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PERFORMANCE BASED DECISIONS UNDER UNCERTAINTY AND RISK

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

---

by
Sundeep Samson
August 2008

Accepted by:
Dr. James A. Reneke, Committee Chair
Dr. Margaret M. Wiecek
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Dr. James R. Brannan

ABSTRACT

Decision making has became increasingly complex as risky decisions are made in uncertain environments. To minimize risk, the problem of quantifying risk becomes very significant. In our review of the literature, we found a number of different risk measures. A closer study reveals that most of these risk measures define both risk and uncertainty in many different ways.

In this dissertation, we model uncertainty and risk based on the Knightian definition of uncertainty and risk. We propose this alternative modeling paradigm and introduce our decision making methodology by solving an illustrative problem presented by a group of decision makers from Sandia Laboratories.

Multicriteria decision problems for complex systems with interacting components require the algebra of operator representations and the separability of these representations to include the feedback due to the interacting uncertainties. However, the sum of these separable representations need not be separable. We propose a separable approximation for the sum and bound the errors due to the approximation.

The issue of consistency of our decision making methodology is also studied by resolving the Ellsberg Paradox where Ellsberg claims that good decision makers should violate Savage's axioms at times. We show that our methodology provides consistent decisions without violating Savage's axioms. This consistency of our methodology is attributed to handling the multiple criteria without aggregation.

Due to this concept of de-aggregated multiple criteria, we believe that the conventional portfolio selection problems may mislead the decision maker by confounding the criteria. We propose an alternate portfolio selection procedure from a given feasible set of portfolios. Real data for Real Estate Investment Trusts, traded in NYSE, was used to create a consistent portfolio. In this dissertation, only the initial work for this portfolio selection problem is presented. Finally, we discuss the directions for future research.

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DEDICATION

To Amma, Appa, Aarthi, Jemmi Ajji, Vimala Ajji, and the two that I never had a chance to meet: John David Thatha and Samuel Thatha.

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CHAPTER 1

INTRODUCTION

Decision making, as old as the human race itself, has always been an integral part of human existence. Many times in reality, the most difficult decisions we make may not only affect us, but may affect someone else, sometimes generations later. As the world evolved, decision making has became increasingly more complex as multiple risky decisions are made simultaneously with conflicting criteria in uncertain environments.

In today's world everyone is constantly presented with decision problems. We all make hundreds of decision everyday, some are straightforward while some are not. In general, many of the decisions that we make everyday are made under a scenario that we are comfortable with. It is usually more difficult to make decisions for new problems in an unknown environment. So all decision problems are not generally treated with the same seriousness. A decision problem can be as simple as deciding what to wear on a particular day with uncertain climatic conditions and can be as complex as how to safely store the nuclear waste with hundreds of uncertain safety issues, some even unknown to the decision makers at the time the decision is made. While a bad decision for the first problem need not be life threatening, the latter has a potential to harm thousands of innocent people living around the storage facility. However small the decision problem, as long as one has to make a decision in an uncertain environment, there is always an element of risk attached to the decision.

This very elemental nature of decision problems, being present in everyone's life almost everyday makes it very important to understand the process of decision making. Every decision maker's objective is to minimize risk in uncertain environments. This objective leads to a few serious questions; What is risk? What is uncertainty? Is uncertainty different from risk? Are uncertainty and risk related? Can uncertainty be quantified? Can risk be quantified? How to minimize risk? Can uncertainty be minimized? Trying to answer these questions was the initial motivation of our work. The following outlines the goals of this dissertation;

- Understanding uncertainty and risk.
- Modeling uncertainty and risk.
- Developing a performance based decision making methodology using the modeled uncertainty and risk.
- Exploring the extension of this decision making methodology to complex systems with multiple interacting uncertainties.
- Illustrating the consistency of the methodology.
- Illustrating the practical usefulness and significance of the decision methodology.
- Presenting a discussion of important future directions.

To understand uncertainty and risk, we present in Chapter 2 a detailed review of the terms uncertainty and risk. We explore different definitions for these terms in various disciplines. We surveyed the literature in financial mathematics, engineering, operations research and other general fields of study. We found definitions that claimed risk to be uncertainty, risk and uncertainty to be very different concepts, risk and uncertainty to be related, etc. In Sections 2.2 and 2.3 we present these findings.

We found a very small number of researchers modeling uncertainty as as interval with no guiding distributions. We propose to model uncertainty as both an interval with unknown governing distributions and if we have enough data to get all possible discrete values we model them as a collection of discrete points. Our model for risk depends on random functions of the uncertainties. We could not find any research modeling risk based on random functions of uncertainties in the literature. A detailed discussion on our modeling paradigm is presented in Section 2.4.

In 2004, a group of researchers from Sandia Laboratories who work extensively with nuclear safety issues, presented the risk community with a Challenge Problem (Oberkampf et al., 2004). *Reliability Engineering and Systems Safety* published a special issue with a good collection of responses to this Challenge Problem. The responses could be broadly divided into two approaches, bayesian analysis and the fuzzy sets analysis. Since our modeling paradigm and methodology do not impose any additional assumptions on the problem presented, which was not the case in general, we felt the need to present our approach. In Section 2.5 we illustrate our decision making methodology using one of the Challenge Problems. We complete Chapter 2 with some concluding remarks on the alternative modeling paradigm, the proposed decision making methodology and the Challenge Problem in Section 2.6.

The Challenge Problem (Oberkampf et al., 2004) discussed in Chapter 2 is a mathematical model of a physical system's response with three uncertainties. However, these three uncertainties were independent of each other, with no interactions among them. The decision process becomes a lot more complex when the uncertainties interact within a problem. In Chapter 3 we address the case of interacting uncertainties. In Section 3.1 we discuss the background mathematics in the form of Hellinger integrable functions and the Wiener fields. We start with the algebra of the operators and separable random fields. The separability of these random fields is the key factor in the performance of our methodology. More special results and the methodology for producing linearizations of such separable random fields are presented in Section 3.2.

A multilevel decision problem with multiple interacting components under uncertain environments is called a complex system. A complex system can be mathematically represented as a network. Each node of this network is modeled by separable representations. The mathematical representation of the interactions in this network would require us to produce sums of these separable representations. However, the sum of these separable representations need not be separable. Since our proposed decision making methodology requires every representation to be separable, in Section 3.3 we propose a separable approximation for these non-separable representations of sums. Finally we provide the concluding remarks in Section 3.4

In Chapter 3 we explored ways to extend our methodology to interacting uncertainties and provided an approximation for the same. Chapter 4 deals with multiple criteria decisions and the consistency of the proposed methodology. In the late 1940s and the early 1950s many researchers were proposing different decision making methodologies. Savage (1954), in his book, suggested that the decision making methodologies should satisfy some basic axioms to produce consistent decisions. The Sure-Thing Principle is one axiom that any decision making methodology should satisfy to be considered consistent. Ellsberg (1961) created two urn problems to challenge Savage's Sure-Thing Principle. He illustrated how experienced decision makers violate the Sure-Thing Principle. He argues that good decision makers sometimes need to violate the Sure-Thing Principle to make good decisions, creating a paradox. We resolve this paradox.

In Section 4.2, we present the Ellsberg's Paradox in detail. In Section 4.3 we map the language of Ellsberg's urn problems to our decision problems. In Section 4.4 we introduce our multicriteria version of risk which handles each criteria separately. In other words, unlike many other decision process which aggregates the criteria before decision making, we keep the criteria de-aggregated until the decision is made. We illustrate how our methodology with the de-aggregated criteria with the proposed preference rule resolves the paradox, i.e., without any additional assumption, our methodology makes consistent choices for the two urn problems devised by Ellsberg.

In Section 4.5 we discuss how our preference rule and methodology relate to Savage's postulates. We show that our framework satisfies the first four of the eight Savage's postulates. This forms the theoretical basis for our methodology to produce consistent decision choices. In Section 4.6 we illustrate how aggregating the criteria results in in-consistent decisions.

Due to this concept of de-aggregated multiple criteria, we believe that the conventional portfolio selection problems may mislead the decision maker by confounding the criteria. To study this in more detail, in Section 4.7, we created a couple of portfolio selection problems similar to the Ellsberg's urn problems and illustrate how using the de-aggregated multiple criteria produce consistent choices while portfolio selection problems with aggregated criteria produce inconsistent choices. The chapter concludes with a detailed discussion, in Section 4.8, of Ellsberg's Paradox and other researchers' stands on the resolution of the paradox.

The issues raised in Chapter 4 regarding the aggregated criteria motivated us to propose a new portfolio selection procedure from a given feasible set of portfolios. In Chapter 5, we propose the initial set-up for creating a consistent portfolio selection procedure. To illustrate this procedure, we chose thirteen Real Estate Investment Trusts (REITs) actively traded in the New York Stock Exchange. Using real data for these thirteen REITs, our initial set-up was to treat each REIT as a portfolio and select the best REIT from among the thirteen REITs. This selection procedure, including the establishment of uncertainty and the preference rule are presented in Section 5.2. In Section 5.3 we analysis the decision of the best REIT choices based on two different confidence levels. Future proposals on how this alternative portfolio selection procedure can be developed are presented in Section 5.4.

Concluding remarks and our thoughts on possible future work is offered in Chapter 6.

CHAPTER 2

A REVIEW OF DIFFERENT PERSPECTIVES ON UNCERTAINTY AND RISK AND AN ALTERNATIVE MODELING PARADIGM

In this cahpter, the literature in economics, finance, operations research, engineering and in general mathematics is first reviewed on the subject of defining uncertainty and risk. The review goes back to 1901. Different perspectives on uncertainty and risk are examined and a new paradigm to model uncertainty and risk is proposed using relevant ideas from this study. This new paradigm is used to represent, aggregate and propagate uncertainty and interpret the resulting variability in a challenge problem developed by Oberkampf et al. (2004). The challenge problem is further extended into a decision problem that is treated within a multicriteria decision making framework to illustrate how the new paradigm yields optimal decisions under uncertainty. The accompanying risk is defined as the probability of an unsatisfactory system response quantified by a random function of the uncertainty.

2.1 Introduction

Due to the recent increased interest in decision making under uncertainty and risk among researchers in engineering and science, we explore the terms uncertainty and risk. Having reviewed a large number of research articles and reports we found that there is no general definition for these terms but rather many discipline or context dependent definitions. In fact, almost every definition we found is problem specific suggesting that every time a new decision problem is stated, new definitions for uncertainty and risk are also introduced for this decision problem. One common aspect of these definitions however is that uncertainty and risk are usually related.

There is an abundance of literature that discusses uncertainty and risk, making it impossible to represent the whole literature. Here, we have subjectively compiled a list of authors who proposed definitions for these terms. Since we deal with very carefully worded

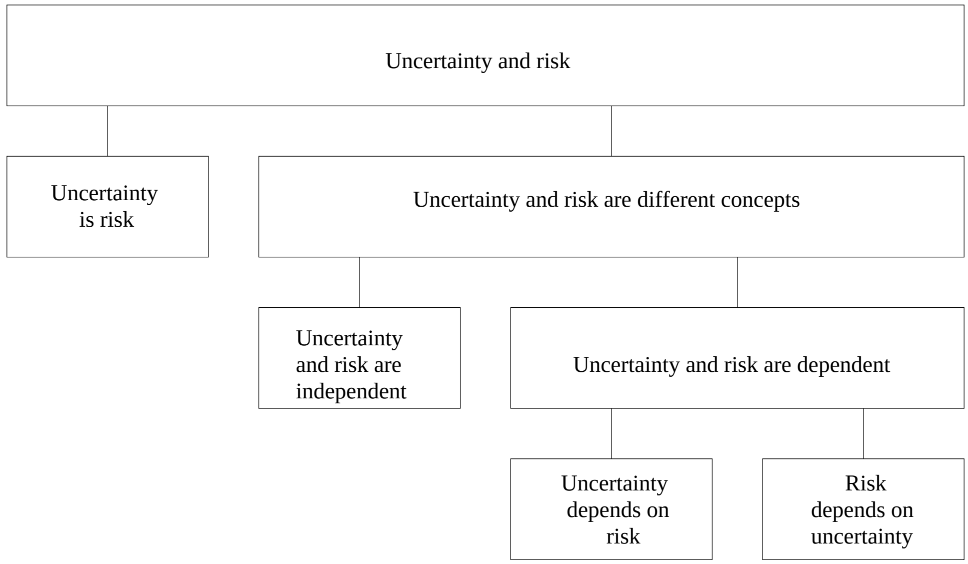

Figure 2.1 Relationships between risk and uncertainty

definitions, we believe it is important to keep the original wording to avoid misrepresentation and misunderstanding. For this reason, most of these proposed definitions are given in this chapter as quotes from original sources.

Figure 2.1 depicts a diagram with different relationships between risk and uncertainty identified in the literature. Every definition considered is also classified into three broad areas including Operations Research (OR), Economics and Finance (EF), and Engineering (E) based on the author's affiliation to the field. Articles collected outside these three areas are not classified with any letters.

Our review of definitions covers two main schools of thought. In Section 2.2, we review papers claiming that uncertainty is risk and, in the subsequent section, we study articles advocating that uncertainty and risk are different concepts. The descriptive overview goes back to 1901 and covers 34 bibliographical sources chosen to reflect the ideas of uncertainty and risk in several fields of science and different time periods.

We also develop a new modeling paradigm, in the form of a two-step approach, to model uncertainty and risk taking into account the approaches and concepts available in the collected set of definitions. This approach is based on the premise that any physical model of a subject can be arranged on a continuum depending on how close the modeler is to the subject. At one extreme the modeler has a frog's viewpoint and at the other a bird's viewpoint. A detailed discussion on how uncertainty and risk can be efficiently modeled using this two-step approach is presented in Section 2.4.

In Section 2.5, a challenge problem posed by Oberkampf et al. (2004, E) is used to illustrate our new modeling paradigm to model uncertainty and risk and quantify the overall variability in the physical system presented. Furthermore, we extend the challenge problem (Oberkampf et al., 2004, E) into a design problem by introducing new design parameters and show how our methodology models uncertainty and quantifies risk in the presence of feasible decision alternatives available to the decision maker. We close Section 2.5 with a discussion on representation, aggregation and propagation of uncertainty, and interpretation of the resulting variability in the physical system presented by Oberkampf et al. (2004, E).

The chapter is concluded in Section 2.6.

2.2 Uncertainty is Risk

We start our review with recognizing a large group of scholars, especially in the world of economics and finance, who define uncertainty as risk and/or risk as uncertainty.

- "..., risk is defined as uncertainty. It has reference to the uncertainty of a financial loss and little to do with the loss itself, the cause of the loss, or the chance of loss. Risk has principally to do with the uncertainty of a loss.... The degree of risk is measured by the probable variation of actual experience from expected experience. The lower the probable percentage of variation, the smaller the risk." (Mehr and Cammack, 1961, EF).

- "The uncertainty of the happening of an unfavorable contingency has been termed *risk*. Risk is present when there is a chance of loss.... The various factors contributing to the uncertainty are termed *hazards*. Ordinarily there are many separate hazards that contribute to the chance or possibility of loss that attach to any particular object or person. The sum total of the hazards constitutes the risk." (Magee, 1961, EF).

- Philippe (2001, EF) in his book defines risk as "...the uncertainty of outcomes. It is best measured in terms of probability distribution functions." He also proposes a measure for risk called value at risk (VaR). "VaR measures the worst expected loss over a given horizon under normal market conditions at a given confidence level. For instance, a bank might say that the daily VaR of its trading portfolio is $35 million at a 99 percent confidence level. In other words, there is only 1 chance in a 100, under normal market conditions, for a loss greater than $35 million to occur."

The most interesting aspect of these definitions is that they assume quantification of uncertainty. More specifically, they assume that uncertainty follows a distribution, or a set of distributions giving rise to a joint distribution, which helps quantify the uncertainties that they define as risk.

In some decision problems however we come across uncertainties that are not quantifiable. They do not follow any specific distribution and usually are defined by a lower bound and an upper bound (an interval). It is then quite natural to look for definitions that assume the non-quantifiability of uncertainty.

In contrast to the perspective that risk is uncertainty, many scholars agree that uncertainty and risk are different concepts and some others define only one of these two concepts even though they discuss both concepts. We examine these issues in the next section.

2.3 Uncertainty and Risk are Different Concepts

One of the earliest definitions of uncertainty and risk found in the literature is given by economist Willett (1901, EF). In his PhD thesis, he defines risk as the "objectified uncertainty regarding the occurrence of an undesirable event" and adds that it can be quantified with probability arguments. The subjective uncertainty "resulting from the imperfection of man's knowledge" is uncertainty.

The following is an argument from Willett's book that clearly explains why uncertainty and risk are different. He explains that the uncertainty is the greatest when the degree of probability is $\frac{1}{2}$, because with this probability there is nothing to show what the outcome will be. But as the probability increases or decreases the uncertainty always decreases and at the end when the probability is either 0 or 1 there is no uncertainty. Clearly uncertainty and risk can not be the same here.

Twenty years later another economist Knight (1921, EF) makes a distinction between quantifiable uncertainty and non-quantifiable uncertainty. He then defines quantifiable uncertainty as risk and non-quantifiable uncertainty as uncertainty.

"But uncertainty must be taken in a sense radically distinct from the familiar notion of risk... It will appear that a measurable uncertainty, or *risk* proper, as we shall use the term, is so far different from an unmeasurable one that it is not in effect an uncertainty at all. We shall accordingly restrict the term *uncertainty* to cases of the non-quantitative type.... The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known, while in the case of uncertainty this is not true, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique."

Over the years there have been many definitions that distinguish uncertainty as unquantifiable and risk as quantifiable, but with different measures. We continue on this observation in subsection 2.3.2. The following are a few definitions given by scholars who choose to define only one of these concepts.

Keynes (1937, EF) defines uncertainty as follows, "By uncertainty... I do not mean merely to distinguish what is known for certain from what is only probable.... About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know."

The following is a definition of risk, where it is not just a probability or any other single value, but a triplet. Kaplan and Garrick (1981, EF) suggest a new way of looking at risk. They argue that when one asks, "What is risk? one is really asking three questions: What can happen? How likely is that to happen? If it does happen, what are the consequences?" They therefore define risk as a triplet. The answer for the first question is one of the possible scenarios. They denotes the i^{th} scenario by S_i and the likelihood of this scenario by L_i and the consequences by X_i. Thus the triplet (S_i, L_i, X_i) defines risk. They believe risk cannot be defined using simple concepts such as numbers, curves and vectors.

Lough et al. (2005, EF) defines risk as "the chance that an undesirable event will occur and the consequences of its possible outcomes."

There are many scholars who believe that uncertainty and risk are two different concepts, but still do not agree on how they are related, if at all. One group of scholars believes that riskiness of a system depends on the uncertainty of the system environment and others believe that the uncertainty in the system depends on the risk in the system. There are some who argue that there need not be any relationship between uncertainty and risk.

2.3.1 Uncertainty and Risk are Independent

The following definition introduces a certainty-risk-uncertainty classification (Luca and Raiffa, 1957, OR) . "Suppose that a choice has to be made between two actions. We shall say that we are in the realm of decision making under:

certainty if each action is known to lead invariably to a specific outcome;

risk if each action leads to one of the set of possible specific outcomes, each outcome occurring with a known probability;

uncertainty if either action or both has as its consequences a set of possible specific outcomes, but where the probabilities of these outcomes are completely unknown are not even meaningful."

This definition is not much different from Knight's in the sense that risk is defined as a quantifiable notion and uncertainty as non-quantifiable one. It however treats uncertainty and risk and as two independent concepts.

Pfeffer (1956, EF) brings in a new perspective to the definition of uncertainty and risk and clearly separates these concepts defining them as follows:

"Risk is a state of the world: uncertainty is a state of the mind .... Risk and uncertainty are counterparts of one another; the one being measured by objective probability; and the other by subjective degree of belief."

By defining risk as a state of the world, he seems to suggest that risk does not depend on a particular person's knowledge. For example, if a new investor is comfortable with an investment then his behavior does not imply that there is low risk involved in the investment, while on the other hand if a seasoned investor feels bad about an investment then his attitude again does not imply that risk is very high, since risk could be very small in both cases.

2.3.2 Uncertainty and Risk are Dependent

Continuing our search for different definitions for uncertainty and risk we explored the engineering community which has worked extensively in the area of risk analysis, especially in the context of nuclear waste management safety (Rechard, 1999; U.S. Nuclear Regulatory Commission, 1975, 1991; Lewis et al., 1978, E). The work initiated by the Atomic Energy Commission in 1972, *Reactor Safety Study* (U.S. Nuclear Regulatory Commission, 1975, E), was published in 1975. After throughly studying the risk assessment in literature and analyzing 40 reactors, the Commission established a definition for risk that was most commonly used in the engineering community as the product of the probability of a bad outcome and the consequence due to the outcome. Engineers acknowledge that uncertainty and risk are two different but related concepts. However, earlier risk assessment strategies failed to handle uncertainty satisfactorily which was pointed out by the Lewis Committee report on risk assessment to the U.S. Nuclear Regulatory Commission (Lewis et al., 1978, E). In the late 80's and early 90's many engineers started working with uncertainty with too little information to be quantified with a distribution. The following are a few of these works.

Apostolakis (1989, E) gives a definition of uncertainty,"The distribution for the uncertainty factor is assessed subjectively, using the different predictions of the various models to indicate the possibility range of variation." Again uncertainty is defined as an interval but possibly with some information on an accompanying probability distribution.

Due to increased interest in uncertainty in the engineering community (Committee on Risk Assessment Methodology, 1993, E) in the early 90's, Helton and Burmaster (1996, E) in a guest editorial article propose two kinds of uncertainties. Later Oberkampf et al. (2004, E) define these two kinds of uncertainty as follows: "The risk assessment community has begun to make a clear distinction between aleatory and epistemic uncertainty in theory and in practice. Aleatory uncertainty is also referred to in the literature as variability, irreducible uncertainty, inherent uncertainty, and stochastic uncertainty. We use the term aleatory uncertainty to describe the inherent variation associated with the physical system or the environment under consideration. Epistemic uncertainty is also termed reducible uncertainty, subjective uncertainty, and model form uncertainty. Epistemic uncertainty derives from some level of ignorance, or incomplete information, of the system or the surrounding environment." For more background on aleatory and epistemic uncertainty we refer the reader to (Apostolakis, 1990; Paté-Cornell, 1996; Helton, 1997, E).

It is interesting to see that these engineers do not define any of these uncertainties as risk but rather prefer to work with these two kinds of uncertainty. They work toward modeling methods to "efficiently represent, aggregate, and propagate different types of uncertainty." In 2004, a special issue of *Reliability Engineering & System Safety* edited by Helton and Oberkampf (2004, E) deals exclusively with epistemic and aleatory uncertainties. Batill et al. (2002, E) define uncertainty as the "variation in measurement" and Mourelatos and Zhou (2004, E) model uncertainty with lower and upper bounds.

In Operations Research the definition of uncertainty given by Zimmermann (2000, OR) also distinguishes uncertainty as objective and subjective uncertainties. He is more concerned about subjective uncertainty and the following definition refers to it. "Uncertainty implies that in a certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behavior or the characteristics." His list of causes for uncertainty includes lack or abundance of information, conflicting evidence, ambiguity measurement and belief. He also strongly believes that uncertainty should not be modeled context free and that there exists no "single method which is able to model all types of uncertainty equally well."

Uncertainty Depends on Risk

We present the only definition we found suggesting that uncertainty depends on risk. Crowe and Horn (1967, EF) along the lines of Pfeffer (1956, EF), view uncertainty as a "state of mind." They then ask the question: "If risk and uncertainty are separate concepts, what relationship exists between the two?" and answer by adding that "...there is no necessary relationship. The two frequently are unrelated." However "...it may be true that situations where risk gives rise to uncertainty are the most important. Here no harm seems to be done if one regards risk as a kind of *proximate cause* of uncertainty..." Also they ask themselves: "Can uncertainty cause, or give rise to, risk?" And they answer negatively. The authors conclude that "the concept of risk...is primarily an objective phenomenon. In no sense is identical to uncertainty, which is a state of mind...however, risk and uncertainty are closely associated, since risk often gives rise to uncertainty."

Risk Depends on Uncertainty

Many definitions suggesting that uncertainty gives rise to risk also seem to define a relationship with the notion of variance. This can be seen in the following definitions dating back to 1901.

Willett (1901, EF), whose definition of risk and uncertainty we stated earlier, is also one of the first to infer that risk is deviation from the mean. He writes: "The greater the probable variation of the actual loss from the average, the greater the degree of uncertainty." Here when he refers to uncertainty, he means objectified uncertainty and therefore risk. Also note that the deviation from the mean is the variance.

In his book about investments in a stock market, Fisher (1906, EF) implies that variance of the outcome of the option is related to the risk of the option. Markowitz

(1952, EF), another economist, writes "…consider expected return a desirable thing and the variance of return an undesirable thing."

In Fisher's definition, the *outcome of an option* can be seen as uncertainty modeled as an interval with the least possible outcome as the lower bound and the highest possible outcome as the upper bound. Also, the *return* in Markowitz's definition can be viewed as uncertainty with just an upper and a lower bound.

Following Willett, Ratcliffe (1963, EF) defines risk using a similar language of deviation from the mean. "Risk is the possibility that actual results may differ from predicted average results. Pure risk is the possibility that actual loss may be greater than predicted average loss." He gives an interesting argument to justify this definition.

