)}$ and $V_{(j,2)} \subseteq V(H_r \square K_2)$, where we define $V_{(j,1)} = \{(a_j, a), (b_j, b)\}$ and $V_{(j,2)} = \{(a_j, b), (b_j, a)\}$, are independent dominating sets in the graph $H_r \square K_2$.*

Proof. Each set $V_{(j,1)}$ and $V_{(j,2)}$ is independent, and for $V_{(j,1)} = \{(a_j, a), (b_j, b)\}$, vertex (a_j, a) dominates every vertex of the form (v, a), except for $v = b_j$, which is dominated by (b_j, b). Similarly, vertex (b_j, b) dominates every vertex of the form (v, b), except for $v = a_j$, which is dominated by (a_j, a). In the same way, we can see that $V_{(j,2)}$ is also a dominating set in $H_r \square K_2$. ■

Lemma 5.4 (**Method 2**): *The sets $V_{(j,1)}$ and $V_{(j,2)} \subseteq V(H_r \square K_2)$, where we define $V_{(j,1)} = \{(a_j, a),\ (b_j, a),\ (a_{j+1}, b),\ (b_{j+1}, b)\}$ and $V_{(j,2)} = \{(a_j, b),\ (b_j, b),\ (a_{j+1}, a),\ (b_{j+1}, a)\}$, are independent dominating sets in the graph $H_r \square K_2$.*

Proof. Since $(a_j, b_j) \notin E(H_r)$ for any j, each set is independent. Additionally, since the set $\{a_j, b_j\}$ dominates H_r, each set is dominating. ■

Lemma 5.5 (**Method 3**): *The sets $V_{(j,1)}$, $V_{(j,2)}$, and $V_{(j,3)} \subseteq V(H_r \square K_2)$, where we define $V_{(j,1)} = \{(a_j, a),\ (b_j, a),\ (a_{j+1}, b),\ (b_{j+1}, b)\}$, $V_{(j,2)} = \{(a_j, b),\ (b_j, b),\ (a_{j+2}, a),\ (b_{j+2}, a)\}$, and $V_{(j,3)} = \{(a_{j+1}, a),\ (b_{j+1}, a),\ (a_{j+2}, b),\ (b_{j+2}, b)\}$, are independent dominating sets in the graph $H_r \square K_2$.*

Proof. Since $(a_j, b_j) \notin E(H_r)$ for any j, each set is independent. Additionally, since the set $\{a_j, b_j\}$ dominates H_r, each set is dominating. ■

This allows us to create new role assignments.

Theorem 5.6: *The graph $H_r \Box K_2$ has a role assignment to K_k, for $r \leq k \leq 2r - 2$ or $k = 2r$.*

Proof. Here, we write $k = r + \ell = (r - \ell) + 2\ell$ for some ℓ, where $\ell = r$ or $0 \leq \ell \leq r - 2$. We begin by considering two cases: $(r - \ell)$ is even, and $(r - \ell)$ is odd.

(1) *$r - \ell$ is even.*

We first consider the $(r-\ell)/2$ sets $S_i = \{(a_{2i}, a), (a_{2i}, b), (b_{2i}, b), (b_{2i}, b), (a_{2i+1}, a), (a_{2i+1}, b), (b_{2i+1}, b), (b_{2i+1}, b)\}$ for $0 < i < ((r - \ell)/2 - 1)$. Method 2 can be used on each of these sets, to generate $(r - \ell)$ disjoint independent dominating sets.

(2) *$r - \ell$ is odd.*

Here, write $r - \ell = 3 + (r - 3 - \ell)$. We can first take the set $\{(a_0, a)$,(a_0, b),(b_0, a), (b_0, b), (a_1, a), (a_1, b), (b_1, a), (b_1, b), (a_2, a), (a_2, b), (b_2, a), $(b_2, b)\}$. This set can be partitioned by Method 3 into three independent dominating sets. Then, since $(r - 3 - \ell)$ is even, we can consider the sets $S_i = \{(a_{2i-1}, a), (a_{2i-1}, b), (b_{2i-1}, a), (b_{2i-1}, b), (a_{2i}, a), (a_{2i}, b), (b_{2i}, a), (b_{2i}, b)\}$ for $2 < i < ((r - 3 - \ell)/2 + 1)$. Method 2 can be used on each of these sets, to generate $(r - 3 - \ell)$ disjoint independent dominating sets.

Now, for the remaining ℓ sets of vertices of the form $T_j = \{(a_j, a), (a_j, b), (b_j, b), (b_j, b)\}$ for $r - \ell - 1 \leq j \leq r - 1$, we can use Method 1 to generate 2 independent dominating sets from each set T_j. This completely partitions the vertex set $V(H_r \Box K_2)$ into $(r-\ell)+2\ell = r + \ell = k$ independent dominating sets. Therefore $H_r \Box K_2$ has a role assignment to K_k. ■

Figure 5.2 gives an example of a role assignment from $H_4 \Box K_2$ to K_6.

