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Portfolio Selection Under Various Risk Measures

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ABSTRACT

Portfolio selection has been a major area of study after Markowitz's ground-breaking paper. Risk quantification for portfolio selection is studied in the literature extensively and many risk measures have been proposed.

In this dissertation we study portfolio selection under various risk measures. After exploring important risk measures currently available we propose a new risk measure, Unequal Prioritized Downside Risk (UPDR). We illustrate the formulation of UPDR for portfolio selection as a mixed-integer program. We establish conditions under which UPDR can be formulated as a linear program.

We study single-period portfolio selection using two risk measures simultaneously. We propose four alternate models for single-period portfolio selection and elucidate their formulation. We discuss a procedure to obtain a set of solutions for the four models and illustrate this procedure with a numerical example. We study these models when chance constraint is included and also examine sensitivity analysis.

Multi-period portfolio selection strives to build an optimal portfolio by doing multiple investment decisions during the investment period. We introduce four alternate models for multi-period portfolio selection under a two-risk measure context. A procedure to solve these four models is outlined with a numerical illustration.

We also propose a new two-step process for portfolio selection. A sample of securities from the NYSE and BSE are taken and an empirical study is conducted to illustrate the two-step process for portfolio selection. Finally, we discuss conclusions based on the models we propose and directions for future research.

DEDICATION

For appa and amma, who are there always for me. Geetha Machi and Paapa, my wonderful cousins who always bring “smiles” to my heart.

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CHAPTER 1

INTRODUCTION

Humans have always looked for ways to enhance one's regular income through good investing. Once a sum of money has been decided as the available amount to invest, one would look for good opportunities to invest. There are two basic criteria one could consider before investing: the investment should fetch the maximum possible return and the return should be stable. The measure of this stability constitutes risk in an investment problem. A portfolio consists of some holding in all different avenues available to invest. Portfolio selection aims to find the optimal portfolio which would achieve the best result based on the criteria put forth. The first step in portfolio selection is to decide on a sample of securities which are the most attractive. The next step is to obtain information regarding the future behavior of these securities. There are two sources to get this information, first would be the past performances and second would be some prediction from security analysts (experts). Then the investor distributes the available investment amount on some of these securities which has the best possibility of maximum return and is stable. Prior to 1952, this type of portfolio selection was done ad hoc with no mathematical basis.

Markowitz (1952) in his ground-breaking paper on portfolio selection, proposed a clear-cut mathematical way to construct the best portfolio. He proposed the best portfolio, among all the portfolios, is the one which has the maximum expected return (E) and minimum variance (V). To obtain this, one has to view the portfolio selection problem as a bi-criteria problem and maximize expected return and minimize variance simultaneously. This method is called E-V method and the fundamental reason for its use is that the investor has to consider how each security co-moves along with other securities to get the most stable expected return from the portfolio. Markowitz (1952) showed that one would be greatly rewarded by utilizing this method to invest.

Markowitz (1959) elucidates more on the portfolio selection problem. He explains that the portfolio selection problem can be viewed in the following way: maximize expected return (mean) and minimize a risk measure. Markowitz suggested variance as a risk measure but also suggested many other measures like semivariance, expected loss, etc. which can be used if needed by the investor. The general portfolio selection problem suggested by Markowitz is called Mean-Risk where the mean (expected return) is maximized and risk is minimized. The general constraint for the problem is that the fractional amount invested in each of the securities should add up to one with the additional requirement that each of the fractions are non-negative. Markowitz was instrumental in developing great deal of interest in portfolio selection. There has been tremendous amount of research on portfolio selection with many varied perspectives.

1.1 Research Motivation and Goal

The motivation for this study was based on several avenues in the literature on portfolio selection. The first area of interest was risk quantification. Markowitz (1959) explained that risk quantification for portfolio selection is an open problem since it depends on the investor's needs and therefore no one risk measure may satisfy different needs of different investors. Many authors have developed different measures to quantify portfolio investment risk. A particular class of measures which quantify possibilities of return below expected return are called downside risk measures. Many authors have theorized that using downside risk measures is appropriate, since it minimizes any possibility of getting below expected return but do not inadvertently penalize any possibility of getting above expected return. Some of the downside risk measures which were developed include Semivariance, Value at risk, Conditional value at risk, Conditional Drawdown at risk, etc.

The need for a new risk measure was felt since no measure in the literature views the region below expected return, downside region, as partitioned regions and prioritizes them. A region closest to the expected return is better than one farther from the expected return, since losses in the closest region are smaller in magnitude when compared to losses in the farther region. So the closest region gets a smaller priority (risk) and the farther region gets a higher priority. In this manner, the downside region is assigned priorities and a new measure is developed to quantify investment risk with these priorities included. Given a set of scenarios for the future behavior of returns the optimal portfolio position may lead to losses in some of the given scenarios. The investor accepts this possibility but he needs a scheme to quantify these losses in a mathematical fashion. The region below the expected return is partitioned and expected value of losses in each of the region is found. The new risk measure value is found by multiplying each of these expected values by the pre-assigned priorities and adding them together. Since the priorities can be changed, this risk measure will help investors get a more sophisticated perspective than the measures already available. The remaining text is organized as follows.

Chapter 2 is an extensive literature review of different areas of portfolio selection. An overview of various areas of research in portfolio selection is given. Chapter 3 details a number of important risk measures available for portfolio selection. These measures are explained in great detail and the pros and cons of each are discussed. A numerical example on how to implement each of the measures for portfolio selection is illustrated. Chapter 4 explains the new risk measure and explains its formulation for portfolio selection as a mixed-integer program. Some of the properties this measure satisfies are proved. We show under certain assumptions portfolio selection using the new risk measure can be formulated as a linear program. A numerical example is shown on how to implement the new risk measure for portfolio selection. The pros and cons of using this measure are also discussed.

Many authors have suggested mean-risk model does not capture the essence of portfolio selection since investors are not sure to present one risk measure. Some authors have suggested the use of two risk measures for portfolio selection instead of one. The main research paper which started multi-risk measure discussion was the paper by Jean (1971) suggesting portfolio selection problem can be extended to a problem with three parameters. There have been many papers along this line of thought with authors approaching the problem with different two risk measures. To the best of our knowledge there is no paper in the literature where semivariance has been used in a two-risk measure context. We aim to address this by incorporating semivariance under a two-risk framework. The models we propose under two-risk measure perspective are dealt in Chapter 5.

The general portfolio selection is conducted on a single-period i.e., a portfolio is bought in the beginning and held throughout the holding period before being sold. In Chapter 5 we propose four alternate models for portfolio selection under a two-risk measure context. A step by step procedure is outlined to solve these models. The formulation of each of these models is explained and solved for a numerical example. We then included chance constraint to these models. Once an expected portfolio return is decided, the portfolio selection problem is solved so that investment risk is minimized. It is required that the portfolio return is greater than or equal to a pre-determined expected return. Since this constraint may not be satisfied, a probable error is allowed and the constraint is formulated as a chance constraint (Charnes and Cooper (1959)). Portfolio selection with chance constraint minimizes risk under the condition that the probability that a portfolio's rate of return is greater than the expected rate of return is no less than a confidence level. A procedure to solve the models with chance constraint is outlined and illustrated with a numerical example. These models give the investor hands-on advantage over the existing models in the literature and can be used as needed.

Portfolio selection requires an input on the future behavior of securities. Since this input is not fixed, one would be better prepared if there is a feedback mechanism scheme to see the sensitivity of the portfolio for changes in input. This type of analysis greatly enhances and gets one prepared with respect to confidence in optimal portfolio composition. Best and Grauer (1991b) conducted sensitivity analysis for the standard E-V portfolio selection problem using a general form of parametric quadratic programming. They showed how the portfolio composition is affected for changes in the mean return of securities and changes in the right hand side of the constraints. A return matrix representing different scenarios for the future return of securities serves as input for portfolio selection in this study. To the best of our knowledge there is no paper on sensitivity analysis for portfolio selection when input is a return matrix. The models we propose will be greatly enhanced if sensitivity analysis is included.

In Chapter 6, we conduct sensitivity analysis for the four alternate models we propose for single-period portfolio selection. An equivalent problem for each model is derived which can be solved to conduct sensitivity for changes in the return matrix. Numerical examples are illustrated to show sensitivity analysis of the four alternate models for changes in return matrix.

Multi-period portfolio selection aims to optimize one's investment by conducting multiple investment decisions during the holding period. In multi-period investment scenario the first step is to decide on the investment period and also the number of investing decisions allowed in this period. Portfolio selection aims to find a portfolio for each of these investing decisions so that at the end of the holding period an optimal result is obtained. Many papers have been published to show this type of investing leads to better portfolio selection. Papers by Merton (1969) and Samuelson (1969) among others initiated research on multi-period portfolio selection. To the best of our knowledge in the literature there is no model for multi-period portfolio selection using two risk measures. A two-risk multi-period model would be a great step forward and would be a good tool to have for the investor to get a better perspective before doing real time portfolio selection.

In Chapter 7, we propose four alternate models for multi-period portfolio selection. A step by step procedure is outlined to solve these models. The formulation of each of these models is explained and illustrated with a numerical example. The pros and cons of using multi-period portfolio selection are discussed.

Portfolio selection aims to find the best portfolio by using the given future behavior of returns. In this type of investing fundamental information about the securities such as sales, profit, growth etc. is not included. Fundamental information about the securities could be a great building block to generate the best portfolio. Greenblatt (2006) showed how to rank securities based on two factors and how it can be used to get a good selection of securities. In Chapter 8, a new two-step process to conduct portfolio selection is introduced. Once a collection of securities to invest is decided, in the first step some important fundamental factors of the securities are used to rank the securities in decreasing order of investment preference. In the second step a sample of ranked securities, best securities, is selected and one of the models we propose is used to conduct portfolio selection. This two-step process would make sure important information about the securities is not missed and is included in portfolio selection. An investor can be assured of making an informed decision by following this process. Numerical examples are illustrated showing portfolio selection under this process.

Chapter 9 concludes the dissertation with a general discussion on the ideas proposed and how it can be easily enhanced by future work.

CHAPTER 2

LITERATURE REVIEW

The seminal works by Markowitz (1952, 1959) were instrumental in developing research on portfolio selection. Markowitz suggests minimizing variance and maximizing mean of the portfolio, simultaneously, to get the best portfolio. Markowitz recommends generating an efficient frontier which represents the best set of solutions available to the investor for different expected returns and the corresponding variance for each of the returns. Once the efficient frontier is generated, the investor decides on a solution that best fits his needs. Markowitz (1959, 1991b) elucidates on risk quantification for portfolio selection and recommends using semivariance instead of variance. Semivariance is the expected value of the squared negative deviations of possible outcomes from the expected return. While variance penalizes any dispersion from the expected return, semivariance only penalizes dispersion below the expected return; hence very intuitive for an investor to view semivariance, rather than variance, as risk. Markowitz also suggests maximizing expected utility instead of expected return and compares several alternative measures of risk: standard deviation, semi-variance, expected value of loss, expected absolute deviation, probability of loss and maximum loss. Roy (1952) independently develops an equation relating portfolio variance of return to the variances of the return of the constituent securities. Roy advises choosing the single portfolio that maximizes $(\mu_p - d)/\sigma_p^2$ where μ_p and σ_p^2 are the mean and variance of the portfolio and d is a "disaster level" return the investor places a high priority of not falling below. The research by Markowitz and Roy generated enormous amount of interest in portfolio selection.

One of the main areas of research interest was risk quantification which many authors aim to resolve by proposing different measures of risk. Many authors have introduced new risk measures and theorized these measures would lead to better quantification of risk and therefore to a better portfolio. Bawa and Lindenberg (1977) and Bawa (1975) suggest lower partial moments of n^{th} order as a measure of risk. The general n^{th} order lower partial moment is derived as follows. Let R and R_τ be the actual portfolio return and the expected (target) return, respectively.

Let

$$(R - R_\tau)^- = \begin{cases} R - R_\tau, & \text{if } (R - R_\tau) \leq 0 \\ 0, & \text{if } (R - R_\tau) > 0 \end{cases}$$

Then LPM_{R_τ} is the expected value of $[(R - R_\tau)^-]^n$. When n is 2, the lower partial moment (LPM) would become the semivariance.

Fishburn (1977) introduces a new kind of risk measure where risk is defined by a probability-weighted function of deviations below a specified target return. Fishburn discuses the advantage of using this measure as compared to the traditional measure variance. Balzer (1990) and others propose that investment risk can be measured by the probability of the return falling below a specified risk benchmark. Value-at-Risk (VaR), another risk measure, describes the magnitude of likely losses a portfolio can be expected to suffer during "normal" market movements (Linsmeier and Pearson (2000)). In plain terms, VaR is a number above which we have only $(1 - \alpha)\%$ of losses and it represents what one can expect to lose with $\alpha\%$ probability, where α is the confidence level.

Konno and Yamazaki (1991) propose Absolute Deviation as a risk measure and show that this makes the portfolio selection problem a linear programming problem, hence easier to solve. Their model does not require the covariances of securities leading to lot of savings in computational time. They also have numerical experiments to show their model generates a portfolio similar to E-V portfolio which can be solved in a fraction of the time needed for E-V. Young (1998) proposes using minimum return as risk rather than variance. In particular, he suggests choosing the portfolio which minimizes the maximum loss over all past observations, for a given level of return. When the portfolio returns are normally distributed, the resulting minimax portfolios would almost be similar to the ones got by E-V.

Value-at-Risk measures the minimum loss corresponding to certain worst number of cases but does not quantify how bad these worst losses are. To overcome this drawback a new risk measure Conditional Value-at-Risk (CVaR) was established (Rockafellar and Ursayev, 2000). CVaR is a measure of the expected loss corresponding to a number of worst cases, depending on the chosen confidence level. Conditional Drawdown-at-Risk (CDaR) is a closely related risk measure to CVaR. A portfolio's drawdown on a sample path is the drop of the uncompounded portfolio value as compared to the maximal value attained in the previous moments on the sample path (Krokhmal et al. (2005)). CDaR, established by Chekhlov et al. (2000), is the expected value of $(1-\alpha)\%$ of the worst drawdowns.

Portfolio selection under lower partial risk (downside risk) is gaining wide popularity lately. The proponents of this method argue that an investor is content if he gets an unexpected gain but not so if he gets a loss. Semivariance suggested by Markowitz is one such risk measure. Value-at-risk(VaR), Conditional Value-at-risk(CVaR) and Conditional Drawdown-at-Risk are some of the other downside risk measures.

Linsmeier and Pearson (2000) explain Value-at-Risk and discuss three methods available to solve the same: historical simulation, the delta-normal method and Monte Carlo simulation. Finally they discuss the pros and cons of using each of the methods. Konno et al. (2002) review some of the downside risk measures: lower semi-variance, lower semi-absolute deviation, first order below target risk and CVaR. They propose a computational scheme to resolve difficulties in a solving a dense linear programming problem. They show that mean-CVaR and mean-lower semi-absolute deviation can control downside risk and are easier to solve mainly because they lead to a linear programming problem. De Girogi (2002) reviews the following risk measures: variance, VaR and expected shortfall. He also shows that under the assumption portfolio returns are normally distributed, the efficient frontiers obtained by taking Value-at-Risk or expected shortfall are subsets of the E-V efficient frontier. He generalizes this result for all risk measures having the form $y\sigma - \mu$ for some positive parameter y, where μ and σ are the mean and standard deviation of the portfolio, respectively. Artzner et al. (1999) present and justify a set of four desirable properties for measures of risk and call the measures satisfying these properties “coherent”.

Some authors propose using just one risk measure for portfolio selection may not be the best approach because investors find it difficult to select only one measure as the one which quantifies risk. Jean (1971) extends the two-parameter portfolio analysis to three or more parameters. He shows that using third or higher moments in the utility analysis makes more sense when the cash returns are non symmetric. Konno et al. (1993) propose a mean-variance-skewness model. Skewness plays an important role if the distribution of the rate of return of assets is asymmetric around the mean. Hence an investor would prefer a portfolio with larger third moment (skewness) if the mean and variance are the same. They propose a practical scheme to obtain a portfolio with a large third moment while placing constraints on mean and variance. The problem is a linear programming problem, so a large problem can be solved rapidly.

Wang (2000) proposes two new models. The first model is a two-stage approach using both E-V and E-VaR approaches in a priority order. In the first stage use one risk measure to get an efficient frontier. In the second stage use the other risk measure to optimize the result from the first stage. The second model is a general E-V-VaR approach using both variance (V) and VaR simultaneously. Roman et al. (2007) propose a model for portfolio optimization using three statistics: the expected value (E), the variance and CVaR at a specified confidence level. The problem is transformed into a single objective problem in which variance is minimized while constraints are imposed on the expected value and CVaR. The E-V and E-CVaR are particular efficient solutions of the new model. The new model also has efficient solutions which may be discarded by E-V and E-CVaR.

Krokhmal et al. (2002) propose a model with CVaR constraints. They show that multiple CVaR constraints with various confidence levels can be used to shape the profit/loss distribution. Vorst (2000) views the optimal portfolio selection problem when a VaR constraint is imposed. He shows that this provides a way to control risk in the optimal portfolio. Kibzun and Kuznetsov (2006) compare Value-at-Risk and Conditional Value-at-Risk. They establish some connection between the two measures and discuss some examples. Alexander et al. (2006) compare the portfolio selection problem under VaR and CVaR and show that solving E-CVaR is a better alternative to E-VaR. Pflug (2000) states some of the properties of Value-at-Risk and Conditional Value-at-Risk, compares them, and studies the structure of the portfolio optimization problem using both measures.

Balzer (1994) compares a set of risk measures for portfolio selection: standard deviation, probability-based measures, minimum shortfall, expected shortfall, moment-based measures and relative semivariance. He suggests some things to consider when dealing with risk quantification, first acknowledge that risk depends entirely on the investor's needs and second measure risk relative to one or more benchmarks. Among the measures he considers, whenever a single measure is needed, he suggests relative semivariance. Rajan and Gnanendran (1998) compare variance and semivariance for portfolio selection. For the data set consisting of 15 to 27 countries they found that the returns are not normally distributed and therefore they suggest using semivariance instead of variance.

Sensitivity analysis of portfolio selection was one area of interest to researchers. Best and Grauer (1991b) do a sensitivity analysis for the E-V portfolio problem using a general form of parametric quadratic programming. Their analysis allows one to examine how parametric changes in means or right hand side of constraints affects the mean, variance and the composition of the optimal portfolio. Best and Grauer (1991a) investigate the change in portfolio for changes in means of individual assets. When there is only a budget constraint they find the portfolio composition, mean and variance are all extremely sensitive to changes in the means of the assets. But when nonnegativity constraints are also imposed they find the portfolio composition is extremely sensitive while the portfolio's expected return and standard deviation are virtually unchanged. Gourieroux et al. (2000) analyze the sensitivity of Value at Risk (VaR) with respect to portfolio selection. They derive analytical expressions for the first and second derivatives and explain how they can be used to perform a local analysis of the VaR. They also give an empirical illustration of this type of analysis for a portfolio of French stocks.

The general portfolio selection problem aims to invest for a single-period but some authors theorized that investing over multiple periods would lead to a better portfolio. Tobin (1965) derives results for a class of multi-period portfolio problems. He presents a proof that the investor's optimal sequence of portfolios through time would be stationary implying a series of portfolios with constant proportionate holdings in each of the included assets. Samuelson (1969) formulates a many-period generalization corresponding to lifetime planning of consumption and investment decisions. He showed how the problem can be formulated and solved it using dynamic programming techniques. Merton (1969) solves the multi-period portfolio selection but unlike Samuelson, he considers the continuous time case.

Li et al. (1998) extend the safety-first approach introduced by Roy to multi-period portfolio selection. They derive an analytical solution that finds an optimal multi-period policy that seeks to minimize the probability that the terminal wealth is below a pre-selected level. Li and Ng (2000) derive an analytical optimal solution to the mean-variance multi-period portfolio selection. They propose an efficient algorithm for finding the optimal portfolio policy that maximizes a utility function of the expected value and the variance of the terminal wealth. Zhou and Li (2000) formulate continuous time portfolio selection as a bi-criteria optimization problem where the objective is to maximize expected terminal return and minimize the variance of the terminal wealth. By putting weights on the two criteria they make it a single objective stochastic control problem and find the efficient frontier in a closed form. Li et al. (2001) formulate mean-variance portfolio selection problems in continuous-time with a constraint that short-selling of stocks is prohibited. The problem is formulated as a stochastic optimal linear-quadratic (LQ) control problem by constructing a continuous function using Riccati functions. An example is illustrated showing all the results.

Lari-Lavassani and Li (2003) propose a very practical dynamic mean-semivariance portfolio optimization with an analytical solution by reducing the multi-dimensional problem to a one-dimensional optimization problem. A numerical comparison of the efficient frontier for mean-variance and mean-semivariance portfolio optimization problem is presented. Wei and Ye (2007) consider a multi-period portfolio selection model on the lines of mean-variance with bankruptcy constraints included in it. A solution scheme is developed to derive an optimal portfolio policy. This policy would help the investor not only achieve an optimal return but will also have a good risk control over bankruptcy.

Portfolio selection being a vast area of research has led many authors to summarize it in review papers. These papers help the researcher get an idea on some of the most interesting and important research done in portfolio selection. Markowitz (1991a) has written a good review paper on foundations of portfolio theory in which he outlines the basic foundations that led him and others to develop portfolio selection. Elton and Gruber (1997) review modern portfolio analysis and outline some areas for future research. They discuss the history of major concepts and suggest areas where further research is needed. Rubenstein (2002) traces the history of portfolio selection after the paper by Markowitz. He elucidates on the origins of using variance as a measure of risk and discusses various contributions made by other authors.

The E-V problem is a quadratic programming which requires all the covariances between the securities being considered. Estimating these covariances is extremely difficult when we are dealing with a large number of securities. Sharpe (1963, 1967) proposes index models to overcome this problem. These models presuppose that inter-relationships among security returns are due to common relationships with one or more indexes. Sharpe (1971) shows how the mean-variance portfolio selection problem can be converted to a linear programming problem by doing some suitable transformation. This helps capture the essence of the mean-variance problem but would be a linear program.

There are a multitude of papers in which empirical studies on portfolio selection is conducted. Papahristodoulou and Dotzauer (2004) consider portfolio selection under three risk measures: variance, maximin and absolute deviation. For a sample of stocks from forty eight months they find that maximin yields the highest return and risk, while variance leads to the lowest return and risk, and absolute deviation leads to a solution similar to variance.

When the expected returns were compared with the actual returns six months later, they find that maximin portfolios are the most robust. Cai et al. (2004) compare stocks from the Hong Kong stock exchange under two risk measures maximin and variance. Under their computational results they find that both risk measures perform similar but maximin is not sensitive to the data.

Byrne and Lee (2004) compare different portfolio compositions got by using different risk measures. In particular, they compare portfolio compositions got by using variance, semivariance, lower partial moment, minimax and absolute deviation. They show that none of these risk measures behave better in the domain of the other measures and therefore they are incomparable. Instead they compare the holdings in terms of composition and find variance and absolute deviation have similar composition. They also find that lower partial moment and maximin have very dissimilar holdings when compared to variance.

There are many papers discussing the distribution of portfolio returns. Tobin (1958) shows that if asset returns are normally distributed then variance is the proper measure of risk. Fama (1965) analyzes the distribution of the thirty securities that comprise the Dow Jones Industrial Average and finds that the security returns are "fat-tailed" and have kurtosis. This leads him to cast doubt on the assumption that returns are normally distributed. Officer (1972) makes a detailed examination of distribution of stock returns and shows that they are "fat-tailed" and so standard deviation is suggested as the well behaved measure of scale. Blattberg and Gonedes (1974) compare Student's t and symmetric-stable distributions for daily rates of returns and find that Student's t has a greater validity than the symmetric-stable model. Hagerman (1978) compares securities from AMEX and NYSE and find that a mixture of normal distributions and the Student's t distribution as the distributions which best fit the data. Aparicio and Estrada (2001) compare daily returns of securities from thirteen European markets and fit four alternative specifications for the data. They find overall support for Student's t and also find partial support for a mixture of two normal distributions; but for monthly returns a normal assumption could be plausible.

Some authors have introduced chance constraint to portfolio selection. Pyle and Turnovsky (1971) discuss the effects of changes in investable wealth on investment behavior when portfolio choices are subject to a chance constraint. Under alternative specifications of the chance constraint with respect to changes in wealth, they find risk aversion is increasing, decreasing or constant. Feiring and Lee (1996) construct mean variance portfolio selection with chance constraint. They assume that the returns are normally distributed and run the model for four confidence levels for a collection of stocks from the Hong Kong stock exchanges. Tang et al. (2001) formulate the chance constrained portfolio selection problem and establish its deterministic equivalent and suggest a new methodology to solve the problem. A numerical example is illustrated.

CHAPTER 3

DIFFERENT RISK MEASURES

Markowitz (1952) showed that the portfolio selection problem can be viewed as bi-criteria problem. His two criteria were expected return and a risk measure. Variance was used as a risk measure by Markowitz with the added suggestion that other risk measures can be used as needed by the investor. Investor's opinions on risk are subjective and vary from person to person, making quantification a difficult task. Thus risk analysis for portfolio selection is an interesting problem with many researchers approaching it in different perspectives. Researchers have worked on this area tremendously and developed various measures to quantify risk and justified why an investor has to use their measure.

This chapter details some of the important risk measures available to an investor and shows how to implement them for portfolio selection. Since the number of measures are many, we have limited our listing to the measures pertinent to our research.

Notations

Let us define some notations we will be using throughout this chapter. Throughout this dissertation, vectors and matrices are denoted using bold font while scalars, constants and random variables are denoted using normal font.

E	—	expected return of a portfolio
R	—	random observed return of a portfolio
V	—	variance of a portfolio
s	—	number of scenarios of information available about the future
n	—	number of securities
α	—	confidence level
$\boldsymbol{r}_{s\times n}$	—	return matrix for the securities
$\boldsymbol{X}_{n\times 1}$	—	the investment vector corresponding to n securities

$\boldsymbol{C}_{n\times n}$	—	the covariance matrix of the securities
$\boldsymbol{\mu}_{n\times 1}$	—	the mean return of the securities
E_0	—	specific expected return for the portfolio
$E_{\min}$	—	minimum possible return for the portfolio
$E_{\max}$	—	maximum possible return for the portfolio
SV	—	semivariance of the portfolio

3.1 Variance

An investor has a sum of money to invest in a certain number of securities and would like to invest so that he gets maximum possible return. The investor would also like his portfolio return to have minimum possible dispersion since it represents risk for him. Variance measures the dispersion from expected return and so can be used to quantify risk for an investor. Markowitz (1952) showed how to do portfolio selection using variance as a risk measure. The method he proposed is commonly known as the E-V method and represents maximizing expected return (E) and minimizing variance (V) simultaneously.

In his book, Markowitz (1959) elucidates the portfolio selection problem. There are two main concerns which quantify risk for the investor. First he wants portfolio return to be as close to the expected return as possible and second would not want to invest in a group of securities that are strongly positively correlated, since that would increase his overall loss if all securities perform badly at the same time. Hence it makes great practical sense to include variance in portfolio selection since it would address both the concerns of the investor. The E-V method will help to develop a portfolio with the same expected return and less risk than a portfolio constructed by ignoring the interactions among securities.

Estimates of expected return and covariance of securities under consideration are required to conduct portfolio selection under E-V method. The main constraint requires that the fractions of the sum invested in all the individual securities add up to one. We also require all the fractions to be non-negative. If we allow short sales, that is borrowed money to invest, then fractions can be negative. In this study we assume we do not have any short sales. Portfolio selection allowing short sales will not be markedly different. We also assume there is an return matrix readily available which represents different scenarios of the future returns.

The portfolio selection problem is a parametric quadratic programming—the parameter being the expected return (E). For any given E, the problem is solved and a corresponding minimum variance (V) is found. An efficient combination of E-V is one which satisfies the constraints and there is no other bigger return with equal or lesser variance or no other bigger or equal return with smaller variance. Portfolio expected return (E) versus portfolio variance (V) is plotted for all efficient combinations and is called an efficient frontier, because each point in the efficient frontier represents the best possible scenario for a particular E and V. The corresponding fractional solution $\boldsymbol{X}$ gives the investor the choice of a portfolio with a fixed E and corresponding V. Here $\boldsymbol{X}$ represents the fractions of the available amount that the investor has to invest in each of the securities.

<u>Standard formulation of portfolio selection problem:</u>

Suppose we have n securities in which we can invest. A portfolio is represented by a vector

$$\boldsymbol{X} = \begin{bmatrix} X_1 & X_2 & . & X_j & . & X_n \end{bmatrix}'$$

where X_j is the fraction invested in the j^{th} security. Assume $X_j \geq 0 \quad , \quad j = 1, 2, \ldots, n.$

The expected return and variance of the portfolio are given by $E = \boldsymbol{X}'\boldsymbol{\mu}$ and $V = \boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}$, respectively. The inputs to the Standard portfolio selection problem are $\boldsymbol{\mu}$ and $\boldsymbol{C}$. The problem is formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & V = \boldsymbol{X}'\boldsymbol{C}\boldsymbol{X} \\ \text{subject to} \quad & \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned} \tag{3.1}$$

The expected return of the portfolio (E_0) is what the investor strives for by investing in these securities and will lie between $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following closely related quadratic programming problem.

$$\begin{aligned} \text{Minimize} \quad & V = \boldsymbol{X}^{'}\boldsymbol{C}\boldsymbol{X} \\ \text{subject to} \quad & \boldsymbol{X}^{'}\boldsymbol{\mu} = E_{\min} \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned}$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem and is the maximum mean return among the mean returns of securities. The original parametric quadratic programming (3.1) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each fixed E_0 the problem is solved to get the minimum variance of the portfolio. Expected returns and variances thus obtained are plotted to get the efficient frontier. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get that particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. Once the problem is solved, the investor is given the set of solutions and he chooses a particular solution that fits his needs on portfolio expected return and variance.

A Numerical Example:

The following example from Markowitz (1991b) is used to illustrate all portfolio selection problems in this dissertation unless stated otherwise. In this example we can invest in the following nine securities: American Tobacco, American Tel.& Tel., United States Steel., General Motors, Atchsion, Topeka & Santa Fe., Coca-Cola, Borden, Firestone, and Sharon Steel. The return for any period is computed the following way:

price change = (closing price for current − previous period)

$$\text{return for current period} = \frac{\text{price change} + \text{dividends for current period}}{\text{closing price of the previous period}}$$

Historical returns for eighteen years from 1937 to 1954 for these securities is computed and listed in the following return matrix $\boldsymbol{r}$. These eighteen yearly returns are assumed as equally likely predictors of the future.

$$\boldsymbol{r} = \begin{bmatrix} -0.305 & -0.173 & -0.318 & -0.477 & -0.457 & -0.065 & -0.319 & -0.4 & -0.435 \\ 0.513 & 0.098 & 0.285 & 0.714 & 0.107 & 0.238 & 0.076 & 0.336 & 0.238 \\ 0.055 & 0.2 & -0.047 & 0.165 & -0.424 & -0.078 & 0.381 & -0.093 & -0.295 \\ -0.126 & 0.03 & 0.104 & -0.043 & -0.189 & -0.077 & -0.051 & -0.09 & -0.036 \\ -0.28 & -0.183 & -0.171 & -0.277 & 0.637 & -0.187 & 0.087 & -0.194 & -0.24 \\ -0.003 & 0.067 & -0.039 & 0.476 & 0.865 & 0.156 & 0.262 & 1.113 & 0.126 \\ 0.428 & 0.3 & 0.149 & 0.225 & 0.313 & 0.351 & 0.341 & 0.58 & 0.639 \\ 0.192 & 0.103 & 0.26 & 0.29 & 0.637 & 0.233 & 0.227 & 0.473 & 0.282 \\ 0.446 & 0.216 & 0.419 & 0.216 & 0.373 & 0.349 & 0.352 & 0.229 & 0.578 \\ -0.088 & -0.046 & -0.078 & -0.272 & -0.037 & -0.209 & 0.153 & -0.126 & 0.289 \\ -0.127 & -0.071 & 0.169 & 0.144 & 0.026 & 0.355 & -0.099 & 0.009 & 0.184 \\ -0.015 & 0.056 & -0.035 & 0.107 & 0.153 & -0.231 & 0.038 & 0 & 0.114 \\ 0.305 & 0.038 & 0.133 & 0.321 & 0.067 & 0.246 & 0.273 & 0.223 & -0.222 \\ -0.096 & 0.089 & 0.732 & 0.305 & 0.579 & -0.248 & 0.091 & 0.65 & 0.327 \\ 0.016 & 0.09 & 0.021 & 0.195 & 0.04 & -0.064 & 0.054 & -0.131 & 0.333 \\ 0.128 & 0.083 & 0.131 & 0.39 & 0.434 & 0.079 & 0.109 & 0.175 & 0.062 \\ -0.01 & 0.035 & 0.006 & -0.072 & -0.027 & 0.067 & 0.21 & -0.084 & -0.048 \\ 0.154 & 0.176 & 0.908 & 0.715 & 0.469 & 0.077 & 0.112 & 0.756 & 0.185 \end{bmatrix}$$

Since all the time periods are equally likely predictors of the future, the mean returns $\boldsymbol{\mu}$ for the securities are the average over these time periods and can be computed using the following formula:

$$\mu_j = \sum_{i=1}^{s} \frac{r_{ij}}{s} \quad j = 1, 2, \ldots, n$$

The mean returns for the nine securities are computed and given here:

$$\boldsymbol{\mu}' = \begin{bmatrix} 0.0659 \\ 0.0616 \\ 0.1461 \\ 0.1734 \\ 0.1981 \\ 0.0551 \\ 0.1276 \\ 0.1903 \\ 0.1156 \end{bmatrix}$$

The covariance matrix $\boldsymbol{C}$ for the securities can be found using the following formula:

$$\boldsymbol{C}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)]$$

The covariance matrix for the nine securities is computed and given here:

$$
\boldsymbol{C} = \begin{bmatrix}
0.0534 & 0.0215 & 0.0287 & 0.049 & 0.0162 & 0.0322 & 0.0243 & 0.04 & 0.0362 \\
0.0215 & 0.0147 & 0.0188 & 0.0244 & 0.008 & 0.01 & 0.0145 & 0.0254 & 0.0208 \\
0.0287 & 0.0188 & 0.0855 & 0.0626 & 0.0444 & 0.0133 & 0.0104 & 0.0686 & 0.042 \\
0.049 & 0.0244 & 0.0626 & 0.0955 & 0.0515 & 0.029 & 0.0208 & 0.09 & 0.0366 \\
0.0162 & 0.008 & 0.0444 & 0.0515 & 0.1279 & 0.0128 & 0.0209 & 0.1015 & 0.045 \\
0.0322 & 0.01 & 0.0133 & 0.029 & 0.0128 & 0.0413 & 0.0113 & 0.0296 & 0.0217 \\
0.0243 & 0.0145 & 0.0104 & 0.0208 & 0.0209 & 0.0113 & 0.0288 & 0.0291 & 0.0174 \\
0.04 & 0.0254 & 0.0686 & 0.09 & 0.1015 & 0.0296 & 0.0291 & 0.1467 & 0.0528 \\
0.0362 & 0.0208 & 0.042 & 0.0366 & 0.045 & 0.0217 & 0.0174 & 0.0528 & 0.0793
\end{bmatrix}
$$

The standard portfolio selection problem (3.1) is solved for this example. $E_{\min}$ and $E_{\max}$ are found to be 0.0668 and 0.1981 respectively. We take ten equidistant values in the range $E_{\min}$ to $E_{\max}$ for E_0 and solve the problem. The efficient frontier plot is given in Figure 3.1. The fractions to invest in the different securities to obtain the ten portfolio returns is given in Table 3.1.

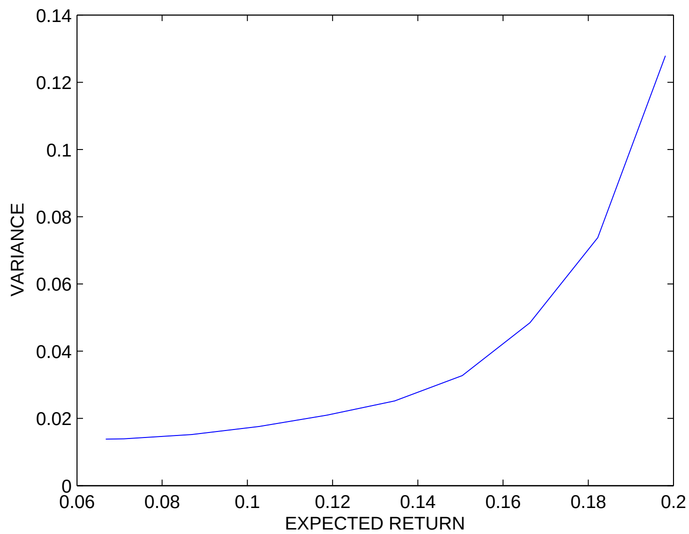

Figure 3.1 Efficient frontier of Mean-Variance.

Table 3.1 Expected return and variance along with corresponding fractions to invest for Mean-Variance.

Portfolio return	Fraction to invest in nine securities									Portfolio variance
0.0668	0	0.838	0	0	0.0437	0.1184	0	0	0	0.0138
0.071	0	0.8013	0	0	0.0605	0.1096	0.0286	0	0	0.0139
0.0869	0	0.6194	0	0	0.0917	0.0865	0.2024	0	0	0.0152
0.1028	0	0.4068	0.0582	0	0.0918	0.076	0.3673	0	0	0.0176
0.1187	0	0.1932	0.1183	0	0.0909	0.0658	0.5318	0	0	0.0209
0.1346	0	0	0.1751	0	0.0956	0.0417	0.6877	0	0	0.0252
0.1504	0	0	0.085	0.1354	0.2136	0	0.566	0	0	0.0327
0.1663	0	0	0	0.2801	0.3671	0	0.3527	0	0	0.0484
0.1822	0	0	0	0.3803	0.5274	0	0.0923	0	0	0.0738
0.1981	0	0	0	0	1	0	0	0	0	0.1279

The investor has two options available to handle standard portfolio selection problem based on the information available. If the investor knows what is the expected return he wants, he can solve the quadratic program (3.1) once and get the required result. Otherwise he can find a set of solutions using the procedure explained and decide on a particular solution which he feels satisfies his needs. Either way he will know what fraction of his initial amount he has to invest in each of the securities to get the optimal result.

3.2 Semivariance (SV)

Variance as a risk measure for portfolio selection is questioned by many researchers because variance penalizes both returns above and below expected return. But for an investor, risk is any possibility of getting below what he expects. Downside risk measures quantify possibilities of return below expected return. Markowitz (1959) suggested a downside risk measure known as semivariance (SV). Semivariance is the expected value of the squared negative deviations of possible outcomes from the expected return. The definition is derived as follows:

Let

$$(R-E)^{-} = \begin{cases} R-E, & \text{if } (R-E) \leq 0 \\ 0, & \text{if } (R-E) > 0 \end{cases}$$

Then Semivariance SV_E is the expected value of $[(R-E)^{-}]^2$.

A portfolio selection problem using semivariance (SV_E) tries to minimize underperformance and does not penalize over-performance with respect to expected return of the portfolio. To conduct portfolio selection using semivariance, it is not required to compute the covariance matrix; but the joint distribution of securities is needed. This risk measure tries to minimize the dispersion of portfolio return from the expected return but only when the former is below the latter. If all distribution returns are symmetric, or have the same degree of asymmetry, then semivariance and variance produces the same set of efficient portfolios (Markowitz (1959)). Given a return matrix $\boldsymbol{r}$ for the future behavior of securities, the E-SV_E problem can be formulated as follows (Markowitz et al. (1993)):

$$\begin{aligned}
\text{Minimize} \quad & \text{SV}_{E_0}(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[E_0 - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned} \tag{3.2}$$

When an observed return is less than the expected return, then the corresponding y variable will be strictly positive. Since we are minimizing the sum of y^2 terms, at optimality the corresponding y variable will take on the exact difference between the expected return and the observed return as needed. When an observed return is greater than the expected return, the y variable will be negative but since we require each of the variables to be non-negative, the variable will take on the value greater than or equal to zero. Since we are minimizing y^2 the optimal value of y will be zero, as needed. Thus the above constraints and objective function will exactly solve for semivariance. The expected return of the portfolio (E_0) will lie between $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following closely related problem.

$$\begin{aligned}
\text{Minimize} \quad & \text{SV}_{E_{\min}}(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[E_{\min} - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_{\min} \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem and is the maximum mean return among the individual returns of securities. The main problem (3.2) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each given E_0 a corresponding semivariance is found for the portfolio. Expected returns and semivariances thus obtained are plotted to get the efficient frontier graph. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get a particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. The investor can look at all the solutions and choose one which best fits his needs.