"When the student is waiting at the airport to board the plane he feels almost certain that he will arrive safely in New York City. He feels practically no uncertainty. Does that mean that there is no risk? Of course not. Subjective uncertainty is not risk. Nor is degree of risk a function of it. Actually the probability of arriving safely in New York is very high; and, of course, the probability of not arriving safely is quite low. Does that mean that there is practically no risk? Of course not. The truth is that there is much risk involved…"

Houston (1964, EF) advocates that "…risk can only be satisfactorily understood as a variance concept…" He continues: "…Since the probability (or chance) of loss is a mean value, equating risk to it is, in effect, defining risk as a mean or average concept and, as will be seen, greatly restricts the analysis. Further, it can be argued that, if risk is equal to the chance of loss, then there is no point in introducing the term *risk* into the discussion, since the term *chance of loss* is already fully defined." He defines two risks, one for the customer and defined as a "standard deviation of the monetary outcome of an action," and the other for the insurer and defined as a "function of variation…defined by the standard error of the mean of the pure premium distribution."

Head (1967, EF) defines risk "as the objective probability that the actual outcome of the event will differ significantly from the expected outcome," and adds "Risk is a probability."

Later Athearn (1969, EF) argues that if risk is uncertainty, then risk analysis techniques available in the literature could be used to successfully deal with uncertainty. But then it is clear that risk analysis alone is not sufficient to solve uncertainty. Hence he concludes that uncertainty can not be risk and defines risk as follows: "Risk may be defined as either (a) the expected possibility of loss or (b) the possibility of an unfavorable deviation from the expectation, because any unfavorable deviation from expectation is a loss."

Another definition of risk that uses variance comes from Pollatsek and Tversky (1970): " The various approaches to the study of risk shares three basic assumptions. 1. Risk is regarded as a property of options (eg., gambles, courses of action) that affects choices among them. 2. Options can be meaningfully ordered with respect to their riskiness. 3. The risk of an option is related in some way to the dispersion, or the variance, of the outcomes."

Mulvey et al. (1995, OR) agree with Markowitz's (Markowitz, 1952, EF) definition of risk and develop a robust optimization (RO) method for risk minimization. They claim that in this RO approach one of the terms in the objective function accounts for reducing variability. They write: "The objective function of this RO formulation has three terms. The first term is expected cost of the operation. The second term is the variance of the cost.... The third term penalizes a norm of the infeasibility..."

While all the definitions collected in this subsection have the use of variance in common, Jia and Dyer (1996, OR) write that "arguments have been made that the mean-variance model (suggested by Markowitz) is appropriate only if ... the joint distribution of return is normal."

Macgill and Siu (2005, EF) define risk descriptively as "the union of the dynamically evolving risk knowledge of the physical and social worlds." The authors also make an interesting observation that making distinction between uncertainty and risk following

Knight's definition produces better decisions when compared to the same problem with no distinction between uncertainty and risk.

We close this section with a quote from Holton (2004, EF) who writes that "...risk entails two essentials components: exposure and uncertainty." He defines *uncertainty* as "a state of not knowing whether a proposition is true or false" and *exposure* as "a self conscious being is exposed to a proposition if the being would care whether or not the proposition is true" then he defines *risk* as the "exposure to a proposition of which one is uncertain." He gives the following example to explain how both exposure and uncertainty is required for any presence of risk: "Suppose a man leaps from an airplane without a parachute. If he is certain to die, he faces no risk."

The above discussed definitions of uncertainty and risk are in no way a complete literature review of this subject. Our discussion is a subjective sample of definitions that interest us in developing our own model of uncertainty and risk. There are a plenitude of definitions for these concepts in areas like fuzzy sets, probability and statistics which we did not cover. Also this discussion does not include works which make use of different concepts related to risk in decision problems. For instance, in OR, *regret, recourse, robustness, sensitivity analysis*, etc. are defined and used in decision making. We chose not to include these in our discussion here. Similarly, it is not our objective to survey the risk assessment literature. For this subject, we refer the reader to a very elaborate study of Rechard (1999).

For any definitions used here, we tried to refer to the oldest paper in which it appeared and do not refer to other papers if they use a similar definition. So, our bibliography includes possible definitions but not all different definitions. We include this review in the first part of this chapter to contrast the different approaches in defining risk and uncertainty in the literature and to show that our approach proposed in the following sections deals with uncertainty and models risk from a completely different perspective.

2.4 Proposed Model for Uncertainty and Risk

This subjectively selected collection of papers gives a flavor of different definitions of uncertainty and risk available in the literature. Almost all of these definitions are problem sensitive, i.e., they may not perform as well if applied to a new problem area. Most scholars believe that uncertainty and risk are two different concepts and that uncertainty gives rise to risk and is vital for decision problems. Some of these scholars suggest that uncertainty can be modeled as an interval even though there is no consensus on whether it is quantifiable or not. Some researchers define risk using the variance concept. However, there is no common modeling method that they all agree upon.

The following is an example to illustrate how we propose to model uncertainty and risk. Consider an airline company. Assume every planned flight that is completed as scheduled incurs a profit and every planned flight that is canceled incurs a loss to the company. One of the major reasons flights are canceled for this airline is weather conditions, especially snow. Planning requires a model of the interactions of snowfall and the number of flights.

Consider snow conditions of a future date, say in winter. If the modeler were to stand near the runway on the above mentioned date, he might observe no snow at all, one inch of snow, 2.5 inches of snow and so on. In other words, the modeler may have a deterministic collection of possible observations. Each of these observations is termed a frog's view and the modeler's objective for modeling uncertainty is to collect all possible frog views. This collection of frog views can be modeled as a subset of an interval with a lower bound suggesting no snow, and an upper bound determining the maximum amount of snow on the runway. Since no one can predict, i.e., estimate the expected value of the amount of snow on the runway for any given day in the future, the amount of snow is random but non-quantifiable and therefore uncertain. This non-quantifiable randomness modeled as an interval represents *uncertainty.*

Due to an unknown amount of snow on a given day, the number of outgoing (incoming) flights is also random. In particular, the number is a function of the amount of snow, i.e., the uncertainty. The number of flights is also random for a particular amount of snow given the randomness in the airline industry due to factors such as equipment delays, availability of runways, crew, etc. The values of this random function for a given amount of snow on the runway has a distribution. A sample drawn from this distribution of the number of flights for a given amount of snow, is termed a bird's view. The bird's view is characterized by the mean, i.e., the expected number of flights for a given amount of snow, and the variance, i.e., the variance in the number of flights for a given amount of snow. The collection of the bird views depending on the frog views gives a global view of the problem. The collection of bird views allows us to estimate the distribution at each value of the uncertainty. This collection of distributions aid us in modeling *risk*, which is discussed in detail later.

In this approach, uncertainty and risk are modeled based on the following two views of the modeler, the frog's view and the bird's view, depending on how close the modeler is to the subject of interest. Clearly, uncertainty is the independent variable and risk is the dependent variable.

We propose several clues, presented in Table 2.1, that help distinguish uncertainty from risk. In the example, the *quantities* are the amount of snow for uncertainty and the distribution for a given amount of snow of the number of flights for risk. The amount of snow is knowable but unknown for a future date. The number of flights which depends on the amount of snow is unknowable due to other random influences. Hence uncertainty is modeled using an interval of deterministic amounts of snow and risk is modeled with a distribution depending on the amount of snow.

Based on this modeling idea, we propose a two-step approach to modeling uncertainty and risk.

Step 1: Determine the frog's view and the bird's view.

Step 2: Collect all possible frog views and bird views. The collection of frog views consisting of a set a deterministic observations is then modeled as a subset of an interval $[a, b]$. Similarly, the collection of bird views determines a random process $\{Y_t, a \leq t \leq b\}$

Table 2.1 Uncertainty and risk characterization

	Uncertainty	**Risk**
View	Frog's	Bird's
Quantities	Knowable but unknown	Unknowable
Model	Deterministic	Random
Number of observations	Few	Many
Observations modeled as	Independent variable	Dependent variable

that can be transformed to $Z_t = \int_a^t Y(s)(ds)^{1/2}$ by the Central Limit Theorem (CLT) to give a normal random process (see Remark 2.1).

As a result of this approach the non-quantifiable randomness modeled as an interval $[a, b]$ is *uncertainty* and the quantifiable randomness modeled by the distributions of a random function's values at each point of the uncertainty aids in modeling *risk*. Of course, the quality of the global model depends on the quality and the quantity of observations. This two step process is illustrated in more detail in the next section.

A similar example on weather effecting the transport and deposition of radio active material released to the atmosphere during a reactor accident is presented in (Helton, 1994). That example motivates the authors to quantify the uncertainty using the available information. In our approach, we assume that the available information does not allow us to quantify uncertainty with a distribution.

Remark 2.1. Let $\{Y_t,\, 0 \leq t \leq 1\}$ be a zero mean random process with finite positive variance $\sigma^2(\cdot)$ and $Y(s)$ and $Y(t)$ be independent for $s \neq t$. Further, suppose that $E(|Y(t)|^3)$ is finite for each t, and both $\sigma^2(\cdot)$ and $E(|Y(\cdot)|^3)$ are Riemann integrable on [0, 1].

Let $\{s_p\}$ be a sequence of partitions of [0, 1] such that for $p = 0$ we have $0 = s_0(0) < s_0(1) < \cdots < s_0(m) = 1$. Also s_{p+1} refines s_p, and $mesh(s_p) \to 0$ as $p \to \infty$ (i.e., $\sup |s_p(j) - s_p(j-1)| = 0$ as $p \to \infty$). Assume that $\{u_i\}_{i=0}^{m}$ is a sequence of nondecreasing integer valued sequences such that $s_p(u_i(p)) = s_0(i)$, for $0 \leq p$ and $0 \leq i \leq m$. Note that $s_p(u_{(\cdot)}(p)) = s_0$. Also, for $0 \leq i \leq m$, $u_i(p) \to \infty$ as $p \to \infty$.

Let $1 \leq i \leq m$ be fixed and $t = s_0(i)$. For each positive integer p and $1 \leq j \leq u_i(p)$, let

$$\hat{Y}_{jp} = (1/\sigma_{ip}) \sqrt{s_p(j) - s_p(j-1)}\, Y(s_p(j))$$

where $(\sigma_{ip})^2 = \sum_{j=1}^{u_i(p)} (s_p(j) - s_p(j-1))\sigma^2(s_p(j))$. Note that $(\sigma_{ip})^2 \to \int_0^t \sigma^2(s)\, ds$ as $p \to \infty$. Let

$$S_p = \sum_{j=1}^{u_i(p)} \hat{Y}_{jp} = (1/\sigma_{ip}) \sum_{j=1}^{u_i(p)} \sqrt{s_p(j) - s_p(j-1)}\, Y(s_p(j))$$

Note that $E(S_p) = \sum_{j=1}^{u_i(p)} E(\hat{Y}_{jp}) = 0$ and $E(S_p^2) = \sum_{j=1}^{u_i(p)} E(\hat{Y}_{jp}^2) = 1$. Consider

$$\sum_{j=1}^{u_i(p)} E(|\hat{Y}_{jp}|^3) \quad = \quad \sum_{j=1}^{u_i(p)} (1/\sigma_{ip})^3 \, (s_p(j) - s_p(j-1))^{3/2}\, E(|Y(s_p(j))|^3)$$

Since $mesh(s_p) \geq (s_p(j) - s_p(j-1))$ for all j, we have

$$\begin{aligned} \sum_{j=1}^{u_i(p)} E(|\hat{Y}_{jp}|^3) \quad \leq \quad & (1/\sigma_{ip})^3 mesh(s_p)^{1/2} \times \\ & \sum_{j=1}^{u_i(p)} (s_p(j) - s_p(j-1)) E(|Y(s_p(j))|^3) \end{aligned}$$

which has the limit $\frac{1}{(\int_0^t \sigma^2(s)\, ds)^{3/2}} \cdot 0 \cdot \int_0^t E(|X(s)|^3)\, ds$ that is equal to zero as $p \to \infty$. By the CLT, which states:

Theorem (Chung, 2001): *If $E\hat{Y}_{jp} = 0$ and $E(|\hat{Y}_{jp}|^3)$ is finite, for $p \geq 1$ and $1 \leq j \leq u(p)$, $\sum_{j=1}^{u(p)} E(\hat{Y}_{jp})^2 = 1$, for $p \geq 1$, and $\sum_{j=1}^{u(p)} E(|\hat{Y}_{jp}|^3)$ has limit 0 as $p \to \infty$ then $\{S_p\}$ converges in distribution to a (0, 1)-normal random variable.*

we have $\{S_p\}$ converging in distribution to a (0, 1)-normal random variable. Further,

$$\sigma_{ip} S_p = \sum_{j=1}^{u_i(p)} \sqrt{s_p(j) - s_p(j-1)}\, Y(s_p(j))$$

converges in distribution to a $(0, (\int_0^t \sigma^2(s)\, ds)^{1/2})$-normal random variable. That is, we obtain

$$\lim_{p\to\infty} \sum_{j=1}^{u_i(p)} \sqrt{s_p(j) - s_p(j-1)}\, Y(s_p(j)) = \int_0^t Y(s)\, (ds)^{1/2}$$

in distribution and denote this limit by Z_t. A detailed discussion on the importance of the square root of a differential in the context of a decision problem is presented in a working paper by Vovk and Shafer (2003).

2.5 Challenge Problem

In their special issue of the of the journal *Reliability Engineering and System Safety*, Helton and Oberkampf (2004, E) collected articles presenting different approaches and methodologies to efficiently model uncertainty. In particular, Oberkampf et al. (2004, E) in their paper called on fellow researchers to continue a dialog among different sciences and engineering communities on this subject. We also would like to participate in this discussion and use the approach proposed in this chapter to model uncertainty and additionally, to make rational decisions under conditions of uncertainty.

Oberkampf et al. (2004, E) distinguished between epistemic and aleatory uncertainties. Epistemic uncertainty is also referred to as *reducible*, *subjective* and *state-of-knowledge* uncertainty and aleatory uncertainty is referred to as *variability*, *irreducible* and *stochastic* uncertainty. Epistemic uncertainty is due to lack of knowledge and is modeled by an interval. The range of the interval gets tighter as the knowledge increases and hence this uncertainty is reducible. On the other hand, having enough information to model random uncertainty with a probability distribution gives rise to aleatory uncertainty. The distribution parameters may be uncertain, i.e., the distribution may depend on the values of epistemic uncertainty. In some sense, epistemic uncertainty is non-quantifiable and aleatory uncertainty is quantifiable in terms of the distribution. That is, by "non-quantifiable" we simply mean that we do not have enough information to assume any distribution for uncertainty.

Based on our research we discuss a challenge problem presented by Oberkampf et al. (2004, E) in terms of uncertainty and risk. Our uncertainty, which is non-quantifiable and modeled as an interval, is the same as epistemic uncertainty discussed in the challenge problem. The distribution associated with aleatory uncertainty in the problem is used in modeling our risk. A discussion on how we incorporate epistemic and aleatory uncertainties into our methodology can be found in the following pages.

We study the algebraic challenge problem given as a "mathematical model describing a physical system" (Oberkampf et al., 2004, E). The mathematical model is

$$y = (a + b)^a \tag{2.1}$$

where y is the system response and the parameters a and b are independent, i.e., knowledge of the value of one parameter does not influence the value of the other. Both a and b are positive real numbers representing uncertainties. It is assumed that the uncertainty in the problem is not due to the algebraic model but it is due to the parametric uncertainties. We concentrate on Problem 5 of the algebraic problem set, in which a models an epistemic uncertainty and b an aleatory uncertainty which follows a *log-normal* distribution with uncertain parameters, mean m and standard deviation s. These parameters m and s also represent epistemic uncertainties modeled by intervals. The task is to quantify the variability of y given the information concerning a and b. The authors of the challenge problem also point out that modeling epistemic uncertainty is much more challenging than modeling aleatory uncertainty which is assumed to follow some known distribution. Our goal is to discuss the challenge problem without assuming any additional information on the epistemic uncertainties.

In the problem of interest, the information concerning a, m and s is given by six independent panels. Each panel specifies closed intervals A_i, M_j and S_j that contains the values for a, m and s respectively. That is, we have

$A_i = [a_L^i, a_U^i]$ for $i = 1, 2, 3$, $M_j = [m_L^j, m_U^j]$ and $S_j = [s_L^j, s_U^j]$ for $j = 1, 2, 3$,

such that $\ln b$ follows $N(m^j, s^j)$ for $j = 1, 2, 3$ where $m^j \in M_j$ and $s^j \in S_j$. The following are the set of epistemic uncertainties as viewed by each panel.

Panel $\mathbf{A}_1$: $a^1 \in [0.5, 1.0]$, **Panel** $\mathbf{MS}_1$: $m^1 \in [0.6, 0.9]$ and $s^1 \in [0.3, 0.45]$.

Panel $\mathbf{A}_2$: $a^2 \in [0.2, 0.7]$, **Panel** $\mathbf{MS}_2$: $m^2 \in [0.1, 0.7]$ and $s^2 \in [0.15, 0.35]$.

Panel $\mathbf{A}_3$: $a^3 \in [0.1, 0.6]$, **Panel** $\mathbf{MS}_3$: $m^3 \in [0.0, 1.0]$ and $s^3 \in [0.1, 0.5]$.

2.5.1 Illustration of the Proposed Modeling Paradigm

We use Problem 5 of the algebraic challenge problem to illustrate our modeling paradigm presented in Section 2.4. We denote stochastic variables with upper-case letters and lower-case letters represent deterministic variables. The system response $\boldsymbol{Y}$,

$$\boldsymbol{Y} = (a + \boldsymbol{B})^a \tag{2.2}$$

is a random function of all three uncertainties a, m and s. This takes into account the log-normal distribution of $\boldsymbol{B}$ which is a function of m and s. The objective is to quantify the variability of $\boldsymbol{Y}$.

In the following section we see how the epistemic uncertainties and aleatory uncertainty of the challenge problem are represented within our framework of uncertainty and risk based on the two-step approach proposed in the previous section. We first discuss the two steps for the frog's view and then for the bird's view.

For the frog's view, the first step of the two-step approach is to identify the view. In other words, it is to identify the uncertainties. From the definition of the challenge problem it is clear that the uncertainties are a, m and s. The second step is to collect all possible frog's views. The authors of the original challenge problem have provided us with the expert panels' opinions on the uncertainties a, m and s as the intervals listed above which we treat as the collection of frog's views.

For every $[a^i, m^j, s^j], a^i \in A_i$, $m^j \in M_j$ and $s^j \in S_j$, the distribution of $\boldsymbol{Y}$ has a mean and variance which constitute the bird's view. The collection of all bird's views over all feasible $[a^i, m^j, s^j]$ yields the mean $\bar{\mu}$ and variance $\bar{\sigma}^2$. Note that $\bar{\mu}$ and $\bar{\sigma}^2$, as functions of three variables $a^i, m^j,$ and s^j describe the distribution of $\boldsymbol{Y}$, which we use to model risk. This completes the two-step process.

We now present details of our approach. The uncertainties a, m and s are discretized for computation as follows:

$$\begin{aligned} A_i &= \{a_1^i, a_2^i, \ldots, a_p^i, \ldots, a_{n_1}^i\}, \\ M_j &= \{m_1^j, m_2^j, \ldots, m_q^j, \ldots, m_{n_2}^j\}, \\ S_j &= \{s_1^j, s_2^j, \ldots, s_t^j, \ldots, s_{n_3}^j\}, \end{aligned}$$

where n_k, $k = 1, 2, 3$, is the number of points in each interval. Since we have three uncertainties, a, m and s, the 3-tuple with values from each interval $[a_p^i, m_q^j, s_t^j]$ is referred to as a *grid point.* A simulation over the uncertainties $[a_p^i, m_q^j, s_t^j]$ yields the mean $\bar{\mu}_{pqt}$ and variance $\bar{\sigma}^2_{pqt}$ of response $\boldsymbol{Y}_{pqt}$ for all p, q and t. Therefore the mean $\bar{\mu}$, variance $\bar{\sigma}^2$ and the response $\boldsymbol{Y}$, are all represented using a three dimensional array of size $n_1 \times n_2 \times n_3$. These three dimensional arrays are also known as tensors of order three. In the following paragraphs, a three dimensional array is always referred to as a tensor of order three.

The last part of the second step in our 2-step process, converts the given random process $\boldsymbol{Y}$ into a normal random process $\boldsymbol{Z}$. As discussed in Remark 2.1, we use the CLT to arrive at this normal random process. We first subtract $E(\boldsymbol{Y})$ from $\boldsymbol{Y}$ to get a zero-mean random process as required by the CLT. Given the region defined by the Cartesian product of the uncertainty intervals A_i, M_j and S_j, the integral

$$\boldsymbol{Z}_1 = \int_{A_i} \int_{M_j} \int_{S_j} [Y - E(Y)](dS)^{\frac{1}{2}} (dM)^{\frac{1}{2}} (dA)^{\frac{1}{2}} \tag{2.3}$$

follows normal distribution (see Remark 2.1). For this normal random process to represent the original random process $\boldsymbol{Y}$, we have to add back the mean $E(\boldsymbol{Y})$. Now consider the following integral which is the mean of $\mathbf{Y}$,

$$\boldsymbol{Z}_2 = \int_{A_i} \int_{M_j} \int_{S_j} E(Y) dS dM dA \tag{2.4}$$

Then by (2.3) and (2.4), $\boldsymbol{Z} = \boldsymbol{Z}_1 + \boldsymbol{Z}_2$ follows normal distribution with mean

$$\mu = \int_{A_i} \int_{M_j} \int_{S_j} \bar{\mu} \; dS dM dA$$

and variance

$$\sigma^2 = \int_{A_i} \int_{M_j} \int_{S_j} \bar{\sigma}^2 dS dM dA$$

Note that simulation of $\boldsymbol{Z}$ is unnecessary because all information pertaining to $\boldsymbol{Z}$ is contained in this normal distribution characterized by μ and σ^2.

We now define *risk* as the probability that the response $\boldsymbol{Z}_{pqt}$ is less than a prefixed value chosen as $\mu_{pqt} - 1.645\sigma_{pqt}$, i.e.,

$$risk = Prob(\boldsymbol{Z}_{pqt} \leq \mu_{pqt} - 1.645\sigma_{pqt})$$

which fixes the probability that the value of $\boldsymbol{Z}_{pqt}$ being less than $\mu_{pqt} - 1.645\sigma_{pqt}$ at 5% for all grid points in the tensor. Fixing risk at 5% level for each uncertainty $[a_p^i, m_q^j, s_t^j]$ means that for each uncertainty we expect the system response $\boldsymbol{Z}_{pqt}$ to be greater than $\mu_{pqt} - 1.645\sigma_{pqt}$, 95% of the times. In general, we can consider any positive α such that the prefixed value is $\mu_{pqt} - \alpha\sigma_{pqt}$. The risk level can be changed by appropriate selection of α. In the rest of the chapter however, risk will be fixed at 5% level, i.e., $\alpha = 1.645$.

Let T be a threshold value for an acceptable system response $\boldsymbol{Z}$. We identify all uncertainties $[a_p^i, m_q^j, s_t^j]$ over all p, q and t such that their performance is satisfactory, i.e.,

$$\mu_{pqt} - 1.645\sigma_{pqt} \geq T$$

which yields all uncertainties $[a_p^i, m_q^j, s_t^j]$ such that

$$Prob(\boldsymbol{Z}_{pqt} \geq T) \geq 95\%$$

and

$$Prob(\boldsymbol{Z}_{pqt} < T) < 5\%$$

Note that only a subset of all uncertainties will satisfy the above conditions. This subset changes as the threshold value T changes. In effect, the prefixed risk level for each uncertainty $[a_p^i, m_q^j, s_t^j]$ has identified the subset of these uncertainties at which the system performs satisfactorily.

2.5.2 Decision-Based Design Problem

The proposed approach is not only capable of modeling uncertainty and risk, but also leads to a methodology for solving decision problems under uncertainty. We illustrate this by converting the mathematical model (2.2) into a design problem by introducing additional design parameters x and z. There is no uncertainty associated with these design variables. Both x and z are positive numbers. The design system is then modeled as

$$\boldsymbol{Y} = (ax + \boldsymbol{B}z)^a \tag{2.5}$$

The tuple (x, z) is defined as a design that yields a system with response $\boldsymbol{Y}$. Given a set of feasible designs, the design problem is understood as the decision problem of finding an optimal design from among this set.

Note that the discussion of the previous section applies to the design $(x, z) = (1, 1)$ and can be repeated for any design (x, z) with x and z being positive real numbers. After modeling the uncertainties and risk as discussed in the last section for each feasible design (x, z), the methodology uses a few important concepts from multicriteria decision making (MCDM) theory to select an optimal design. The following are the definitions of these concepts.

Given a feasible design, we fix α at 1.645 so that for each uncertainty $[a^i_p, m^j_q, s^j_t]$ the probability that the system response $\boldsymbol{Z}_{pqt}$ for this design is less than $\mu_{pqt} - \alpha\sigma_{pqt}$ is 5%. The collection of all values $\mu_{pqt} - \alpha\sigma_{pqt}$ over all p, q and t gives $\mu - \alpha\sigma$, the *outcome*, that is available as a tensor of order 3.

Given a set of feasible designs and the associated outcomes, we define *nondominated* outcomes among them. An outcome associated with a feasible design is nondominated if there does not exist another feasible design whose outcome is bigger (better) at one or more grid points, while it is not smaller (worse) at any grid point. The designs producing the nondominated outcomes are called *efficient*. Note that these outcomes are obtained for a fixed risk level based on the α that has been chosen. This constant risk level across the designs enables us to compare the outcomes with each other.

A collection of all the maximum values at each grid point $[a^i_p, m^j_q, s^j_t]$ over all the designs is termed an *utopia* outcome. In essence, this is the best possible solution for the decision problem if all the maximum values are produced by a single design. However, in most cases this tensor has values produced by more then one design making it an infeasible solution.

The goal of the decision problem within the MCDM framework is to choose a *preferred* design (and its outcome) from the set of all efficient designs based on the decision maker's knowledge, experience and preferences. Note that the decision maker could be a person overlooking the problem above the panels that suggest the uncertainty intervals.