Corollary 5.8: *Consider graphs G, H and I. Let G and I be bipartite, and suppose G and H have a role assignment to K_r. Then $G \Box H \Box I$ has a role assignment to K_k, for all values of k where $r \leq k \leq 2r - 2$ or $k = 2r$.*

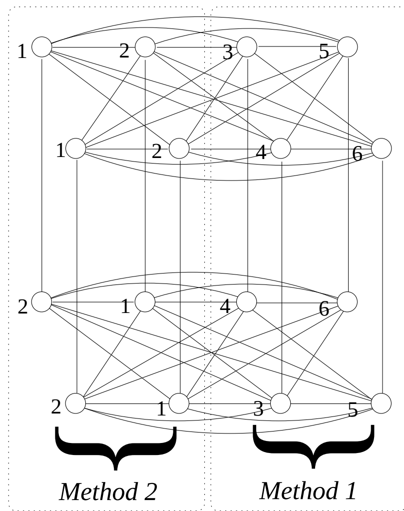

FIGURE 5.2. A role assignment of $H_4 \Box K_2$ to K_6, where $k = 6$, $r = 4$, and $\ell = 2$

PROOF. From Corollary 5.7, there exists a role assignment $f_1 : G \Box H \to H_r$, and since I is bipartite, we can take the natural role assignment $f_2 : I \to K_2$. Then by Observation 4.2, there exists a role assignment $f_3 : (G \Box H) \Box I \to H_r \Box K_2$. From Theorem 5.6, there exists a role assignment $f_4 : H_r \Box K_2 \to K_k$, where $r \leq k \leq 2r - 2$ or $k = 2r$. Then, by Observation 4.1 there exists a final role assignment $f_5 : G \Box H \Box I \to K_k$. ■

COROLLARY 5.9: *For every $k \geq 2$, $k \neq 3$, there is some d such that Q_d has a role assignment to K_k.*

PROOF. We proceed by using induction on k, where $d = 2^k - 1$, considering the hypercubes Q_d. As a base case, for $k = 2$, we know that Q_3 has role assignments to K_2 and K_4. Now we assume that Q_d has a role assignment to K_k for each $k \leq d + 1$ and $k \neq 3$ or d. We can then take $Q_d \Box Q_d \Box K_2$ to form Q_{2d+1} where $2d + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$. Then from Corollary 5.8, using the fact that Q_d has a role assignment to K_{d+1}, we know that Q_{2d+1} has a role assignment to K_k for any k where $d \leq k \leq 2d + 2$, $k \neq 2d + 1$. From Corollary 5.1 and our induction hypothesis, Q_{2d+1} also has a role assignment to K_k for any k such that $k \leq d + 1$ and $k \neq 3$ or d. Finally, we can get that Q_{2d+1} has a role assignment to K_d by using Corollary 5.8, and the fact that Q_d has a role assignment to K_{d-1}. This shows that for every $k \geq 2, k \neq 3$, there is a number d such that Q_d has a role assignment to K_k. ■

Combining Corollaries 5.3 and 5.9, we can restate the results in the following theorem, which answers an open question posed in [**18**], "What fall colorings do the n-cubes have?", which is equivalent to asking "For what values k do the d-dimensional hypercubes have a role assignment to K_k?"

THEOREM 5.7: *The hypercubes Q_d do not have a role assignment to K_3 for any value d. However, for each $k > 3$, there is some d such that Q_d has a role assignment to K_k.*

Chapter 6

Conclusions and Open Questions

1. Conclusions

In Chapters 2 and 3, the classes of K_r-free graphs were considered and results were obtained which describe the structure of dense graphs in these classes. We were able to extend results on the colorings and homomorphisms of dense triangle-free graphs to K_r-free graphs. Additionally, we looked at the size of independent sets, and we were able to improve some bounds on the binding number of K_r-free graphs.

In Chapters 4 and 5, we focused on role assignments as the description of the structure of a graph, and considered the classes of chordal graphs, strongly chordal graphs, trees, and iterated cartesian products of graphs. In each class, we were either able to describe the conditions for role assignments to a complete graph, or answer more general questions concerning role assignments.

2. Open Questions

Many questions about the structure of dense K_r-free graphs and role assignments of graphs remain. The open questions listed here represent problems encountered in attempting to either extend a result or view the results of others in a consistent fashion. When there is some evidence to support a particular result, we offer a conjecture.

(1) *Removal of edges.* Let H be a graph such that $\chi(H) = r + 1$ and $n_H = |V(H)|$. In [**3**], Alon and Sudakov proved that for H-free graphs G (of large enough size), where $\delta(G) \geq ((3r-7)/(3r-4)+\varepsilon)\, n$, at most $O(n^{2-1/(4r^{2/3}n_H)})$ edges need to be

removed to make G r-colorable. Can any H-free graph (of large enough size) with $\delta(G) > Cn$ and $\chi(H) = r$ be made K_r-free by the removal of $O(n^{2-1/(4r^{2/3}n_H)})$ edges?