A Numerical Example:

The Mean-Semivariance problem is solved for the example given in the beginning of the chapter. $E_{\min}$ and $E_{\max}$ are found to be 0.0666 and 0.1981 respectively. The problem is solved for ten equidistant values in this range and efficient frontier along with the solutions are given in Figure 3.2 and Table 3.2 respectively.

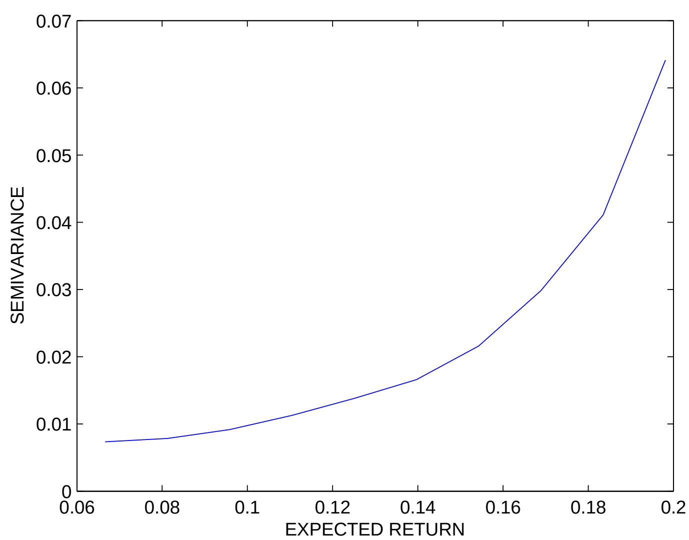

Figure 3.2 Efficient frontier of Mean-Semivariance.

Table 3.2 Expected return and semivariance along with corresponding fractions to invest for Mean-Semivariance.

Portfolio return	Fraction to invest in nine securities									Portfolio semivariance
0.0666	0	0.768	0	0	0.0343	0.1747	0.023	0	0	0.0073
0.0812	0	0.5815	0	0	0.0718	0.1796	0.1671	0	0	0.0078
0.0958	0	0.4015	0	0	0.1187	0.1878	0.292	0	0	0.0092
0.1105	0	0.2605	0.0962	0	0.1395	0.1594	0.3444	0	0	0.0113
0.1251	0	0.1632	0.1922	0	0.1428	0.0741	0.4277	0	0	0.0138
0.1397	0	0	0.3472	0	0.1157	0.0345	0.5026	0	0	0.0166
0.1543	0	0	0.2539	0.0296	0.2611	0	0.4199	0.0355	0	0.0216
0.1689	0	0	0.008	0.1276	0.3102	0	0.3403	0.2139	0	0.0298
0.1835	0	0	0	0.1757	0.3961	0	0.1107	0.3175	0	0.0411
0.1981	0	0	0	0	1	0	0	0	0	0.0641

Portfolio selection using semivariance penalizes only the cases that perform below the expected return. Since we use scenarios to represent the future returns, in general, we can expect half these scenarios to perform better than what was expected (E_0). Hence portfolio selection using semivariance requires more information regarding future behavior of returns than variance. But an investor can conduct portfolio selection using semivariance and can be assured that he is not penalizing any possibility of getting more than what he expects. For any investor, using semivariance should make practical sense as it minimizes downside (actual) risk.

3.3 Absolute Deviation

Konno and Yamazaki (1991) developed a new risk measure called absolute deviation. This measure quantifies the deviation from the expected return making its formulation a linear programming problem leading to tremendous savings in computational time. Konno and Yamazaki (1991) showed that we can solve a problem with more than thousand securities in a reasonable amount of time. The other advantage is that we do not have to compute the covariance matrix to do portfolio selection using absolute deviation. The authors also showed using numerical experiments the model generates a portfolio very similar to that of the standard portfolio selection problem (E-V method).

An optimal solution of a mean-variance portfolio selection may have many non-zero elements and could be as many as n the number of securities, since it is a quadratic program (Konno and Yamazaki (1991)). We would have to cut-off some of these fractions from our investment scheme, since investing in many securities at the same time would lead to huge transaction costs. On the other hand, Konno and Yamazaki (1991) showed that the optimal solution using mean-absolute deviation portfolio selection would have at most $2 \times s + 2$ positive components. So s can be used as a control variable to make sure that we do not have to invest in impractically huge number of securities.

Absolute deviation is the expected value of the absolute deviation of the expected return and the random observed return. It is derived as follows:

Let

$$|R - E| = \begin{cases} R - E, & \text{if } R > E \\ E - R, & \text{if } R \leq E \end{cases}$$

Then absolute deviation is given by expected value of $|R - E|$.

Given a return matrix $\boldsymbol{r}$ regarding the future behavior of returns, the portfolio selection problem using E versus absolute deviation is formulated as the following linear program.

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{s}\sum_{i=1}^{s} y_i \\ \text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[(r_{ij}X_j) - E_0] : i = 1, 2, \ldots, s \\ & y_i \geq \sum_{j=1}^{n}[E_0 - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\ & y_i \geq 0 : i = 1, 2, \ldots, s \\ & \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned} \tag{3.3}$$

When an observed return is greater/less than the expected return, then the first constraint will imply y variable is strictly positive/negative and the second constraint will imply otherwise. Since we require the y variable to be non-negative and are minimizing the sum of y terms, at optimality the corresponding y variable will take on the exact difference between the expected return and the observed return as needed. Thus the above constraints and objective function will exactly solve for absolute deviation. The above linear programming is solved for different values of E_0 in the range $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following linear programming problem.

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\sum_{i=1}^{s} y_i \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[(r_{ij}X_j) - E_{\min}] : i = 1, 2, \ldots, s \\
& y_i \geq \sum_{j=1}^{n}[E_{\min} - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_{\min} \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem which is the maximum mean return among the mean returns of securities. The original parametric linear programming (3.3) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each given E_0 a corresponding absolute deviation is found for the portfolio. Expected returns and absolute deviations thus obtained are plotted to get the efficient frontier graph. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get that particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. Once the investor looks at the range of solutions available to him, he chooses one which best fits his needs.

A Numerical Example:

The Mean-Absolute deviation portfolio selection is solved for the given example. $E_{\min}$ and $E_{\max}$ are found to be 0.0641 and 0.1981 respectively. The problem is solved for ten equidistant values in this range and efficient frontier along with the solutions are given in Figure 3.3 and Table 3.3 respectively.

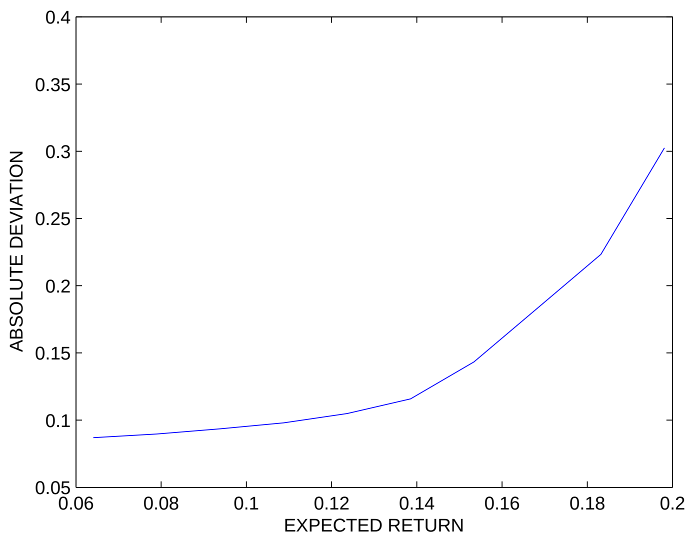

Figure 3.3 Efficient frontier of Mean-Absolute deviation.

Table 3.3 Expected return and absolute deviation along with corresponding fractions to invest for Mean-Absolute deviation.

Portfolio return	Fraction to invest in nine securities									Absolute deviation
0.0641	0	0.8806	0	0	0	0.0743	0.0451	0	0	0.087
0.079	0	0.6871	0	0	0	0.0421	0.2523	0	0.0186	0.0897
0.0938	0	0.436	0	0	0.01	0.0596	0.3822	0	0.1123	0.0936
0.1087	0	0.3066	0	0	0.0467	0	0.4876	0	0.159	0.098
0.1236	0	0.0558	0	0	0.102	0.0706	0.5752	0	0.1963	0.1049
0.1385	0	0	0	0.0002	0.1795	0	0.6754	0	0.1449	0.1159
0.1534	0	0	0	0.1871	0.2446	0	0.5684	0	0	0.1433
0.1683	0	0	0	0.3275	0.3646	0	0.308	0	0	0.1833
0.1832	0	0	0	0.4678	0.4846	0	0.0476	0	0	0.2233
0.1981	0	0	0	0	1	0	0	0	0	0.3025

Portfolio selection using absolute deviation is an easy linear program and does not require us to compute covariance matrix. The main drawback is that it does not address the issue of penalizing the upside, treating deviation above and below the expected return as equally bad. An investor can use absolute deviation for portfolio selection when he is dealing with numerous securities since he will be dealing with a linear program and can use s as a control variable to limit how many securities he has to buy.

3.4 Value at risk (VaR)

Value at Risk is one of the very popular risk measures widely used in the financial industry. Value at risk (VaR) describes the magnitude of likely losses a portfolio can be expected to suffer during "normal" market movements (Linsmeier and Pearson (2000)). In plain terms, VaR is a number above which we have only $(1-\alpha)\%$ of losses and it represents what one can expect to lose with $\alpha\%$ probability, where α is the confidence level.

Someone holding a portfolio may report that their portfolio has a one-year VaR of $1 million at the 99% confidence level. This means under normal trading conditions the holder can expect with a probability of 99% that their portfolio value would not decrease more than $1 million during one-year. It also means there is a 1% chance that the portfolio value will decrease by $1 million or more during one-year.

There are three ways to compute VaR: variance covariance, historical returns and Monte Carlo simulation. The variance covariance method uses information on the volatility and correlation of stocks to compute the VaR of a portfolio. The Monte Carlo simulation is done by generating random scenarios for the future returns and computing VaR for these varied scenarios.

In our research we will illustrate how to compute VaR using historical returns or any future projected returns of securities. Let us assume we have scenarios of information available to us regarding the future behavior of the returns. Based on this information VaR would be the loss that will be exceeded only by $(1-\alpha)\%$ of the cases. VaR is derived for losses adjusted for returns using the following approach. Usually losses are in monetary terms, but we list losses in terms of returns (percentage).

Let V_t -market value at time t

V_{t+h}- market value at time $t+h$

Define Loss $L = \frac{V_t - V_{t+h}}{V_t} = -\boldsymbol{r}\boldsymbol{X}$

Find VaR_α by the requirement $P(L > \text{VaR}_\alpha) = 1 - \alpha$

The following non-convex integer program would exactly solve for VaR.

$$
\begin{aligned}
\text{Minimize} \quad & \text{VaR} = M_{\lfloor\lfloor(1-\alpha)s\rfloor:s\rfloor}(-\boldsymbol{r}\boldsymbol{X}) \\
\text{subject to} \quad & \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{3.4}
$$

Here the function $M_{\lfloor k:N \rfloor}$ denotes k^{th} largest among the N numbers.

If the portfolio returns are assumed to follow normal distribution, then VaR formulation is a nonlinear programming problem and can be formulated as follows. Suppose there are n securities in which we can invest and their mean return is given by ξ a random variable. Let us suppose that the mean return of the securities ξ has a normal distribution $N(\boldsymbol{\mu}, \boldsymbol{C})$, where $\boldsymbol{C}$ is positive definite symmetric matrix. Then we can use some of the properties of normal distribution to formulate VaR.

Since $\xi \sim N(\boldsymbol{\mu}, \boldsymbol{C})$, then $-\boldsymbol{X}'\xi = \sum_{i=1}^{n} -X_i\xi_i \sim N(\text{E}(\boldsymbol{X}), \sigma(\boldsymbol{X}))$. Here $\text{E}(\boldsymbol{X}) = -\boldsymbol{X}'\boldsymbol{\mu}$ and $\sigma(\boldsymbol{X}) = \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}$. Figure 3.4 illustrates VaR computation for a given confidence level α.

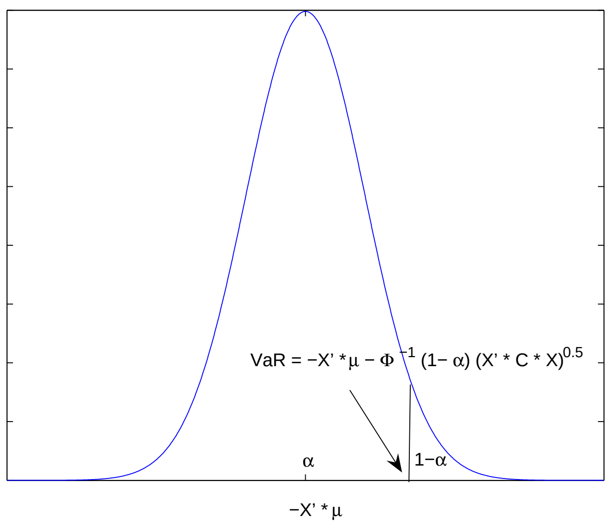

Figure 3.4 Value at risk computation.

The following problem can be solved to compute VaR.

$$\begin{aligned}
\text{Minimize} \quad & -(\boldsymbol{X}'\boldsymbol{\mu}) - \Phi^{-1}(1-\alpha)\sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}} \\
\text{subject to} \quad & \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned} \tag{3.5}$$

Here $\Phi(.)$ is the standard normal value. The above problem is solved for different values of E_0 in the range $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following closely related problem.

$$\begin{aligned} \text{Minimize} \quad & -(\boldsymbol{X}^{'}\boldsymbol{\mu}) - \Phi^{-1}(1-\alpha)\sqrt{\boldsymbol{X}^{'}\boldsymbol{C}\boldsymbol{X}} \\ \text{subject to} \quad & \boldsymbol{X}^{'}\boldsymbol{\mu} = E_{\min} \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned}$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem which is the maximum mean return among the mean returns of securities. The original parametric nonlinear problem (3.5) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each given E_0 a corresponding VaR is found for the portfolio. Expected returns and VaRs' thus obtained are plotted to get the efficient frontier graph. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get that particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. Once the investor looks at the range of solutions available to him, he chooses one which best fits his needs.

A Numerical Example:

The Mean-VaR portfolio selection is solved for the given example assuming the portfolio returns are normally distributed with a confidence level of 95%. $E_{\min}$ and $E_{\max}$ are found to be 0.0942 and 0.1981 respectively. The problem is solved for ten equidistant values in this range and efficient frontier along with the solutions are given in Figure 3.5 and Table 3.4 respectively.

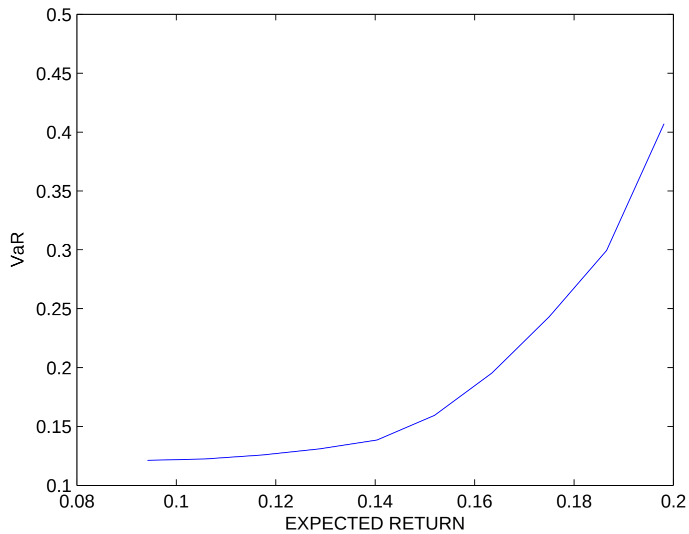

Figure 3.5 Efficient frontier of Mean-Value at risk.

Table 3.4 Expected return and VaR along with corresponding fractions to invest for Mean-VaR.

Portfolio return	Fraction to invest in nine securities									VaR
0.0942	0	0.5219	0.026	0	0.0922	0.0818	0.2781	0	0	0.1212
0.1057	0	0.3674	0.0691	0	0.0914	0.0736	0.3985	0	0	0.1224
0.1173	0	0.2116	0.1132	0	0.0908	0.0669	0.5176	0	0	0.1258
0.1288	0	0.0568	0.1569	0	0.0904	0.0594	0.6365	0	0	0.1309
0.1404	0	0	0.1617	0.0319	0.118	0	0.6884	0	0	0.1385
0.1519	0	0	0.0758	0.1482	0.2286	0	0.5474	0	0	0.1593
0.1635	0	0	0	0.262	0.3382	0	0.3997	0	0	0.1953
0.175	0	0	0	0.3349	0.4547	0	0.2104	0	0	0.2432
0.1866	0	0	0	0.4077	0.5711	0	0.0212	0	0	0.2995
0.1981	0	0	0	0	1	0	0	0	0	0.4072

VaR for portfolio selection is apt, the main reason being it does not penalize the upside with respect to expected return. It has certain drawbacks, we require the portfolio returns to be normally distributed which may not be an appropriate assumption. If we do not use this assumption, computing VaR is a very complicated integer program or we would have to use extensive Monte Carlo simulation; either of these require a lot of computing. For these reasons, the investor should use VaR only when he knows that his portfolio returns follow normal distribution.

3.5 Conditional Value at risk (CVaR)

Rockafellar and Ursayev (2000) established a new risk measure called Conditional value at risk (CVaR). Value at risk measures the minimum loss corresponding to certain worst number of cases but it does not quantify how bad these worst losses are. An investor may need to know the magnitude of these worst losses to discern whether there are possibilities of losing huge sums of money. CVaR quantifies this magnitude and is a measure of the expected loss corresponding to a number of worst cases, depending on the chosen confidence level. Using CVaR makes the portfolio selection problem linear and when we solve it a minimum VaR is found since CVaR $\geq$ VaR (Rockafellar and Ursayev (2000)). CVaR is derived as follows:

Let $f(X, \xi)$ be the loss function of the portfolio. Usually losses are in monetary terms, but we list losses in terms of returns (percentage). Given a confidence level α, CVaR is the expected value of all $(1-\alpha)\%$ losses and can be found using the following function:

$$\text{CVaR}_\alpha(X, \eta) = \eta + (1-\alpha)^{-1} \int_{\xi \epsilon R^n} [f(X, \xi) - \eta]^+ p(\xi) d\xi$$

η — VaR

ξ — random variable

$z^+ = \max\{z, 0\}$

The Mean-CVaR portfolio selection can be formulated as a linear programming problem when scenarios of future returns is available. Since $\boldsymbol{r}$ is the return matrix, $\boldsymbol{rX}$

is the portfolio returns. Therefore the losses will be $-\boldsymbol{rX}$. The problem tries to find the expected value of all the worst (1-α)% losses. The following linear program would solve the problem:

$$
\begin{aligned}
\text{Minimize} \quad & \eta + \frac{1}{(1-\alpha)s}\sum_{i=1}^{s}(y_i) \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[(-r_{ij}X_j) - \eta] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{3.6}
$$

If a loss scenario is greater than VaR (η), then the y variable will take on the exact difference between the loss scenario and VaR (η). If a loss scenario is less than VaR, then the y variable will take on the value zero. Since the distribution of y_i represents the tail distribution of losses exceeding VaR, the mean can be found by computing the weighted sum divided by $1 - \alpha$. Then CVaR is this mean added to VaR which the objective function computes as required. The above linear program is solved for different values of E_0 in the range $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following closely related problem.

$$
\begin{aligned}
\text{Minimize} \quad & \eta + \frac{1}{(1-\alpha)s}\sum_{i=1}^{s}(y_i) \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[(-r_{ij}X_j) - \eta] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_{\min} \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem which is the maximum mean return among the mean returns of securities. The original parametric linear program (3.6) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each given E_0 a corresponding CVaR is found for the portfolio. Expected returns and CVaRs' thus obtained are plotted to get the efficient frontier graph. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get that particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. Once the investor looks at the range of solutions available to him, he chooses one which best fits his needs.

A Numerical Example:

The Mean-CVaR portfolio selection is solved for the given example. The confidence level is taken to be 95%. $E_{\min}$ and $E_{\max}$ are found to be 0.0692 and 0.1981 respectively. The problem is solved for ten equidistant values in this range and efficient frontier along with the solutions are given in Figure 3.6 and Table 3.5 respectively.

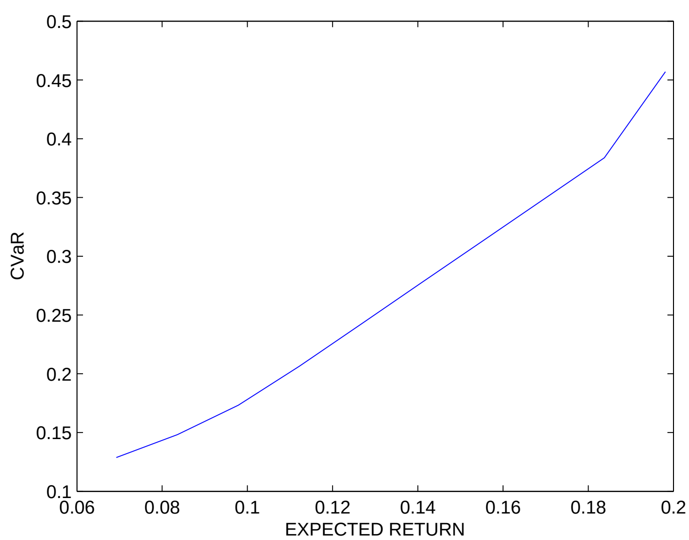

Figure 3.6 Efficient frontier of Mean-Conditional Value at risk.

Table 3.5 Expected return and CVaR along with corresponding fractions to invest for Mean-CVaR.

Portfolio return	Fraction to invest in nine securities									CVaR
0.0692	0	0.2074	0	0	0.0321	0.6474	0.1131	0	0	0.1287
0.0836	0	0	0	0	0.1422	0.752	0.0988	0.007	0	0.1482
0.0979	0	0	0	0	0.0101	0.6718	0.027	0.2912	0	0.1733
0.1122	0	0	0	0	0	0.5778	0	0.4222	0	0.2064
0.1265	0	0	0	0	0	0.4719	0	0.5281	0	0.2419
0.1408	0	0	0	0	0	0.366	0	0.634	0	0.2774
0.1552	0	0	0	0	0	0.2602	0	0.7398	0	0.3128
0.1695	0	0	0	0	0	0.1543	0	0.8457	0	0.3483
0.1838	0	0	0	0	0	0.0484	0	0.9516	0	0.3838
0.1981	0	0	0	0	1	0	0	0	0	0.457

Portfolio selection using CVaR is an easy linear program and does not require assumption of normality for the portfolio returns. It is a downside risk measure, so is better suited than absolute deviation. CVaR can be used instead of VaR by investors, since the solutions are closely related and can be solved faster.

3.6 Conditional Drawdown-at-Risk (CDaR)

Conditional Drawdown-at-Risk (CDaR) is a closely related risk measure to CVaR. CDaR was established by Chekhlov et al. (2000) who showed how to implement it for portfolio selection. Portfolio's drawdown on a sample path is the drop of the uncompounded portfolio value as compared to the maximal value attained in the previous moments on the sample path (Krokhmal et al. (2005)).

Suppose we start observing a portfolio in January and record its uncompounded portfolio value every month. Let the portfolio value be \$100 in January and becomes \$120 in February. Then the portfolio drawdown for February is \$0. Suppose the portfolio value in march drops to \$80, then the drawdown for march is \$40(in absolute terms) or 33.3% in percentage terms. Mathematically the drawdown function is given as follows:

$$f(X,j) = \max_{1\le k\le j}\left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{k} r_{ti})X_i\right\} - \left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{j} r_{ti})X_i\right\} \quad (3.7)$$

CDaR is the expected value of $(1-\alpha)\%$ of the worst drawdowns and can be found using the following function:

$\mathrm{CDaR}_\alpha(X,\eta) = \eta + (1-\alpha)^{-1}\sum_{j=1}^{s}[f(X,j)-\eta]^+$

η — threshold exceeded by $(1-\alpha)s$ drawdowns

ξ — random variable

$z^+ = \max\{z,0\}$

The Mean-CDaR portfolio selection can be formulated as a linear programming problem when scenarios of future returns is available. We assume some historical sample path of returns is available and given by $\boldsymbol{r}$. The formulation of CDaR follows directly from CVaR formulation with one difference—the drawdown function is used to quantify losses. The problem tries to find the expected value of the worst $(1-\alpha)\%$ drawdowns. The following linear program would solve the problem:

$$\begin{aligned}
\text{Minimize} \quad & \eta + \frac{1}{(1-\alpha)s}\sum_{j=1}^{s}(y_j) \\
\text{subject to} \quad & y_j \geq \left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{k} r_{ti})X_i\right\} - \left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{j} r_{ti})X_i\right\} - \eta \\
& k = 1, 2, \ldots, j \\
& y_j \geq 0 \\
& j = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned} \tag{3.8}$$

The above linear program is solved for different values of E_0 in the range $E_{\min}$ and $E_{\max}$. $E_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following closely related problem.

$$\begin{aligned}
\text{Minimize} \quad & \eta + \frac{1}{(1-\alpha)s}\sum_{j=1}^{s}(y_j) \\
\text{subject to} \quad & y_j \geq \left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{k} r_{ti})X_i\right\} - \left\{\sum_{i=1}^{n}(1+\sum_{t=1}^{j} r_{ti})X_i\right\} - \eta \\
& k = 1, 2, \ldots, j \\
& y_j \geq 0 \\
& j = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} = E_{\min} \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

$E_{\max}$ represents the maximum possible portfolio return for the given problem which is the maximum mean return among the mean returns of securities. The original parametric linear program (3.8) is solved for different values of E_0 in the range $E_{\min}$ to $E_{\max}$. For each given E_0 a corresponding CDaR is found for the portfolio. Expected returns and CDaRs' thus obtained are plotted to get the efficient frontier graph. The corresponding $\boldsymbol{X}$ values gives the fractions to invest in each security to get that particular E_0 and represents a particular portfolio. Any linear interpolation of adjacent pairs of portfolios is efficient. Once the investor looks at the range of solutions available to him, he chooses one which best fits his needs.

<u>A Numerical Example:</u>

The Mean-CDaR portfolio selection is solved for the given example. The confidence level is taken to be 95%. $E_{\min}$ and $E_{\max}$ are found to be 0.1419 and 0.1981 respectively. The problem is solved for ten equidistant values in this range and efficient frontier along with the solutions are given in Figure 3.7 and Table 3.6 respectively.

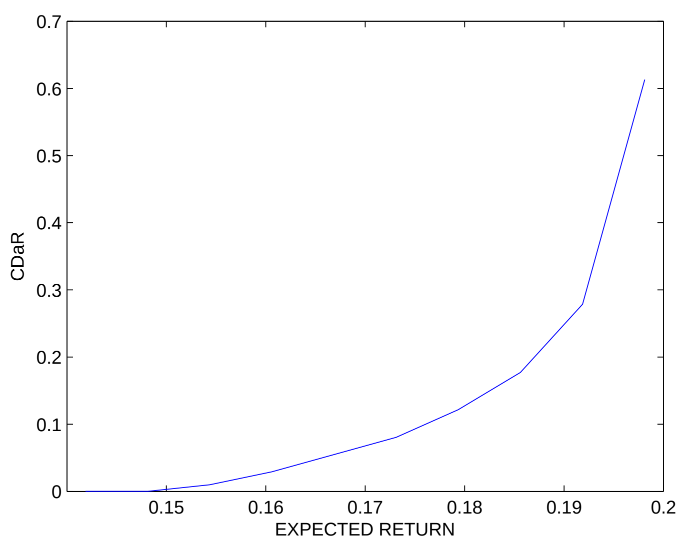

Figure 3.7 Efficient frontier of Mean-Conditional Drawdown at risk.

Table 3.6 Expected return and CDaR along with corresponding fractions to invest for Mean-CDaR.

Portfolio return	Fraction to invest in nine securities									CDaR
0.1419	0.0028	0.0172	0.4386	0.0253	0.0873	0.0021	0.4128	0.0064	0.0075	0
0.1481	0.003	0.0061	0.4802	0.0025	0.1703	0.0021	0.3262	0.0051	0.0046	0
0.1544	0	0	0.4217	0	0.135	0	0.2609	0.1558	0.0265	0.0099
0.1606	0	0	0.3307	0.0151	0.14	0	0.2537	0.2605	0	0.0291
0.1669	0	0	0.1905	0.1124	0.1618	0	0.2295	0.3058	0	0.0548
0.1731	0	0	0.0502	0.2097	0.1836	0	0.2054	0.3512	0	0.0806
0.1794	0	0	0	0.3288	0.2275	0	0.1146	0.3291	0	0.1218
0.1856	0	0	0	0.4057	0.274	0	0	0.3203	0	0.1771
0.1919	0	0	0	0.0178	0.2352	0	0	0.7469	0	0.2787
0.1981	0	0	0	0	1	0	0	0	0	0.613

Portfolio selection using CDaR is similar to CVaR but measure's risk in terms of drawdown, in other words more conservatively. It compares the most "unfavorable" moment in the past with respect to the current discrete moment. An investor may allow small drawdowns but if there is a large drawdown, it means there is a problem with the current pool, maybe time to move the securities. Drawdown accounts for not only losses but also the sequence of losses; it is a loss measure with memory taking into account the time sequence of losses (Krokhmal et al. (2005)).

In this chapter, we presented six different risk measures currently available for portfolio selection. These measures give a varied perspective to an investor and depending on his risk quantification he can use one which best fits his needs. In the next chapter we introduce a new risk measure.

CHAPTER 4

UNEQUAL PRIORITIZED DOWNSIDE RISK

Chapter 3 details a wide range of risk measures available to an investor, but none of these measures prioritize losses based on their location from the expected return. Investors know that a possibility of loss is inherent in any investment and are willing to accept it as long as it satisfies their pre-determined criteria on loss. A conservative investor may not want many possible loss scenarios farther from the expected return and so would like to treat them as a grave problem. A risk taking investor on the other hand would be willing to accept possibilities of extreme losses for an added benefit. So investors have some idea on how their losses should be treated. The risk measure we establish utilizes this information from the investor regarding how losses need to be treated based on their location from the expected return.

Investors input a set of priorities based on which the loss (downside) region is divided; a higher priority implying they do not want many losses in that region and lower otherwise. The new risk measure utilizes this input of priorities from the investor and establishes a quantification process to measure risk and find the best portfolio. The new risk measure is called Unequal Prioritized Downside Risk (UPDR). Unequal Prioritized Downside Risk is the expected loss of the downside given unequal priorities for losses in different downside regions.

The investor inputs a set of priorities and weights. The priorities should sum up to one and should be non-decreasing because they represent losses progressively farther from the expected return. Priority 1 represents the closest region to expected return; priority 2 represents the next closest region and so on. The weights divide the whole downside region into parts and are strictly increasing. The expectation of all the downside losses is found but loss in each particular region is multiplied by a corresponding priority.

Section 4.1 explains priorities and weights of UPDR and details the derivation of UPDR. Section 4.2 shows the properties UPDR satisfies and section 4.3 shows the formulation of UPDR as a mixed-integer program. Section 4.4 shows under certain conditions UPDR can be formulated as a linear programming problem and the procedure to generate the efficient frontier is explained in detail. In Section 4.5 a numerical example is presented. Finally some conclusions and the need to use UPDR are discussed.

Notations

Let us define some notations we will be using throughout this chapter.

η_1	—	expected return of a portfolio
R	—	random observed return of a portfolio
s	—	number of scenarios of information available about the future
n	—	number of securities
α	—	confidence level
$\boldsymbol{r}_{s\times n}$	—	return matrix for the securities
$\boldsymbol{X}_{n\times 1}$	—	the investment vector corresponding to n securities
$\boldsymbol{\mu}_{n\times 1}$	—	the mean return of the securities
ξ	—	random variable
$f(X,\xi)$	—	loss function of the portfolio
$\boldsymbol{p}$	—	priorities
$\boldsymbol{w}$	—	weights
M	—	a large constant

4.1 Derivation

UPDR is the expected value of the downside given priorities for losses in different regions. Usually losses are in monetary terms, but we list losses in terms of returns (percentage). We need to find the expected value of all losses which are above portfolio expected return η_1. The loss region is divided into parts by $\eta_1, \eta_2, \ldots, \eta_k$ which are strictly non-decreasing. Losses in any region $[\eta_m, \eta_{m+1})$ is given a priority of p_m where

$m =\ 1, 2, \ldots, k-1$. Finally losses in $[\eta_k, \infty)$ are given a priority of p_k. The priorities follow the convention that their sum adds up to one.

The investor specifies a confidence level α which implies that the loss region above η_k should have $(1-\alpha)\%$ of all losses. The region between η_1 and η_k is divided based on an input of weights given by the investor. The weights correspond to the percentage of region each of $\eta_2, \eta_3, \ldots, \eta_{k-1}$ should have between η_1 and η_k starting from η_1. The weights have to be strictly increasing and can take only values between zero and one. The region between η_1 and η_k has a weight of one.

Then UPDR is given as follows:

$$\begin{aligned} &\mathrm{UPDR}_\alpha(X, \eta_1, \eta_2, \ldots, \eta_k) = \\ &\sum_{i=1}^{k-1} p_i \times \left\{ \eta_i + \int_{\xi \epsilon R^n} [f(X,\xi) - \eta_i / (f(X,\xi) < \eta_{i+1})]^+ p(\xi) d\xi \right\} \\ &+ p_k \times \left\{ \eta_k + \frac{\int_{\xi \epsilon R^n} [f(X,\xi) - \eta_k]^+ p(\xi) d\xi}{(1-\alpha)} \right\} \end{aligned}$$

Equivalently we can write UPDR in the following notation:

$$\begin{aligned} &\mathrm{UPDR}_\alpha(X, \eta_1, \eta_2, \ldots, \eta_k) = \\ &\sum_{i=1}^{k-1} p_i \times \{ \eta_i + \mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+ \} \\ &+ p_k \times \left\{ \eta_k + \frac{\mathrm{E}[f(X,\xi) - \eta_k]^+}{(1-\alpha)} \right\} \end{aligned}$$

η_k — VaR

$z^+ = \max\ \{z, 0\}$

$\eta_{i+1} = \eta_1 + (\eta_k - \eta_1) \times w_i,\ i = 1, 2, \ldots, k-2$

The above derivation finds the expectation of the losses in each of the downside regions multiplied by a given priority for that region. A return matrix regarding the future behavior of securities serves as input for the portfolio selection problem with the assumption that each of these scenarios is an equally likely predictor of the future. The loss function is derived in terms of returns as follows. Since $\boldsymbol{rX}$ is the portfolio return, then $-\boldsymbol{rX}$

represents the loss. Since our objective function is minimization, our losses are written as positive values and in particular loss values greater than portfolio expected return η_1 represents loss for our portfolio selection problem. The derivation explained above can be used to solve for UPDR once we have these scenarios of future returns.

The following figure gives a clear representation of priorities, weights and the way the downside region is divided for UPDR.

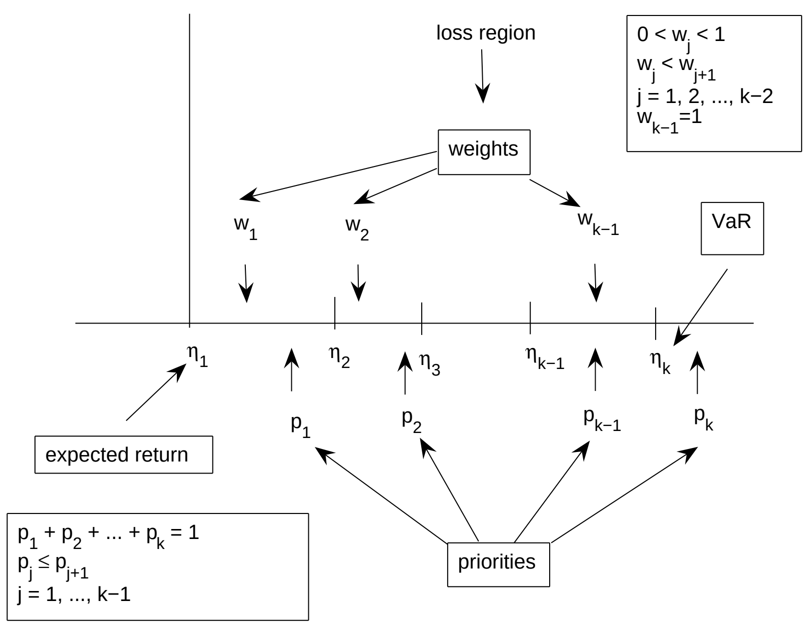

Figure 4.1 Illustration of UPDR.

4.2 Properties

UPDR satisfies the following properties translation-equivariance and positive homogeneity.