We now consider four randomly selected feasible designs which yield the system response $\boldsymbol{Z}$.

$$d^1 = (14, 15),\ d^2 = (10, 11),\ d^3 = (2, 9), \text{ and } d^4 = (14, 18).$$

For each design $d^l, l = 1, 2, 3, 4$, we produce a tensor that represents the outcome $O^l = (\mu - 1.645\sigma)^l$ associated with this design. More specifically, the outcome O^l is found as follows:

$$O^l = (\{\mu_{pqt} - 1.645\sigma_{pqt}\}, \forall p, q, t)^l \text{ for } l = 1, 2, 3, 4.$$

Since a nondominated outcome is a tensor of order three, it is not very instructional to plot it. However, if we fix a parameter, say a, we can plot the values of m and s for that layer of the three dimensional tensor. In other words, since we discretize uncertainties, by fixing a at a value, we only consider a function of two variables which can be easily plotted. Figure 2.2 depicts a layer of an utopia tensor for a_6 fixed at 0.6. Each entry in the plot represents the efficient design that yields O_{6qt}'s maximum values at each grid point $[a_p^i, m_q^j, s_t^j]$. The three boxes represent the feasible region according to the three panels' suggested uncertainty intervals for m and s. The feasible region suggested by Panel MS_1 is represented by the smallest box in Figure 2.2. The medium box and the large box represent the feasible regions suggested by Panel MS_2 and Panel MS_3, respectively. It can be seen that design d^3 does not produce maximum values at any grid point in this slice of the utopia tensor. We have n such slices, if we discretize a into n values.

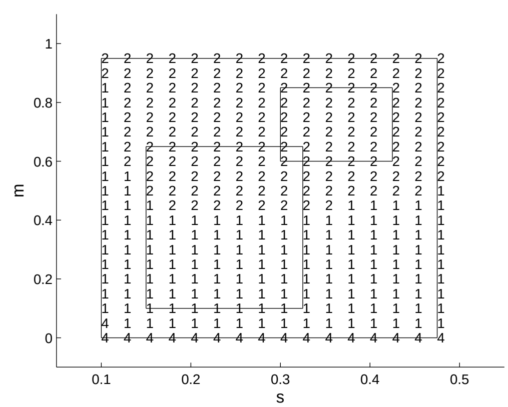

Figure 2.2 A sample Matlab output representing a slice of the utopia tensor

2.5.3 Decision Analysis

Given the set of nondominated outcomes and the utopia tensor, the decision maker uses his preferences to choose a preferred (optimal) design. The decision maker not only makes a decision on which design to choose but also on which feasible region R to work. Since there are three panels to suggest intervals for a, and three panels to suggest intervals for m and s, a combination of these two groups of panels' suggestions forms the feasible regions R. The proposed methodology allows the decision maker to arrive at a preferred design under each panel's choice. However, it leaves the decision maker the freedom of choice in terms of which panel's suggestion and/or combination of suggestions to consider. This is however a different decision problem, involving the issues of aggregation, solved subjectively by the decision maker. A few conceptual ideas for dealing with this problem are discussed in this section.

The decision maker has a rich class of choices to consider, for instance, the basic combination of Panel A_i's interval can be considered with Panel MS_j's interval. This gives rise to nine basic feasible regions. Also, the decision maker can consider the union of all the Panel A_i's intervals for a with one of the Panel MS_j's interval or the union of Panel A_i's intervals with the union of Panel MS_j's intervals. The latter gives equal importance to all the panels, while the former gives more emphasis to one of the three panels' intervals for m and s. If the intersection of panels' choices exists, then this option can also be considered, i.e., $\cap A_i$, $i = 1, 2, 3$ and $\cap M_j$ and $\cap S_j$ $j = 1, 2, 3$.

We consider twelve scenarios of which the first nine are simple combinations of the panels' choices of uncertainty. Scenario J is a biased combinations of the panels' choices because the decision maker picks one panel's choice over the others for m and s. Scenario K is an unbiased combination in the form of the union of all the suggested intervals. The last scenario is the strongest and unbiased, but there is a chance for this set to be empty leading the decision maker nowhere.

Scenario A : feasible region suggested by Panel A_1 and Panel MS_1;

Scenario B : feasible region suggested by Panel A_1 and Panel MS_2;

Scenario C : feasible region suggested by Panel A_1 and Panel MS_3;

Scenario D : feasible region suggested by Panel A_2 and Panel MS_1;

Table 2.2 Efficient designs associated with each scenario

Scenario	d^1	d^2	d^3	d^4
A	x			x
B	x	x		x
C	x	x		x
D	x			x
E	x	x		x
F	x	x		x
G	x	x		x
H	x	x		x
I	x	x		x
J	x	x		x
K	x	x		x
L	x			x

Scenario E : feasible region suggested by Panel A_2 and Panel MS_2;

Scenario F : feasible region suggested by Panel A_2 and Panel MS_3;

Scenario G : feasible region suggested by Panel A_3 and Panel MS_1;

Scenario H : feasible region suggested by Panel A_3 and Panel MS_2;

Scenario I : feasible region suggested by Panel A_3 and Panel MS_3;

Scenario J : feasible region suggested by $\cup\mathbf{A}_i$, $i = 1, 2, 3.$ and Panel MS_2;

Scenario K : feasible region suggested by $\cup\mathbf{A}_i$, $i = 1, 2, 3.$, $\cup\mathbf{M}_j$ and $\cup\mathbf{S}_j$, $j = 1, 2, 3.$;

Scenario L : feasible region suggested by $\cap\mathbf{A}_i$, $i = 1, 2, 3.$, $\cap\mathbf{M}_j$ and $\cap\mathbf{S}_j$, $j = 1, 2, 3.$;

In Table 2.2, the rows represent scenarios A, B, ..., L and the columns represent designs d^1, d^2, d^3, and d^4. According to scenario A given by Panel A_1 and Panel MS_1, designs d^1 and d^4 are efficient and yield nondominated outcomes. Similarly, for scenarios D and L, designs d^1 and d^4 yield nondominated outcomes, while for scenario B, C, E, F, G, H, I, J, and K designs d^1, d^2 and d^4 yield nondominated outcomes. Design d^3 is never efficient.

Table 2.3 Distance between the utopia outcome and the nondominated outcome of each efficient design

Scenario	d^1	d^2	d^3	d^4
A	26.49	-	-	139.08
B	39.09	196.62	-	296.56
C	43.32	463.50	-	240.91
D	25.55	186.23	-	5.83
E	3.48	-	-	26.59
F	18.91	141.01	-	18.85
G	18.04	108.25	-	0.76
H	2.57	39.98	-	9.76
I	13.46	83.54	-	6.90
J	22.81	121.03	-	166.69
K	28.28	28.08	-	135.16
L	0.06	-	-	7.23

To make this illustration complete, we propose the following method from MCDM to make a choice of a preferred design from the efficient set of designs. An efficient design whose outcome produces the minimum distance, calculated, for example, according to L_2 norm, from the utopia outcome, is chosen as the preferred design.

Table 2.3 gives the distance measured with the L_2 norm between the utopia outcome and the nondominated outcome tensors associated with each of the efficient designs. The preferred design is the efficient design with the smallest norm. In this example, design d^1 is preferred over designs d^2 and d^4 for scenario B, C H, and J and design d^1 is preferred over design d^4 for scenario A, E and L. While design d^4 is preferred over designs d^1 and d^2 for scenario D, F, G and I. Also for scenario K the norm value of design d^2 is a lot better than the value of design d^1 making d^4 the preferred design for this scenario. If the decision maker wants to make an unbiased decision, he might favor scenarios K or L and choose design d^1 or d^2, respectively, since it outperforms every other design.

The scenarios discussed above are not the only possibilities. They were selected to illustrate how the decision maker can arrive at a preferred design using the information he has.

2.5.4 Discussion

We now discuss our methodology in the context of certain tasks that are of interest to Oberkampf et al. (2004, E). These tasks include representation, aggregation and propagation of uncertainties and interpretation of resultant uncertainty in the system response.

In the proposed modeling paradigm, intervals suggested by the panels for A_i, M_j and S_j for the parameters a, m and s respectively, are treated as epistemic uncertainties. Note that aleatory uncertainties are represented as a random function of epistemic uncertainties. In the algebraic problem discussed, $\boldsymbol{B}$ is a random function of the epistemic uncertainties $m \in M_j$ and $s \in S_j$ and hence an aleatory uncertainty. Moreover, the system response $\boldsymbol{Y}$ is a random function of all three epistemic uncertainties and hence the uncertainty in the system response is represented as an aleatory uncertainty which is the building block to model risk that quantifies the uncertainty in the system response $\boldsymbol{Y}$. For this reason, in our paradigm we refer to epistemic uncertainty as uncertainty and we use risk which is modeled using both epistemic and aleatory uncertainty. In effect, we do not explicitly refer to aleatory uncertainty.

Aggregation of uncertainties given by different sources gives rise to the scenarios discussed in Section 2.5.3 on decision analysis. A few examples of aggregation of uncertainties are shown in that section. For instance, if scenario F is not empty, i.e., the three panel's intervals intersect, then there is some agreement among the sources in the uncertainties prediction which may mean that the decision maker needs to concentrate more on these common uncertainties. However, scenario G is the union of all intervals creating a larger interval for each uncertainty. This may mean that the decision maker is careful in considering all the sources with equal importance, but it could be too expensive in problems with large complexity.

The simulation of $\boldsymbol{Y}$ as a random function of the uncertainties $[a_p^i, m_q^j, s_t^j]$ and its transformation to $\boldsymbol{Z}$, as discussed in Section 2.5.1, propagates these uncertainties through the system.

The system response $\boldsymbol{Z}$ is not explicitly available but its distribution is normal with mean μ and standard deviation σ that are numerically available for each uncertainty $[a_p^i, m_q^j, s_t^j]$. We use this μ and σ to define risk, which is how we interpret the variability in the system response and identify the uncertainties $[a_p^i, m_q^j, s_t^j]$ for which the system performance is satisfactory with respect to a certain desired threshold.

The proposed methodology always considers the system response as a random function of uncertainties which allows to deal with other versions of Problem 5 of the challenge problem. We chose to work with Problem 5 because the other problems can be seen as its special cases. In particular, in Problem 5, we define the system response $\boldsymbol{Y}$ as a random function of uncertainties a, m, and s which are all modeled as intervals. Working within this scheme enables us to deal with all the six problems without any change to our methodology. Problem 4 is similar to Problem 5 but has only one panel's suggestion and so we do not need to aggregate any information. Also Problem 6 is the easiest in the sense that there is only one uncertainty a. The distribution of b is incorporated in the random function $\boldsymbol{Y}$. Problems 1, 2 and 3 involve only two uncertainties represented as intervals and propagated through $Y = (a + b)^a$, where both a and b are deterministic variables. While aggregation of different scenarios can be done for Problems 2 and 3 as discussed earlier, Problem 1 is a single scenario problem. For all these problems we go through the same steps and quantify the variability in the system response by risk and the threshold value T of the acceptable system response.

The original challenge problem (Oberkampf et al., 2004, E) can be treated as a design problem (2.5) with design parameters $(x, z) = (1, 1)$. We extended this problem into a design problem by defining the design parameters (x, z) as real numbers. While the representation, aggregation and propagation of uncertainties are the same in both problems, the interpretation of the resultant uncertainty in the system response is different. Since in the design problem the interest is to arrive at a design that produces an optimal system response, we do not define any threshold but rather compare the outcomes of all designs and choose a preferred design from among the efficient designs according to the MCDM framework. Here each design's outcome depends on a fixed risk level which enables us to compare these outcomes and arrive at a preferred one.

2.6 Conclusion

In this chapter, we first surveyed the literature to explore how risk and uncertainty have been perceived in different disciplines. We collected some of the most interesting and some of the earliest definitions we could find in engineering, operations research, finance and economics. Some of these notions were of a philosophical nature that stimulated researchers while others did not define risk and uncertainty but rather modeled them. A subjectively constructed list of these works forms the first part of this chapter.

We also modeled uncertainty and risk in an efficient manner. The conceptual model, consisting of bird and frog views is discussed with an example. We modeled nonquantifiable uncertainties by intervals and used the distribution of a random function of these uncertainties to model risk. A two-step modeling paradigm was proposed to model uncertainty and risk.

We also applied our modeling paradigm and the resulting methodology to one of the challenge problems presented in Oberkampf et al. (2004, E). In effect, the epistemic uncertainty in the challenge problem is the nonquantifiable uncertainty in our model in which the aleatory uncertainty quantifies the random system responses dependent upon epistemic uncertainty. The distribution of the random system response values are used to model risk. We also converted the challenge problem into a decision-based design problem to illustrate further capabilities of our approach.

The proposed approach models uncertainty and risk and leads to a decision problem offering the decision maker freedom to arrive at a decision influenced by his knowledge and experience. The decision maker is not given one 'correct' decision but rather a small collection of choices from which the final decision is selected. We believe modeling uncertainty and risk as proposed in this chapter will make the decision making more dynamic and environment specific because of the detailed knowledge of the decision maker engaged in the decision situation of interest.

CHAPTER 3

MODELS AND RISK ANALYSIS OF UNCERTAIN COMPLEX SYSTEMS

In the last chapter, we worked with three uncertainties which did not interact with each other. In this chapter, we investigate and extend our methodology to handle interactions between uncertainties. These interactions usually occur in problems involving complex systems. A modeling methodology is developed for complex systems, systems of several interacting components, with observable component performance normal fields that are also separable over the space of uncertainties. The algebra of operator representations of system components is completed by using separable equivalents in place of sums which are in general nonseparable. The end product is a computation of the mean and variance of the performance of the modeled system at each point in the the space of system uncertainties.

3.1 Wiener Fields and Spaces of Hellinger Integrable Functions

An emerging class of decision problems under conditions of uncertainty and risk requires a flexible modeling methodology for representing multiple component systems (Reneke et al., 2002; Reneke and Wiecek, 2002). Providing a framework or setting for these models in the mathematical literature motivates this chapter. Certain tools used in the methodology have a long history: reproducing kernel Hilbert spaces (Aronszajn, 1950), Hellinger integrals (Hellinger, 1907; Rovnyak, 1990), Wiener fields (Chentsov, 1956), representations of operators on spaces of functions of several variables (Mac Nerney, 1980), and The Central Limit Theorem (Chung, 2001). However, the requirements of a modeling methodology exposes new problems to be addressed in this setting. In particular, the representation of separable random fields (Vanmarcke, 1983), the class of fields most like continuous random processes, and separable approximations for sums of separable fields, which in general are nonseparable, are addressed.

Our approach to random fields is from a modeling viewpoint, as opposed to a data driven approach, using theoretical means and covariances of random fields to construct more complex models out of simpler component models. The models are not dynamic, in fact time is usually not a system variable, but rather linear operators on appropriate inner product spaces of functions. The end product is a computation of the mean and variance of the modeled field at each point in the domain of the random field (the space of uncertainties).

Representation of discrete surfaces: Suppose each of S and T is a positive number and F is a field defined on $[0, S] \times [0, T]$. If $0 = s_1 < s_2 < \cdots < s_m = S$ and $0 = t_1 < t_2 < \cdots < t_n = T$ then let F_{st} denote the discrete surface defined by

$$F_{st}(i, p) = F(s_i, t_p)$$

F_{st} can also be thought of as an $m \times n$ matrix. Let K_{ss} and K_{tt} denote the discrete covariance kernels of the standard Wiener process W on $[0, S]$ and $[0, T]$, respectively. That is, $K_{ss}(i, j) = EW(s_i)W(s_j)$ and $K_{tt}(p, q) = EW(t_p)W(t_q)$. For the rest of the chapter we fix S and T as positive numbers, m and n as positive integers, and $s = \{s_p\}_{p=1}^{m}$ and $t = \{t_q\}_{q=1}^{n}$ as partitions of $[0, S]$ and $[0, T]$, respectively.

3.1.1 Definition of a Random Wiener Field

Random Wiener fields play a fundamental role in the representation of more general random fields. We can use Wiener fields to define the basic inner product spaces in which more general random fields with mean zero can at first be identified with positive definite operators.

The standard Wiener field (Chentsov, 1956; Ossiander and Waymire, 1989), also denoted by W, on $[0, S] \times [0, T]$ has the following defining properties:

- sample fields are continuous
- $W(a, 0) = W(0, b) = 0$
- $E[W(a, b)] = 0$
- $E[(W(b, d) - W(a, d) - W(b, c) + W(a, c))^2] = E[(dW(a, b, c, d))^2] = (b - a)(d - c)$

- if $[a,b]\times[c,d]\cap[z,w]\times[x,y]$ is empty or has no interior then $E[dW(a,b,c,d)dW(z,w,x,y)]=0$

where $E[\cdot]$ is the expectation operator.

Notice that for $0<a,\, b$ the process $\frac{1}{\sqrt{a}}W(a,\ )$ is the standard Wiener process on $[0,\, T]$ and $\frac{1}{\sqrt{b}}W(\ ,\, b)$ is the standard Wiener process on $[0,\, S]$.

The Wiener kernel for fields: Suppose F is a field defined on $[0,\, S]\times[0,\, T]\times[0,\, S]\times[0,\, T]$, such as the covariance kernel of the Wiener field. Let $F_{(st)^2}$ denote the discrete function defined by

$$F_{(st)^2}(i,\, p,\, j,\, q)=F(s_i,\, t_p,\, s_j,\, t_q)$$

Let K_{ss}^C and K_{tt}^C be upper triangular matrices with nonnegative diagonals such that $(K_{ss}^C)^TK_{ss}^C=K_{ss}$ and $(K_{tt}^C)^TK_{tt}^C=K_{tt}$. In an extension of classic usage we refer to K_{ss}^C and K_{tt}^C as *upper Cholesky factors* of K_{ss} and K_{tt}, respectively. We can obtain simulations of W_{st} in terms of K_{ss}^C and K_{tt}^C as follows:

$$W_{st}=(K_{ss}^C)^TZ_{st}K_{tt}^C$$

where Z_{st} is a (0, 1)-normal $m\times n$ matrix of independent random variables.

Theorem 3.1. *W_{st} is a discretization of the standard Wiener field.*

Lemma 3.1. *If U_{ss} is a nonnegative $m\times m$ matrix with upper Cholesky factor U_{ss}^C in the above sense and V_{tt} is a nonnegative $n\times n$ matrix with upper Cholesky factor V_{tt}^C then*

$$E([(U_{ss}^C)^TZ_{st}V_{tt}^C](i,\, p)(U_{ss}^C)^TZ_{st}V_{tt}^C](j,\, q)=U_{ss}(i,\, j)V_{tt}(p,\, q)$$

for $1\le i,\, j\le m$ and $1\le p,\, q\le n$.

Proof of the lemma.

$$
\begin{aligned}
&E([(U_{ss}^C)^T Z_{st} V_{tt}^C](i,\, p)(U_{ss}^C)^T Z_{st} V_{tt}^C](j,\, q) \\
&= E\left(\sum_{\ell_1=1}^{m}\sum_{\ell_2=1}^{n}(U_{ss}^C)^T(i,\,\ell_1)Z_{st}(\ell_1,\,\ell_2)V_{tt}^C(\ell_2,\,p)\right)\cdot \\
&\qquad\qquad \left(\sum_{\bar\ell_1=1}^{m}\sum_{\bar\ell_2=1}^{n}(U_{ss}^C)^T(j,\,\bar\ell_1)Z_{st}(\bar\ell_1,\,\bar\ell_2)V_{tt}^C(\bar\ell_2,\,q)\right) \\
&= \sum_{\ell_1=1}^{m}\sum_{\ell_2=1}^{n}(U_{ss}^C)^T(i,\,\ell_1)(U_{ss}^C)^T(j,\,\ell_1)V_{tt}^C(\ell_2,\,p)V_{tt}^C(\ell_2,\,q) \\
&= \sum_{\ell_1=1}^{m}(U_{ss}^C)^T(i,\,\ell_1)U_{ss}^C(\ell_1,\,j)\sum_{\ell_2=1}^{n}(V_{tt}^C)^T(p,\,\ell_2)V_{tt}^C(\ell_2,\,q) \\
&= U_{ss}(i,\,j)V_{tt}(p,\,q)
\end{aligned}
$$

Proof of the theorem. First, $EW_{st} = 0$, the $m \times n$ zero matrix, and $W_{st}(1,\ ) = W_{st}(\ ,1) = 0$. Also from Lemma 3.1 with $U_{ss} = K_{ss}$ and $V_{tt} = K_{tt}$

$$
\begin{aligned}
&EW_{st}(i,\,p)W_{st}(j,\,q) \\
&= E[(K_{ss}^C)^T Z_{st} K_{tt}^C](i,\,p)[(K_{ss}^C)^T Z_{st} K_{tt}^C](j,\,q) \\
&= K_{ss}(i,\,j)K_{tt}(p,\,q)
\end{aligned}
$$

Therefore $K_{(st)^2}(i,\,p,\,j,\,q) = K_{ss}(i,\,j)K_{tt}(p,\,q)$. Furthermore, for $i \le j$ and $p \le q$,

$$
\begin{aligned}
&E(W_{st}(j,\, q) - W_{st}(i,\, q) - W_{st}(j,\, p) + W_{st}(i,\, p))^2 \\
&\quad = \; E(W_{st}(j,\, q)^2 + EW_{st}(i,\, q)^2 + EW_{st}(j,\, p)^2 + EW_{st}(i,\, p)^2 \\
&\qquad - 2EW_{st}(j,\, q)W_{st}(i,\, q) - \; 2EW_{st}(j,\, q)W_{st}(j,\, p) \\
&\qquad + 2EW_{st}(j,\, q)W_{st}(i,\, p) + 2EW_{st}(i,\, q)W_{st}(j,\, p) \\
&\qquad - 2EW_{st}(i,\, q)W_{st}(i,\, p) - 2EW_{st}(j,\, p)W_{st}(i,\, p) \\
&\quad = \; K_{ss}(j,\, j)K_{tt}(q,\, q) + K_{ss}(i,\, i)K_{tt}(p,\, p) \\
&\qquad - K_{ss}(i,\, i)K_{tt}(q,\, q) - K_{ss}(j,\, j)K_{tt}(p,\, p) \\
&\quad = \; (K_{ss}(j,\, j) - K_{ss}(i,\, i))(K_{tt}(q,\, q) - K_{tt}(p,\, p)) \\
&\quad = \; (s_j - s_i)(t_q - t_p)
\end{aligned}
$$

Finally, suppose $[i,\, j] \times [p,\, q] \cap [\bar{i},\, \bar{j}] \times [\bar{p},\, \bar{q}] = \phi$. Assume to be definite that $i < j < \bar{i} < \bar{j}$. The other cases work the same way. Then

$$
\begin{aligned}
&E(W_{st}(j,\, q) - W_{st}(i,\, q) - W_{st}(j,\, p) + W_{st}(i,\, p)) \cdot \\
&\qquad (W_{st}(\bar{j},\, \bar{q}) - W_{st}(\bar{i},\, \bar{q}) - W_{st}(\bar{j},\, \bar{p}) + W_{st}(\bar{i},\, \bar{p})) \\
&\quad = \; (K_{ss}(j,\, \bar{j}) - K_{ss}(j,\, \bar{i}) - K_{ss}(i,\, \bar{j}) + K_{ss}(i,\, \bar{i})) \cdot \\
&\qquad (K_{tt}(q,\, \bar{q}) - K_{tt}(q,\, \bar{p}) - K_{tt}(p,\, \bar{q}) + K_{tt}(p,\, \bar{p})) \\
&\quad = \; 0
\end{aligned}
$$

The discrete representations and the method of proof in Theorem 3.1 are the result of our modeling viewpoint. Implementation of the theory to be developed, because of the matrix emphasis, becomes almost immediate in MATLAB, our computational environment. Although most of the material in Section 3.2 is familiar, we will present everything in terms of finite matrix manipulations.

In addition to the role of the standard Wiener field W in the structure of the theory taken up in the next subsection, the Wiener field is the prototypical of the random fields of interest. The Wiener kernel can be factored, i.e., $K_{(st)^2}(i,\, p,\, j,\, q) = K_{ss}(i,\, j)K_{tt}(p,\, q)$, for $0 \le i,\, j \le m$ and $0 \le p,\, q \le n$, and K_{ss} and K_{tt} can in turn be factored and the factors used to simulate the discrete Wiener field W_{st}. See the note in Section 3.2.2 for more general Wiener fields (Chentsov, 1956).

Separable random fields: For scalar fields defined on two-dimensional rectangles, sufficient conditions on the system covariance kernel are given for the development of a system linearization based on a factorization of the system covariance kernel. The mathematical question of limitations imposed by the covariance condition can be discussed in terms of properties of the resulting linearizations. A set of reasonable conditions on linear systems can be shown to be equivalent to the covariance condition. Thus the methods apply to a rich class of random fields.

The central idea in this subsection is to produce a condition on the covariance kernel of a random field (function of two variables) which for fields F generated by linear systems, i.e., of the form $F = AW$, is sufficient to produce a representation of A. (Here W is the standard Wiener field, discussed earlier.) If F is generated by a nonlinear system $\hat{A}$ then the method produces a linearization of $\hat{A}$, namely A. In this case the utility of the linearization depends upon the particular application and how nearly the assumption of the condition (perhaps uncheckable) is to holding.

The condition is the following: for each pair of points (a, b) and (c, d) in $[0, S] \times [0, T]$

$$\begin{aligned} &\mathrm{cov}(F(a,b),\, F(c,d)) = \\ &\quad \mathrm{cov}(F(a,T),\, F(c,T)) \cdot \mathrm{cov}(F(S,b),\, F(S,d))/var(F(S,\, T)) \end{aligned} \tag{3.1}$$

where $\mathrm{cov}(\cdot)$ is the covariance operator. (See Vanmarcke (1983, page 82)) Random fields satisfying the condition are said to be *separable*. Note that the condition is on the observation field F rather than on the underlying unmodeled system. Also note that the standard Wiener field is separable.