(2) *Forbidding a family of graphs.* In [**4**] it was additionally pointed out that a graph with odd girth at least $2k-1$ and minimum degree at least $\delta(G) > \frac{2}{2k+1}n$ is bipartite. In [**61**], Xu, Jin, and Liu proved that a graph which does not contain a 5-wheel W_5 or a K_4 and has $\delta(G) > \frac{7}{12}n$ is 3-colorable. We can consider both of these results in a more general question. Consider the class of graphs such that any subgraph induced by k vertices has chromatic number less than r. Can we determine the minimum value of $C_{k,r}$ such that $\delta(G) > C_{k,r}n$ is $(r-1)$-colorable? (Note that the results above imply that $C_{2k,3} = \frac{2}{2k+1}$ and $C_{6,4} = \frac{7}{12}$.)

(3) *Similarity sets.* In the case of maximal K_r-free graphs for $r \geq 4$, $\delta(G) \geq \frac{2r-5}{2r-3}n$ certainly indicates the presence of large sets of mutually similar vertices, but we were not able to determine the minimum degree conditions where we first begin to see similar vertices. Therefore, we offer the following conjecture for K_4-free graphs.

Conjecture 6.1: *A maximal K_4-free graph G has a set of size $2\delta(G) - n$ of mutually similar vertices.*

(4) *Binding Number.* We were able to make progress determining the binding number of K_r-free graphs, but the bounds which we determined do not seem to be sharp. In particular, we are interested in the minimum value $f(r)$ such that a graph with binding number larger than $f(r)$ must contain a K_r. Alternatively, is there a sequence of K_r-free graphs with binding number tending toward $f(r)$?

Conjecture 6.2:

$$f(4) = \frac{\sqrt{33}-3}{7-\sqrt{33}}$$

Conjecture 6.3:

$$f(r) = r - 2 + o\left(\frac{1}{r}\right)$$

(5) *Sharpness.* One important open question to consider is the sharpness of the minimum degree bound in Theorem 3.1, especially in the case of partitioning a K_r-free graph into an independent set and a K_{r-1}-free graph.

(6) *Planar Graphs.* A *subdivision* of a graph G is a graph resulting from the subdivision of edges in G. Subdividing an edge $(u, v) \in E(G)$ yields a new graph G' such that G' contains a new vertex w, where the edge (u, v) is replaced with the edges (u, w) and (w, v). A graph is *planar* if it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph with 3 vertices in each bipartition). Let H be a planar graph, and let $\mathcal{R}(H)$ be the set of graphs G for which there exists a role assignment $r : G \to H$. Is it true that $G \in \mathcal{R}(H)$ must be planar? We can also ask a more general question. Suppose $r : G \to H$ for some role assignment r, where Γ is a subgraph of H. For what graphs Γ, beyond $\Gamma = C_k$, can we conclude that G contains a subdivision of Γ?

(7) *Chordal Graphs.* Let G be a chordal graph, with $\delta(G) + 1 = \omega(G) = k$. If G has no $(k-2)$-sun, then can we conclude that G has a role assignment to K_k?

(8) *Chordal Graphs.* In [**52**], 2-role assignments of chordal graphs where considered, and it was shown that any chordal graph which contains at most one leaf has a role assignment to a K_2, with a leaf on one vertex, or alternatively stated, the vertex set can be partitioned into an independent set and a total dominating set. This is by no means a characterization, as the corona $G \circ K_1$ of a chordal graph G is chordal and there is a trivial partition of the vertices into an independent dominating set and a total dominating set. Can we determine "broader" sufficient conditions for a chordal graph G to be partitioned into an independent dominating set and a total dominating set?

(9) *Hypercubes.* Any graph G which is contained in $\mathcal{R}(Q)$ does not have a role assignment to K_3. Can the bipartite graphs which are contained in $\mathcal{R}(Q)$ be characterized? At present, the only result is that cartesian products of paths and stars are contained in $\mathcal{R}(Q)$.

(10) *Cartesian Products.* In Chapter 5, we showed that a graph G has a role assignment to K_3 if and only if $G \square K_2$ has a role assignment to K_3. This can be relaxed to say G has a role assignment to K_3 if and only if $G \square H$ has a role assignment to K_3 for any bipartite graph $H \in \mathcal{R}(Q)$. We conjecture that this can be strengthened in the following way.

Conjecture 6.4: *Let H be a graph such that G has a role assignment to K_3 if and only $G \square H$ has a role assignment to K_3. Then $H \in \mathcal{R}(Q)$.*

Furthermore, does there exist a sequence of graphs G_k such that a graph G has a role assignment to K_k if and only if $G \square G_k$ has a role assignment to K_k?

(11) *Hypercubes.* It was shown that given k, there exists an d-dimensional hypercube which has a role assignment to K_k for a large enough value of d. However, if we are given d, determining for which values of k a d-dimensional hypercube has a role assignment to K_k is more difficult to determine. In particular, finding the largest value of k is difficult, which leads to the following question: Given a hypercube Q_d, what is the largest value k such that there exists a role assignment $r : Q_d \to K_k$? That is, what is the domatic number of Q_d?

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