Proposition 4.1. UPDR is translation-equivariant i.e.

$$\mathrm{UPDR}_\alpha(Y+a) = \mathrm{UPDR}_\alpha(Y)+a$$

$$a \in \Re$$

Proof. : $\mathrm{UPDR}_\alpha(Y+a) =$

$$\sum_{i=1}^{k-1}\left\{p_i(\eta_i + a + \mathrm{E}[(f(X,\xi)+a)-(\eta_i+a)/(f(X,\xi)+a<\eta_{i+1}+a)]^+\right\}$$
$$+\,p_k\left\{\eta_k + a + \frac{\mathrm{E}[(f(X,\xi)+a)-(\eta_k+a)]^+}{(1-\alpha)}\right\} \tag{4.1}$$

Since $(f(X,\xi)+a<\eta_{i+1}+a) = (f(X,\xi)<\eta_{i+1})$, (4.1) implies

$$\sum_{i=1}^{k-1}\left\{p_i(\eta_i + a + \mathrm{E}[(f(X,\xi)+a)-(\eta_i+a)/(f(X,\xi)<\eta_{i+1})]^+\right\}$$
$$+\,p_k\left\{\eta_k + a + \frac{\mathrm{E}[(f(X,\xi)+a)-(\eta_k+a)]^+}{(1-\alpha)}\right\} \tag{4.2}$$

Since $(f(X,\xi)+a)-(\eta_i+a) = (f(X,\xi)-\eta_i)$, (4.2) implies

$$\sum_{i=1}^{k-1}\left\{p_i(\eta_i + a + \mathrm{E}[(f(X,\xi)-\eta_i)/(f(X,\xi)<\eta_{i+1})]^+\right\}$$
$$+\,p_k\left\{\eta_k + a + \frac{\mathrm{E}[f(X,\xi)-\eta_k]^+}{(1-\alpha)}\right\}$$

$$=$$

$$\sum_{i=1}^{k-1}(p_i a) + \sum_{i=1}^{k-1}\left\{p_i(\eta_i + \mathrm{E}[(f(X,\xi)\eta_i)/(f(X,\xi)<\eta_{i+1})]^+\right\}$$
$$+\,(p_k a) + p_k\left\{\eta_k + \frac{\mathrm{E}[f(X,\xi)-\eta_k]^+}{(1-\alpha)}\right\}$$

$=$

$$\left(\sum_{i=1}^{k} p_i\right) a + \sum_{i=1}^{k-1} \left\{p_i(\eta_i + \mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+\right\}$$
$$+ p_k \left\{\eta_k + \frac{\mathrm{E}[f(X,\xi) - \eta_k]^+}{(1-\alpha)}\right\} \tag{4.3}$$

Since $\sum_{i=1}^{k}(p_i) = 1$, (4.3) implies

$$a + \sum_{i=1}^{k-1} \left\{p_i(\eta_i + \mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+\right\}$$
$$+ p_k \left\{\eta_k + \frac{\mathrm{E}[f(X,\xi) - \eta_k]^+}{(1-\alpha)}\right\}$$
$$= a + \mathrm{UPDR}(Y)$$

□

Proposition 4.2. UPDR is positively-homogeneous, i.e.

$$\mathrm{UPDR}_\alpha(cY) = c\ \mathrm{UPDR}_\alpha(Y)$$
$$c > 0$$

Proof. : $\mathrm{UPDR}_\alpha(cY) =$

$$\sum_{i=1}^{k-1} \left\{p_i(\eta_i\, c + \mathrm{E}[((f(X,\xi)\, c) - (\eta_i\, c))/(f(X,\xi)\, c < \eta_{i+1}\, c)]^+)\right\}$$
$$+ p_k \left\{\eta_k\, c + \frac{\mathrm{E}[(f(X,\xi)\, c) - (\eta_k\, c)]^+}{(1-\alpha)}\right\} \tag{4.4}$$

Since $(f(X,\xi)\, c < \eta_{i+1}\, c) = (f(X,\xi) < \eta_{i+1})$, (4.4) implies

$$\sum_{i=1}^{k-1} \left\{p_i(\eta_i\, c + \mathrm{E}[((f(X,\xi)\, c) - (\eta_i\, c))/(f(X,\xi) < \eta_{i+1})]^+)\right\}$$
$$+ p_k \left\{\eta_k\, c + \frac{\mathrm{E}[(f(X,\xi)\, c) - (\eta_k\, c)]^+}{(1-\alpha)}\right\} \tag{4.5}$$

Since $(f(X,\xi)\, c) - (\eta_i\, c) = c\,(f(X,\xi) - \eta_i)$, (4.5) implies

$$\sum_{i=1}^{k-1} \left\{p_i(\eta_i\, c + \mathrm{E}[c\,(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+)\right\}$$
$$+ p_k \left\{\eta_k\, c + \frac{\mathrm{E}[c\,(f(X,\xi) - \eta_k)]^+}{(1-\alpha)}\right\} \tag{4.6}$$

Since $\mathrm{E}(c\,X) = c\,\mathrm{E}(X)$ for any constant c, (4.6) implies

$$\sum_{i=1}^{k-1} \{p_i(\eta_i\, c + c\,\mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+)\}$$
$$+\, p_k \left\{\eta_k\, c + c\,\frac{\mathrm{E}[(f(X,\xi) - \eta_k)]^+}{(1-\alpha)}\right\}$$
$$=$$
$$\sum_{i=1}^{k-1} c\,\{p_i(\eta_i + \mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+)\}$$
$$+\, p_k\, c \left\{\eta_k + \frac{\mathrm{E}[(f(X,\xi) - \eta_k)]^+}{(1-\alpha)}\right\}$$
$$=$$
$$c \left\{\sum_{i=1}^{k-1} \{p_i(\eta_i + \mathrm{E}[(f(X,\xi) - \eta_i)/(f(X,\xi) < \eta_{i+1})]^+)\right\}$$
$$+ p_k \left\{\eta_k + \frac{\mathrm{E}[(f(X,\xi) - \eta_k)]^+}{(1-\alpha)}\right\}$$
$$= c\ \mathrm{UPDR}(Y) \qquad \square$$

4.3 Formulation

Portfolio selection under Mean-UPDR can be formulated as a mixed integer program. We aim to minimize our risk measure for a particular portfolio expected return, hence the objective function is minimization. There is one main assumption we need before solving this problem—for any partitioned region in the downside all the losses cannot be exactly at the beginning endpoint of the interval. We need this assumption because if all the losses in a region are exactly at the beginning end point, then the formulation would incorrectly treat it as if there is no loss in that region. The investor can make sure that this does not happen by providing a return matrix that has reasonably many scenarios so that losses are spread out. The following steps explain the formulation:

Step 1 All losses above η_k should have a priority of p_k and should have $(1-\alpha)\%$ of losses. This is achieved by having these constraints and the following objective function

$y_i^k \geq -\boldsymbol{r}\ \boldsymbol{X} - \eta_k$, $\forall i$

$y_i^k \geq 0$, $\forall i$

Objective function: Minimize $p_k \times \{\eta_k + (1-\alpha)^{-1} \sum_{i=1}^{s} [\frac{y_i^k}{s}]\}$

The above objective function and constraints will ensure that only $(1-\alpha)\%$ of losses are above η_k and their expectation is found and multiplied by priority p_k.

Step 2 All losses in a region $[\eta_m, \eta_{m+1})$ should get a priority of p_m. This is achieved by having these constraints:

$y_i^m \geq -\boldsymbol{r}\ \boldsymbol{X} - \eta_m$, $\forall i$

$y_i^m \geq 0$, $\forall i$

These constraints would include all losses above the starting end point of the current region η_m. We need to make sure losses from the regions above the current region are not included again. The variable y_i^m finds the difference between the loss and the starting end point of the current region η_m. If this difference y_i^m added to η_m is greater than or equal to η_{m+1} it means the loss is actually above the current region. We add the following constraint to make sure losses from the regions above are not included again. A binary variable δ is introduced with the condition that if it is 1 it means the loss is in the current region and 0 otherwise. The following constraint would satisfy our condition on loss.

$y_i^m + \eta_m - \eta_{m+1} + \mathrm{M}\delta_i^m \geq 0$, $\forall i$

Consider the case where the loss is in a higher region, then the difference y_i^m added to η_m will be greater than η_{m+1} thus the quantity $y_i^m + \eta_m - \eta_{m+1}$ would be strictly greater than zero. This would mean a 0 or 1 for δ_i^m would satisfy the previous constraint, but 0 would be optimal since our objective function is minimization. Thus all losses in regions above the current region will not be considered.

The other case we are concerned about is when there are no losses in a region. Then the difference y_i^m multiplied δ_i^m for all i would be zero. We introduce a new binary variable I^m with the condition that a 0 for it would mean there are no losses in that region and 1 otherwise. The following constraint would satisfy this requirement.

$\sum_{i=1}^{s} [y_i^m \delta_i^m] - \mathrm{M}I^m \leq 0$

If $\sum_{i=1}^{s}[y_i^m \delta_i^m]$ is zero then it means there is no loss in the current region and the above constraint would be satisfied when I^m is zero or one. Since our objective function is minimization, a value of zero for I^m would be optimal. When $\sum_{i=1}^{s}[y_i^m \delta_i^m]$ is greater than zero, the above constraint will be satisfied if and only if I^m is one. This constraint correctly handles regions with no losses.

For step 2, the objective function and constraints would be:

$$\text{Minimize} \quad \sum_{j=1}^{k-1}\left\{\sum_{i=1}^{s} p_{k-1} \times \left[\eta_j I^j + \frac{y_i^j \delta_i^j}{s}\right]\right\}$$
$$y_i^m \geq -\boldsymbol{rX} - \eta_m, \forall i$$
$$y_i^m \geq 0, \forall i$$
$$y_i^m + \eta_m - y_i^{m+1} + \mathrm{M}\delta_i^m \geq 0, \forall i$$
$$\sum_{i=1}^{s}[y_i^m \delta_i^m] - \mathrm{M}I^m \leq 0$$

The final mixed integer optimization to solve UPDR is got by adding steps 1 and 2.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j=1}^{k-1}\left\{\sum_{i=1}^{s} p_{k-1} \times \left[\eta_j I^j + \frac{y_i^j \delta_i^j}{s}\right]\right\} \\
& + p_k \times \left\{\eta_k + (1-\alpha)^{-1}\sum_{i=1}^{s}[\frac{y_i^k}{s}]\right\} \\
\text{subject to} \quad & \\
& y_i^k \geq -\boldsymbol{rX} - \eta_k, \forall i \\
& y_i^k \geq 0, \forall i \\
& m = 1, 2, \ldots, k-1 \\
& y_i^m \geq -\boldsymbol{rX} - \eta_m, \forall i \\
& y_i^m \geq 0, \forall i \\
& y_i^m + \eta_m - \eta_{m+1} + \mathrm{M}\delta_i^m \geq 0, \forall i \\
& \sum_{i=1}^{s}[y_i^m \delta_i^m] - \mathrm{M}I^m \leq 0 \\
& \sum_{i=1}^{n} X_i = 1
\end{aligned} \tag{4.7}$$

$$\eta_{i+1} = \eta_1 + (\eta_k - \eta_1) \times w_i, i = 1, \ldots, k-2$$
$$\sum_{i=1}^{n} X_i \mu_i = \eta_1$$
$$\boldsymbol{X} \geq 0$$

To generate the efficient frontier, for different values of η_1, the optimization problem (4.7) is solved and a particular minimum UPDR is found. Since the above is a mixed integer problem, we may not get an optimal solution. Thus we cannot generate the efficient frontier, since for each expected return we need the minimum possible UPDR. To conduct portfolio selection using UPDR if we have s scenarios and k priorities, then we would need $(s+1) \times (k-1)$ binary variables. This makes the problem computationally intensive. For these two reasons, an investor can go for a linear program to solve UPDR provided priorities satisfy certain additional condition. The next section explains formulation of Mean-UPDR as a linear program.

4.4 Linear Programming Formulation

UPDR can be solved as a linear programming problem if the priorities satisfy the following additional condition:

$$p_j \quad > \quad \sum_{i=1}^{j-1} p_i : j = 1, 2, \ldots, k-1 \tag{4.8}$$

We assumed previously for any region not all scenarios can be at the beginning end point; since we do not want to use integer variables we also assume that no region is empty. The following example explains UPDR's formulation as a linear programming problem when priorities satisfy condition (4.8). Let the portfolio expected return η_1 be 0.1. Since we define loss as positive, the loss region is all possible returns above η_1. The investor would like to divide his loss region into three parts by $[\eta_1, \eta_2)$, $[\eta_2, \eta_3)$ and $[\eta_3, \infty)$. So he inputs three priorities for the corresponding parts $\boldsymbol{p}$=[0.1 0.2 0.7] and a weight of $\boldsymbol{w} = 0.5$.

The priorities correspond to how the investor feels about losses in a particular region. The weight corresponds to the percentage of region η_2 should have between η_1 and η_3 starting from η_1. So a weight of 0.5 implies η_2 is equidistant from η_1 and η_3.

Based on the given weight η_2=η_1+$(\eta 3 - \eta 1) \times 0.5$. The losses between $[\eta_1, \eta_2)$ and $[\eta_2, \eta_3)$ are given priorities 0.1 and 0.2 respectively. The losses above η_3 are given a priority of 0.7. Finally η_3 is left to vary so that it has $(1-\alpha)\%$ losses above it. We solve this problem using the following approach which leads to a linear program.

The input for our problem is the return matrix $\boldsymbol{r}_{s\times n}$ where s is the number of scenarios and n is the number of securities. Our problem is to run for s different scenarios. The loss is defined as $L = -\boldsymbol{r}\boldsymbol{X}$, where $\boldsymbol{X}_{n\times 1}$ is the fractional investment in each of the n securities. Only loss values greater than η_1 are loss scenarios for our problem.

First we need $(1-\alpha)\%$ of losses above η_3. Our objective function needs the expected value of all losses above η_3 multiplied by given priority 0.7. This can be achieved by having the following objective function and constraints:

$$\begin{aligned} \text{Minimize} \quad & 0.7 \times (\eta_3 + (1-\alpha)^{-1} \sum_{i=1}^{s} [\frac{y_i^3}{s}]) \\ \text{subject to} \quad & y_i^3 \geq -\boldsymbol{r}\boldsymbol{X} - \eta_3 \\ & y_i^3 \geq 0 \end{aligned}$$

The losses between $[\eta_1, \eta_2)$ should get a priority of 0.1. This can be achieved by the following objective function and constraints:

$$\begin{aligned} \text{Minimize} \quad & 0.1 \times (\eta_1 + \sum_{i=1}^{s} [\frac{y_i^1}{s}]) \\ \text{subject to} \quad & y_i^1 \geq -\boldsymbol{r}\boldsymbol{X} - \eta_1 - \mathrm{M}y_i^3 \\ & y_i^1 \geq 0 \end{aligned}$$

The constraint would insure that a particular scenario which lies in the region above η_3 is not looked at again. Since we minimize the objective function, this part will make sure only losses between $[\eta_1, \eta_3]$ are found and their expected value is computed and multiplied by priority 0.1. While doing this we will include losses in the region $[\eta_2, \eta_3]$ with a priority of 0.1.

The losses in the region $[\eta_2, \eta_3)$ should get a priority of 0.2. Since these losses have already got a priority of 0.1, we add the following objective function and constraints.

$$\begin{aligned}
\text{Minimize} \quad & 0.1 \times (\eta_2 + \sum_{i=1}^{s}[\frac{y_i^2}{s}]) \\
\text{subject to} \quad & y_i^2 \geq -\boldsymbol{rX} - \eta_2 - \mathrm{M}y_i^3 \\
& y_i^2 \geq 0
\end{aligned}$$

The above will make sure losses in $[\eta_2, \eta_3]$ get a priority of 0.1 but since these losses already got a priority of 0.1, they would finally get a priority of 0.2 as required. Since η_3 gets included in all three objective functions, it would get a priority of 1. Combine all the information to get the full optimization problem:

$$
\begin{aligned}
\text{Minimize} \quad & 0.1 \times (\eta_1 + \sum_{i=1}^{s}[\frac{y_i^1}{s}]) + 0.1 \times (\eta_2 + \sum_{i=1}^{s}[\frac{y_i^2}{s}]) \\
& + 0.7 \times (\eta_3 + \frac{\sum_{i=1}^{s}[\frac{y_i^3}{s}]}{(1-\alpha)}) \\
\text{subject to} \quad & \\
& i = 1, 2, \ldots, s \\
& y_i^3 \geq -\boldsymbol{r}\boldsymbol{X} - \eta_3 \\
& y_i^3 \geq 0 \\
& y_i^2 \geq -\boldsymbol{r}\boldsymbol{X} - \eta_2 - \mathrm{M}y_i^3 \\
& y_i^2 \geq 0 \\
& y_i^1 \geq -\boldsymbol{r}\boldsymbol{X} - \eta_1 - \mathrm{M}y_i^3 \\
& y_i^1 \geq 0 \\
& \eta_2 = \eta_1 + (\eta_3 - \eta_1) \times 0.5 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \sum_{i=1}^{n} X_i \mu_i = \eta_1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

4.4.1 General formulation and Efficient frontier

Given a priority vector $\boldsymbol{p}$ satisfying condition (4.8), reconvert the priorities before formulation. The new priority vector $\boldsymbol{p}'$ would be found the following way:

$$
p_i' = \begin{cases} p_1 & i = 1 \\ p_2 - p_1 & i = 2 \\ p_i - \sum_{t=1}^{i-1} p_t' & i = 3, 4, \ldots, k-1 \\ p_k & i = k \end{cases}
$$

The linear program formulation of Mean-UPDR would be as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{k-1} p_i^{'} \times \left\{ \eta_i + \sum_{j=1}^{s} \frac{y_i^j}{s} \right\} + p_k^{'} \times \left\{ \eta_k + \frac{\sum_{j=1}^{s} [\frac{y_k^j}{s}]}{(1-\alpha)} \right\} \\
\text{subject to} \quad & \\
& y_i^j \geq -\boldsymbol{rX} - \eta_i - \mathrm{M} y_k^j : i = 1, 2, \ldots, k-1, \forall j \\
& y_k^j \geq -\boldsymbol{rX} - \eta_k : \forall j \\
& y_i^j \geq 0, \forall i, j \\
& \eta_{i+1} = \eta_1 + \{\eta_k - \eta_1\} \times w_i, i = 1, \ldots, k-2 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \sum_{i=1}^{n} X_i \mu_i = \eta_1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{4.9}
$$

To generate the efficient frontier, we need to find $\eta_1^{\min}$ and $\eta_1^{\max}$ which represents the minimum possible portfolio return and the maximum possible portfolio return for mean-UPDR portfolio selection. To find $\eta_1^{\min}$ we solve the following closely related linear program:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{k-1} p_i^{'} \times \left\{ \eta_i + \sum_{j=1}^{s} \frac{y_i^j}{s} \right\} + p_k^{'} \times \left\{ \eta_k + \frac{\sum_{j=1}^{s} [\frac{y_k^j}{s}]}{(1-\alpha)} \right\} \\
\text{subject to} \quad & \\
& y_i^j \geq -\boldsymbol{rX} - \eta_i - \mathrm{M} y_k^j : i = 1, 2, \ldots, k-1, \forall j \\
& y_k^j \geq -\boldsymbol{rX} - \eta_k : \forall j \\
& y_i^j \geq 0, \forall i, j \\
& \eta_{i+1} = \eta_1 + \{\eta_k - \eta_1\} \times w_i, i = 1, \ldots, k-2 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \sum_{i=1}^{n} X_i \mu_i = \eta_1^{\min} \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

$\eta_1^{\max}$ corresponds to the maximum mean return among the available securities. Portfolio expected return will lie in interval $[\eta_1^{\min}, \eta_1^{\max}]$. We can select values in this interval and solve (4.9) to find the corresponding UPDR value. The portfolio returns along with UPDR values are plotted to represent the efficient frontier. The investor then has the choice to decide among the available solutions and decide on one which fits his needs.

4.5 A Numerical Example

The example stated in Chapter 3 is used to illustrate portfolio selection in this chapter. For this example we assume $\boldsymbol{p}$=[0.1 0.2 0.7] and a weight of $\boldsymbol{w} = 0.5$. $\eta_1^{\min}$ and $\eta_1^{\max}$ are found to be 0.0692 and 0.1981 respectively. Ten equidistant values in [0.0692, 0.1981] are selected and the problem (4.9) is solved. The portfolio returns along with UPDR values obtained are plotted to represent the efficient frontier. The investor then has the choice to decide among the available solutions the one that fits his needs. Figure 4.2 gives the efficient frontier and Table 4.1 gives the solutions.

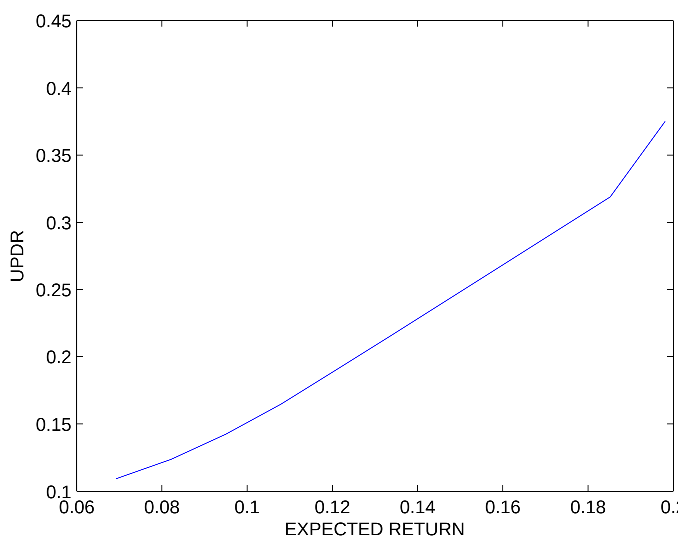

Figure 4.2 Efficient frontier of Mean-UPDR.

Table 4.1 Expected return and UPDR along with corresponding fractions to invest for Mean-UPDR.

Portfolio return	Fraction to invest in nine securities									UPDR
0.0692	0	0.2074	0	0	0.0321	0.6474	0.1131	0	0	0.1092
0.0836	0	0	0	0	0.1422	0.752	0.0988	0.007	0	0.1254
0.0979	0	0	0	0	0.0101	0.6718	0.027	0.2912	0	0.147
0.1122	0	0	0	0	0	0.5778	0	0.4222	0	0.1731
0.1265	0	0	0	0	0	0.4719	0	0.5281	0	0.2014
0.1408	0	0	0	0	0	0.366	0	0.634	0	0.2299
0.1552	0	0	0	0	0	0.2602	0	0.7398	0	0.2586
0.1695	0	0	0	0	0	0.1543	0	0.8457	0	0.2873
0.1838	0	0	0	0	0	0.0484	0	0.9516	0	0.316
0.1981	0	0	0	0	1	0	0	0	0	0.375

UPDR is established and the formulation of Mean-UPDR is explained in detail. This risk measure builds up on CVaR and semivariance by including all the downside information but with different priorities. UPDR unlike CVaR will use all downside information before making a decision. This way the investor can be sure of not missing any useful information which can be got from downside loss scenarios. UPDR becomes CVaR when a priority vector of all zeros for the regions except one for the final region is used, provided we have at least two regions. To solve UPDR we need to do a mixed-integer program but when priorities satisfy certain additional condition we can solve it using a linear program.

UPDR maps all the downside information to generate the optimal portfolio. Unlike the currently available risk measures, for the same expected return two investors can get different portfolios since their priorities could be different. This gives the investor more sophistication. Investors have some idea on their priority setting for downside losses which UPDR includes to build the best possible portfolio. This information (prioritization) serves as a tool for the investor to generate multiple portfolios and see the effects of his prioritization.

In this chapter, we established a new risk measure and showed its implementation for a numerical example. In the next chapter, portfolio selection is handled with multiple risk measures simultaneously and new models are introduced.

CHAPTER 5

SINGLE-PERIOD MULTI-OBJECTIVE PORTFOLIO SELECTION

Investors use standard portfolio selection to buy securities for a single period. Information about the future behavior of the individual securities about this period is assumed to be known. Based on the available information about the securities, the standard portfolio selection problem aims to maximize return and minimize a risk measure for this single period. The main assumption under this type of investing is that the investor will hold the securities for a pre-determined single period—the period being one day, one week, one month, etc. The investment decision is then made based on an analysis of the future behavior of securities for the pre-determined single period that satisfies the investor's criteria on return and risk. Chapter 3 and Chapter 4 list risk measures available for the investor among which he can select one which best fits his needs.

Portfolio selection using just one risk measure may not be the best way to solve the problem. Deciding which measure is "best" for all problems is still unresolved (see e.g. Stone (1973)). The main reason being each measure performs best in its domain but not so when considered in the domain of an alternate measure. For example using Mean-Semivariance will lead to portfolios having minimum semivariance but if we calculate absolute deviation from the solution it may be bigger than what we would get by solving Mean-Absolute deviation. Investors when asked to choose may list more than one measure which they feel quantifies their risk but may not be able to come up with just one measure. Many authors have showed that using more than one risk measure simultaneously helps the investor get a better perspective. Konno et al. (1993) show a portfolio model using mean, variance and skewness where variance and skewness are treated as two risk measures. Roman et. al. (2007) show a portfolio model using mean, variance and CVaR with variance and CVaR minimized simultaneously. To the best of our knowledge, there is no paper which addresses semivariance in the context of multiple risk measures.

Semivariance is a downside risk measure and is similar to variance but addresses the dispersion of the expected return on the downside. We decided to use semivariance as the reference risk measure, and another risk measure simultaneously for portfolio selection, since most investors would like to include semivariance in their analysis as it quantifies the entire downside risk. The portfolio selection problem then becomes multi-objective with three criteria—mean, semivariance and another risk measure (RM). One of the four risk measures absolute deviation, CVaR, CDaR or UPDR can be used as the other risk measure (RM). We choose these four measures since all of them can be formulated linearly and so are comparable in computational difficulty. We assume the priorities of UPDR satisfy the additional condition (4.8) so that it can also be formulated as a linear program.

In this type of portfolio selection the investor can choose one of the other risk measures we have listed along with semivariance and vary the weight given to each of the measures to get a multi-dimensional perspective. Using this approach we can get results which may not be available just by using any of the two risk measures separately. It also lets the investor choose more than one measure to satisfy his risk criterion. We call the model we propose as Mean-Semivariance-RM. Since we need to optimize all three objectives simultaneously multiobjective programming can be used to do the same.

A general multi-objective program consists of multiple objective functions which need to optimized simultaneously. These objective functions are not comparable and hence need to be solved simultaneously. Consider the following general multi-objective problem:

$$\begin{aligned} \text{Maximize} \quad & \{(f_1(x), f_2(x), \ldots, f_T(x))\} \\ \text{subject to} \quad & x \in A \end{aligned} \tag{5.1}$$

The optimal solution of the multi-objective problem (5.1) is characterized by pareto preference relation. A solution x^1 pareto dominates another feasible solution x^2 if $f_i(x^1) \geq f_i(x^2)$ for all i and $f_i(x^1) > f_i(x^2)$ for at least one i.

There are many methods available to solve general multi-objective problems. We use the ϵ-constrained method to solve the problem for several reasons. This is very intuitive for an investor who does not have much knowledge on multi-objective optimization and it is very easy to solve. In the ϵ-constrained method one of the objective functions is left in the objective and the other functions are brought to the constraint region and constraints are placed for them. The general form of the ϵ-constrained method is given as follows:

$$\begin{aligned} \text{Maximize} \quad & f_j(x) \\ \text{subject to} \quad & f_k(x) \geq \epsilon_k : k = 1, 2, \ldots, T \ : k \neq j \\ & x \in A \end{aligned} \tag{5.2}$$

Consider the following proposition from Roman et al. (2007), which relates the optimal solutions of (5.1) and (5.2).

Proposition 5.1. (Roman et al. (2007)) Let f_1, f_2, f_3: $\mathbb{R}^n \rightarrow \mathbb{R}$ and $X \subseteq \mathbb{R}^n$. A point $x^* \in X$ is a pareto efficient solution of the multi-objective problem

$$\begin{aligned} \text{(MO): Maximize} \quad & (f_1(x), f_2(x), f_3(x)) \\ \text{subject to:} \quad & x \in X \end{aligned}$$

if and only if x^* is also an optimal solution of the single objective problem:

$$\begin{aligned} \text{(SO): Maximize} \quad & f_1(x) \\ \text{subject to:} \quad & f_2(x) \geq a \\ & f_3(x) \geq b \\ & x \in X \end{aligned}$$

with $a = f_2(x^*)$ and $b = f_3(x^*)$

Proof. : ($\Rightarrow$) Let x^* be a pareto efficient solution of the multi-objective problem (MO). Assume that x^* is not an optimal solution of the single objective problem (SO). This means that there exists an $x^{'} \in X$ such that $f_1(x^{'}) > f_1(x^*)$, $f_2(x^{'}) \geq a = f_2(x^*)$, $f_3(x^{'}) \geq b = f_3(x^*)$. This means that $(f_1(x^{'}), f_2(x^{'}), f_3(x^{'}))$ pareto dominates $(f_1(x^*), f_2(x^*), f_3(x^*))$ which is a contradiction with x^* being a pareto efficient solution of (MO).

($\Leftarrow$) Let x^* be an optimal solution of the single objective problem (SO) with $a = f_2(x^*)$ and $b = f_3(x^*)$. Assume that x^* is not a pareto efficient solution of (MO). This means there exists an $x^{'} \in X$ such that $f_1(x^{'}) \geq f_1(x^*)$, $f_2(x^{'}) \geq f_2(x^*)$ and $f_3(x^{'}) \geq f_3(x^{'})$ with at least one strict inequality. However the inequality $f_1(x^{'}) \geq f_1(x^*)$ cannot be strict, because $f_1(x^*)$ is the optimal value of the objective function (SO). The only possibility left is that $f_1(x^{'}) = f_1(x^*)$ and at least one of the inequalities $f_2(x^{'}) \geq f_2(x^*)$ and $f_3(x^{'}) \geq f_3(x^{'})$ is strict. This would imply that in the problem (SO) at least one of the constraints $f_2(x) \geq a, f_3(x) \geq b$ is not active, which is a contradiction. Thus x^* is a pareto efficient solution of (MO). This completes the proof. □

The multi-objective problem we need to solve is as follows:

$$\begin{aligned} \text{Minimize} \quad & [\text{Semivariance}(X), \text{RM}(X), -\text{E}(X)] \\ \text{subject to} \quad & X \in A \end{aligned} \tag{5.3}$$

We use semivariance as the reference risk measure, hence it is left in the objective function and constraints are placed for the expected return and the other risk measure on the lines of the ϵ-constrained method. The single-objective problem we need to solve is.

$$\begin{aligned} \text{Minimize} \quad & \text{Semivariance}(X) \\ \text{subject to:} \quad & \text{RM}(X) \leq z \\ & \text{E}(X) \geq d \\ & X \in A \end{aligned} \tag{5.4}$$

The general constraints are $A = \{\sum_{i=1}^{n} X_i = 1, \boldsymbol{X} \geq 0\}$. Any additional constraints can be added to the set A depending on the problem.

Using proposition (5.1), a point X^* is an optimal solution of (5.3) if and only if it is also an optimal solution of (5.4) with $z = \text{RM}(X^*)$ and $d = \text{E}(X^*)$. Therefore to get all the efficient solutions of the mean-semivariance-RM model, we solve the problem (5.4) by varying z and d such that the constraints on the risk measure (RM) and expected return are active.

We need to obtain the efficient solutions of the mean-semivariance-RM model and hence we need to choose the right hand values d and z for the expected return and risk measure(RM), such that the constraints are active. We have extended the procedure proposed by Roman et al. (2007) to solve our models. This procedure is explained here to get a set of efficient solutions to (5.4).

Step 1: The first step is to find the possible values of d which will lie in the interval $[d_{\text{min}}, d_{\text{max}}]$. The minimum possible return $d_{\text{min}} = \max[d_{\text{minSV}}, d_{\text{minRM}}]$, where d_{minSV} and d_{minRM} are the expected returns of the minimum semivariance portfolio and minimum RM portfolio respectively. They are found by solving the following two problems:

Problem 1:

$$\begin{aligned} \text{Minimize} \quad & \text{M*Semivariance}(X) - d_{\text{minSV}} \\ \text{subject to} \quad & \\ & \text{E}(X) = d_{\text{minSV}} \\ & X \in A, d_{\text{minSV}} \in \Re \end{aligned}$$

Problem 2:

$$\begin{aligned} \text{Minimize} \quad & \text{M*RM}(X) - d_{\text{minRM}} \\ \text{subject to} \quad & \\ & \text{E}(X) = d_{\text{minRM}} \\ & X \in A, d_{\text{minRM}} \in \Re \end{aligned}$$

Here M is a large constant which makes sure the minimization of semivariance and RM is pre-emptive. Then $d_{\text{min}} = \max\{d_{\text{minSV}}, d_{\text{minRM}}\}$. This will make sure we are not looking at any inefficient expected return values.

The maximum possible return $d_{\max}$ corresponds to the maximum possible mean return among the component securities. Then $d \in [d_{\min}, d_{\max}]$. Any value of d in this interval will be efficient for the problem (5.4).

Step 2: For a fixed value of d, say d^*, the possible values of z must lie in the interval $[z_{d^*_{\min}}, z_{d^*_{\max}}]$ where $z_{d^*_{\min}}$ is the minimum RM for the expected return $d*$ and $z_{d^*_{\max}}$ is the RM of the portfolio that minimizes semivariance for the expected return $d*$. $z_{d^*_{\min}}$ is found by solving problem 3.

Problem 3:

$$\begin{aligned} \text{Minimize} \quad & \text{RM}(X) \\ \text{subject to} \quad & \text{E}(X) = d^* \\ & X \in A \end{aligned}$$

$z_{d^*_{\min}}$ is given by the objective function value of problem 3. $z_{d^*_{\max}}$ is found by solving problem 4.

Problem 4:

$$\begin{aligned} \text{Minimize} \quad & \text{Semivariance}(X) \\ \text{subject to} \quad & \text{E}(X) = d^* \\ & X \in A \end{aligned}$$

Then $z_{d^*_{\max}} = \text{RM}(X^*)$, where X^* is the optimal value from problem 4. Finally $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$. This makes sure that the problem (5.4) is not infeasible because the limit on z is greater than $z_{d^*_{\min}}$ and since the limit on z is lesser than $z_{d^*_{\max}}$ will make sure the constraint on risk measure (RM) is active.

Step 3: The main problem (5.4) is solved for a fixed level of expected return d^* and z value so that the constraints on expected return and risk measure(RM) are active. This procedure can be repeated for different fixed levels of expected return d^* and z values to get a set of efficient solutions to the problem.

While solving the problem (5.4) for a fixed expected return d^* and a risk measure (RM) value $z_{d^*_{\max}}$, we will obtain a mean-semivariance efficient portfolio. Similarly when we solve (5.4) for a fixed expected return d^* and a risk measure (RM) value $z_{d^*_{\min}}$, we will obtain a mean-risk measure(RM) efficient portfolio. So solutions obtained using this method will also have efficient solutions of the two separate mean-semivariance and mean-risk measure(RM) problems. For a fixed expected return, the efficient solutions of mean-semivariance-RM model will form a three dimensional space. We plot the curve in a semivariance-RM space for a given expected return. In this curve the lower end and upper end is represented by the mean-RM and mean-semivariance efficient solution, respectively.

Step 4: The investor has two options available to him. He can use the above procedure and find a set of efficient solutions for different expected return values. The investor can then choose one of the solutions which best fits his needs in terms of different weights assigned to semivariance and the other risk measure (RM). On the other hand if he clearly knows his expected return and what value he wants to assign for his risk measure, he can solve the model once to get the corresponding solution.

The flowchart in Figure 5.4 gives a pictorial description of the procedure we outlined to solve Mean-SV-RM model.

We considered four different models and used the above procedure to obtain a set of efficient solutions to the models.

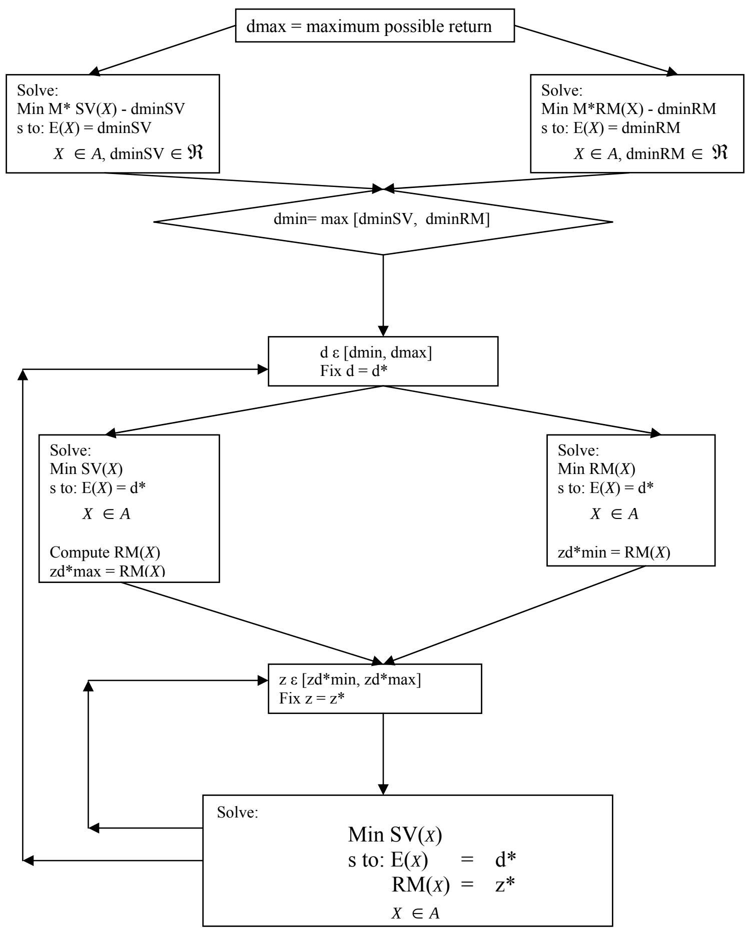

Figure 5.1 Solution Procedure to solve Single-period Mean-Semivariance-RM models

Notations

Let us define some notations we will be using throughout this chapter.

s	—	number of scenarios of information available about the future
n	—	number of securities
k	—	number of partitioned regions for UPDR
α	—	confidence level
$\boldsymbol{r}_{s\times n}$	—	return matrix for the securities
$\boldsymbol{X}_{n\times 1}$	—	the investment vector corresponding to n securities
$\boldsymbol{\mu}_{n\times 1}$	—	the mean return of the securities
$\boldsymbol{p}_{k\times 1}$	—	the priority vector for UPDR
$\boldsymbol{w}_{k-2\times 1}$	—	the weight vector for UPDR

The example stated in Chapter 3 is used to illustrate portfolio selection problems in this chapter. There are eighteen scenarios of returns and each of these are assumed to be equally likely predictors of the next year's return. Throughout this chapter the confidence level α is assumed to be 0.95 unless otherwise stated.

5.1 Mean-Semivariance-Absolute deviation

The first model we considered was Mean-Semivariance-Absolute deviation. Semivariance quantifies the downside risk whereas absolute deviation measures the absolute deviation of the expected return. An investor can use these two measures for portfolio selection and use the procedure we outlined to get a set of solutions. The investor can decide on what weight he wants to assign the different measures and get corresponding solutions based on that.

Semivariance is left in the objective function and constraints are placed for absolute deviation and expected return. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[(r_{ij}X_j) - d] : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& a_i \geq 0 : i = 1, 2, \ldots, s \\
& \frac{1}{s}\sum_{i=1}^{s} a_i \leq z \qquad (5.5) \\
& \boldsymbol{X}'\boldsymbol{\mu} \geq d \qquad (5.6) \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

Let us represent the variables as follows

$$x = [Y_{1\times s} \qquad A_{1\times m}]$$

Here Y represents the semivariance variables and A represents all other variables. The following proposition is needed before we solve the problem.

Proposition 5.2. The objective function is convex

Proof. : A multi-dimensional function $f(x)$ is convex if and only if $Z^{'}\nabla^2 f(x)Z \geq 0$ at every point $x \in S$, for all Z. Here S is the set of constraints. (Nash and Sofer (1996)). Consider

$$\begin{aligned} Z^{'}\nabla^2 f(x)Z &= \begin{bmatrix} z_1 & z_2 & \dots & z_{s+m} \end{bmatrix} \frac{1}{s} \begin{bmatrix} 2 & 0 & \dots & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{s+m} \end{bmatrix} \\ &= \frac{2}{s}\sum_{i=1}^{s} z_i^2 \geq 0 \qquad (5.7) \end{aligned}$$

The above inequality holds for any subset of $\Re^{s+m}$. Hence the objective function is convex.

□

Remark 5.1. Since all the constraints are linear, any non-empty feasible region will be convex. Since the objective function and the feasible region are both convex, the problem is convex. For the other three models to show that the optimization problem is convex, note that the constraint set consists of all linear constraints and hence the feasible region is convex. The objective function is the same and is convex implies the problems are convex. Since all the four problems are convex, we are guaranteed a global optimal solution(Nash and Sofer (1996)).