In general, the independence expressed in the condition might not hold. However, such independence is implicit in the common engineering practice of exploring a given physical system by allowing only one quantity to vary at a time.

Vanmarcke (1983), with an extensive bibliography, provides a foundation reference for separable fields. Separability in Vanmarcke is given in terms of correlation functions rather than covariance kernels. See Yaglom (1986), also using correlations rather than covariances and with a large bibliography, for alternative representations for random fields. Models incorporating time are different because time is different from a space variable, i.e., for time there is a past, a present, and a future. Space variables are not linearly ordered. While related to our random fields, random space-time functions have a different flavor (Kyriakidis and Journel, 1999). In fact our random fields would correspond to the limited case where spatial behavior is the same at all time instants.

Wiener fields were introduced by Chentsov (1956). A Wiener field is *not* Lévy's Brownian motion process of two parameters (Lévy, 1940, 1945; Berman, 1967). Our use of "separable" is not to be confused with the use for abstract metric spaces or infinite dimensional spaces, i.e., the existence of a countable dense subset. In this use separable processes are processes whose sample paths are separable subsets of a function space (Naresh and Monrad, 1983; Monrad, 1983).

3.1.2 Reproducing Kernel Hilbert (RKH) Spaces

Covariance kernels for random fields have the nonnegative function property that makes them reproducing kernels of some complete inner product space of functions of two variables (Aronszajn, 1950). Of interest are those kernels which can be associated with positive linear operators on the space determined by the Wiener kernel. A scalar function R on $[0, S] \times [0, T] \times [0, S] \times [0, T]$ is said to be nonnegative definite provided for each sequence $\{\mathbf{u}^p\}_{p=1}^{\ell}$, where $\mathbf{u^p}$ is an ordered pair in $[0, S] \times [0, T]$, and sequence $\{a_p\}_{p=1}^{n}$ of nonzero real numbers

$$\sum_{p=1}^{n}\sum_{q=1}^{n} R(\mathbf{u}^p, \mathbf{u}^q) a_p a_q \geq 0$$

The space of Hellinger integrable functions of two variables provide an explicit representation of the RKH space determined by the Wiener kernel (Mac Nerney, 1980). Functions f defined on $[0, S] \times [0, T]$ are said to be Hellinger integrable provided the set of approximating sums

$$\left\{ \sum_{s}\sum_{t} \frac{(df(u, v))^2}{du\, dv} = \sum_{i=2}^{m}\sum_{p=2}^{n} \frac{(f(s_i, t_p) - f(s_{i-1}, t_p) - f(s_i, t_{p-1}) + f(s_{i-1}, t_{p-1}))^2}{(s_i - s_{i-1})(t_p - t_{p-1})} \right\}$$

is bounded. The least upper bound is denoted by $\int\int_{[0, S]\times[0, T]} \frac{(df(u,v))^2}{du\, dv}$.

The covariance kernel for the standard Wiener surface W is

$$E(W(u, v)W(x, y)) = min(u, v)min(x, y) = K(u, v)K(x, y)$$

The complete inner product space of functions $\{G, Q\}$ determined by this kernel is the space of Hellinger integrable functions f on $[0, S] \times [0, T]$ such that $f(0, \cdot) = 0$ and $f(\cdot, 0) = 0$. Further, the inner product is defined by $Q(f, g) = \int\int_{[0, S]\times[0, T]} \frac{df(u,v)\, dg(u,v)}{du\, dv}$.

Approximating sums and discrete representations: We can consider approximating sums as the results of matrix products.

Theorem 3.2.

$$\begin{aligned} &\sum_{t}\sum_{s} \frac{df(s, t)\, dg(s, t)}{ds\, dt} \\ &= \quad trace((K_{tt}^{C})^{-T} f_{st}^{T} K_{ss}^{-1} g_{st} (K_{tt}^{C})^{-1}) \end{aligned}$$

Proof of the theorem. Recall that for $2 \leq i,\, p$

$$\begin{aligned}
&[(K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1}](i,\, p) \\
&\quad = \frac{\{f(s_i,\, t_p) - f(s_{i-1},\, t_p) - f(s_i,\, t_{p-1}) + f(s_{i-1},\, t_{p-1})\}}{(\sqrt{s_i - s_{i-1}}\sqrt{t_p - t_{p-1}})}
\end{aligned}$$

Let $A = (K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1}$ and $B = (K_{ss}^C)^{-T} g_{st} (K_{tt}^C)^{-1}$. Then

$$\begin{aligned}
&trace((K_{tt}^C)^{-T} f_{st}^T K_{ss}^{-1} g_{st} (K_{tt}^C)^{-1}) \\
&\quad = trace(((K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1})^T ((K_{ss}^C)^{-T} g_{st} (K_{tt}^C)^{-1})) \\
&\quad = trace(A^T B) \\
&\quad = \sum_{p=1}^{n} [A^T B](p,\, p) \\
&\quad = \sum_{p=1}^{n} \sum_{i=1}^{m} A(i,\, p) B(i,\, p) \\
&\quad = \sum_{t} \sum_{s} \frac{df(s,\, t)\, dg(s,\, t)}{ds\, dt}
\end{aligned}$$

From the general theory of RKH spaces obtained using Hellinger integrals (Mac Nerney, 1980), we know that $Q(f,\, K(\ ,\, u)K(\ ,\, v)) = f(u,\, v)$, i.e., the Wiener kernel is the reproducing kernel for $\{G,\, Q\}$, but it is useful to use our representation of Q to establish this fact in our context.

Theorem 3.3. *For f in $\{G,\, Q\}$ and $(u,\, v)$ in $[0,\, S] \times [0,\, T]$*

$$Q(f,\, K(\ ,\, u)K(\ ,\, v)) = f(u,\, v)$$

Proof of the theorem. Suppose $s_i = u$ and $t_p = v$. Then

$$\begin{aligned}
&Q(f,\, K(\ ,u)K(\ ,v)) \\
&\sim\ trace((K^C_{tt})^{-T} f^T_{st} K^{-1}_{ss} K_{ss}(\ ,i) K_{tt}(\ ,p)^T (K^C_{tt})^{-1}) \\
&=\ trace((K^C_{tt})^{-T} f^T_{uv} I_{uu}(\ ,i) K^C_{vv}(\ ,p)^T) \\
&=\ trace((K^C_{tt})^{-T} f_{st}(i,\ )^T K^C_{tt}(\ ,p)^T) \\
&=\ f_{st}(i,\ )(K^C_{tt})^{-1} K^C_{tt}(\ ,p) \\
&=\ f_{st}(i,\ ) I_{tt}(\ ,p) \\
&=\ f_{st}(i,\, p) \\
&=\ f(u,\, v)
\end{aligned}$$

Again, from the general theory we can characterize all continuous linear operators on $\{G,\, Q\}$ and their matrix representations (Mac Nerney, 1980). Our interest is in a special class of operators which the following theorem enables us to characterize in terms of their matrix representations.

Theorem 3.4. *Suppose L is the matrix representation of a continuous linear transformation A on $\{G,\, Q\}$. These are equivalent:*

1. *There is an $m \times m$ matrix A_L and an $n \times n$ matrix A_R such that $[Af]_{st}(j,\, q) = ((K^C_{ss})^{-1} A_L(\ ,j))^T f_{st} (K^C_{tt})^{-1} A_R(\ ,q)$.*
2. *There is an $m \times m$ matrix L^1_{ss} and an $n \times n$ matrix L^2_{tt} such that $L_{(st)^2}(i,\, j,\, p,\, q) = [L^1_{ss}(\ ,j)(L^2_{tt}(\ ,q))^T](i,\, p)$.*

Proof of the theorem. Suppose $(Af)_{st} = ((K^C_{ss})^{-1} A_L)^T f_{st} (K^C_{tt})^{-1} A_R$, where A_L is an $m \times m$ matrix, f_{st} is an $m \times n$ matrix, and A_R is an $n \times n$ matrix. Then

$$\begin{aligned}
&(AK(\ ,u)K(\ ,v))_{st} \\
&=\ ((K^C_{ss})^{-1} A_L)^T K_{ss}(\ ,i) K_{tt}(\ ,p)^T (K^C_{tt})^{-1} A_R \\
&=\ A^T_L (K^C_{ss})^{-T} (K^C_{ss})^T K^C_{ss}(\ ,i)((K^C_{tt})^T (K^C_{tt})(\ ,p))^T (K^C_{tt})^{-1} A_R \\
&=\ A^T_L K^C_{ss}(\ ,i) K^C_{tt}(\ ,p)^T A_R
\end{aligned}$$

Therefore

$$
\begin{aligned}
& L_{(st)^2}(i, j, p, q) \\
&= Q(AK(\ , s_i)K(\ , t_p), K(\ , s_j)K(\ , t_q)) \\
&= trace((K_{tt}^C)^{-T} A_R^T K_{tt}^C(\ , p) K_{ss}^C(\ , i)^T A_L K_{ss}^{-1} K_{ss}(\ , j) K_{tt}(\ , q)^T (K_{tt}^C)^{-1}) \\
&= trace((K_{tt}^C)^{-T} A_R^T K_{tt}^C(\ , p) K_{ss}^C(\ , i)^T A_L(\ , j) K_{tt}(\ , q)^T (K_{tt}^C)^{-1}) \\
&= (A_L(\ , j))^T K_{ss}^C(\ , i) K_{tt}^C(\ , p)^T A_R (K_{tt}^C)^{-1} K_{tt}^C(\ , q) \\
&= (A_L(\ , j))^T K_{ss}^C(\ , i) K_{tt}^C(\ , p)^T A_R(\ , q) \\
&= [L_{ss}^1(\ , j)(L_{tt}^2(\ , q))^T](i, p)
\end{aligned}
$$

On the other hand, suppose that

$$L_{(st)^2}(i, j, p, q) = [L_{ss}^1(\ , j)(L_{tt}^2(\ , q))^T](i, p)$$

and

$$[(Af)_{st}](j, q) = trace((K_{tt}^C)^{-T} f_{st}^T K_{ss}^{-1} L_{(st)^2}(\ , j, \ , q)(K_{tt}^C)^{-1})$$

Let $A_L = (K_{ss}^C)^{-T} L_{ss}^1$ and $A_R = (K_{tt}^C)^{-T} L_{tt}^2$. Then

$$
\begin{aligned}
& [(Af)_{st}](j, q) \\
&= trace((K_{tt}^C)^{-T} (f_{st})^T K_{ss}^{-1} L_{ss}^1(\ , j)(L_{tt}^2(\ , q))^T (K_{tt}^C)^{-1}) \\
&= trace((K_{tt}^C)^{-T} (f_{st})^T (K_{ss}^C)^{-1} A_L(\ , j) A_R(\ , q)^T) \\
&= ((K_{ss}^C)^{-1} A_L(\ , j))^T f_{st} (K_{tt}^C)^{-1} A_R(\ , q)
\end{aligned}
$$

See (Rovnyak, 1990) for a survey of Hellinger's contributions to integration and operator theory. The collection of benchmark papers edited by Wei (1982) contains the foundation papers for applications of reproducing kernel Hilbert space methods to signal analysis. Note the small change in notation. We use RKH space representations meaning operator representations on the reproducing kernel Hilbert space. For the Parzen/Kailath meaning ($RKHS$ representations) the random process is represented by the reproducing kernel Hilbert space itself.

3.2 Linear Operators Associated with Separable Random Fields

The method of stochastic linearization (Bhartia and Vanmarcke, 1991) is considered in the context of separable random surfaces. The factorization of the surface kernels following from this assumption enables us to present a representation of linear operators generating the random surfaces.

3.2.1 Producing Linearizations of Separable Random Fields.

We provide a recipe for representations of linear systems generating separable random fields. The recipe works for a large class of systems. Finite dimensional approximations for a linear operator A require two matrices. For instance,

$$AW_{st} = ((K_{ss}^C)^{-1} A_L)^T W_{st} (K_{tt}^C)^{-1} A_R$$

where A_L is an $m \times m$ matrix, W_{st} is an $m \times n$ matrix, the discretization of the standard Wiener field, and A_R is an $n \times n$ matrix. The representation problem for a linear transformation A in the general case reduces to a search for appropriate matrices A_L and A_R.

Theorem 3.5. *If F is a separable random field defined on $[0, S] \times [0, T]$ then there are appropriate matrices A_L and A_R such that*

$$AW_{st} = ((K_{ss}^C)^{-1} A_L)^T W_{st} (K_{tt}^C)^{-1} A_R$$

has the mean and covariance of F_{st}.

Proof of the theorem. Let U_{ss} and V_{tt} be matrices defined by

$$U_{ss}(i,\, j) = \mathrm{cov}(F_{st}(i,\, n),\, F_{st}(j,\, n))/std(F(S,\, T))$$

and

$$V_{tt}(p,\, q) = \text{cov}(F_{st}(m,\, p),\, F_{st}(m,\, q))/std(F(S,\, T))$$

Then $U_{ss} = (U_{ss}^C)^T U_{ss}^C$ and $V_{tt} = (V_{tt}^C)^T V_{tt}^C$, where each of U_{ss}^C and V_{tt}^C is an upper triangular matrix with nonnegative entries on the main diagonal. For $A_L = U_{ss}^C$ and $A_R = V_{tt}^C$ we have, using Lemma 3.1,

$$\begin{aligned}
&\text{cov}([AW_{st}](i,\, p),\, [AW_{st}](j,\, q)) \\
&= U_{ss}(i,\, j)V_{tt}(p,\, q) \\
&= \text{cov}([AW_{st}](i,\, n),\, [AW_{st}](j,\, n))/std(F(S,\, T)) \cdot \\
&\quad \text{cov}([AW_{st}](m,\, p),\, [AW_{st}](m,\, q))/std(F(S,\, T))
\end{aligned}$$

Hence, AW_{st} has the mean and covariance of F_{st} and so A is the stochastic linearization of the system generating F_{st}.

Corollary 3.1. *For* $1 \le i \le m$ *and* $1 \le p \le n$

$$cov([AW_{st}](i,\, p),\, [AW_{st}](i,\, p)) = U_{ss}(i,\, i)V_{tt}(p,\, p)$$

That is, the variance of AW_{st} at each point can be readily computed from the diagonals of U_{ss} and V_{tt}.

Corollary 3.2. *The risk surface for* F *is specified by two increasing functions* k_1 *and* k_2 *with* $k_1(0) = k_2(0) = 0$.

Basic results for linearizations: Our goal is risk analysis for complex systems constructed of simpler components. Combining components requires algebraic rules for combining linearizations. From the basic representation of a separable field H, we say an operator A is separable provided $H_{st} = AF_{st} = (A_L)^T (K_{ss}^C)^{-T} F_{st} (K_{tt}^C)^{-1} A_R$, for each field F in the domain of A. Recall from Theorem 3.4 that the definition could have been given equivalently in terms of factorization of the matrix representation. Further, the definition results in the random field $H_{st} = AW_{st}$, using an extension of A to the continuous functions, being separable. Composition (or products) and inverses of the separable linear transformations yield separable linear transformations. A similar result fails for addition. In general, $(A+B)_L \neq A_L + B_L$ and $(A+B)_R \neq A_R + B_R$. See discussion below for approximation results.

Theorem 3.6. *If each of A and B is a separable linear transformation then*

$$(AB)_L = B_L(K_{ss}^C)^{-1}A_L \text{ and } (AB)_R = B_R(K_{tt}^C)^{-1}A_R$$

Furthermore,

$$(A^{-1})_L = K_{ss}^C(A_L)^{-1}K_{ss}^C \text{ and } (A^{-1})_R = K_{tt}^C(A_R)^{-1}K_{tt}^C$$

Proof of the theorem.

$$\begin{aligned} & E\left\{[ABW_{st}](i,\,p)[ABW_{st}](i,\,p)\right\} \\ &= [A_L^T(K_{ss}^C)^{-T}B_L^T B_L(K_{ss}^C)^{-1}A_L](i,\,i)\cdot \\ & \quad [A_R^T(K_{tt}^C)^{-T}B_R^T B_R(K_{tt}^C)^{-1}A_R](p,\,p) \end{aligned}$$

i.e., $U_{ss} = A_L^T(K_{ss}^C)^{-T}B_L^T B_L(K_{ss}^C)^{-1}A_L$ or $U_{ss}^C = (AB)_L = B_L(K_{ss}^C)^{-1}A_L$. In the same way, $(AB)_R = B_R(K_{tt}^C)^{-1}A_R$.

3.2.2 Special Results

Formulas for K_{ss}, U_{ss}, and V_{ss}:

The nonnegative definite matrices K_{ss}, U_{ss}, and V_{ss} have a special form and combine in nice ways. For instance

$$U_{ss} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & k(s_2) & k(s_2) & k(s_2) & \dots \\ 0 & k(s_2) & k(s_3) & k(s_3) & \dots \\ 0 & k(s_2) & k(s_3) & k(s_4) & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix}$$

$$U_{ss}^C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{k(s_2)} & \sqrt{k(s_2)} & \sqrt{k(s_2)} & \dots \\ 0 & 0 & \sqrt{k(s_3)-k(s_2)} & \sqrt{k(s_3)-k(s_2)} & \dots \\ 0 & 0 & 0 & \sqrt{k(s_4)-k(s_3)} & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix}$$

Starting with U_{ss} we get U_{ss}^C and conclude, since the principal subdeterminants of U_{ss}^C are nonnegative, that U_{ss} is nonnegative definite. Assuming equal increments for the partition $\{s_p\}_{p=0}^n$, we have

$$(K_{ss}^C)^{-1} = \frac{1}{\sqrt{s_1}} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix}$$

$$U^C_{ss}(K^C_{ss})^{-1} = \frac{1}{\sqrt{s_1}} \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & \sqrt{k(s_2)} & 0 & 0 & \ldots \\ 0 & 0 & \sqrt{k(s_3)-k(s_2)} & 0 & \ldots \\ 0 & 0 & 0 & \sqrt{k(s_4)-k(s_3)} & \ldots \\ . & . & . & . & \ldots \\ . & . & . & . & \ldots \\ . & . & . & . & \ldots \end{bmatrix}$$

$$[U^C_{ss}(K^C_{ss})^{-1}V^C_{ss}](i,\, j) =$$

$$\begin{cases} 0 & \text{if } i = 1 \text{ or } j < i \\ \frac{1}{\sqrt{s_1}}\,\sqrt{k_1(s_i) - k(s_{i-1})}\sqrt{k_2(s_i) - k(s_{i-1})} & \text{otherwise} \end{cases}$$

$$[(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})'(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})](i,\, j) =$$

$$\begin{cases} 0 & \text{if } i = 1 \text{ or } j < i \\ \frac{1}{s_1}\,\sum_{\ell=2}^{i}(k_1(s_\ell) - k(s_{\ell-1}))(k_2(s_\ell) - k(s_{\ell-1})) & \text{otherwise} \end{cases}$$

Note for $i \leq j$ that

$$\begin{aligned} &[(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})'(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})](i,\, j) \\ &\quad = [(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})'(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})](j,\, i) \\ &\quad = [(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})'(U^C_{ss}(K^C_{ss})^{-1}V^C_{ss})](i,\, i) \end{aligned}$$

A note on kernels for fields with more that two independent variables: Suppose d is a positive integer and $\{k_i\}_1^d$ is a sequence of increasing functions on $[0, 1]$ with $k_i(0) = 0$, for $i = 1, \ldots, d$. Let

$$R(\mathbf{u}, \mathbf{v}) = \Pi_{i=1}^d k_i(u_i \wedge v_i)$$

for d-sequences $\mathbf{u} = \{u_i\}_{i=1}^d$ and $\mathbf{v} = \{v_i\}_{i=1}^d$ with values in $[0, 1]$, where $a \wedge b = min(a, b)$.

Theorem 3.7. *R is positive definite, i.e., if $\{\mathbf{u}^p\}_{p=1}^n$ is a sequence of d-sequences $\mathbf{u}^p$ with values in $(0, 1]$ and $\{a_p\}_{p=1}^n$ is a nonzero sequence of real numbers then*

$$\sum_{p=1}^n \sum_{q=1}^n R(\mathbf{u}^p, \mathbf{u}^q) a_p a_q > 0$$

Example. The Wiener field (Chentsov, 1956; Ossiander and Waymire, 1989) on $[0, 1]^d$ has covariance kernel

$$K(\mathbf{u}, \mathbf{v}) = \Pi_{i=1}^d (u_i \wedge v_i)$$

Lemma 3.2. *If M is an $n \times n$ symmetric matrix such that*

1. *$M(1, 1) > 0$,*
2. *$M(p, q) = M(q, p) = M(p, p)$, for $1 \leq p \leq q \leq n$, and*
3. *$\{M(p, p)\}_{p=1}^n$ is an increasing sequence,*

then M is positive definite.

Proof of the lemma. Note that

$$M^C =$$

$$\begin{bmatrix} \sqrt{M(1,1)} & \sqrt{M(1,1)} & \sqrt{M(1,1)} & \sqrt{M(1,1)} & \cdots \\ 0 & \sqrt{M(2,2)-M(1,1)} & \sqrt{M(2,2)-M(1,1)} & \sqrt{M(2,2)-M(1,1)} & \cdots \\ 0 & 0 & \sqrt{M(3,3)-M(2,2)} & \sqrt{M(3,3)-M(2,2)} & \cdots \\ 0 & 0 & 0 & \sqrt{M(4,4)-M(3,3)} & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{bmatrix}$$

is the upper Cholesky factor of M. Therefore the principal subdeterminants of M are positive and so M is positive definite.

Offered in proof for the theorem: Suppose $d = 2$. Let $U(p, q) = k_1(u_p \wedge u_q)$ and $V(p, q) = k_2(v_p \wedge v_q)$ for $1 \leq p, q \leq n$. Let $\mathbf{u}^p = (u_p, v_p)$ for $p = 1, \ldots, n$ and

$$M(p, q) = R(\mathbf{u}^p, \mathbf{u}^q) = k_1(u_p \wedge u_q)k_2(v_p \wedge v_q) = U(p, q)V(p, q)$$

for $1 \leq p, q \leq n$. By Lemma 3.2 M is positive definite and the theorem follows for $d = 2$. The extension to $d > 2$ is immediate.

Interest in building more complicated models from simpler components also arises in data driven studies. This pursuit is well represented by (De Iaco et al., 2001) which goes over some of the same ground covered in this chapter. The viewpoint is quite different, however. Rather than considering the object random field as the result of a "visible" interaction of components, the modeling viewpoint, the field is analyzed in terms of available observables of the field itself. The goal of the research thrust is the most general representation of a random field that can be unraveled in terms of the observables of the field.

Our starting point is the class of separable random fields, the most tractable case. Since the envisioned application is to decision problems with uncertainty and risk, we are able to restrict our attention to an even smaller class of fields. The principal difficulty is developing an algebra for combining the simpler models. Dealing with this difficulty is a principal focus of this chapter.

3.2.3 Covergence Results

Central Limit Theorem: Assume that $\{u_p\}$ is a sequence of positive integers with limit ∞ and $\{\hat{X}_{jp}\}$ is a double sequence such that, for each positive integer p, $1 \leq j \leq u(p)$. Let $S_p = \sum_{j=1}^{u(p)} \hat{X}_{jp}$.

Theorem (Chung, 2001): *If $E\hat{X}_{jp} = 0$ and $E(|\hat{X}_{jp}|^3)$ is finite, for $p \geq 1$ and $1 \leq j \leq u(p)$, $\sum_{j=1}^{u(p)} E(\hat{X}_{jp})^2 = 1$, for $p \geq 1$, and $\sum_{j=1}^{u(p)} E(|\hat{X}_{jp}|^3)$ has limit 0 as $p \to \infty$ then $\{S_p\}$ converges in distribution to a (0, 1)-normal random variable.*

Suppose that $\{s_p\}$ is a sequence of partitions of [0, 1], $0 = s_0(0) < s_0(1) < \cdots < s_0(m) = 1$, s_{p+1} refines s_p, and $mesh(s_p) \to 0$ as $p \to \infty$. Assume that $\{u_i\}_{i=0}^{m}$ is a sequence of nondecreasing integer valued sequences such that $s_p(u_i(p)) = s_0(i)$, for $0 \le p$ and $0 \le i \le m$. Note that $s_p(u_{(\cdot)}(p)) = s_0$. Also, for $0 \le i \le m$, $u_i(p) \to \infty$ as $p \to \infty$. Further, suppose that $\{t_q\}$ is a sequence of partitions of [0, 1], $0 = t_0(0) < t_0(1) < \cdots < t_0(n) = 1$, t_{q+1} refines t_q, and $mesh(t_q) \to 0$ as $q \to \infty$. Assume that $\{v_j\}_{j=0}^{n}$ is a sequence of nondecreasing integer valued sequences such that $t_q(v_j(q)) = t_0(j)$, for $0 \le q$ and $0 \le j \le n$.

Suppose that

1. Z is a random function on $[0, 1] \times [0, 1]$ such that
   (a) $Z(s, t)$ is (0, 1)-normal.
   (b) $Z(s, t)$ and $Z(u, v)$ are independent for $(s, t) \neq (u, v)$.
2. Each of k_L and k_R is a continuous increasing function on [0, 1], $k_L(0) = k_R(0) = 0$, and $k_L(1) = k_R(1) = 1$.
3. R_L and R_R are nonnegative definite functions on $[0, 1] \times [0, 1]$ defined
   (a) $R_L(s, t) = k_L(min(s, t))$
   (b) $R_R(s, t) = k_R(min(s, t))$

Theorem 3.8. *If i and j are fixed, $s = s_0(i)$, and $t = t_0(j)$ then*

$$S_{pq} = \sum_{\alpha=1}^{u_i(p)} \sum_{\beta=1}^{v_j(q)} \sqrt{k_L(s_p(\alpha)) - k_L(s_p(\alpha-1))}\sqrt{k_R(t_q(\beta)) - k_R(t_q(\beta-1))}\, Z_{s_p t_q}(\alpha, \beta)$$

has limit in distribution the $(0, (k_L(s)k_R(t))^{1/2})$-normal random variable $\int_0^t \int_0^s Z(\alpha, \beta)\,(dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$ as $p, q \to \infty$.

Proof of the theorem. If q is fixed and $1 \le \beta \le v_j(q)$ then the Central Limit Theorem allows us to conclude that

$$\sum_{\alpha=1}^{u_i(p)} \sqrt{k_L(s_p(\alpha)) - k_L(s_p(\alpha-1))} Z_{s_p t_q}(\alpha, \beta)$$

has limit in distribution the $(0,\, k_L(s)^{1/2})$-normal random variable

$$\int_0^s Z(\alpha,\, t_q(\beta))\, (dk_L(\alpha))^{1/2}$$

as $p \to \infty$. Further, we can conclude that

$$\sum_{\beta=1}^{v_j(q)} \sqrt{k_R(t_q(\beta)) - k_R(t_q(\beta - 1))} \int_0^s Z(\alpha,\, t_q(\beta))\, (dk_L(\alpha))^{1/2}$$

has limit in distribution the $(0,\, (k_L(s)k_R(t))^{1/2})$-normal random variable $\int_0^t \int_0^s Z(\alpha,\, \beta)\, (dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$ as $q \to \infty$. Hence the result.

Let $U = R_L$ and $V = R_R$. Then, for $0 \le i \le m$ and $0 \le j \le n$, $S_{pq} = [(U^C_{s_p s_p})^T Z_{s_p t_q} V^C_{t_q t_q}](u_i(p),\, v_j(q))$ has limit in distribution

$$\int_0^{t_0(j)} \int_0^{s_0(i)} Z(\alpha,\, \beta)\, (dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$$

as $p,\, q \to \infty$.

Note that a more general result is available, i.e., for more general R_L and R_R, but the limit will lack the nice integral representation. Assuming the setup above, i.e., $\{\{s_p\},\, \{u_i\},\, \{t_q\},\, \{v_j\},\, Z\}$, $U = R_L$, and $V = R_R$, then, for i and j fixed, $s = s_0(i)$, and $t = t_0(j)$,

$$S_{pq} = [(U^C_{s_p s_p})^T Z_{s_p t_q} V^C_{t_q t_q}](u_i(p),\, v_j(q))$$

has limit in distribution a $(0,\, (R_L(s,\, s)R_R(t,\, t))^{1/2})$ - normal random variable.

We can extend a continuous linear operator A on $\{G,\, Q\}$ with associated discretized covariances $U_{s_p s_p}$ and $V_{t_q t_q}$ to a domain including random functions as follows:

$$[\hat{A}F](s,\, t) = [AE(F)](s,\, t) + \\ \lim_p \lim_q [(U^C_{s_p s_p})^T (K^C_{s_p s_p})^{-T} (F - E(F))_{s_p t_q} (K^C_{t_q t_q})^{-1} V^C_{t_q t_q}](u_i(p),\, v_j(q))$$

We have to impose conditions on $F - E(F)$ in order satisfy the hypothesis of the Central Limit Theorem but we do not have to assume normality. On the other hand, for simulations we can substitute a zero mean normal field for $F - E(F)$ simplifying the numerics and arrive at the same limit. Because these extensions are always available, we concentrate on continuous linear operators defined on $\{G, Q\}$ for decision models developed in the next section.

3.3 Risk Analysis of an Uncertain Complex System

A system with at least two interacting components is a complex system. Broadly defined risk is the probability of an undesirable decision outcome. In our setup the distribution of outcomes depends on the uncertainties, our independent variables. The goal of decision making for complex systems is a subjective balance of expected payoff and risk. Risk is only half of the equation.

In a recent INFORMS Tutorial, Rockafellar (2007) surveyed a variety of approaches to risk and the axioms of risk. The goal of the axioms is a coherent risk assessment, which we interpret as a consistent assessment leading to a consistent decision methodology. More on consistency later.

Rockafellar assumes uncertainties can be quantified, i.e., the randomness he chooses to categorize as uncertainties has a distribution. Our investigation of Ellsburg's paradox (Samson and Reneke, 2008) leads us in a different direction. Uncertainties should be modeled as independent variables (no distributions) and decision problems with uncertainties should be resolved using multi-criteria decision methods.

The goal of balance can be achieved by either maximizing $\mu - \alpha\,\sigma$ or minimizing $\mu + \alpha\,\sigma$, where μ is the mean of a component performance, σ^2 is the variance of the performance, and α is a positive parameter set by the decision maker reflecting his/her risk tolerance. (See Samson et al. (2008).) Note that the probability that performance is less than $\mu - \sigma$, an undesirable event, is approximately 0.1586.

3.3.1 Network Representations of Complex Systems

The following example, Figure 3.1, has all of the essential features of more complex systems and will serve to illustrate general results. We want to study the interaction of random fields, in particular, estimates of $\mu_i - \alpha\,\sigma_i$, α a real parameter, computed from G_i,

$i = 1, 2$. See Figure 3.1. Our general approach is through the construction of more complex models by combining simpler components. Notice that the nodes (M and N) represent linear operators and the arcs (F_i and G_i) represent random fields. The model could be used for a simplified investment decision (Reneke and Wiecek, 2004) or some other multicriteria decision problem (Reneke et al., 2002; Reneke and Wiecek, 2002).

In these, and other decision problems, the domain of the random fields is interpreted as the space of uncertainties and the decision variable $\mu - \alpha\,\sigma$ as an acceptable balance of expected payoff and risk. Formally, taking M and N as linear operators on an appropriate space of fields, we have

$$
\begin{aligned}
&G_1 = M(F_1 + G_2) \\
&G_2 = N(F_2 + G_1) \\
&(I - MN)G_1 = M(F_1 + NF_2) \\
&G_1 = (I - MN)^{-1} M(F_1 + NF_2)
\end{aligned}
$$

Thus we need to be able to deal with sums of fields and products and inverses of sums of operators.

In general, the sum of two separable fields is not separable. At one level, this is not a problem since we can still express the covariance of the sum in terms of the covariances of each of the summands. However, for model building we need a single linear operator representation of the sum and this representation should be separable so we can combine the result with other operator representations of fields. We have to be content with separable approximations to the sum rather than separable representations.

3.3.2 Separable Approximations of Sums of Fields

There are two cases, i.e., we need separable approximations for a discrete random field H_{st} when either

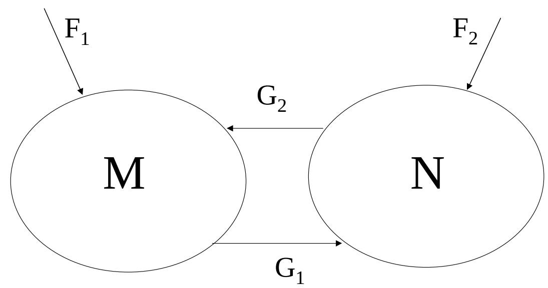

Figure 3.1 A two component model representing the interaction of two random fields F_1 and F_2

1. $H_{st} = (A_L)^T Z_{st} A_R + (B_L)^T \bar{Z}_{st} B_R$, where Z_{st} and $\bar{Z}_{st}$ are independent (0, 1)-normal, and
2. $H_{st} = (A_L)^T Z_{st} A_R + (B_L)^T Z_{st} B_R$, where Z_{st} is (0, 1)-normal.

Referencing the example the first case arises, for instance in G_1, when $A_L = F_L^1$, $A_R = F_R^1$, $B_L = F_L^2 (K_{ss}^C)^{-1} N_L$ and $B_R = F_L^2 (K_{tt}^C)^{-1} N_R$. The second case arises, again in G_1, when $A_L = A_R = K_{ss}^C$, $B_L = N_L (K_{ss}^C)^{-1} M_L$, and $B_R = N_R (K_{tt}^C)^{-1} M_R$.

For the first case,

$$
\begin{aligned}
&\operatorname{cov}([H_{st}](i,\, p),\, [H_{st}](j,\, q)) \\
&\quad = \; [(A_L)^T A_L](i,\, j)[(A_R)^T A_R](p,\, q) + [(B_L)^T B_L](i,\, j)[(B_R)^T B_R](p,\, q)
\end{aligned}
$$

Note that for $i \leq j$ and $p \leq q$ we have

$$
\operatorname{cov}([H_{st}](i,\, p),\, [H_{st}](j,\, q)) = \operatorname{cov}([H_{st}](i,\, p),\, [H_{st}](i,\, p))
$$

Therefore we seek increasing functions k_1 and k_2 on [0, 1] such that $k_1(0) = k_2(0) = 0$ and $max_{i,\,p}(|\text{cov}([H_{st}](i,\,p),\,[H_{st}](i,\,p)) - k_1(s_i)k_2(t_p)|)$ is as small as possible.

Suppose each of F and G is a random field on $[0,\,S] \times [0,\,T]$ with risk fields R_F and R_G, respectively. Our modeling assumption is that we only "observe" $F(\,\cdot,\,T)$ and $F(S,\,\cdot\,)$, i.e., we only "know" $R_F(\,\cdot,\,T)$ and $R_F(S,\,\cdot\,)$. Something similar holds for G. We say that F and G are *equivalent* if $R_F(\,\cdot,\,T) = R_G(\,\cdot,\,T)$ and $R_F(S,\,\cdot\,) = R_G(S,\,\cdot\,)$. Further, we say G is the *separable equivalent* of F if $R_G(s,\,t) = R_F(s,\,T)R_F(S,\,t)/R_F(S,\,T)$, for $(s,\,t)$ in $[0,\,S] \times [0,\,T]$. Note that the separable equivalent is always available.

Turning to the second case, suppose

$$H_{st} = (A_L)^T (K^C_{ss})^{-T} W_{st} (K^C_{tt})^{-1} A_R + (B_L)^T (K^C_{ss})^{-T} W_{st} (K^C_{tt})^{-1} B_R$$

We begin by computing the covariance for H_{st}.

Theorem 3.9.

$$\begin{aligned} &cov([H_{st}](i,\,p),\,[H_{st}](j,\,q)) \\ &= [(A_L)^T A_L](i,\,j)[(A_R)^T A_R](p,\,q) + [(B_L)^T A_L](i,\,j)[(B_R)^T A_R](p,q) + \\ &\quad [(A_L)^T B_L](i,\,j)[(A_R)^T B_R](p,q) + [(B_L)^T B_L](i,\,j)[(B_R)^T B_R](p,\,q) \end{aligned}$$

Proof of the theorem.. Using Lemma 3.1

$$
\begin{aligned}
&\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](j,\,q)) \\
&= E([(A_L)^T Z_{st} A_R + (B_L)^T Z_{st} B_R](i,\,p)[(A_L)^T Z_{st} A_R + \\
&\quad (B_L)^T Z_{st} B_R](j,\,q)) \\
&= E([(A_L)^T Z_{st} A_R](i,\,p)[(A_L)^T Z_{st} A_R](j,\,q)) + \\
&\quad E([(A_L)^T Z_{st} A_R](i,\,p)[(B_L)^T Z_{st} B_R](j,\,q)) + \\
&\quad E([(B_L)^T Z_{st} B_R](i,\,p)[(A_L)^T Z_{st} A_R](j,\,q)) + \\
&\quad E([(B_L)^T Z_{st} B_R](i,\,p)[(B_L)^T Z_{st} B_R](j,\,q)) \\
&= [(A_L)^T A_L](i,\,j)[(A_R)^T A_R](p,\,q) + [(B_L)^T A_L](i,\,j)[(B_R)^T A_R](p,\,q) + \\
&\quad [(A_L)^T B_L](i,\,j)[(A_R)^T B_R](p,\,q) + [(B_L)^T B_L](i,\,j)[(B_R)^T B_R](p,\,q)
\end{aligned}
$$

3.3.3 Separable Approximations of Inverses of Sums of Operators

In the modeling context we require a separable approximation of $(I - A)^{-1}$, where A is separable. If we use the separable approximation B to $I - A$ then one of the basic results gives a separable representation of B^{-1}.

Separable representation of the inverse of a sum - The dependent case: Suppose

$$H_{st} = W_{st} - (A_L)^T (K_{ss}^C)^{-T} W_{st} (K_{tt}^C)^{-1} A_R$$

then

$$
\begin{aligned}
&\mathrm{cov}([H_{st}](i,\, p),\, [H_{st}](i,\, p)) \\
&= [K_{ss}(i,\, i)[K_{tt}(p,\, p) - 2[(A_L)^T K_{ss}^C](i,\, i)[(A_R)^T K_{tt}^C](p, p) + \\
&\quad [(A_L)^T A_L](i,\, i)[(A_R)^T A_R](p,\, p) \\
&= [diag(K_{ss}) diag(K_{tt})^T](i,\, p) - \\
&\quad 2[diag((A_L)^T K_{ss}^C) diag((A_R)^T K_{tt}^C)^T](i,\, p) + \\
&\quad [diag((A_L)^T A_L) diag((A_R)^T A_R)^T](i,\, p)
\end{aligned}
$$

Thus given F_1 and F_2 we are able to simulate the discrete version of

$$G_1 = (I - MN)^{-1} M(F_1 + NF_2)$$

and hence can estimate $\mu_1 = EG_1$ and $\sigma_1^2 = var(G_1)$. In the same way we can estimate μ_2 and σ_2. These are simple tasks in the MatLab environment. We are ready to take up the decision process.

The decision process: The discussion of the example of a complex decision problem can be completed by briefly indicating how the modeling methodology which we have developed can be employed to resolve a hypothetical decision problem, choosing from a finite set of alternatives.

1. An outline for comparing decision alternatives for the example
    (a) We assume the criteria interact with the interactions modeled by Figure 3.1.
    (b) Associated with the jth alternative will be a pair of continuous linear operators M^j and N^j.
    (c) We assume the random external influences F_1 and F_2 are the same for all alternatives.
    (d) The evaluation of the jth alternative results in two random fields G_1^j and G_2^j, different for each alternative.
    (e) We compute the mean and standard deviation fields μ_i^j and σ_i^j for G_i^j, $i = 1, 2$.
2. Two steps to problem resolution

(a) **Multi-criteria optimization.** Since the goal is to balance μ and σ, we would either consider maximizing $\mu - \sigma$ or minimizing $\mu + \sigma$ based on the example's objective, where μ is the mean of a component performance and σ^2 is the variance of the performance. Let us assume that we are interested in maximizing $\mu - \sigma$ for our decision problem represented by Figure 3.1. Therefore for each of the finite set of alternative, we compute $R_i^j = (\mu_i^j - \sigma_i^j)$, $i = 1, 2$.
Given two alternatives, in Multi-criteria decision making for a minimization problem, Alternative 1 is said to *dominate* Alternative 2 if for all uncertainties $R_i^1 \leq R_i^2$ and for at least one discretized uncertainty $R_i^1 < R_i^2$. Otherwise, both alternatives are said to be *nondominated*.
The first step of problem resolution involves in finding all nondominated alternatives. If the first step provides an unique nondominated alternative, this is the preferred alternative. On the other hand, if at the end of first step we have multiple nondominated alternatives, we go to the second step of the problem resolution.

(b) **Identifying the preferred choice.** If more than one alternative is nondominated, we do not have a clear preferred choice. To choose a preferred alternative we first define an *ideal alternative*. An ideal alternative is any infeasible alternative which clearly dominates all the nondominated alternatives from step one. Then, we use L_2 norm to pick a preferred alternative from the set of nondominated alternatives which is closest to the ideal alternative.

3. Questions of consistency for our approach are resolved (Reneke and Samson, 2008) by establishing a mapping between Savage's informal probability structure (Savage, 1954) and our preference rules. Only the first four of Savage's Postulates hold but those suffice to show the methodology is rational.

The models open a new field of application to decision problems. Requirements for decision problems dictate our approach at various points: the discretization/matrix representations ease the computational burden, two dimensional domains represent a significant enlargement of the class of possible applications, the emphasis on risk comes from decision problems. Decisions in an environment of uncertainty and risk are ubiquitous. The modeling tools discussed in this chapter enable us to consider larger, more complex problems. Problems for developing a decision making methodology are not discussed but would include efficient multi-criteria optimization algorithms and implementation of decision preference rules for multi-criteria decisions.

An engineering approach to risk analysis of complex systems: Engineering systems, such as submarines or power generation plants, can be composed of many parts. A method for estimating risk of failure for such complex systems proceeds as follows (Leitch, 1995). Estimate expected time to failure for each part, perhaps based on experiment. The possible failures are partitioned by significance, for example as major or minor. Risk for each category is the probability of failure during an operating period, for example the probability of a major failure during the next year. The probabilities increase over time limiting the useful "safe" lifetime of the system. Part of the engineering art is deciding when a series of minor failures can cascade into a major failure, for example the pattern of failures in the Three Mile Island incident. Risk estimates in this methodology are not functions of "uncertainties" although the estimates of future risk can be updated using operating histories. Many aircraft operate long after the end of the original designed "lifespan" because of hardware updates and refurbishment. Whether our "soft" approach has anything to offer to "hard" engineering problems remains to be seen.

3.4 Conclusions

The chapter is a mixture of hard results and plausible methods based on numerical experiments. The results are firmly based in RKH space theory but theory which is restricted to special cases. The numerical experiments need to be followed up with a more systematic study.

Separable approximations enable us to build complex models from simpler components. We have a complete algebra of separable components, at least in terms of approximations. Separable approximations to nonseparable fields (operators) obtained through addition seem unavoidable. The approximations are not heavily penalized, at least in numerical experiments, and so seem to be useful for modeling.

The approximations fit within an established framework, i.e., RKH spaces. Discretizations of various objects, for instance fields and covariance kernels, are not approximations but exact. The random fields of interest are not elements of the underlying RKH space. However, the covariance kernel of a random field provides the matrix representation of a linear transformation of the RKH space which can be extended to a larger space including the Wiener field enabling the representation of the random field. The representations provide a complete characterization of the zero mean random fields. The representations are more tractable, at least for our purposes, than distribution function representations, etc..

Convergence is a natural question for continuous parameter models. From a data viewpoint, the usual collection methodology starts with discrete random functions but some representations (transform methods) immediately progress to continuous parameter models. Denied the possibility of infinite sampling, the convergence question is mute.

Some optimization problems, at first glance, seem to be approachable as large mathematical programming problems after a suitable discretization. Such an approach leaves open the problem of convergence of the optimal solutions of the subproblems. A difficulty which is often overlooked. Our multi-criteria optimization problems on the discrete models might leave us open to similar criticisms. However, our discretizations leave us in the original RKH space setting, i.e., we can claim that the discrete problem is the “real” problem.

Are we losing something by only modeling risk? Of course we are. Many different system models, models based on different covariances, can share the same risk profiles. However, if the decision is to turn on analysis of risk then we might as well choose the simpler models.

CHAPTER 4
A MULTICRITERIA APPROACH RESOLVING ELLSBERG'S PARADOX WITH COMMENTS ON CRITERIA AGGREGATION

In this chapter, we show that the methodology developed in chapter one is consistent. Ellsberg (1961) proposed two urn problems to show that the rational decision makers frequently violate Savage's axioms (Savage, 1954) yielding irrational decisions. We show that our methodology involving stochastic analysis and performance based multicriteria decision making under uncertainty and risk, yields a rational decision to Ellsberg's urn problems without any additional conditions on the uncertainty. We also identify as possible reason for the violations of Savage's axioms the aggregation of the multiple criteria, intended to reduce the complexity of the decision making problem.

4.1 Introduction

In developing our performance based multicriteria decision making methodology under uncertainty and risk, we want to show that our methodology yields consistent and rational decisions. We discuss in our terms a decision problem with clear roles for uncertainty and risk. We use a well known urn problem from the literature which raised issues when it was published by Ellsberg (1961) and is still being investigated with over 37 citations in 2007 alone. Researchers from diverse areas of science are studying Ellsberg's paradox in their context of interest. Many scholars recognize this paradox as an important benchmark in validating their decision making methodology. The following is a subjectively chosen set of works in the resent past to illustrate the wide interest in Ellsberg's paradox. We found researchers in the areas such as management (Nau, 2006), business (Cabantous, 2007), economics (Mukerji and Tallon, 2003; Klibanoff et al., 2005) and psychology (Kuhberger and Perner, 2003) addressing Ellsberg's paradox. Later in this chapter we refer to various contributions published during the last sixty years which are of relevance to our study.

The general outline of this chapter includes, in Section 4.2, the introduction of Ellsberg's paradox and the controversy it created in the world of decision making. We take a closer look at the urn problems presented by Ellsberg (1961) to understand the paradox. In Section 4.3, Ellsberg's urn problem is mapped to our modeling terminology and in Section 4.4 we illustrate the resolution of the paradox by solving the urn problem using our decision making methodology. In Section 4.5, we map our methodology's preference rule to Savage's framework of the problem and justify that our methodology preserves Savage's first four postulates (Savage, 1954). In Section 4.6, we discuss the problem of aggregation and how it misleads the decision maker to arrive at an inconsistent decision. In Section 4.7, an illustrative portfolio selection problem is solved using the proposed methodology to show that the choices made where consistent with respect to Savage. Finally, in Section 4.8, we discuss different resolutions of Ellsberg's paradox and some supporting arguments for this paradox in contrast to our work. While some believe that Savage's Sure-Thing principle should be softened, in some cases even dropped, others believe that a more clear presentation of Ellsberg's urn problems will avoid any inconsistent decisions. Section 4.8 explores more of these arguments. The following is the Ellsberg's Paradox.

4.2 Ellsberg's Paradox

Ellsberg (1961) proposed two urn problems to illustrate that rational decision makers frequently violate Savage's axioms (Savage, 1954) yielding irrational decisions. He goes on to claim that these so called irrational decisions, according to Savage, are indeed the best decision and hence the paradox. The following is a detailed view of this urn problem.

4.2.1 The Urn Problem

Consider an urn known to contain 30 red balls and 60 black and yellow balls, the latter in unknown proportion. The proportion of black and yellow balls is taken as uncertain and is modeled as an interval, whose values are denoted by λ. The distribution of the

Table 4.1 Payoffs for drawing a red, black, or yellow ball for Options I and II

	30	60	
	Red	Black	Yellow
I	\$100	\$0	\$0
II	\$0	\$100	\$0

occurrence of the red, black, and yellow balls can be modeled as $(1/3,\ \lambda,\ 2/3 - \lambda)$, where $\lambda \in [0, 2/3]$.

One ball is to be drawn at random from the urn. The random result of the draw can be quantified with probabilities for drawing red, black, and yellow balls depending on the value of the uncertainty.

Ellsberg proposes two decision problems with two decision *options* for each problem. He then challenges the reader to choose an option for each problem. The following are the decision problems proposed by Ellsberg:

- Problem 1: One ball is drawn at random from the urn. Table 4.1 gives the payoff of drawing a particular ball. Option I is to bet on a red ball and Option II is to bet on a black ball. A bet on a red ball. i.e., Option I, will result in a \$100 win if the player draws a red ball. Similarly, a bet on Option II will win the player \$100 if he draws a black ball. Figure 4.1 depicts the total expected payoffs of Option I, represented by the dotted line, and Option II, represented by the solid line. The question is which option would one prefer.

- Problem 2: Now for the same urn problem, the payoff for drawing a particular ball is slightly changed as given in Table 4.2. Option III is to bet on a red or a yellow ball and Option IV is to bet on a black or a yellow ball. In this problem a player's Option III bet will win him \$100 if he draws a red or a yellow ball and his Option IV bet will win him \$100 with a draw of a black or yellow ball. Figure 4.2 depicts the total expected payoffs of Option III, represented by the dotted line, and Option IV, represented by the solid line. Again the question is which option would one prefer.

Ellsberg also surveyed and collected a few opinions from decision makers. He reports the finding which raises the issue of consistency in decision making. A discussion of this issue of consistency follows.

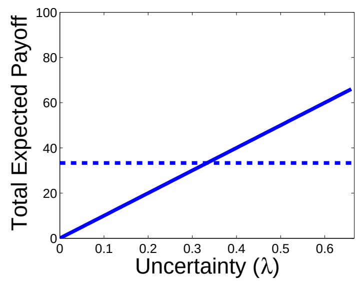

Figure 4.1 Total Expected Payoffs for Option I (dotted) and Option II (solid).

Table 4.2 Payoffs for drawing a red, black, or yellow ball for Options III and IV

	30	60	
	Red	Black	Yellow
III	\$100	\$0	\$100
IV	\$0	\$100	\$100

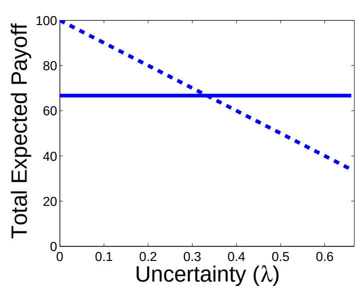

Figure 4.2 Total Expected Payoffs for Option III (dotted)and Option IV (solid).

4.2.2 The Problem of Consistency

Ellsberg reports that the most frequent resolutions of the two decision problems are: Option I is preferred to Option II and Option IV is preferred to Option III.

We do not know the reasoning behind Ellsberg's respondents' choices. The thinking developed in Figures 4.1 and 4.2 might not have played a role but the graphs do make the reported preferences reasonable. However, this approach only makes use of the expected payoffs and neglects other statistics available from the quantified randomness.

This pattern of most frequent resolutions of the two decision problems by Ellsberg's respondents violates Savage's Sure-Thing principle, i.e., since the payoff for Yellow is the same for both options in each of the problems, the decision should depend only on the payoffs for Red and Black. Thus, if Option I is preferred to Option II, then Option III should be preferred to Option IV.