We solve this problem using the procedure outlined for the given example. Using the procedure we outlined d_{minSV} and $d_{\text{minAbsdev}}$ were found to be 0.0666 and 0.0641 respectively. $d_{\min} = \max\{d_{\text{minSV}}, d_{\text{minAbsdev}}\}$ and is given by 0.0666. The maximum expected return $d_{\max}$ is found to be 0.1981. Therefore expected return $d \in [0.0666, 0.1981]$. In this interval six equidistant expected return values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The optimization problem is solved for different values of z and d^* so that constraints on absolute deviation (5.5) and expected return (5.6) are active. These solutions are plotted on a semivariance-absolute deviation space for each given expected return and is given in Figure 5.2. The corresponding solutions are given in Table 5.1

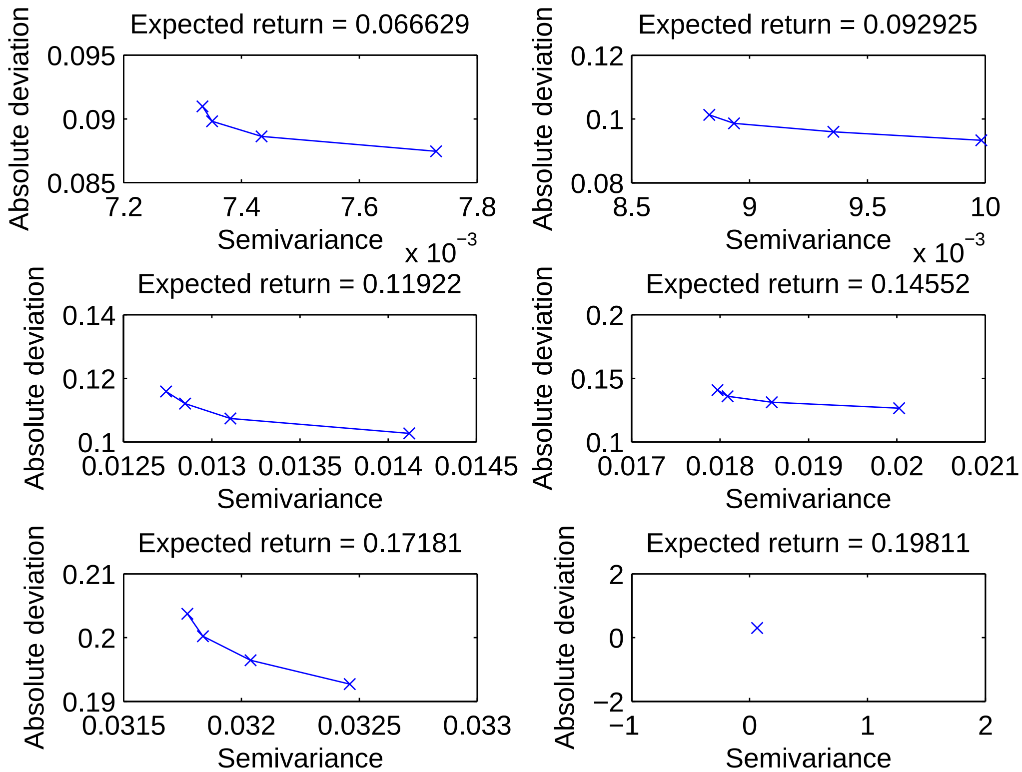

Figure 5.2 Efficient frontier of Mean-Semivariance-Absolute deviation.

Table 5.1 Semivariance, Expected return and Absolute deviation along with corresponding fractions to invest for Mean-Semivariance-Absolute deviation.

Semivariance	Absolute deviation	Expected return	Fraction to invest in nine securities								
0.0077	0.0875	0.0666	0	0.8498	0	0	0	0.0669	0.0833	0	0
0.0074	0.0886	0.0666	0	0.8224	0	0	0.0295	0.1205	0.0275	0	0
0.0074	0.0898	0.0666	0	0.7959	0	0	0.0379	0.1529	0.0133	0	0
0.0073	0.091	0.0666	0	0.7698	0	0	0.0369	0.1757	0.0176	0	0
0.01	0.0933	0.0929	0	0.4414	0	0	0.0079	0.0656	0.3743	0	0.1107
0.0094	0.096	0.0929	0	0.4585	0	0	0.0571	0.1041	0.3066	0	0.0737
0.0089	0.0986	0.0929	0	0.4754	0	0	0.0957	0.1364	0.2808	0	0.0116
0.0088	0.1013	0.0929	0	0.4406	0	0	0.1145	0.1883	0.2567	0	0
0.0141	0.1027	0.1192	0	0.1259	0	0	0.0907	0.0547	0.5203	0	0.2083
0.0131	0.1074	0.1192	0	0.0604	0	0	0.1312	0.1723	0.5398	0	0.0963
0.0128	0.112	0.1192	0	0.1524	0.062	0	0.1548	0.1384	0.4636	0	0.0289
0.0127	0.1159	0.1192	0	0.19	0.1419	0	0.1398	0.1147	0.4136	0	0
0.02	0.1265	0.1455	0	0	0	0.0893	0.2082	0	0.6306	0	0.0719
0.0186	0.1312	0.1455	0	0	0.1955	0.0246	0.1869	0	0.593	0	0
0.0181	0.1359	0.1455	0	0	0.2928	0	0.1774	0	0.5298	0	0
0.018	0.1407	0.1455	0	0	0.367	0	0.158	0	0.475	0	0
0.0325	0.1927	0.1718	0	0	0	0.3604	0.3927	0	0.2469	0	0
0.032	0.1965	0.1718	0	0	0	0.2629	0.3749	0	0.2709	0.0913	0
0.0318	0.2002	0.1718	0	0	0	0.1853	0.3539	0	0.2892	0.1715	0
0.0318	0.2037	0.1718	0	0	0	0.1502	0.3284	0	0.2955	0.2259	0
0.0641	0.3025	0.1981	0	0	0	0	1	0	0	0	0

5.2 Mean-Semivariance-CVaR

The second model we considered was Mean-Semivariance-CVaR. Semivariance quantifies the downside risk whereas CVaR measures the expected value of the worst losses. This model is for an investor who believes semivariance and CVaR are the two best measures which quantify risk. Let us suppose the investor wants to solve this model for different expected returns and risk values. He can solve the model using the procedure we outlined to get a set of solutions.

Semivariance is left in the objective function and constraints are placed for CVaR and expected return. The main problem is given as follows:

$$\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[(-r_{ij}X_j) - \eta] : i = 1, 2, \ldots, s \\
& \eta + \frac{1}{(1-\alpha)s}\sum_{i=1}^{s}(a_i) \leq z \qquad (5.8) \\
& a_i \geq 0 : i = 1, 2, \ldots, s \\
& \boldsymbol{X}'\boldsymbol{\mu} \geq d \qquad (5.9) \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

We solved this problem using the procedure outlined for the given data. Using the procedure we outlined d_{minSV} and d_{minCVaR} were found to be 0.0666 and 0.0692 respectively. $d_{\text{min}} = \max\{d_{\text{minSV}}, d_{\text{minCVaR}}\}$ and is given by 0.0692. The maximum expected return d_{max} is found to be 0.1981. Therefore expected return $d \in [0.0692, 0.1981]$. In this interval six equidistant expected return values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\text{min}}}, z_{d^*_{\text{max}}}]$ and solved the problem for four equidistant values in this interval. The optimization problem is solved for different values of z and d^*

so that constraints on CVaR (5.8) and expected return (5.9) are active. These solutions are plotted on a semivariance-CVaR space for each given expected return and is given in Figure 5.3. The corresponding solutions are given in Table 5.2

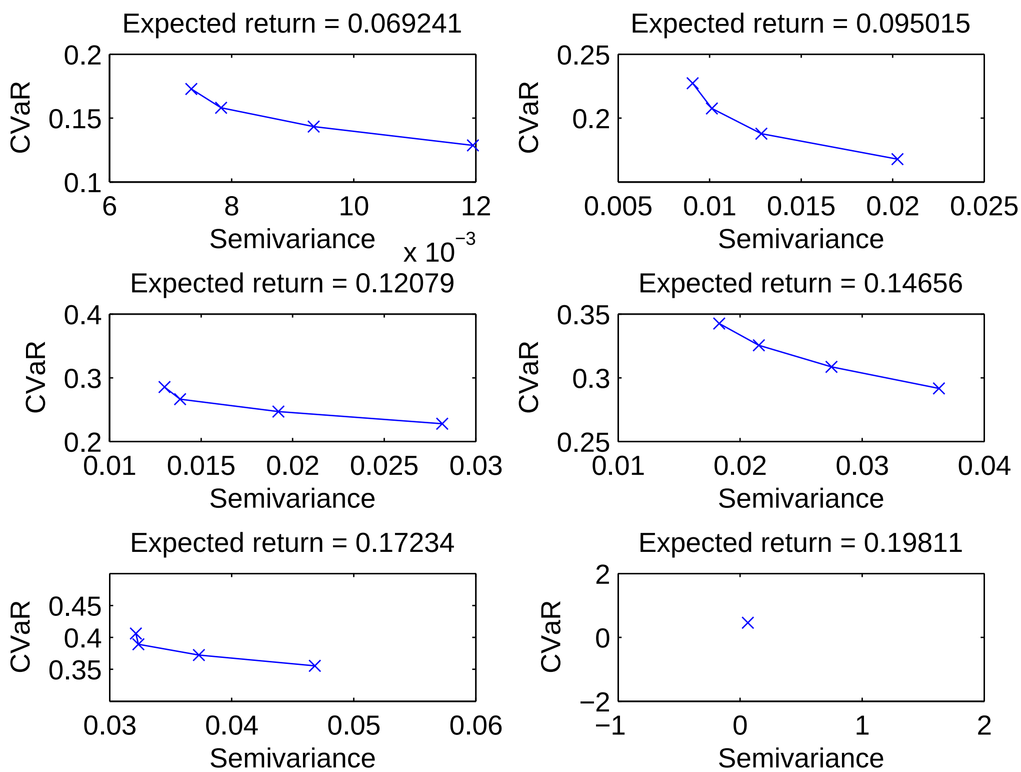

Figure 5.3 Efficient frontier of Mean-Semivariance-CVaR.

Table 5.2 Semivariance, Expected return and CVaR along with corresponding fractions to invest for Mean-Semivariance-CVaR.

Semivariance	CVaR	Expected return	Fraction to invest in nine securities								
0.012	0.1287	0.0692	0	0.2074	0	0	0.0321	0.6474	0.1131	0	0
0.0093	0.1434	0.0692	0	0.439	0	0	0.079	0.4819	0	0	0
0.0078	0.1581	0.0692	0	0.6015	0	0	0.0717	0.3268	0	0	0
0.0073	0.1727	0.0692	0	0.7529	0	0	0.0649	0.1822	0	0	0
0.0203	0.1679	0.095	0	0	0	0	0.0088	0.6822	0.05	0.259	0
0.0128	0.1877	0.095	0	0	0.1204	0	0.0843	0.5622	0.2331	0	0
0.0101	0.2075	0.095	0	0.2111	0.0403	0	0.1163	0.3806	0.2518	0	0
0.0091	0.2273	0.095	0	0.4027	0	0	0.1202	0.1996	0.2775	0	0
0.0282	0.2277	0.1208	0	0	0	0	0	0.5143	0	0.4857	0
0.0192	0.247	0.1208	0	0	0.1333	0	0.1399	0.4146	0.1381	0.1741	0
0.0139	0.2662	0.1208	0	0	0.1962	0	0.147	0.287	0.3698	0	0
0.013	0.2854	0.1208	0	0.1558	0.1666	0	0.1325	0.1233	0.4218	0	0
0.0363	0.2916	0.1466	0	0	0	0	0	0.3237	0	0.6763	0
0.0275	0.3085	0.1466	0	0	0.1886	0	0.1248	0.2252	0.0947	0.3667	0
0.0215	0.3255	0.1466	0	0	0.2901	0	0.1414	0.1084	0.2768	0.1832	0
0.0183	0.3425	0.1466	0	0	0.3663	0	0.173	0	0.4607	0	0
0.0468	0.3554	0.1723	0	0	0	0	0	0.1331	0	0.8669	0
0.0373	0.3723	0.1723	0	0	0.2527	0	0.1425	0.0301	0.0613	0.5135	0
0.0324	0.3893	0.1723	0	0	0.0223	0	0.2783	0	0.3057	0.3937	0
0.0321	0.4062	0.1723	0	0	0	0.0785	0.3096	0	0.3042	0.3077	0
0.0641	0.457	0.1981	0	0	0	0	1	0	0	0	0

5.3 Mean-Semivariance-CDaR

The third model we considered was Mean-Semivariance-CDaR. Semivariance quantifies the downside risk whereas CDaR measures the expected value of the worst downside at risk losses. Since CDaR measure losses more conservatively than CVaR, this model could be used by a more conservative investor. The procedure outlined can be used to solve this model for different expected returns.

Semivariance is left in the objective function and constraints are placed for CDaR and expected return. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \dots, s \\
& y_i \geq 0 : i = 1, 2, \dots, s \\
& a_j \geq \left\{\sum_{i=1}^{n}(1 + \sum_{t=1}^{k} r_{ti})X_i\right\} - \left\{\sum_{i=1}^{n}(1 + \sum_{t=1}^{j} r_{ti})X_i\right\} - \eta \\
& k = 1, 2, \dots, j \\
& a_j \geq 0 \\
& j = 1, 2, \dots, s \\
& \eta + \frac{1}{(1-\alpha)s}\sum_{j=1}^{s}(a_j) \leq z \qquad (5.10) \\
& \boldsymbol{X}'\boldsymbol{\mu} \geq d \qquad (5.11) \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

We solved this problem using the procedure outlined for the given data. Using the procedure we outlined d_{minSV} and d_{minCDaR} were found to be 0.0666 and 0.1419 respectively. Therefore $d_{\min} = \max\{d_{\text{minSV}}, d_{\text{minCDaR}}\}$ and is given by 0.1419. The maximum expected return $d_{\max}$ is found to be 0.1981. Therefore expected return $d \in [0.1491, 0.1981]$. In this interval six equidistant expected return values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The optimization problem is solved for different values of z and d^* so that constraints on CDaR (5.10) and expected return (5.11) are active. These solutions are plotted on a semivariance-CDaR space for each given expected return and is given in Figure 5.4. The corresponding solutions are given in Table 5.3

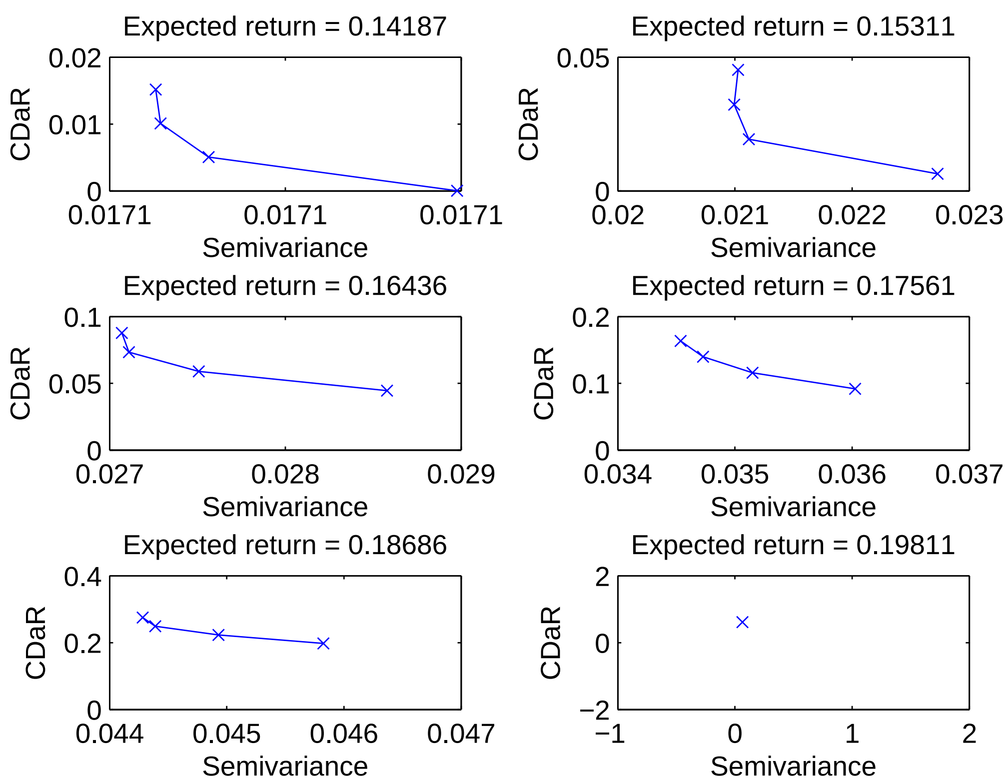

Figure 5.4 Efficient frontier of Mean-Semivariance-CDaR.

Table 5.3 Semivariance, Expected return and CDaR along with corresponding fractions to invest for Mean-Semivariance-CDaR.

Semivariance	CDaR	Expected return	Fraction to invest in nine securities								
0.0171	0	0.1419	0	0	0.4129	0	0.0942	0	0.4929	0.0000	0
0.0171	0.005	0.1419	0	0.0006	0.3871	0	0.102	0.0005	0.5098	0	0
0.0171	0.0101	0.1419	0	0.0035	0.3738	0	0.1086	0.0009	0.5132	0	0
0.0171	0.0151	0.1419	0	0.0045	0.3626	0	0.1132	0.0016	0.5181	0	0
0.0227	0.0064	0.1531	0	0	0.4381	0	0.1347	0	0.2617	0.1327	0.0328
0.0211	0.0193	0.1531	0	0	0.4231	0.0006	0.2433	0	0.3247	0.0082	0
0.021	0.0323	0.1531	0	0	0.3534	0.0047	0.2592	0	0.3748	0.0079	0
0.021	0.0452	0.1531	0	0	0.2795	0.0189	0.2575	0	0.4229	0.0211	0
0.0286	0.0445	0.1644	0	0	0.2466	0.0735	0.1531	0	0.2392	0.2877	0
0.0275	0.059	0.1644	0	0	0.1947	0.1428	0.2413	0	0.2681	0.1532	0
0.0271	0.0734	0.1644	0	0	0.1396	0.1119	0.2849	0	0.3207	0.1429	0
0.0271	0.0878	0.1644	0	0	0.1099	0.0958	0.3015	0	0.348	0.1447	0
0.036	0.0919	0.1756	0	0	0	0.25	0.1937	0	0.1914	0.3649	0
0.0352	0.1158	0.1756	0	0	0	0.2629	0.2649	0	0.1967	0.2755	0
0.0347	0.1397	0.1756	0	0	0	0.2235	0.3029	0	0.2121	0.2615	0
0.0345	0.1636	0.1756	0	0	0	0.184	0.3396	0	0.2272	0.2492	0
0.0458	0.1974	0.1869	0	0	0	0.3281	0.2662	0	0	0.4056	0
0.0449	0.2232	0.1869	0	0	0	0.2992	0.3235	0	0.0149	0.3624	0
0.0444	0.249	0.1869	0	0	0	0.2199	0.3577	0	0.0405	0.3819	0
0.0443	0.2748	0.1869	0	0	0	0.2219	0.3938	0	0.0444	0.3399	0
0.0641	0.613	0.1981	0	0	0	0	1	0	0	0	0

5.4 Mean-Semivariance-UPDR

The final model we considered was Mean-Semivariance-UPDR. Semivariance quantifies the downside risk whereas UPDR measures the expected value of the downside but priorities are assigned to the losses. Let us suppose the investor wants to solve this model for different expected returns and different risk values. The procedure outlined is used to solve the model to obtain a set of solutions.

Semivariance is left in the objective function and constraints are placed for UPDR and expected return. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_i^j \geq -\boldsymbol{r}\boldsymbol{X} - \eta_i - \mathrm{M}\, y_k^j : i = 1, 2, \ldots, k-1, \forall j \\
& a_k^j \geq -\boldsymbol{r}\boldsymbol{X} - \eta_k : \forall j \\
& a_i^j \geq 0, \forall i, j \\
& \eta_{i+1} = \eta_1 + (\eta_k - \eta_1) \times w_i, i = 1, \ldots, k-2 \\
& \sum_{i=1}^{k-1} p_i^{'} \times \left\{ \eta_i + \sum_{j=1}^{s} \frac{y_i^j}{s} \right\} + p_k^{'} \times \left\{ \eta_k + \frac{\sum_{j=1}^{s}[\frac{y_k^j}{s}]}{(1-\alpha)} \right\} \leq z \qquad (5.12) \\
& \boldsymbol{X}^{'}\boldsymbol{\mu} \geq d \qquad (5.13) \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

We solved this problem using the procedure outlined for the given data. The priority vector $\boldsymbol{p}$ is assumed to be [0.1 0.2 0.7] and the weight vector $\boldsymbol{w}$ is given as 0.5. Using the procedure we outlined d_{minSV} and d_{minUPDR} were found to be 0.0666 and 0.1092 respectively. Therefore $d_{\min} = \max\{d_{\text{minSV}}, d_{\text{minUPDR}}\}$ and is given by 0.1092. The maximum expected return $d_{\max}$ is found to be 0.1981. Therefore expected return $d \in [0.10922, 0.1981]$. In this interval six equidistant expected return values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The optimization problem is solved for different values of z and d^* so that constraints on UPDR (5.12) and expected return (5.13) are active. These solutions are plotted on a semivariance-UPDR space for each given expected return and is given in Figure 5.5. The corresponding solutions are given in Table 5.4

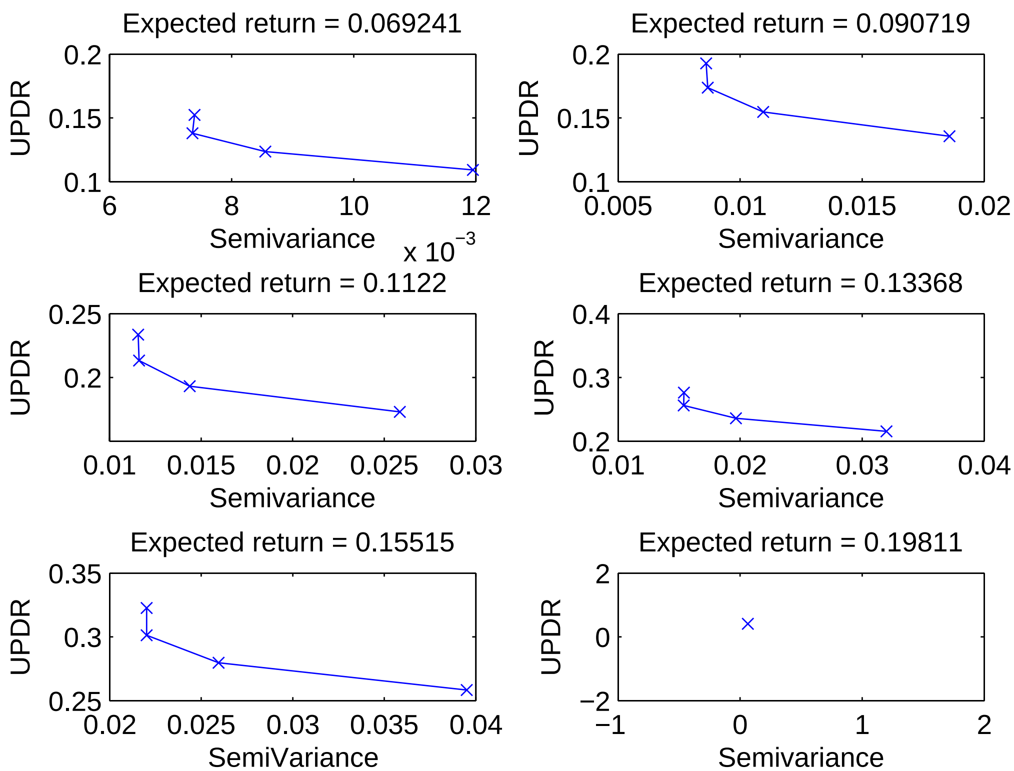

Figure 5.5 Efficient frontier of Mean-Semivariance-UPDR.

Table 5.4 Semivariance, Expected return and UPDR along with corresponding fractions to invest for Mean-Semivariance-UPDR.

Semivariance	UPDR	Expected return	Fraction to invest in nine securities								
0.012	0.1092	0.0692	0	0.2074	0	0	0.0321	0.6474	0.1131	0	0
0.0086	0.1236	0.0692	0	0.5113	0	0	0.0758	0.413	0	0	0
0.0074	0.1379	0.0692	0	0.72	0	0	0.0664	0.2136	0	0	0
0.0074	0.1523	0.0692	0	0.7305	0	0	0.0323	0.1709	0.0663	0	0
0.0186	0.1356	0.0907	0	0	0	0	0.007	0.6958	0.0889	0.2082	0
0.0109	0.1547	0.0907	0	0.1425	0.0661	0	0.0891	0.4825	0.2199	0	0
0.0087	0.1737	0.0907	0	0.4548	0	0	0.1432	0.2338	0.1682	0	0
0.0086	0.1928	0.0907	0	0.4621	0	0	0.1006	0.1857	0.2516	0	0
0.0258	0.173	0.1122	0	0	0	0	0	0.5778	0	0.4222	0
0.0144	0.1931	0.1122	0	0	0.159	0	0.1309	0.4144	0.2564	0.0393	0
0.0116	0.2133	0.1122	0	0.1601	0.1069	0	0.119	0.2096	0.4043	0	0
0.0116	0.2335	0.1122	0	0.2507	0.1002	0	0.14	0.1458	0.3633	0	0
0.032	0.2154	0.1337	0	0	0	0	0	0.419	0	0.581	0
0.0197	0.2357	0.1337	0	0	0.1962	0	0.1436	0.2587	0.2249	0.1766	0
0.0154	0.2559	0.1337	0	0.0052	0.2801	0	0.1135	0.0932	0.508	0	0
0.0154	0.2762	0.1337	0	0.0482	0.3032	0	0.1153	0.0616	0.4716	0	0
0.0395	0.2583	0.1552	0	0	0	0	0	0.2602	0	0.7398	0
0.026	0.2798	0.1552	0	0	0.2494	0	0.1555	0.0877	0.2151	0.2923	0
0.022	0.3012	0.1552	0	0	0.2649	0	0.2551	0	0.4055	0.0745	0
0.022	0.3226	0.1552	0	0	0.2493	0.0299	0.2621	0	0.4093	0.0494	0
0.0641	0.4063	0.1981	0	0	0	0	1	0	0	0	0

5.5 Discussion of Solutions

As discussed in the previous sections the investor has a choice of four models Mean-SV-Absolute deviation, Mean-SV-CVaR, Mean-SV-CDaR and Mean-SV-UPDR for investing. These four models were solved for the same numerical example and results were given in the previous sections.

Among the four methods, Mean-SV-Absolute deviation gives the solution with the minimum expected return of 0.0666. Among the four methods only Mean-SV-Absolute deviation solutions invest a certain fraction in security nine. All other methods do not invest in security nine. All the four methods do not invest any fraction in security one. We can safely assume that security one does not play any significance in the investment decision. Under Mean-SV-CVaR and Mean-SV-UPDR, security four is a part of only one solution whereas for Mean-SV-CDaR it is part of nearly all solutions. Security four is part of some solutions of Mean-SV-Absolute deviation. Securities five and six are part of nearly all solutions under the methods expect under Mean-SV-CDaR.

Solutions of Mean-SV-CVaR and Mean-SV-UPDR are very similar to each other. Solutions of Mean-SV-Absolute deviation, Mean-SV-CVaR and Mean-SV-UPDR have similar solutions with Mean-SV-Absolute deviation investing certain fractions in security nine whereas the other two do not invest any in that security. Mean-SV-CDaR is dissimilar from the other three models.

The investor can look at all these models and then make a decision. As we saw in this small example the solutions are different for each of the methods. If the investor has a better idea on the different securities he can include that in his analysis and then make a decision. An investor may want a particular security to be included in his portfolio always and which may not be included in one of the methods, then the investor can safely avoid that method. Portfolio selection using these models gives the investor a general perspective with respect to portfolio composition and makes good sense to analyze them always before making a decision.

5.6 Probabilistic or chance constrained portfolio selection

An investor decides on a particular portfolio return and then solves the portfolio problem so that his investment risk is minimized. The investor requires that the portfolio return is greater than or equal to a pre-determined expected return. Since this constraint may not be satisfied, a probable error is allowed and the constraint is formulated as a chance constraint (Charnes and Cooper (1959)). Portfolio selection under chance constraint aims to minimize a risk measure under the condition that the probability that a portfolio's rate of return is greater than the expected rate of return is no less than a confidence level. Since the confidence level and expected return varies among investors the decision making will be investor driven.

Suppose there are n securities in which we can invest and their mean return is given by ξ a random variable. The chance constraint for a specified expected return E_0 and confidence level $\alpha_c > 0.5$ is given as follows:

$$\Pr\left\{\boldsymbol{X}'\,\xi \geq E_0\right\} \geq \alpha_c$$

Let us suppose that the mean return of the securities ξ has a normal distribution $N(\boldsymbol{\mu}, \boldsymbol{C})$, where $\boldsymbol{C}$ is positive definite symmetric matrix. Then we can use some of the results of normal distribution to convert the chance constraint to a non linear constraint.

Since $\xi \sim N(\boldsymbol{\mu}, \boldsymbol{C})$, then $\boldsymbol{X}'\,\xi = \sum_{i=1}^{n} X_i \xi_i \sim N(\mathrm{E}(\boldsymbol{X}), \sigma(\boldsymbol{X}))$

where $\mathrm{E}(\boldsymbol{X}) = \boldsymbol{X}'\boldsymbol{\mu}$ and $\sigma(\boldsymbol{X}) = \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}$

Define a new random variable U as follows

$$U = \frac{\sum_{i=1}^{n} X_i \xi_i - \mathrm{E}(\boldsymbol{X})}{\sigma(\boldsymbol{X})}$$

Then $U \sim N(0,1)$. Now consider the chance constraint

$$\begin{aligned}
& \Pr\{\boldsymbol{X}'\xi \geq E_0\} \geq \alpha_c \\
\Rightarrow\ & \Pr\{\frac{\boldsymbol{X}'\xi - \mathrm{E}(\boldsymbol{X})}{\sigma(\boldsymbol{X})} \geq \frac{E_0 - \mathrm{E}(\boldsymbol{X})}{\sigma(\boldsymbol{X})}\} \geq \alpha_c \\
\Rightarrow\ & \Pr\{U \geq \frac{E_0 - \mathrm{E}(\boldsymbol{X})}{\sigma(\boldsymbol{X})}\} \geq \alpha_c \\
\Rightarrow\ & \Pr\{U \leq \frac{\mathrm{E}(\boldsymbol{X}) - E_0}{\sigma(\boldsymbol{X})}\} \geq \alpha_c \\
\Rightarrow\ & \Phi(\frac{\mathrm{E}(\boldsymbol{X}) - E_0}{\sigma(\boldsymbol{X})}) \geq \alpha_c \\
\Rightarrow\ & \frac{\mathrm{E}(\boldsymbol{X}) - E_0}{\sigma(\boldsymbol{X})} \geq \Phi^{-1}(\alpha_c) \\
\Rightarrow\ & -\mathrm{E}(\boldsymbol{X}) + E_0 + \sigma(\boldsymbol{X})\Phi^{-1}(\alpha_c) \leq 0 \qquad (5.14)
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal value. Consider the following equation set A:

$$A = \begin{cases} \sum_{i=1}^{n} X_i = 1 \\ -\boldsymbol{X}'\boldsymbol{\mu} + d + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\ \boldsymbol{X} \geq 0 \end{cases}$$

The following lemma shows that the set A is convex.

Lemma 5.1. (Tang et al. (2001)) A is a convex set.

Proof. : Consider the set $D = \{\boldsymbol{X} | -\mathrm{E}(\boldsymbol{X}) + E_0 + \sigma(\boldsymbol{X})\,\Phi^{-1}(\alpha_c) \leq 0\}$. Let us rewrite the chance constraint as follows:

$$-\mathrm{E}(\boldsymbol{X}) + E_0 + \sigma(\boldsymbol{X})\Phi^{-1}(\alpha_c) \leq 0 \quad \Rightarrow \mathrm{E}(\boldsymbol{X}) - E_0 \geq \sigma(\boldsymbol{X})\Phi^{-1}(\alpha_c)$$

Since $\mathrm{E}(\boldsymbol{X})$ is a linear function, thus for any $\lambda \in$ (0,1) and $X_1, X_2 \in D$ we have the following:

$$\mathrm{E}(\lambda X_1) - \lambda E_0 \geq \lambda\sigma(X_1)\Phi^{-1}(\alpha_c) \qquad (5.15)$$

$$\mathrm{E}((1-\lambda)X_2) - (1-\lambda)E_0 \geq ((1-\lambda)\sigma(X_2))\Phi^{-1}(\alpha_c) \qquad (5.16)$$

$$\mathrm{E}(\lambda X_1 + (1-\lambda)X_2) = \lambda\mathrm{E}(X_1) + (1-\lambda)\mathrm{E}(X_2) \qquad (5.17)$$

Combining 5.15, 5.16 and 5.17

$$\begin{aligned}
\mathrm{E}(\lambda X_1 + (1-\lambda)X_2) - E_0 &= \lambda \mathrm{E}(X_1) + (1-\lambda)\mathrm{E}(X_2) - E_0 \\
&\geq \lambda\sigma(X_1)\Phi^{-1}(\alpha_c) + ((1-\lambda)\sigma(X_2))\Phi^{-1}(\alpha_c) \\
&= (\lambda\sigma(X_1) + (1-\lambda)\sigma(X_2))\Phi^{-1}(\alpha_c) \qquad (5.18)
\end{aligned}$$

Since $\sigma(\boldsymbol{X})$ is a strict convex function on D, for $\lambda \in (0,1)$ we have

$$\lambda\sigma(X_1) + (1-\lambda)\sigma(X_2) \geq \sigma(\lambda X_1 + (1-\lambda)X_2) \qquad (5.19)$$

Combining (5.18) and (5.19) we get the desired result

$$\begin{aligned}
\mathrm{E}(\lambda X_1 + (1-\lambda)X_2) - E_0 &\geq (\lambda\sigma(X_1 + (1-\lambda)\sigma(X_2))\Phi^{-1}(\alpha_c) \\
&\geq \sigma(\lambda X_1 + (1-\lambda)X_2)\Phi^{-1}(\alpha_c) \in D
\end{aligned}$$

Since D is a convex set, A is also a convex set. □

A plot of expected returns and the maximum possible confidence level is first generated for a portfolio selection problem with chance constraint. This plot is called the permission set and is found by solving the following problem for different expected returns.

$$\begin{aligned}
\text{Maximize} \quad & \alpha_c \\
\text{subject to} \quad & -\boldsymbol{X}'\boldsymbol{\mu} + E_0 + \sqrt{\boldsymbol{X}'C\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

Here E_0 is the expected return. A plot of the permission set for the given numerical example is solved and given in Figure 5.6.

The permission set gives a basic idea about the problem with chance constraint. We can see the maximum confidence level the investor can expect is 0.69 but then he has to be satisfied with a very low expected return of 0.0551. On the other hand if he wants a higher

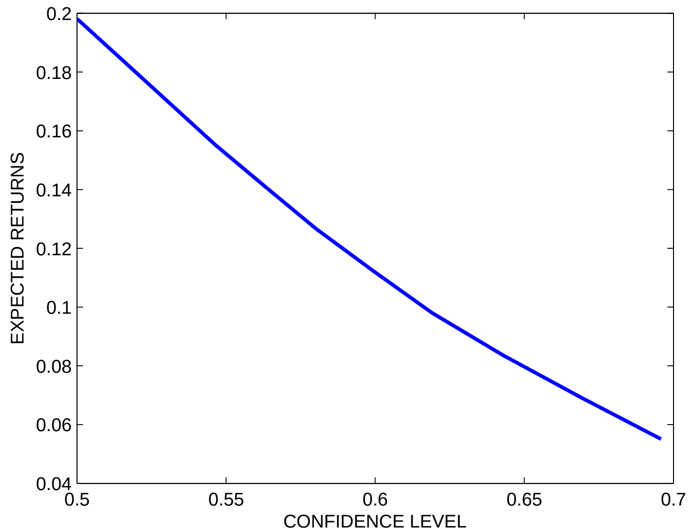

Figure 5.6 Permission set for chance constraint

expected return, he has to sacrifice on his confidence level. Based on the permission set the investor can decide on his confidence level α_c.

The next step would be to find the efficient frontier for the standard portfolio selection problem with chance constraint. The expected return and variance of the portfolio are given by $E = \boldsymbol{X}^{'}\boldsymbol{\mu}$ and $V = \boldsymbol{X}^{'}\boldsymbol{C}\boldsymbol{X}$ respectively. Problem (5.20) is solved for different expected returns E_0 to find the efficient frontier.

$$
\begin{aligned}
\text{Minimize} \quad & V = \boldsymbol{X}^{'}\boldsymbol{C}\boldsymbol{X} \\
\text{subject to} \quad & -\boldsymbol{X}^{'}\boldsymbol{\mu} + E_0 + \sqrt{\boldsymbol{X}^{'}\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned} \tag{5.20}
$$

The expected return of the portfolio (E_0) will lie between $d_{\min}$ and $d_{\max}$. $d_{\min}$ represents the minimum possible portfolio return for the given problem and can be found by solving the following problem.

$$\begin{aligned}
\text{Minimize} \quad & d_{\min} \\
\text{subject to} \quad & -\boldsymbol{X}'\boldsymbol{\mu} + d_{\min} + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

$d_{\max}$ represents the maximum possible portfolio return for the given problem and can be found by solving the following problem.

$$\begin{aligned}
\text{Maximize} \quad & d_{\max} \\
\text{subject to} \quad & -\boldsymbol{X}'\boldsymbol{\mu} + d_{\max} + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

Problem (5.20) is solved for different expected returns and the corresponding variances are found. The different expected returns and their corresponding variances are plotted to get the efficient frontier.

The given numerical example is solved for $\alpha_c = 0.6$ and the efficient frontier is given in Figure 5.7.

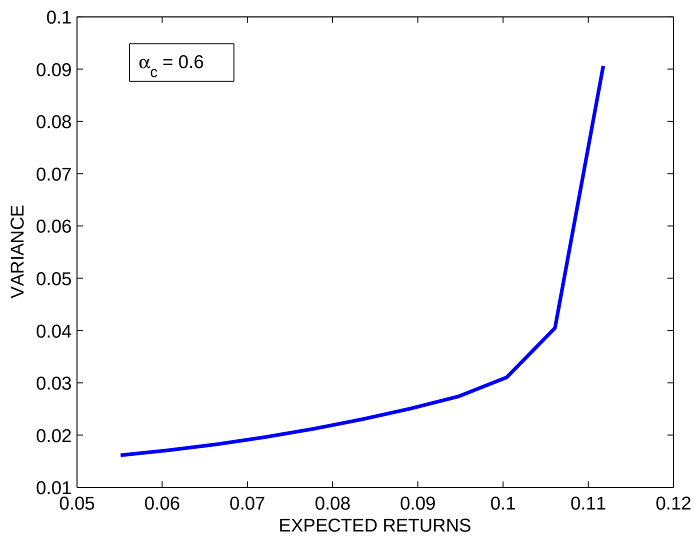

Figure 5.7 Efficient frontier of Mean-Variance with chance constraint.

Table 5.5 Variance, Expected return along with corresponding fractions to invest for Mean-Variance with chance constraint.

Variance	Expected return	Fraction to invest in nine securities								
0.0161	0.0551	0	0.6134	0	0	0.0924	0.087	0.2073	0	0
0.0171	0.0608	0	0.5238	0.0309	0	0.0881	0.0811	0.2761	0	0
0.0183	0.0664	0	0.4329	0.0551	0	0.0893	0.0779	0.3449	0	0
0.0196	0.0721	0	0.3378	0.0825	0	0.0869	0.0737	0.4191	0	0
0.0212	0.0778	0	0.2521	0.1015	0	0.0928	0.0649	0.4887	0	0
0.023	0.0834	0	0.1517	0.1299	0	0.0934	0.0649	0.5601	0	0
0.025	0.0891	0	0.0548	0.1573	0	0.0939	0.0595	0.6345	0	0
0.0274	0.0948	0	0	0.174	0.0068	0.1016	0.0221	0.6955	0	0
0.0311	0.1004	0	0	0.1257	0.0803	0.1628	0	0.6311	0	0
0.0405	0.1061	0	0	0.0345	0.2037	0.2766	0	0.4852	0	0
0.0906	0.1118	0	0	0	0.4084	0.5916	0	0	0	0

5.6.1 Mean-Semivariance-RM models with chance constraint

An investor may want to include chance constraint to his model to get confidence in his returns. For such an investor we wanted to investigate the Mean-Semivariance-RM models we proposed with chance constraint included in them. Since the set A is convex, the same procedure outlined earlier can be used to get all the efficient solutions with two main differences. Based on the permission set, the investor has to decide on a confidence level α_c for chance constraint. The maximum return unlike the previous model may not correspond to the highest mean return and so has to be found by solving the following optimization problem.

$$
\begin{aligned}
\text{Maximize} \quad & d_{\max} \\
\text{subject to} \quad & -\boldsymbol{X}'\boldsymbol{\mu} + d_{\max} + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

The main problem we need to solve for Mean-Semivariance-RM models is as follows

$$
\begin{aligned}
\text{Minimize} \quad & [\text{Semivariance}(X), \text{RM}(X)] \\
\text{subject to} \quad & X \in A
\end{aligned}
\tag{5.21}
$$

We use semivariance as the reference risk measure, hence it is left in the objective function and constraint is placed for the other risk measure on the lines of the ϵ-constrained method. The single-objective problem we need to solve is.

$$
\begin{aligned}
\text{Minimize} \quad & \text{Semivariance}(X) \\
\text{subject to:} \quad & \text{RM}(X) \leq z \\
& X \in A
\end{aligned}
\tag{5.22}
$$

Using proposition (5.1), a point X^* is an optimal solution of (5.21) if and only if it is also an optimal solution of (5.22) with $z = \text{RM}(X^*)$. We skip the rest of the explanation but give a general outline in the Figure 5.8.