We strongly believe that making a decision considering only the expected payoff is the reason for this inconsistency and that considering all the available information carefully will yield a consistent choice. The following relates the elements of Ellsberg's problems with our methodology.

4.3 Decision Language Mapping

In this section, we map our terminologies onto Ellsberg's terminologies. We do not change any values just the labels.

The basic building blocks of our decision model are the *criteria* that determines the decision, the *contingencies* that influence the criteria values, the available *alternatives* for making a decision and most importantly the *uncertainties*, the unquantifiable randomness under which a decision will have to be made.

We first replace the $100 payoff in the original urn problem to $1, because it will not make any difference in the decision making. Table 4.3 gives the new payoff of $1 for drawing a red, black, or yellow ball for Options I, II, III and IV.

Table 4.3 Payoffs for drawing a red, black, or yellow ball for Options I, II, III and IV

	30	60	
	Red	Black	Yellow
I	\$1	\$0	\$0
II	\$0	\$1	\$0
III	\$1	\$0	\$1
IV	\$0	\$1	\$1

Ellsberg's urn problem has three mutually exclusive contingencies: the event of an occurrence of a red ball, the event of an occurrence of a black ball and finally the event of an occurrence of a yellow ball. That is,

- Contingency 1 $(C1)$: the event of an occurrence of a red ball.
- Contingency 2 $(C2)$: the event of an occurrence of a black ball.
- Contingency 3 $(C3)$: the event of an occurrence of a yellow ball.

The options discussed in Ellsberg's problems are referred to as alternatives in our decision language. Ellsberg's urn problem has three criteria labeled as R, B and Y. The value of each criterion applied to the alternative depends on which contingency occurs. For instance, for the criterion R, the payoff will be \$1 for the first alternative if the first contingency occurs and the payoff will be \$0 if either $C2$ or $C3$ occurs. However, for the same criterion R, the payoff will be \$0 for the second alternative under contingencies, $C1$, $C2$, and $C3$. Similarly, for each criterion, a payoff value is found based on the alternatives and the contingencies. These payoffs for each criterion are represented in Tables 4.4-4.6. Our decision will be made based on these three criteria.

In the urn problem there is randomness found in the unknown proportion of the black and yellow balls. Since this randomness is not quantifiable and is modeled by an interval, the elements of this interval λ are taken as our uncertainty.

Table 4.4 The result of drawing a red ball for each payoff option and criterion

Criterion ⟹	R	B	Y
Option I	\$1	0	0
Option II	0	0	0
Option III	\$1	0	0
Option IV	0	0	0

Table 4.5 The result of drawing a black ball for each payoff option and criterion

Criterion ⟹	R	B	Y
Option I	0	0	0
Option II	0	\$1	0
Option III	0	0	0
Option IV	0	\$1	0

Table 4.6 The result of drawing a yellow ball for each payoff option and criterion

Criterion ⟹	R	B	Y
Option I	0	0	0
Option II	0	0	0
Option III	0	0	\$1
Option IV	0	0	\$1

Table 4.7 Decision maker's assessment for each of the alternative's performance

	Criterion $\Longrightarrow$	R	B	Y
P1	Alternative I	$(X = C1)$	0	0
	Alternative II	0	$(X = C2)$	0
P2	Alternative III	$(X = C1)$	0	$(X = C3)$
	Alternative IV	0	$(X = C2)$	$(X = C3)$

Now we have all the tools to assess the performance of each of the alternatives enabling us to formulate a decision model for the decision problem. Once a decision model is formulated, applying our methodology is an easy last step in the decision process.

Consider a random variable X with values in the set $\{C1, C2, C3\}$. Also consider the logical statement $(X = Ci)$ that takes the value one if the random variable X equals Contingency $i, i = 1, 2, 3$. and zero otherwise. Table 4.7 gives the decision maker's assessment of the alternative's performance of the two problems, where a one means winning a dollar and zero otherwise. For instance, if a person bets on a red ball, i.e., Alternative I of Problem 1, he will win a dollar if Contingency 1 ($C1$) occurs, i.e., the event of a red ball occurs. He will not make any money if either $C2$ or $C3$ occurs. This completes the mapping of the basic elements and the assessment of the alternatives performances. In the following section we apply our methodology to the decision problem which leads to the resolution of the paradox

4.4 Resolution of the Paradox

In this section we apply the performance based multicriteria decision making techniques developed by us (Samson et al., 2008) to the two urn problem defined by Ellsberg. The urn problem is special in the sense that there is no cost to place a bet and the payoff is 0 or 1 with a simple binomial distribution for each criteria. Also, we know the distribution of the contingencies as a function of uncertainty. That is, we know $prob(X = C1) = 1/3$,

Table 4.8 Probability of the expected payoffs

	Criterion $\Longrightarrow$	R	B	Y
P1	Alternative I	1/3	0	0
	Alternative II	0	λ	0
P2	Alternative III	1/3	0	$2/3-\lambda$
	Alternative IV	0	λ	$2/3-\lambda$

Table 4.9 Decision Model in terms of risk defined as probability of failure

	Criterion $\Longrightarrow$	R	B	Y
P1	Alternative I	2/3	1	1
	Alternative II	1	$1-\lambda$	1
P2	Alternative III	2/3	1	$1/3+\lambda$
	Alternative IV	1	$1-\lambda$	$1/3+\lambda$

$prob(X = C2) = \lambda$ and $prob(X = C3) = 2/3 - \lambda$. Therefore, the probability of the expected payoffs for each alternative as a function of λ can be modeled. Table 4.8 depicts these probabilities of the expected payoffs.

4.4.1 Decision Model

Define *risk* as the probability of failure for this urn problem, i.e., the probability that the bet will yield zero payoff . Then intuitively, choosing the bet with a minimum risk will be the preferred choice. We would like to make a choice with minimum risk for each problem. Table 4.9 gives the probabilities of failure, risk, for each alternative as a function of λ, the uncertainty.

Risk is our *decision variable* for this urn problem and Table 4.9 is our *decision model*. The objective is to minimize the decision variable and arrive at a preferred choice in a multicriteria decision making framework.

4.4.2 A Multicriteria Version of Risk

Let P_{ij}^F be the probability of failure for the i^{th} alternative and j^{th} criteria, $i = 1, 2, 3, 4.$ and $j = 1, 2, 3$. Then the triple $\phi_i(\lambda) = (P_{i1}^F, P_{i2}^F, P_{i3}^F)$, is the multicriteria version of the risk for the i^{th} alternative as a function of uncertainty λ , $i = 1, 2, 3, 4$. That is, risk of an alternative is a vector of individual risks under each criteria for that alternative.

Therefore, $\phi_1(\lambda) = (2/3, 1, 1)$, $\lambda \in [0, 2/3]$ is the risk of Alternative I and $\phi_2(\lambda) = (1, 1 - \lambda, 1)$, $\lambda \in [0, 2/3]$ is the risk of Alternative II. Figure 4.3, depicts these *risks of alternatives*.

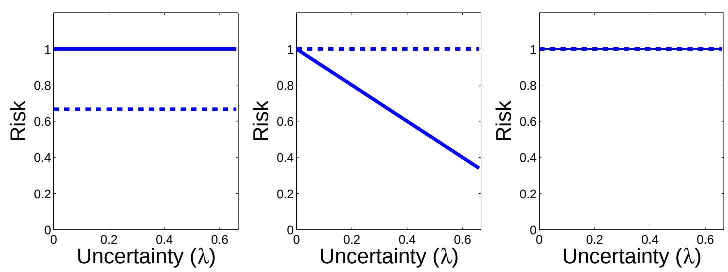

Figure 4.3 The dotted lines are the probabilities of failure for Alternative I and the solid lines are the probabilities of failure for Alternative II.

Similarly $\phi_3(\lambda) = (2/3, 1, 1/3 + \lambda)$, $\lambda \in [0, 2/3]$ is the risk of Alternative III and $\phi_4(\lambda) = (1, 1 - \lambda, 1/3 + \lambda)$, $\lambda \in [0, 2/3]$ is the risk of Alternative IV. Figure 4.4, depicts this set of risk of alternatives.

Defining this multicriteria version of risk enables us to choose an alternative as a preferred choice. The following section develops a preference rule to make this choice.

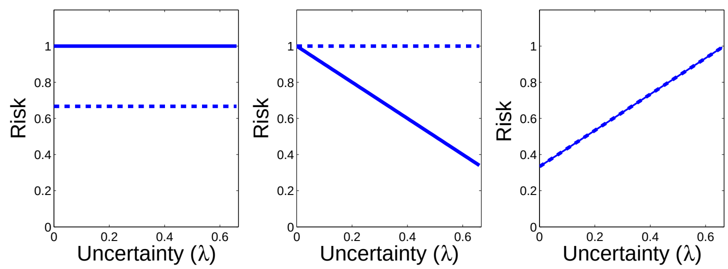

Figure 4.4 The dotted lines are the probabilities of failure for Alternative III and the solid lines are the probabilities of failure for Alternative IV.

4.4.3 Preference Rule

Our *preference rule* is developed as a two step process as follows: A simple analysis of Figures 4.3 and 4.4, reveals that for each problem there is no one single alternative that has the lowest risk for all criteria under the uncertainty. That is, if Problem 1 is considered, we see that both alternatives have equal risk for the third criteria. However, for the first criteria Alternative I yields minimum risk while Alternative II yields the minimum risk for criteria two. Therefore, in multicriteria decision making language, we say that neither alternative *dominates* the other, i.e., both alternatives are *nondominated*. Alternative A dominates B if for all uncertainty, the risk of Alternative A is less than or equal to the risk of Alternative B under all criteria and the risk of Alternative A is strictly less than the risk of Alternative B for at least one discrete value of the uncertainty. Similarly, both alternatives are nondominated in Problem 2. Finding these nondominated alternatives consist of the first step of the *preference rule*. If this step produces a unique nondominated alternative, we have found the preferred alternative. If more than one nondominated alternative is found in the first step, we continue with the second step.

Since both alternatives are nondominated, there is no clear solution for the problems. Therefore we define a second part of the preference rule to make a preferred choice. To define

Table 4.10 Norm values of distance between the utopia and the risk of an alternative

		*	$norm(*)^2$
P1	Alternative I	$U - \phi_1(\lambda)$	132/81
	Alternative II	$U - \phi_2(\lambda)$	134/81
P2	Alternative III	$U - \phi_3(\lambda)$	104/81
	Alternative IV	$U - \phi_4(\lambda)$	106/81

this part of the preference rule we first define a *utopia* (Ehrgott, 2005) alternative. A utopia alternative is created so that both Alternatives I and II would clearly be dominated by this alternative. In other words, this is an infeasible (too good to be true) solution for the defined problem. Since we are concerned with minimizing the risk for both these urn problems, a suitable utopia alternative would be a zero risk alternative. That is, if we denote utopia alternative for both the urn problems as U, then $U(\lambda) = (0, 0, 0)$, is a constant function for $\lambda \in [0, 2/3]$.

Then to make a consistent choices, our preference rule is to choose an alternative that is closest to the utopia alternative, measured in terms of L_2 norm.

$$\begin{aligned} norm(U - \phi_i(\lambda))^2 &= norm(\phi_i(\lambda))^2 \\ &= \sum_{j=1}^{3} \int_0^{2/3} |P_{ij}^F(\lambda)|^2 \, d\lambda, i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3. \end{aligned}$$

The alternative chosen using this preference rule will be our preferred choice. Table 4.10 gives the norm values of the distance between the alternatives and the utopia found analytically.

Clearly we see Alternative I is preferred over Alternative II for Problem 1 and Alternative III is preferred over Alternative IV for Problem 2. These choices preserve Savage's Sure-Thing principle and hence are consistent. We continue our decision analysis in

Section 4.6. In the following section we relate our framework with Savage's first four postulates to justify (Shafer, 1986) that our preference rule preserves the Sure-Thing principle and generates a consistent preferred choice.

4.5 Our framework and Savage's Postulates

We will provide a map between Savage's framework and first four postulates and the elements of our methodology establishing the consistency of our approach to decision making. Since everything in Savage is in terms of *preferences*, we accomplish this proof of consistency by checking whether our preference rules satisfy Savage's Postulates I-IV. The preference rule postulated by Savage applies to the decision variables, functions from the alternatives to numerical functions of the uncertainties. In our case the decision variables are the probability of failure, i.e., risk, and we want to minimize the decision variables.

The preference rule in our methodology is developed in two steps:

- Step 1: Preference based on dominance, i.e., picking the alternatives that are non-dominated.
- Step 2: Distance of the alternatives from the utopia alternative for resolving non-dominance among alternatives.

Our preference rule satisfies Savage's Postulates I-IV based on the following map between Savage's structure and our methodology.

As we mapped our terminologies onto Ellsberg's terminologies in Section 4.3, we now map Savage's terminologies onto ours. The *states* S in Savage's terms are mapped to our criteria $\{R, B, Y\}$. The *consequences* F are functions on the interval of uncertainties [0, 2/3]. The *acts* are functions $\mathbf{f}$ from the set of criteria S into the set of consequences F, which are our alternatives. In particular, the bets are given by $[\mathbf{f}(s)](\lambda)$, the probability of failure for s in $S = \{R, B, Y\}$ and $0 \leq \lambda \leq 2/3$.

The elements of F are partially ordered in our methodology, so we have to introduce a little different notation, i.e., we will replace $\mathbf{f} \leq \mathbf{f}'$ used by Savage ($\mathbf{f}$ is not preferred to $\mathbf{f}'$) by $\mathbf{f} \preceq \mathbf{f}'$. Note that if $[\mathbf{f}(s)](\lambda) \geq [\mathbf{f}'(s)](\lambda)$, for all s in S and $0 \leq \lambda \leq 2/3$, then

$\mathbf{f} \preceq \mathbf{f}'$. In general, $\mathbf{f}$ is not preferred to $\mathbf{f}'$ provided $norm(\mathbf{f})^2 = \sum_{s \in S} norm(\mathbf{f}(s))^2 = \sum_{s \in S} \int_0^{2/3} |[\mathbf{f}(s)](\lambda)|^2 \, d\lambda \geq norm(\mathbf{f}')^2$.

In Sections 4.5.1 - 4.5.4, we show that our preference rule satisfies Savage's postulates. Note that our preference rule will satisfy Savage's postulates only if risk is defined as a triple as modeled in Section 4.4.2. We will discuss other risk models in the decision analysis section later.

4.5.1 Postulate 1

Postulate 1: The relation $\preceq$ is a simple ordering. To clarify the meaning of "simple ordering" we formulate Postulate 1 explicitly as follows:

Postulate 1': Given alternatives $\mathbf{f}$, $\mathbf{f}'$, and $\mathbf{f}''$, i) either $\mathbf{f} \preceq \mathbf{f}'$ or $\mathbf{f}' \preceq \mathbf{f}$ and ii) if $\mathbf{f} \preceq \mathbf{f}'$ and $\mathbf{f}' \preceq \mathbf{f}''$ then $\mathbf{f} \preceq \mathbf{f}''$.

Proof of the postulate follows: Either $norm(\mathbf{f}) \geq norm(\mathbf{f}')$ or $norm(\mathbf{f}') \geq norm(\mathbf{f})$. If $norm(\mathbf{f}) \geq norm(\mathbf{f}')$ and $norm(\mathbf{f}') \geq norm(\mathbf{f}'')$ then $norm(\mathbf{f}) \geq norm(\mathbf{f}'')$.

Definition 4.1. For alternatives $\mathbf{f}$ and $\mathbf{g}$, $\mathbf{f} \preceq \mathbf{g}$ given the set of criteria B if and only if $\mathbf{f}' \preceq \mathbf{g}'$ for every $\mathbf{f}'$ and $\mathbf{g}'$ that agree with $\mathbf{f}$ and $\mathbf{g}$, respectively, on B and with each other on $\sim B$ and $\mathbf{g}' \preceq \mathbf{f}'$ either for all such pairs or for none.

The definition allows for the possibility that a given pair of alternatives $\mathbf{f}$ and $\mathbf{g}$ might not be comparable given B. We say alternative $\mathbf{f}$ is not preferred to alternative $\mathbf{f}'$ for criterion s by considering only the second components of the graphs with first components s, the values of the decision variables, the probabilities of failure. In effect, we are "extending" the decision rule to apply to single criteria.

4.5.2 Postulate 2

Postulate 2: For every $\mathbf{f}$ and $\mathbf{g}$ and B either $\mathbf{f} \preceq \mathbf{g}$ given B or $\mathbf{g} \preceq \mathbf{f}$ given B.

This is Savage's Sure-Thing principle. The second part of our preference rule must force one or the other alternative. Proof of the second postulate follows. Either

$$\sum_{s \in B} norm(\mathbf{f}(s))^2 \geq \sum_{s \in B} norm(\mathbf{g}(s))^2$$

or

$$\sum_{s \in B} norm(\mathbf{g}(s))^2 \geq \sum_{s \in B} norm(\mathbf{f}(s))^2$$

If the former holds then

$$\begin{aligned}
norm(\mathbf{f}')^2 &= \sum_{s \in S} norm(\mathbf{f}'(s))^2 \\
&= \sum_{s \in B} norm(\mathbf{f}'(s))^2 + \sum_{s \in \sim B} norm(\mathbf{f}'(s))^2 \\
&= \sum_{s \in B} norm(\mathbf{f}(s))^2 + \sum_{s \in \sim B} norm(\mathbf{f}'(s))^2 \\
&\geq \sum_{s \in B} norm(\mathbf{g}(s))^2 + \sum_{s \in \sim B} norm(\mathbf{f}'(s))^2 \\
&= \sum_{s \in B} norm(\mathbf{g}'(s))^2 + \sum_{s \in \sim B} norm(\mathbf{g}'(s))^2 \\
&= \sum_{s \in S} norm(\mathbf{g}'(s))^2 \\
&= norm(\mathbf{g}')^2
\end{aligned}$$

i.e., $\mathbf{f} \preceq \mathbf{g}$ given B. If the latter case holds then $\mathbf{g} \preceq \mathbf{f}$ given B.

Definition 4.2. $g \preceq g'$ if and only if $\mathbf{f} \preceq \mathbf{f}'$, when $\mathbf{f}(s) = g$, $\mathbf{f}'(s) = g'$ for every $s \in S$.

We move from an ordering of alternatives to an implied ordering of elements of F. This implied ordering allows Savage the freedom to consider nonnumerical consequences. It allows us to extend the natural partial order on F considered as real functions of the uncertainties to a simple order. Also note that, if $g(u) \leq g'(u)$ for every $u \in \Omega$ then by Postulate 0, $g \preceq g'$ and if there is a value of the uncertainty u such that $g(u) > g'(u)$ and another value v such that $g(v) < g'(v)$ then we still have either $g \preceq g'$ or $g' \preceq g$.

Definition 4.3. B is null, if and only if $\mathbf{f} \preceq \mathbf{g}$ given B for every $\mathbf{f}$ and $\mathbf{g}$.

In our methodology a criterion is irrelevant if it does not distinguish between at least one pair of alternatives, i.e., for each criterion s there is at least one pair of alternatives for which $\mathbf{f} \preceq \mathbf{g}$ given $B = \{s\}$ and it is false that $\mathbf{g} \preceq \mathbf{f}$ given $B = \{s\}$. A similar statement holds for any set of criteria. The effect of assuming that B is not null is that B is not the empty subset of S.

4.5.3 Postulate 3

Postulate 3: If $\mathbf{f}(s) = g$, $\mathbf{f}'(s) = g'$ for every $s \in B$ and B is not null, then $\mathbf{f} \preceq \mathbf{f}'$ given B, if and only if $g \preceq g'$.

We now have, with the extension of the preference rule, that any two elements in F are comparable. If B were empty then $\mathbf{f} \preceq \mathbf{f}'$ given B would be satisfied vacuously and so have no relevance for the comparison of g and g'. Also, we assume that $\mathbf{f}$ and $\mathbf{f}'$ agree on $\sim B$. To prove Postulate 3, we first need to introduce the following lemma.

Lemma 4.1. *$g \preceq g'$ if and only if $norm(g) \geq norm(g')$.*

Proof of the lemma: Suppose g and g' are in F and, for each $s \in S$, $\mathbf{f}(s) = g$ and $\mathbf{f}'(s) = g'$. Then $norm(\mathbf{f})^2 = \sum_{s \in S} norm(\mathbf{f}(s))^2 = |S|\, norm(g)^2$ and $norm(\mathbf{f}')^2 = |S|\, norm(g')$. Hence $g \preceq g'$, i.e., $\mathbf{f} \preceq \mathbf{f}'$, if and only if $norm(g) \geq norm(g')$.

Proof of Postulate 3: If $\mathbf{f} \preceq \mathbf{f}'$ given B then

$$\begin{aligned}
norm(\mathbf{f})^2 &= \sum_{s \in S} norm(\mathbf{f}(s))^2 \\
&= \sum_{s \in B} norm(\mathbf{f}(s))^2 + \sum_{s \in \sim B} norm(\mathbf{f}(s))^2 \\
&= |B|\, norm(g)^2 + \sum_{s \in \sim B} norm(\mathbf{f}(s))^2 \\
&\geq norm(\mathbf{f}')^2 \\
&= \sum_{s \in S} norm(\mathbf{f}'(s))^2 \\
&= |B|\, norm(g')^2 + \sum_{s \in \sim B} norm(\mathbf{f}(s))^2
\end{aligned}$$

Thus $norm(g) \geq norm(g')$, *i.e.,* $g \preceq g'$. *If* $g \preceq g'$ *then*

$$\begin{aligned} norm(\mathbf{f})^2 &= \sum_{s\in S} norm(\mathbf{f}(s))^2 \\ &= \sum_{s\in B} norm(\mathbf{f}(s))^2 + \sum_{s\in\sim B} norm(\mathbf{f}'(s))^2 \\ &= |B|\, norm(g)^2 + \sum_{s\in\sim B} norm(\mathbf{f}(s))^2 \\ &\geq |B|\, norm(g')^2 + \sum_{s\in\sim B} norm(\mathbf{f}(s))^2 \\ &= \sum_{s\in S} norm(\mathbf{f}'(s))^2 \\ &= norm(\mathbf{f}')^2 \end{aligned}$$

i.e., $\mathbf{f} \preceq \mathbf{f}'$ *given* B.

Definition 4.4. $A \preceq B$; if and only if $\mathbf{f}_A \preceq \mathbf{f}_B$ or $g \preceq g'$ for every $\mathbf{f}_A$, $\mathbf{f}_B$, g, g' such that: $\mathbf{f}_A(s) = g$ for $s \in A$, $\mathbf{f}_A(s) = g'$ for $s \in (\sim A)$, $\mathbf{f}_B(s) = g$, for $s \in B$, $\mathbf{f}_B(s) = g'$ for $s \in (\sim B)$.

For our methodology, we move from an ordering of alternatives to an implied ordering of subsets of S. Following Definition 4.4, we have

$$\begin{aligned} norm(f_A)^2 &= \sum_{s\in S} \int_0^{2/3} |[f_A(s)](\lambda)|^2\, d\lambda \\ &= \sum_{s\in A} norm(f_A(s))^2 + \sum_{s\in(\sim A)} norm(f_A(s))^2 \\ &= \sum_{s\in A} norm(g)^2 + \sum_{s\in(\sim A)} norm(g')^2 \\ &= |A|\, norm(g)^2 + |\sim A|\, norm(g')^2 \end{aligned}$$

and

$$norm(f_B)^2 = |B|\, norm(g)^2 + |\sim B|\, norm(g')^2$$

Hence $A \preceq B$ if and only if, for every g, $g' \in F$, either

$$|A|\, norm(g)^2 + |\sim A|\, norm(g')^2 \geq |B|\, norm(g)^2 + |\sim B|\, norm(g')^2$$

or

$$norm(g) \geq norm(g')$$

Suppose that $|A| \leq |B|$ and $norm(g) < norm(g')$ then $a = \frac{norm(g)}{norm(g')} < 1$

$$| \sim A| \geq | \sim B|$$

$$(1-a)| \sim A| \geq (1-a)| \sim B|$$

$$(1-a)| \sim A|\, norm(g')^2 \geq (1-a)| \sim B|\, norm(g')^2$$

$$-| \sim A|\, norm(g)^2 + | \sim A|\, norm(g')^2 \geq -| \sim B|\, norm(g)^2 + | \sim B|\, norm(g')^2$$

$$(|S| - | \sim A|)\, norm(g)^2 + | \sim A|\, norm(g')^2$$

$$\geq (|S| - | \sim B|)\, norm(g)^2 + | \sim B|\, norm(g')^2$$

$$|A|\, norm(g)^2 + | \sim A|\, norm(g')^2 \geq |B|\, norm(g)^2 + | \sim B|\, norm(g')^2$$

i. e., $A \preceq B$.

4.5.4 Postulate 4

Postulate 4: For every A, B, $A \preceq B$ or $B \preceq A$.

For Savage, Postulate 4 establishes the independence of probabilities and payoffs. The relation "$\preceq$" enables us to compare any two subsets of S. The proof of this postulate follows, either $|A| \leq |B|$ or $|B| \leq |A|$ and so either $|A| \preceq |B|$ or $|B| \preceq |A$.

According to Savage, the first four postulates are sufficient for the decision methodology to be considered "rational". These postulates guide our methodology's reasoning to deal with each criteria separately as discussed in the decision analysis section.

4.6 Decision Analysis

We continue our analysis from Section 4.4.3 to see how a multicriteria approach with all criteria treated independently yields consistent choices, as demonstrated in Section 4.4.3, while the same problem solved as a single criterion or a bi-criteria problem will

Table 4.11 Decision model of a single criterion problem and the norm values

	Prob of payoffs	Risk	Norm
I	1/3	2/3	24/81
II	λ	$1-\lambda$	26/81
III	$1-\lambda$	λ	8/81
IV	2/3	1/3	6/81

yield inconsistent choices. It is our belief that most of Ellsberg's respondents solved the urn problem as a single criterion problem and hence the inconsistency and the paradox.