Remark 5.2. For all the four problems the objective function is convex (Proposition 5.2). From lemma 5.1 we know set A is convex and noting that the other constraints are linear, implies the constraint set is composed of two convex sets. Since the intersection of convex sets is convex, the constraint set is convex. Hence all the four problems are convex problems with convex objective function. Thus we are guaranteed a global optimal solution.

The four models Mean-SV-Absolute Deviation, Mean-SV-CVaR, Mean-SV-CDaR and Mean-SV-UPDR are solved with chance constraint included in them. Each of the models is solved for six different expected returns and four different risk measure values. The investor can look at all these solutions and then pick one which best fits his needs. The confidence level for the risk measures α_{RM} is assumed to be 0.95 and for the chance constraint α_c is assumed to be 0.60.

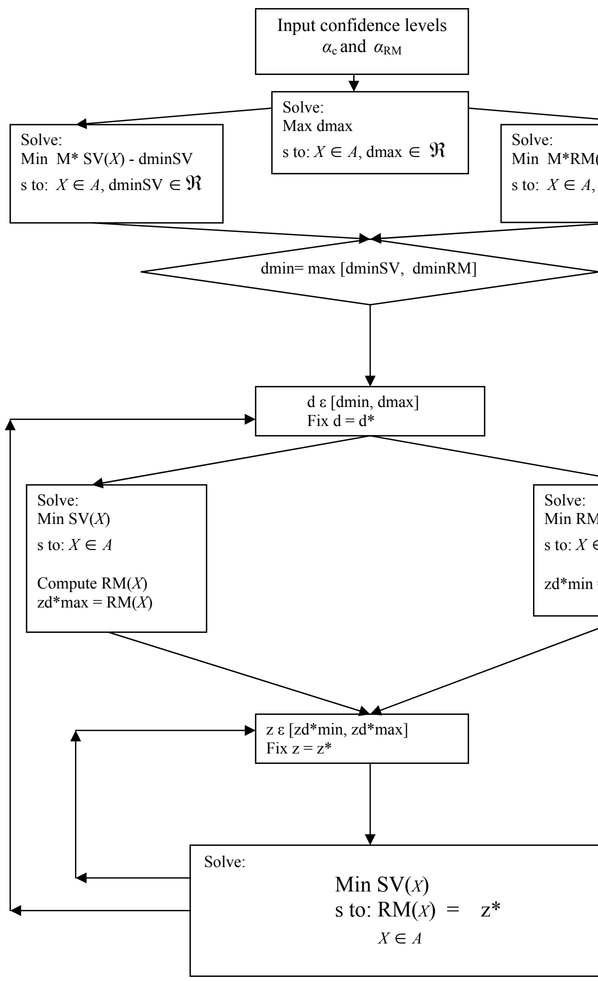

Figure 5.8 Solution Procedure to solve Mean-Semivariance-RM models with chance constraint.

The main problem for Mean-Semivariance-Absolute Deviation is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[(r_{ij}X_j) - d] : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& a_i \geq 0 : i = 1, 2, \ldots, s \\
& \frac{1}{s}\sum_{i=1}^{s} a_i \leq z \\
& -\boldsymbol{X}'\boldsymbol{\mu} + d + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{5.23}
$$

This problem is solved using procedure outlined and the different solutions are given in Table 5.6

Table 5.6 Semivariance, Expected return and Absolute deviation along with corresponding fractions to invest for Mean-Semivariance-Absolute deviation with chance constraint.

Semivariance	Absolute deviation	Expected return	Fraction to invest in nine securities								
0.0067	0.0921	0.0551	0	0.5062	0	0	0.0028	0.0699	0.3339	0	0.0872
0.0058	0.0955	0.0551	0	0.5776	0	0	0.0696	0.1105	0.2422	0	0
0.0056	0.099	0.0551	0	0.5103	0	0	0.0967	0.1796	0.2133	0	0
0.0056	0.1021	0.0551	0	0.4319	0	0	0.0953	0.2267	0.246	0	0
0.0086	0.096	0.0669	0	0.3854	0	0	0.0295	0.0021	0.4562	0	0.1267
0.0074	0.0988	0.0669	0	0.3401	0	0	0.0632	0.1011	0.4193	0	0.0764
0.0069	0.1016	0.0669	0	0.3805	0	0	0.1105	0.1338	0.3682	0	0.007
0.0068	0.1043	0.0669	0	0.3167	0.0383	0	0.1004	0.1911	0.3535	0	0
0.0104	0.102	0.0786	0	0.1389	0	0	0.0684	0.0509	0.5685	0	0.1733
0.0091	0.1062	0.0786	0	0.175	0.0181	0	0.1131	0.1077	0.5173	0	0.0688
0.0085	0.1105	0.0786	0	0.1879	0.0747	0	0.1158	0.1377	0.484	0	0
0.0085	0.1118	0.0786	0	0.2206	0.1052	0	0.1157	0.117	0.4414	0	0
0.0129	0.1111	0.0904	0	0	0	0.0264	0.1224	0.0322	0.6279	0	0.191
0.0111	0.1161	0.0904	0	0	0.1233	0	0.1081	0.0921	0.6146	0	0.0619
0.0105	0.1212	0.0904	0	0	0.2016	0	0.0988	0.1175	0.582	0	0
0.0105	0.1245	0.0904	0	0.0098	0.2535	0	0.1076	0.118	0.5111	0	0
0.015	0.1294	0.1022	0	0	0.086	0.066	0.207	0	0.6162	0	0.0249
0.0138	0.1335	0.1022	0	0	0.2023	0.0314	0.194	0	0.5723	0	0
0.0136	0.1376	0.1022	0	0	0.2483	0.0196	0.2015	0	0.5307	0	0
0.0135	0.1412	0.1022	0	0	0.281	0.0095	0.2124	0	0.4971	0	0
0.0319	0.2421	0.1139	0	0	0	0.3972	0.6028	0	0	0	0

The minimum and maximum expected return for Mean-Semivariance-Absolute deviation with chance constraints is 0.0551 and 0.1139 respectively.

The main problem for Mean-Semivariance-CVaR with chance constraint is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_i \geq \sum_{j=1}^{n}[(-r_{ij}X_j) - \eta] : i = 1, 2, \ldots, s \\
& \eta + \frac{1}{(1-\alpha_{\mathrm{RM}})s}\sum_{i=1}^{s}(a_i) \leq z \\
& a_i \geq 0 : i = 1, 2, \ldots, s \\
& -\boldsymbol{X}'\boldsymbol{\mu} + d + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{5.24}
$$

This problem is solved using procedure outlined and the different solutions are given in Table 5.7.

Table 5.7 Semivariance, Expected return and CVaR along with corresponding fractions to invest for Mean-Semivariance-CVaR with chance constraint.

Semivariance	CVaR	Expected return	Fraction to invest in nine securities								
0.0082	0.1878	0.0551	0	0	0.1738	0	0.0716	0.5548	0.1998	0	0
0.0058	0.2007	0.0551	0	0.3384	0.0059	0	0.0923	0.3215	0.242	0	0
0.0056	0.2135	0.0551	0	0.409	0	0	0.0723	0.2224	0.2964	0	0
0.0056	0.2264	0.0551	0	0.4091	0	0	0.0703	0.2307	0.2899	0	0
0.0084	0.2213	0.0669	0	0	0.1819	0	0.0956	0.4359	0.2866	0	0
0.0068	0.2342	0.0669	0	0.2658	0.0469	0	0.1101	0.2407	0.3365	0	0
0.0068	0.2471	0.0669	0	0.3372	0.0208	0	0.1285	0.1925	0.321	0	0
0.0068	0.26	0.0669	0	0.3421	0.0259	0	0.1243	0.1867	0.321	0	0
0.0091	0.2583	0.0786	0	0	0.1954	0	0.1274	0.3073	0.3699	0	0
0.0084	0.2715	0.0786	0	0.1666	0.1361	0	0.1043	0.1587	0.4343	0	0
0.0085	0.2846	0.0786	0	0.2069	0.1457	0	0.1228	0.1394	0.3852	0	0
0.0085	0.2978	0.0786	0	0.2231	0.1464	0	0.13	0.1306	0.3698	0	0
0.0106	0.3002	0.0904	0	0	0.2129	0	0.1689	0.1648	0.4534	0	0
0.0105	0.311	0.0904	0	0.0267	0.2586	0	0.113	0.1067	0.4949	0	0
0.0105	0.3218	0.0904	0	0.0384	0.2624	0	0.1158	0.0981	0.4853	0	0
0.0105	0.3326	0.0904	0	0.0407	0.2615	0	0.1169	0.0969	0.484	0	0
0.0136	0.3485	0.1022	0	0	0.2369	0	0.2222	0.0037	0.5371	0	0
0.0135	0.3603	0.1022	0	0	0.2684	0.0106	0.2097	0	0.5113	0	0
0.0135	0.372	0.1022	0	0	0.2788	0.0017	0.2175	0	0.502	0	0
0.0135	0.3838	0.1022	0	0	0.2845	0.0017	0.2185	0	0.4953	0	0
0.0319	0.4649	0.1139	0	0	0	0.397	0.603	0	0	0	0
0.0319	0.4751	0.1139	0	0	0	0.3972	0.6028	0	0	0	0
0.0319	0.4853	0.1139	0	0	0	0.3972	0.6028	0	0	0	0
0.0319	0.4955	0.1139	0	0	0	0.3972	0.6028	0	0	0	0

The minimum and maximum expected return for Mean-Semivariance-CVaR with chance constraint is 0.0551 and 0.1139 respectively. The minimum and the maximum CVaR value are 0.1878 and 0.4955 respectively while the minimum and maximum semivariance values are 0.0082 and 0.0319 respectively.

The main problem for Mean-Semivariance-CDaR is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d - (r_{ij}X_j)] : i = 1, 2, \ldots, s \\
& y_i \geq 0 : i = 1, 2, \ldots, s \\
& a_j \geq \left\{\sum_{i=1}^{n}(1 + \sum_{t=1}^{k} r_{ti})X_i\right\} - \left\{\sum_{i=1}^{n}(1 + \sum_{t=1}^{j} r_{ti})X_i\right\} - \eta \\
& k = 1, 2, \ldots, j \\
& a_j \geq 0 \\
& j = 1, 2, \ldots, s \\
& \eta + \frac{1}{(1 - \alpha_{\mathrm{RM}})s}\sum_{j=1}^{s}(a_j) \leq z \qquad (5.25) \\
& -\boldsymbol{X}'\boldsymbol{\mu} + d + \sqrt{\boldsymbol{X}'\boldsymbol{C}\boldsymbol{X}}\,\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

This problem is solved using procedure outlined and the different solutions are given in Table 5.8.

Table 5.8 Semivariance, Expected return and CDaR along with corresponding fractions to invest for Mean-Semivariance-CDaR with chance constraint.

Semivariance	CDaR	Expected return	Fraction to invest in nine securities								
0.0099	0	0.0853	0	0.1705	0.3361	0	0.1012	0.0361	0.3561	0	0
0.0097	0.0102	0.0853	0	0.1249	0.285	0	0.0837	0.0757	0.4306	0	0
0.0096	0.0205	0.0853	0	0.1049	0.2422	0	0.0928	0.1039	0.4562	0	0
0.0096	0.0307	0.0853	0	0.1016	0.2004	0	0.1144	0.1228	0.4607	0	0
0.0109	0	0.091	0	0.0738	0.3731	0	0.089	0.0203	0.4438	0	0
0.0107	0.0092	0.091	0	0.0564	0.3324	0	0.0945	0.0489	0.4679	0	0
0.0106	0.0185	0.091	0	0.0135	0.2972	0	0.0891	0.0889	0.5113	0	0
0.0106	0.0277	0.091	0	0.0095	0.2593	0	0.1075	0.1079	0.5157	0	0
0.012	0	0.0968	0	0	0.4292	0	0.1125	0	0.4583	0	0
0.0119	0.0071	0.0968	0	0	0.3826	0	0.1097	0.01	0.4977	0	0
0.0119	0.0142	0.0968	0	0	0.353	0	0.1249	0.0259	0.4962	0	0
0.0119	0.0213	0.0968	0	0	0.329	0	0.1252	0.03	0.5159	0	0
0.015	0.0228	0.1025	0	0	0.3663	0.119	0.2137	0	0.3011	0	0
0.014	0.0282	0.1025	0	0	0.3296	0.0644	0.2089	0	0.3971	0	0
0.0137	0.0337	0.1025	0	0	0.301	0.0311	0.2144	0	0.4535	0	0
0.0137	0.0391	0.1025	0	0	0.2783	0.0095	0.2225	0	0.4897	0	0
0.0203	0.0613	0.1082	0	0	0.188	0.2673	0.3016	0	0.2431	0	0
0.0178	0.0671	0.1082	0	0	0.1524	0.213	0.3003	0	0.3344	0	0
0.0174	0.0729	0.1082	0	0	0.1226	0.186	0.3073	0	0.384	0	0
0.0173	0.0787	0.1082	0	0	0.0958	0.1687	0.3178	0	0.4177	0	0
0.0319	0.3211	0.1139	0	0	0	0.3972	0.6028	0	0	0	0

The minimum and maximum expected return for Mean-Semivariance-CDaR with chance constraint is 0.0853 and 0.1139 respectively.

The main problem for Mean-Semivariance-UPDR is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & SV_d(\boldsymbol{X}) = \frac{1}{s}\sum_{i=1}^{s} y_i^2 \\
\text{subject to} \quad & y_i \geq \sum_{j=1}^{n}[d-(r_{ij}X_j)] : i = 1,2,\ldots,s \\
& y_i \geq 0 : i = 1,2,\ldots,s \\
& a_i^j \geq -\boldsymbol{rX} - \eta_i - \mathrm{M}y_k^j : i = 1,2,\ldots,k-1, \forall j \\
& a_k^j \geq -\boldsymbol{rX} - \eta_k : \forall j \\
& a_i^j \geq 0, \forall i,j \qquad\qquad (5.26) \\
& \sum_{i=1}^{k-1} p_i^{'} \times \left\{\eta_i + \sum_{j=1}^{s}\frac{y_i^j}{s}\right\} + p_k^{'} \times \left\{\eta_k + \frac{\sum_{j=1}^{s}[\frac{y_k^j}{s}]}{(1-\alpha)}\right\} \leq z \\
& -\boldsymbol{X}^{'}\boldsymbol{\mu} + d + \sqrt{\boldsymbol{X}^{'}\boldsymbol{CX}}\Phi^{-1}(\alpha_c) \leq 0 \\
& \sum_{j=1}^{n} X_j = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

This problem is solved using procedure outlined and the different solutions are given in Table 5.9. The priority vector $\boldsymbol{p}$ and weight vector $\boldsymbol{w}$ are assumed to be [0.1 0.2 0.7] and 0.5 respectively.

Table 5.9 Semivariance, Expected return and UPDR along with corresponding fractions to invest for Mean-Semivariance-UPDR with chance constraint.

Semivariance	UPDR	Expected return	Fraction to invest in nine securities								
0.0081	0.15	0.0551	0	0	0.1579	0	0.0747	0.5564	0.2109	0	0
0.0058	0.1593	0.0551	0	0.3641	0.0062	0	0.1075	0.3139	0.2083	0	0
0.0056	0.1687	0.0551	0	0.4974	0	0	0.1165	0.2102	0.1758	0	0
0.0056	0.178	0.0551	0	0.4979	0	0	0.107	0.2109	0.1843	0	0
0.0084	0.1763	0.0669	0	0	0.1806	0	0.0956	0.4359	0.2879	0	0
0.0069	0.1856	0.0669	0	0.2503	0.0515	0	0.0998	0.2458	0.3526	0	0
0.0068	0.1948	0.0669	0	0.3263	0.0247	0	0.1158	0.1924	0.3408	0	0
0.0068	0.2041	0.0669	0	0.3209	0.0185	0	0.1195	0.1979	0.3432	0	0
0.0091	0.2058	0.0786	0	0	0.1954	0	0.1274	0.3073	0.3699	0	0
0.0084	0.2148	0.0786	0	0.1676	0.1238	0	0.1122	0.1622	0.4343	0	0
0.0085	0.2237	0.0786	0	0.1593	0.1164	0	0.1129	0.1684	0.4429	0	0
0.0085	0.2327	0.0786	0	0.0711	0.1159	0	0.0759	0.2093	0.5278	0	0
0.0107	0.239	0.0904	0	0	0.212	0	0.1708	0.1658	0.4514	0	0
0.0105	0.2459	0.0904	0	0.019	0.2479	0	0.1128	0.1137	0.5067	0	0
0.0105	0.2529	0.0904	0	0.0064	0.2488	0	0.1045	0.1187	0.5216	0	0
0.0105	0.2598	0.0904	0	0.0008	0.2373	0	0.1208	0.135	0.506	0	0
0.0136	0.277	0.1022	0	0	0.2371	0	0.2222	0.0037	0.5369	0	0
0.0135	0.2846	0.1022	0	0	0.2877	0.0017	0.2191	0	0.4914	0	0
0.0135	0.2922	0.1022	0	0	0.2864	0.0028	0.2181	0	0.4927	0	0
0.0135	0.2997	0.1022	0	0	0.2835	0.0027	0.2176	0	0.4962	0	0
0.0319	0.3669	0.1139	0	0	0	0.3971	0.6029	0	0	0	0
0.0319	0.3764	0.1139	0	0	0	0.3972	0.6028	0	0	0	0
0.0319	0.3859	0.1139	0	0	0	0.3972	0.6028	0	0	0	0
0.0319	0.3953	0.1139	0	0	0	0.3972	0.6028	0	0	0	0

The minimum and maximum expected return for Mean-semivariance-UPDR with chance constraint is 0.0551 and 0.1139 respectively.

The four models give different solutions for different expected returns. For the highest possible return the solution(s) is got by investing certain fractions in security 4 and security 5 for all the four models. But for lower returns the solutions in the four models are different from each other. The minimum possible return for Mean-SV-Absolute deviation, Mean-SV-CVaR and Mean-SV-UPDR is 0.0551, but the four different solutions we looked at are different for each of the three models. The solutions tend to get closer to each other as the expected return gets closer to the maximum possible return. The investor can decide on a model and then make his choice based on the different solutions or can look at all four models for the same expected return and then make his decision.

In this chapter we have discussed four different models available to the investor. The investor can decide on a particular risk measure along with semivariance and then generate solutions based on that. The investor can also generate solutions based on all the four methods and then choose a solution. An investor can also include chance constraint in his models if needed.

In the next chapter sensitivity analysis of the four models is derived and illustrated with numerical examples.

CHAPTER 6

SENSITIVITY ANALYSIS

Portfolio selection models we propose require an input of scenarios regarding the future behavior of securities and optimizes the model based on this input. Since this input is not fixed, the investor would be better prepared if there is a feedback mechanism scheme wherein he can see the sensitivity of the portfolio for changes in input. If the investor sees that the solutions are not stable for small perturbations in the return matrix, it warns him that he cannot place great confidence on his portfolio composition. This type of analysis greatly enhances the investor and gets him prepared with respect to the confidence in his portfolio composition.

Best and Grauer (1991b) conducted sensitivity analysis for the standard E-V portfolio selection problem using a general form of parametric quadratic programming. They showed how the portfolio composition is affected for changes in the mean return and changes in the right hand side of the constraints. To the best of our knowledge there is no paper on sensitivity analysis for portfolio selection when the input is a return matrix.

We have showed how to conduct sensitivity analysis for the four models we propose for single-period portfolio selection. For each of the models we have derived the problem to solve for sensitivity analysis and in the last section of the chapter numerical examples have been dealt in detail to illustrate sensitivity analysis.

Notations

Let us define some notations we will be using throughout the chapter.

s	—	number of scenarios of information available about the future
n	—	number of securities
m	—	number of priorities for UPDR
α	—	confidence level
$\boldsymbol{r}_{s\times n}$	—	return matrix for the securities
$\boldsymbol{X}_{n\times 1}$	—	investment vector corresponding to n securities
$\boldsymbol{Y}_{s\times 1}$	—	variables corresponding to semivariance
$\boldsymbol{A}_{s\times 1}$	—	variables corresponding to absolute deviation or CVaR or CDaR
d	—	target expected return
z	—	target second risk measure value
$g,\ h$	—	positive finite integers
$\mathbf{I}_{g\times g}$	—	identity matrix
$\mathbf{0}_{g\times g}$	—	matrix of zeros
$\mathbf{1}_{g\times h}$	—	matrix of ones
$\mathbf{e}^i_{g\times 1}$	—	i^{th} unit vector
$\mathbf{F}^j_{g\times h}$	—	$\frac{1}{j}\mathbf{1}_{g\times 1}\mathbf{1}_{h\times 1}{}'$
$\mathbf{SP}^1_{s\times s}$	—	$\mathbf{F}^s_{s\times s} - \mathbf{I}_{s\times s}$
$\mathbf{SP}^2_{s\times s}$	—	$\mathbf{I}_{s\times s} - \mathbf{F}^s_{s\times s}$

Throughout this chapter when matrix $\mathbf{1}$ is used without any index for the size, it is assumed it is a vector of size $s \times 1$. Similarly the matrices $\mathbf{I}$, $\mathbf{0}$, $\mathbf{SP}^1$ and $\mathbf{SP}^2$ when used without any index the size is assumed to be $s \times s$.

6.1 Sensitivity Analysis of Mean-Semivariance-Absolute Deviation

The first model we solved for the single period portfolio selection was Mean-SV-Absolute deviation. Let us suppose we have solved the main optimization problem (6.1) for a particular Absolute deviation value z and Expected return d and the solutions are known.

$$\begin{aligned} \text{Minimize} \quad & \text{Semivariance}(\boldsymbol{X}) \\ \text{subject to} \quad & \text{Absdev}(\boldsymbol{X}) = z \\ & \text{E}(\boldsymbol{X}) = d \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned} \tag{6.1}$$

Since $\boldsymbol{r}$ is the only input for our problem, an investor would like to see how the solutions behave for changes in $\boldsymbol{r}$. We explain how to conduct the sensitivity analysis by solving a closely related problem which is derived using the following steps. First let us rewrite the optimization problem in terms of the input $\boldsymbol{r}$:

$$\text{Minimize} \quad \text{SV}_d(\boldsymbol{X}) = \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y}$$

$$\text{subject to} \quad \boldsymbol{Y} \geq \boldsymbol{1}d - \boldsymbol{r}\boldsymbol{X} \tag{6.2}$$

$$\boldsymbol{Y} \geq 0 \tag{6.3}$$

$$\boldsymbol{A} \geq \boldsymbol{r}\boldsymbol{X} - \boldsymbol{1}d \tag{6.4}$$

$$\boldsymbol{A} \geq \boldsymbol{1}d - \boldsymbol{r}\boldsymbol{X} \tag{6.5}$$

$$\frac{1}{s}\boldsymbol{1}'\boldsymbol{A} = z \tag{6.6}$$

$$\boldsymbol{A} \geq 0 \tag{6.7}$$

$$\frac{1}{s}\boldsymbol{1}'\boldsymbol{r}\boldsymbol{X} = d \tag{6.8}$$

$$\boldsymbol{1}'\boldsymbol{X} = 1 \tag{6.9}$$

$$\boldsymbol{X} \geq 0 \tag{6.10}$$

Equations 6.2 and 6.3 represent the constraints for semivariance. Equations 6.4, 6.5 and 6.6 represent the constraints for absolute deviation. Equation 6.8 represents the constraint for expected return. Finally equations 6.9 and 6.10 represent the budget and non negativity constraints respectively. Use the left hand side of constraint 6.8 to replace d in all other constraints where it appears to get the following optimization problem:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} & (6.11) \\
& \boldsymbol{Y} \geq 0 \\
& \boldsymbol{A} \geq \boldsymbol{rX} - \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} & (6.12) \\
& \boldsymbol{A} \geq \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} & (6.13) \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{A} = z \\
& \boldsymbol{A} \geq 0 \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{rX} = d \\
& \mathbf{1}'\boldsymbol{X} = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

Let us consider all the inequality constraints which have $\boldsymbol{r}$ and simplify them. Equation 6.11 can be simplified using basic properties of matrix algebra as follows:

$$\begin{aligned}
& \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \frac{1}{s}\mathbf{1}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}^{s}_{ss}\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}^{s}_{ss}\boldsymbol{rX} - \mathbf{I}\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{Y} \geq [\mathbf{F}^{s}_{ss} - \mathbf{I}]\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{SP}^{1}\boldsymbol{rX} & (6.14)
\end{aligned}$$

Equation 6.12 can be simplified as follows:

$$
\begin{aligned}
& \boldsymbol{A} \geq \boldsymbol{rX} - \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \boldsymbol{rX} - \frac{1}{s}\mathbf{1}\mathbf{1}'\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \boldsymbol{rX} - \mathbf{F}^{s}_{ss}\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \mathbf{I}\boldsymbol{rX} - \mathbf{F}^{s}_{ss}\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq [\mathbf{I} - \mathbf{F}^{s}_{ss}]\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \mathbf{SP}^{2}\boldsymbol{rX} \qquad (6.15)
\end{aligned}
$$

Equation 6.13 can be simplified as follows:

$$
\begin{aligned}
& \boldsymbol{A} \geq \mathbf{1}\frac{1}{s}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \frac{1}{s}\mathbf{1}\mathbf{1}'\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \mathbf{F}^{s}_{ss}\boldsymbol{rX} - \boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \mathbf{F}^{s}_{ss}\boldsymbol{rX} - \mathbf{I}\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq [\mathbf{F}^{s}_{ss} - \mathbf{I}]\boldsymbol{rX} \\
\Rightarrow \quad & \boldsymbol{A} \geq \mathbf{SP}^{1}\boldsymbol{rX} \qquad (6.16)
\end{aligned}
$$

We can replace equations 6.11, 6.13 and 6.12 by 6.14, 6.16 and 6.15 respectively in the optimization problem to get the following simplified optimization problem:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \\
& \boldsymbol{Y} \geq 0 \\
& \boldsymbol{A} \geq \mathbf{SP}^2\boldsymbol{r}\boldsymbol{X} \\
& \boldsymbol{A} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{A} = z \\
& \boldsymbol{A} \geq 0 \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d \\
& \mathbf{1}'\boldsymbol{X} = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

Consider all the inequality constraints other than the non-negativity constraints

$$\boldsymbol{Y} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X}$$

$$\boldsymbol{A} \geq \mathbf{SP}^2\boldsymbol{r}\boldsymbol{X}$$

$$\boldsymbol{A} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X}$$

We can bring all the variables to one side to get the following set of equations

$$-\boldsymbol{Y} + \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \leq 0$$

$$-\boldsymbol{A} + \mathbf{SP}^2\boldsymbol{r}\boldsymbol{X} \leq 0$$

$$-\boldsymbol{A} + \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \leq 0$$

These constraints can be written in array format as follows:

$$\left[\begin{array}{cc|c} -\mathbf{I} & \mathbf{0} & \mathbf{SP}^1\boldsymbol{r} \\ \mathbf{0} & -\mathbf{I} & \mathbf{SP}^2\boldsymbol{r} \\ \mathbf{0} & -\mathbf{I} & \mathbf{SP}^1\boldsymbol{r} \end{array}\right] \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \hline \boldsymbol{X} \end{array}\right] \leq \left[\begin{array}{c} \mathbf{0}_{(s+s+s)\times 1} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{c|c} \mathbf{B}^1 & \mathbf{B}^2\ \boldsymbol{r} \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] \leq \left[\begin{array}{c} \mathbf{0}_{(s+s+s)\times 1} \end{array}\right]$$

$$\Rightarrow \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s+s)\times 1} \tag{6.17}$$

where

$$\mathbf{B}^1_{(s+s+s)\times(s+s)} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{B}^2_{(s+s+s)\times(n)} = \begin{bmatrix} \mathbf{SP}^1 \\ \mathbf{SP}^2 \\ \mathbf{SP}^1 \end{bmatrix}$$

$$\mathbf{C}^1_{(s+s)\times 1} = \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{A} \end{bmatrix} \quad \text{and} \quad \mathbf{C}^2_{(n)\times 1} = \begin{bmatrix} \boldsymbol{X} \end{bmatrix}$$

Equation 6.17 represents the inequality constraints in array form. Consider all the equality constraints:

$$\frac{1}{s}\mathbf{1}'\boldsymbol{A} = z$$
$$\frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d$$
$$\mathbf{1}'\boldsymbol{X} = 1$$

These can be written in array form as follows:

$$\left[\begin{array}{cc|c} \mathbf{0}_{1\times s} & \frac{1}{s}\mathbf{1}' & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & \frac{1}{s}\mathbf{1}'\boldsymbol{r} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & \mathbf{1}_{1\times n} \end{array}\right] \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \hline \boldsymbol{X} \end{array}\right] = \left[\begin{array}{c} z \\ \hline d \\ \hline 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \mathbf{Beq}^1\mathbf{C}^1 \\ \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 \\ \mathbf{Beq}^3\mathbf{C}^2 \end{bmatrix} = \begin{bmatrix} z \\ d \\ 1 \end{bmatrix} \tag{6.18}$$

where

$$\mathbf{Beq}^1_{1\times(s+s)} = \begin{bmatrix} \mathbf{0}_{1\times s} & \frac{1}{s}\mathbf{1}' \end{bmatrix} \quad , \quad \mathbf{Beq}^2_{1\times(n)} = \begin{bmatrix} \frac{1}{s}\mathbf{1}' \end{bmatrix}$$

$$\text{and} \qquad \mathbf{Beq}^3_{1\times(n)} = \begin{bmatrix} \mathbf{1}_{1\times n} \end{bmatrix}$$

Equation 6.18 represents the equality constraints in array form. The optimization problem can be rewritten using 6.18 and 6.17 to get the final problem in terms of arrays:

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\ \text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s+s)\times 1} \\ & \mathbf{Beq}^1\mathbf{C}^1 = z \\ & \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 = d \\ & \mathbf{Beq}^3\mathbf{C}^2 = 1 \\ & \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \geq \mathbf{0}_{(s+s+n)\times 1} \end{aligned}$$

The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem and the scalars z and d are also known. To conduct the sensitivity analysis with respect to changes in $\boldsymbol{r}$ the following closely related problem 6.19 can be solved.

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s+s)\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = z \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = d \\
& \mathbf{Beq}^3\mathbf{C}^2 = 1 \\
& \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+s+n)\times 1}
\end{aligned} \tag{6.19}
$$

The term $t\ \mathbf{q}$ captures the change in $\boldsymbol{r}$, allowing the investor to perform the sensitivity analysis. The parameter t varies between specified lower and upper bounds, which could be $-\infty$ to ∞ respectively and $\mathbf{q}$ is a matrix of zeros with the same size as $\boldsymbol{r}$. To do sensitivity analysis for any particular entry of $\boldsymbol{r}$, that corresponding entry in $\mathbf{q}$ will have 1 in it.

6.2 Sensitivity Analysis of Mean-Semivariance-CVaR

The second model we solved for single-period portfolio selection was Mean-SV-CVaR. Let us suppose we have solved the main optimization problem (6.20) for a particular CVaR value z and Expected return d and the solutions are known.

$$
\begin{aligned}
\text{Minimize} \quad & \text{Semivariance}(\boldsymbol{X}) \\
\text{subject to} \quad & \text{CVaR}(\boldsymbol{X}) = z \\
& \text{E}(\boldsymbol{X}) = d \\
& \sum_{i=1}^{n} X_i = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned} \tag{6.20}
$$

Since $\boldsymbol{r}$ is the only input for our problem, an investor would like to see how the solutions behave for changes in $\boldsymbol{r}$. We explain how to conduct the sensitivity analysis by solving a closely related problem which is derived using the following steps. First let us rewrite the optimization problem in terms of the input $\boldsymbol{r}$:

$$
\begin{aligned}
\text{Minimize} \quad & \mathrm{SV}_d(\boldsymbol{X}) = \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}d - \boldsymbol{r}\boldsymbol{X} && (6.21) \\
& \boldsymbol{Y} \geq 0 && (6.22) \\
& \boldsymbol{A} \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta && (6.23) \\
& \eta + \frac{1}{s(1-\alpha)}\mathbf{1}'\boldsymbol{A} = z && (6.24) \\
& \boldsymbol{A} \geq 0 && (6.25) \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d && (6.26) \\
& \mathbf{1}'\boldsymbol{X} = 1 && (6.27) \\
& \boldsymbol{X} \geq 0 && (6.28)
\end{aligned}
$$

Equations 6.21 and 6.22 represent the constraints for semivariance. Equations 6.23, 6.24 and 6.25 represent the constraints for CVaR. Equation 6.26 represents the constraint for expected return. Finally equations 6.9 and 6.10 represent the budget and non-negativity constraints respectively. Use the left hand side of constraint 6.26 to replace d in all other

constraints where it appears to get the following optimization problem:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}^{'}\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
& \boldsymbol{Y} \geq 0 \\
& \boldsymbol{A} \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta \\
& \eta + \frac{1}{s(1-\alpha)}\mathbf{1}^{'}\boldsymbol{A} = z \\
& \boldsymbol{A} \geq 0 \\
& \frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} = d \\
& \mathbf{1}^{'}\boldsymbol{X} = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
\tag{6.29}
$$

Let us consider all the inequality constraints which have $\boldsymbol{r}$ and simplify them. Equation 6.29 can be simplified using basic steps of matrix algebra as follows:

$$
\begin{aligned}
& \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \frac{1}{s}\mathbf{1}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}^{s}_{ss}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}^{s}_{ss}\boldsymbol{r}\boldsymbol{X} - \mathbf{I}\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq [\mathbf{F}^{s}_{ss} - \mathbf{I}]\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{SP}^{1}\boldsymbol{r}\boldsymbol{X}
\end{aligned}
\tag{6.30}
$$

Replace equation 6.29 by 6.30 in the optimization problem to get the following simplified optimization problem:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \\
& \boldsymbol{Y} \geq 0 \\
& \boldsymbol{A} \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta \\
& \eta + \frac{1}{s(1-\alpha)}\mathbf{1}'\boldsymbol{A} = z \\
& \boldsymbol{A} \geq 0 \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d \\
& \mathbf{1}'\boldsymbol{X} = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}
$$

Consider all the inequality constraints other than the non-negativity constraints

$$
\begin{aligned}
& \boldsymbol{Y} \geq \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \\
& \boldsymbol{A} \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta
\end{aligned}
$$

We can bring all the variables to one side to get the following set of equations

$$
\begin{aligned}
& -\boldsymbol{Y} + \mathbf{SP}^1\boldsymbol{r}\boldsymbol{X} \leq 0 \\
& -\boldsymbol{A} - \boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta \leq 0
\end{aligned}
$$

These constraints can be written in an array format as follows.

$$
\left[\begin{array}{ccc|c} -\mathbf{I} & \mathbf{0} & \mathbf{0}_{s\times 1} & \mathbf{SP}^1\boldsymbol{r} \\ \mathbf{0} & -\mathbf{I} & -\mathbf{1} & -\boldsymbol{r} \end{array}\right]
\left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \\ \hline \boldsymbol{X} \end{array}\right]
\leq
\left[\begin{array}{c} \mathbf{0}_{(s+s)\times 1} \end{array}\right]
$$

$$\Rightarrow \quad \left[\begin{array}{c|c} \mathbf{B}^1 & \mathbf{B}^2\ \boldsymbol{r} \end{array}\right] \quad \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] \quad \leq \quad \left[\mathbf{0}_{(s+s)\times 1} \right]$$

$$\Rightarrow \quad \mathbf{B}^1\mathbf{C}^1 \quad + \quad \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \quad \leq \quad \mathbf{0}_{(s+s)\times 1} \tag{6.31}$$

where

$$\mathbf{B}^1_{(s+s)\times(s+s+1)} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0}_{s\times 1} \\ \mathbf{0} & -\mathbf{I} & -\mathbf{1} \end{bmatrix} \quad \text{and} \quad \mathbf{B}^2_{(s+s)\times(n)} = \begin{bmatrix} \mathbf{SP}^1 \\ -\mathbf{I} \end{bmatrix}$$

$$\mathbf{C}^1_{(s+s+1)\times 1} = \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \end{bmatrix} \quad \text{and} \quad \mathbf{C}^2_{n\times 1} = \left[\boldsymbol{X} \right]$$

Equation 6.31 represents the inequality constraints in array form. Consider all the equality constraints:

$$\eta + \frac{1}{s(1-\alpha)}\mathbf{1}'\boldsymbol{A} = z$$
$$\frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d$$
$$\mathbf{1}'\boldsymbol{X} = 1$$

These can be written in array form as follows:

$$\left[\begin{array}{ccc|c} \mathbf{0}_{1\times s} & \frac{1}{s(1-\alpha)}\mathbf{1}' & 1 & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & 0 & \frac{1}{s}\mathbf{1}'\boldsymbol{r} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & 0 & \mathbf{1}_{1\times n} \end{array}\right] \quad \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \\ \hline \boldsymbol{X} \end{array}\right] \quad = \quad \left[\begin{array}{c} z \\ \hline d \\ \hline 1 \end{array}\right]$$

$$\Rightarrow \quad \begin{bmatrix} \mathbf{Beq}^1\mathbf{C}^1 \\ \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 \\ \mathbf{Beq}^3\mathbf{C}^2 \end{bmatrix} \quad = \quad \begin{bmatrix} z \\ d \\ 1 \end{bmatrix} \tag{6.32}$$

where

$$\mathbf{Beq}^1_{1\times(s+s+1)} = \left[\begin{array}{ccc} \mathbf{0}_{1\times s} & \frac{1}{s(1-\alpha)}\mathbf{1}' & 1 \end{array}\right] \qquad \text{and} \qquad \mathbf{Beq}^2_{1\times n} = \left[\begin{array}{c} \frac{1}{s}\mathbf{1}' \end{array}\right]$$

$$\text{and} \qquad \mathbf{Beq}^3_{1\times(n)} = \left[\begin{array}{c} \mathbf{1}_{1\times n} \end{array}\right]$$

Equation 6.32 represents equality constraints in array form. The optimization problem can be rewritten using 6.32 and 6.31 to get the final problem in terms of arrays:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s)\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = z \\
& \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 = d \\
& \mathbf{Beq}^3\mathbf{C}^2 = 1 \\
& \left[\begin{array}{c} \mathbf{C}^1 \\ \mathbf{C}^2 \end{array}\right] \quad \geq \quad \mathbf{0}_{(s+s+1+n)\times 1}
\end{aligned} \tag{6.33}$$

The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem and the scalars z and d are also known. To conduct the sensitivity analysis with respect to changes in $\boldsymbol{r}$ the following closely related problem 6.34 can be solved.

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s)\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = z \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = d \\
& \mathbf{Beq}^2\mathbf{C}^2 = 1 \\
& \left[\begin{array}{c} \mathbf{C}^1 \\ \mathbf{C}^2 \end{array}\right] \quad \geq \quad \mathbf{0}_{(s+s+1+n)\times 1}
\end{aligned} \tag{6.34}$$

The term $t\,\mathbf{q}$ captures the change in $\boldsymbol{r}$, allowing the investor to perform the sensitivity analysis. The parameter t varies between specified lower and upper bounds, which could be $-\infty$ to ∞ respectively and $\mathbf{q}$ is a matrix of zeros with the same size as $\boldsymbol{r}$. To do sensitivity analysis for any particular entry of $\boldsymbol{r}$, that corresponding entry in $\mathbf{q}$ will have 1 in it.

6.3 Sensitivity Analysis of Mean-Semivariance-CDaR

The third model we solved for single period portfolio selection was Mean-SV-CDaR. Let us suppose we have solved the main optimization problem (6.35) for a particular CDaR value z and Expected return d and the solutions are known.

$$\begin{aligned} \text{Minimize} \quad & \text{Semivariance}(\boldsymbol{X}) \\ \text{subject to} \quad & \text{CDaR}(\boldsymbol{X}) = z \\ & \text{E}(\boldsymbol{X}) = d \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned} \tag{6.35}$$

Since $\boldsymbol{r}$ is the only input for our problem, an investor would like to see how the solutions behave for changes in $\boldsymbol{r}$. We explain how to conduct the sensitivity analysis by solving a closely related problem which is derived using the following steps. First let us rewrite the optimization problem in terms of the input $\boldsymbol{r}$:

$$
\begin{aligned}
\text{Minimize} \quad & \mathrm{SV}_d(\boldsymbol{X}) = \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}d - \boldsymbol{r}\boldsymbol{X} && (6.36) \\
& \boldsymbol{Y} \geq 0 && (6.37) \\
& A_j \geq \left\{\sum_{i=1}^{n}(1+\sum_{h=1}^{k} r_{hi})X_i\right\} - \left\{\sum_{i=1}^{n}(1+\sum_{h=1}^{j} r_{hi})X_i\right\} - \eta && (6.38) \\
& k = 1,2,\ldots,j \text{ for } j = 1,2,\ldots,s \\
& \eta + \frac{1}{s(1-\alpha)}\mathbf{1}'\boldsymbol{A} = z && (6.39) \\
& \boldsymbol{A} \geq 0 && (6.40) \\
& \frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d && (6.41) \\
& \mathbf{1}'\boldsymbol{X} = 1 && (6.42) \\
& \boldsymbol{X} \geq 0 && (6.43)
\end{aligned}
$$

Equations 6.36 and 6.37 represent the constraints for semivariance. Equations 6.38, 6.39 and 6.40 represent the constraints for CDaR. Equation 6.41 represents the constraint for expected return. Finally equations 6.42 and 6.43 represent the budget and non-negativity constraints respectively. Use the left hand side of constraint 6.41 to replace d in all other

constraints where it appears to get the following optimization problem:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}^{'}\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} && (6.44) \\
& \boldsymbol{Y} \geq 0 \\
& A_j \geq \left\{\sum_{i=1}^{n}(1+\sum_{h=1}^{k} r_{hi})X_i\right\} - \left\{\sum_{i=1}^{n}(1+\sum_{h=1}^{j} r_{hi})X_i\right\} - \eta && (6.45) \\
& k = 1, 2, \ldots, j \text{ for } j = 1, 2, \ldots, s \\
& \eta + \frac{1}{s(1-\alpha)}\mathbf{1}^{'}\boldsymbol{A} = z \\
& \boldsymbol{A} \geq 0 \\
& \frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} = d \\
& \mathbf{1}^{'}\boldsymbol{X} = 1 \\
& \boldsymbol{X} \geq 0
\end{aligned}$$

Let us consider all the inequality constraints which have $\boldsymbol{r}$ and simplify them. Equation 6.44 can be simplified using basic properties of matrix algebra as follows:

$$\begin{aligned}
& \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \frac{1}{s}\mathbf{1}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}_{ss}^{s}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}_{ss}^{s}\boldsymbol{r}\boldsymbol{X} - \mathbf{I}\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq [\mathbf{F}_{ss}^{s} - \mathbf{I}]\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{SP}^{1}\boldsymbol{r}\boldsymbol{X} && (6.46)
\end{aligned}$$

Consider 6.45 which can be further simplified as follows:

$$
\begin{aligned}
& A_j \geq \left\{ \sum_{i=1}^{n} (1 + \sum_{h=1}^{k} r_{hi}) X_i \right\} - \left\{ \sum_{i=1}^{n} (1 + \sum_{h=1}^{j} r_{hi}) X_i \right\} - \eta \\
\Rightarrow \quad & A_j \geq \left\{ \sum_{i=1}^{n} (\sum_{h=1}^{k} r_{hi}) X_i \right\} - \left\{ \sum_{i=1}^{n} (\sum_{h=1}^{j} r_{hi}) X_i \right\} - \eta \\
\Rightarrow \quad & A_j \geq \left\{ \sum_{i=1}^{n} (\sum_{h=1}^{k} r_{hi} - \sum_{h=1}^{j} r_{hi}) X_i \right\} - \eta
\end{aligned}
$$

The above equation is realized for every $j = 1, 2, \ldots, s$ where $k = 1, 2, \ldots, j$. Therefore the number of constraints would be $\frac{s(s+1)}{2}$. The constraints can be written in array format as follows:

$$
\begin{bmatrix} \mathbf{e}^1 \\ \hline \mathbf{e}^2 \\ \mathbf{e}^2 \\ \hline \vdots \\ \hline \mathbf{e}^s \\ \mathbf{e}^s \\ \vdots \\ \mathbf{e}^s \end{bmatrix}
\begin{bmatrix} \boldsymbol{A} \end{bmatrix} \geq
\begin{bmatrix} (\mathbf{e}^1) - (\mathbf{e}^1) \\ \hline (\mathbf{e}^1) - (\mathbf{e}^1 + \mathbf{e}^2) \\ (\mathbf{e}^1 + \mathbf{e}^2) - (\mathbf{e}^1 + \mathbf{e}^2) \\ \hline \vdots \\ \hline (\mathbf{e}^1) - (\mathbf{e}^1 + \mathbf{e}^2 + \cdots + \mathbf{e}^s) \\ (\mathbf{e}^1 + \mathbf{e}^2) - (\mathbf{e}^1 + \mathbf{e}^2 + \cdots + \mathbf{e}^s) \\ \vdots \\ (\sum_{h=1}^{s} \mathbf{e}^h) - (\sum_{h=1}^{s} \mathbf{e}^h) \end{bmatrix}
\begin{bmatrix} \boldsymbol{r} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \end{bmatrix} - \eta \begin{bmatrix} \mathbf{1}_{\frac{s(s+1)}{2} \times 1} \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} \mathbf{NM}^1 \end{bmatrix} \boldsymbol{A} \geq \begin{bmatrix} \mathbf{NM}^2 \end{bmatrix} \boldsymbol{rX} - \eta \begin{bmatrix} \mathbf{1}_{\frac{s(s+1)}{2} \times 1} \end{bmatrix}
$$

$$
\Rightarrow \quad \begin{bmatrix} -\mathbf{NM}^1 \end{bmatrix} \boldsymbol{A} + \begin{bmatrix} \mathbf{NM}^2 \end{bmatrix} \boldsymbol{rX} - \eta \begin{bmatrix} \mathbf{1}_{\frac{s(s+1)}{2} \times 1} \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}_{\frac{s(s+1)}{2} \times 1} \end{bmatrix} \tag{6.47}
$$

The semivariance constraint 6.46 and CDaR constraint 6.47 can be written in array format as follows:

$$\Rightarrow \left[\begin{array}{ccc|c} -\mathbf{I} & \mathbf{0} & \mathbf{0}_{s\times 1} & \mathbf{SP}_1\boldsymbol{r} \\ \hline \mathbf{0}_{\frac{s(s+1)}{2}\times s} & -\mathbf{NM}^1 & -\mathbf{1}_{\frac{s(s+1)}{2}\times 1} & \mathbf{NM}^2\boldsymbol{r} \end{array}\right] \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \\ \hline \boldsymbol{X} \end{array}\right] \leq \left[\begin{array}{c} \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{c|c} \mathbf{B}^1 & \mathbf{B}^2\ \boldsymbol{r} \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] \leq \left[\begin{array}{c} \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \end{array}\right]$$

$$\Rightarrow \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \leq \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \tag{6.48}$$

where

$$\mathbf{B}^1_{(s+\frac{s(s+1)}{2})\times(s+s+1)} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0}_{s\times 1} \\ \mathbf{0}_{\frac{s(s+1)}{2}\times s} & -\mathbf{NM}^1 & -\mathbf{1}_{\frac{s(s+1)}{2}\times 1} \end{bmatrix}$$

$$\mathbf{B}^2_{(s+\frac{s(s+1)}{2})\times n} = \begin{bmatrix} \mathbf{SP}^1 \\ \mathbf{NM}^2 \end{bmatrix}$$

$$\mathbf{C}^1_{(s+s+1)\times 1} = \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \end{bmatrix} \quad \text{and} \quad \mathbf{C}^2_{n\times 1} = \begin{bmatrix} \boldsymbol{X} \end{bmatrix}$$

Consider the equality constraints:

$$\eta + \frac{1}{s(1-\alpha)}\mathbf{1}'\boldsymbol{A} = z$$

$$\frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d$$

$$\mathbf{1}'\boldsymbol{X} = 1$$

The equality constraints can be written in array form as follows:

$$\left[\begin{array}{ccc|c} \mathbf{0}_{1\times s} & \frac{1}{s(1-\alpha)}\mathbf{1}' & 1 & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & 0 & \frac{1}{s}\mathbf{1}'\boldsymbol{r} \\ \hline \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & 0 & \mathbf{1}_{1\times n} \end{array}\right] \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A} \\ \eta \\ \hline \boldsymbol{X} \end{array}\right] = \left[\begin{array}{c} z \\ \hline d \\ \hline 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{c} \mathbf{Beq}^1\mathbf{C}^1 \\ \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 \\ \mathbf{Beq}^3\mathbf{C}^2 \end{array}\right] = \left[\begin{array}{c} z \\ d \\ 1 \end{array}\right] \tag{6.49}$$

where

$$\mathbf{Beq}^1_{1\times(s+s+1)} = \left[\begin{array}{ccc} \mathbf{0}_{1\times s} & \frac{1}{s(1-\alpha)}\mathbf{1}' & 1 \end{array}\right], \quad \mathbf{Beq}^2_{1\times n} = \left[\begin{array}{c} \frac{1}{s}\mathbf{1}' \end{array}\right]$$

$$\text{and} \quad \mathbf{Beq}^3_{1\times n} = \left[\begin{array}{c} \mathbf{1}_{1\times n} \end{array}\right]$$

The optimization problem can be rewritten using 6.49 and 6.48 to get the final problem in terms of arrays:

$$\begin{array}{rl} \text{Minimize} & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\ \text{subject to} & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \le \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \\ & \mathbf{Beq}^1\mathbf{C}^1 = z \\ & \mathbf{Beq}^2\ \boldsymbol{r}\ \mathbf{C}^2 = d \\ & \mathbf{Beq}^3\mathbf{C}^2 = 1 \\ & \left[\begin{array}{c} \mathbf{C}^1 \\ \mathbf{C}^2 \end{array}\right] \ge \mathbf{0}_{(s+s+1+n)\times 1} \end{array}$$

The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem and the scalars z and d are also known. To conduct the sensitivity analysis with respect to changes in $\boldsymbol{r}$ the following closely related problem 6.50 can be solved.

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = z \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = d \\
& \mathbf{Beq}^3\mathbf{C}^2 = 1 \\
& \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+s+1+n)\times 1}
\end{aligned} \tag{6.50}
$$

The term $t\ \mathbf{q}$ captures the change in $\boldsymbol{r}$, allowing the investor to perform the sensitivity analysis. The parameter t varies between specified lower and upper bounds, which could be $-\infty$ to ∞ respectively and $\mathbf{q}$ is a matrix of zeros with the same size as $\boldsymbol{r}$. To do sensitivity analysis for any particular entry of $\boldsymbol{r}$, that corresponding entry in $\mathbf{q}$ will have 1 in it.

6.4 Sensitivity Analysis of Mean-Semivariance-UPDR

The final model we solved for single period portfolio selection was Mean-SV-UPDR. Let us define some notations needed for Mean-SV-UPDR.

M — a large positive constant

$\boldsymbol{p}_{m\times 1}$ — priority vector for UPDR

$\boldsymbol{w}_{m-2\times 1}$ — weight vector for UPDR

$\boldsymbol{\eta}_{m\times 1}$ — the vector which divides the downside region

η_i — a particular value from the vector $\boldsymbol{\eta}_{m\times 1}$

For each priority of UPDR there will be s variables corresponding to the scenarios of the return matrix. The variables are defined in the following array

$$\boldsymbol{A}_{(m\,s\times 1)} = \begin{bmatrix} \boldsymbol{A}^1_{s\times 1} \\ \boldsymbol{A}^2_{s\times 1} \\ \vdots \\ \boldsymbol{A}^m_{s\times 1} \end{bmatrix}$$

The new priority vector $\boldsymbol{p}'_{m\times 1}$ which would be used to solve UPDR is computed as follows:

$$p'_i = \begin{cases} p_1 & i = 1 \\ p_2 - p_1 & i = 2 \\ p_i - \sum_{t=1}^{i-1} p'_t & i = 3, 4, \ldots, m-1 \\ p_m & i = m \end{cases}$$

Let us suppose we have solved the main optimization problem (6.51) for a particular UPDR value z and Expected return d and the solutions are known.

$$\begin{aligned} \text{Minimize} \quad & \text{Semivariance}(\boldsymbol{X}) \\ \text{subject to} \quad & \text{UPDR}(\boldsymbol{X}) = z \\ & \text{E}(\boldsymbol{X}) = d \\ & \sum_{i=1}^{n} X_i = 1 \\ & \boldsymbol{X} \geq 0 \end{aligned} \tag{6.51}$$

Since $\boldsymbol{r}$ is the only input for our problem, an investor would like to see how the solutions behave for changes in $\boldsymbol{r}$. We explain how to conduct the sensitivity analysis by solving a closely related problem which is derived using the following steps. First let us rewrite the optimization problem in terms of the input $\boldsymbol{r}$:

$$\begin{aligned}
\text{Minimize} \quad & \mathrm{SV}_d(\boldsymbol{X}) = \frac{1}{s}\boldsymbol{Y}^{'}\boldsymbol{Y} \\
\text{subject to} \quad & \boldsymbol{Y} \geq \mathbf{1}d - \boldsymbol{r}\boldsymbol{X} && (6.52) \\
& \boldsymbol{Y} \geq 0 && (6.53) \\
& \boldsymbol{A}^i \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_i - \mathrm{M}\boldsymbol{A}^m : i = 1, 2, \ldots, m-1 && (6.54) \\
& \boldsymbol{A}^m \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_m && (6.55) \\
& \boldsymbol{A}^i \geq 0, i = 1, 2, \ldots, m \\
& \eta_{i+1} = \eta_1 + (\eta_m - \eta_1) \times w_i, i = 1, 2, \ldots, m-2 && (6.56) \\
& \sum_{i=1}^{m-1} \left[p_i^{'} \times (\eta_i + \frac{\mathbf{1}^{'}\boldsymbol{A}^i}{s}) \right] + p_m^{'} \times \left[\eta_m + \frac{\mathbf{1}^{'}\boldsymbol{A}^m}{(1-\alpha)s} \right] = z && (6.57) \\
& \frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} = d && (6.58) \\
& \mathbf{1}^{'}\boldsymbol{X} = 1 && (6.59) \\
& \boldsymbol{X} \geq 0 && (6.60)
\end{aligned}$$

Equations 6.52 and 6.53 represent the constraints for semivariance. Equations 6.54, 6.55, 6.56 and 6.57 represent the constraints for UPDR. Equation 6.58 represents the constraint for expected return. Finally equations 6.59 and 6.60 represent the budget and non-negativity constraints. Use the left hand side of constraint 6.58 to replace d in all other

constraints where it appears to get the following optimization problem:

$$\text{Minimize} \quad \frac{1}{s}\boldsymbol{Y}^{'}\boldsymbol{Y}$$

$$\text{subject to} \quad \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \tag{6.61}$$

$$\boldsymbol{Y} \geq 0 \tag{6.62}$$

$$\boldsymbol{A}^{i} \geq -\boldsymbol{r}X - \mathbf{1}\eta_i - \text{M}\boldsymbol{A}^{m} : i = 1, 2, \ldots, m-1 \tag{6.63}$$

$$\boldsymbol{A}^{m} \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_m \tag{6.64}$$

$$\boldsymbol{A}^{i} \geq 0, i = 1, 2, \ldots, m$$

$$\eta_{i+1} = \eta_1 + (\eta_m - \eta_1) \times w_i, i = 1, 2, \ldots, m-2 \tag{6.65}$$

$$\sum_{i=1}^{m-1}\left[p_i^{'} \times (\eta_i + \frac{\mathbf{1}^{'}\boldsymbol{A}^{i}}{s})\right] + p_m^{'} \times \left[\eta_m + \frac{\mathbf{1}^{'}\boldsymbol{A}^{m}}{(1-\alpha)s}\right] = z \tag{6.66}$$

$$\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} = d \tag{6.67}$$

$$\mathbf{1}^{'}\boldsymbol{X} = 1 \tag{6.68}$$

$$\boldsymbol{X} \geq 0 \tag{6.69}$$

Let us consider all the inequality constraints which have $\boldsymbol{r}$ and simplify them. Equation 6.61 can be simplified using basic properties of matrix algebra as follows:

$$\begin{aligned}
& \boldsymbol{Y} \geq \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \frac{1}{s}\mathbf{1}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}_{ss}^{s}\boldsymbol{r}\boldsymbol{X} - \boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{F}_{ss}^{s}\boldsymbol{r}\boldsymbol{X} - \mathbf{I}\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq [\mathbf{F}_{ss}^{s} - \mathbf{I}]\boldsymbol{r}\boldsymbol{X} \\
\Rightarrow \quad & \boldsymbol{Y} \geq \mathbf{SP}^{1}\boldsymbol{r}\boldsymbol{X}
\end{aligned} \tag{6.70}$$

Consider the inequality constraint (6.63) for UPDR when $i = 1$, we know that $\eta_1 = d$. We can use the value of d from equation 6.58 to simplify as follows:

$$
\begin{aligned}
& \boldsymbol{A}^1 \geq -\boldsymbol{r}X - \mathbf{1}\eta_1 - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq -\boldsymbol{r}\boldsymbol{X} - \frac{1}{s}\mathbf{1}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{F}_s^{s\times s}\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq -\mathbf{I}\boldsymbol{r}\boldsymbol{X} - \mathbf{F}_s^{s\times s}\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq [-\mathbf{I} - \mathbf{F}_s^{s\times s}]\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & \boldsymbol{A}^1 \geq [\mathbf{UP}^1]\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \\
\Rightarrow \quad & -\boldsymbol{A}^1 + [\mathbf{UP}^1]\boldsymbol{r}\boldsymbol{X} - \mathrm{M}\boldsymbol{A}^m \leq \mathbf{0}_{s\times 1}
\end{aligned}
$$

Consider 6.63 for $i = 2, 3, \ldots, m-1$.

$$
\begin{aligned}
& \boldsymbol{A}^i \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_i - \mathrm{M}\boldsymbol{A}^m : i = 2, 3, \ldots, m-1 \\
\Rightarrow \quad & -\boldsymbol{A}^i - \boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_i - \mathrm{M}\boldsymbol{A}^m \leq \mathbf{0}_{s\times 1} : i = 2, 3, \ldots, m-1
\end{aligned}
$$

Consider 6.63 for $i = k$.

$$
\begin{aligned}
& \boldsymbol{A}^m \geq -\boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_m \\
\Rightarrow \quad & -\boldsymbol{A}^m - \boldsymbol{r}\boldsymbol{X} - \mathbf{1}\eta_m \leq \mathbf{0}_{s\times 1}
\end{aligned}
$$

Consider all the variables of the problem and write in them in array form

$$\left[\begin{array}{c} \mathbf{C}^1_{(s+m*s+s)\times 1} \\ \hline \mathbf{C}^2_{n\times 1} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{Y} \\ \boldsymbol{A}^1 \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^m \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \\ \hline \boldsymbol{X} \end{array}\right]$$

Define the following arrays to represent the inequality constraints of semivariance and UPDR

$$\mathbf{B}^1_{(s+m\,s)\times(s+m\,s+m)} = \left[\begin{array}{ccccccccccc} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \dots & \mathbf{0}_{s\times 1} \\ \hline \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & -\mathrm{M}\mathbf{I} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \dots & \mathbf{0}_{s\times 1} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \dots & -\mathrm{M}\mathbf{I} & \mathbf{0}_{s\times 1} & -\mathbf{1} & \mathbf{0}_{s\times 1} & \dots & \mathbf{0}_{s\times 1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \dots & -\mathrm{M}\mathbf{I} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & -\mathbf{1} & \mathbf{0}_{s\times 1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \dots & -\mathbf{I} & -\mathrm{M}\mathbf{I} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \dots & -\mathbf{1} & \mathbf{0}_{s\times 1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{I} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \dots & \mathbf{0}_{s\times 1} & -\mathbf{1} \end{array}\right]$$

$$\mathbf{B}^2_{(s+m\,s)\times n} = \begin{bmatrix} \mathbf{SP}_1 \\ \hline \mathbf{UP}_1 \\ -\mathbf{I} \\ -\mathbf{I} \\ \vdots \\ -\mathbf{I} \\ -\mathbf{I} \end{bmatrix}$$

The semivariance and UPDR constraints can be written in array form as follows:

$$\left[\begin{array}{c|c} \mathbf{B}^1 & \mathbf{B}^2\ \boldsymbol{r} \end{array} \right] \qquad \begin{bmatrix} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{bmatrix} \qquad \leq \mathbf{0}_{(s+m\,s)\times 1}$$

$$\Rightarrow \qquad \mathbf{B}^1\mathbf{C}^1 \qquad + \qquad \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \qquad \leq \qquad \mathbf{0}_{(s+s*m)\times 1} \tag{6.71}$$

Consider all the equality constraints:

$$\begin{aligned} &\eta_{i+1} = \eta_1 + (\eta_m - \eta_1) \times w_i, i = 1, \ldots, m-2 \\ &\sum_{i=1}^{m-1} [p_i^{'} \times (\eta_i + \frac{\mathbf{1}^{'}\boldsymbol{A}^i}{s})] + p_m^{'} \times [\eta_m + \frac{\mathbf{1}^{'}\boldsymbol{A}^m}{(1-\alpha)s}] = z \\ &\frac{1}{s}\mathbf{1}^{'}\boldsymbol{r}\boldsymbol{X} = d \\ &\mathbf{1}^{'}\boldsymbol{X} = 1 \end{aligned}$$

Consider the first equality constraint:

$$\begin{aligned} & \eta_{i+1} = \eta_1 + (\eta_m - \eta_1) \times w_i, i = 1, \ldots, m-2 \\ \Rightarrow \quad & \eta_{i+1} = \eta_1 - \eta_1 w_i + \eta_m w_i, i = 1, \ldots, m-2 \\ \Rightarrow \quad & \eta_{i+1} = \eta_1(1 - w_i) + \eta_m w_i, i = 1, \ldots, m-2 \\ \Rightarrow \quad & \eta_1(1 - w_i) - \eta_{i+1} + \eta_m w_i = 0, i = 1, \ldots, m-2 \end{aligned}$$

In array form these constraints can be written as follows:

$$\left[\begin{array}{cccccc|c} \mathbf{0}_{m-2\times s} & \mathbf{0}_{m-2\times(m\,s)} & \mathbf{1}_{m-2\times 1}-\mathbf{w} & -\mathbf{I}_{m-2\times m-2} & \mathbf{w} & \mathbf{0}_{m-2\times n} \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] = \mathbf{0}_{m-2\times 1} \tag{6.72}$$

Consider the second equality constraint:

$$\begin{aligned} & \sum_{i=1}^{m-1}[p_i^{'}\times(\eta_i+\frac{\mathbf{1}^{'}\boldsymbol{A}^i}{s})]+p_m^{'}\times[\eta_m+\frac{\mathbf{1}^{'}\boldsymbol{A}^m}{(1-\alpha)s}]=z \\ \Rightarrow\quad & \sum_{i=1}^{m-1}[p_i^{'}\eta_i+\frac{p_i^{'}\mathbf{1}^{'}\boldsymbol{A}^i}{s}]+[p_m^{'}\eta_m+\frac{p_m^{'}\mathbf{1}^{'}\boldsymbol{A}^m}{(1-\alpha)s}]=z \end{aligned}$$

Let $\mathbf{e}^i_{1\times m}$ be the i^{th} unit vector of size $1\times m$. Then in array form this constraint can be written as follows:

$$\left[\begin{array}{cccccc|c} \mathbf{0}_{1\times s} & \frac{\mathbf{e}^1\boldsymbol{p}^{'}\mathbf{1}^{'}}{s} & \ldots & \frac{\mathbf{e}^{m-1}\boldsymbol{p}^{'}\mathbf{1}^{'}}{s} & \frac{\mathbf{e}^{m}\boldsymbol{p}^{'}\mathbf{1}^{'}}{(1-\alpha)s} & \boldsymbol{p}^{'} & \mathbf{0}_{1\times n} \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] = z \tag{6.73}$$

Consider the fourth equality constraint:

$$\mathbf{1}^{'}\boldsymbol{X}=1$$

This can written in array form as follows:

$$\left[\begin{array}{ccc|c} \mathbf{0}_{1\times s} & \mathbf{0}_{1\times(m\,s)} & \mathbf{0}_{1\times m} & \mathbf{1}_{1\times n} \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] = 1 \tag{6.74}$$

Combine the first, second and fourth equality constraints to write in the following array form

$$\left[\begin{array}{c|c} \mathbf{Beq}^1 & \mathbf{Beq}^2 \end{array}\right] \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] = \left[\begin{array}{c} \mathbf{0}_{m-2\times 1} \\ z \\ 1 \end{array}\right]$$

$$\Rightarrow \quad \mathbf{Beq}^1\mathbf{C}^1 \quad + \quad \mathbf{Beq}^2\mathbf{C}^2 \quad = \quad \begin{bmatrix} \mathbf{0}_{m-2\times 1} \\ z \\ 1 \end{bmatrix} \tag{6.75}$$

where $\mathbf{Beq}^1_{(m-2+1+1)\times(s+m\,s+s+n)}$ and $\mathbf{Beq}^2_{1\times n}$ are as partitioned in equations 6.72, 6.73 and 6.74.

Consider the third equality constraint:

$$\frac{1}{s}\mathbf{1}'\boldsymbol{r}\boldsymbol{X} = d$$

This can be written in array form as follows:

$$\left[\begin{array}{ccc|c} \mathbf{0}_{1\times s} & \mathbf{0}_{1\times(m\,s)} & \mathbf{0}_{1\times m} & \frac{1}{s}\mathbf{1}'\boldsymbol{r} \end{array}\right] \quad \left[\begin{array}{c} \mathbf{C}^1 \\ \hline \mathbf{C}^2 \end{array}\right] \quad = d$$

$$\Rightarrow \quad \mathbf{Beq}^3\ \boldsymbol{r}\ \mathbf{C}^2 = \quad \begin{bmatrix} d \end{bmatrix} \tag{6.76}$$

where

$$\mathbf{Beq}^3_{1\times n} = \begin{bmatrix} \frac{1}{s}\mathbf{1}' \end{bmatrix}$$

The optimization problem can be rewritten using 6.75, 6.71 and 6.76 to get the final problem in terms of arrays as follows:

$$\begin{array}{rl} \text{Minimize} & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\ \text{subject to} & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ \boldsymbol{r}\ \mathbf{C}^2 \leq \mathbf{0}_{(s+m\,s)\times 1} \\ & \mathbf{Beq}^1\mathbf{C}^1 + \mathbf{Beq}^2\mathbf{C}^2 = \begin{bmatrix} \mathbf{0}_{m-2\times 1} \\ z \\ 1 \end{bmatrix} \\ & \mathbf{Beq}^3\ \boldsymbol{r}\ \mathbf{C}^2 = d \\ & \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \geq \mathbf{0}_{(s+m\,s+s+n)\times 1} \end{array}$$

The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem and the scalars z and d are also known. To conduct the sensitivity analysis with respect to changes in $\boldsymbol{r}$ the following closely related problem 6.77 can be solved.

$$\begin{array}{rl} \text{Minimize} & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\ \text{subject to} & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+m\,s)\times 1} \\ & \mathbf{Beq}^1\mathbf{C}^1 + \mathbf{Beq}^2\mathbf{C}^2 = \begin{bmatrix} \mathbf{0}_{m-2\times 1} \\ z \\ 1 \end{bmatrix} \\ & \mathbf{Beq}^3\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = d \\ & \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \geq \mathbf{0}_{(s+m\,s+s+n)\times 1} \end{array} \tag{6.77}$$

The term $t\ \mathbf{q}$ captures the change in $\boldsymbol{r}$, allowing the investor to perform the sensitivity analysis. The parameter t varies between specified lower and upper bounds, which could be $-\infty$ to ∞ respectively and $\mathbf{q}$ is a matrix of zeros with the same size as $\boldsymbol{r}$. To do sensitivity analysis for any particular entry of $\boldsymbol{r}$, that corresponding entry in $\mathbf{q}$ will have 1 in it.

6.5 Numerical Examples of Sensitivity Analysis

The example stated in Chapter 3 is used to illustrate sensitivity analysis. Throughout this chapter the confidence level α is assumed to be 0.95. For UPDR, priority vector $\boldsymbol{p}$ is assumed to be [0.1 0.2 0.7] and the weight vector $\boldsymbol{w}$ is given as [0.5].

6.5.1 Mean-Semivariance-Absolute deviation

Let us suppose for the given example we have solved Mean-SV-Absolute deviation for an expected return (d) of 0.0929 and Absolute deviation value (z) of 0.096. The fractional solution to invest in securities is given in Table 6.1.

Table 6.1 Semivariance and corresponding fractions to invest for Mean-SV-Absolute deviation.

Semivariance	Fraction to invest in nine securities								
0.0094	0	0.4585	0	0	0.0571	0.1041	0.3066	0	0.0737

There is no investment in securities 1, 3, 4 and 8. Let us suppose we would like to see what happens if we vary the highest return of security 5 which is in the portfolio with the lowest fraction. The sixth scenario of security 5 has the highest return and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (6,5) place a 1. Now the following problem can be solved to conduct the sensitivity analysis.

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s+s)\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = 0.096 \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = 0.0929 \\
& \mathbf{Beq}^3\mathbf{C}^2 = 1 \\
& \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+s+n)\times 1}
\end{aligned}
$$

We found that the solution was sensitive for changes in t and in particular for the following small range $[-0.001, 0.001]$, the solution remained the same. As t was increased more than 0.001, we found that the investment fraction in securities 2, 5 and 6 reduced and investment in securities 7 and 9 increased.

Let us consider varying the worst loss of security 5. The first scenario of security 5 has the worst loss and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (1,5) place a 1. The above problem can be solved to conduct the sensitivity analysis. When t is in the range $[-0.1, 0.01]$ and increasing, the composition of the portfolio remains the same with security 7 losing some of its resources to the other securities. When t is less than -0.1, security 5 leaves the portfolio and security 3 becomes part of the portfolio with all the other securities remaining.

6.5.2 Mean-Semivariance-CVaR

Let us suppose for the given example we have solved Mean-SV-CVaR for an expected return (d) of 0.095 and CVaR value (z) of 0.1877. The fractional solution to invest in securities is given in Table 6.2.

There is no investment in securities 1, 2, 4, 8 and 9. Let us suppose we would like to see what happens if we increase the highest return of security 5 which is in the portfolio with the lowest fraction. The sixth scenario of security 5 has the highest return and hence

Table 6.2 Semivariance and corresponding fractions to invest for Mean-SV-CVaR.

Semivariance	Fraction to invest in nine securities								
0.0128	0	0	0.1204	0	0.0843	0.5622	0.2331	0	0

to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (6,5) place a 1. The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem.

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}^{'}\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+s)\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = 0.1877 \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = 0.095 \\
& \mathbf{Beq}^2\mathbf{C}^2 = 1 \\
& \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+s+1+n)\times 1}
\end{aligned}
$$

We found that the solution was sensitive for changes in t and in particular for the following small range $[-0.001, 0.001]$, the solution remained the same. When t was less than -0.1 we found that there was no more investment in security 5.

Let us consider varying the worst loss of security 5. The first scenario of security 5 has the worst loss and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (1,5) place a 1. The above problem can be solved to conduct the sensitivity analysis. When t is in the range $[-0.01, 0.01]$ and increasing, the composition of the portfolio remains the same with security 3 losing some of its resources to the other securities. When t is less than -0.1, security 5 leaves the portfolio and security 8 takes its place and all the other securities remain.

6.5.3 Mean-Semivariance-CDaR

Let us suppose for the given example we have solved Mean-SV-CDaR for an expected return (d) of 0.1531 and CDaR value (z) of 0.0193. The fractional solution to invest in securities is given in Table 6.3.

Table 6.3 Semivariance and corresponding fractions to invest for Mean-SV-CDaR.

Semivariance	Fraction to invest in nine securities								
0.0211	0	0	0.4231	0.0006	0.2433	0	0.3247	0.0082	0

There is no investment in securities 1, 2, 6 and 9. Let us suppose we would like to see what happens if we increase the highest return of security 4 which is in the portfolio with the lowest fraction. The eighteenth scenario of security 4 has the highest return and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (18,4) place a 1. The matrices $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem.

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} \quad & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+\frac{s(s+1)}{2})\times 1} \\
& \mathbf{Beq}^1\mathbf{C}^1 = 0.0193 \\
& \mathbf{Beq}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = 0.1531 \\
& \mathbf{Beq}^3\mathbf{C}^2 = 1 \\
& \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+s+1+n)\times 1}
\end{aligned}
$$

We found that the solution was sensitive for changes in t and in particular for the following small range $[-0.001, 0.001]$, the solution remained the same. When t is less than -0.1 there is no more investment in security 4. When t was increased more than 0.001 investment in security 4 and 7 increased and investment in security 3, 5 and 8 decreased.

Let us consider varying the worst loss of security 4. The first scenario of security 4 has the worst loss and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (1,4) place a 1. The above problem can be solved to conduct the sensitivity analysis. When t is less than 0.01, the composition of the portfolio remains the same but security 4 leaves the portfolio. When t is in the range $[0.01, 1]$ and increasing, the composition remains the same with securities 4 and 5 increasing and security 3 decreasing its stake in the portfolio.

6.5.4 Mean-Semivariance-UPDR

Let us suppose for the given example we have solved Mean-SV-UPDR for an expected return (d) of 0.15 and UPDR value (z) of 0.31. The fractional solution to invest in securities is given in Table 6.4.

Table 6.4 Semivariance and corresponding fractions to invest for Mean-SV-UPDR.

Semivariance	Fraction to invest in nine securities								
0.0196	0	0	0.3578	0	0.2240	0	0.4183	0	0

There is no investment in securities 1, 2, 4, 6, 8 and 9. Let us suppose we would like to see what happens if we increase the highest return of security 5 which is in the portfolio with the lowest fraction. The sixth scenario of security 5 has the highest return and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (6,5) place a 1. The matrices

$\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{Beq}^1$, $\mathbf{Beq}^2$ and $\mathbf{Beq}^3$ can be computed before we solve the optimization problem.

$$\begin{array}{rl}
\text{Minimize} & \frac{1}{s}\boldsymbol{Y}'\boldsymbol{Y} \\
\text{subject to} & \mathbf{B}^1\mathbf{C}^1 + \mathbf{B}^2\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 \leq \mathbf{0}_{(s+m*s)\times 1} \\
 & \mathbf{Beq}^1\mathbf{C}^1 + \mathbf{Beq}^2\mathbf{C}^2 = \begin{bmatrix} \mathbf{0}_{m-2\times 1} \\ 0.31 \\ 1 \end{bmatrix} \\
 & \mathbf{Beq}^3\ [\boldsymbol{r} + t\ \mathbf{q}]\ \mathbf{C}^2 = 0.15 \\
 & \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \end{bmatrix} \quad \geq \quad \mathbf{0}_{(s+m\,s+s+n)\times 1}
\end{array}$$

We found that the solution was sensitive for changes in t and in particular for the following small range $[-0.001, 0.001]$, the solution remained the same. When t is less than -5 there is no more investment in security 5.

Let us consider varying the worst loss of security 5. The first scenario of security 5 has the worst loss and hence to conduct the sensitivity analysis in the $\mathbf{q}$ matrix in position (1,5) place a 1. The above problem can be solved to conduct the sensitivity analysis. When t is in the range $[-0.01, 0.1]$ and increasing, the composition of the portfolio remains the same with security 3 losing some of its resources to securities 5 and 7. When t is less than -0.5, security 5 leaves the portfolio and the other securities remain.

Sensitivity analysis has been explained in detail and examples have been shown to implement it for the different models we propose. The models were extremely sensitive for changes in the input when we considered changing the highest return but not so sensitive for changes in the worst loss, of the security with the least presence. This could be expected since we solved the problem for optimality, hence for any new market changes the problem tries to optimize the portfolio to increase (decrease) the gain (loss). Sensitivity analysis is a good tool for the investor to have when undertaking portfolio selection since it gives a reworking mechanism to change the portfolio for changes in input. It also helps the investor get a better perspective as to the stability of his portfolio composition.

In the next chapter, portfolio selection is handled for multiple risk measures in a multi-period context.

CHAPTER 7

MULTI-PERIOD MULTI-OBJECTIVE PORTFOLIO SELECTION

Investors using single-period portfolio selection buy securities, hold them for a fixed time, and sell them. Some investors may want to do multiple investment decisions in this holding period for additional gain. If detailed information is available about securities in the holding period, it can be used for revision of portfolio to improve return and/or reduce loss since it may not be the best decision to hold the same securities for the entire holding period. The investor who would like to do multiple investing decisions in the holding period should view the problem as Multi-period portfolio selection. Since we are dealing with multiple periods of investment the optimal result would be obtained by optimizing over all time periods, rather than optimizing individual time periods. But doing a revision of the portfolio would lead to additional transaction costs. Transaction costs are usually a percentage of the amount when the amount invested is reasonably large. We assume the investor has a reasonable sum of money to invest and therefore the transaction costs do not play a major role in his investment decision, making multi-period investing attractive.

The model we considered for multi-period portfolio selection was based on three parameters: the expected value, semivariance and a second risk measure (RM). These three parameters are solved with the condition that the expected value is maximized, the semivariance (SV) and the second risk measure are minimized with respect to terminal wealth. In multi-period portfolio selection expected value is used instead of expected return by which we denote the terminal wealth at the end of final period. In other words expected value is the value of the portfolio at the end of final period. Any one of Absolute deviation, CVaR, CDaR and UPDR is the second risk measure (RM). Like the single-period model we wanted to use semivariance as the reference and so kept it in the objective function and placed constraints for expected value and the second risk measure(RM).

Dynamic programming, which aims to optimize a multi-stage problem with respect to the terminal stage, can be used to solve multi-period portfolio selection. Dynamic programming problems are solved sequentially with respect to stages and so requires separability of the objective function. The stages in our model would be the different investment time periods. Consider a dynamic programming problem with N stages with following objective function:

$$g[r_N(X_N, D_N), r_{N-1}(X_{N-1}, D_{N-1}), \ldots, r_1(X_1, D_1)]$$

where $r_i(X_i, D_i)$ represents the contribution from the i^{th} stage. The function has to be separable and so has to satisfy the following condition.

$$g[r_N(X_N, D_N), r_{N-1}(X_{N-1}, D_{N-1}), \ldots, r_1(X_1, D_1)] =$$
$$g_1[r_N(X_N, D_N), g_2(r_{N-1}(X_{N-1}, D_{N-1}), r_{N-2}(X_{N-2}, D_{N-2}), \ldots, r_1(X_1, D_1))]$$

where g_1 and g_2 are real valued functions. For our optimization problem semivariance is in the objective function. Semivariance is given by the following function $SV = \mathrm{E}[(X - \mathrm{E}(X))^-]^2$. This function is not separable and therefore dynamic programming cannot be used to solve our model. We viewed the multi-period problem as a single optimization problem and placed constraints to establish changes between time periods to solve the model. The procedure and formulation of the problem is explained in detail here.

Let us suppose we have T investment periods so that a decision has to be made at the beginning of each period. The number of investment decisions will be T with 0 representing the initial investment decision and so on upto $T-1$ for the last period. Let the number of securities available to invest be n and let $\boldsymbol{e}_0$, $\boldsymbol{e}_1$,... $\boldsymbol{e}_{T-1}$ and $\boldsymbol{X}_0$, $\boldsymbol{X}_1$,... $\boldsymbol{X}_{T-1}$ be the vectors of mean returns and fractional vectors of investment in different securities for the time periods $0, 1, \ldots, T-1$ respectively. Let u_0, u_1,... u_{T-1} be the amount available for investing in the time periods $1, 2, \ldots, T$ and u_T is the value of the portfolio at the end of the last time period. We assume there is one unit of amount available to invest in the beginning period i.e., $u_0 = 1$. There are three main indexes in this problem $i = 0, 1, \ldots, T-1$,

$j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, s$ where i, j and k represent the investment time periods, number of securities and the number of scenarios of returns for last time period, respectively.