Some look upon the criteria $\{R, B, Y\}$ as "primitive criteria" out of which the criteria for making decisions are constructed. For instance, a single criterion might be the payoff for the bet which is the sum of R, B, and Y values. The way the problem is posed makes this a natural approach. There is also a "hard headed" appeal to the single criterion. The decision maker is keeping his eye on the bottom line. Another possibility is to introduce $R+Y$ and $B+Y$ as two criteria. We will examine each of these possibilities.

4.6.1 One Criterion Problem

In a single criterion approach one would find the total expected payoffs of each alternative bet. Figure 4.1 and 4.2, represent these expected payoffs. As discussed in Section 4.2.2, analyzing these figures one could justify choosing Alternative I for Problem I and Alternative IV for Problem 2. The total expected payoffs for Alternative I and Alternative IV do not have any uncertainty. So a uncertainty averse person would bet on these Alternatives thinking that there is no uncertainty. Since the probability of an alternative payoff is the sum of probabilities of individual criteria payoffs, the risk is one minus the probability of a payoff. This is also reflected in the norm values as seen in Table 4.11.

Table 4.12 Decision model of the two criterion problem and the norm values

	Prob R+Y	Prob B+Y	Risk R+Y	Risk B+Y	Norm
I	1/3	0	2/3	1	78/81
II	0	λ	1	$1-\lambda$	80/81
III	$1-\lambda$	$2/3-\lambda$	λ	$1/3+\lambda$	34/81
IV	$2/3-\lambda$	2/3	$1/3+\lambda$	1/3	32/81

Again as was inferred with the figures, we can see Alternative I is preferred to Alternative II for Problem 1 and Alternative IV is preferred to Alternative III for Problem 2, violating Savage's Sure-Thing principle. Now let's consider a two criteria problem.

4.6.2 Two Criteria Problem

Here we consider the case where the decision maker could consider $R+Y$ and $B+Y$ as two criteria. The expected payoff for each criteria is just the sum of the two primitive criteria. A similar analysis as above gives Table 4.12 which includes the probabilities of payoffs, the risk and the norm values. Clearly, the preferred choice again violates Savage's Sure-Thing principle because Alternative I and IV are chosen as preferred choices based on the norm values. Thus using one or two criteria leads to inconsistent choices, i.e., violations of the Sure-thing principle. As Section 4.6.1 and Section 4.6.2 illustrate, aggregating the criteria misleads the decision analysis and hence leads to an inconsistent choice. The preferred choices identified in the three criteria case, which was done first, are consistent with the Sure-thing principle.

In the original problems we only get one draw. With one draw, we could have a bad outcome for the "right" choice of alternatives and a good outcome for the "wrong" choice of alternatives. However, we believe that the right choice should be the one which gives better results over the course of many draws.

To illustrate the relevance of these findings in the real life problems, we have considered a simple, but specially created, portfolio selection problem with three criteria. This portfolio selection problem is constructed so that the decision model of this problem and the decision model of the Ellsberg's problem are the same. By doing this we can see the resolution of Ellsberg's paradox helps making consistent and rational decisions for a "real world" portfolio problem.

4.7 Illustrating the Relevance of the Resolution of Ellsberg's Paradox

A portfolio selection problem is constructed to illustrate Ellsberg's paradox and its resolution. This carefully created portfolio problem has the same basic elements as that of Ellsberg's urn problem. No extra information was assumed either on the uncertainty or any other element.

4.7.1 Basic Elements

The basic elements of these two comparable problems are the criteria affecting the decision, the contingencies that could occur which may affect the criteria, the uncertainty, the alternatives and the distribution of these contingencies based on the uncertainty. We will discuss each of these elements in detail in the following paragraphs.

Financial criteria affecting the decision

In Ellsberg's problem, the criteria were {R, B, Y}. For the portfolio problem we are constructing, we also consider three criteria on which a decision would be made. We consider all financial criteria for this portfolio problem as follows:

- Yield, which is the stock's annual dividend payments divided by its current closing price. We are interested in the portfolio's combined yield.
- Growth rate, which is the current price minus the previous price divided by the previous price of the stock. Again we calculate the portfolio's growth rate.
- Safety of the portfolio, which is subjectively measured in terms of the portfolio's capital preserving nature under different contingencies.

Contingency

The second element we are interested in is composed of the contingencies (see Figure 4.5) that could occur which may affect the criteria. In Ellsberg's Urn problem the contingencies are mutually exclusive events. That is, the event of an occurrence of a red ball, the event of an occurrence of a black ball and the event of an occurrence of a yellow ball.

Since a country's Economic Growth Rate (EGR) affects the investment's yield, growth rate, and safety, we have decided to use the EGR to construct our contingencies.

For this portfolio problem different levels of mutually exclusive economic growth rate are considered to be the contingencies that could occur. Three possible contingencies are considered ; *normal growth rate*: $0\% \leq$ EGR $\leq 4\%$, *recession*: EGR is below 0%, and *unsustainable growth rate*: EGR is above 4%. That is,

- Contingency 1 (C1) : Normal EGR (i.e., $0\% \leq$ EGR $\leq 4\%$).
- Contingency 2 (C2): Recession (i.e., EGR $< 0\%$).
- Contingency 3 (C3): Unsustainable EGR (i.e., EGR $> 4\%$).

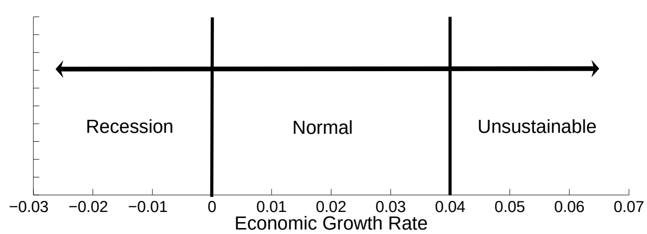

Figure 4.5 Contingencies

Uncertainty

The most significant parameter in our portfolio problem is the *prime interest rate* "set" by the Federal Reserve. Since we can model EGR as a random function of prime interest rate and since prime interest rates do not follow any known distribution, we select the rate to be the uncertainty. For simplicity the prime interest rate is scaled to be in the interval $[0, 2/3]$ and is denoted by λ.

Similar to the uncertainty of the proportion of black and yellow balls described in the Ellsberg's Urn problem, which is modeled by the interval $[0, 2/3]$, our portfolio problem's uncertainty also is modeled by the interval $[0, 2/3]$, keeping the uncertainties in both the problems similar and assuming nothing additional.

Alternatives

Ellsberg in his paper defines *actions* that a person could choose. He defines two actions for each of the two problems he presents. In our portfolio problem, we refer to these actions as *alternatives*.

We consider two alternative portfolios from which one is selected under uncertainty. The first alternative is a portfolio created based on *conservative* attitude toward the market and the second alternative is a portfolio created based on an *aggressive* attitude toward the market. A portfolio created with a conservative attitude concentrating on yield would be designed to perform satisfactorily in the normal EGR conditions while a portfolio created with an aggressive attitude concentrating on growth would take advantage of the unsustainable EGR.

Distribution of Contingencies

Given λ, the prime interest rate, EGR would react in each of the three contingencies as follows:

1. High prime interest rate λ. See Figure 4.6.
    - If EGR is in recession, a high prime interest rate could push the EGR further into recession which does not favor either of the considered alternatives.
    - If EGR is normal, a high prime interest rate could push the EGR into recession hurting the conservative portfolio.

- If EGR is unsustainable, a high prime interest rate could push the EGR into the normal range favoring the conservative portfolio while hurting the aggressive portfolio.

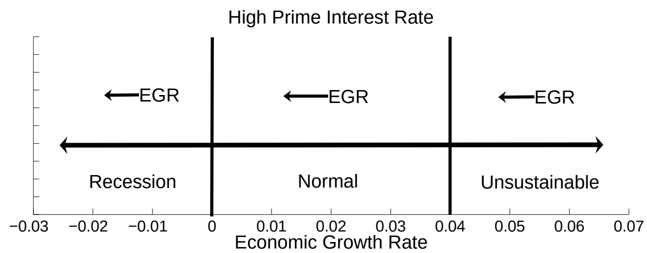

Figure 4.6 Relationship between the High Prime Interest Rates and the EGR

2. Low prime interest rate λ. See Figure 4.7.

    - If EGR is in recession, a low prime interest rate could push the EGR into the normal range favoring the conservative portfolio.
    - If EGR is normal, a low prime interest rate could push the EGR into unsustainable highs favoring the aggressive portfolio while hurting the conservative portfolio.
    - If EGR is unsustainable, a low interest rate could push the EGR further into unsustainable highs making the aggressive portfolio very attractive.

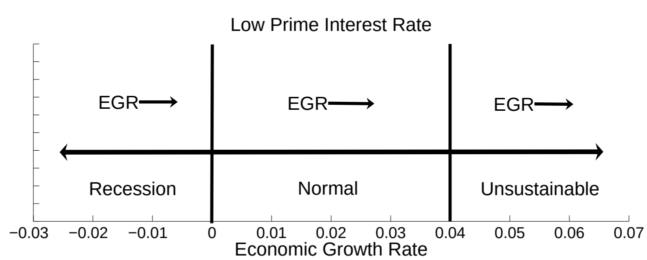

Figure 4.7 Relationship between the Low Prime Interest Rates and the EGR

This above listed relationship between the prime interest rates and the EGR helps the Federal Reserve to balance the EGR by carefully changing the prime interest rate as it sees fit. The distribution of these contingencies as a function of the uncertainty λ can be found using historical data provided enough data is available. In this chapter we assume this distribution of contingencies $(C1, C2, C3)$ to be $(1/3,\ \lambda,\ 2/3 - \lambda)$, where $\lambda \in [0, 2/3]$, the scaled prime interest rate. Note that this is the same distribution assumed by Ellsberg for the contingencies in his Urn problem.

4.7.2 Example to Illustrate Rational and Consistent Decision Making under Uncertainty

Now using these elements in an example we illustrate how we can make rational and consistent decisions in a portfolio problem. Consider two portfolio selection problems with two alternatives each.

- Problem 1
  - Conservative 1: This portfolio constructed giving priority to yield, reacts positively for normal EGR conditions, i.e., in the occurrence of contingency 1.
  - Aggressive 1: This portfolio, on the other hand, is constructed giving priority to portfolio growth rate and produces the largest portfolio growth rates when EGR is greater than 4%, i.e., if contingency 3 occurs.
- Problem 2
  - Conservative 2: This portfolio is constructed giving priority to both yield and safety of the portfolio. The portfolio is not only created to react positively in normal EGR conditions, it is also created to preserve capital under contingency 2, i.e., recession.
  - Aggressive 2: This portfolio is constructed giving priority to both portfolio growth rate and safety. So this portfolio reacts positively under unsustainable EGR and is still safe if contingency 2 occurs.

Decision Model

Consider a random variable X with values in the set $\{C1, C2, C3\}$. Also consider the logical statement $(X = Ci)$ that takes the value one if the random variable X equals Contingency $i, i = 1, 2, 3.$ and zero otherwise. The decision maker's assessment of the

Table 4.13 Decision maker's assessment of the portfolio's performance

	Criterion $\Longrightarrow$	Yield	Growth Rate	Safety
P1	Conservative 1	$(X = C1)$	0	0
	Aggressive 1	0	$(X = C3)$	0
P2	Conservative 2	$(X = C1)$	0	$(X = C2)$
	Aggressive 2	0	$(X = C3)$	$(X = C2)$

Table 4.14 Probability of expected Payoff

	Criterion $\Longrightarrow$	Yield	Growth Rate	Safety
P1	Conservative 1	1/3	0	0
	Aggressive 1	0	$2/3 - \lambda$	0
P2	Conservative 2	1/3	0	λ
	Aggressive 2	0	$2/3 - \lambda$	λ

portfolio's performance of the two portfolio problems considered is given in Table 4.13, where a one means the performance is satisfactory and zero otherwise.

Given the distribution of the contingencies, one can find the expected payoff of each criteria in a portfolio. Since the distribution of the contingencies is known to be (1/3, λ, $2/3 - \lambda$), where $\lambda \in [0, 2/3]$, the scaled prime interest rate, we have $prob(X = C1) = 1/3$, $prob(X = C2) = \lambda$ and $prob(X = C3) = 2/3 - \lambda$. The probabilities of the expected payoff is given in Table 4.14. Also, *risk*, defined as the probability of failure for these problems, forms the decision model of the two problems and is given in Table 4.15.

Referring back at Section 4.4.3, we can see that the decision model of Ellsberg's problem and our portfolio problem are isomorphic. The only difference is that the probabilities, prob(X = C2) and prob(X = C3) are switched. This however does not affect the

Table 4.15 Decision Model in terms of risk defined as probability of failure

	Criterion $\Longrightarrow$	Yield	Growth Rate	Safety
P1	Conservative 1	$2/3$	1	1
	Aggressive 1	1	$1/3+\lambda$	1
P2	Conservative 2	$2/3$	1	$1-\lambda$
	Aggressive 2	1	$1/3+\lambda$	$1-\lambda$

decision making in any way because λ is the uncertainty and we consider all discretized values of $\lambda \in [0, 2/3]$.

Discussion and Preferred Portfolio

Since the above decision model's expected payoff and risk are the same as presented by Ellsberg for his urn problem. Ellsberg's question for the reader, in terms of our portfolio problem, is which alternative they would choose for Problem 1 and Problem 2. According to his findings, the most common response is that people choose Alternative 1 for Problem 1 and Alternative 2 for Problem 2. In our case, with the given payoff, Conservative 1 will be chosen for Problem 1 and Aggressive 2 for Problem 2.

As we discussed earlier in Section 4.2.2, this is a paradox because the common approach violates Savage's Sure-Thing principle, i.e., since the payoff for safety is the same for both alternatives within a problem, this should not play a role in the decision making. Therefore if a person chooses the Conservative alternative for Problem 1 he should choose the Conservative alternative for Problem 2 as well.

The conventional portfolio selection problem converts the multicriteria problem to a single criteria problem before making a decision as discussed in Section 4.6.1. While our proposed methodology makes a decision based on all three criteria separately before finding the expected payoff of the portfolio as illustrated in Section 4.4, which clearly shows that by solving the portfolio problem keeping the multiple criteria separate until a decision is made, resolves the violation of the Sure-Thing principle.

4.8 Discussion

The problem of consistency and rationality in the field of decision making has been an issue since around the time von Neumann and Morgenstern (1947) presented the utility function approach to decision making. Scholars like Savage (Friedman and Savage, 1948) in the same time frame advocated a standard set of axiom-based procedures to make statistically rational and consistent decisions. At the same time other scholars such as Allais (1953) and Ellsberg (1961) challenged statistical rationality and consistency derived from Savage's axioms. They suggested that most decision makers violate these axioms when asked to make a choice, implying that decision making based on Savage's axioms may be rational and consistent, but need not be generally accepted and may lead to less attractive decisions. In this chapter, we have concentrated on Ellsberg's paradox because we believe Savage's discussion of Allais' Paradox in his book (Savage, 1954) is sufficient for the resolution of this paradox. In Allais' paradox the problem is presented with criteria already aggregated. We strongly discourage any aggregation because we have shown in Section 4.6 that decision makers violate Savage's axioms by looking at the aggregated information. The violation of Savage's axiom can be easily averted if we de-aggregate Allais' problem. Sarri (1995) in his paper on aggregation paradoxes discusses more problems raised by aggregating information in a decision framework.

Slovic and Tversky (1974) conducted an interesting experiment to test if any subject who completely understands Ellsberg's urn problem still violates the Sure-Thing principle. In the experiment, after the group of subjects have made a decision on the Ellsberg's problem, if they violated the Sure-Thing principle they were presented with a case for the Sure-Thing principle and given a chance to revise their choice. Also, if they had not violated the Sure-Thing principle, they were presented with a case against the Sure-Thing principle and were allowed to revise their choice. It was observed that the number of subjects that changed their choice the second time is almost the same for both groups suggesting that the argument for not violating the Sure-Thing principle is as strong as the argument against the principle. This study raises questions about how convincing the Sure-Thing principle is.

The decision making community seems to be divided in their level of belief when it comes to Savage's axioms. While some believe that Savage's axioms are important for make consistent choices, some suggest the there needs to be some sort of revision/review of these axioms to make use of them efficiently. For instance, Raiffa (1961) writes about the importance of Savage's axioms to make consistent decisions. He claims that he made the 'mistake' of violating the Sure-Thing principle when Ellsberg challenged him with the urn problem. But after a little consideration and with the help of Savage's axioms, he believes, he would have made a consistent and right choice. Hazen (1992) shows that decisions with multiple implications are consistent and does not violate Savage's axioms. Maher and Kashima (1997) also believe that if Ellsberg's problems are 'formulated in a different but equivalent way,' most of the subjects will cease to violate the Sure-Thing principle.

However, many have taken the issue of consistency in decision making raised by Ellsberg (1961) and tried to resolve the paradox by modifying Savage's axioms, especially the second postulate, the Sure-Thing principle. Shafer (1986) while being very defensive of Savage's work suggests that the second postulate is not very convincing. He agrees with Allais' example and suggests that there needs to be a revision of this second postulate. Machina and Schmeidler (1992) also suggest that by dropping the second postulate and the forth postulate(Weak Comparative Axiom) and by introducing a new postulate (Strong Comparative Axiom) they can resolve the paradox. They believe that it makes more sense for Savage's axioms to replace the second postulate by a new postulate. Fishburn (1983) also resolves the paradox with new set of axioms which does not include anything similar to the Sure-Thing principle. Bell (1982) resolves the Allais' paradox which also violates the Sure-Thing principle by introducing a 'regret' factor to the utility function.

Roberts (1963) comments on Ellsberg's urn problem and believes that one of the main reason for the violation of Savage's axiom by many subjects is the subjects' 'vagueness'. He writes, 'A person is vague about the probability assigned to a single trial if he cannot obtain for himself a clear answer as to what probability to assign to it. He is vague about the probability distribution if introspection fails to reveal clearly what the distribution is.' His 'vagueness' is Ellsberg's 'ambiguity' (Ellsberg, 1963) which in turn is our 'uncertainty'. In our work, as discussed in the previous sections, we do not believe that there needs to be any change in Savage's axioms. The performance-based decision making methodology equipped with our preference rule makes consistent choices resolving Ellsberg's paradox with no additional assumptions.

CHAPTER 5

INITIAL SET-UP FOR CREATING A CONSISTENT REAL ESTATE INVESTMENT TRUST PORTFOLIO

The concept of de-aggregated criteria in the last chapter resolving Ellsberg's Paradox (Ellsberg, 1961) lead us to explore the portfolio selection problem more seriously. We concluded Chapter 4 with concerns toward the conventional portfolio selection problems because they generally aggregate the criteria before the decision making step. This motivated us to create an alternative portfolio selection procedure with de-aggregated criteria. In this chapter, we present the initial set-up for this procedure. We selected portfolios of Real Estate Investment Trusts (REITs) traded in New York Stock Exchange for this real data-based study.

5.1 Introduction

Mullaney (1998) in his book defines Real Estate Investment Trusts (REITs) as "publicly-traded companies which invest in and manage portfolios of commercial properties or mortgage loans". REITs enable even small-time investors to own a piece of real estate by buying into the stocks of these companies. Even though REITs are traded in the stock market as a regular stock, they are more like bonds. Unlike regular stocks, REITs are not double taxed. That is, the companies that trade REITs are not taxed on profits. This special concession however requires the companies to invest at least 75% of total assets in real estate and derive at least 75% of their gross income as rents or interest mortgages of real properties and distribute 90% of the profits (95% before 2001) to the share-holders, who then pay the tax.

REITs are not new to the market, they have been around from the 1960s. However they did not become popular until the 1990s. In 2008, 127 REITs are traded on the New York Stock Exchange (NYSE) alone, with hundreds more traded globally. The National

Association of Real Estate Investment Trusts (NAREIT, www.reit.com) estimates $600 billion of commercial real estate assets to be owned by REITs which is almost 15 percent of the total institutionally owned commercial real estate in the United States with a market capitalization of $322 billion. According to Datamonitor (www.datamonitor.com) published by the Global Real Estate Investment Trusts in April 2008, the market capitalization of the global REITs industry is valued at $559.1 billion which is over a 20% compounded growth in the last five years. Thus REITs are high dividend yielding stocks that are more like bonds with a variable rate. REITs are not as volatile as regular stocks and they usually have a good return, mostly in form of regular quarterly dividends. REITs are considered one of the best forms of investment for long term, less risky investment options.

REITs are classified into three major categories: those which invest in mortgages, those which invest directly in real property, and those which do both.

Mortgage REITs. Datamonitor defines mortgage REITs as the market that "covers companies or trusts that service, originate, purchase and/or securitize residential and/or commercial mortgage loans, including trusts that invest in mortgage-backed securities and other mortgage related assets". Most of today's REITs loan money to the owners of existing properties. Their portfolio usually consists of first mortgage loans and/or investments in groups of mortgage loans. In general mortgage REITs generate a higher dividend yield than equity REITs but have a limited potential to increase in value.

Normally, a mortgage REITs' price increases only if the interest rate decreases. Alternatively if the interest rate increases the price decreases. Therefore the primary risk of investing in mortgage REITs is due to the changes in the interest rate. This fact is of interest to us because we are making our decision under uncertain interest rates.

Equity REITs. NAREIT reports that more than 90% of today's REITs are equity REITs. Equity REITs consist of a variety of companies that engage in acquisition, development, ownership, renovation, leasing, management and sale of real estate. Some of the common Equity REITs are Residential REITs, Retail REITs, Office and Industry

REITs, Hotel and Resort REITs and Health Care REITs. Most focus their investment on one type of property. Diversified REITs are REITs invested in different types of property. For instance, a REIT that has investments in both residential property as well as hotels.

Some equity trusts provide a high current dividend, but have limited growth potential. For example a triple net lease, where the tenant pays the rent, all taxes, insurance and maintenance expenses that was incured by use of the property to the landlord. Trusts which engage in a significant amount of new property development provide a lower current dividend, but have greater appreciation potential.

The share price of equity REITs will fluctuate with changing interest rates, but, because these REITs have the ability to increase their cash flow (by increasing rent, etc), they are less susceptible to interest rate risk!

Hybrid REITs provide a dual investment strategy. There are about seven Hybrid REITs traded in the US. Since a Hybrid is a mix of equity and mortgage REITs, They were created for diversification of risk. So when the interest rate changes, there is a slight balance between the two types of REITs. But since the diversified REITs address this issue more efficiently, Hybrid REITs are not as popular.

5.2 Outline of the Selection Procedure

As a first analysis of a portfolio of REITs, we use our methodology to pick the best REIT from among thirteen considered REITs. We analyze these REITs using two *criteria*; dividend yield and growth rate. Since we are interested in long term investments in REITs, and since the prime interest rate influences the price and hence the dividends and the growth rates, we believe that it is natural to choose the prime interest rate as the *uncertainty*. We considered daily data from Google Finance (http://finance.google.com) and Yahoo Finance (http://finance.yahoo.com) for our analysis and the data includes information on closing prices, quarterly dividends, growth rates and splits from August 1995 to June 2007. Since we are interested in the closing price without adjustment, we used Google Finance to download these daily closing prices and used Yahoo Finance to get the information on the dividends and the splits. We then combined this information with prices and dividends adjusted for splits. For instance if a REIT pays out a dividend of \$0.60 before a stock split date and the if the stock splits 3:2, then the adjustment factor of the dividend is 2/3. Therefore our adjusted dividend will be $\$0.60 \times (2/3) = \$0.40$ and the adjusted price is Previous Price $\times$ (2/3). All prices and dividends before a split are adjusted to reflect a split.

5.2.1 Uncertainty

As suggested earlier, our uncertainty for the REIT selection problem is prime interest rate set indirectly by the Federal Reserve. The Federal Reserve changes key interest rates when they deem fit by a multiple of $\pm 0.25\%$. These rate changes affect the prime interest rate. Sorting these interest rates provides an interval of rates modeling the uncertainty. Within the considered range of data the minimum for the rate is 4% and the maximum is 9.5%. Since the interest rate always changes by a multiple of $\pm 0.25\%$, a discrete collection of points $[4, 4.25, 4.5, ...9, 9.25, 9.5]$ models our uncertainty.

Sorting the prime interest rates, i.e., uncertainties, creates clusters of the data, a cluster for each value of the uncertainty. Figure 5.1 shows both the prime interest rate as they changed in tome and the sorted rates. The longer the horizontal line in the sorted graph, the larger the cluster. Since we work with real data our clusters are of different sizes. The more uniform the cluster size the better the analysis. We will discuss data clusters again in the next section.

5.2.2 Criteria and it's Mean and Variance

We require two criteria in our analysis. The two criteria we use to measure the quality of the REITs are daily *dividend yield* calculated using the regular quarterly dividends and the daily *growth rate*. Thus,

$$\text{Daily Dividend Yield (DDY)} = \frac{4 \times \text{Current Quarterly Dividend}}{\text{Current Price}}$$
$$\text{Daily Growth Rate (DGR)} = \frac{\text{(Current Price - Previous Price)}}{\text{Previous Price}}$$

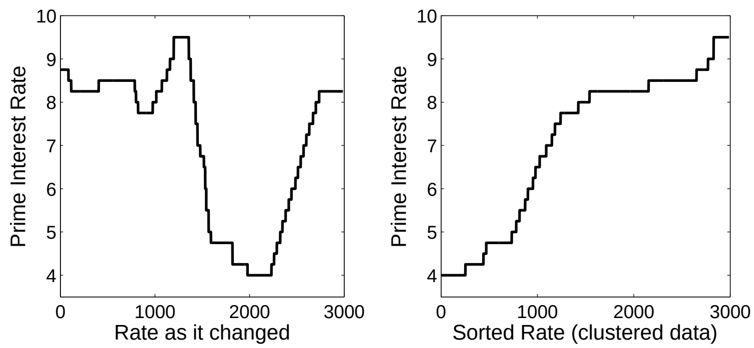

Figure 5.1 Prime Interest Rate as time series and as sorted clusters

Some REITs payout extra/special dividends in between the regular specified quarterly dividends, due to various reasons including selling a large property. In this preliminary analysis, we have assumed no such payouts were made, i.e., we have removed any such payout from the data. We need further research on how to incorporate such extra dividends.