Notations

Let us rewrite the notations we will be using throughout this chapter.

s	—	number of scenarios of information available about last time period
n	—	number of securities
T	—	number of time periods
α	—	confidence level
i	=	$0, 1, \ldots T-1$
j	=	$1, 2, \ldots n$
k	=	$1, 2, \ldots s$
$\boldsymbol{e}_i$	—	mean returns for the securities in each time period
u_i	—	amount available to invest in the beginning of the time period
u_T	—	amount at the end of the final time period
$\boldsymbol{X}_i$	—	investment vector in the securities in each time period
$\boldsymbol{r}_{s \times n}$	—	return matrix for the securities in the last period

The formulation of the main problem is as follows:

$$\begin{aligned} \text{Minimize} \quad & [\text{Semivariance}(u_T), RM(u_T), -E(u_T)] \\ & A = 0 \\ & X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1 \end{aligned} \tag{7.1}$$

$$\begin{aligned} A = \quad & u_i - \sum_{j=1}^{n} X_i^j : i = 0, 1, \ldots, T-1 \\ & u_{i+1} - (u_i + \sum_{j=1}^{n} [X_i^j e_i^j]) : i = 0, 1, \ldots, T-1 \end{aligned}$$

The first set of equations in A makes sure that whatever invested in any period equals the amount available at the beginning of that time period and the second set of equations makes sure that whatever amount available to invest at any period is the amount available at the beginning of the previous period plus the amount earned (lost) in the previous period. The mean returns of securities for the first $T-1$ time periods and s scenarios of returns of securities for the last time period T is assumed to be known. Semivariance and the second risk measure are minimized by computing them based on these scenarios and using the investment fraction and the expected return of the last period. The expected value, which represents the terminal wealth, is maximized with respect to all time periods.

There is one main hurdle the investor needs to be prepared for before attempting multi-period portfolio selection. The actual returns may be markedly different from what was predicted leading to a loss if reworking of the portfolio is undertaken as proposed by the multi-period model. Under this scenario the investor has to make a decision and decide either to continue investing or conduct a new multi-period portfolio selection.

We use semivariance as the reference risk measure, hence it is left in the objective function and constraints are placed for the expected value and the other risk measure on the lines of the ϵ-constrained method. The multi-objective problem we need to solve.

$$
\begin{aligned}
\text{Minimize} \quad & \text{Semivariance}(u_T) \\
\text{subject to} \quad & \\
& RM(u_T) \leq z \qquad (7.2) \\
& u_T \geq d \\
& A = 0 \\
& X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1
\end{aligned}
$$

From proposition 5.1, a point u_T^* is an optimal solution of (7.1) if and only if it is also an optimal solution of (7.2) with $z = \text{RM}_{u_T^*}$ and $d = u_T^*$. Therefore to get all the efficient solutions of the mean-semivariance-RM model for multi-period, we solve the problem (7.2) by varying z and d such that the constraints on the risk measure (RM) and expected value are active.

Remark 7.1. Consider the optimization problem 7.2, proposition 5.2 implies the objective function is convex. The constraint set A comprises of linear constraints, hence is a convex set. Since all other constraints are linear they will comprise a convex set. The intersection of both these convex sets will lead to a convex set. Thus any non-empty feasible region is convex. Since the objective function is convex and the feasible region is a convex set, the problem is convex for all four models. Thus we are guaranteed a global optimal solution.

To find all the solutions of the above multi-objective problem we used the same approach as the one we used for the single-period case. The maximum possible expected value $d_{\max}$ is computed using the following formula: $\prod_{i=0}^{T-1}(1+\max_{1\leq j\leq n} e_i^j)$. The procedure is similar to the single-period model and so we skip the explanation but the general outline is given in the following flowchart.

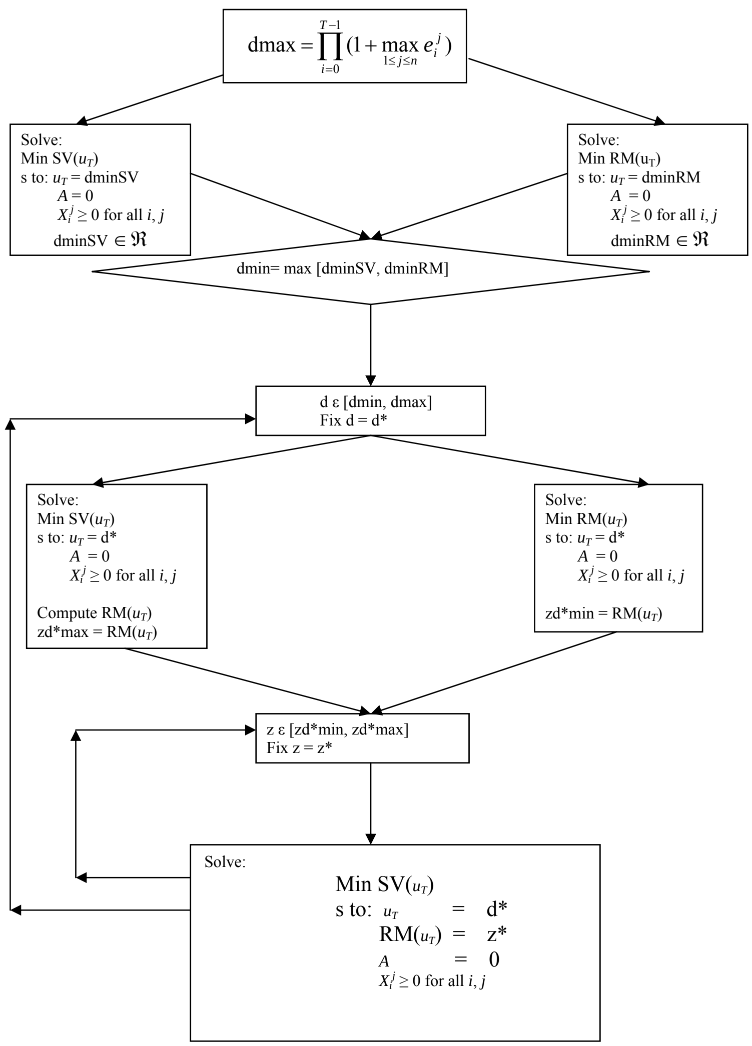

Figure 7.1 Solution Procedure to solve Multi-period Mean-SV-RM models

The following example from Markowitz (1991b) is used to illustrate all portfolio selection problems in this chapter. In this example we can invest in nine securities. The nine securities are American Tobacco, American Tel. & Tel., United States Steel., General Motors, Atchsion, Topeka & Santa Fe., Coca-Cola, Borden, Firestone, and Sharon Steel. The return for any period is computed the following way:

price change = (closing price for current - previous period)

return for current period = $\frac{\text{price change+(dividends for current period)}}{\text{closing price of the previous period}}$

Historical returns are computed between 1937 and 1954 and the data is used as follows for the multi-period model. We assume that there are three periods in which we can invest. The mean returns for the first and second period and sixteen possible scenarios of returns for the third time period are given in the following table.

mean returns for first time period	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
mean returns for second time period	0.055	0.2	-0.047	0.165	-0.424	-0.078	0.381	-0.093	-0.295
return scenarios for third time period, $\boldsymbol{r}$=	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
	-0.126	0.03	0.104	-0.043	-0.189	-0.077	-0.051	-0.09	-0.036
	-0.28	-0.183	-0.171	-0.277	0.637	-0.187	0.087	-0.194	-0.24
	-0.003	0.067	-0.039	0.476	0.865	0.156	0.262	1.113	0.126
	0.428	0.3	0.149	0.225	0.313	0.351	0.341	0.58	0.639
	0.192	0.103	0.26	0.29	0.637	0.233	0.227	0.473	0.282
	0.446	0.216	0.419	0.216	0.373	0.349	0.352	0.229	0.578
	-0.088	-0.046	-0.078	-0.272	-0.037	-0.209	0.153	-0.126	0.289
	-0.127	-0.071	0.169	0.144	0.026	0.355	-0.099	0.009	0.184
	-0.015	0.056	-0.035	0.107	0.153	-0.231	0.038	0	0.114
	0.305	0.038	0.133	0.321	0.067	0.246	0.273	0.223	-0.222
	-0.096	0.089	0.732	0.305	0.579	-0.248	0.091	0.65	0.327
	0.016	0.09	0.021	0.195	0.04	-0.064	0.054	-0.131	0.333
	0.128	0.083	0.131	0.39	0.434	0.079	0.109	0.175	0.062
	-0.01	0.035	0.006	-0.072	-0.027	0.067	0.21	-0.084	-0.048
	0.154	0.176	0.908	0.715	0.469	0.077	0.112	0.756	0.185

All the sixteen scenarios for the third time period are assumed as equally likely predictors and their average is the mean return for that period. The mean returns for the three time periods are given as follows:

returns for first time period	returns for second time period	returns for third time period
0.513	0.055	0.0878
0.098	0.2	0.0754
0.285	-0.047	0.1734
0.714	0.165	0.2117
0.107	-0.424	0.2366
0.238	-0.078	0.0622
0.076	0.381	0.1539
0.336	-0.093	0.2251
0.238	-0.295	0.1480

We solved the Multi-period model using any of Absolute deviation, CVaR, CDaR or UPDR as the second risk measure (RM). So we have four different models Mean-SV-Absolute deviation, Mean-SV-CVaR, Mean-SV-CDaR and Mean-SV-UPDR to use for Multi-period portfolio selection. The four models were solved for the given numerical example with a confidence level α of 0.95.

7.1 Mean-Semivariance-Absolute Deviation

The first model we considered was Mean-Semivariance-Absolute deviation. Semivariance quantifies the downside risk whereas absolute deviation measures the absolute deviation of the expected return. An investor can use these two measures for his portfolio selection and use the procedure we outlined to get a set of solutions. Semivariance is left in the objective function and constraints are placed for absolute deviation and expected value. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\sum_{k=1}^{s} y_k^2 \\
\text{subject to} \quad & y_k \geq \sum_{j=1}^{n}[e_{T-1}^j - r_{kj}]X_{T-1}^j : k = 1, 2, \ldots, s \\
& y_k \geq 0 : k = 1, 2, \ldots, s \\
& a_k \geq \sum_{j=1}^{n}[r_{kj} - e_{T-1}^j]X_{T-1}^j : k = 1, 2, \ldots, s \\
& a_k \geq \sum_{j=1}^{n}[e_{T-1}^j - r_{kj}]X_{T-1}^j : k = 1, 2, \ldots, s \\
& a_k \geq 0 : k = 1, 2, \ldots, s \\
& \frac{1}{s}\sum_{k=1}^{s} a_k \leq z \qquad (7.3) \\
& u_T \geq d \qquad (7.4) \\
& A = 0 \\
& X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1
\end{aligned}
$$

We can use the procedure outlined to solve this problem for different solutions. The minimum expected value $d_{\min}$ and maximum expected value $d_{\max}$ was found to be 0.7546 and 2.9272 respectively. Therefore expected value $d \in [0.7546, 2.9272]$. In this interval six equidistant expected values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The problem is solved for different values of z and d^* so that constraints on absolute deviation (7.3) and expected value (7.4) are active. These solutions are plotted on a semivariance-absolute deviation space for each given expected value and is given in Figure 7.2. The corresponding solutions are given in Table 7.1.

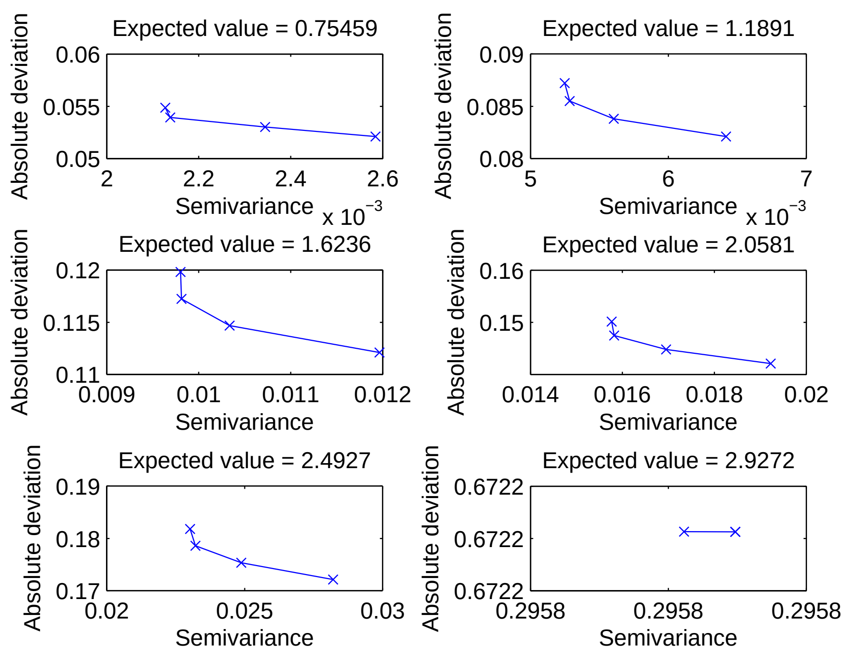

Figure 7.2 Efficient frontier of Mean-Semivariance-Absolute deviation for Multi-Period.

Table 7.1 Some solutions of Mean-Semivariance-Absolute deviation along with corresponding fractions to invest for different time periods.

Semivariance	Absolute deviation	Expected value	Fraction to invest in nine securities for three different periods								
0.0026	0.0521	0.7546	0	0.321	0	0	0.2953	0	0.3837	0	0
			0	0	0	0	0.6573	0	0	0.0243	0.4106
			0	0.5375	0	0	0.0056	0	0.147	0	0
0.0064	0.0821	1.1891	0	0.2636	0.024	0	0.2521	0.0842	0.2918	0	0.0842
			0.1222	0.1171	0.1258	0.1183	0.139	0.1269	0.1108	0.1274	0.1345
			0	0.8471	0	0	0.0088	0	0.2317	0	0
0.012	0.1121	1.6236	0.0536	0.1604	0.1123	0.0019	0.158	0.1243	0.166	0.0991	0.1243
			0.1406	0.2893	0.036	0.2534	0	0.0042	0.475	0	0
			0	1.1566	0	0	0.012	0	0.3163	0	0
0.0192	0.1421	2.0581	0.2302	0.0085	0.1084	0.3375	0.0133	0.0833	0	0.1356	0.0833
			0.0812	0.3689	0	0.2994	0	0	0.7279	0	0
			0	1.4661	0	0	0.0152	0	0.401	0	0
0.0282	0.1721	2.4927	0.0356	0	0	0.9644	0	0	0	0	0
			0	0.3001	0	0.1068	0	0	1.2999	0	0
			0	1.7757	0	0	0.0184	0	0.4857	0	0
0.2958	0.6722	2.9272	0	0	0	1	0	0	0	0	0
			0	0	0	0	0	0	1.714	0	0
			0	0	0	0	2.367	0	0	0	0

7.2 Mean-Semivariance-CVaR

The second model we considered was Mean-Semivariance-CVaR. Semivariance quantifies the downside risk whereas CVaR measures the expected value of the worst losses. Let us suppose the investor wants to use these two measures for his risk quantification and wants to do multi-period portfolio selection. Semivariance is left in the objective function and constraints are placed for CVaR and expected value. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\sum_{k=1}^{s} y_k^2 \\
\text{subject to} \quad & y_k \geq \sum_{j=1}^{n} [e_{T-1}^j - r_{kj}] X_{T-1}^j : k = 1, 2, \ldots, s \\
& y_k \geq 0 : k = 1, 2, \ldots, s \\
& a_k \geq \sum_{j=1}^{n} -r_{kj} X_{T-1}^j - \eta : k = 1, 2, \ldots, s \\
& \eta + \frac{1}{(1-\alpha)s}\sum_{k=1}^{s}(a_k) \leq z \\
& a_k \geq 0 : k = 1, 2, \ldots, s \qquad (7.5) \\
& u_T \geq d \qquad (7.6) \\
& A = 0 \\
& X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1
\end{aligned}
$$

We can use the procedure outlined to solve this problem for different solutions. The minimum expected value $d_{\min}$ and maximum expected value $d_{\max}$ was found to be 0.7771 and 2.9272 respectively. Therefore expected value $d \in [0.7771, 2.9272]$. In this interval six equidistant expected values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The problem is solved for different values of z and d^* so that constraints on CVaR (7.5) and expected value (7.6) are active. These solutions are plotted on a semivariance-CVaR space for each given expected return and is given in Figure 7.3. The corresponding solutions are given in Table 7.2.

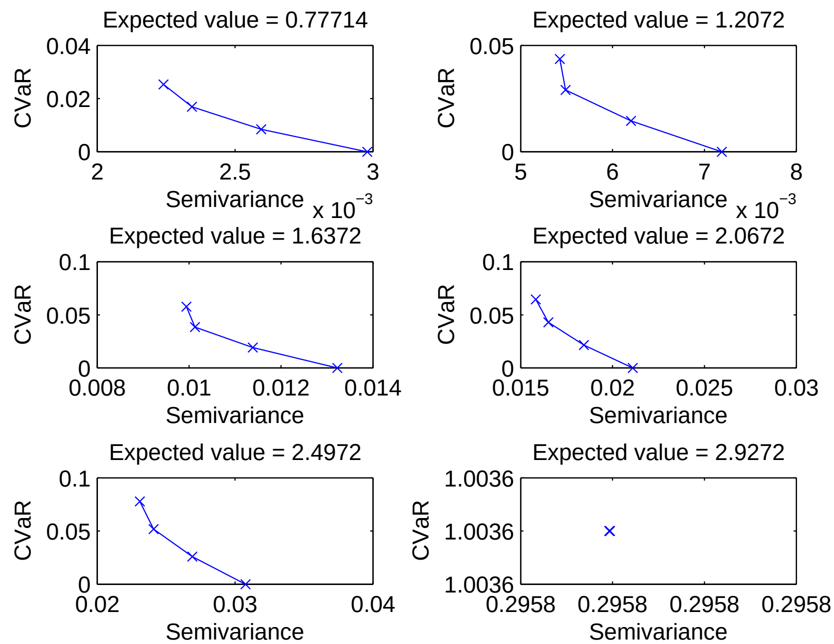

Figure 7.3 Efficient frontier of Mean-Semivariance-CVaR for Multi-Period.

Table 7.2 Some solutions of Mean-Semivariance-CVaR along with corresponding fractions to invest for different time periods.

Semivariance	CVaR	Expected value	Fraction to invest in nine securities for three different periods								
0.003	0	0.7771	0	0.3187	0	0	0.2883	0	0.393	0	0
			0	0	0	0	0.7213	0	0	0	0.3707
			0	0.1712	0.1959	0	0.0707	0	0.2359	0	0.0031
0.0072	0	1.2072	0.0017	0.2681	0.0153	0.0032	0.2559	0.0789	0.2978	0.0003	0.0789
			0.1171	0.1016	0.128	0.1053	0.1684	0.1313	0.0822	0.1329	0.1546
			0	0.2668	0.3041	0	0.1101	0	0.3656	0	0.0047
0.0132	0	1.6372	0.0059	0.1848	0.1008	0.0113	0.1808	0.1219	0.1947	0.0778	0.1219
			0.1443	0.264	0.0601	0.2351	0.0016	0.0345	0.4134	0.0221	0.0011
			0	0.3378	0.4168	0	0.1408	0	0.5213	0	0.0081
0.0211	0	2.0672	0.1925	0.0414	0.1095	0.2657	0.0447	0.0924	0.0334	0.1281	0.0924
			0.0951	0.3531	0.0002	0.2908	0.0016	0.0004	0.6752	0.0004	0.0012
			0	0.4067	0.5299	0	0.1707	0	0.6792	0	0.0115
0.0158	0.0644	2.0672	0	0.6826	0	0	0.1696	0.2038	0.5885	0	0.1986
			0.2511	0	0	0.7423	0	0	0	0.0066	0
			0	0.3721	0	0.2416	0	0	1.0474	0	0
0.2958	1.0036	2.9272	0	0	0	1	0	0	0	0	0
			0	0	0	0	0	0	1.714	0	0
			0	0	0	0	2.367	0	0	0	0

7.3 Mean-Semivariance-CDaR

The third model we considered was Mean-Semivariance-CDaR. Semivariance quantifies the downside risk whereas CDaR measures the expected value of the worst downside at risk losses. Let us suppose the investor wants to use these two measures for risk quantification and undertake multi-period portfolio selection. Semivariance is left in the objective function and constraints are placed for CDaR and expected value. The main problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\sum_{k=1}^{s} y_k^2 \\
\text{subject to} \quad & y_k \geq \sum_{j=1}^{n}[e_{T-1}^j - r_{kj}]X_{T-1}^j : k = 1, 2, \ldots, s \\
& y_k \geq 0 : k = 1, 2, \ldots, s \\
& a_k \geq \left\{ \sum_{j=1}^{n}(1 + \sum_{t=1}^{l} r_{tj})X_{T-1}^j \right\} - \left\{ \sum_{j=1}^{n}(1 + \sum_{t=1}^{k} r_{tj})X_{T-1}^j \right\} - \eta \\
& l = 1, 2, \ldots, k \\
& a_k \geq 0 \\
& k = 1, 2, \ldots, s \\
& \eta + \frac{1}{(1-\alpha)s}\sum_{k=1}^{s}(a_k) \leq z \qquad (7.7) \\
& u_T \geq d \qquad (7.8) \\
& A = 0 \\
& X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1
\end{aligned}
$$

We can use the procedure outlined to solve this problem for different solutions. The minimum expected value $d_{\min}$ and maximum expected value $d_{\max}$ was found to be 0.7761 and 2.9272 respectively. Therefore expected value $d \in [0.7761, 2.9272]$. In this interval six equidistant expected values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The problem is solved for different values of z and d^* so that constraints on CDaR (7.7) and expected value (7.8) are active. These solutions are plotted on a semivariance-CDaR space for each given expected value and is given in Figure 7.4. The corresponding solutions are given in Table 7.3.

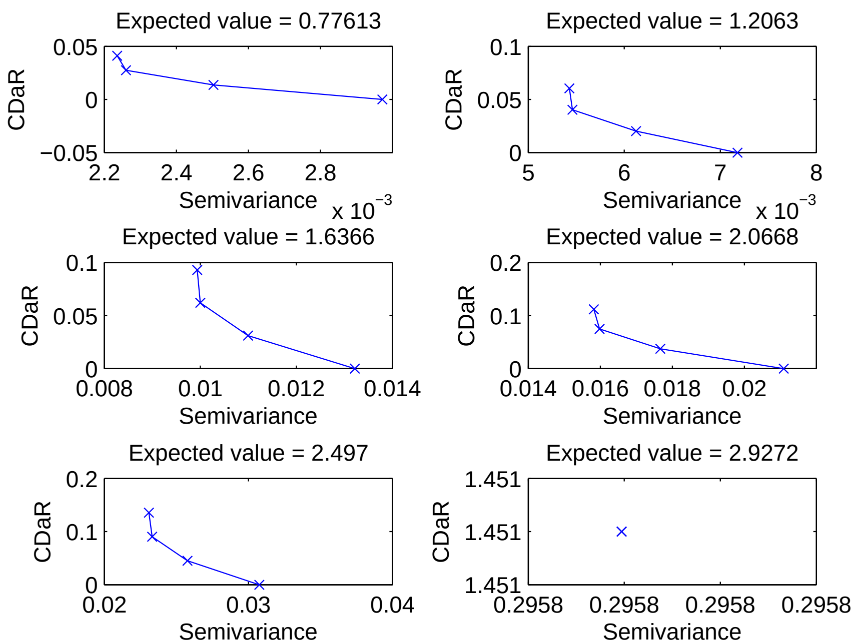

Figure 7.4 Efficient frontier of Mean-Semivariance-CDaR for Multi-Period.

Table 7.3 Some solutions of Mean-Semivariance-CDaR along with corresponding fractions to invest for different time periods.

Semivariance	CDaR	Expected value	Fraction to invest in nine securities for three different periods								
0.003	0	0.7761	0	0.3184	0	0	0.2874	0	0.3942	0	0
			0	0	0	0	0.7279	0	0	0	0.364
			0	0.1712	0.1956	0	0.0707	0	0.2354	0	0.0031
0.0072	0	1.2063	0.0017	0.2682	0.0152	0.0032	0.256	0.0788	0.2979	0.0003	0.0788
			0.117	0.1013	0.1281	0.1051	0.1689	0.1314	0.0817	0.133	0.1549
			0	0.2666	0.3039	0	0.1101	0	0.3653	0	0.0047
0.0132	0	1.6366	0.0059	0.185	0.1006	0.0113	0.181	0.1218	0.195	0.0776	0.1218
			0.1443	0.2637	0.0603	0.2349	0.0016	0.0348	0.4128	0.0224	0.0011
			0	0.3377	0.4167	0	0.1408	0	0.521	0	0.0081
0.0211	0	2.0668	0.1924	0.0415	0.1095	0.2654	0.0448	0.0924	0.0335	0.128	0.0924
			0.0952	0.3531	0.0002	0.2908	0.0016	0.0004	0.675	0.0004	0.0012
			0	0.4066	0.5298	0	0.1706	0	0.6791	0	0.0115
0.0308	0	2.497	0.2523	0	0	0.7417	0	0	0	0.0059	0
			0	0.3725	0	0.2421	0	0	1.0464	0	0
			0	0.542	0.6308	0	0.2243	0	0.7665	0	0.0105
0.2958	1.451	2.9272	0	0	0	1	0	0	0	0	0
			0	0	0	0	0	0	1.714	0	0
			0	0	0	0	2.367	0	0	0	0

7.4 Mean-Semivariance-UPDR

The final model we considered was Mean-Semivariance-UPDR. Semivariance quantifies the downside risk whereas UPDR measures the expected value of the downside but priorities are assigned to the losses. Let us suppose the investor wants to use these two measures for risk quantification and undertake multi-period portfolio selection. The investor can decide on what weight he wants to assign the different measures and get corresponding solutions based on that. Let us suppose there are c priorities for portfolio selection and a new index $l = 1, 2, \ldots, c$.

Semivariance is left in the objective function and constraints are placed for UPDR and expected value. The main problem is given as follows:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{s}\sum_{k=1}^{s} y_k^2 \\
\text{subject to} \quad & y_k \geq \sum_{j=1}^{n}[e_{T-1}^{j} - r_{kj}]X_{T-1}^{j} : k = 1, 2, \ldots, s \\
& y_k \geq 0 : k = 1, 2, \ldots, s \\
& a_k^l \geq \sum_{j=1}^{n}[-r_{kj}X_{T-1}^{j}] - \eta_l - \mathrm{M}\, a_k^c : k = 1, 2, \ldots, s \text{ and } l = 1, 2, \ldots, c-1 \\
& a_k^c \geq \sum_{j=1}^{n}[-r_{kj}X_{T-1}^{j}] - \eta_c : k = 1, 2, \ldots, s \\
& a_k^l \geq 0, k = 1, 2, \ldots, s \text{ and } l = 1, 2, \ldots, c \\
& \eta_{l+1} = \eta_1 + (\eta_c - \eta_1) \times w_l, l = 1, \ldots, c-2 \\
& \sum_{l=1}^{c-1}\left\{p_l^{'} \times (\eta_l + \sum_{k=1}^{s}\frac{a_k^l}{s})\right\} + p_c^{'} \times \left\{\eta_c + \sum_{k=1}^{s}\frac{a_k^c}{(1-\alpha)s}\right\} \leq z \qquad (7.9) \\
& u_T \geq d \qquad (7.10) \\
& A = 0 \\
& X_i^j \geq 0 : j = 1, 2, \ldots, n \text{ and } i = 0, 1, \ldots, T-1
\end{aligned}$$

We can use the procedure outlined to solve this problem for different solutions. The priority vector $\boldsymbol{p}$=[0.1 0.2 0.7] and the weight vector $\boldsymbol{w}$ = [0.6] was used to solve the example. The minimum expected value $d_{\min}$ and maximum expected value $d_{\max}$ was found to be 0.7546 and 2.9272 respectively. Therefore expected value $d \in [0.7546, 2.9272]$. In this interval six equidistant expected values were chosen to solve our problem. For each of these values d^* we found the bound for $z \in [z_{d^*_{\min}}, z_{d^*_{\max}}]$ and solved the problem for four equidistant values in this interval. The problem is solved for different values of z and d^* so that constraints on UPDR (7.9) and expected value (7.10) are active. These solutions are plotted on a semivariance-UPDR space for each given expected value and is given in Figure 7.5. The corresponding solutions are given in Table 7.4.

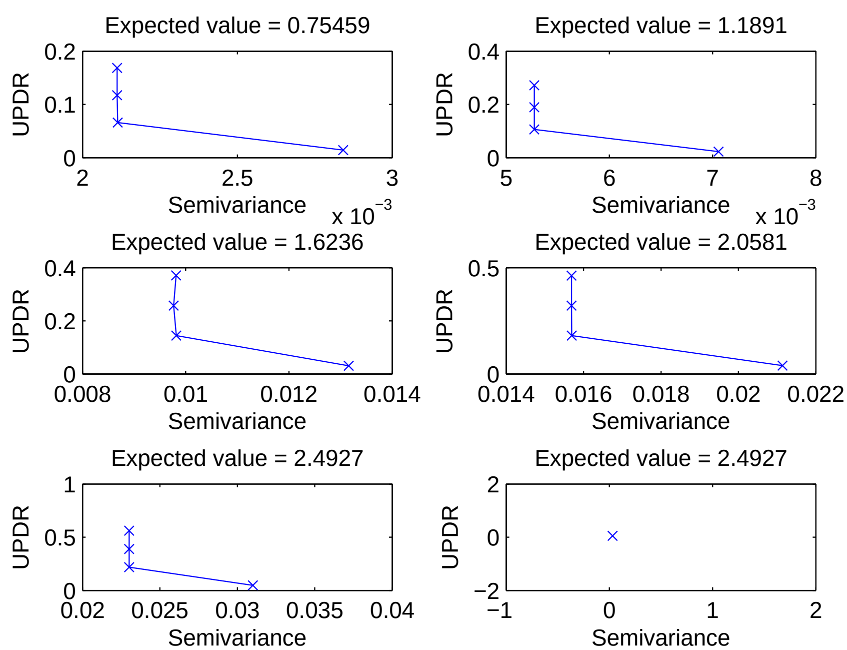

Figure 7.5 Efficient frontier of Mean-Semivariance-UPDR for Multi-Period.

Table 7.4 Some solutions of Mean-Semivariance-UPDR along with corresponding fractions to invest for different time periods.

Semivariance	CDaR	Expected value	Fraction to invest in nine securities for three different periods								
0.0028	0.0143	0.7546	0	0.313	0	0	0.2708	0	0.4162	0	0
			0	0	0	0	0.8532	0	0	0	0.2381
			0	0.214	0.1816	0	0.086	0	0.1777	0	0
0.0071	0.0225	1.1891	0	0.2701	0.0145	0	0.2578	0.0787	0.3002	0	0.0787
			0.1154	0.0972	0.1283	0.1016	0.1758	0.1322	0.0744	0.1341	0.1595
			0	0.3372	0.2862	0	0.1355	0	0.2801	0	0
0.0132	0.0308	1.6236	0.0023	0.1864	0.1034	0	0.1824	0.1243	0.1962	0.0808	0.1243
			0.1438	0.2633	0.0597	0.2345	0	0.0341	0.4125	0.0218	0
			0	0.4604	0.3908	0	0.185	0	0.3824	0	0
0.0211	0.039	2.0581	0.1917	0.0421	0.1095	0.2642	0.0453	0.0926	0.0341	0.1279	0.0926
			0.0962	0.3538	0	0.2916	0	0	0.6752	0	0
			0	0.5836	0.4954	0	0.2345	0	0.4847	0	0
0.031	0.0472	2.4927	0.252	0	0	0.748	0	0	0	0	0
			0	0.3718	0	0.2405	0	0	1.0511	0	0
			0	0.7068	0.6	0	0.2839	0	0.5871	0	0
0.2958	0.8384	2.9272	0	0	0	1	0	0	0	0	0
			0	0	0	0	0	0	1.714	0	0
			0	0	0	0	2.367	0	0	0	0

The Multi-period model gives the investor an investing scheme using which he can revise his portfolio multiple number of times in one holding period and aim for a better result. The main drawback is that the model requires exhaustive information about the future behavior of the returns, which may not be readily available. This can be overcome by using historical returns as representing the future.

The investor has four models using which he can make his investment decision. The four models Mean-SV-Absolute deviation, Mean-SV-CVaR, Mean-SV-CDaR and Mean-SV-UPDR were solved for the same numerical example and results were given in the previous sections. The solutions for the four models are different in that they invest different fractions in the nine securities. The investor can do a small study based on expected value and use these models and see their composition before making a decision on which model to use. On the other hand the investor can pick a model based on his risk measure preference. Investors would be better prepared if they conduct an exhaustive study using Multi-period model before deciding whether to use it or go for a Single-period model.

In the next chapter a new two-step process for portfolio selection is established and examples are illustrated for a sample of securities.

CHAPTER 8

SECURITY SELECTION

Investors have the choice of using one of the models we propose for portfolio selection. Our models require an input of projected future returns of securities which may not be readily available. In the absence of which the investor can use past historical returns as representing the future. This type of investing scheme does not include any fundamental information about the underlying securities. Value investing is a sophisticated type of investing tool which builds on the fundamental value of securities. In other words it aims to find securities which have strong return potential by analyzing the fundamentals of securities. Investors would get a better portfolio if fundamental information about the securities is also incorporated into the model.

Greenblatt (2006) explains how to use factors about the securities as a ranking mechanism to find the best subset of securities to invest in. He proposed two factors earnings yield and return on capital based on which the securities are ranked in ascending order. Any security which has a lower rank is better than one with a higher rank with respect to a particular factor. In the next step the sum of ranks of each security for the two factors is found. Finally the sum of ranks can be used to rank the securities in ascending order. The investor can then choose a subset of the securities which have low ranks and invest on these securities. Greenblatt (2006) showed that investing using this ranking scheme will lead to higher return.

On the lines of Greenblatt, we decided on certain factors which we feel adequately cover the basics of the securities and can easily be computed using freely available information. Though there are many factors which can be used, we limited our listing to three which we think adequately covers the fundamentals of the securities.

The investor can do the following two-step process once he has decided on a sample of securities in which he can invest in. In the first step consider the three factors we have proposed to rank the securities in ascending order. Then sum the ranks of the securities and rank the rank sum in ascending order. The investor can then decide on a small subset of the available securities based on this final rank choosing ones which have low ranks. The investor can decide either on the single-period or the multi-period model for investing and in the second step the investor can use any one of the four available models that meets his risk criteria and solve the model to find the fractions to invest in the securities.

8.1 Underlying factors about the securities

There are many underlying factors available about the securities which can be used for analysis. Since we wanted to rank securities based on these factors, only factors which can be used to compare two securities uniformly were included. We decided on three factors which we felt adequately covers the fundamentals of securities and solved our investment model. The investor can chose other factors if needed but the investment process would remain the same.

The first factor we considered was current Price to earnings to ratio (P/E). This represents how much one is willing to pay to get $1 from the company. This can be used as a standard to compare different companies from different markets since this ratio is independent of currency and always represents how much one is willing to pay for 1 unit of currency from the company. A smaller P/E is better than a larger P/E since it means for smaller investment one can get higher return from the company. The ranking will be in ascending order where lower rank is better than higher rank. The current P/E is usually calculated as a trailing P/E since the information used is based on the price of the security from last four quarters and is found by using the following formula:

$$\text{current P/E} = \frac{\text{current market value per share}}{\text{EPS}}$$

Here EPS represents the earnings per share and is usually calculated from the last four quarters. It is calculated by dividing the income from continuing operations by the average number of shares outstanding during the period. In the EPS calculation, it is more accurate to use a weighted average number of shares outstanding over the reporting term, because the number of shares outstanding can change over time. For example, assume that a company has a net income of \$10 million. If the company pays out \$1 million in preferred dividends and has 1 million shares for half the year and 2 million shares for the other half, EPS would be:

$$\$6 \quad = \quad \frac{10-1}{\frac{1+2}{2}}$$

Finally if two securities have a P/E of \$20 and \$30 respectively, it is preferable to invest in the first company since it requires lesser investment to get \$1 from the company. Based on the P/E the securities are ranked so that securities with smaller P/E have lesser rank than securities with larger P/E.

The second factor we considered was return on asset (ROA) which is a percentage and represents how much money is used to convert into profit. A security with a higher percentage ROA means the security uses lesser assets to create more income and hence the securities can be ranked higher to lower in percentage of ROA. It is usually found for the last year and is found by using the following formula:

$$\text{ROA} = \frac{\text{Net Income}}{\text{Total Assets}} \times 100$$

If two securities have ROA of 5% and 6% respectively, then it is preferable to invest in the second security since it requires lesser assets to create more income. After computing ROA for all securities they are ranked so that ones with larger ROA get a lesser rank than the securities with smaller ROA.

The final factor we used was dividend yield which measures how much a company pays out in dividend each year relative to its share price. A higher dividend yield is better than a lower yield and so the ranking goes from higher to lower. Dividend yield is calculated using the following formula:

$$\text{Dividend yield} = \frac{\text{Annual dividends per share}}{\text{current price per share}} \times 100$$

For example, if two companies A and B both pay annual dividends of $1 per share, but the stock prices of company A and B are $10 and $20 respectively. Then dividend yield for the two companies would be 10% and 5% respectively. Thus an investor would prefer company A to company B, since that would supplement his annual income more. So rank the securities for dividend yields so that securities that have larger yield get lesser rank than securities with smaller yield.

For the available sample of securities the investor can rank the securities for the factors. Then the investor can find the sum of these ranks and rank the securities based on this sum. The securities with low rank sum are better than the securities with high rank sum, hence the investor can choose securities which have low rank sum. Choosing a small subset of securities with low ranks will give the best securities with respect to the fundamental factors. Then the models we proposed can then be used to find the fraction to invest in these securities.

An analysis was done for thirty securities of the Dow Jones Industrial Average (NYSE: DJI). The Dow Jones Industrial Average is one of several stock market indexes created by nineteenth century Wall Street Journal editor and Dow Jones & Company co-founder Charles Dow. He compiled the index as a mathematical way to gauge the performance of the industrial component of America's stock markets. It is one of the oldest continuing market indexes and consists of thirty of the largest and most widely held public companies in the U. S. The DJI along with other indexes are used to represent the market as a whole. The reader is referred to the website of the New York Stock Exchange which has more exhaustive information on this index and its component securities.

The yearly data between 1990 and 2006 of the thirty securities was taken from Wharton Research Data Services (WRDS) which is available to access through Clemson University Library. The securities were ranked based on the three factors and were put in three groups. Group one for any year represents the top ten rank securities, group two for any year represents the middle ten rank securities and finally group three for any year represents the bottom ten rank securities. The cumulative returns for the three groups were plotted for sixteen years and is given in Figure 8.1.