Before we sort the data, we find the daily dividend yield and growth rate for each day from August 1995 to June 2007. Sorting the data will give us clusters of daily dividend yields and growth rates for each prime interest rate $r = [4, 4.25, 4.5, ...9, 9.25, 9.5]$. Since the clusters are of different sizes, we consider the mean ($\bar{\mu}_r$) and the variance ($\bar{\sigma}_r^2$) of the dividend yield and the growth rate within each cluster. Therefore, for each REIT, we find the mean and the variance of the dividend yield and the growth rate for discrete uncertainty value from $r = [4, 4.25, 4.5, ...9, 9.25, 9.5]$. Figure 5.2 shows how the sorted clustered data is distributed for a specific REIT.

Clearly, these sorted data do not follow a normal distribution. To make the analysis practical, using the Central Limit Theorem, we consider the integral of the mean and the variance to average out the errors from the estimates due to the limited data and to normalize the data. The integrated mean (μ) and variance (σ) are approximated using MATLAB

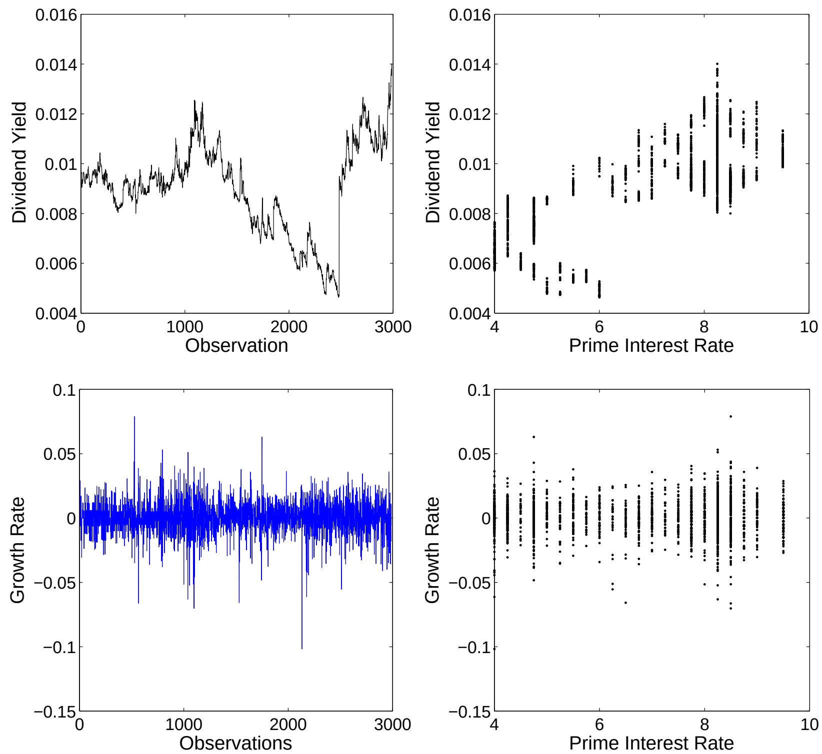

Figure 5.2 Dividend yield as announced and the sorted clusters of dividend yields and growth rates as announced and the sorted clusters of growth rate for a REIT

command *cumsum*. This integrated μ and σ define approximating normal distributions. Figure 5.3 shows an example of a REIT's $\bar{\mu}$ and the integrated μ.

Using the integrated data instead of the observed data reduce the error in μ and σ^2. Since, μ and σ^2 represent normal distributions, our methodology based on normal random distributions can be applied without any difficulties.

This completes the "massaging" part of our methodology. To summarize, given the daily closing prices and the regular quarterly dividends, we first calculate the daily dividend yield (DDY) and the growth rate (DGR). Then using the prime interest rate we cluster the data and find the mean dividend yield $\bar{\mu}_Y$ and the variance of the dividend yield $\bar{\sigma}_Y^2$ of each

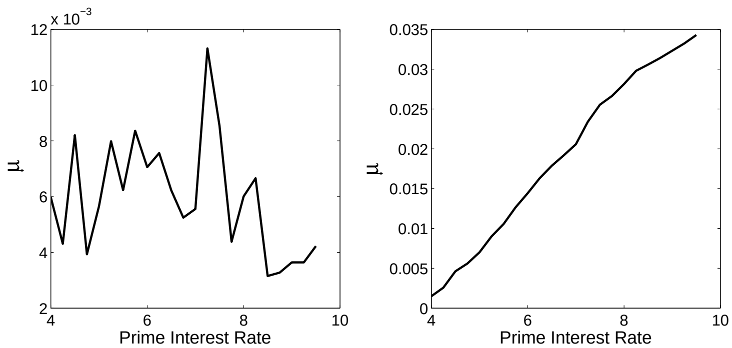

Figure 5.3 REIT's $\bar{\mu}$ and the integrated μ

cluster. Finally, to normalize the data, we integrate this mean and variance to get μ_Y and σ_Y^2 for each REIT. We do the same calculations for the growth rates. Now we have all the needed data to analyze and pick the best REIT. We consider thirteen alternative REITs which include, equity, mortgage and diversified REITs. Our objective it to define *risk* and make a choice based on the two criteria such that the choice is a REIT alternative with the optimal balance of expected payout and risk.

5.2.3 Preferred Choice

Since the integrated statistic defines normal distributions, we can use our two step preference rule 4.4.3. For a given REIT 'i', we find μ_Y^i, μ_{GR}^i, σ_Y^i and σ_{GR}^i as discussed in Section 5.2.2. Note that these are integrated means and square roots of the integrated variances. The decision variable of the two criteria problem is then defined in terms of the $\mu^i = (\mu_Y^i \ \mu_{GR}^i)'$ and $\sigma^i = (\sigma_Y^i \ \sigma_{GR}^i)'$ as $(\mu^i - \alpha\sigma^i)$ where α fixes the level of risk tolerance. For instance, if $\alpha = 1$ the decision maker's acceptable level of risk is 16% and if $\alpha = 1.645$ the decision maker's acceptable level of risk is 5%. The larger value of $(\mu^i - \alpha\sigma^i)$ is better.

After finding $(\mu^i - \alpha\sigma^i)$ for every REIT alternative, our preference rule to choose the best REIT consists of two steps. First, we check for a nondominated $(\mu^i - \alpha\sigma^i)$ from among the thirteen alternatives. If the decision variable of Alternative A dominates Alternative B, then Alternative A is preferred over Alternative B. That is, Alternative A is preferred to Alternative B if for all uncertainty, the decision variable of Alternative A is larger than or equal to the decision variable of Alternative B for all criteria and values of uncertainty and the decision variable of Alternative A is strictly larger than the decision variable of Alternative B for at least one value of the uncertainty. If either one or both of the above inequalities fails, in multicriteria decision making terminology, neither decision variable *dominates* the other, i.e., both decision variables are *nondominated*, yielding no clear Alternative which is preferred. Therefore, the first step is to check if there exists a single preferred REIT. If we find a unique REIT whose decision variable dominating every other alternative's decision variables, we have a preferred choice and we are done. If we can not find a unique preferred REIT, we proceed to the second step of our preference rule.

Since more than one REIT is nondominated, there is no clear solution for the problem. Therefore we define a second part of the preference rule to produce a preferred choice. To define this part of the preference rule we first define an *Ideal* (Ehrgott, 2005) Alternative. An Ideal Alternative is created so that all REITs would clearly be dominated by this Alternative. In other words, this is an infeasible (too good to be true) solution for the defined problem. Since we are concerned with maximizing $(\mu - \alpha\sigma)$ for the investment, a suitable ideal Alternative would be the upper envelope of $(\mu^i - \alpha\sigma^i)$ for all nondominated i. Then to make a consistent choice, our preference rule is to choose the alternative that is closest to the Ideal alternative, measured in terms of the L_2 norm.

For our illustration we choose thirteen REITs actively traded in the market. We selected them mostly at random with the exception that we picked only the REITs that has been traded from before 1995. Using 1995 as a starting date gave us enough data for different analyses that we had planned. Even though our methodology produces better results with longer runs of data, we do not require the longer runs to successfully implement our methodology. In this problem we consider the following REITs for the analysis;

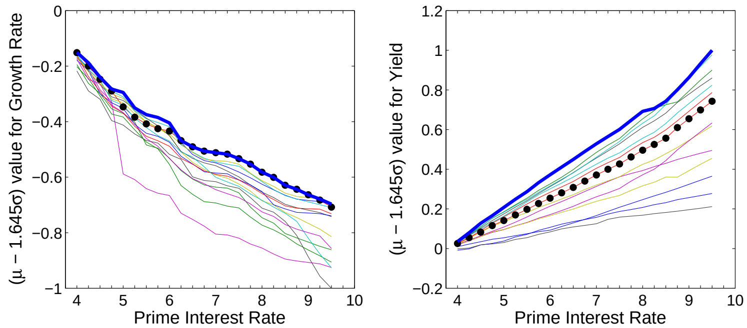

Figure 5.4 $(\mu - 1.645\sigma)$ of all the alternate REITs

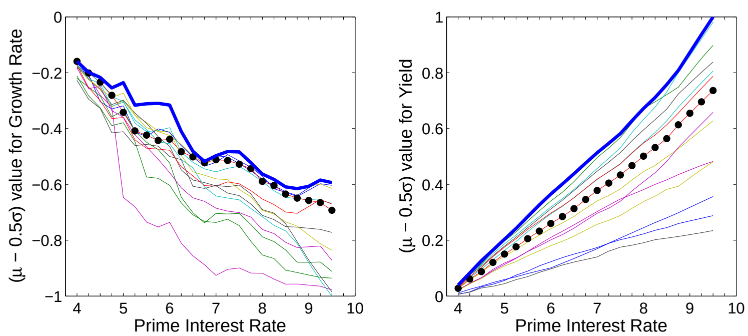

Figure 5.5 $(\mu - 0.5\sigma)$ of all the alternate REITs

- Industrial
  EastGroup Properties, Inc (EGP)
  First Industrial Realty Trust (FR)
  ProLogis (PLD)

- Office
  Highwoods Properties Inc (HIW)
  Hrpt Properties Trust (HRP)
  Mack Cali Realty Corporation (CLI)

Table 5.1 Values of distance between the Ideal Alternative and $(\mu - \alpha\sigma)$ of each REIT

REIT Alternatives	$(norm(*)^2)_{\alpha=1.645}$	$(norm(*)^2)_{\alpha=0.5}$
CBL	0.88	0.87
CLI	0.37	0.38
CLP	0.24	**0.28**
CUZ	0.84	1.05
EGP	0.54	0.54
FR	**0.20**	0.34
GGP	0.96	0.92
GRT	0.39	0.49
HIW	0.33	0.36
HRP	0.26	0.41
PCL	0.65	0.67
PLD	0.73	0.69
RYN	1.09	1.07

- Regional Malls
  CBL & Associates Properties, Inc (CBL)
  General Growth Properties Inc (GGP)
  Glimcher Realty Trust (GRT)
- Diversified
  Colonial Properties Trust (CLP)
  Cousins Properties Inc (CUZ)
- Timber
  Plum Creek Timber Company, Inc (PCL)
  Rayonier Inc (RYW)

Also, to make our analyses more interesting, we work with two levels of confidence by fixing the α value in $(\mu - \alpha\sigma)$ at $\alpha = 1.645$ to allow a 95% confidence level and then fixing $\alpha = 0.5$ to allow approximatly a 70% confidence level for the decision.

The first step looks for nondominated decision variables $(\mu - \alpha\sigma)$ for alternative REITs. In this set of REITs, no one company performed better than the others at all times for either risk level. That is the decision variables of all the thirteen companies were nondominated. Figure 5.4 and 5.5 shows the $(\mu - 1.645\sigma)$ and $(\mu - 0.5\sigma)$ values of growth rate and dividend yield for all REITs respectively. The dotted lines are the preferred choice based on the second step of the preference rule with the thick lines representing the Ideal alternative REIT. By the defined preference rule, since the norm distance to the Ideal REIT is the smallest for FR for the 5% risk case and CLP for the 30% risk case as seen in Table 5.1, FR is the preferred REIT with a strong 95% confidence level and CLP is the preferred REIT with a weaker 70% confidence level.

5.3 Decision Analysis

In Section 5.2, we illustrated the decision process with data collected up to June 2007. We claimed FR and CLP to be the preferred choices for the two α levels considered. In this section, we check the performance of FR and CLP from July 2007 to June 2008 with respect to the other twelve REITs respectively. However, unlike in Section 5.2 where the decision process used time series with respect to uncertainty, in this section all the series will be with respect to time. We use Yahoo Finance to collect the monthly closing price and the dividends.

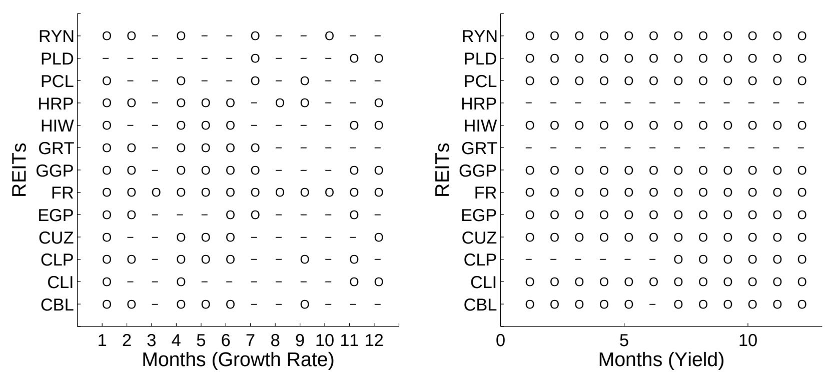

Figure 5.6 Growth Rate and Yield's performance of FR with respect to other REITs

In this analysis, we compare the growth rate and yield as defined in Section 5.2.2 for every month from July 2007 to June 2008. In Figure 5.6 the columns are months starting

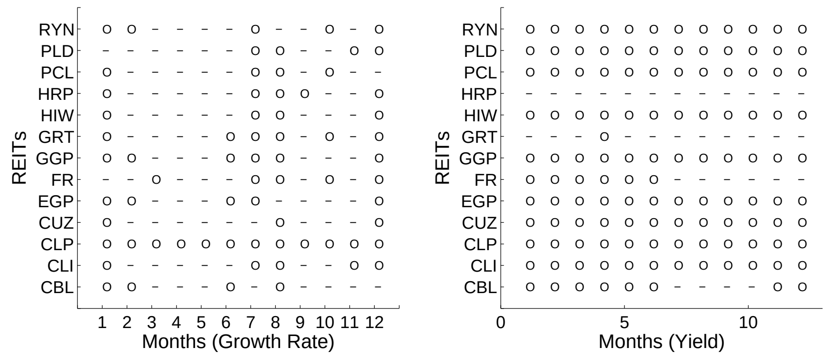

Figure 5.7 Growth Rate and Yield's performance of CLP with respect to other REITs

from July 2007 and the rows are the REITs. The graph on the left side represents the Growth Rate data and the graph on the right represents the Yield data. A circle represents better performance of FR. For instance, a circle in the grid (CBL,1) in the right side graph implies that FR's yield is better than CBL's yield in July 2007. Similarly, a dash in grid (CLP,6) in the right side graph implies that in December 2007, CLP's yield outperformed FR's yield. From this graph we can see that REITs HRP and GRT outperformed FR in every month that we considered and CLP performed better for half a year with respect to yield.

However, since we considered two criteria for decision making, we look at the second criteria also. The graph on the left side of Figure 5.6 has information on the growth rate of the REITs. Since REITs HRP and GRT outperformed FR in all the 12 months with respect to yield, it is of interest to see how they fared with respect to growth rate. As it can be seen in Figure 5.6, FR had better growth rate than HRP for 8 of the 12 months and FR and GGP both had better growth rates for six months. Also, CLP did better in only one of the six months in which it performed better with respect to yield.

Table 5.2 gives the number of months FR performed better than the other REIT. For example, if we consider the first row CBL, we see that FR's yield is better than CBL's

Table 5.2 Number of months FR performed better than REIT *

*	Both	Yield	Growth Rate
CBL	5	11	6
CLI	4	12	4
CLP	2	6	7
CUZ	5	12	5
EGP	5	12	5
FR	12	12	12
GGP	8	12	8
GRT	0	0	6
HIW	6	12	6
HRP	0	0	8
PCL	4	12	4
PLD	3	12	3
RYN	5	12	5

yield in 11 of the 12 months. Also, the growth rate of FR is better in 6 of the 12 months. The first column finds when both yield and growth rate outperforms the other REIT. FR performs better in both yield and growth rate in 5 of the 12 months. This also means that the one month that CBL performed better in Yield, FR did better in Growth Rate. Similar analysis for each REIT shows that FR is better than the other REITs except GRT. As it is the nature of portfolio selection problem, our preferred choice was better in most cases, but not all the time. Also, the norm value in Table 5.1 suggests that the second and third best option would be CLP and HRP which was also reflected in this analysis.

A similar analysis can be done for CLP. Figure 5.7 shows the months CLP performed better with respect to Growth Rate and Yield in the left and right side graphs respectively. In this case also, interestingly HRP and GRT out perform in every month with respect to Yield while CLP performs better in Growth Rate. Table 5.3 gives the number of months CLP performed better than the other REIT which is very similar to FR's performance.

Table 5.3 Number of months CLP performed better than REIT *

*	Both	Yield	Growth Rate
CBL	3	8	4
CLI	5	12	5
CLP	12	12	12
CUZ	3	12	3
EGP	5	12	5
FR	1	6	5
GGP	6	12	6
GRT	0	1	6
HIW	4	12	4
HRP	0	0	5
PCL	4	12	4
PLD	4	12	4
RYN	5	12	5

A survey of the closing prices for HRP and GRT reveals the reason as to why the yields of these REITs outperformed every other REIT. For both the REITs the price falls by almost 50% over the period of interest (July 2007 to June 2008) of the original price paid for the REIT in June 2007. Such steep fall in the closing price directly imply higher annual yield. Since high yielding REIT by itself does not imply better REIT, neither of these REITs were our preferred choice.

Since the Prime Interest Rate changed drastically in the period of interest (July 2007 to June 2008), we would like to see how our preferred REIT performed at each quarter of that year. The Prime Interest Rate was changed six times, starting at 8.25% in July 2007 and ending up at 5% in June 2008. There has never been over a 3% rate fall with-in a 12 month period in the last decade. In Table 5.4 we have listed the value of a dollar invested in July 2007 in the REIT FR and CLP. Value of the July 2007 dollar in September 2007, i.e., after three months of holding, is the sum of the current closing price of the REIT in

Table 5.4 Value of a dollar invested in the REIT FR and CLP for 3, 6, 9 and 12 months

Holding Time Months	FR Value of $1	CLP Value of $1
3	1.021	0.960
6	0.929	0.653
9	0.852	0.706
12	0.850	0.703

September 2007 and the total dividend yearned from July to September 2007 divided by the price at which the REIT was originally purchased in July 2007.

Because of the bad economy, all REITs considered lost money over the period of last one year. So we consider the minimum loss as opposed to the maximum profit. We can clearly see from Table 5.4 that FR has a better value for a dollar invested than CLP. This is interesting, because it follows the general pattern of the market. That is, less risk implies less profit or loss and similarly higher risk would imply higher profit or loss. Since the whole market is performing badly, both our preferred choices of different levels of risk produced losses. However, FR which is the preferred choice with 95% confidence level, i.e., with 5% risk produced a lesser loss than CLP which is the preferred choice with only 70% confidence level, i.e., with a 30% risk. Clearly, due to the bad market, the conservative choice FR with the tighter constraint has performed better than the less conservative choice CLB.

5.4 Concluding Remarks

In the Section 5.3 we established the efficiency of the methodology for the selected REITs. Can we claim that the methodology will always produce good portfolio selection choices for any given set of REITs or portfolio? We don't know yet. We believe that the de-aggregated criteria which is used to define risk has influenced this outcome. However, we do not have any proof to provide at this time to show that this procedure will always produce good choices. We will consider this issue in our future work also.

Choosing the best REIT to invest in is just the first step in choosing a best portfolio of REITs. Even though our objective is to find the best portfolio of REITs, due to time constraints, we stop with finding the best REIT. Producing the best portfolio of REITs will be a problem for the future. In a regular portfolio selection problem, the best portfolio is selected for a specified expected return. In our methodology we do not specify any expected return. This makes our problem very complex.

An easier problem to solve would be to consider a finite set of predefined portfolios among which a best portfolio is selected. Since we have already completed the problem of choosing the best REIT, we have information on the ordering of these REITs which could be somehow used efficiently to compile the finite set of portfolios. Once a finite set of portfolios have been created, choosing the best portfolio would be the same as choosing the best REIT as discussed in Section 5.2.

CHAPTER 6

SUMMARY

Decision problems have always been challenging. There are many different perspectives on how a 'good' decision is made and what a 'good' decision means. Multicriteria decision problems become more challenging and adding the element of surprise by introducing uncertainties to these problems takes the level of complexity to a new high. The concept of 'good' decision changes for multicriteria decision problems under uncertainty to *preferred* choice from among a set of *efficient* choices. That is, as the complexity increases, we do not want the decision model to provide a single optimal choice. Instead, we would like the decision methodology to provide a set of efficient choices. The decision maker can then use other preference rules to choose his preferred choice.

Therefore, in developing our decision methodology, we wanted to model the complex system and not the decision maker. For instance, if we consider the problem of storing nuclear waste, the decision methodology should provide the Department of Energy with a set of possible options with reduced risk instead of dictating to the department by providing only the 'good' choice.

For our decision making methodology, we modeled uncertainty as an interval or a set of discrete points. In our methodology, uncertainty is the independent variable and we model risk as a dependent variable. Random functions of uncertainty are used to model risk. In Chapter 2 we proposed our uncertainty and risk modeling paradigm and the resulting decision making methodology with the help of a Challenge Problem (Oberkampf et al., 2004). Here the decision maker is not given a single 'good' decision but rather a collection of choices from which the final decision is made.

In Chapter 2, we define and model uncertainty and risk efficiently and develop a performance based decision making methodology using the modeled uncertainty. This Chapter was accepted as a paper in the journal of *Reliability Engineering and System Safety* as an response to the Challenge Problem published by researchers from Sandia Laboratories in the same journal.

Decision problems involving large industries contain multiple components. For instance, decision problems for airline industry could be affected by many different components within the the management, and even more components on the flights themselves. All such components will have to be taken into account for an airline decision problem. While some of these components might be independent of the others, some interact. Consider rain conditions affecting component A in the flight and temperature affecting component B in the flight. Where the rain conditions and the temperature are two different uncertainties. These two uncertainties together could create some serious problems by freezing the wings. So understanding how the uncertainties interact and including these interactions in the decision model is very vital to any decision methodology.

We extended our methodology to handle the interacting components of the complex systems in Chapter 3. We introduced the operator representation and the separability of these representations. However, if the uncertainties' interactions produce a sum in their mathematical model, the representation of the sum was not always separable. We proposed a separable approximation that enabled us to handle sums of representations. In terms of separable approximations, we have developed a complete set of algebra for separable components which allowed us to build complex models from simpler components. Chapter 3 was published in the *International Journal of Pure and Applied Mathematics*.

Another important aspect of a decision problem are consistent and rational decisions. This is a very current issue discussed by many scholars as illustrated in Chapter 4. In this chapter, we tackled one of the most popular paradoxes in the decision analysis world and validated how our methodology with its two step preference rule and the multicriteria version of risk using the de-aggregated criteria to produce consistent and rational decisions. In the latter part of the chapter we promoted our belief, motivated by the resolution of Ellsberg's Paradox (Ellsberg, 1961), that multicriteria portfolio selection problems might need an alternative selection procedure to produce consistent and rational choices. We have submitted Chapter 4 as a paper on resolution of Ellsberg's Paradox with comments on de-aggregated criteria to be reviewed.

In Chapter 5 we presented the initial set-up of an applied portfolio selection problem dealing with Real Estate Investment Trusts (REITs) that are are currently traded in the New York Stock Exchange. We used real data to choose a best REIT from among a group of REITs. This was presented as an outline to a more general portfolio selection procedure. Even though this general procedure is not complete, we presented possible options to develop our procedure. Given the time constraint, this part is left as one of our future research problems.

Since each REIT by it self is a portfolio of many different investments of the REIT's company, we know that our portfolio selection procedure will provide consistent choices if we are presented with a pre-defined feasible set of portfolios. The future research would involve constructing these feasible sets of portfolios. A few suggestions were presented in Chapter 5.

A more difficult future direction would be to bound the errors of the separable approximation discussed or to provide a better approximation. Also, the problem of large numbers of uncertainties interacting within a component could be explored.

The preference rule presented through out this dissertation always balanced the expected return and risk in one way or the other because we strongly believe that the tradeoff between expected return and risk is vital for any decision making procedure. However, the efficiency and consistency of other preference rules can also be explored.

As we have expressed many times in this dissertation, decision making is part of every human endeavor including this work presented here. It is our hope that we have minimized the risk of confusing the readers and presented an easy-to-read and an easy-to-understand dissertation without any uncertainties.

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