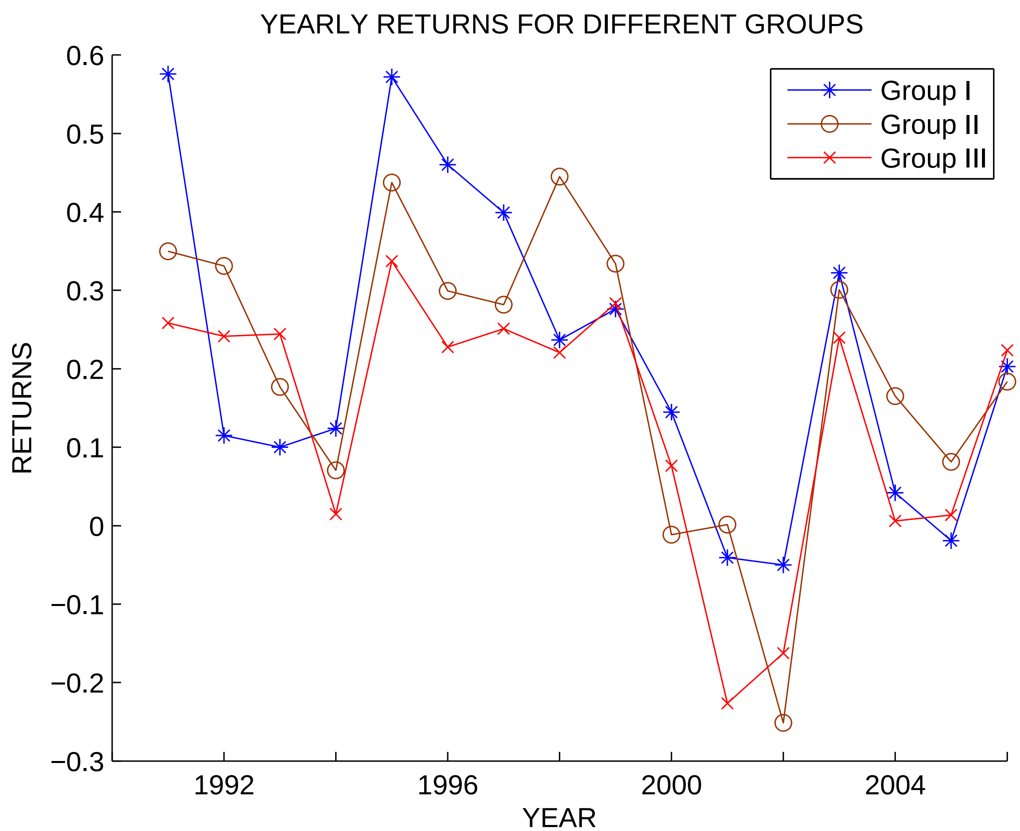

Figure 8.1 Yearly returns of three groups for sixteen years.

As can be clearly seen in Figure 8.1 group one gives the best return for eight years, group two gives the best return for six years and group three gives the best return for two years. Thus the investor would get a good pool of securities by using the ranking scheme and picking ones which have the best ranks.

8.2 Numerical Examples

We first applied our investing process to a set of securities from the U. S. stock market. Let us suppose the investor would like a small subset of securities to invest which is part of some index. Let us consider an investor whose investment pool consists of the thirty securities from DJI. Let us also suppose the investor has \$10,000 to invest. He has a pool of thirty securities available to him, but because of transaction costs a subset of ten securities is the maximum in which he can invest in. Let us use our ranking model for these thirty securities and rank them. Table 8.1 gives a rank of the thirty securities for various factors and also the final ranks of securities. We assumed the investor wants to invest in the beginning of 2007. All the ranks are done by using information from 2006.

Let us suppose the investor wants to invest for the single month of January 2007. Based on the ranking of securities the ten securities which have the lowest ranks are MMM, AA, MO, CAT, DD, XOM, HD, INTC, JNJ and JPM. The monthly returns of January 2003 through November 2006 are assumed as equally likely predictors for January 2007 and computed as follows:

price change = (closing price for current - previous period)

$$\text{return for current period} = \frac{\text{price change+dividends for current period}}{\text{closing price of the previous period}}$$

Let us also suppose the investor would like to invest based on all the four different models for single-period selection. Let $\alpha = 0.95$, $\boldsymbol{p} = [0.1\ 0.2\ 0.7]$ and $\boldsymbol{w} = 0.5$. The four single-period models were solved for an expected return of 2% and minimum possible risk measure (RM) value. The different solutions are given in Table 8.2. The closing price of December 2006 was taken as the buying price of the securities and three sell dates were considered January 30, January 31 and February 1 of 2007 and the closing prices of the securities on these days were used to calculate the actual return. The actual returns and the expected returns are plotted in Figure 8.2.

Table 8.1 Ranks of thirty NYSE securities.

Securities by Symbol	Rank by P/E	Rank by ROA	Rank by Div Yield	Sum of ranks	Final rank
MMM	10	6	14	30	4
AA	3	2	18	23	2
MO	14	3	5	22	1
AXP	7	24	25	56	23
AIG	1	28	28	57	25
T	22	23	4	49	17
BA	12	16	20	48	16
CAT	6	19	19	44	10
C	29	30	3	62	30
KO	26	5	16	47	14
DD	8	20	6	34	6
XOM	4	4	24	32	5
GE	13	25	8	46	12
GM	30	22	7	59	27
HPQ	15	11	30	56	24
HD	5	12	11	28	3
HON	21	17	22	60	29
INTC	19	8	15	42	8
IBM	11	14	26	51	19
JNJ	20	10	13	43	9
JPM	2	29	9	40	7
MCD	27	7	12	46	13
MRK	28	9	10	47	15
MSFT	17	1	27	45	11
PFE	23	26	1	50	18
PG	25	15	17	57	26
UTX	18	13	23	54	21
VZ	24	27	2	53	20
WMT	16	18	21	55	22
DIS	9	21	29	59	28

Table 8.2 Semivariance and risk measure value along with corresponding fractions to invest for Single-period Mean-SV-RM for NYSE securities.

Semivariance	RM	Fraction to invest in ten securities									
0.0005	0.0254	0.3798	0	0	0	0.1637	0	0.0757	0	0.1396	0.2412
0.0005	0.0321	0.2838	0	0	0	0.218	0	0.1856	0.0119	0.0693	0.2315
0.0005	0.0411	0.2539	0	0	0.0323	0.208	0	0.1412	0.0523	0.0416	0.2706
0.0005	0.027	0.2844	0	0	0	0.2177	0	0.187	0.0083	0.0708	0.2318

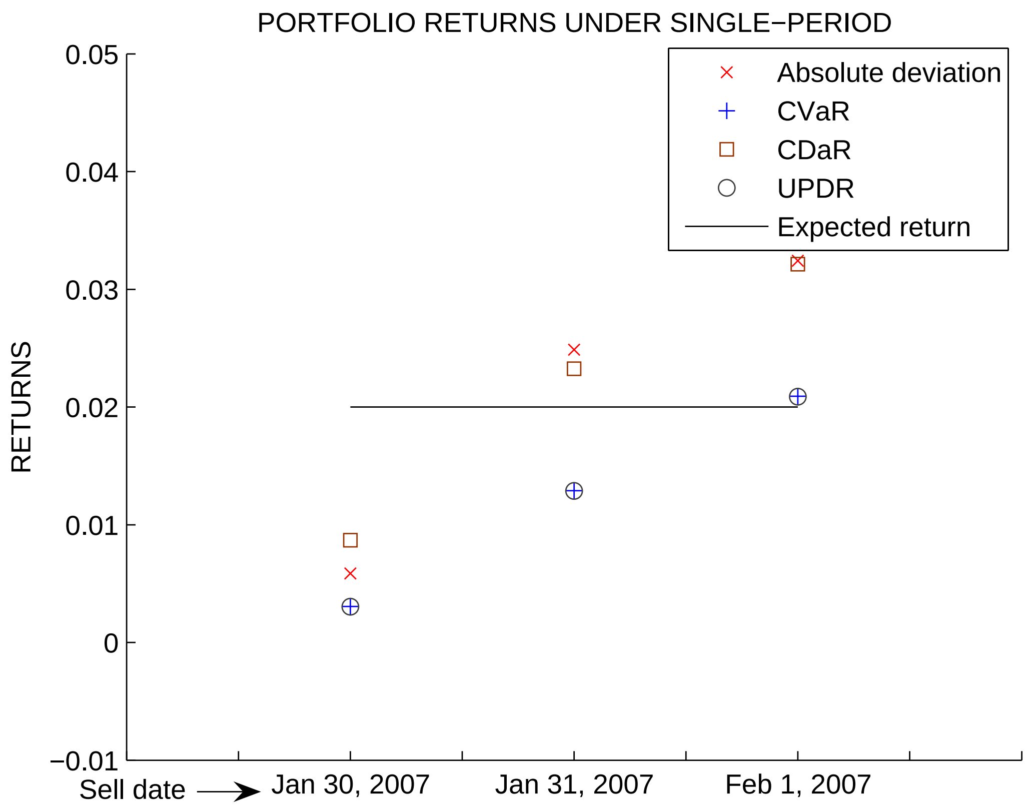

Figure 8.2 Expected returns and actual returns for NYSE securities using single-period Mean-SV-RM models.

All the four models for the three sell dates give a positive return and when the sell date is February 1 all the four models give a return better than what was expected. For all the sell dates the returns of CVaR and UPDR are close to each other. Absolute deviation and CDaR returns are close to each other with the former being better than the latter for the sell dates of January 31 and February 1.

Let us suppose the investor wants to consider multi-period portfolio selection. Let us suppose the investor has $10000 to invest and would like to invest for three consecutive months of January, February and March of 2007. The average returns of the four January and February months of 2003 through 2006 are taken as representing returns of January and

February of 2007 respectively. Finally the returns of January 2004 through November 2006 are taken as different projected return scenarios for March 2007. Since the ranking has been done before, we use same set of ten securities to invest in. With all this information the four multi-period models are solved with the condition that the risk measure (RM) values are at a minimum for an expected return of 5% so that the expected value (terminal wealth) is $10,500. The closing price of December 2006, January 2007, February 2007 and March 2007 are considered for the analysis. The actual returns and the solutions for the four models are given in Table 8.3.

Table 8.3 Example 1: Semivariance and risk measure value along with corresponding fractions to invest for Multi-period Mean-SV-RM for NYSE securities.

SV	RM	Fraction to invest in ten securities in three months										Actual return%
0.0001	0.0101	0.1006	0.0992	0.0995	0.1014	0.0984	0.0991	0.1011	0.1002	0.1014	0.0991	1.17
		0.0431	0.0726	0.1208	0.1474	0.1052	0.1124	0.118	0.0534	0.1364	0.1201	
		0.1227	0.0948	0.0878	0.0847	0.1314	0.0963	0.117	0.0834	0.0911	0.1229	
0.0002	0.0133	0.0963	0.1052	0.1034	0.091	0.11	0.1055	0.0929	0.0989	0.0911	0.1058	1.39
		0.0195	0.0608	0.128	0.1652	0.1063	0.1163	0.124	0.0339	0.1498	0.1271	
		0.2798	0	0.1247	0.0004	0.1097	0	0.0653	0	0.2778	0.18	
0.0002	0.0138	0.0971	0.1041	0.1027	0.093	0.1078	0.1043	0.0944	0.0991	0.0931	0.1045	1.59
		0.024	0.063	0.1266	0.1618	0.1061	0.1155	0.1229	0.0376	0.1473	0.1258	
		0.2476	0	0.0103	0	0.1177	0	0.1716	0.001	0.3149	0.1735	
0.0001	0.0114	0.0934	0.1093	0.106	0.0841	0.1177	0.1098	0.0873	0.098	0.0842	0.1102	0.96
		0.0037	0.0528	0.1328	0.1771	0.107	0.1189	0.1281	0.0208	0.1589	0.1317	
		0.2464	0	0.1956	0	0	0	0.0698	0	0.2244	0.305	

The actual returns were lower than the expected return for all the four models. UPDR gave the least return and CDaR produced the best return. We considered another multi-period example in which we invest in different months. Let us suppose the investor has \$10000 to invest and would like to invest for three consecutive months of November and December of 2006 and January 2007. The average returns of the three November and December months of 2003 through 2005 are taken as representing returns of November and December of 2006 respectively. Finally the returns of January 2005 through October 2006 are taken as different projected return scenarios for January 2007. Since the ranking has been done before, we use same set of ten securities to invest in. With all this information the four multi-period models are solved with the condition that the risk measure (RM) values are at a minimum for an expected return of 5%. The closing price of October 2006, November 2006, December 2006 and January 2007 are considered for the analysis. The actual returns and the solutions for the four models are given in Table 8.4.

Table 8.4 Example 2: Semivariance and risk measure value along with corresponding fractions to invest for Multi-period Mean-SV-RM for NYSE securities.

SV	RM	Fraction to invest in ten securities in three months										Actual returns%
0.0001	0.0101	0.1141	0.1119	0.0757	0.0999	0.1011	0.1159	0.1029	0.0715	0.096	0.1111	6.18
		0.1428	0.0692	0.1046	0.0649	0.1907	0.0371	0.1485	0	0.1301	0.1478	
		0.0036	0.0041	0.0501	0	0	0	0.0921	0	0.6913	0.2009	
0.0002	0.0133	0.1078	0.1066	0.0865	0.1	0.1006	0.1088	0.1016	0.0842	0.0978	0.1061	5.61
		0.1378	0.0748	0.1051	0.0712	0.1788	0.0474	0.1427	0.0069	0.1269	0.1421	
		0.2771	0	0.1236	0	0.1058	0	0.0672	0	0.2795	0.185	
0.0002	0.0137	0.1058	0.1049	0.0899	0.1	0.1004	0.1066	0.1012	0.0882	0.0983	0.1046	6.27
		0.136	0.0762	0.105	0.0728	0.1748	0.0503	0.1406	0.0119	0.1257	0.14	
		0.2298	0	0.015	0	0.0718	0	0.1749	0	0.3154	0.2298	
0.0002	0.0114	0.1125	0.1106	0.0784	0.0999	0.1009	0.1141	0.1026	0.0747	0.0964	0.1098	6.44
		0.1417	0.0708	0.1049	0.0667	0.1879	0.0399	0.1472	0	0.1295	0.1466	
		0.045	0	0.1963	0	0	0	0.0698	0	0.4243	0.3059	

The actual returns were higher than the expected return for all the four models. UPDR gave the highest return and CVaR produced the least return. Compared to the single-period model, multi-period model requires information farther into the future which may be difficult to obtain. Because of this drawback multi-period portfolio selection should be used only if the investor is confident of predicting farther into the future. As clearly seen by the two examples we illustrated, multi-period models should be handled with great caution because the models invest based on predictions farther into the future than single-period models.

Next we considered securities from the Bombay Stock Exchange (BSE) for analysis. The BSE is the main stock exchange in India and was started in 1875. One of the main indexes used in BSE is SENSEX, first compiled in 1986 comprises of thirty component securities representing a sample of large, well-established and financially sound companies. The base year of SENSEX is 1978-79. The reader is referred to the website of the Bombay Stock Exchange for a more detailed discussion of this index and its component securities.

Let us suppose the investor has 10000 rupees (Rs.) available to invest which at current market prices is approximately \$ 250. We consider an investor who would like to invest in a maximum of ten securities which are part of the index SENSEX. Let us use our ranking model for these thirty securities and rank them. Table 8.5 gives a rank of the thirty securities for various factors and also the final ranks of securities. The data to compute the ranks and returns was taken from the websites of Wharton Research Data Services (WRDS) and Mergent Online, which are available to access through the Clemson University library. These databases provide information which is more than one year old from a current date for securities from emerging markets. Hence we assumed the investor wants to invest in the beginning of 2006. All the ranks are done by using information from 2005. Only securities which were part of the index from December 2002 were included in the ranking and the other securities were assumed to have the worst ranks. We did this because the returns from 2003 to 2005 were used for the analysis.

Table 8.5 Ranks of thirty BSE securities.

Number	Securities by name	Rank by P/E	Rank by ROA	Rank by Div Yield	Final rank
1	ACC Ltd.	18	22	12	21
2	Ambuja Cements Ltd.	17	11	5	8
3	Bajaj Auto Ltd.	21	14	11	18
4	Bharat Heavy Electricals Ltd.	25	18	20	24
5	Bharti Airtel Ltd.	26	26	25	27
6	Cipla Ltd.	11	9	19	13
7	DLF Ltd.	28	28	28	28
8	Grasim Industries Ltd.	12	16	17	17
9	HDFC	22	27	7	22
10	HDFC Bank Ltd.	24	23	21	26
11	Hindalco Industries Ltd.	4	15	14	9
12	Hindustan Unilever Ltd.	23	3	3	4
13	ICICI Bank Ltd.	1	24	9	10
14	Infosys Technologies Ltd.	19	1	24	16
15	ITC Ltd.	5	6	8	3
16	Larsen & Toubro Ltd.	10	7	15	6
17	Mahindra & Mahindra Ltd.	8	17	6	5
18	Maruti Suzuki India Ltd.	2	10	26	12
19	NTPC Ltd.	29	29	29	29
20	ONGC Ltd.	7	8	1	2
21	Ranbaxy Laboratories Ltd.	27	20	10	23
22	Reliance Communications Ltd.	3	19	27	20
23	Reliance Energy Ltd.	20	21	23	25
24	Reliance Industries Ltd.	13	12	18	15
25	Satyam Computer Services Ltd.	14	5	16	11
26	State Bank of India	9	25	13	19
27	Tata Consultancy Services Ltd.	30	30	30	30
28	Tata Motors Ltd.	15	13	4	7
29	Tata Steel Ltd.	6	4	2	1
30	Wipro Ltd.	16	2	22	14

Securities 7, 18, 19, 22 and 27 are not considered for investing since they were not traded from December 2002 but were included in SENSEX some time later. So these securities have the worst ranks and will not get included in the best ten securities. Let us suppose the investor wants to invest for the single month of February 2006. Based on the ranking of securities the ten securities which have the lowest ranks are 2, 11, 12, 13, 15, 16 17, 20, 28 and 29. The monthly returns of January 2003 through December 2005 are assumed as equally likely predictors for February 2006 and computed as explained before. Let us also suppose the investor would like to invest based on all the four different models. The four single-period models were solved for an expected return of 2% and minimum possible risk measure (RM) value. The different solutions are given in Table 8.6. The closing price of January 2006 was taken as the buying price of the securities and three sell dates were considered, February 28, March 1 and March 2 of 2006, and the closing prices of the securities on these days were used to calculate the actual return. The actual returns and the expected returns are plotted in Figure 8.3.

Table 8.6 Semivariance and risk measure value along with corresponding fractions to invest for Single-period Mean-SV-RM for BSE securities.

Semivariance	RM	Fraction to invest in ten securities									
0.0026	0.0607	0.1407	0.0767	0.6542	0	0.0826	0	0	0	0	0.0458
0.0029	0.1061	0.174	0	0.826	0	0	0	0	0	0	0
0.0026	0.326	0.0593	0.1129	0.6204	0	0.207	0	0	0.0004	0	0
0.0026	0.0777	0.2995	0	0.676	0	0.0122	0	0	0.0123	0	0

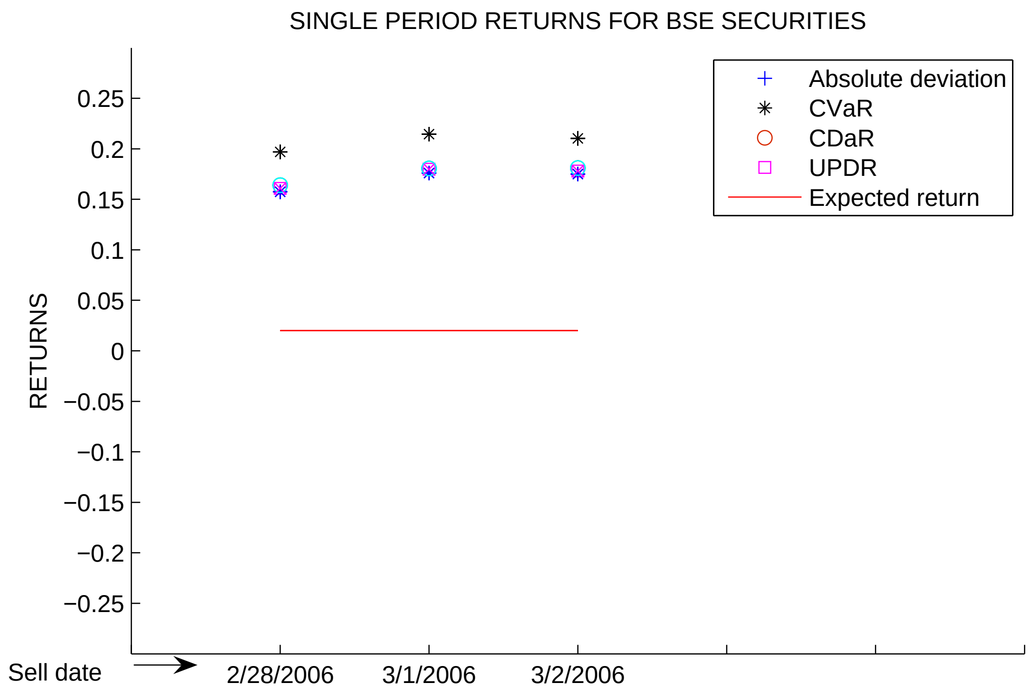

Figure 8.3 Expected returns and actual returns for BSE securities using single-period Mean-SV-RM models.

All the four models for the three sell dates give a very high return compared to the expected return. CVaR gives the best return for the three sell dates and the other models produce almost same returns. Since the returns were much higher than expected we further analyzed SENSEX. The following figure is a plot of the BSE SENSEX for different trading days between January 31 and March 2 of 2006, the buying and the final selling date considered for our analysis.

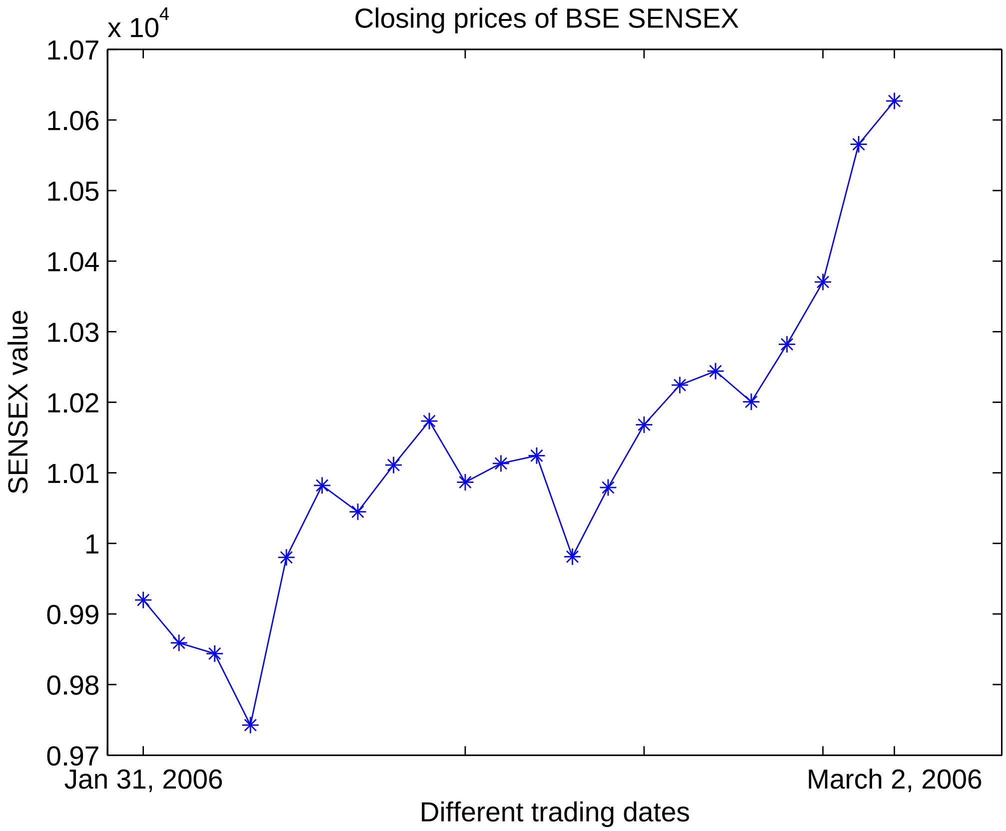

Figure 8.4 Plot of BSE SENSEX for different trading days.

The SENSEX had a good growth in the period of at least 6%. Since our model selects the best ten securities, as expected the returns were better than the overall SENSEX growth. The following figure plots the monthly returns for both the indexes between January 2003 and March 2006.

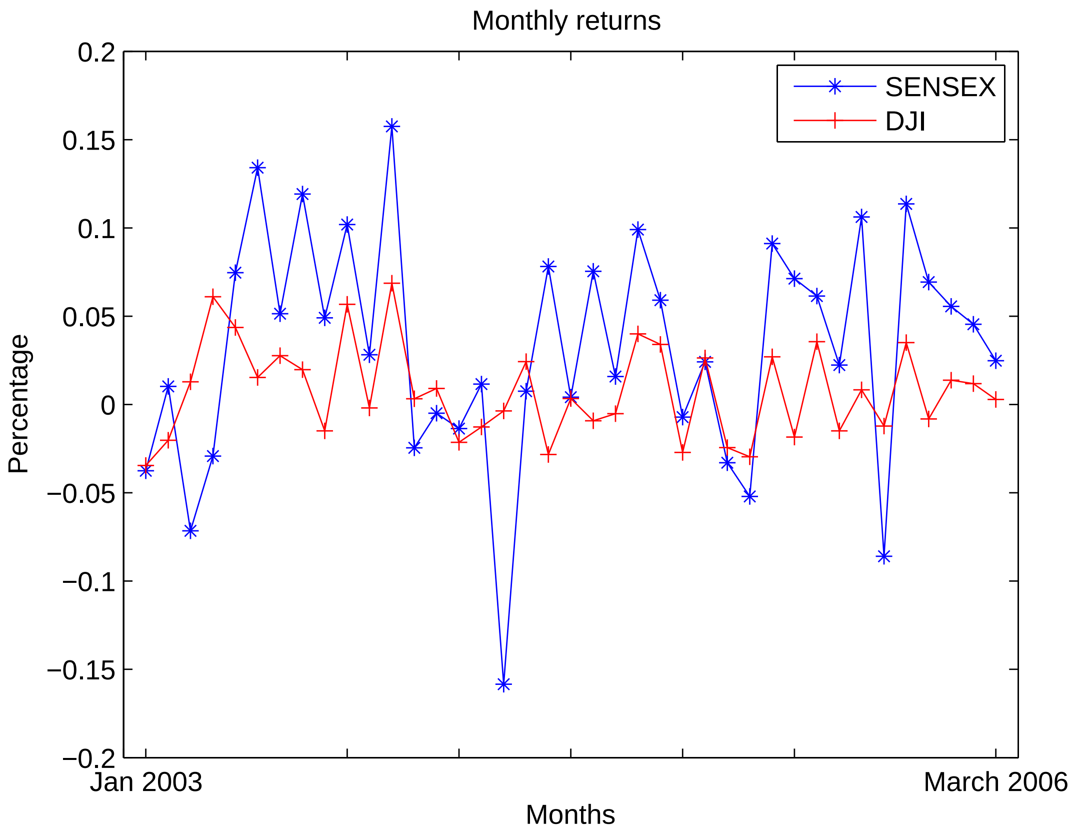

Figure 8.5 Monthly returns of SENSEX and DJI.

The SENSEX is very volatile as compared to the DJI as can be seen in the above figure. Since we could see the volatility very clearly we did not do further analysis. The emerging markets are not stable in general, hence we may not be able to emulate high returns with certainty. Therefore investing in emerging markets requires close scrutiny and better understanding of the underlying market. Since the returns were very volatile we did not attempt multi-period portfolio selection for the SENSEX securities.

The investor can do his investment decision based on the investment process we outlined. Once the investor decides on the securities and has a way to find projected future returns he can solve the models and find his investment fraction which would give the optimal return. Some of the issues the investor has to decide are the risk measure(s) he wants to use, the sample of securities and the maximum number of securities he wants in his final portfolio. He should also decide on single-period or multi-period model. Investing using this process would make sure he addresses the risk for his portfolio as well as include fundamental information about the securities. This will always fetch him the optimal portfolio.

In the next chapter we conclude the dissertation with a general discussion on the ideas proposed and how it can be easily enhanced by future work.

CHAPTER 9

CONCLUSIONS AND FUTURE WORK

The objective of our study was to study portfolio selection and enhance existing risk measures and portfolio selection models. In the present chapter, we summarize this study and discuss some directions for future research.

The first two chapters provided a brief introduction and literature review on portfolio selection. In the third chapter, we discussed some important risk measures currently available for portfolio selection. We believed that there was need for a new risk measure which would quantify risk in a more sophisticated fashion than the measures already available. A new risk measure Unequal Prioritized Downside risk (UPDR) was established in Chapter 4. This risk measure quantifies the possibilities of returns below the expected return based on a set of priorities given by the investor. This input of priorities can be used as a tool to develop varied portfolios. Since no two investors may believe in the same level of risk tolerance, their priorities will be different and hence each one will get a portfolio that directly meets their risk tolerance level. For a same expected return using UPDR two investors could get different portfolios based on prioritization which none of the measures currently available can emulate.

In the next part of our study, we considered single-period portfolio selection under multi-risk. In the past multi-risk portfolio selection has not been constructed in the context of semivariance. We presented four alternate models for portfolio selection and called them Mean-SV-RM models. Here RM represents the second risk measure which could be any one of Absolute deviation, CVaR, CDaR and UPDR. A procedure was outlined to solve these models. Next, we considered the four models when chance constraint is included and presented a procedure to solve the models. Numerical examples are illustrated for each of the models. These models enhance risk quantification and help in get a different perspective than the models currently available.

Portfolio selection is largely dependent on input and hence sensitivity analysis is of great concern for investors. In Chapter 6, we derive the problems to conduct sensitivity analysis for the four alternate models we proposed for single-period. Numerical examples are illustrated on how to conduct sensitivity analysis for changes in the input. The investor can conduct sensitivity analysis when needed to check the confidence in his portfolio composition.

In Chapter 7, we considered multi-period portfolio selection. There was a need to consider multi-period portfolio selection in a multi-risk context since this has not been dealt in the literature. We developed four models for multi-period portfolio selection and call them Mean-SV-RM models where RM represents any one of the following risk measures Absolute deviation, CVaR, CDaR and UPDR. We outlined a step by step procedure to solve these models and illustrate them with numerical examples. The multi-period models give the investor a multi-risk perspective which is not available in the literature.

In Chapter 8, we propose a two-step portfolio selection process. Most investors make their investment decision based on readily available information on securities. For these investors the models we developed serve as good tools. The two-step portfolio selection process builds on the fundamentals of securities and the sophisticated models we developed. The investor has to do the following before undertaking this process—decide on a set of securities as the investment pool, the amount of money to invest and the model to use. In the first step, three important fundamental information of the securities are used to rank the securities in decreasing order of investment preference. In the second step, select the best ranked securities and apply the decided model to handle portfolio selection. A portfolio built using this process would include fundamental information of securities as well as incorporate risk preference of the investor, leading to a better portfolio.

9.1 Future Work

The multi-period models we developed could not be solved using dynamic programming since semivariance is not separable. An enriching area to work could be to consider separable approximations for semivariance which would allow us to use dynamic programming techniques to solve the models. All the portfolio selection models we considered assume an arbitrary sell date for the securities. This may not lead to the best return making the study on when to sell securities an exciting problem.

Our methodology can handle chance constraint when portfolio returns are normally distributed. A more challenging and interesting problem would be the extension of the methodology to handle chance constraint without the assumption of normality of portfolio returns. Since increasing number of markets allow foreign investors, an interesting area of study could be comparing the models' performance with respect to securities from different markets of the world. Though we illustrated small examples using securities from NYSE and BSE, a more detailed study can be attempted to compare securities over a longer period of time and from different markets.

Many more areas could open for future work as our research progresses.

BIBLIOGRAPHY

Alexander, S., Coleman, T. F., and Li, Y. (2006). Minimizing CVaR and VaR for Portfolio of Derivatives. *Journal of Banking and Finance*, 30(2):538–605.

Aparicio, F. M. and Estrada, J. (2001). Empirical distributions of stock returns: European securities markets, 1990-95. *European Journal of Finance*, 7(1):1–21.

Artzner, P., Delbaen, F., and Eber, J. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Balzer, L. A. (1990). How to measure risk. In *Conference on Investment Performance, AIC Conferences*, Sydney, Australia.

Balzer, L. A. (1994). Measuring investment risk. *The Journal of Investing*, 3(3):47–58.

Bawa, V. S. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2(1):95–121.

Bawa, V. S. and Lindenberg, E. B. (1977). Capital market equilibrium in a mean-lower partial moment framework. *Journal of Financial Economics*, 5(2):189–200.

Best, M. J. and Grauer, R. R. (1991a). On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results. *Review of Financial Studies*, 4(2):315–342.

Best, M. J. and Grauer, R. R. (1991b). Sensitivity analysis for mean-variance portfolio problems. *Management Science*, 37(8):980–989.

Blattberg, R. C. and Gonedes, N. J. (1974). A comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business*, 47(2):244–280.

Byrne, P. and Lee, S. (2004). Different risk measures: Different portfolio compositions. *Journal of Property Investment and Finance*, 22(6):501–511.

Cai, X., Teo, K. L., Yang, X. Q., and Zhou, X. Q. (2004). Minimax portfolio optimization: empirical numerical study. *Journal of the Operations Research Society*, 55(1):65–72.

Charnes, A. and Cooper, W. W. (1959). Chance-constrained programming. *Management Science*, 6(1):73–79.

Chekhlov, A., Ursayev, S., and Zabarankin, M. (2000). Portfolio optimization with drawdown constraints. *Research Report 2000-5, ISE Department, University of Florida.*

De Giorgi, E. (2002). A note on portfolio selection under various risk measures. *Working Paper 122, Institute of Empirical Research in Economics, University of Zurich.*

Elton, E. J. and Gruber, M. J. (1997). Modern portfolio theory, 1950 to date. *Journal of Banking & Finance*, 21(11-12):1743–1759.

Fama, E. F. (1965). The behavior of stock prices. *Journal of Business*, 38(1):34–105.

Feiring, B. R. and Lee, S. W. (1996). A chance-constrained approach to stock selection in hong kong. *International Journal of Systems Science*, 27(1):33–41.

Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *The American Economic Review*, 67(2):116–125.

Gourieroux, C., Laurent, J. P., and Scaillet, O. (2000). Sensitivity analysis of Values at Risk. *Journal of Empirical Finance*, 7(3-4):225–245.

Greenblatt, J. (2006). *The Little Book That Beats the Market.* John Wiley & Sons, Inc., Hoboken, US.

Hagerman, R. (1978). More evidence on the distribution of security returns. *The Journal of Finance*, 33(4):1213–1221.

Jean, W. H. (1971). The extension of portfolio analysis to three or more parameters. *Journal of Financial and Quantitative Analysis*, 6(1):505–515.

Kibzun, A. I. and Kuznetsov, E. A. (2006). Analysis of criteria VaR and CVaR. *Journal of Banking & Finance*, 30(2):779–796.

Konno, H., Shirakawa, H., and Yamazaki, H. (1993). A mean-absolute deviation-skewness portfolio optimization model. *Annals of Operations Research*, 45(2):205–220.

Konno, H., Waki, H., and A., Y. (2002). Portfolio optimization under lower partial risk measures. *Asia-Pacific Financial Markets*, 9(2):127–140.

Konno, H. and Yamazaki, H. (1991). Mean-absolute deviation portfolio optimization model and its applications to tokyo stock market. *Management Science*, 37(5):519–531.

Krokhmal, P., Palmquist, J., and Ursayev, S. (2002). Portfolio optimization with conditional value at risk objective and constraints. *Journal of Risk*, 4(2):43–68.

Krokhmal, P., Ursayev, S., and Zrazhevsky, G. (2005). Numerical Comparison of CVaR and CDaR approaches: Application to Hedge Funds. *Applications of Stochastic Programming, MPS-SIAM Series on Optimization*, 1:609–631.

Lari-Lavassani, A. and Li, X. (2003). Dynamic mean semi-variance portfolio selection. *Lecture notes in computer science*, 2657:95–104.

Li, D., Chan, T., and Ng, W. (1998). Safety-first dynamic portfolio selection. *Dynamics of Continuous Discrete and Impulsive Systems*, 4(4):585–600.

Li, D. and Ng, W. (2000). Optimal dynamic portfolio selection: Multi-period mean-variance formulation. *Mathematical Finance*, 10(3):387–406.

Li, X., Y., Z. X., and Lim, A. E. B. (2001). Dynamic mean-variance portfolio selection with no-shorting constraints. *SIAM Journal on Control and Optimization*, 40(5):1540–1555.

Linsmeier, T. J. and Pearson, N. D. (2000). Value at Risk. *Financial Analysts Journal*, 56(2):47–67.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, 7(1):77–91.

Markowitz, H. (1959). *Portfolio Allocation: Efficient Diversification of Investments*. John Wiley & Sons, Inc., New York. A Cowles Foundation Monograph.

Markowitz, H. (1991a). Foundations of portfolio theory. *Journal of Finance*, 46(2):469–477.

Markowitz, H. (1991b). *Portfolio Allocation: Efficient Diversification of Investments*. Blackwell, Oxford, UK, second edition. reprinted in a 2^{nd} edition with Markowitz's hindsight comments on several chapters and with an additional bibliography supplied by Mark Rubenstein.

Markowitz, H., Todd, P., Xiu, G., and Yamane, Y. (1993). Computation of mean-semivariance efficient sets by the critical line algorithm. *Annals of Operations Research*, 45(1):307–317.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *Review of Economics and Statistics*, 51(2):247–257.

Nash, S. G. and Sofer, A. (1996). *Linear and Nonlinear Programming*. The McGraw-Hill Companies, Inc., New York.

Officer, R. R. (1972). The distribution of stock returns. *Journal of the American Statistical Association*, 67(340):317–335.

Papahristodoulou, C. and Dotzauer, E. (2004). Optimal portfolios using linear programming problems. *Journal of the Operations Research Society*, 55(11):1169–1177.

Pflug, G. C. (2000). Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk. *Probabilistic Constrained Optimization: Methodology and Applications*, 1:272–281.

Pyle, D. and Turnovsky, S. J. (1971). Risk aversion in chance constrained portfolio selection. *Management Science*, 18(3):218–225.

Rajan, M. and Gnanendran, K. (1998). Alternative risk measures in portfolio selection: Variance vs semivariance. In *Proceedings of the annual Meeting-Decision Sciences Institute*, volume 1, pages 259–261, San Diego, CA.

Rockafellar, R. T. and Ursayev, S. (2000). Optimization of Conditional Value-at-Risk. *The Journal of Risk*, 2(3):21–41.

Roman, D., Dowman, K. D., and Mitra, G. (2007). Mean risk models using two risk measures: A multi-objective approach. *Quantitative Finance*, 7(4):443 – 458.

Roy, A. D. (1952). Safety first and the holding of assets. *Econometrica*, 20(3):431–449.

Rubenstein, M. (2002). Markowitz's "portfolio selection" : A fifty-year perspective. *The Journal of Finance*, 57(3):1041–1045.

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics*, 51(3):239–2246.

Sharpe, W. F. (1963). A simplified model for portfolio analysis. *Management Science*, 9(2):277–293.

Sharpe, W. F. (1967). A linear programming algorithm for mutual fund portfolio selection. *Management Science*, 13(7):499–510.

Sharpe, W. F. (1971). A linear programming approximation for the general portfolio analysis problem. *Journal of Financial and Quantitative Analysis*, 6(5):1263–1275.

Stone, B. (1973). A general class of three-parameter risk measures. *The Journal of Finance*, 28(3):675–685.

Tang, W., Han, Q., and Li, G. (2001). The portfolio selection problems with chance-constrained. In *Systems, Man, and Cybernetics IEEE International Conference on*, volume 4, pages 2674–2679.

Tobin, J. (1958). Liquidity preference as behavior toward risk. *Review of Economic Studies*, 26(2):65–86.

Tobin, J. (1965). *"The Theory of Portfolio Selection" The Theory of Interest Rates*. Hahn, F. H. and Brechling, F. P. R. (eds.), MacMillan, London, UK.

Vorst, T. (2000). Optimal Portfolios Under a Value at Risk Constraint. In *European Congress of Mathematics*, volume 2, pages 1317–1334, Barcelona, Spain.

Wang, J. (2000). Mean-Variance-VaR Based Portfolio Optimization. www.gloriamundi.org.

Wei, S. and Ye, Z. (2007). Multi-period optimization portfolio with bankruptcy control in stochastic market. *Applied Mathematics and Computation*, 186(1):414–425.

Young, M. R. (1998). A minimax portfolio selection rule with linear programming solution. *Management Science*, 44(5):673–683.

Zhou, X. Y. and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Journal Applied Mathematics and Optimization*, 42(1):19–33.